NPS55-80-011

NAVAL POSTGRADUATE SCHOOL

Monterey, California

A NEW AUTOREGRESSIVE TIME SERIES MODEL IN EXPONENTIAL VARIABLES (NEAR(1))

by

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and

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March 1980

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Prepared for:

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, Ca. 93940

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1. REPORT NUMBER: NPS55-80-011	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): A New Autoregressive Time Series Model in Exponential Variables (NEAR(1))	5. TYPE OF REPORT & PERIOD COVERED: Technical
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): A. J. Lawrance and P. A. W. Lewis	8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Naval Postgraduate School, Monterey, CA 93940	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61152N, RR000-01-10, N0001480WR00054
11. CONTROLLING OFFICE NAME AND ADDRESS: Naval Postgraduate School, Monterey, CA 93940	12. REPORT DATE: March 1989
	13. NUMBER OF PAGES: 39
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Autoregressive model in exponential variables; Negative correlation; Cross-coupled processes; Antithetic variables; Correlated uniform process; Time series; Point process; Simulation.

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20. Abstract Cont.

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A NEW AUTOREGRESSIVE TIME SERIES MODEL

IN EXPONENTIAL VARIABLES

(NEAR(1))

by

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SUMMARY

A new time series model for exponential variables having first order autoregressive structure is presented. Unlike the recently studied standard autoregressive model in exponential variables (EAR(1)), runs of constantly scaled values are avoidable, and the two parameter structure allows some adjustment of time nonreversibility effects in sample path behavior. The model is further developed by the use of cross-coupling and antithetic ideas to allow negative dependency. Joint distributions and autocorrelations are investigated. A transformed version of the model has a uniform marginal distribution and its correlation and regression structures are also obtained. Estimation aspects of the models are briefly considered.

KEYWORDS: Autoregressive model in exponential variables; Negative correlation; Cross-coupled processes; Antithetic variables; Correlated uniform process: Time series; Point process; Simulation.

1. INTRODUCTION

In this paper we begin by introducting a new two-parameter model, to be called NEAR(1), first mentioned in Lawrance (1979), for a first-order autoregressive time series with exponentially distributed marginals. The model is a first-order Markov process. Suitably choosing one of the parameters as a function of the other produces a one-parameter first-order autoregressive process which can give any value of the lag one autocorrelation between zero and one. One particular model produced in this way is the EAR(1) model introduced by Gaver and Lewis (1980); this model had the problem that a "zero-defect" caused successive values of the process to be, at times, fixed multiples of the previous values. The NEAR(1) model does not have this defect except for the EAR(1) special case and thus seems much more suitable than the EAR(1) model for the modelling of real data. In addition, the fact that there are two parameters indexing the dependency structure of the model allows one to consider sample path behavior as well as the customary fitting of the first and second order moments to the data. The model is defined in Section 2.

At another extreme from the EAR(1) model, a one-parameter model (TEAR(1)) is produced which is much easier to extend to higher order autoregressive structures than is the EAR(1) model (Lawrance and Lewis, 1980). However while it has no zero defect, this TEAR(1) model produces realizations which, for high serial correlation, tend to run up most of the time; for the general NEAR(1) model these aspects can be adjusted. A one-parameter model which can mimic some of the time-reversible character of normal AR(1) processes is produced from the NEAR(1) model by requiring either that the probability of a jump up from one value to the next be one-half or requiring that the first directional moments be equal. A property which the NEAR(1) model does not share with its special EAR(1) case is additivity, so that extensions to Gamma marginals are not automatic; other marginal distributions are possible with the NEAR(1) structure but these are not discussed here.

An important property of the NEAR(1) models is that they are simple random linear combinations of independent exponential variables and therefore easy to simulate. This simplicity is bought at the price of autocorrelations which are nonnegative.

The second thrust of the paper concerns alternation and negativity of autocorrelations; this will be achieved by a scheme coupling two antithetic NEAR(1) sequences, a scheme introduced by Gaver and Lewis (1980) for the negatively correlated EAR(1) process. The resulting model, to be called the NEARA(1), includes both the NEAR(1) and hence TEAR(1) as special cases; it has autocorrelations which alternate into negativity under a geometrically decaying envelope. However, simulation of the negatively dependent models involves random linear combinations from independent pairs of negatively dependent exponential variables, and this can be complicated. Most developments in the paper are undertaken for the general NEARA(1) model, and further detailing of results are given separately for the positive and negative dependency cases. In particular, the paper deals with the allowable range of lag one autocorrelations, lag r bivariate distributions, exponentiation of the models to have uniform marginal distribution, and aspects of time reversibility, sample path behavior and estimation.

Simulation aspects of the models are discussed in Lawrance and Lewis (1980); detailed graphical representations of different sample path behaviors are also given there.

2. CONSTRUCTION OF THE MODELS

The conventional linear autoregressive model (AR(1)) with exponential(λ) marginal distributions (Gaver and Lewis, 1980) takes the form

$$X_n = \rho X_{n-1} + \begin{cases} 0 & \text{w.p. } \rho \\ E_n & \text{w.p. } 1-\rho \end{cases} \qquad n = 0,1,2,\ldots , \tag{2.1}$$

where ρ is a parameter $(0 \leq \rho < 1)$ and the E_n, $n = 0,1,2,\ldots$ are independent exponential variables with parameter $\lambda > 0$. This EAR(1) model has serial correlations of order r, $\rho_r = \text{corr}(X_n, X_{n+r})$, given by ρ^r and generates sample paths in which large values are followed by runs of falling values with geometrically distributed run-length. The large values arise when E_n is included, while the falling values stem from the selection in (2.1) giving only $X_n = \rho X_{n-1}$. This behavior is likely to limit the broad applicability of the model, although it can be overcome the more complicated moving-average and mixed moving average-autoregressive developments (Lawrance and Lewis, 1977, 1980a; Jacobs and Lewis, 1977).

An alternative exponential first-order autoregressive Markov model is obtained by interchanging the independent and identically distributed variables X_{n-1} and E_n in (2.1); this can have no effect on the exponential(λ) marginal distribution of X_n's. Proceeding this way, with ρ replaced by $1-\alpha$, we have the model

$$X_n = (1-\alpha)E_n + \begin{cases} X_{n-1} & \text{w.p. } \alpha \\ 0 & \text{w.p. } 1-\alpha \end{cases} \qquad n = 0.k,2,\ldots . \tag{2.2}$$

This exponential AR(1) model, called TEAR(1), is again Markovian and has the α^r correlation structure of the EAR(1) model; it is, as will be shown later, particularly tractable analytically. The characteristic behavior of realizations generated by this model (particularly distinct when α is large) is that of runs of rising values (with geometrically distributed run length) when the selection $(1-\alpha)E_n + X_{n-1}$ is being made, followed by a sharp fall when the selection $(1-\alpha)E_n$ is made without inclusion of the previous value. Illustrations of these effects both for the EAR(1) and TEAR(1) models are given in the simulations of Fig. 1a and Fig. 1b. These simulated sample paths use the same simulated exponential error sequence $\{E_n\}$.

Broader behavior in realizations generated by an exponential model can be obtained from the model in which the X_{n-1} of (2.2) is scaled by a coefficient β. This gives the proposed NEAR(1) model (Lawrance, 1980) as

$$X_n = \varepsilon_n + \begin{cases} \beta X_{n-1} & \text{w.p. } \alpha \\ 0 & \text{w.p. } 1-\alpha \end{cases} \qquad n = 0,1,2,\ldots , \tag{2.3}$$

where the existence and distribution of the i.i.d. $\{\varepsilon_n\}$ sequence which makes the X_n's in the stationary case have exponential(λ) distributions, needs to be established afresh. We now show that ε_n must have a particular mixed exponential distribution.

Let the Laplace-Stieltjes transforms of the X and ε variables be denoted by

$$\phi_X(s) = E\{e^{-sX}\} \quad \text{and} \quad \phi_\varepsilon(s) = E\{e^{-s\varepsilon}\} . \tag{2.4}$$

Then (2.3) gives, if we assume stationarity,

$$\phi_\varepsilon(s) = \frac{\phi_X(s)}{\alpha\phi_X(\beta s) + (1-\alpha)} = \frac{\lambda + \beta s}{\lambda + s}\,\frac{\lambda}{\lambda + (1-\alpha)\beta s}\ , \qquad (2.5)$$

on using $\phi_X(s) = \lambda/(\lambda+s)$. Thus, providing α and β are not both equal to one, ε_n can be generated from an E_n by the exponential mixture

$$\varepsilon_n = \begin{cases} E_n & \text{w.p. } \dfrac{1-\beta}{1-(1-\alpha)\beta} \\ (1-\alpha)\beta E_n & \text{w.p. } \dfrac{\alpha\beta}{1-(1-\alpha)\beta} \end{cases} \qquad n = 0,1,\ldots\ . \qquad (2.6)$$

When $\alpha = 0$ or $\beta = 0$ the $\{X_n\}$ are exponential i.i.d., whereas with $\alpha = 1$ the EAR(1) model (2.1) is obtained with $\rho_1 = \beta$. When $\beta = 1$ the TEAR(1) model is obtained. Thus the two-parameter exponential, first-order, autoregressive Markov NEAR(1) model can be expected, for fixed serial correlation of lag 1, $\rho_1 = \alpha\beta$, to model broader behavior than is obtained in the extreme cases ($\alpha = 1$ or $\beta = 1$). In particular α and β can be chosen to produce both runs of ascending and descending values, intermediate to the profiles of EAR(1) and TEAR(1) models, as was illustrated in Fig. 1a or Fig. 1b. Figure 1c represents an intermediate case which will be discussed in Section 8. Note that the correlation is the same, 0.75, in all three figures.

It is also clear from the Markovian nature of the model (i.e. that conditional on $X_{n-1} = x_{n-1}$ the distribution of subsequent values $X_n, X_{n+1}, \ldots$ is independent of $X_{n-2}, X_{n-3}, \ldots$), that if X_0 is exponential(λ) and independent of $E_1, E_2, \ldots$, then the process X_n, $n = 1,2,\ldots$ is stationary. Note too that the NEAR(1) model is, by definition, explicitly (physically) autoregressive and thus not only autoregeressive in the sense that $E(X_n | X_{n-1} = x)$ is a linear function of x.

We note too that the NEAR(1) model gives a solution to the (random) stochastic difference equation

$$X_n = A_n X_{n-1} + B_n , \qquad n = 0,1,2,\ldots \tag{2.7}$$

discussed by Vervaat (1979) in which $A_n = \beta$ w.p. α and $A_n = 0$ w.p. $(1-\alpha)$: Vervaat's paper discusses questions of existence and infinite divisibility applying to the model (2.7).

In the NEAR(1) model the parameters α and β are nonnegative. Therefore the autocorrelations $\rho_k = (\alpha\beta)^k$ are positive and geometrically decreasing. This is unlike the standard AR(1) model with, say, normal marginals, where ρ_1 can be negative, so that the autocorrelations can alternate between positive and negative values with a geometrically decreasing envelope. To extend the exponential models to the situation where there is a possibility of alternation in the autocorrelations and negative correlation requires some sacrifice of simplicity. As noted in Section 1, the primary idea here is to cross-couple two sequences $\{X_n\}$ and $\{X'_n\}$ with identically exponentially distributed marginal distributions across an independent bivariate sequence $\{\varepsilon_n, \varepsilon'_n\}$ of negatively correlated and marginally identical variables. This final development produces our so-called NEARA(1) model; it is specified by the equations

$$X_n = \varepsilon_n + \beta V_n X'_{n-1} , \qquad V_n = \begin{cases} 1 & \text{w.p. } \alpha \\ 0 & \text{w.p. } 1-\alpha, \end{cases}$$

$$n = 0,1,2,\ldots \tag{2.8}$$

$$X'_n = \varepsilon'_n + \beta V'_n X_{n-1} , \qquad V'_n = \begin{cases} 1 & \text{w.p. } \alpha \\ 0 & \text{w.p. } 1-\alpha \end{cases}$$

where the serially independent binary pairs V_n and V'_n generally have negative dependency. Some insight into the model comes from seeing that X_n is positively dependent on X'_{n-1}; this is negatively dependent on X_{n-1}, so making X_n and X_{n-1} negatively dependent. Though defined compactly in terms of the two processes, the interest here is in the marginal process X_n. A univariate description of X_n is possible and given at equation (3.2).

The special case of the bivariate sequences $\{\varepsilon_n, \varepsilon'_n\}$ and $\{V_n, V'_n\}$ in which $\varepsilon_n = \varepsilon'_n$ and $V_n = V'_n$ recovers the NEAR(1) model. The special case when $\beta = 1$ will be called the TEARA(1) model.

Detailed aspects of the sequence $\{X_n\}$ depend on the joint distributions of $\{\varepsilon_n, \varepsilon'_n\}$ and $\{V_n, V'_n\}$, though the marginal distributions of ε_n and ε'_n must be as at (2.6) for X_n to be marginally exponential. For instance for the TEARA(1) model, strongest alternation in serial correlations is obtained when the $\{\varepsilon_n, \varepsilon'_n\}$ are maximally negatively correlated exponential variables and therefore are antithetic pairs, and similarly for the binary pairs $\{V_n, V'_n\}$. For the broader NEARA(1) model, (2.8), negatively correlated mixed exponential variables $\{\varepsilon_n, \varepsilon'_n\}$ are required. Some of these aspects of the model are explored in general and for specific $\{\varepsilon_n, \varepsilon'_n\}$ and $\{V_n, V'_n\}$ distributions in Sections 4 and 5. In this respect this paper extends results and details for the negatively correlated EAR(1) model given by Gaver and Lewis (1980).

Note that while $\{X_n, X'_n\}$ is a bivariate Markovian model, the full Markovian property of X_n individually is lost unless it reduces to the NEAR(1) model; that is to be expected from the cross-dependency built into the model.

3. AUTOCORRELATION STRUCTURE OF THE MODEL

The simple autocorrelation structure of the NEARA(1) model, as given at (2.8) by

$$X_n = \varepsilon_n + \beta V_n X'_{n-1} , \qquad X'_n = \varepsilon'_n + \beta V'_n X_{n-1} \tag{3.1}$$

is best approached by recursively expressing the dependency of X_n on either X'_{n-1}, X_{n-2}, X'_{n-3}, X_{n-4}, ... , and so on. Directly from (3.1) it can be noted that the distribution of (X_n, X_{n-1}) is simply expressed in terms of the distribution of (X_{n-1}, X'_{n-1}): in fact, this latter joint distribution, equivalently (X_n, X'_n), plays a central role in the process. However, substituting for X'_{n-1} in the first equation (3.1) from the second, gives

$$X_n = \varepsilon_n + \beta V_n \varepsilon'_{n-1} + \beta^2 V'_{n-1} V_n X_{n-2} . \tag{3.2}$$

Hence the joint distribution of (X_n, X_{n-2}) does not need to be expressed in terms of (X_n, X'_n) and this is very convenient. Generally, there is this distinction between the odd-r and even-r cases of (X_n, X_{n-r}). This is shown in the following key expressions which are obtained by repeated substitutions;

$$X_n = \varepsilon_n + \beta V_n \varepsilon'_{n-1} + \beta^2 V'_{n-1} V_n \varepsilon_{n-2} + \cdots + (\beta^{r-1} V'_{n-r+2} \cdots V'_{n-1} \varepsilon_{n-r+1})$$

$$+ (\beta^r V_{n-r+1} \cdots V'_{n-1} V_n X'_{n-r}); \qquad (r \text{ odd}) \tag{3.3}$$

$$X_n = \varepsilon_n + \beta V_n \varepsilon'_{n-1} + \beta^2 V'_{n-1} V_n \varepsilon_{n-2} + \cdots + (\beta^{r-1} V_{n-r+2} \cdots V'_{n-1} V_n \varepsilon'_{n-r+1})$$

$$+ (\beta^r V'_{n-r+1} \cdots V'_{n-1} V_n X_{n-r}) . \qquad (r \text{ even}) \tag{3.4}$$

The autocovariances of $\{X_n\}$ follow easily from (3.3) and (3.4). On noting that the indicator variables $\{V_i\}$ occur independently in the products, we have

$$E(V'_{n-r+1} \cdots V'_{n-1} V_n) = \alpha^r \tag{3.5}$$

and hence

$$\mathrm{Cov}(X_n, X_{n-r}) = \begin{cases} (\alpha\beta)^r \ \mathrm{Var}(X_{n-r}) & (r \ \text{even}) \\ (\alpha\beta)^r \ \mathrm{Cov}(X_{n-r}, X'_{n-r}) & (r \ \text{odd}) \ . \end{cases} \tag{3.6}$$

In terms of correlations, this central result becomes

$$\mathrm{Corr}(X_n, X_{n-r}) = \begin{cases} (\alpha\beta)^r & (r \ \text{even}) \\ (\alpha\beta)^r \ \mathrm{Corr}(X_n, X'_n) & (r \ \text{odd}) \ . \end{cases} \tag{3.7}$$

Alternation of these autocorrelations under a geometric envelope is evident; negativity of the odd lag correlations requires the negativity of $\mathrm{Corr}(X_n, X'_n)$. For the simpler NEAR(1) model in which $X_n = X'_n$ the Markov $(\alpha\beta)^r$ correlation structure is evident.

For the NEARA(1), an investigation of $\mathrm{Corr}(X_n, X'_n)$ is required. To this end, multiply together the respective sides of the two equations (3.1), giving

$$X_n X'_n = \varepsilon_n \varepsilon'_n + \beta V'_n \varepsilon_n X_{n-1} + \beta V_n \varepsilon'_n X'_{n-1} + \beta^2 V_n V'_n X_{n-1} X'_{n-1} \tag{3.8}$$

and take expectations. Let

$$\epsilon = \mathrm{Cov}(\varepsilon_n, \varepsilon'_n) \qquad \text{and} \qquad \nu = \mathrm{Cov}(V_n, V'_n) \tag{3.9}$$

and assume stationarity. Then following from (3.8) there is the result

$$\mathrm{Corr}(X_n, X'_n) = (\epsilon + \beta^2 \nu)/\{1 - (\alpha^2 + \nu)\beta^2\}. \tag{3.10}$$

The important conclusion is that maximum negativity of $\mathrm{Corr}(X_n, X'_n)$ is obtained, for any fixed values of α and β, for maximum negative correlations within the pairs $(\varepsilon_n, \varepsilon'_n)$ and (V_n, V'_n). The proof is omitted. Obtaining this maximum negative correlation by the use of antithetic variables is developed in the next section.

4. ANTITHETIC ASPECTS OF THE MODEL.

It is simplest to deal first with the binary (V_n, V'_n) variables; the basic antithetic idea is to relate the distribution of V_n to a monotonic transformation of a uniform variable U on $(0,1)$; then V'_n is the same transformation of $1 - U$ which also has a $(0,1)$ uniform distribution. The variables V_n and V'_n are then maximally negatively correlated. Thus, we define

$$\left\{\begin{array}{lll} V_n = 1 & \text{if} & U_n \leq \alpha \\ V_n = 0 & \text{if} & U_n > \alpha \end{array}\right\}, \quad \left\{\begin{array}{lllll} V'_n = 1 & \text{if} & 1-U_n \leq \alpha & \text{or} & U_n \geq 1-\alpha \\ V'_n = 0 & \text{if} & 1-U_n > \alpha & \text{or} & U_n < 1-\alpha \end{array}\right\}.$$

The resulting joint distribution takes one of two forms, as given below, depending on whether $\alpha < 1/2$ or $\alpha > 1/2$:

$V_n =$	1	0	V'_n
$V'_n = 1$	0	α	α
0	α	$1-2\alpha$	$1-\alpha$
V_n	α	$1-\alpha$	1

$(\alpha \leq 1/2)$

$V_n =$	1	0	V'_n
$V'_n = 1$	$2\alpha-1$	$1-\alpha$	α
0	$1-\alpha$	0	$1-\alpha$
V_n	α	$1-\alpha$	1

$(\alpha \geq 1/2)$

(4.1)

The resulting covariances are $\nu = -\alpha^2$ for $0 \leq \alpha \leq 1/2$ and $\nu = (1-\alpha)^2$ for $1/2 \leq \alpha < 1$; the corresponding correlations are thus $-\alpha/(1-\alpha)$ if $0 \leq \alpha \leq 1/2$ and $-(1-\alpha)/\alpha$ if $1/2 \leq \alpha < 1$. The $\alpha = 1$ case is exceptional and is excluded since it leads to the negatively correlated EAR(1) model treated in Gaver and Lewis (1980).

Next we consider how to obtain a bivariate mixed exponential distribution for $(\varepsilon_n, \varepsilon'_n)$ having maximum negative dependency. In the case of positive continuous random variables, the maximum negative correlation is obtained by the antithetic pair (Moran, 1967). However, with $(\varepsilon_n, \varepsilon'_n)$ having mixed exponential marginals, the full antithetic distributions cannot be obtained explicitly since the inverse distribution function $F^{-1}(\cdot)$ cannot be obtained explicitly. An alternative way of obtaining negatively correlated $(\varepsilon_n, \varepsilon'_n)$ begins by noting that

$$\varepsilon_n = K_n E_n \qquad \text{and} \qquad \varepsilon'_n = K'_n E'_n \tag{4.2}$$

where marginally, from (2.6),

$$K_n,\ K_n' = \begin{cases} 1 & \text{w.p.} \quad (1-\beta)/\{1-(1-\alpha)\beta\} \\ (1-\alpha)\beta & \text{w.p.} \quad \alpha\beta/\{1-(1-\alpha)\beta\}\ , \end{cases} \tag{4.3}$$

and E_n, E_n' are exponential(λ) variables, marginally. The dependency of ε_n and ε_n' is then given by

$$\mathrm{Cov}(\varepsilon_n,\varepsilon_n') = (\lambda^{-2} + C_E)C_K + [E(K)]^2\ C_E\ , \tag{4.4}$$

where C_K is the covariance of K_n and K_n' and C_E is the covariance of E_n and E_n'. Although any negatively correlated bivariate exponential can be used for (E,E'), the most negative correlation is attained in the (degenerate) antithetic case. The antithetic choice for (K_n,K_n'), whose distribution follows (4.1) with α replaced by $(1-\beta)/\{1-(1-\alpha)\beta\}$, does not involve degeneracy. These antithetic choices should give a negatively correlated mixed exponential pair $(\varepsilon_n,\varepsilon_n')$ whose correlation is almost as negative as the true antithetic bivariate mixed exponential pair. Note that the distribution of this bivariate mixed exponential pair $(\varepsilon_n,\varepsilon_n')$ is a little complicated in view of the break in form of the antithetic distribution of the binary pair (K_n,K_n') at $(1-\beta)/\{1-(1-\alpha)\beta\} = 1/2$ or $\beta = 1/(1+\alpha)$. Covariance calculations using (4.4) then give the result

$$\lambda^{-2}\mathrm{Cov}(\varepsilon_n,\varepsilon_n') = \begin{cases} (1-2\alpha\beta)(1-\pi^2/6) - (\alpha\beta)^2 & \text{for } \beta < 1/(1+\alpha) \\ (1-\alpha)\beta(2-\alpha\beta-\beta)(1-\pi^2/6)-(1-\beta)^2 & \text{for } \beta > 1/(1+\alpha). \end{cases} \tag{4.5}$$

This expression will now be used in determining explicit results for the first autocorrelation of the NEARA(1) model. Other less degenerate negatively correlated exponential random variables can be used; the simplest and most easily utilized one is given by Gaver (1972).

5. THE FIRST AUTOCORRELATION

The first autocorrelation of the NEARA(1) model can now be obtained, and its range of values will be determined, both generally and in the $\beta = 1$ case, the so-called TEARA(1) model. Interest is in the degree to which negativity can be attained, bearing in mind that with exponential marginal distributions there is a theoretical lower bound of $(1-\pi^2/6) = -0.6449$ on the correlation. From (3.7), (3.10) and (4.1) we have

$$\rho_1 = \mathrm{Corr}(X_n, X_{n-1}) = \begin{cases} \alpha\beta\epsilon - (\alpha\beta)^3 & \text{for } 0 \le \alpha \le 1/2 \\ \alpha\beta\{\epsilon - (1-\alpha)^2\beta^2\}/\{1-(2\alpha-1)\beta^2\} & \text{for } 1/2 \le \alpha < 1. \end{cases} \tag{5.1}$$

This result is combined with ϵ from (4.5) to give the most general expression

$$\rho_1 = \begin{cases} \alpha\beta(1-2\alpha\beta)(1-\pi^2/6) - 2(\alpha\beta)^3, & 0 \le \alpha \le 1/2,\ \beta < 1/(1+\alpha) \\ \alpha\beta^2(1-\alpha)(2-\alpha\beta-\beta)(1-\pi^2/6) - \alpha\beta(1-\beta)^2 - (\alpha\beta)^3, & 0 \le \alpha \le 1/2,\ \beta > 1/(1+\alpha) \\ \{\alpha\beta(1-2\alpha\beta)(1-\pi^2/6) - (\alpha\beta)^3 - \alpha(1-\alpha)^2\beta^3\}/\{1-(2\alpha-1)\beta^2\} & \\ & 1/2 \le \alpha < 1,\ \beta < 1/(1+\alpha) \\ \{\alpha(1-\alpha)\beta^2(2-\alpha\beta-\beta)(1-\pi^2/6) - \alpha\beta(1-\beta)^2 - \alpha(1-\alpha)^2\beta^3\}/\{1-(2\alpha-1)\beta^2\} & \\ & 1/2 \le \alpha < 1,\ \beta > 1/(1+\alpha). \end{cases} \tag{5.2}$$

It is worth stressing that the negativity of ρ_1 implies the negativity of $\mathrm{Corr}(X_n, X_n')$; then by virtue of the general result (3.7) there is strong alternation in the autocorrelations which parallels the usual $\{\rho_1^r\}$ Markov correlation structure when ρ_1 is negative. However, in general the marginal NEARA(1) process is not a first-order Markov process.

6. THE LAG ONE JOINT DISTRIBUTION

Following on from the first autocorrelation, the full joint distribution of (X_n, X_{n-1}) is of interest in describing the process and matching it with data. This joint distribution can be obtained from the NEARA(1) model equations (2.8) with the use of Laplace-Stieltjes transforms; thus

$$\phi_{X_n, X_{n-1}}(s,t) = E\{\exp(-SX_n - tX_{n-1})\} \tag{6.1}$$

$$= E\{\exp[-s(\varepsilon_n + \beta V_n X'_{n-1}) - tX_{n-1}]\}$$

$$= E\{\exp(-s\varepsilon_n - tX_{n-1} - \beta s V_n X'_{n-1})\}. \tag{6.2}$$

Writing $\phi_{\varepsilon_n}(s)$ for $E\{\exp(-s\varepsilon_n)\}$ and taking expectations with respect to V_n, we have

$$\phi_{X_n, X_{n-1}}(s,t) = \phi_\varepsilon(s)\{\alpha\phi_{X,X'}(t,\beta s) + (1-\alpha)\phi_X(t)\}. \tag{6.3}$$

The suffix n has been dropped from the right-hand side of (6.3) in view of the stationary assumption; again the joint distribution of (X,X') is required. However, when the simpler NEAR(1) model allowing only positive dependency is considered so that formally $X = X'$, there is the simpler result

$$\phi_{X_n, X_{n-1}}(s,t) = \phi_\varepsilon(s)\ \{\alpha\phi_X(\beta s + t) + (1-\alpha)\ \phi_X(t)\}. \tag{6.4}$$

where $\phi_X(t) = \lambda/(\lambda+t)$. This can be inverted but the overall behavior can be seen immediately. With probability $1-\alpha$ there is a scatter of values $X_n = \varepsilon_n$ independent of the X_{n-1} variable, where as with probability α, $X_n = \varepsilon_n + \beta X_{n-1}$ and is always above the line $X_n = \beta X_{n-1}$.

Returning now to the NEARA(1) model, the joint distribution of (X,X') is required. By constructing Laplace-Stieltjes transforms from each side of the model equations (2.8), it follows that

$$\begin{aligned}\phi_{X_n, X'_n}(s,t) &= E\{\exp[-s(\varepsilon_n + \beta V_n X'_{n-1}) - t(\varepsilon'_n + \beta V'_n X_{n-1})]\} \\ &= \phi_{\varepsilon,\varepsilon'}(s,t)\ E\{\exp(-\beta t V'_n X_{n-1} - \beta s V_n X'_{n-1})\} \quad .\end{aligned} \tag{6.5}$$

Now the joint distribution of (V_n, V'_n) is available from (4.1) and so

$$\phi_{X_n, X'_n}(s,t)$$
$$= \phi_{\varepsilon,\varepsilon'}(s,t) \begin{cases} 1 - 2\alpha + \alpha\phi_X(\beta s) + \alpha\phi_X(\beta t), & 0 \le \alpha \le 1/2 \\ (2\alpha-1)\phi_{X_{n-1}, X'_{n-1}}(\beta t, \beta s) + (1-\alpha)\ \phi_X(\beta s) + (1-\alpha)\ \phi_X(\beta t) & \\ & 1/2 \le \alpha < 1\ . \end{cases} \tag{6.6}$$

It is seen that, for $0 \le \alpha \le 1/2$, $\phi_{X_n, X'_n}(s,t)$ is immediately available in terms of $\phi_{\varepsilon,\varepsilon'}(s,t)$ whereas for $1/2 \le \alpha < 1$ a recursive calculation is required. This simplifies somewhat if it can be assumed that the joint distribution of $(\varepsilon,\varepsilon')$ and hence the joint distribution of (X_n, X_n') are symmetric in s and t; there would be no point in assuming otherwise for univariate modelling of $\{X_n\}$. A certain amount of calculation then gives the final form of (6.6) as

$$\phi_{X_n,X_n'}(s,t) = \begin{cases} [(1-2\alpha)+\alpha\{\phi_X(\beta s)+\phi_X(\beta t)\}]\phi_{\varepsilon,\varepsilon'}(s,t), & 0 \leq \alpha \leq 1/2 \\ (1-\alpha)\sum_{j=0}^{\infty}(2\alpha-1)^j[\phi_X(\beta^{j+1}s)+\phi_X(\beta^{j+1}t)]\prod_{i=0}^{j}\phi_{\varepsilon,\varepsilon'}(\beta^i s,\beta^i t), & 1/2 \leq \alpha < 1. \end{cases} \quad (6.7)$$

The series here can be summed in the $\beta = 1$, TEARA(1) case when the joint distribution of $(\varepsilon,\varepsilon')$ is a bivariate exponential. In the NEARA(1) case, the bivariate distribution which has been proposed at (4.2) gives

$$\phi_{\varepsilon,\varepsilon'}(s,t) = E\{\exp(-sK_nE_n - tK_n'E_n')\} \quad (6.8)$$

and this can be expressed in terms of the joint Laplace-Stieltjes transform $\phi_{E,E'}(s,t)$ of the underlying bivariate exponentials. Thus

$$\begin{aligned} \phi_{\varepsilon,\varepsilon'}(s,t) &= 2\,\frac{1-\beta}{1-(1-\alpha)\beta}\{\phi_{E,E'}(s,(1-\alpha)\beta t)+\phi_{E,E'}((1-\alpha)\beta s,t)\} \\ &\quad + \left\{1-2\,\frac{1-\beta}{1-(1-\alpha)\beta}\right\}\phi_{E,E'}((1-\alpha)\beta s,\,(1-\alpha)\beta t)\,, \quad (6.9) \\ &\qquad \beta < 1/(1+\alpha) \end{aligned}$$

and

$$\phi_{\varepsilon,\varepsilon'}(s,t) = 2\,\frac{1-\beta}{1-(1-\alpha)\beta}\;\phi_{E,E'}(s,t)$$

$$+\left\{1 - 2\,\frac{1-\beta}{1-(1-\alpha)\beta}\right\}\;[\phi_{E,E'}(s,(1-\alpha)\beta t) + \phi_{E,E'}((1-\alpha)s,t)],$$

$$\beta > 1/(1+\alpha). \tag{6.10}$$

The joint Laplace-Stieltjes transform of the distribution of (E,E') in the antithetic case is given, with U a uniform (0,1) random variable, by

$$\phi_{E,E'}(s,t) = E\{\exp[s \log U + t \log (1-U)]\} = \int_{u=0}^{1} u^s(1-u)^t \, du \tag{6.11}$$

which is a Beta function.

Both regressions from (X_n,X_{n-1}) of the NEARA(1) model are non-linear; directly from the model equations (2.8), the forward conditional expectation is

$$E(X_n|X_{n-1} = x) = (1-\alpha\beta)\lambda^{-1} + \alpha\beta E(X'_{n-1}|X_{n-1} = x) . \tag{6.12}$$

The regression on the right-hand side is complicated but can be obtained in the $\beta = 1$ TEARA(1) case. With positive dependency only, (6.12) applies for the NEAR(1) with formally $X'_{n-1} = X_{n-1}$, and so there is linear regression in this case with

$$E(X_n|X_{n-1} = x) = (1-\alpha\beta)\lambda^{-1} + \alpha\beta x. \tag{6.13}$$

7. THE (X_n,X_{n-r}) JOINT DISTRIBUTIONS; THE $\Sigma_1^r X_{n-1}$ SUM DISTRIBUTIONS

These distributions follow directly from the basic expressions (3.3) and (3.4) where X_n is expressed in terms of X_{n-r} or X'_{n-r}. Expectations are taken over the independent V_n, V'_{n-1}, ... in turn, with the following results

$$\phi_{X_n, X_{n-r}}(s,t) = E\{\exp(-sX_n - tX_{n-r})\}$$

$$= \alpha^r \prod_{i=0}^{r-1} \phi_\varepsilon(\beta^i s) \begin{cases} \phi_{X,X'}(t,\beta^r s) & r \text{ odd} \\ \phi_X(\beta^r s+t) & r \text{ even} \end{cases} + \sum_{j=0}^{r-1} \alpha^j(1-\alpha) \prod_{i=0}^{j} \phi_\varepsilon(\beta^i s)\, \phi_X(t). \quad (7.1)$$

In the $\beta = 1$, TEARA(1) case, there is the more explicit expression,

$$\phi_{X_n, X_{n-r}}(s,t) = \alpha^r [\phi_E\{(1-\alpha)s\}]^r \begin{cases} \phi_{X,X'}(s,t) & r \text{ odd} \\ \phi_X(s+t) & r \text{ even} \end{cases}$$

$$+ (1-\alpha)\, \phi_E\{(1-\alpha)s\}\, \phi_X(t) \frac{1 - [\alpha\phi_E\{(1-\alpha)s\}]^r}{1 - \alpha\phi_E\{(1-\alpha)s\}}. \quad (7.2)$$

In Section 8 these expressions are used to derive the autocorrelations of the sequence after transformation to a uniform (0,1) marginal distribution.

The distribution of the sums $\Sigma_{i=1}^{r} X_{n-i}$ can in principle be obtained from the expressions (3.3) and (3.4) in a similar way; for instance,

$$X_n + X_{n-1} = \varepsilon_n + X_{n-1} + \beta V_n X'_{n-1} \quad (7.3)$$

$$X_n + X_{n-1} + X_{n-2} = \varepsilon_n + \varepsilon_{n-1} + \beta V_n \varepsilon'_{n-1} + (1 + \beta^2 V'_{n-1} V_n) X_{n-2} + \beta V_n X'_{n-2}. \quad (7.4)$$

Generating functions for these two sums can be written down, but the results get progressively more complicated. There does not appear to be any simple general result, even with the NEAR(1) model.

8. RUN PROBABILITIES AND A PARTIALLY REVERSIBLE PROCESS, PREAR(1)

We have already indicated in Figures 1a, 1b, 1c that the sample path behavior of NEAR(1) processes can be distinctive, and is adjustable through the two parameters α and β. This distinctive behavior makes the model very rich and is principally observed as runs of increasing values (up-runs) or runs of decreasing values (down-runs) or both (peaks). Such behavior is not possible with Gaussian AR(1) models. In the discussion which follows we will explain the parameterization of the process illustrated in Figure 1c, which exhibits a partial time reversibility.

A simple quantification of sample path behavior is given by $P(X_n < X_{n-1})$, which is related to the average length of up-run sequences. Calculation of $P(X_n < X_{n-1})$ follows from (2.3) as

$$\begin{aligned} P(X_n < X_{n-1}) &= (1-\alpha)\, P(X_{n-1} > \varepsilon_n) + \alpha P(X_{n-1} > \varepsilon_n + \beta X_{n-1}) \\ &= (1-\alpha)\, P(X_{n-1} > \varepsilon_n) + (1-\alpha)\, P(X_{n-1} > \varepsilon_n/(1-\beta)) \; . \end{aligned} \tag{8.1}$$

By using the definition of ε_n given at (2.6) and the independence of X_{n-1} and ε_n, the probabilities in (8.1) are easily calculated and give

$$\begin{aligned} P(X_n < X_{n-1}) &= \frac{1-\alpha}{1-(1-\alpha)\beta}\left[\frac{1-\beta}{2} + \frac{\alpha\beta}{1+(1-\alpha)\beta}\right] \\ &\quad + \frac{\alpha}{1-(1-\alpha)\beta}\left[\frac{1}{1+(1-\beta)^{-1}} + \frac{\alpha\beta}{1+(1-\beta)^{-1}(1-\alpha)\beta}\right] \end{aligned} \tag{8.2}$$

$$= \frac{(1-\alpha)(1+\beta)}{2[1+(1-\alpha)\beta]} + \frac{\alpha(1-\beta)}{(2-\alpha)(1-\alpha\beta)} \; . \tag{8.3}$$

For the TEAR(1) process this probability (with $\beta = 1$) reduces to $(1-\alpha)/(2-\alpha)$ and is thus always less than one-half, so indicating an excess of up-runs; this is clearly illustrated in Figure 1b. A grid of values of this probability for $\alpha,\beta = 0.0(0.1)1.0$ is given in Table 2.

The asymmetry of up-run and down-run sequences for most NEAR(1) processes is evidence enough of their irreversibility in time. The value of $P(X_n < X_{n-1})$ and its difference from one-half gives one measure of this; another possible measure could be based on the difference between the directional correlations $\mathrm{Corr}(X_n, X_{n-1}^2)$ and $\mathrm{Corr}(X_n^2, X_{n-1})$; from (2.3) these may straightforwardly be obtained as

$$\mathrm{Corr}(X_n, X_{n-1}^2) = \alpha\beta \tag{8.4}$$

$$\mathrm{Corr}(X_n^2, X_{n-1}) = \alpha\beta(1 - \alpha\beta + 2\beta) \;. \tag{8.5}$$

The equality of these two correlations suggests one definition of partial reversibility, and for NEAR(1) processes gives the condition $\beta = 1/(2-\alpha)$. The simulations in Figure 1c are for this parametrization. Another partial characterization of time reversibility would simply be that $P(X_n < X_{n-1}) = 1/2$; surprisingly, for NEAR(1) processes, this second definition also leads to the condition $\beta = 1/(2-\alpha)$. Hence we shall refer to the NEAR(1) process with $\beta = 1/(2-\alpha)$ as the <u>partially reversible</u> or PREAR(1) process. It is not fully reversible, even as far as the joint distribution of (X_n, X_{n-1}) is concerned, but it seems somewhat remarkable that it is reversible in both the run-probability and directional-correlation aspects.

9. TRANSFORMATION TO A MULTIPLICATIVE PROCESS WITH UNIFORM MARGINALS

One useful aspect of exponential processes is that they provide a suitable base from which to transform to other processes of positive variables; they are particularly convenient for transforming to a multiplicative uniform process; thus the transformed process $\{\exp(-\lambda X_n)\}$ is now considered, with derivations of the autocorrelations and autoregressions.

When X_n has an exponential marginal distribution with parameter λ, the variable $U_n = \exp(-\lambda X_n)$ has a uniform $(0,1)$ marginal distribution. The autocorrelations of the $\{U_n\}$ sequence are easily obtained from the joint Laplace-Stieltjes transform of the joint distribution of (X_n, X_{n-r}); thus

$$\begin{aligned}\mathrm{Corr}(U_n, U_{n-r}) &= \{E(U_n U_{n-r}) - 1/4\}/(1/12) \\ &= 12E\{\exp[-\lambda(X_n + X_{n-r})]\} - 3 \\ &= 12\phi_{X_n, X_{n-r}}(\lambda,\lambda) - 3\ . \qquad (9.1)\end{aligned}$$

Working from (7.1) a reasonably explicit result for (9.1) is obtained; the first expression to be considered is

$$12\alpha^r \prod_{i=0}^{r-1} \phi_\varepsilon(\beta^i\lambda)\ ,$$

where $\phi_\varepsilon(s)$ is given by (2.5). After some cancellations, we get

$$12\alpha^r \prod_{i=0}^{r-1} \phi_\varepsilon(\beta^i\lambda) = 6\alpha^r(1+\beta^r) \prod_{i=1}^{r} \{1 + (1-\alpha)\beta^i\}^{-1}\ . \qquad (9.2)$$

The second term of (7.1) and (9.1) is

$$12(1-\alpha) \sum_{j=0}^{r-1} \alpha^j \prod_{i=0}^{j} \phi_\varepsilon(\beta^i\lambda)\, \phi_X(\lambda) - 3$$

$$= 3(1-\alpha) \sum_{j=0}^{r-1} \alpha^j (1+\beta^{j+1}) \prod_{i=0}^{j} \{1 + (1-\alpha)\beta^{i+1}\}^{-1} - 3 \,. \tag{9.3}$$

This does not look promising, at least not until the $j = 0$ term is taken out and combined with the -3; the expression then becomes

$$3(1-\alpha) \sum_{j=1}^{r-1} \alpha^j (1 + \beta^{j+1}) \prod_{i=0}^{j} \{1 + (1-\alpha)\beta^{i+1}\}^{-1} - 3\alpha\{1 + (1-\alpha)\beta\}^{-1} \,.$$

Next the term $j=1$ is taken out and combined with the last term; this yields

$$3(1-\alpha) \sum_{j=2}^{r-1} \alpha^j (1+\beta^{j+1}) \prod_{i=0}^{j} \{1+(1-\alpha)\beta^{i+1}\}^{-1} - 3\alpha^2 \prod_{i=0}^{1} \{1 + (1-\alpha)\beta^{i+1}\}^{-1} \,.$$

Continuing in this fashion gives the final expression

$$-3\alpha^r \prod_{i=1}^{r} \{1 + (1-\alpha)\beta^i\}^{-1} \,.$$

Bringing together (9.1), (7.1), (9.2) and (9.4) gives

$$\operatorname{Corr}(U_n,U_{n-r}) = \left[6\alpha^r(1+\beta^r) \begin{Bmatrix} \phi_{X,X'}(\lambda,\beta^r\lambda) & r \text{ even} \\ 1/(2+\beta^r) & r \text{ odd} \end{Bmatrix} - 3\alpha^r \right] \prod_{i=1}^{r} \{1+(1-\alpha)\beta^i\}^{-1}$$

$$= \begin{cases} 3\{2(1+\beta^r)\phi_{X,X'}(\lambda,\beta^r\lambda)-1\} \displaystyle\prod_{i=1}^{r} \left(\frac{\alpha}{1+(1-\alpha)\beta^i}\right) & (r \quad \text{odd}) \\ \dfrac{3}{2+\beta^r} \displaystyle\prod_{i=1}^{r} \left(\frac{\alpha\beta}{1+(1-\alpha)\beta^i}\right) & (r \text{ even}). \end{cases} \tag{9.5}$$

This is the required result; it is computationally explicit in several cases: the $\beta = 1$ TEARA(1) model, the NEARA(1) model for $0 \leq \alpha \leq 1/2$ and the NEAR(1) model for the full parameters region. This latter model has as its transformed autocorrelation function

$$\operatorname{Corr}(U_n,U_{n-r}) = \frac{3}{2+\beta^r} \prod_{i=1}^{r} \left(\frac{\alpha\beta}{1+(1-\alpha)\beta^i}\right) \quad , \quad r = 1,2,\ldots \tag{9.6}$$

The only case of (9.5) which is not available in closed form is the NEARA(1) model for $1/2 \leq \alpha \leq 1$. The series expansion from (6.7) for $\phi_{X,X'}(\lambda,\beta^r\lambda)$ would require detailed examination; the lower bound of the $r=1$ case would be interesting.

We now derive the forward regression $E(U_n|U_{n-1})$ of the variables in this uniform process; it has previously been remarked, equation (6.13), that for exponential NEAR(1) variables this is linear. As for the auto-correlations of transformed exponential processes, equation (9.1), a a general result is available. Without going into details, this can be written

$$E[\exp(-\lambda X_n)|X_{n-1} = x]^* = \lambda^{-1} \phi_{X_n,X_{n-1}}(\lambda,s-\lambda) \ , \tag{9.7}$$

where asterisk denotes Laplace-Stieltjes transform with respect to x of argument s. Inversion in the NEAR(1) case gives the desired result

$$E(U_n | U_{n-1} = u) = \frac{1}{2} \frac{1+\beta}{1+(1-\alpha)\beta} (1 - \alpha + \alpha u^{\beta}) . \tag{9.8}$$

As to be expected it is non-linear. The corresponding backward regression is also available from (9.7).

Finally, we note that the results from Section 8 on run behavior apply here since the transformation used is monotonic; in particular the uniform process is reversible in its run behavior under the condition $\beta = 1/(2-\alpha)$. However, reversibility of the directional correlations will not be achieved under this condition. The directional correlations can be obtained by similar methods to those used to obtain the ordinary correlations.

10. ASPECTS OF ESTIMATION

Formal methods of estimation are rather intractable with the NEAR(1) models: as an illustration, in the NEAR(1) case with just one observation x_1 after the initial value x_0, the likelihood takes the form

$$L(\alpha,\beta;x_1,x_0) = (1-\alpha)\ f_{\varepsilon}(x_1) + \begin{cases} \alpha f_{\varepsilon}(x_1-\beta x_0), & \beta < x_1/x_0 , \\ 0 & \beta > x_1/x_0 , \end{cases} \tag{10.1}$$

where $f_{\varepsilon}(.)$ is the mixed exponential pdf of the independent ε variables given at (2.6). With more observations, the full likelihood becomes, in view of the first order Markov structure of this model, the product of similar terms. The maximization needs to be done numerically and because of singularities in the parameter space, the standard asymptotic theory of maximum likelihood is inapplicable. A discussion of the problems in the $\alpha = \beta$ case of the NEAR(1) model is given by Raftery (1979). When $\alpha = 1$ the estimator proposed in Gaver and Lewis (1980) is the maximum likelihood estimate (personal communcation from G. Weiss). In this section we limit ourselves to ad hoc possibilities for estimation when $\alpha \neq 1$.

The method of moments can be developed for the NEAR(1) model: use can be made of the directional correlations (8.4) and (8.5). The product $\alpha\beta$ in (8.4) is best estimated by the first serial correlation, rather than the sample directional coorelations. Then using the sample directional correlation based on (8.4) an estimate of β can be obtained and hence an estimate of α. Methods of improving the efficiency of these moment estimates are being studied. Use of the run probability given by (8.3) is also a possible tool for estimation.

11. FURTHER DEVELOPMENTS

Further work on this topic is being directed at the estimation, simulation and sample path aspects. Extensions of the model to mixed exponential variables are also being developed.

ACKNOWLEDGMENTS

Dr. P. A. W. Lewis was supported by the United States Office of Naval Research (Grant NR-42-284) and in part by the Naval Postgraduate School Foundation. Dr. A. J. Lawrance was partially supported by the Naval Postgraduate School Foundation.

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TABLE 1. FIRST AUTOCORRELATIONS ρ_1 FROM EQUATION (5.2) FOR THE NEARA(1) MODEL WITH ANTITHETIC EXPONENTIAL AND INDICATOR VARIABLES AS A FUNCTION OF α AND β

α \ β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	-0.0063	-0.0124	-0.0182	-0.0239	-0.0293	-0.0345	-0.0395	-0.0444	-0.0491	-0.0532
0.2	-0.0124	-0.0239	-0.0345	-0.0444	-0.0536	-0.0623	-0.0705	-0.0784	-0.0845	-0.0906
0.3	-0.0182	-0.0345	-0.0491	-0.0623	-0.0745	-0.0860	-0.0971	-0.1006	-0.1134	-0.1218
0.4	-0.0239	-0.0444	-0.0623	-0.0784	-0.0934	-0.1081	-0.1234	-0.1327	-0.1430	-0.1569
0.5	-0.0293	-0.0536	-0.0745	-0.0934	-0.1119	-0.1314	-0.1494	-0.1626	-0.1805	-0.2056
0.6	-0.0344	-0.0618	-0.0842	-0.1038	-0.1225	-0.1427	-0.1524	-0.1602	-0.1737	-0.1974
0.7	-0.0394	-0.0694	-0.0929	-0.1126	-0.1316	-0.1502	-0.1486	-0.1480	-0.1535	-0.1727
0.8	-0.0441	-0.0764	-0.1006	-0.1207	-0.1407	-0.1504	-0.1399	-0.1282	-0.1212	-0.1316
0.9	-0.0487	-0.0829	-0.1078	-0.1287	-0.1516	-0.1493	-0.1297	-0.1050	-0.0804	-0.0740

TABLE 2. PROBABILITY THAT X_n IS LESS THAN X_{n-1} FOR THE NEAR(1) MODEL AS A FUNCTION OF α AND β. THE VALUE OF ONE-HALF HOLDS FOR $\alpha = 0$ OR $\beta = 0$, THE INDEPENDENCE CASES, AND FOR THE CASE $\beta = 1/(2-\alpha)$, THE PARTIALLY REVERSIBLE PREAR(1) EXPONENTIAL MODEL. NOTE THAT THE MODEL IS NOT DEFINED IF BOTH α AND β EQUAL 1.

α \ β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.5020	0.5030	0.5031	0.5023	0.5006	0.4979	0.4941	0.4890	0.4824	0.4737
0.2	0.5041	0.5064	0.5070	0.5058	0.5026	0.4974	0.4896	0.4787	0.4640	0.4444
0.3	0.5063	0.5103	0.5118	0.5107	0.5065	0.4989	0.4870	0.4696	0.4453	0.4118
0.4	0.5087	0.5147	9.5177	0.5173	0.5128	0.5033	0.4874	0.4629	0.4269	0.3750
0.5	0.5112	0.5196	0.5248	0.5260	0.5222	0.5118	0.4923	0.4603	0.4102	0.3333
0.6	0.5139	0.5253	0.5334	0.5374	0.5357	0.5259	0.5044	0.4650	0.3980	0.2857
0.7	0.5167	0.5316	0.5438	0.5521	0.5546	0.5482	0.5275	0.4829	0.3964	0.2308
0.8	0.5197	0.5387	0.5561	0.5708	0.5808	0.5824	0.5687	0.5255	0.4208	0.1667
0.9	0.5229	0.5466	0.5708	0.5947	0.6169	0.6345	0.6408	0.6190	0.5178	0.0909
1.0	0.5263	0.5556	0.5882	0.6250	0.6667	0.7143	0.7692	0.8333	0.9091	---

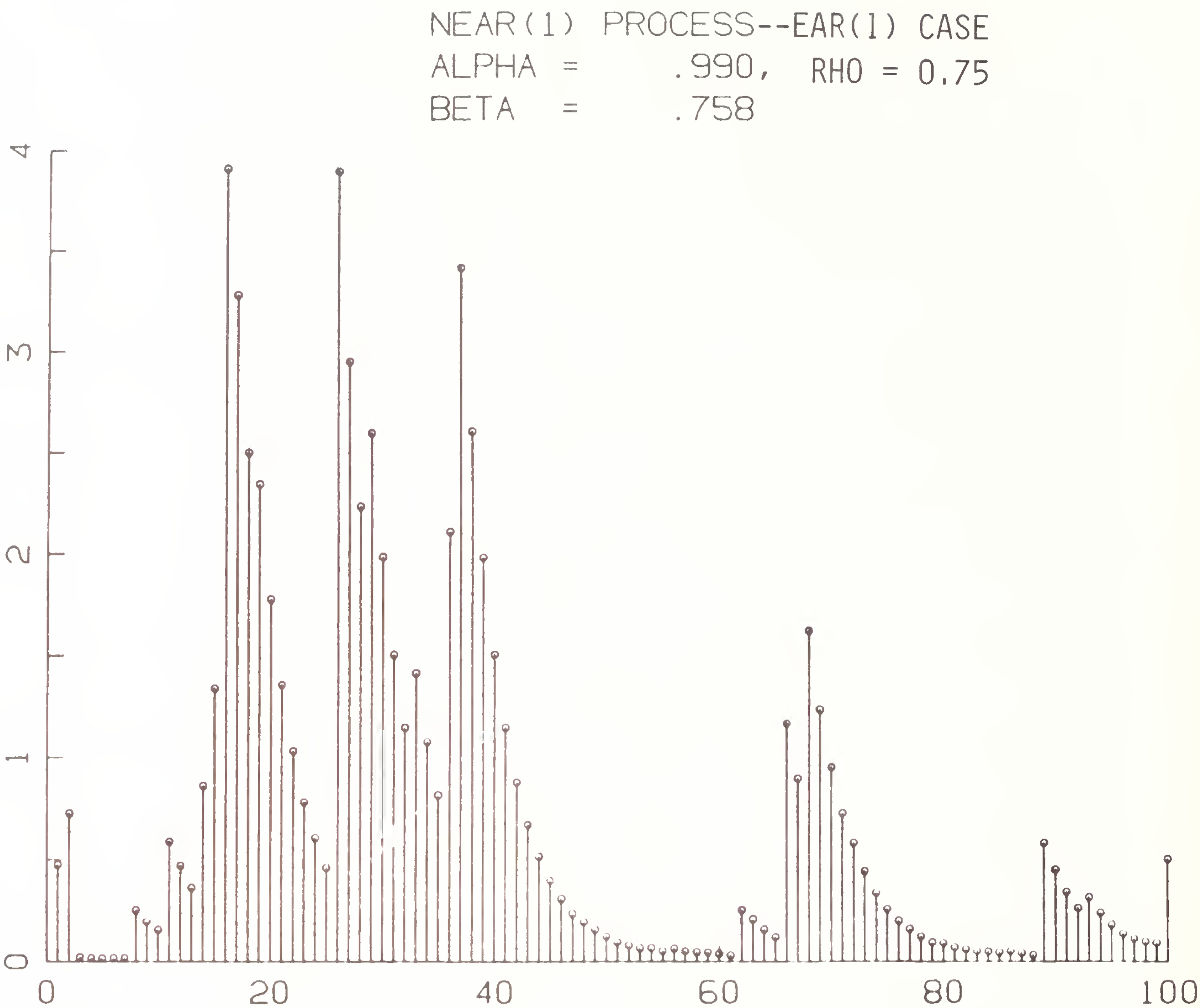

FIGURE 1a. Simulated sample path for the EAR(1) process of Gaver and Lewis (1980) which is the special case NEAR(1) process in which $\alpha = 1.0$. (Simulation done with $\alpha = .99$ to avoid computation problems.) For this case $P\{X_n < X_{n-1}\} = .78$ and the runs of falling values are clearly discernible.

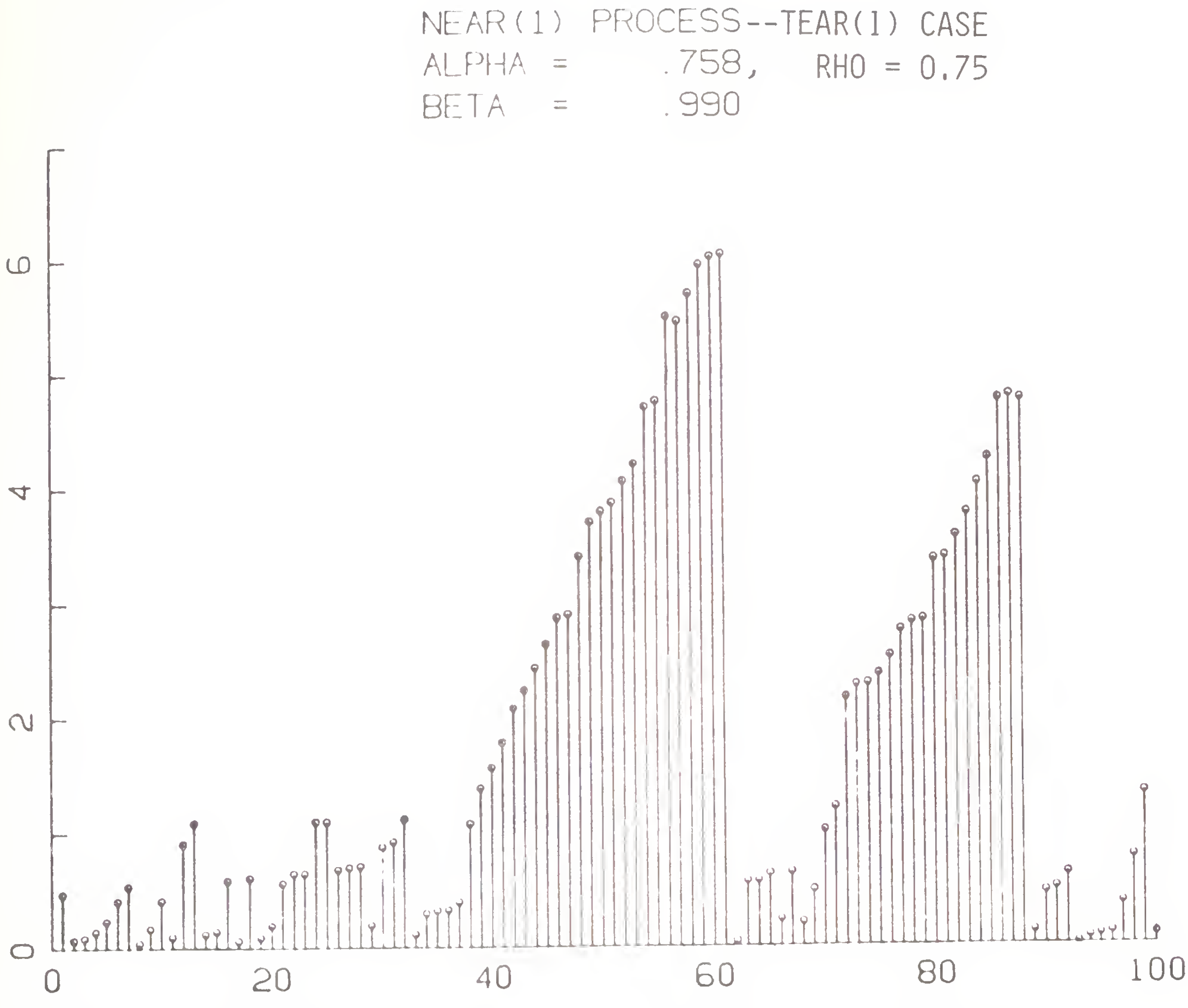

Figure 1b. Simulated sample path for the TEAR(1) process, the special case NEAR(1) process in which $\beta = 1$. (Simulation done with $\beta = .99$ to avoid computational problems.) For this case $P\{X_n < X_{n-1}\} = 0.22$ and the predominance of runs of ascending values is clearly discernible.

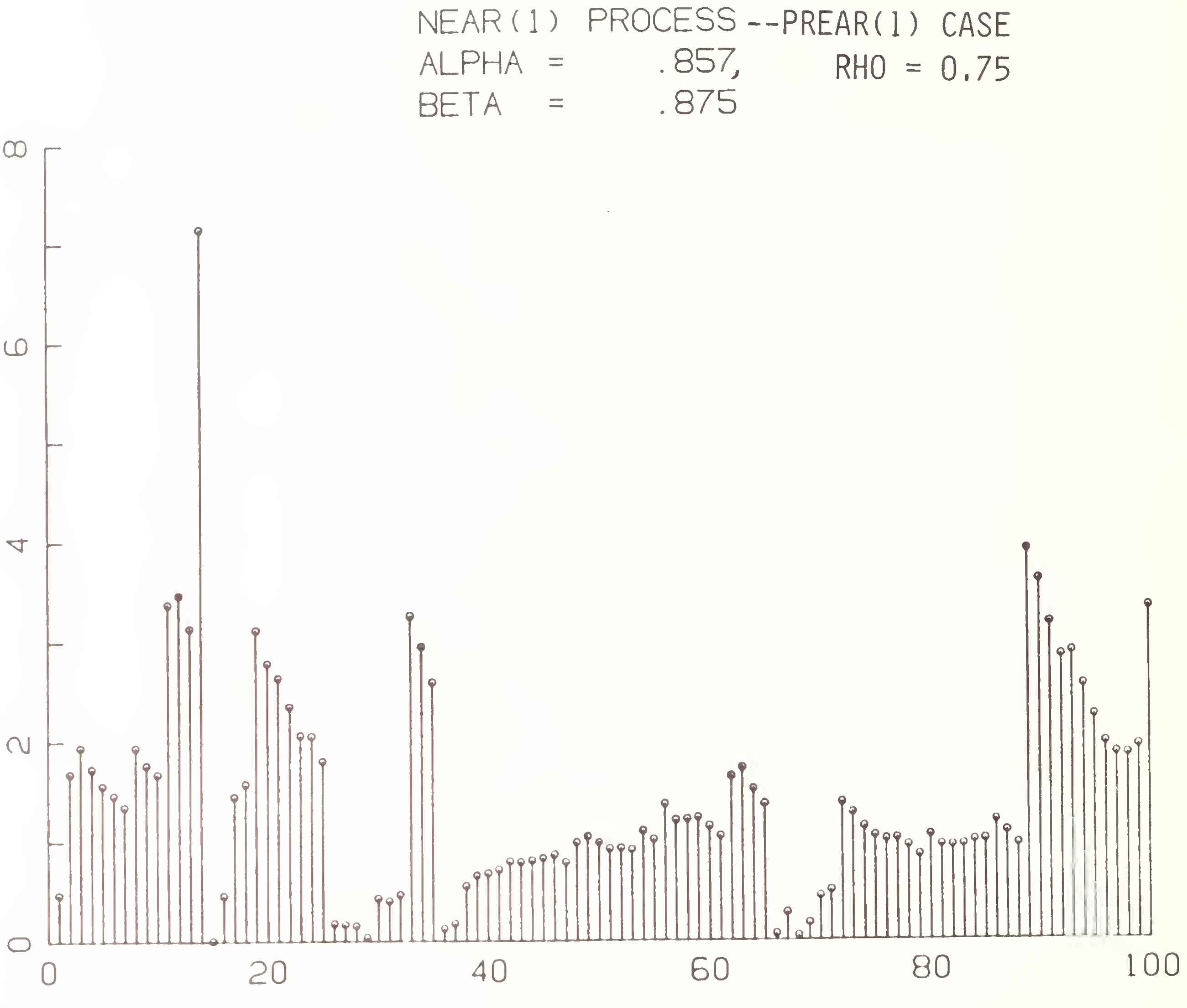

Figure 1c. Simulated sample path for the NEAR(1) process which is partially time-reversible in that the directional correlations are equal and $P\{X_n < X_{n-1}\} = 1/2$. The parametrization for this PREAR(1) process is $\beta = 1/(2-\alpha)$. Note that the same i.i.d. exponential sequence $\{E_n\}$ was used in the three simulations of Figures 1a, 1b, 1c.

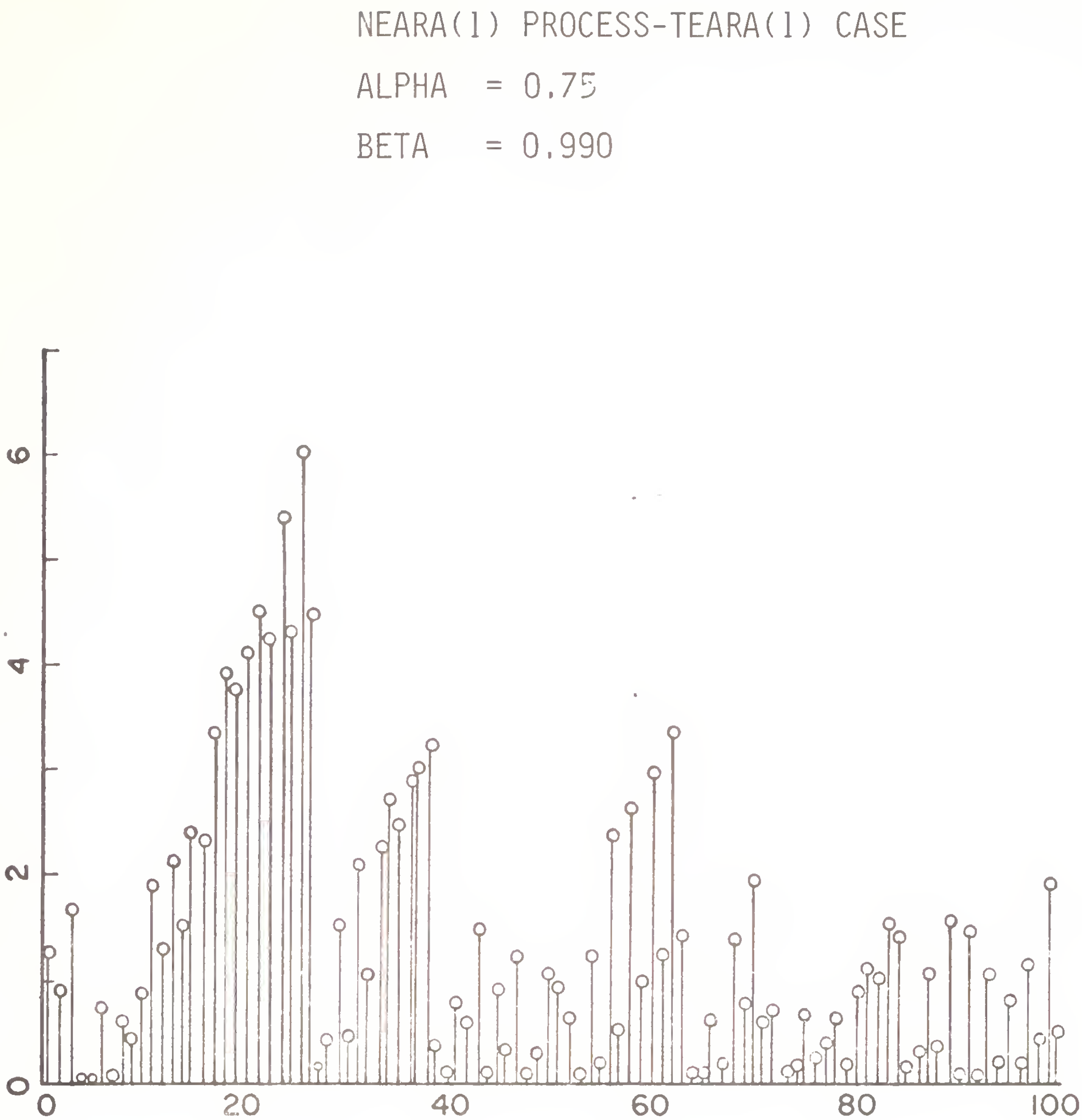

Figure 1d. Simulated sample path for the TEARA(1) process, the special case NEARA(1) process in which $\beta = 1$. (Simulations done with $\beta = 0.99$ to avoid computational problems.) Runs of alternating ascending values can be discerned, and are produced by the negative dependency in the model; this compares with the smoother run-up sequences in the TEAR(1) simulation of Fig. 1b.

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