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NAVAL POSTGRADUATE SCHOOL
Monterey, California

MULTIVARIATE GEOMETRIC DISTRIBUTIONS

GENERATED BY A CUMULATIVE DAMAGE PROCESS

by

J. D. Esary

and

A. W. Marshall

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NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral M. B. Freeman
Superintendent

M. U. Clauser
Provost

ABSTRACT

Two (narrow and wide) multivariate geometric analogues of the Marshall-Olkin multivariate exponential distribution are derived from the following cumulative damage model. A set of devices is exposed to a common damage process. Damage occurs in discrete cycles. On each cycle the amount of damage is an independent observation on a nonnegative random variable. Damages accumulate additively. Each device has its own random breaking threshold. A device fails when the accumulated damage exceeds its threshold. Thresholds are independent of damages, and have a Marshall-Olkin multivariate exponential distribution. The joint distribution of the random numbers of cycles up to and including failure of the devices has the wide multivariate geometric distribution. It has the narrow multivariate geometric distribution if the damage variable is infinitely divisible.

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1. Introduction

Suppose that we have a device for which exposure to failure occurs in discrete cycles, that on each cycle the device is damaged by an amount which is an observation on a nonnegative random variable X, and that damages, which are independent from cycle to cycle, accumulate additively. The device fails when the accumulated damage reaches $Y > 0$, its breaking threshold.

Let N be the number of cycles up to and including failure of the device. Then

$$N = \min\{k: X_1 + \ldots + X_k \geq Y\}, \tag{1.1}$$

where $X_1, X_2, \ldots$ are independent and identically distributed as X. If the component is to eventually fail, it must be that $P[X>0] > 0$.

Suppose now that the breaking threshold Y is a random variable, independent of the damages $X_1, X_2, \ldots,$ and with the exponential survival function

$$\bar{G}(y) = P[Y>y] = e^{-\lambda y}, \qquad \lambda > 0, \quad y \geq 0. \tag{1.2}$$

Since $N > k \geq 1$ if and only if $Y > X_1 + \ldots + X_k$, then

$$\begin{aligned} P[N>k] &= P[Y>X_1+\ldots+X_k] = E\bar{G}(X_1+\ldots+X_k) \\ &= Ee^{-\lambda(X_1+\ldots+X_k)} = \prod_{j=1}^{k} Ee^{-\lambda X_j} \\ &= \{Ee^{-\lambda X}\}^k, \qquad k = 1,2,\ldots . \end{aligned}$$

Thus since N has the positive integers as its values, N has the geometric survival function

$$(1.3) \qquad \bar{F}(k) = P[N>k] = \theta^k, \qquad 0 \le \theta < 1, \quad k = 0,1,\ldots,$$

with $\theta = Ee^{-\lambda X}$. That $\theta < 1$ follows from $\lambda > 0$ and $P[X>0] > 0$.

This paper is devoted to the properties of multivariate geometric distributions that can be generated by the process outlined above--subjecting a set of devices with different breaking thresholds to a common sequence of additive damages. The results are a step in the systematic study of the discrete multivariate life distributions that can be derived from cumulative damage models, and relate to the study of the continuous multivariate life distributions that can be derived from compound Poisson processes. A discussion of the general problem setting, univariate results, and a bibliography can be found in Esary, Marshall, and Proschan (1970).

2. Two bivariate geometric distributions

To place a discussion of bivariate geometric distributions in a context similar to that with which we began suppose that we have two devices for which exposure to failure occurs in discrete cycles, and are concerned with the joint distribution of K_1 and K_2, the numbers of cycles up to and including failure of the devices.

One could assume that in each cycle there is a shock to the first device which it survives with probability θ_1, a shock to the second device which it survives with probability θ_2, and a shock to both devices which both survive with probability θ_{12} and neither survives with probability $1-\theta_{12}$, and that the events of surviving the three kinds of shocks are independent of each other and from cycle to cycle. If each device is to eventually fail, it must be that $\theta_1\theta_{12} < 1$ and $\theta_2\theta_{12} < 1$. Then the joint survival function of K_1,K_2 is

$$\text{(2.1)} \qquad \bar{F}(k_1,k_2) = P[K_1>k_1,K_2>k_2] = \theta_1^{k_1}\theta_2^{k_2}\theta_{12}^{\max(k_1,k_2)}$$

$$0 \le \theta_i \le 1, \quad i = 1,2, \quad 0 \le \theta_{12} \le 1,$$

$$\theta_1\theta_{12} < 1 \text{ and } \theta_2\theta_{12} < 1, \quad k_1,k_2 = 0,1,\ldots .$$

We will say that positive integer valued random variables K_1,K_2 whose joint distribution is given by a survival function of the form (2.1) have a <u>bivariate geometric distribution in the narrow sense</u> (BVG-N). A BVG-N distribution has geometric marginals, an intuitive genesis similar to that for the univariate geometric, and is a discrete analogue of the bivariate exponential distribution introduced by Marshall and Olkin (1967).

A wider class of bivariate geometric distributions can be generated if one assumes that on each cycle there is a shock to both devices with probabilities p_{11} that both devices survive, p_{10} that the first device survives and the second device does not survive, p_{01} that the first device does not survive and the second device survives, and p_{00} that both devices do not survive, and that the events of surviving the shocks are independent from cycle to cycle. If each device is to eventually fail, it must be that $p_{10} + p_{11} < 1$ and $p_{01} + p_{11} < 1$. Then the joint survival function of K_1, K_2 is

$$(2.2)\quad \bar{F}(k_1,k_2) = P[K_1>k_1,K_2>k_2] = \begin{cases} p_{11}^{k_1}(p_{01}+p_{11})^{k_2-k_1} & \text{if } k_1 \le k_2, \\ p_{11}^{k_2}(p_{10}+p_{11})^{k_1-k_2} & \text{if } k_2 \le k_1, \end{cases}$$

$$0 \le p_{ij} \le 1,\quad i,j = 0,1,\quad \Sigma_{i,j=0}^{1}\, p_{ij} = 1,$$

$$p_{10} + p_{11} < 1 \text{ and } p_{01} + p_{11} < 1,\quad k_1,k_2 = 0,1,\ldots\,.$$

We will say that positive integer valued random variables K_1, K_2 whose joint distribution is given by a survival function of the form (2.2) have a <u>bivariate geometric distribution in the wide sense</u> (BVG-W). Again a BVG-W distribution has geometric marginals, an appropriate genesis, and as will be established later, is also a discrete analogue of the Marshall-Olkin bivariate exponential distribution.

The survival function of a BVG-W distribution can be written in a form similar to that of a BVG-N distribution by introducing parameters $\theta_1, \theta_2, \theta_{12}$ that are the solutions of the equations

$$
\begin{aligned}
(2.3) \qquad \theta_1\theta_2\theta_{12} &= p_{11} \\
\theta_1\theta_{12} &= p_{10} + p_{11} \\
\theta_2\theta_{12} &= p_{01} + p_{11}
\end{aligned}
$$

(See Figure 1).

$p_{11} = \theta_1\theta_2\theta_{12}$	$p_{10} = \theta_1(1-\theta_2)\theta_{12}$	$p_{1.} = \theta_1\theta_{12}$
$p_{01} = (1-\theta_1)\theta_2\theta_{12}$	$p_{00} = 1 - \theta_1\theta_{12} - \theta_2\theta_{12} + \theta_1\theta_2\theta_{12}$	$p_{0.} = 1 - \theta_1\theta_{12}$
$p_{.1} = \theta_2\theta_{12}$	$p_{.0} = 1 - \theta_2\theta_{12}$	1

Figure 1.

Since

$$0 \le p_{11} \begin{matrix} \le p_{10} + p_{11} \\ \le p_{01} + p_{11} \end{matrix} \le p_{10} + p_{01} + p_{11} \le 1,$$

then

$$0 \le \theta_1\theta_2\theta_{12} \begin{array}{l} \le \theta_1\theta_{12} \\ \le \theta_2\theta_{12} \end{array} \le \theta_{12}(\theta_1+\theta_2-\theta_1\theta_2) \le 1,$$

i.e. θ_1, θ_2, θ_{12} must satisfy conditions which reduce to

$$(2.4) \qquad 0 \le \theta_1 \le 1, \quad 0 \le \theta_2 \le 1, \quad 0 \le \theta_{12}(\theta_1+\theta_2-\theta_1\theta_2) \le 1.$$

Conversely, if the θ's satisfy the conditions (2.4), then through (2.3) and $p_{00} = 1 - p_{10} - p_{01} - p_{11}$ they define p_{ij}, $i,j = 0,1$, that are probabilities that add to 1. Also θ_1, θ_2, θ_{12} must satisfy the additional conditions.

$$(2.5) \qquad \theta_1\theta_{12} < 1, \quad \theta_2\theta_{12} < 1.$$

It follows that the survival function (2.2) of a BVG-W distribution can be expressed in the equivalent form

$$(2.6) \quad \bar{F}(k_1,k_2) = P[K_1 > k_1, K_2 > k_2] = \theta_1^{k_1}\theta_2^{k_2}\theta_{12}^{\max(k_1,k_2)},$$

$$0 \le \theta_i \le 1, \quad i = 1,2, \quad 0 \le \theta_{12}(\theta_1+\theta_2-\theta_1\theta_2) \le 1,$$

$$\theta_1\theta_{12} < 1 \text{ and } \theta_2\theta_{12} < 1, \quad k_1,k_2 = 0,1,\ldots .$$

Example 2.1. If $\theta_1 = \theta_2 = \frac{1}{2}$ and $\theta_{12} = \frac{4}{3}$, then the distribution defined by the survival function (2.6) is BVG-W but not BVG-N. □

Since $0 \le \theta_i \le 1$, $i = 1,2$, implies $0 \le \theta_1 + \theta_2 - \theta_1\theta_2 \le 1$, it is apparent from (2.1) and (2.6) that a BVG-N survival function must always be BVG-W.

By contrast with the BVG-N and BVG-W distributions, the more familiar bivariate geometric (negative binomial) distribution described in Mardia (1970), Section 10.4, can be viewed as arising from a sequence of three outcome trials; success of type 1 occurring with probability p_1, success of type 2 occurring with probability p_2, and failure occurring with probability $1 - p_1 - p_2$, with K_1 and K_2 defined respectively to be the numbers of successes of types 1 and 2 prior to the first failure.

3. A bivariate cumulative damage process.

We can now consider the bivariate case of the problem which motivates this paper. Suppose that on each cycle both devices are damaged by the same amount, which is an observation on a nonnegative random variable X, and that damages, which are independent from cycle to cycle, accumulate additively. The first device fails when the accumulated damage reaches $Y_1 > 0$, its breaking threshold. The second device fails when the accumulated damage reaches $Y_2 > 0$, its breaking threshold. As before, if each device is to eventually fail, it must be that $P[X > 0] > 0$.

Let N_1, N_2 be the number of cycles up to and including failure of the two devices. Then as in (1.1)

$$N_i = \min\{k: X_1 + \ldots + X_k \geq Y_i\}, \qquad i = 1,2, \tag{3.1}$$

where $X_1, X_2, \ldots$ are independent and identically distributed as X.

We will be concerned with the case in which the breaking thresholds Y_1, Y_2 are random variables that are independent of the damages, and in particular will suppose that Y_1, Y_2 have a Marshall-Olkin bivariate exponential distribution, i.e.

$$\bar{G}(y_1,y_2) = P[Y_1 > y_1,\ Y_2 > y_2] = e^{-\lambda_1 y_1 - \lambda_2 y_2 - \lambda_{12}\max(y_1,y_2)}, \tag{3.2}$$

$$\lambda_i \geq 0,\quad i = 1,2,\quad \lambda_{12} \geq 0,\quad \lambda_1 + \lambda_{12} > 0$$

$$\text{and}\quad \lambda_2 + \lambda_{12} > 0,\quad y_1, y_2 \geq 0.$$

The survival function (3.2) includes the case in which Y_1, Y_2 are independent and exponentially distributed.

Since $N_i > k \geq 1$ if and only if $Y_i > X_1 + \dots + X_k$, $i = 1,2$, then if $1 \leq k_1 \leq k_2$

$$P[N_1 > k_1,\ N_2 > k_2] = P[Y_1 > X_1 + \dots + X_{k_1},\ Y_2 > X_1 + \dots + X_{k_2}]$$

$$= E\ \bar{G}(X_1 + \dots + X_{k_1},\ X_1 + \dots + X_{k_2})$$

$$= E\ e^{-\lambda_1(X_1+\dots+X_{k_1}) - \lambda_2(X_1+\dots+X_{k_2}) - \lambda_{12}(X_1+\dots+X_{k_2})}$$

$$= E\ e^{-(\lambda_1+\lambda_2+\lambda_{12})(X_1+\dots+X_{k_1}) - (\lambda_2+\lambda_{12})(X_{k_1+1}+\dots+X_{k_2})}$$

$$= \{E\ e^{-(\lambda_1+\lambda_2+\lambda_{12})X}\}^{k_1} \{E\ e^{-(\lambda_2+\lambda_{12})X}\}^{k_2-k_1} .$$

Similarly, if $1 \leq k_2 \leq k_1$, then

$$P[N_1 > k_1,\ N_2 > k_2] = \{E\ e^{-(\lambda_1+\lambda_2+\lambda_{12})X}\}^{k_2} \{E\ e^{-(\lambda_1+\lambda_{12})X}\}^{k_1-k_2} .$$

Letting

(3.3)
$$\theta_1\theta_2\theta_{12} = E\ e^{-(\lambda_1+\lambda_2+\lambda_{12})X}$$

$$\theta_1\theta_{12} = E\ e^{-(\lambda_1+\lambda_{12})X}$$

$$\theta_2\theta_{12} = E\ e^{-(\lambda_2+\lambda_{12})X} ,$$

the survival function of N_1, N_2 becomes

$$(3.4)\qquad \bar{F}(k_1,k_2) = \theta_1^{k_1}\theta_2^{k_2}\theta_{12}^{\max(k_1,k_2)}, \qquad k_1,k_2 = 0,1,\ldots .$$

We will show that θ_1, θ_2, θ_{12} satisfy the conditions that make (3.4) a BVG-N survival function.

It is immediate from (3.3) that $\theta_i \geq 0$, $i = 1,2$, and that $\theta_{12} \geq 0$. Since $\lambda_1 + \lambda_{12} > 0$, $\lambda_2 + \lambda_{12} > 0$, and $P[X > 0] > 0$, it also follows that $\theta_1\theta_{12} < 1$ and $\theta_2\theta_{12} < 1$. We need to show that $\theta_i \leq 1$, $i = 1,2$, and $\theta_{12} \leq 1$.

Let $\omega(\lambda) = E\, e^{-\lambda X}$, $\lambda \geq 0$, be the Laplace transform of X, and $\psi(\lambda) = -\log \omega(\lambda)$. Then $\psi(0) = 0$ and ψ is concave and increasing in λ. It follows that ψ is subadditive, i.e. $\psi(\lambda+\nu) \geq \psi(\lambda) + \psi(\nu)$, $\lambda \geq 0$, $\nu \geq 0$.

To introduce some convenient notation let

$$(3.5)\qquad \mu_{12} = \psi(\lambda_1+\lambda_2+\lambda_{12}), \qquad e^{-\mu_{12}} = \omega(\lambda_1+\lambda_2+\lambda_{12}) = \theta_1\theta_2\theta_{12}$$

$$\mu_1 = \psi(\lambda_1+\lambda_{12}) \quad , \qquad e^{-\mu_1} = \omega(\lambda_1+\lambda_{12}) \qquad = \theta_1\theta_{12}$$

$$\mu_2 = \psi(\lambda_2+\lambda_{12}) \quad , \qquad e^{-\mu_2} = \omega(\lambda_2+\lambda_{12}) \qquad = \theta_2\theta_{12},$$

and define α_1, α_2, α_{12} by

$$\alpha_1 + \alpha_2 + \alpha_{12} = \mu_{12} \tag{3.6}$$

$$\alpha_1 + \alpha_{12} = \mu_1$$

$$\alpha_2 + \alpha_{12} = \mu_2$$

Then $\theta_{12} = e^{-\alpha_{12}}$, $\theta_1 = e^{-\alpha_1}$, $\theta_2 = e^{-\alpha_2}$. Thus N_1, N_2 have a BVG-N distribution, i.e. $\theta_i \le 1$, $i = 1,2$, $\theta_{12} \le 1$, if and only if $\alpha_i \ge 0$, $i = 1,2$, and $\alpha_{12} \ge 0$.

Theorem 3.1. N_1, N_2 <u>have a</u> BVG-N <u>distribution</u>.

<u>Proof</u>. From (3.6), $\alpha_1 = \mu_{12} - \mu_2$, $\alpha_2 = \mu_{12} - \mu_1$, $\alpha_{12} = \mu_1 + \mu_2 - \mu_{12}$. Then $\alpha_1 \ge 0$, since ψ is increasing and $\mu_{12} = \psi(\lambda_1+\lambda_2+\lambda_{12}) \ge \psi(\lambda_2+\lambda_{12}) = \mu_2$. Similarly $\alpha_2 \ge 0$. Also $\alpha_{12} \ge 0$, since ψ is subadditive, increasing and

$$\mu_1 + \mu_2 = \psi(\lambda_1+\lambda_{12}) + \psi(\lambda_2+\lambda_{12}) \ge \psi(\lambda_1+\lambda_2+2\lambda_{12})$$

$$\ge \psi(\lambda_1+\lambda_2+\lambda_{12}) = \mu_{12}.$$

Thus N_1, N_2 are BVG-N. $\square$

The balance of the paper is devoted to the multivariate version of the problem just considered. While the definitions and approach generalize, it will appear that Theorem 3.1 is peculiar to the bivariate case. Figure 2 introduces a point of view towards the equations (3.6) which will be useful.

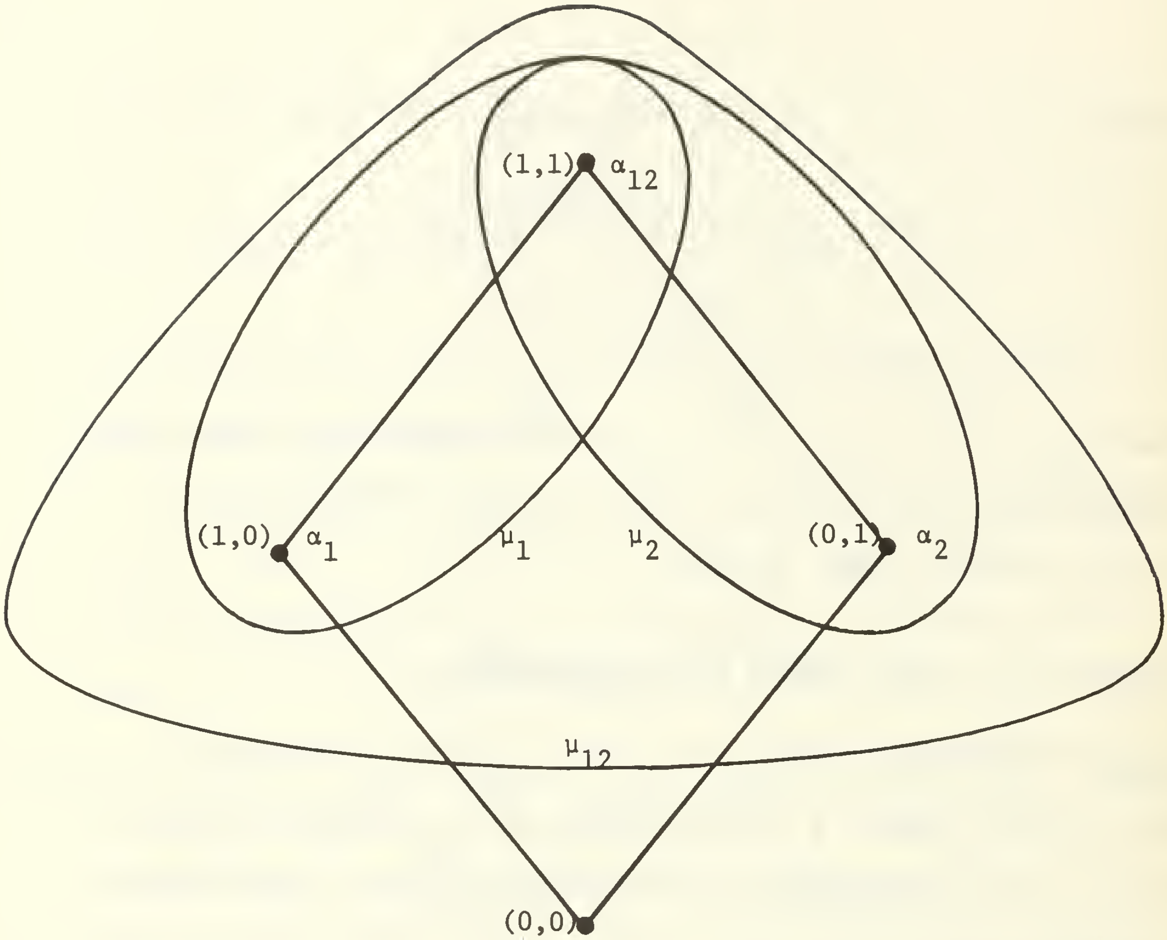

Figure 2.

In the figure α_1, α_2, α_{12} define point masses on all the vertices of a unit square except (0,0), and μ_1, μ_2, μ_{12} are the corresponding masses of the sets where the increasing Boolean functions $\phi(x_1,x_2) = x_1$, $\phi(x_1,x_2) = x_2$, $\phi(x_1,x_2) = x_1 \vee x_2 = x_1 + x_2 - x_1x_2$, $x_1 = 0$ or 1, $i = 1,2$, are equal to 1. The random variables N_1,N_2 have a BVG-N distribution if and only if the point masses are all nonnegative.

4. Two multivariate geometric distributions

It is apparent that K_1, K_2 have the BVG-N survival function (2.1) if and only if

$$K_1 = \min(M_1, M_{12})$$
$$(4.1)$$
$$K_2 = \min(M_2, M_{12})$$

where M_1, M_2, M_{12} are independent, positive integer valued random variables with the distributions $P[M_1 > k] = \theta_1^k$, $P[M_2 > k] = \theta_2^k$, $P[M_{12} > k] = \theta_{12}^k$, $k = 0,1,\ldots$, where $0 \le \theta_i \le 1$, $i = 1,2$, $0 \le \theta_{12} \le 1$, $\theta_1\theta_{12} < 1$ and $\theta_2\theta_{12} < 1$. If a θ is less than 1, then the corresponding M has a geometric distribution. If a θ is equal to 1, then the corresponding M can be regarded as degenerate at infinity, <u>or simply can be omitted from the representation</u> (4.1).

We will say that positive integer valued random variables $K_1,\ldots,K_n$ have a <u>multivariate geometric distribution in the narrow sense</u> (MVG-N) if $K_1,\ldots,K_n$ are distributed as though

$$K_i = \min\{M_J : i \in J\}, \qquad i = 1,\ldots,n, \tag{4.2}$$

where:

(a) The sets J are elements of a class $\mathcal{J}$ of nonempty subsets of $\{1,\ldots,n\}$ having the property that

for each $i \in \{1,\ldots,n\}$, $i \in J$ for some $J \in \mathcal{J}$.

(b) The random variables M_J are independent and geometrically distributed, i.e. M_J is positive integer valued and

$$P[M_J > k] = \theta_J^k, \qquad k = 0,1,\ldots ,$$

for some $0 \le \theta_J < 1$.

This definition is a discrete analogue of a characterization of the Marshall-Olkin multivariate exponential distribution (See Marshall and Olkin, 1967, Theorem 3.2 and p. 41).

Next we consider a multivariate version of the BVG-W distribution. It is also apparent that K_1, K_2 have the BVG-W survival function (2.2) if and only if

$$(4.3) \qquad \begin{aligned} P[\min(K_1,K_2) > k] &= p_{11}^k \\ P[K_1 > k] &= (p_{10}+p_{11})^k \\ P[K_2 > k] &= (p_{01}+p_{11})^k, \qquad k = 0,1,\ldots , \end{aligned}$$

where $0 \le p_{ij} \le 1$, $i,j = 0,1$, $p_{10} + p_{11} < 1$ and $p_{01} + p_{11} < 1$, and

$$(4.4) \quad P[K_1>k_1,K_2>k_2] = \begin{cases} P[\min(K_1,K_2)>k_1]P[K_2>k_2-k_1] & \text{if } 0 \le k_1 \le k_2 \\ P[\min(K_1,K_2)>k_2]P[K_1>k_1-k_2] & \text{if } 0 \le k_2 \le k_1 . \end{cases}$$

Let $\mathcal{I}$ be the class of nonempty subsets of $\{1,\ldots,n\}$, and for each $I \in \mathcal{I}$ let $K_I = \min_{i \in I} K_i$. We will say that the joint distribution of positive integer valued random variables $K_1,\ldots,K_n$ has <u>geometric minimums</u> if

$$(4.5)\qquad P[K_I > k] = \rho_I^k, \qquad 0 \le \rho_I < 1, \quad k = 0,1,\ldots ,$$

for each $I \in \mathcal{I}$.

Given a simplex $0 \le k_{i_1} \le \ldots \le k_{i_n}$, let $I_1 = \{i_1,\ldots,i_n\} = \{1,\ldots,n\}$, $I_2 = \{i_2,\ldots,i_n\}$, ..., $I_n = \{i_n\}$. We will say that positive integer valued random variables $K_1,\ldots,K_n$ have a <u>multivariate geometric distribution in the wide sense</u> (MVG-W) if:

(a) The joint distribution of $K_1,\ldots,K_n$ has geometric minimums.

(b) On each simplex $0 \le k_{i_1} \le \ldots \le k_{i_n}$

$$(4.6)\quad P[K_{i_1} > k_{i_1}, \ldots, K_{i_n} > k_{i_n}] = \prod_{j=1}^{n} P[K_{I_j} > k_{i_j} - k_{i_{j-1}}],$$

where $k_{i_0} = 0$.

This definition is also a discrete analogue of a characterization of the Marshall-Olkin multivariate exponential distribution (See Esary and Marshall, 1970, Application 5.1).

It is easy to see that MVG-N distributions are also MVG-W. Example 2.1 shows that there are MVG-W distributions that are not MVG-N. Both the MVG-N and MVG-W classes of distributions have the following properties:

(P_1) If the joint distribution of $K_1,\ldots,K_n$ is in the class, then the joint distribution of any subset of $K_1,\ldots,K_n$ is in the class.

(P_2) If the joint distribution of $K_1,\ldots,K_n$ is in the class, the joint distribution of $L_1,\ldots,L_m$ is in the class and $(K_1,\ldots,K_n)$ and $(L_1,\ldots,L_m)$ are independent, then the joint distribution of $K_1,\ldots,K_n;\ L_1,\ldots,L_m$ is in the class.

(P_3) If the joint distribution of $K_1,\ldots,K_n$ is in the class, then each K_i, $i = 1,\ldots,n$, has a geometric distribution.

(P_4) If the joint distribution of $K_1,\ldots,K_n$ is in the class and $K_{I_j} = \min_{i\in I_j} K_i$, $j = 1,\ldots,m$, where $I_1,\ldots,I_m$ are nonempty subsets of $1,\ldots,n$, then the joint distribution of $K_{I_1},\ldots,K_{I_m}$ is in the class.

If the joint distribution of $K_1,\ldots,K_n$ has geometric minimums, it will be convenient to let $\mu_I = -\log \rho_I$ for each $I \in \mathcal{I}$, i.e. let $e^{-\mu_I} = \rho_I$. Since $\rho_I < 1$, then $\mu_I > 0$.

Theorem 4.1. <u>Let</u> $K_1,\ldots,K_n$ <u>have a</u> MVG-W <u>distribution</u>. <u>Then</u> $K_1,\ldots,K_n$ <u>have a</u> MVG-N <u>distribution if and only if there exists an</u> $\alpha_J \geq 0$ <u>for each</u> $J \in \mathcal{I}$ <u>such that</u>

$$\mu_I = \Sigma_{J:I\cap J\neq\emptyset}\ \alpha_J$$

<u>for each</u> $I \in \mathcal{I}$.

<u>Proof</u>. Suppose $K_1,\ldots,K_n$ have a MVG-N distribution. Let

$$e^{-\alpha_J} = \begin{cases} \theta_J & \text{if } J \in \mathcal{J} \\ 1 & \text{if } J \in \mathcal{I} - \mathcal{J}. \end{cases}$$

If $J \in \mathcal{J}$, then $\alpha_J > 0$ since $\theta_J < 1$. If $J \in \mathcal{I} - \mathcal{J}$, then $\alpha_J = 0$. Since

$$e^{-\mu_I} = P[N_I > 1] = \prod_{J:I\cap J\neq\emptyset} \theta_J = e^{-\sum_{J:I\cap J\neq\emptyset} \alpha_J},$$

then $\mu_I = \sum_{J:I\cap J\neq\emptyset} \alpha_J$.

Suppose for each $I \in \mathcal{I}$, $\mu_I = \sum_{I:I\cap J\neq\emptyset} \alpha_J$, where $\alpha_J \geq 0$, $J \in \mathcal{I}$. Let $\mathcal{J}$ consist of the sets J in $\mathcal{I}$ such that $\alpha_J > 0$. We have noted that $\mu_I > 0$ for each $I \in \mathcal{I}$. If $I = \{i\}$, then $\mu_{\{i\}} = \sum_{J:i\in J} \alpha_J$. Thus $\alpha_J > 0$ for some J such that $i \in J$, i.e. $i \in J$ for some $J \in \mathcal{J}$. For each $J \in \mathcal{J}$ construct a positive integer valued random variable M_J with the geometric distribution $P[M_J > k] = \theta_J^k$, $k = 0,1,\ldots,$ where $\theta_J = e^{-\alpha_J}$. Since $\alpha_J > 0$, then $\theta_J < 1$. Since $K_1,\ldots,K_n$ have a MVG-W distribution, then on the simplex $0 \leq k_{i_1} \leq \ldots \leq k_{i_n}$

$$P[K_{i_1} > k_{i_1}, \ldots, K_{I_n} > k_{i_n}]$$

$$= \exp[-\mu_{I_1} k_{i_1}]\exp[-\mu_{I_2}(k_{i_2} - k_{i_1})]\ldots\exp[-\mu_{I_n}(k_{i_n} - k_{i_{n-1}})]$$

$$= \exp[-k_{i_1} \textstyle\sum_{J:I_1\cap J\neq\emptyset}\alpha_J]\exp[-(k_{i_2} - k_{i_1})\textstyle\sum_{J:I_2\cap J\neq\emptyset}\alpha_J]$$

$$\ldots \exp[-(k_{i_n} - k_{i_{n-1}})\textstyle\sum_{J:I_n\cap J\neq\emptyset}\alpha_J]$$

$$= \exp[-\textstyle\sum_{J\in\mathcal{J}} k_j\alpha_j] = \prod_{J\in\mathcal{J}} \theta_J^{k_J},$$

where $k_J = \max\{k_{i_j} : i_j \in J\}$. Thus $K_1,\ldots,K_n$ are distributed as if $K_i = \min\{M_J : i \in J\}$, $i = 1,\ldots,n$, i.e. $K_1,\ldots,K_n$ have a MVG-N distribution. $\square$

5. A multivariate cumulative damage process.

In keeping with the damage model that we have previously described, let

$$(5.1) \qquad N_i = \min\{k: X_1 + \dots + X_k \geq Y_i\}, \qquad i = 1,\dots,n,$$

where $X_1, X_2, \dots$ are independent and identically distributed as a nonnegative random variable X such that $P[X > 0] > 0$. Assume that $(Y_1,\dots,Y_n)$ is independent of $\{X_1, X_2, \dots\}$ and that $Y_1,\dots,Y_n$ have a Marshall-Olkin multivariate exponential distribution, i.e. that $Y_1,\dots,Y_n$ are distributed as if

$$(5.2) \qquad Y_i = \min\{S_J: i \in J\}, \qquad i = 1,\dots,n,$$

where the sets J are elements of a class $\mathcal{J}$ of nonempty subsets of $\{1,\dots,n\}$ such that for each i, $i \in J$ for some $J \in \mathcal{J}$, and the random variables S_J are independent with the exponential distributions $P[S_J > s] = e^{-\lambda_J s}$, $\lambda_J > 0$, $s \geq 0$.

For each $I \in \mathcal{I}$, let $N_I = \min_{i \in I} N_i$ and $Y_I = \min_{i \in I} Y_i$. Then by computations parallel to those that led to (1.3) and (3.3) it is easy to see that for $k \geq 1$,

$$P[N_I > k] = P[Y_I > X_1 + \dots + X_k] = \{E\, e^{-\eta_I X}\}^k,$$

where $\eta_I = \sum_{J: I \cap J \neq \emptyset} \lambda_J$, and that for $1 \leq k_{i_1} \leq \dots \leq k_{i_n}$,

$$P[N_{i_1} > k_{i_1}, \dots, N_{i_n} > k_{i_n}]$$

$$= P[Y_{i_1} > X_1 + \dots + X_{k_{i_1}}, \dots, Y_{i_n} > X_1 + \dots + X_{k_{i_n}}]$$

$$= \{E\, e^{-\eta_{I_1} X}\}^{k_{i_1}} \{E\, e^{-\eta_{I_2} X}\}^{k_{i_2} - k_{i_1}} \dots \{E\, e^{-\eta_{I_n} X}\}^{k_{i_n} - k_{i_{n+1}}},$$

where $I_1 = \{i_1, \dots, i_n\} = \{1, \dots, n\}$, $I_2 = \{i_2, \dots, i_n\}, \dots, I_n = \{i_n\}$. Letting $\rho_I = E\, e^{-\eta_I X^n}$, $I \in \mathcal{I}$, the survival function of $N_1, \dots, N_n$ becomes

$$\bar{F}(k_1, \dots, k_n) = P[N_1 > k_1, \dots, N_n > k_n] \tag{5.3}$$

$$= \rho_{I_1}^{k_{i_1}} \rho_{I_2}^{k_{i_2} - k_{i_1}} \dots \rho_{I_n}^{k_{i_n} - k_{i_{n-1}}}$$

on the simplex $0 \le k_{i_1} \le \dots \le k_{i_n}$. The content of the preceding remarks is summarized by the following theorem.

Theorem 5.1. $N_1, \dots, N_n$ <u>have a</u> MVG-W <u>distribution</u>. □

Now let $\mu_I = -\log \rho_I$, $I \in \mathcal{I}$, i.e. $e^{-\mu_I} = \rho_I = E\, e^{-\eta_I X}$. The following definitions and lemmas are directed towards finding conditions on X for which the equations $\mu_I = \Sigma_{J: I \cap J \neq \emptyset}\, \alpha_J$, $I, J \in \mathcal{I}$, have a set of nonnegative solutions α_J. Then by Theorem 4.1, $N_1, \dots, N_n$ will have a MVG-N distribution.

A coherent structure function of order n is an increasing binary function $\phi(\underset{\sim}{x}) = \phi(x_1,\ldots,x_n) = 0$ or 1 of binary arguments $x_i = 0$ or 1, $i = 1,\ldots,n$, such that $\phi(0,\ldots,0) = 0$ and $\phi(1,\ldots,1) = 1$. The <u>coherent life function</u> $\tau(\underset{\sim}{t}) = \tau(t_1,\ldots,t_n)$, $t_i \geq 0$, $i = 1,\ldots,n$, that corresponds to ϕ is defined by

$$\tau(\underset{\sim}{t}) = \sup\{u\colon \phi\{x(u,t_1),\ldots,x(u,t_n)\} = 1\},$$

where $x(u,t) = 1$ if $u < t$, $x(u,t) = 0$ if $u \geq t$ (cf. Esary and Marshall (1970b). The <u>dual</u> of ϕ is the coherent structure function $\phi^D(x_1,\ldots,x_n) = 1 - \phi(1-x_1,\ldots,1-x_n)$, and τ^D is the life function that corresponds to ϕ^D. The coherent structure function $\phi_1\phi_2$ has $\min(\tau_1,\tau_2)$ as its corresponding life function. The coherent structure function $\phi_1 \vee \phi_2 = \phi_1 + \phi_2 - \phi_1\phi_2$ has $\max(\tau_1,\tau_2)$ as its corresponding life function. The dual of $\phi_1\phi_2$ is $\phi_1^D \vee \phi_2^D$ and the dual of $\phi_1 \vee \phi_2$ is $\phi_1^D\phi_2^D$.

The following lemma holds for $Y_1,\ldots,Y_n$ with an arbitrary joint distribution.

Lemma 5.2. <u>For each coherent structure function</u> ϕ <u>of order</u> n, <u>let</u> $m(\phi) = P[\phi^D(\underset{\sim}{Y}) \leq X]$. <u>Then</u>:

(a) $m(\phi) \geq 0$.

(b) $\phi_1 \leq \phi_2$ implies $m(\phi_1) \leq m(\phi_2)$.

(c) $m(\phi_1 \vee \phi_2) = m(\phi_1) + m(\phi_2) - m(\phi_1\phi_2)$.

Proof. That (a) holds is immediate since $m(\phi)$ is a probability. To show (b), note that

$$\phi_1 \le \phi_2 \Rightarrow \phi_1^D \ge \phi_2^D \Rightarrow \tau_1^D \ge \tau_2^D$$

$$\Rightarrow P[\tau_1^D(\underset{\sim}{Y}) \le X] \le P[\tau_2^D(\underset{\sim}{Y}) \le X]$$

$$\Rightarrow m(\phi_1) \le m(\phi_2).$$

To show (c), note that

$$m(\phi_1 \vee \phi_2) = P[\min\{\tau_1^D(\underset{\sim}{Y}),\ \tau_2^D(\underset{\sim}{Y})\} \le X]$$

$$= P[\tau_1^D(\underset{\sim}{Y}) \le X,\ \tau_2^D(\underset{\sim}{Y}) \le X]$$

$$= P[\tau_1^D(\underset{\sim}{Y}) \le X] + P[\tau_2^D(\underset{\sim}{Y}) \le X]$$

$$- P[\max\{\tau_1^D(\underset{\sim}{Y}),\ \tau_2^D(\underset{\sim}{Y})\} \le X]$$

$$= m(\phi_1 + m(\phi_2) - m(\phi_1\phi_2).$$

Thus, (a), (b) and (c) all hold. □

Each coherent structure function has a representation

$$\phi(\underset{\sim}{x}) = \prod_{i\in P_1} x_i \vee \ldots \vee \prod_{i\in P_p} x_i,$$

where $P_1,\ldots,P_p$ are the minimal path sets of ϕ, i.e. the minimal subsets P of $\{1,\ldots,n\}$ such that $x_i = 1$ for all $i \in P$ implies $\phi(\underset{\sim}{x}) = 1$. The equivalent representation for the life function corresponding to ϕ is

(5.4) $$\tau(\underset{\sim}{t}) = \max_{j=1,\ldots,p} \min_{i \in P_j} t_i .$$

The random variable X is infinitely divisible if X is distributed as if $X = X_{1,r} + \ldots + X_{r,r}$ for each $r = 1,2,\ldots,$ where $X_{1,r},\ldots,X_{r,r}$ are independent and identically distributed as a random variable X_r. Since X is nonnegative and $P[X > 0] > 0$, then X_r is nonnegative and $P[X_r > 0] > 0$. As before let $\omega(\lambda) = E\, e^{-\lambda X}$ be the Laplace transform of X, and $\psi(\lambda) = -\log \omega(\lambda)$. Let $\omega_r(\lambda) = E\, e^{-\lambda X_r} = \omega(\lambda)^{1/r}$ be the Laplace transform of X_r. Then $r\{1 - \omega_r(\lambda)\} \to \psi(\lambda)$ as $r \to \infty$.

The following lemma uses the assumption that $Y_1,\ldots,Y_n$ have a Marshall-Olkin multivariate exponential distribution to the extent that then Y_I has an exponential distribution for each $I \in \mathcal{I}$, i.e. $Y_1,\ldots,Y_n$ have exponential minimums.

Lemma 5.3. Let X be infinitely divisible. For each coherent structure function ϕ of order n, and each $r = 1,2,\ldots,$ define $m_r(\phi) = P[\tau^D(\underset{\sim}{Y}) \le X_r]$. Then

$$m(\phi) = \lim_{r\to\infty} r m_r(\phi)$$

exists for each ϕ, and m satisfies (a), (b) and (c) of Lemma 5.2.

Proof. From (5.4)

$$m_r(\phi) = P[\tau^D(\underset{\sim}{Y}) \le X_r] = P[Y_{P_1} \le X_r , \ldots , Y_{P_p} \le X_r],$$

where $P_1,\ldots,P_p$ are the minimal path sets of ϕ^D. Then by a standard inclusion and exclusion argument

$$m_r(\phi) = \Sigma_{j=1}^{p} \{1-P[Y_{P_j} > X_r]\} - \Sigma_{\substack{j,k=1 \\ j<k}}^{p} \{1-P[Y_{P_j} > X_r, Y_{P_k} > X_r]\}$$

$$+ \ldots \pm \{1-P[Y_{P_1} > X_r , \ldots , Y_{P_p} > X_r]\}$$

$$= \Sigma_{j=1}^{p} \{1-\omega_r(\eta_{P_j})\} - \Sigma_{\substack{j,k=1 \\ j<k}}^{p} \{1-\omega_r(\eta_{P_j \cup P_k})\}$$

$$+ \ldots \pm \{1-\omega_r(\eta_{P_1 \cup \ldots \cup P_p})\}.$$

Since for each λ, $r\{1 - \omega_r(\lambda)\} \to \psi(\lambda)$ as $r \to \infty$, it follows that $m(\phi)$, the limit of $rm_r(\phi)$ exists. Since for each r, m_r satisfies (a), (b) and (c) of Lemma 5.2, so does m. □

For each $I \in \mathcal{I}$, let $\phi_I = \bigvee_{i \in I} x_i$, where $\bigvee_{i=1}^{n} x_i = x_1 \vee \ldots \vee x_n$. Then I is the only minimal path set of ϕ_I^D. Embedded in the proof of Lemma 5.3 is the observation that $m_r(\phi_I) = 1 - E\, e^{-\eta_I X_r} = 1 - \omega_r(\eta_I)$ and

(5.5) $$m(\phi_I) = \lim_{r\to\infty} rm_r(\phi_I) = \psi(\eta_I)$$

$$= -\log E\, e^{-\eta_I X} = \mu_I .$$

<u>Theorem</u> 5.4. <u>If</u> X <u>is infinitely divisible</u>, <u>then</u> $N_1,\ldots,N_n$ <u>have</u> <u>a</u> MVG-N <u>distribution</u>.

<u>Proof</u>. By Theorem 5.1 $N_1,\ldots,N_n$ have a MVG-W distribution. Then by Theorem 4.1 it is sufficient to show that for each $I \in \mathcal{J}$ $\mu_I = \sum_{J:I\cap J=\emptyset} \alpha_J$, where $\alpha_J \geq 0$, $J \in \mathcal{J}$.

Let m be defined as in Lemma 5.3. Since m satisfies (a), (b) and (c) of Lemma 5.2, it follows from Lemma 3.1, Esary and Marshall (1970a) that there exists a nonnegative function $\alpha(\underset{\sim}{x})$ such that

$$m(\phi) = \sum_{\underset{\sim}{x}} \alpha(\underset{\sim}{x})\phi(\underset{\sim}{x})$$

for each coherent structure function ϕ of order n.

Let the i^{th} coordinate of $\underset{\sim}{x}^J$ be 1 if $i \in J$ and 0 if $i \in J$. Then $\phi_I(\underset{\sim}{x}^J) = 1$ if and only if $I \cap J \neq \emptyset$. Let $\alpha_J = \alpha(\underset{\sim}{x}^J) \geq 0$, $J \in \mathcal{I}$. Then from (5.5)

$$\mu_I = m(\phi_I) = \sum_{J:I\cap J\neq\emptyset} \alpha_J .$$

Thus $N_1,\ldots,N_n$ have a MVG-N distribution. □

For the purpose of the following theorem, assume that $Y_1,\ldots,Y_n$ are independent and that Y_i has the exponential distribution $P[Y_i > y] = e^{-\lambda_i y}$, $y \geq 0$, $\lambda_i > 0$, i.e. that $Y_1,\ldots,Y_n$ have a special case of the Marshall-Olkin multivariate exponential distribution.

Theorem 5.5 (Converse to Theorem 5.4). _If_ $N_1,\ldots,N_n$ _have a_ MVG-N _distribution for each_ n _and all_ $\lambda_1 > 0,\ldots,\lambda_n > 0$, _then_ X _is infinitely divisible_.

Proof. Since $N_1,\ldots,N_n$ have a MVG-N distribution, it follows from Theorem 4.1 that for each $J \in \mathcal{I}$ there exists an $\alpha_J \geq 0$ such that

$$\mu_I = \Sigma_{J:I\cap J\neq\emptyset}\, \alpha_J$$

for each $I \in \mathcal{I}$. Let $\alpha(\underset{\sim}{x}) = \alpha_J$ where $J = \{i\colon x_i = 1\}$, $\underset{\sim}{x} \neq (0,\ldots,0)$, and define $m(\phi) = \Sigma_{\underset{\sim}{x}}\, \alpha(\underset{\sim}{x})\phi(\underset{\sim}{x})$ for each coherent structure function ϕ of order n. Then m satisfies conditions (a), (b) and (c) of Lemma 5.2. Also

$$m(\phi_I) = \Sigma_{\underset{\sim}{x}}\, \alpha(\underset{\sim}{x}) \bigvee_{i\in I} x_i = \Sigma_{J:I\cap J\neq\emptyset}\, \alpha_J$$

$$= \mu_I = \psi(\eta_I) = \psi(\Sigma_{i\in I}\, \lambda_i).$$

Then, with the incidental use of an inclusion-exclusion argument based on condition (c) of Lemma 5.2, for $n \geq 2$ (letting $\phi_i = \phi_{\{i\}}$, $\phi_{ij} = \phi_{\{ij\}}$, etc.),

$$- \alpha_{2\ldots n} = m(\Pi_{i=1}^{n} x_i) - m(\Pi_{i=2}^{n} x_i)$$

$$= \Sigma_{i=1}^{n} m(\phi_i) - \Sigma_{\substack{i,j=1 \\ i<j}}^{n} m(\phi_{ij}) + \ldots \pm m(\phi_{1\ldots n})$$

$$- \Sigma_{i=2}^{n} m(\phi_i) + \Sigma_{\substack{i,j=2 \\ i<j}}^{n} m(\phi_{ij}) + \ldots \pm m(\phi_{2\ldots n})$$

$$= m(\phi_1) - \Sigma_{1=2}^{n} m(\phi_{1i}) + \ldots \pm m(\phi_{1\ldots n}).$$

Thus

$$(-1)^n \alpha_{2\ldots n} = m(\phi_{1\ldots n}) - \ldots \pm \Sigma_{1=2}^{n} m(\phi_{1i}) \mp m(\phi_1)$$

$$= \psi(\lambda_1+\ldots+\lambda_n) - \ldots \pm \Sigma_{i=2}^{n} \psi(\lambda_1+\lambda_i) \mp \psi(\lambda_1)$$

$$= \Delta_{\lambda_n} \ldots \Delta_{\lambda_2} \psi(\lambda_1),$$

where $\Delta_y f(x) = f(x+y) - f(x)$. Since $\alpha_{2\ldots n} \geq 0$, it follows that

$$(-1)^n \psi^{(n-1)}(\lambda_1) \geq 0, \qquad n = 2,3,\ldots ,$$

where $\psi^{(n)}(\lambda)$ is the n^{th} derivative of $\psi(\lambda)$ with respect to λ. Thus

$$(-1)^n \frac{d^n\psi^{(1)}(\lambda)}{d\lambda^n} \geq 0, \qquad n = 0,1,\ldots, \quad \lambda > 0,$$

i.e. $\psi^{(1)}(\lambda)$ is a completely monotone function. It follows from Theorem 1, p. 425, Feller (1966) that $\omega(\lambda) = e^{-\psi}$ is the Laplace transform of an infinitely divisible random variable, i.e. that X is infinitely divisible. □

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Two (narrow and wide) multivariate geometric analogues of the Marshall-Olkin multivariate exponential distribution are derived from the following cumulative damage model. A set of devices is exposed to a common damage process. Damage occurs in discrete cycles. On each cycle the amount of damage is an independent observation on a nonnegative random variable. Damages accumulate additively. Each device has its own random breaking threshold. A device fails when the accumulated damage exceeds its threshold. Thresholds are independent of damages, and have a Marshall-Olkin multivariate exponential distribution. The joint distribution of the random numbers of cycles up to and including failure of the devices has the wide multivariate geometric distribution. It has the narrow multivariate geometric distribution if the damage variable is infinitely divisible.

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