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MINIMAL POINT CUBATURES OF PRECISION SEVEN FOR SYMMETRIC PLANAR REGIONS, REVISITED

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The key theorem concerning a lower bound for the number of points required by cubature formulas of precision seven for certain symmetric planar regions is extended to arbitrary symmetric planar regions, with a simplified proof.

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1.0 Introduction

The purpose of this paper is to give a simplified proof of a theorem that was previously stated in a more restricted form [1]. In that paper it was shown how to construct 12 point cubature formulas having precision seven for symmetric planar regions. For fully symmetric regions and rectangular regions the construction cannot fail, and the resulting formulas use the minimum number of points. In the next section we establish 12 points as a lower bound for all symmetric (in each variable) planar regions. The existence of 12 point formulas has not been established for arbitrary symmetric regions. The notation of this paper is the same as that of [1].

2.0 Minimal Point Formulas

Let R be a symmetric (about each axis) planar region and let $w(x,y)$ be symmetric in each variable. We consider cubature formulas of the form

$$(1) \qquad \int_R wf \simeq \sum_{k=1}^{N} A_k \, f(\nu_k)$$

As a consequence of the characterization theorem given by Stroud [2], or [3, p.111], we have the following result, also given in [1].

Proposition 1: Let there be given a formula of type (1), with precision 7, for a region in the plane. Suppose that $N < 15$. Then the points ν_k, $k=1,\ldots,N$, are zeros of $15 - N$ linearly independent polynomials of degree 4, each of which is orthogonal, over R with respect to w, to all polynomials of degree ≤ 3.

Our approach, as before, is to show that for the regions under consideration, there do not exist five orthogonal polynomials of degree four with 10 common zeros, nor four with 11 common zeros.

Theorem 2: Let the cubature formula (1) have precision seven, where R and w are symmetric. Then $N \geq 12$. Proof: We first establish that no 10 point formula exists, that being the minimum possible for any region. Assume the con-contrary. Then the basic orthogonal polynomials, which have the form

$$
\begin{aligned}
&P^{(4,0)} = x^4 + a_4x^2 + b_4y^2 + c_4 \\
&P^{(3,1)} = x^3y + a_3xy \\
(2)\quad &P^{(2,2)} = x^2y^2 + a_2x^2 + b_2y^2 + c_2 \\
&P^{(1,3)} = xy^3 + a_1xy \\
&P^{(0,4)} = y^4 + a_0\,x^2 + b_0\,y^2 + c_0
\end{aligned}
$$

must have the ten nodes as common zeros. Inspection of $P^{(3,1)}$ and $P^{(1,3)}$ reveal that their common zeros are $(\pm\alpha,\pm\beta)$, where $\alpha^2 = -a_3$ and $\beta^2 = -a_1$, and the axes. However, $P^{(2,2)}$ has at most two finite zeros on each axis, hence $P^{(3,1)}$, $P^{(2,2)}$, and $P^{(1,3)}$ can have at most eight common zeros. Hence no ten point formula can exist.

We now show that no 11 point formula exists. If one did there would exist four linearly independent orthogonal polynomials of which the nodes would be common zeros. This would mean that all the ν_k lie on some nontrivial linear combination of any two of the orthogonal polynomials (2). For some μ_1 and λ_1 they lie on $P_1 = \mu_1P^{(3,1)} + \lambda_1P^{(1,3)} = xy(\mu_1x^2 + \lambda_1y^2 + \mu_1a_3 + \lambda_1a_1)$. Thus the nodes lie on the axes or the conic $Q_2 = \mu_1x^2 + \lambda_1y^2 + \mu_1a_3 + \lambda_1a_1$. We now consider two possibilities: (i) one of μ_1 or λ_1 is zero, i.e., the nodes lie on $P^{(3,1)}$ or $P^{(1,3)}$; (ii) $\mu_1\lambda_1 \neq 0$.

Case (i): We show this cannot happen. We assume the nodes lie on $P^{(3,1)}$, and the argument is dual if they lie on $P^{(1,3)}$. The nodes also lie on $P_2 = \mu_2P^{(4,0)} + \lambda_2P^{(2,2)}$ for some μ_2 and λ_2, not both zero, and on

$P_3 = \mu_3 P^{(0,4)} + \lambda_3 P^{(2,2)}$ for some μ_3 and λ_3 , not both zero. We see that P_2 has at most two zeros on the x axis. The remaining nodes must lie on $x^2 + a_3$ and P_2 which can have at most four finite common zeros, hence P_1 , P_2 and P_3 can have at most eight finite common zeros in this case.

Case (ii): We will show in this instance that the eleven nodes must be a subset of twelve points of the form $(\pm\alpha, 0)$, $(0, \pm\beta)$, $(\pm x_1 , \pm y_1)$, $(\pm x_2 , \pm y_2)$. We have both μ_1 and λ_1 nonzero. We first suppose the nodes all lie on $P^{(2,2)}$. Now, $P^{(2,2)}$ and P_1 cannot have a common component; this is easily established by an argument on the coefficients. $P^{(2,2)}$ has at most two zeros on each axis, of the form $(\pm\alpha, 0)$ and $(0, \pm\beta)$, as claimed. If $c_2 = 0$, the zeros coalesce at the origin, but we will no longer have enough common zeros. Q_2 and $P^{(2,2)}$ are seen to have at most eight common zeros, symmetrically located with respect to the axis, i.e., $(\pm x_1 , \pm y_1)$, $(\pm x_2 , \pm y_2)$.

Now, suppose the nodes do not all lie on $P^{(2,2)}$. Then they lie on each of $P_2 = \mu_2 P^{(4,0)} + \lambda_2 P^{(2,2)}$, where $\mu_2 \neq 0$ and $P_3 = \mu_3 P^{(0,4)} + \lambda_3 P^{(2,2)}$, where $\mu_3 \neq 0$. We see that P_2 and P_3 can have at most four common zeros on the axis. We consider Q_2, P_2, and P_3 . If the three have a common component, we see by symmetry that Q_2 must be a factor of both P_2 and P_3 . Then

$$P_2 = (ax^2 + b)Q_2$$
$$P_3 = (cy^2 + d)Q_2 \quad .$$

Now, P_2 and P_3 have $(\pm\alpha, 0)$ and $(0, \pm\beta)$ as common zeros, so $bQ_2(0, \pm\beta) = 0$ and $dQ_2(\pm\alpha, 0) = 0$. If $b = 0$, P_2 has only the origin as a zero on the axis. This would not be enough zeros. Likewise if $d = 0$, P_3 has only the origin as a zero on the axis, which is impossible. Thus, $(\pm\alpha, 0)$ and $(0, \pm\beta)$ lie on Q_2 . But then all the nodes lie on Q_2 , which is impossible. Thus Q_2 cannot be a factor of both P_2 and P_3 . With no loss of generality, assume Q_2 is not a factor of P_2 . Then Q_2 and P_2 can have at most 8 common zeros, and they are of the stated form.

It now remains to be shown that no proper subset of twelve points of the form $(\pm\alpha,0)$, $(0,\pm\beta)$, $(\pm x_1 , \pm y_1)$, $(\pm x_2 , \pm y_2)$ can be used for a cubature formula of precision seven for the regions under consideration. Let $A^{(\alpha,0)}$ be the weight associated with the node $(\alpha,0)$. Consider $Q^{(\alpha,0)} = Q_2 \cdot x \cdot (x + \alpha)$. Now, $Q^{(\alpha,0)}$ is zero at all the nodes except $(\alpha,0)$, so $\int_R wQ^{(\alpha,0)} = A^{(\alpha,0)} Q^{(\alpha,0)}_{(\alpha,0)}$. Similarly, we have $\int_R wQ^{(-\alpha,0)} = A^{(-\alpha,0)} Q^{(-\alpha,0)}_{(-\alpha,0)}$. Now we note that because of symmetry $\int_R wQ^{(\alpha,0)} = \int_R 2w\alpha x^2$, $\int_R wQ^{(-\alpha,0)} = -\int_R 2w\alpha x^2$, and $Q^{(\alpha,0)}_{(\alpha,0)} = -Q^{(-\alpha,0)}_{(-\alpha,0)}$. Thus we have $A^{(\alpha,0)} = A^{(-\alpha,0)}$ and since a ten point formula cannot exist $A^{(\alpha,0)} \neq 0$. Similar consideration of $Q^{(0,\pm\beta)} = Q_2 \cdot y \cdot (y \neq \beta)^2$ yields $A^{(0,\beta)} = A^{(0,-\beta)} \neq 0$.

Now consider

$$Q^{(x_i,y_i)} = (x^2-x_j^{\,2} + y^2-y_j^{\,2}) \cdot x \cdot y \cdot (x + x_i)(y + y_i) \quad , \text{where } j = 3 - i \quad ,$$

for $i = 1,2$. It is easily seen that $Q^{(x_i,y_i)}$ is zero at all nodes except (x_i,y_i) . Let $A^{(x_i,y_i)}$ be the weight associated with $(x_i y_i)$, then we have $\int_R wQ^{(x_i,y_i)} = A^{(x_i,y_i)} Q^{(x_i,y_i)}_{(x_i,y_i)}$. Then we note that

$$Q^{(-x_i,y_i)}_{(-x_i,y_i)} = Q^{(x_i,-y_i)}_{(x_i,-y_i)} = Q^{(-x_i,-y_i)}_{(-x_i,-y_i)} = Q^{(x_i,y_i)}_{(x_i,y_i)}$$

and that $\int_R wQ^{(\pm x_i,\pm y_i)} = \int_R w(x^2 - x_j^{\,2} + y^2 - y_j^{\,2})x^2y^2$, and thus $A^{(x_i,y_i)} = A^{(-x_i,y_i)} = A^{(x_i,-y_i)} = A^{(-x_i,-y_i)} \neq 0$. Thus we have shown that all 12 points have nonzero weights, and thus no 11 point formula can exist. This concludes the proof of the theorem.

3.0 Conclusions

We have shown that 12 is a lower bound for the number of nodes in a cubature formula of precision seven for a symmetric planar region. As noted previously the existence of a 12 point formula for an arbitrary symmetric region has not been shown. To show that the author's construction [1] does not fail it is necessary to show that certain determinants involving the moments are not zero. This problem has been quite intractable, and it seems unlikely the approach of the author can be made to work. Some of the key inequalities used in [1] for fully symmetric regions fail to hold for arbitrary symmetric regions. A considerable effort has been expended in attempting to find a region for which the construction fails, but none has been found. As before the author believes the construction cannot fail, but is unable to prove it.

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