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MANPOWER PLANNING MODELS - I:
BASIC CONCEPTS

by

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and

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This report is the first in a series which present methods used in analyzing manpower planning problems. The concepts of stocks, flows, manpower classification and timing convention are introduced, together with basic flow conservation concepts and equilibrium analysis.

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1. Introduction.

This report is the first of a series and presents the basic concepts of manpower flow models. The notation, conventions, and definitions that will be used in this and subsequent reports is introduced. The three main concepts described in this report are (i) conservation of flow, (ii) equilibrium, and (iii) the relation between waiting times and the flow process. The usefulness of these ideas is not limited to manpower planning and we hope the reader will find these basic concepts useful in wide variety of other areas.

The report concludes with a brief discussion of the possible uses of manpower flow models and of the need for more structured models to provide useful answers to planning questions. Subsequent reports present models with more structure and examples of their application.

2. Manpower Classification.

Consider a manpower system which consists of N different classes of manpower $n = 1,2,...,N$. To prevent undue repetitions the words, type kind, and category are used as synonyms for class. In some applications we shall employ more specific and descriptive names such as rank, grade, state, or status.

The particular method of manpower classification is not important as long as each individual in the organization can be identified as a member of one and only class. Thus any classification scheme partitions the members of the organization into disjoint groups whose union is the entire organization.

The manpower classification $n = 0$, is special. People in class zero are not part of the organization, and we typically consider that an infinite pool of manpower exists outside the organization. It is convenient to explicitly consider class 0, since most manpower organizations have significant interactions with the external manpower pool.

There are a large number of ways to partition the members of any organization. The exact partitioning rule should be related to the eventual purpose of the manpower flow model. Several partitioning rules are listed to stimulate the reader's imagination.

Partitioning Rules:

(i) By common institutional rule: e.g. job, rank, pay level, etc.

(ii) By common personal characteristic, e.g. age, skill, test performance, etc.

(iii) By common past and future career patterns, e.g. length of service, final status, initial status, total time in organization, or entire career pattern.

Under (iii) one can have interesting classification rules which are not immediately obvious. For example, in planning models where alternative policies are to be tried the input of manpower to the organization can be classified by the career path it takes before it leaves. It is not known a priori which path a given individual will take, but such models still have important uses in manpower planning as we shall see in later reports.

Example 1: Students in a two year college can be classified by ranks F and S for freshmen and sophomores. They can also be partitioned by career pattern as follows:

Type	Career Pattern
1	FS
2	FFS
3	FSS
4	F

Type 1 students complete in two years, type 2 and 3 complete in three years, and type 4 does not complete.

Problem 1: Suppose for the past ten years three students of each type were admitted to the college. How many freshmen and sophomores do we have now? From now on, we follow the new policy shown below:

	Admissions/year	
Type	Old Policy	New Policy
1	3	4
2	3	0
3	3	4
4	3	8

The new policy is to dismiss those who need to repeat the freshman year. This makes more admissions possible. We have assumed that the four new admissions include one in type 1, one in type 3, and two in type 4. Trace the evolution of the system under the new policy for four years.

□

An essential part of modeling manpower systems is the interaction between different classes through time. The next section introduces the notion of manpower flow between classifications.

3. Stocks, Flows, and the Timing Convention.

This section presents definitions of manpower stocks and flows along with a timing convention. The definitions and conventions are to some degree arbitrary; however, a great deal of unnecessary confusion can be avoided if a convention is agreed upon and used throughout.

The manpower system evolves over time. New individuals join the system and individuals in the system remain in one classification for a time, then either move to another classification or leave the system. At certain points in time $t = 0,1,2,\ldots$ we imagine that all motion in the system stops and we count the number of individuals in each classification. These instants at which we observe and count the people in each classification are referred to in various ways, as time t, inventory point t, observation time t, and accounting point t.*

Definition

> Let $s_i(t)$ be the number of people in classification i at observation time t; $s_i(t)$ is the <u>stock</u> of class i manpower at accounting point t. The N-vector $s(t) = [(s_1(t), s_2(t), \ldots, s_N(t))]$ gives the stock of the manpower system at time t.

Notice that class zero manpower has been omitted from our definition. We shall find that the role of manpower outside the organization will differ greatly from one application to the next.

The interval of time between observation points $t - 1$ and t is defined to be <u>period t</u>. To be more precise, period t is the time interval $(t-1,t]$; thus time t is the last instant in period t, and time $t - 1$ marks the beginning of period t, although it is not included in the period.

Definition:

> Let $f_{ij}(t)$ be the number of individuals that start period t in classification i and finish period t in classification j. The variable $f_{ij}(t)$ is called the <u>flow from i to j in period t.</u>

---

*The observation points t are not necessarily evenly spaced in time. The index t does count the number of observation points since time zero. We generally assume, however, that the observation points are evenly spaced in time.

The timing conventions are depicted in Figure I.1, below. The stock $s_i(t-1)$ is divided into several flows. One of the flows moves to classification j during the period. The stock $s_j(t)$ is the sum of all individuals that flow into class j during the period. Note that $f_{jj}(t)$ is simply the number of individuals who start and end period t in class j. Clearly the flows $f_{ij}(t)$ must be non-negative.

The flow $f_{i0}(t)$ and $f_{0i}(t)$ are respectively the number of class i individuals who leave the system in period t, and the number of people who join the system in period t and are first counted in class i.

The $(N+1)$-component vectors $f_{i*}(t) = [f_{i0}(t), f_{i1}(t), \ldots, f_{iN}(t)]$ and $f_{*j}(t) = [f_{0j}(t), f_{1j}(t), \ldots, f_{Nj}(t)]$ are respectively the flows from class i and to class j in period t. Finally, let $f(t)$ be the $(N+1)^2$-vector $f(t) = [f_{0*}(t), f_{1*}(t), \ldots, f_{N*}(t)]$.

Example 2: Consider a two class organization over ten days and the histories of four individuals A, B, C, D. At $t = 0$ the system is taken to be empty.

		Observation Time									
		1	2	3	4	5	6	7	8	9	10
Individual	A	0	1	1	2	1	2	2	1	0	0
	B	0	0	2	2	2	1	2	1	1	2
	C	2	2	1	1	1	1	2	1	1	1
	D	1	1	2	2	2	2	2	1	2	2

The table shows the classification of each individual at the observation times. To be precise, we assume flow between classifications takes place from 8:00 am - 5:00 pm each day. Class zero indicates the individual is outside the organization. The organization has two internal states.

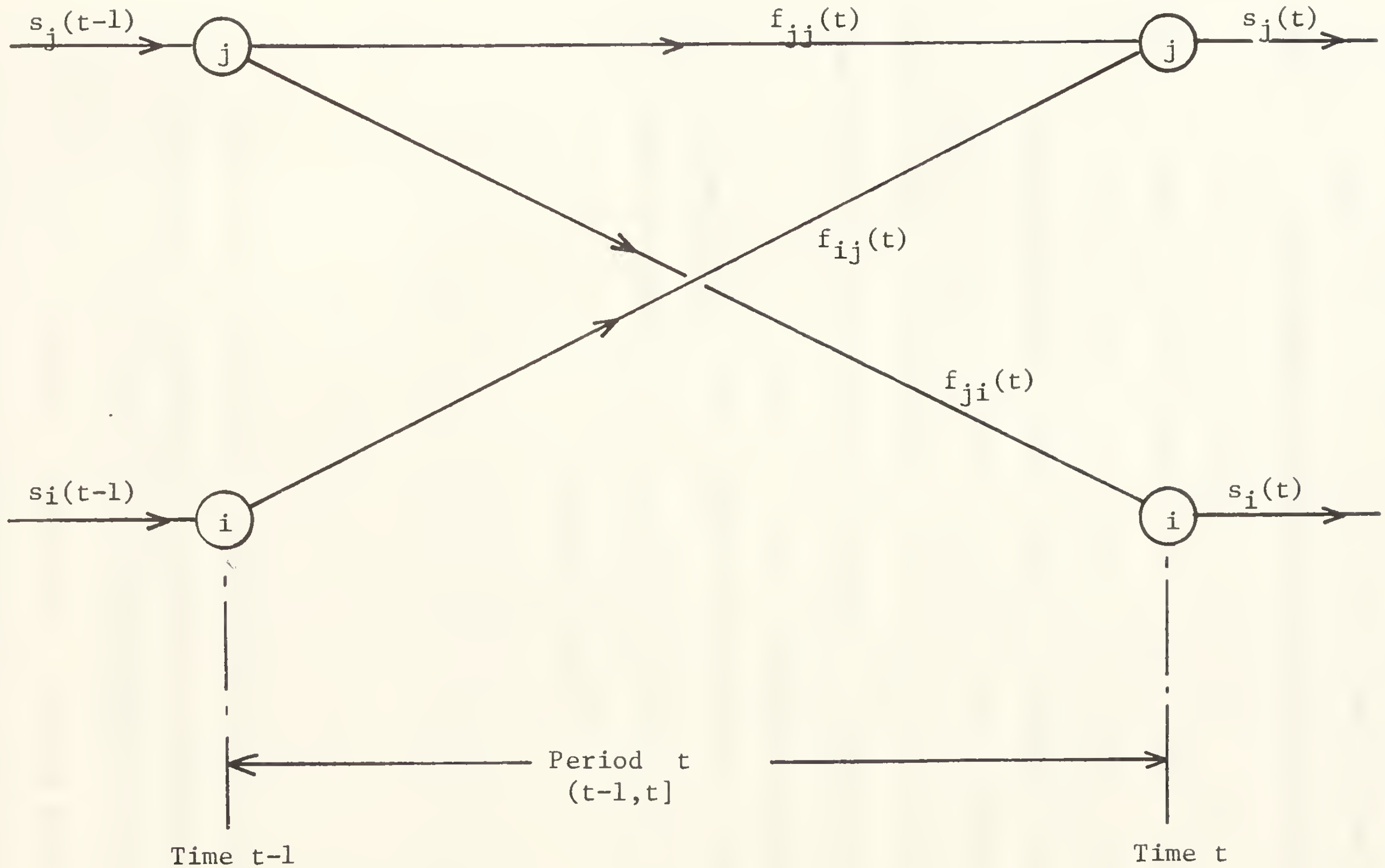


Figure I.1 : The Timing and Flow Conventions

With this history the stocks and flows are:

Observation Time

		0	1	2	3	4	5	6	7	8	9	10
Stock	$s_1(t)$	0	1	2	2	1	2	2	0	4	2	1
	$s_2(t)$	0	1	1	2	3	2	2	4	0	1	2

Period

		1	2	3	4	5	6	7	8	9	10
Flow	$f_{01}(t)$	1	1								
	$f_{02}(t)$	1		1							
	$f_{10}(t)$									1	
	$f_{11}(t)$		1	1	1	1	1			2	1
	$f_{12}(t)$			1	1		1	2		1	1
	$f_{20}(t)$										
	$f_{21}(t)$			1		1	1		4		
	$f_{22}(t)$		1		2	2	1	2			1

Blank entries represent zero flows.

<u>Example 3</u>: Given the same data as example 2, suppose we inventory every other day at 6:00 pm. The following stocks and flows are counted.

Observation Time

		0	1	2	3	4	5
Stock	$s_1(t)$	0	2	1	2	4	1
	$s_2(t)$	0	1	3	2	0	2

	Period				
	1	2	3	4	5
$f_{01}(t)$	2				
$f_{02}(t)$	1	1			
$f_{10}(t)$					1
$f_{11}(t)$			1	2	1
$f_{12}(t)$		2			2
$f_{20}(t)$					
$f_{21}(t)$		1	1	2	
$f_{22}(t)$			2		

Again, blank entries represent zero flows.

Problem 2: Find another history of four individuals that leads to the same stocks and flows calculated in example 3. Do not use the same histories as in example 2.

4. Conservation of Flow

A simple accounting relation must hold between the stocks and flows introduced above. Every individual classified in the system (classes 1 through N) at observation time t, must be in some class at observation times $t - 1$ and $t + 1$. Thus it is possible to evaluate $s_i(t)$, the number of people in class i at time t, either by conditioning on their prior class or their subsequent class. The equations are

$$(1) \qquad \sum_{j=0}^{N} f_{ji}(t) = s_i(t) = \sum_{j=0}^{N} f_{ij}(t+1) ,$$

$$f_{ij}(t) \geq 0.$$

Equation (1) is the fundamental conservation of flow relation.

In many applications certain flows are equal to zero. When this is the case it is more convenient to use a matrix form of the flow conservation law. We illustrate this point by an example.

Example 4: The faculty of a university is partitioned into three classes: 1-non-tenured, 2-tenured, and 3-retired. Due to institutional restrictions. there is no flow from the tenured class to the non-tenured class; thus $f_{21}(t) = 0$. In addition, there is no flow from the retired class to either tenure or non-tenure; thus $f_{31}(t) = f_{32}(t) = 0$ for all t. Similarly, $f_{03}(t)$ is zero, the flow $f_{13}(t)$ is likely to be zero, and there seems to be no reason to consider the flow $f_{00}(t)$. With these points in mind we form the 10 component flow vector

$$f(t) = [f_{01}(t), f_{02}(t), f_{10}(t), f_{11}(t), f_{12}(t), f_{20}(t), f_{22}(t), f_{23}(t), f_{30}(t), f_{33}(t)].$$

The matrices B and A below are used to partition and sum the flows in order to enumerate the members of the class by their class at the observation point before (B) and after (A) time t.

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The conservation of flow relations are:

(2) $$Bf(t) = s(t) = Af(t+1),$$

$$f(t) \geq 0 .$$

In the general case, let $k = 1,2,\ldots,K$ index the possible flows. For each k there is an $i(k)$ and $j(k)$ which indicates that flow k moves from class i to class j. Then B and A are $N \times K$ matrices with

$$B_{ik} = \begin{cases} 1 & \text{if } j(k) = i \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_{ik} = \begin{cases} 1 & \text{if } i(k) = i \\ 0 & \text{otherwise.} \end{cases}$$

With this definition equation (2) describes the conservation of flow for any system.

Example 5: (Continuation of example 4.) The functions $i(k)$ and $j(k)$ are tabulated below

k	1	2	3	4	5	6	7	8	9	10
i(k)	0	0	1	1	1	2	2	2	3	3
j(k)	1	2	0	1	2	0	2	3	0	3

□

The general flow of manpower is described by a system of equations

$$
\begin{array}{llr}
 & Af(1) & = s(0) \\
 & -Bf(1) + Is(1) & = 0 \\
(3) & \quad -Is(1) + Af(2) & = 0 \\
 & \quad\quad -Bf(2) + Is(2) & = 0 \\
 & \quad\quad\quad -Is(2) + Af(3) & = 0 \\
 & \quad\quad\quad\quad \vdots \quad \vdots & \vdots
\end{array}
$$

where I is an N x N idenity matrix, and $s(t)$ and $f(t)$ are non-negative vectors. These equations are network flow conservation relations. The columns associated with $f_{ij}(t)$ or $s_i(t)$ contain exactly one positive element (+1) and one negative element (-1). If K flow combinations are possible, then the system of equations (3), over T time periods has $T \times (K+N)$ variables and $2 \times T \times N$ constraints. The flow network for the faculty system presented in example four is depicted below in Figure I.2. Notice that time elapses during the flow phase, but that zero time elapses during the counting phase.

Problem 3. Construct the A and B matrices for example 2, and show that the flows calculated in example 2 satisfy the conservation relations.

Problem 4. In a hierarchy the manpower classifications are ranked so that position i is dominated by j if $j > i$. If we assume $f_{ij}(t) = 0$ if $j < i$ and $j \neq 0$, then what will be the dimension of A and B for an N classification hierarchy?

The hierarchy is strict if $f_{ij}(t)$ differs from zero only when $j = i$, $i + 1$, or 0. What are the dimensions of A and B for an N classification strict hierarchy?

□

5. Equilibrium

The notion of equilibrium is important in the study of physical social, and economic processes and it will play a central role in our study of manpower flow systems. We do not believe that many manpower systems are in equilibrium. However, the simplifications that result in analyzing an equilibrium system make for a useful approximation to the actual system and the examination of the equilibrium consequences of any fixed (stationary) policy is essential in uncovering the direction of change implied by the policy and for discovering the policy's long run implications.

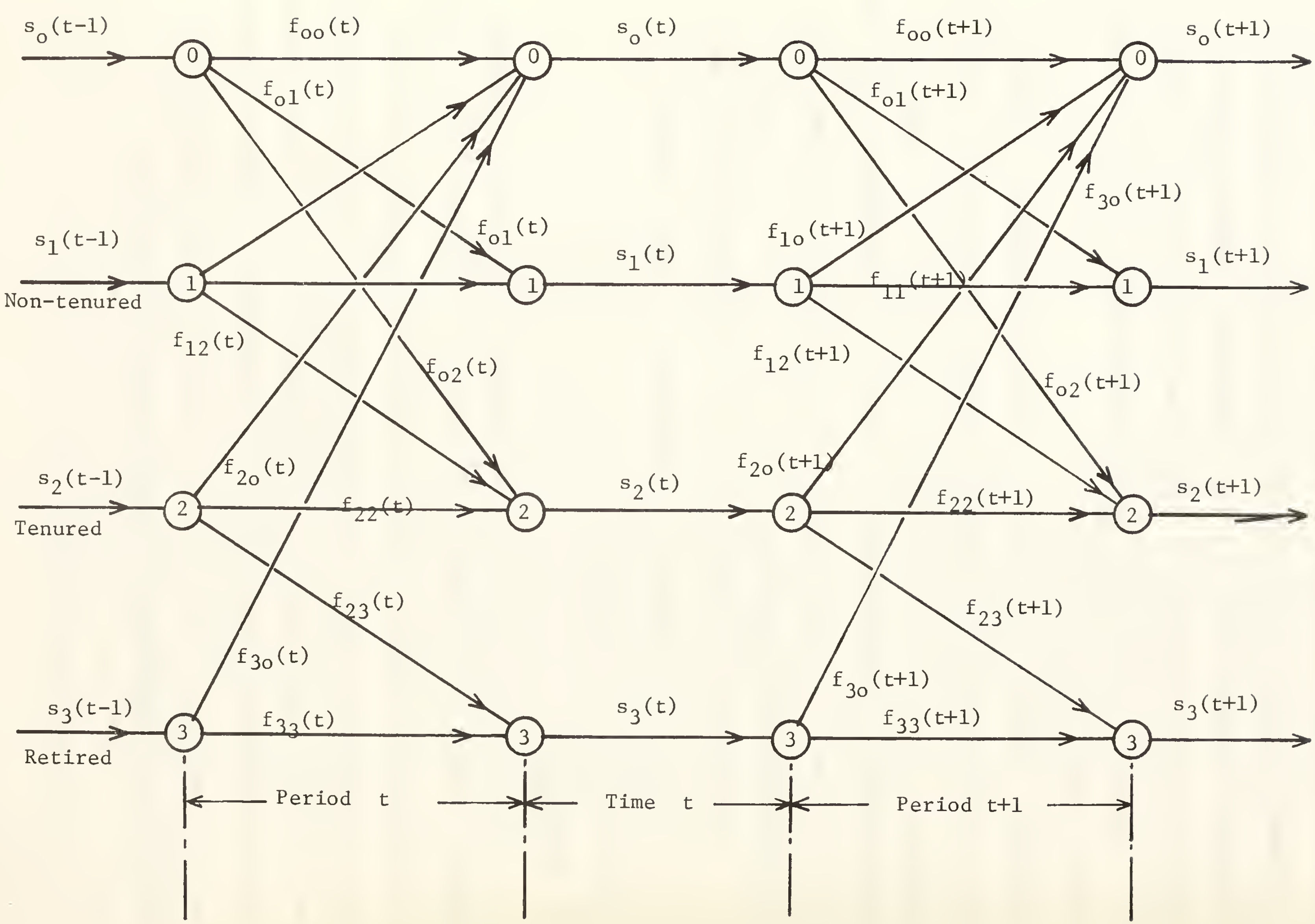


Figure I.2 : Flow Network for Example 4, a University Faculty

Equilibrium indicates some degree of regularity over time. It will be useful to define some terms that describe systems over time. First, a <u>transitory</u> system is one which moves from one trivial equilibrium to another. An example is a one time operation like an election campaign (which starts with no one in it and ends after the election with no one in it). A <u>transient</u> system is one on its way to equilibrium. The policy governing the system is an equilibrium policy, however. The initial conditions are such that equilibrium is not immediately obtained. We shall consider several types of equilibrium; constant size, expanding or contracting, geometric or arithmetic growth. Finally, we shall use the term <u>steady-state</u> interchangeably with equilibrium.

This section will characterize the equilibria that describe a constant size system. Transitions between equilibria and expanding (or contracting) systems will be examined in a later section.

The system (3) is defined to be in equilibrium if $f(t) = f$ for all t. It follows that $s(t) = s$ for all t, and that

(4) $$s = Bf = Af, \qquad f \geq 0.$$

The equations that characterize the possible equilibria can also be interpreted as network flow equations,

(5) $$\begin{bmatrix} A & -I \\ -B & I \end{bmatrix} \begin{bmatrix} f \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$f \geq 0.$$

In general, (5) will contain $2N$ independent equations and $N + K$ nonnegative variables (f and s).

Example 6: The faculty system presented in example 4, has the following equilibrium flow equations. There are 6 equations and 13 unknown.

f_{01}	f_{02}	f_{10}	f_{11}	f_{12}	f_{20}	f_{22}	f_{23}	f_{30}	f_{33}	s_1	s_2	s_3	
		1	1	1						-1			= 0
					1	1	1				-1		= 0
								1	1			-1	= 0
-1			-1							1			= 0
	-1			-1		-1					1		= 0
							-1		-1			1	= 0

The network corresponding to an equilibrium flow in this system is presented below in figure 3.

Example 7: The flows and stocks given below satisfy the conservation equations for the faculty system

f_{01}	f_{02}	f_{10}	f_{11}	f_{12}	f_{20}	f_{22}	f_{23}	f_{30}	f_{33}	s_1	s_2	s_3
17	4	6	80	11	10	320	5	5	95	97	335	100

Problem 5: Write out the equilibrium flow equations for the network of problem 3. Fix $s_1 = 10$, $s_2 = 5$, and $f_{01} = f_{02} = 1$, $f_{00} = f_{10} = 0$. Show this implies $f_{20} = 2$. Now solve for f_{12}, f_{11}, and f_{21} in terms of f_{22}. Calculate the flows for $f_{22} = 0,1,2,3$.

□

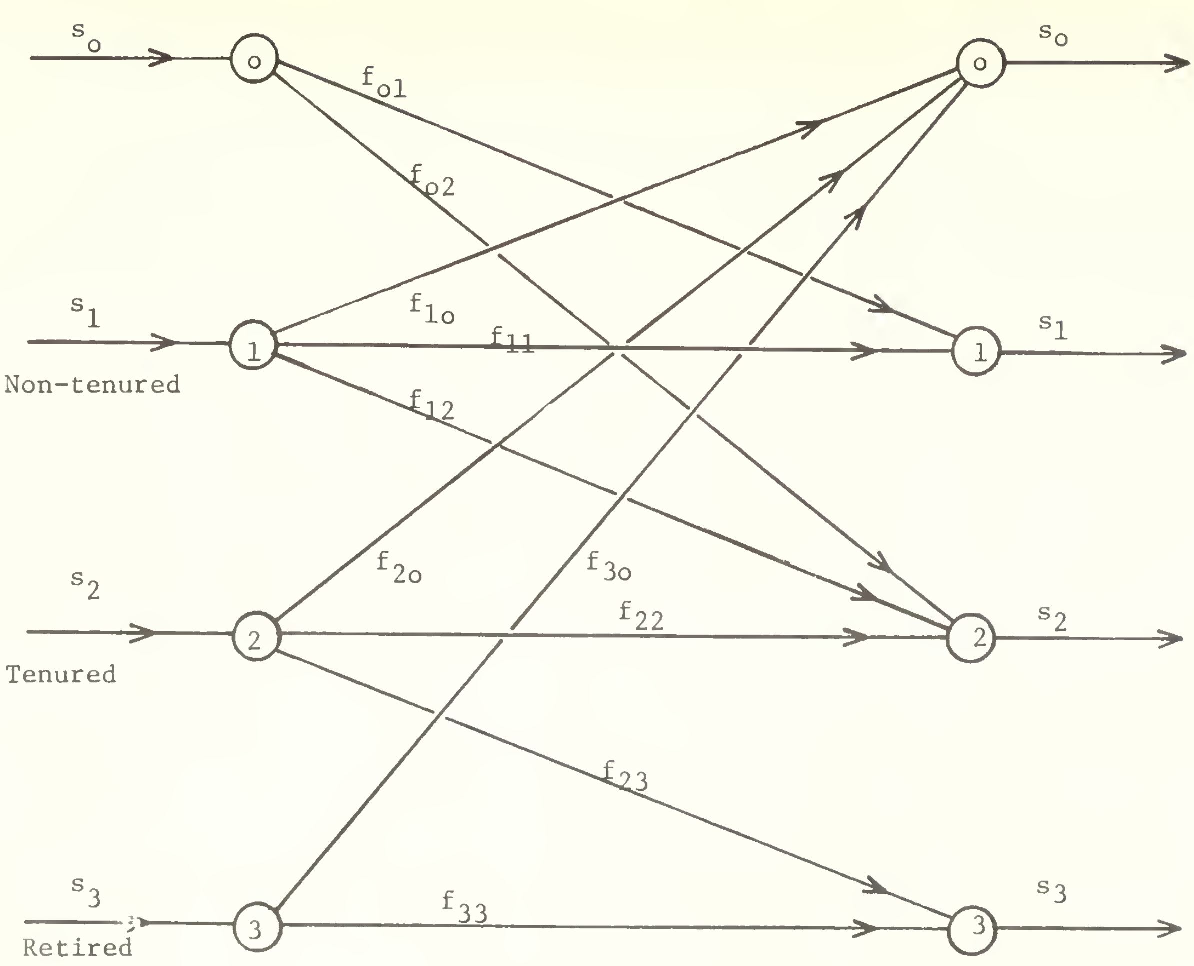


Figure I.3: Equilibrium Flow in the 3-class Faculty Example

6. Average Lifetime in a Class.

We define the lifetime (or term) of an individual in a manpower class as the number of consecutive time points $t = 1,2,3,\ldots$ in which the individual is counted as a member of that class. In this section we derive a simple expression for the average lifetime in any class when the system is in equilibrium.

Example 8. Suppose the time point t marks the start of a month. An individual who joins a manpower class on March 13th and leaves September 20th of the same year has a lifetime of six months in the manpower classification, since the individual was counted in that class at the beginning of April, May,..., September.

□

It follows that the lifetime of any individual in a class is a positive integer. Let ℓ_i be the average lifetime in class i, where the average is over the group of individuals that arrive in class i in any period. We shall show that

$$\ell_i = s_i/(s_i - f_{ii}) = 1/1 - (f_{ii}/s_i). \tag{6}$$

In each period there are $s_i - f_{ii}$ new arrivals in class i. Let $k = 1,2,\ldots,s_i-f_{ii}$ index these arrivals and let $\ell_{i,k}$ be the lifetime in class i of arrival k. The average lifetime is thus

$$\ell_i = \left(\frac{1}{s_i-f_{ii}}\right) \sum_{k=1}^{s_i-f_{ii}} \ell_{i,k}\,.$$

Now for $m = 1,2,\ldots$ let n_m be the number of arrivals with lifetime equal to m. It follows that

$$s_i - f_{ii} = \sum_{m=1}^{\infty} n_m\,,$$

and

(7) $$\ell_i = \frac{1}{s_i - f_{ii}} \sum_{m=1}^{\infty} m n_m \, .$$

Now we exploit the assumption of equilibrium. If the system is in equilibrium then $\ell_{i,k}$ is the same in each period. The individuals in class i at any inventory time t can be identified by their eventual lifetime in class i. Let h_m be the number with lifetime equal to m. It follows that $h_1 = n_1$; i.e., all those entering period t with lifetime equal to 1. Moreover $h_2 = n_2 + n_2 = 2n_2$; i.e., all individuals with $\ell_{ik} = 2$ that joined in periods $t - 1$ and t. Earlier arrivals with $\ell_{ik} = 2$ have already departed. In general then we see that $h_m = mn_m$. However, h_m is also a partition of s_i according to duration, thus

(8) $$s_i = \sum_{m=1}^{\infty} h_m = \sum_{m=1}^{\infty} m n_m \, .$$

When (8) is substituted into (7) equation (6) results.

<u>Problem 6</u>: Show that no equilibrium exists if ℓ_{ik} is not finite.

□

Notice from the argument above that there is another tempting way to talk about average lifetime. Suppose at time t, we determine the durations of the individuals in class i. There will be s_i individuals, and there will be $h_m = mn_m$ with duration equal to m. The average duration, λ_i, over this group is clearly

(9) $$\lambda_i = \frac{1}{s_i} \sum_{m=1}^{\infty} m h_m = \frac{\sum_{m=1}^{\infty} m^2 n_m}{\sum_{m=1}^{\infty} m n_m} \, .$$

Problem 7: Let M be the largest m such that $n_m > 0$. If $M > 1$, show that $\lambda_i > \ell_i$. When $M = 1$, show that $\lambda_i = \ell_i$. □

We see that except in the trivial case $(M = 1)$, the second method strictly overestimates average lifetime. This phenomena is known as the inspection paradox. In the first averaging method we sample flow into the classification; in the second method we sample the stock in the classification. The stock necessarily contains a larger proportion of the individuals with long lifetime in the class. This phenomenon is also known as "length-biased sampling."

Example 9: Consider the equilibrium flow for our three class faculty system that was calculated in Example 7. The average lifetimes are

$$\ell_1 = 5.7 ,$$

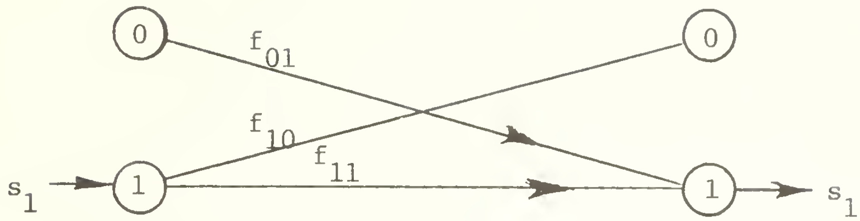
$$\ell_2 = 22.3 ,$$

$$\ell_3 = 20.0 .$$

Example 10: Each year 1000 new students arrive at a university. The breakdown of the 1000 entrants by duration is:

m	n_m
1	200
2	100
3	200
4	500 .

We see that $\ell_1 = 3$. This implies that $f_{11}/s_1 = 2/3$ or $f_{11} = 2/3\ s_i$. The equilibrium flow relations are



$$s_1 = f_{10} + f_{11} = f_{11} + f_{01}$$

$$f_{01} = 1000 = f_{10} \Rightarrow s_1 = 1000 + f_{11}$$

$$\Rightarrow f_{11} = 2000 \quad \text{and} \quad s_1 = 3000.$$

Moreover, the breakdown of the stock of students by lifetime is

m	h_m
1	200
2	200
3	600
4	2000 .

The average lifetime among students is $\frac{1}{3000}$ (200+400+1800+8000) = 3.46. Note also that 50% of all entrants have lifetime of 4 years while 67% of all students have a lifetime of 4 years.

Problem 8: Continuation of problem 5. Express the lifetime as a function of f_{22}. Calculate for $f_{22} = 0,1,2,3$.

Problem 9: You ask a consultant for a simple way to measure how long on the average a person stays with your organization. He suggests you "randomly select" 10% of your current personnel, determine from their records when they entered your organization, and follow their records until they leave. Averaging these lifetimes will give you a good estimate of the average lifetime of your personnel. Is his advice good? If not, what do you think is wrong and what would you do to improve it?

7. Arithmetic Expansion.

It is possible for the system to expand in an arithmetic manner. Let $f(t)$ be determined by the difference equation

$$f(t+1) = f(t) + g = f(1) + tg, \tag{10}$$

where $f(1)$ and g are given vectors. It follows from the general flow equations (3), that

$$Af(1) = s(0) \quad \text{and} \quad Bf(1) = s(1).$$

In general we have for $t \geq 1$, that

$$s(t) = Af(t+1) = Af(1) + Ag + (t-1)Ag = s(0) + Ag + (t-1)Ag,$$

$$s(t) = Bf(t) = Bf(1) + (t-1)Ag = s(1) + (t-1)Bg.$$

It follows, for $t = 1$, that $Ag = s(1) - s(0)$. In addition, we must have $Ag = Bg$. Thus $f(1)$ and g must satisfy

$$Bg = Ag = (B-A)f(1), \tag{11}$$

$$g \geq 0, \quad f(1) \geq 0.$$

Example 11: A solution of (11) for the three class faculty example is given below.

	01	02	10	11	12	20	22	23	30	33
f(1)	10	4	3	19	4	1	26	4	1	8
g	2	1	1	1	1	1	1	1	1	2

Problem 10: Calculate $f(t)$ and $s(t)$ for five periods given the data in example 11.

Problem 11: If some element of g is negative, derive an expression for the first time such that an element of $f(t)$ or $s(t)$ becomes negative.

Problem 12: Given $s(0)$, and the knowledge that (2) has a solution with $g \neq 0$, is there a maximum growth solution of (2)?

8. Geometric Expansion and Contraction.

In contrast to the static equilibrium described by equation (4) or the arithmetic growth determined by equations (10) and (11) we can consider a geometric change in the system where, for some scalar θ, $f(t) = \theta^t f$. The system is expanding if $\theta > 1$, contracting if $\theta < 1$, and of constant size if $\theta = 1$. The basic flow equations (2) become

$$s(t) = \theta^t Bf = \theta^{t+1} Af = \theta^t s$$

This implies f and s must satisfy

$$\begin{bmatrix} \theta A & -I \\ -B & I \end{bmatrix} \begin{bmatrix} f \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$f \geq 0 .$$

Example 12: For the three class faculty model of example 4 the equilibrium equations are

	f_{01}	f_{02}	f_{10}	f_{11}	f_{12}	f_{20}	f_{22}	f_{23}	f_{30}	f_{33}	s_1	s_2	s_3	
			θ	θ	θ						-1			= 0
						θ	θ	θ				-1		= 0
									θ	θ			-1	= 0
(13)	-1			-1							1			= 0
		-1			-1		-1					1		= 0
								-1		-1			1	= 0.

Note that $\theta = 1$, reduces to the constant size case and that for $\theta \neq 1$, the network flow interpretation of the equilibrium equations (10) is lost. A feasible growth equilibrium for $\theta = 1.05$ is given by

f_{01}	f_{02}	f_{10}	f_{11}	f_{12}	f_{20}	f_{22}	f_{23}	f_{30}	f_{33}	s_1	s_2	s_3
54	33	20	156	24	30	258	12	7	93	210	315	105

<u>Problem 13</u>: Will (12) have a nontrivial $(f,s) \neq 0$, solution for any $\theta > 0$?

<u>Problem 14</u>: Find a solution for (13) when $\theta = .95$ and s_1 and s_2 are in the same proportion (3/2) as in example 12.

□

The formula (6) for average lifetime in each class does not apply. However, it is possible to obtain an approximate expression for average duration.

The inflow into a class in period t is $\theta^t[s_i - f_{ii}]$. Assume that in period t there are $\theta^t n_m$ entrants with a lifetime of m periods. As before the average duration is $\sum_{m=1}^{M} mn_m/(s_i - f_{ii})$. However, the sum $\sum_{m=1}^{M} mn_m$ is not equal to s_i when $\theta \neq 1$. Using the same logic as before we can determine that

$$s_i(t) = \theta^t \sum_{m=1}^{M} n_m \left(\sum_{j=0}^{m-1} \theta^{-j} \right) = \theta^t \; s_i \; .$$

The linear Taylor expansion for s_i around the value $\theta = 1$ is given by

$$s_i = \sum_{m=1}^{M} n_m \left(\sum_{j=0}^{m-1} \theta^{-j} \right) \cong \sum_{m=1}^{M} mn_m - \frac{(\theta-1)}{2} \sum_{m=1}^{M} (m^2-m)n_m \, ,$$

or

$$\sum_{m=1}^{M} mn_m \cong \left(\frac{2}{\theta+1}\right) s_i + s\left(\frac{\theta-1}{\theta+1}\right) \sum_{m=1}^{M} m^2 n_m \; .$$

From this we see that

$$\ell_i \cong \left(\frac{2}{\theta+1}\right) \frac{s_i}{s_1 - f_{ii}} + 2\left(\frac{\theta-1}{\theta+1}\right) \sum_{m=1}^{M} m^2 n_m \; .$$

If we use (9), and substitute $\lambda_i \sum_{m=1}^{M} mn_m$ for $\sum_{m=1}^{M} m^2 n_m$, we obtain

$$\ell_i \cong \left(\frac{2}{1+\theta-(\theta-1)\lambda_i}\right) \left(\frac{s_i}{s_i - f_{ii}}\right) \quad \text{when} \quad \theta \cong 1 \; .$$

This equation gives a useful approximation to the lifetime ℓ_i, which is simple to calculate and is in terms of the growth (or decay) rate θ.

9. Uses and Need for Structure.

To this point we have presented a rather general model of the manpower flow process and by making special assumptions we have been able to characterize constant size, arithmetic, and geometric equilibria and have obtained an expression for the average lifetime in a manpower class for the constant size equilibrium.

Manpower planning models are useful to the extent that they can show the impact of alternate policy decisions on indicators of system performance. System performance is generally some function of the stocks and flows, while manpower policy is concerned with hiring, promotion, termination, and remuneration. To this point our model does not relate policy and performance in any meaningful way. The conservation of flow relations, (3), merely tell us, in the broadest sense, what stocks and flows are permissible. In fact, as example 13 below will attest, they admit some unrealistic possibilities.

To obtain results that link policy and performance it is necessary to make some assumptions. Our scant results to this point follow from the equilibrium assumption. In the reports that follow we shall describe several assumptions about manpower flow processes and extract as much theoretical and operational information from these assumptions as possible.

Example 13: For the three class faculty example, the following equilibrium stocks and flows are feasible:

f_{01}	f_{02}	f_{10}	f_{11}	f_{12}	f_{20}	f_{22}	f_{23}	f_{30}	f_{33}	s_1	s_2	s_3
10,000	1	10,000	0	0	0	0	1	1	0	10,000	1	1

The reader can see, using Figure I.2, that f_{01}, f_{10} and s_1 can be made arbitrarily large with all other flows and stocks fixed, and equations (3) will still be satisfied.

10. Notes and Comments

Manpower flow models have been analyzed by a number of authors in the 1950's and 1960's, but most results have appeared in research reports and papers. Recently a number of textbooks have appeared which cover various aspects of manpower planning. A list of some of these is given below. The list is not intended to be a complete bibliography, but each text is a valuable source for further reading and references.

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