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[Lambda]-Laplace processes

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ℓ-LAPLACE PROCESSES

by

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EDWARD MCKENZIE

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A broad family of symmetric, thick tailed distributions, the ℓ-Laplace distributions, is described. They are natural generalizations of the Laplace distribution. A family of random coefficient ARMA processes with ℓ-Laplace marginal distributions is constructed and its properties are explored. Extensions to ARIMA processes are considered. The first order autoregression with a standard Laplace marginal distribution is examined in detail and different estimates of the autoregressive parameter are compared theoretically and by simulation.

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1. INTRODUCTION

Random coefficient models and ordinary linear models with non-Gaussian marginal distributions have been developed for a variety of interesting situations in time series analysis to offer viable alternatives to the standard Gaussian assumptions (see Lewis, 1985, for a survey). Gastwirth and Wolff (1965) in an unpublished manuscript developed a stationary linear first order autoregressive process (i.e., one which satisfies the equation $X_n = \rho X_{n-1} + \epsilon_n$, $n \geq 1$; $-1 < \rho < 1$) with a Laplace marginal distribution for the X_n's (called LAR(1)). Independently Gaver and Lewis (1980) developed a linear AR(1) process satisfying the same process but with a Gamma marginal distribution, called GAR(1). Subsequently both of these processes have been shown to be special cases of more general discrete random coefficient autoregressive models (Dewald and Lewis, 1985 and Lawrance and Lewis, 1981 and 1985). Other time series models using continuous random coefficients with a specified marginal distribution are, for Gamma distributions, due to Lewis (1981), Hugus (1982), and Lewis, McKenzie and Hugus (1989), and, with Beta distributions, due to McKenzie (1985). A first-order autoregressive process with the symmetric thick tailed hyperbolic secant distribution has been discussed by Rao and Johnson (1988). Nicholls and Quinn (1982) discuss general random coefficient autoregressive processes without reference to a particular marginal distribution.

The Laplace LAR(1) model, and its generalizations to higher order correlation structures, was put forward by Dewald and Lewis, (1985) as a model where two-sided symmetrical random variables had larger kurtosis and longer tails than could be expected from Gaussian time series. A particular example is that of position errors in a large navigation system (Hsu, 1979) which were

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found to have Laplace distributions. Again the N-S or E-W components of wind velocity data are often symmetric and long-tailed, especially in the tropics. For a summary of these applications and a summary of methods of generating non-normal time series, see Lewis (1985) and Dewald and Lewis (1985).

Another application of the Laplace models arises when positive-valued Gamma time series are differenced to remove trends. The resulting marginal random variables are two-sided and result in the ℓ-Laplace family of distributions which we consider in this paper.

In Section 2, we introduce this ℓ-Laplace family of distributions. These infinitely divisible, additive, symmetric distributions have extremely thick tails for small values of the parameter ℓ and approach a normal distribution as ℓ increases. A square-root-Beta-Laplace transform is introduced which allows us to transform one member of the ℓ-Laplace family into another in a simple manner.

Section 3 introduces ℓ-Laplace processes with first order autoregressive structure. These are Markov processes with the geometric autocorrelation function which is typical of the Gaussian, first order autoregressive (AR(1)) process. The basic structure is a random coefficient autoregression, and we generalize the structure of this model in Section 4 to encompass moving average (MA) and mixed first-order autoregressive, qth-order moving average (ARMA) processes. Again, both the structure of the process and their autocorrelation functions mirror those of the Gaussian ARMA processes. We also briefly compare these processes with others which may be obtained by differencing the random coefficient Gamma processes of Lewis, McKenzie and Hugus (1989). Finally, we show that a most important application of ℓ-Laplace processes lies in the modelling of differenced, possibly non-stationary Gamma

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processes.

In Sections 4 and 5, we consider the very important and particular case $\ell = 1$, i.e. the Laplace distribution. The first order autoregression, the 1-BELAR(1) process, is discussed in some detail and parameter estimates and their properties are discussed and assessed both theoretically and using simulation.

2.   THE ℓ-LAPLACE FAMILY OF DISTRIBUTIONS

2.1 ℓ-Laplace Random Variables

The probability density function of a Laplace distributed random variable, L, has two parameters --- a location parameter, $-\infty < \mu < +\infty$, and a scale parameter, $\lambda > 0$ --- so that

$$f_L(x, u, \lambda) = \frac{1}{2\lambda} \exp\{-|x-\mu|/\lambda\} , \qquad -\infty < x < +\infty. \quad (2.1)$$

We assume without loss of generality that $\mu = 0$ and $\lambda = 1$ throughout the remainder of this discussion.

The characteristic function of the standard Laplace random variable ($\mu = 0$ and $\lambda = 1$) is

$$\phi_L(\omega) = (1 + \omega^2)^{-1} , \qquad -\infty < \omega < +\infty. \quad (2.2)$$

It is well-known that the standard Laplace distribution belongs to the class of infinitely divisible distributions. Thus

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$$\phi_X(\omega) = (1+\omega^2)^{-\ell}, \qquad \ell > 0, \ -\infty < \omega < +\infty. \quad (2.3)$$

is the characteristic function of a random variable. In fact it is the
difference of two independent, identically distributed, i.i.d, Gamma(ℓ,1)
random variables, where ℓ is the shape parameter and 1 is the value of the
scale parameter, k, of the Gamma variable. If X has a characteristic function
given by (2.3), then X is said to be an ℓ-Laplace random variable.

Since (2.3) is real-valued in ω, X is a symmetric random variable. It is
also easily verified that

$$E(X^n) = \begin{cases} 0 & \text{if n is odd,} \\ (k+1)_k(\ell)_k & \text{if } n = 2k, \ k=1,2,\ldots, \end{cases} \quad (2.4)$$

where $(b)_k = b(b+1)\ldots(b+k-1)$ for $b > 0$.

Thus, in particular, var(X) = 2ℓ. From (2.4) we have

$$\frac{E(X^4) - [E(X)]^4}{[Var(X)]^2} = 3 + 3/\ell, \quad (2.5)$$

and the kurtosis approaches 3, as $\ell \to \infty$, which corresponds to the kurtosis of a
normal distribution.

We note too that we are dealing with an additive family of infinitely
divisible distributions, in the sense that the shape parameter is additive.
In particular, if X and Y are independent ℓ_1-Laplace and ℓ_2-Laplace
respectively, then X+Y is $(\ell_1+\ell_2)$-Laplace. This is easily verified using the
characteristic function (2.3). Equally important is the obvious fact that X-Y
is $(\ell_1+\ell_2)$-Laplace. Thus differencing keeps one in the same family of

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distributions.

2.2 The ℓ-Laplace Density

Since X is the difference of two i.i.d. Gamma(ℓ,1) random variables, we can obtain the following density for X;

$$f_X(x,\ell) = \int_{L(x)}^{\infty} \frac{1}{\Gamma^2(\ell)} \left[\frac{1}{g(g+x)} \right]^{1-\ell} \exp\{-(2g+x)\}dg, \qquad (2.6)$$

where

$$L(x) = \text{Max}(0,-x) \qquad (2.7)$$

Now if ℓ is a positive integer, (2.6) and (2.7) can be evaluated analytically. If $\ell = 1$, we obtain the density of the standard Laplace distribution. For $\ell = 2$, 3 and 4 the densities are also well-known derivations given, for example, as textbook problems in Feller (1971). Figure 2.1 displays the densities for $\ell = 1,2,3$ and 4. Note how the graphs approach the shape of a normal density with $\sigma^2 = 2\ell$ as ℓ increases.

When ℓ is not an integer, (2.6) must be evaluated numerically. Figure (2.2) displays examples of ℓ-Laplace densities for non-integral ℓ. Note that for $\ell \leq 1$, the density is not absolutely continuous at zero. In fact for $\ell \leq 0.5$, the density is infinite at zero. For $\ell > .5$ we have

$$f_X(0,\ell) = \Gamma(2\ell-1)/\{\Gamma^2(\ell)2^{2\ell-1}\} < \infty. \qquad (2.8)$$

For details on integrating (2.6) and (2.7) numerically, see Dewald (1985).
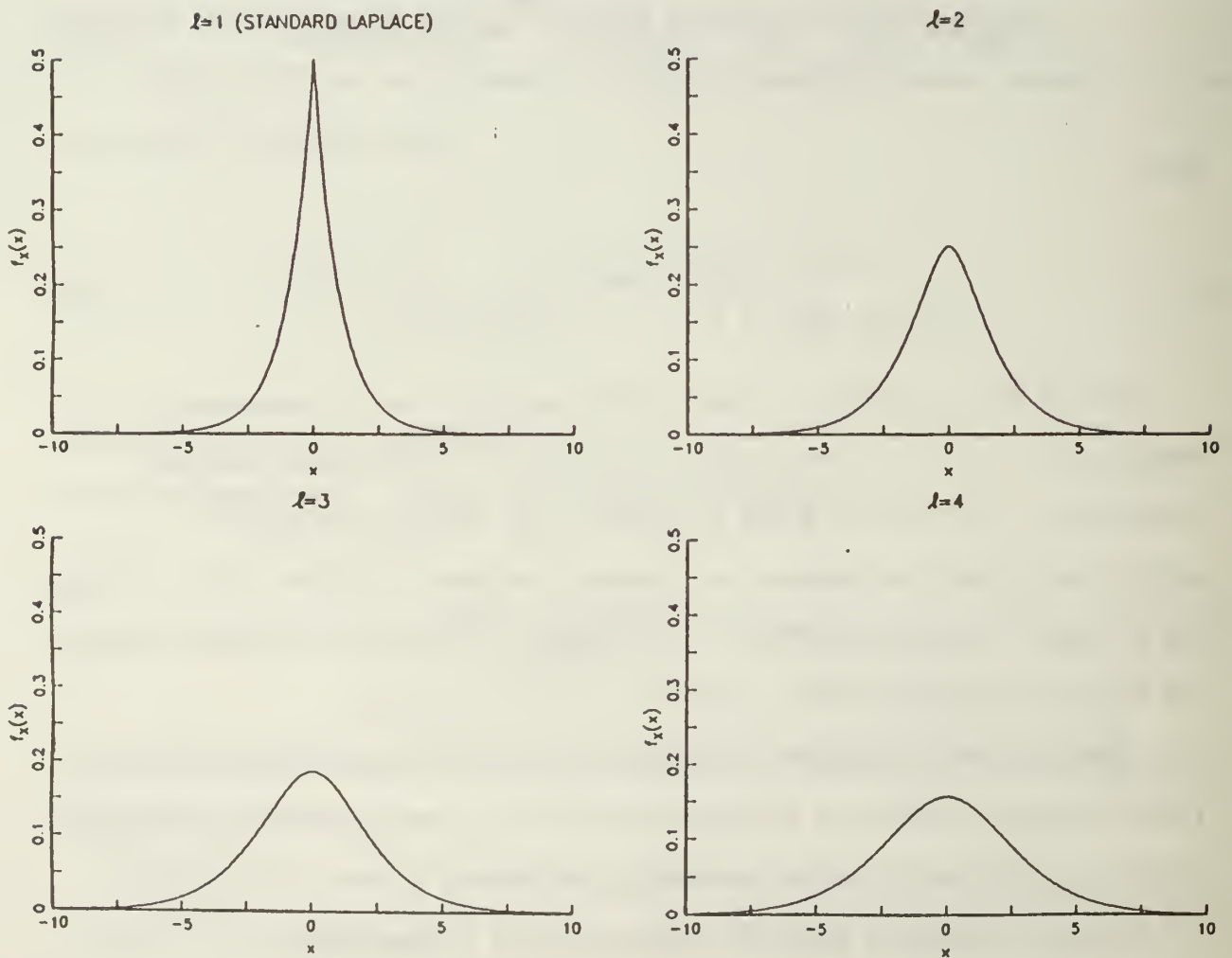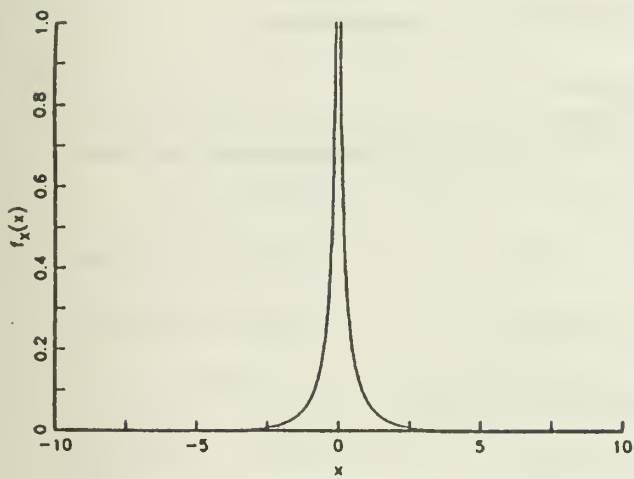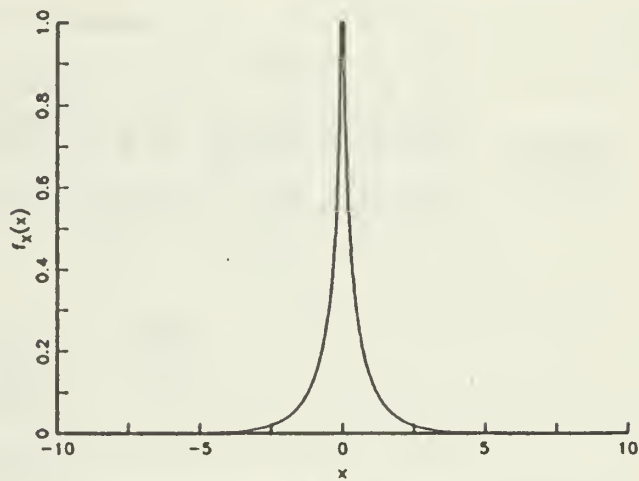
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ℓ−LAPLACE DENSITIES FOR INTEGRAL ℓ

Figure 2.1   Probability density functions of the ℓ−Laplace random variables
when ℓ=1,2,3 and 4.

ℓ−LAPLACE DENSITIES FOR NON−INTEGRAL ℓ

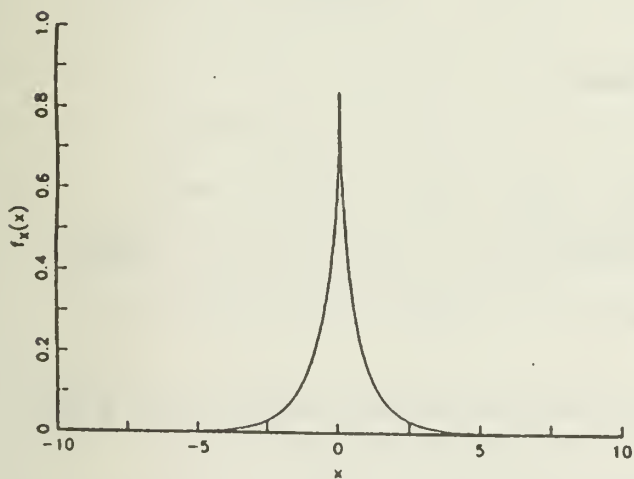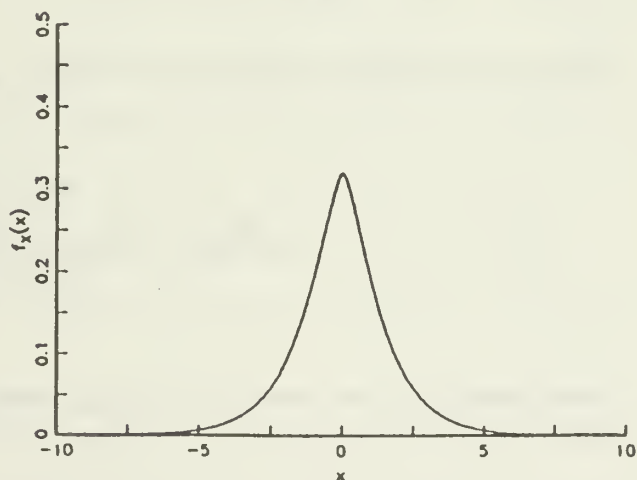Figure 2.2   Probability density functions of the ℓ-Laplace random variables
when ℓ=0.25, ℓ=0.50, ℓ=0.75, ℓ=1.50.

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2.3 The Square Root Beta-Laplace Transformation.

Much of the sequel uses the square-root-Beta-Laplace transformation. By this technique, an ℓ_1-Laplace random variable can be transformed into an ℓ_2-Laplace random variable, where $\ell_2 \leq \ell_1$.

<u>Theorem</u>: Let X ~ ℓ-Laplace and B ~ Beta $(\ell\alpha, \ell\bar{\alpha})$ where $\ell > 0$, $0 < \alpha < 1$, a and $\bar{\alpha} = 1 - \alpha$. It is assumed that X and B are independent.
If $Y = B^{1/2}X$, then Y ~ $(\ell\alpha)$-Laplace.

<u>Proof</u>: By conditioning on B, we obtain the following expression for the characteristic function of Y:

$$
\begin{aligned}
\phi_Y(\omega) &= E\{\exp(-iB^{1/2}X\omega)\} \\
&= E_B[E\{\exp(-ib^{1/2}\omega X)\}] \\
&= E_B[(1+b\omega^2)^{-\ell}].
\end{aligned}
\tag{2.9}
$$

Using the binomial expansion in (2.9) we have

$$
\phi_Y(\omega) = E_B\left[\sum_{k=0}^{\infty} \frac{(\ell)_k}{k!}(-\omega^2)^k\, b^k\right].
\tag{2.10}
$$

Interchanging the expectation and summation in a convergent power series gives

$$
\phi_Y(\omega) = \sum_{k=0}^{\infty} \frac{(\ell)_k}{k!}(-\omega^2)^k\, E(B^k).
\tag{2.11}
$$

From Johnson and Kotz (1970) we have, for integral k,

$$E(B^k) = (\ell\alpha)_k / (\ell)_k \qquad\qquad (2.12)$$

Substituting back into (2.11) completes the proof thus

$$\phi_\gamma(w) = \sum_{k=0}^{\infty} \frac{(\ell\alpha)_k}{k!}(-\omega^2)^k = (1+\omega^2)^{\ell\alpha}. \qquad\qquad (2.13)$$

3.   THE ℓ-BETA-LAPLACE FIRST-ORDER AUTOREGRESSIVE PROCESS, ℓ-BELAR(1)

The ℓ-Laplace random variable is not only infinitely divisible but also self-decomposable (Loeve, 1963).   Thus a linear, constant coefficient autoregression can be defined, as was done in Gaver and Lewis (1980) for the first-order autoregressive process, GAR(1), in Gamma-distributed random variables.   Moreover the innovation random variables are differences of independent Gamma AR1 innovation random variables (Gaver and Lewis, 1980) and can therefore be generated by methods of Lawrance (1982) and McKenzie (1987).

Unfortunately this linear ℓ-Laplace process has the same "zero-defect" as the Gamma GAR(1) process, which arises from the fact that the innovation random variable takes on the value 0 with positive probability, and this probability can be large if ℓ is small and the lag-one corrrelation, $\rho(1)$, is large.

For this reason we exploit, in this section, the square-root Beta-Laplace transform to define a different 2-parameter first-order autoregressive process in ℓ-Laplace variables.   The first parameter, ℓ, determines the non-Gaussian symmetric marginal distribution.   The second parameter, α, given the value of ℓ, determines uniquely the lag-one serial correlation.   Since the random

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coefficient model is shown to be first-order Markovian. α determines the entire autocorrelation function up to the sign. The process does not have the same "zero-defect" as the linear model.

We define the stationary process $\{X_n(\ell)\}$ by means of an additive, random coefficient equation:

$$X_n(\ell) = A_n^{1/2}(\ell\alpha, \ell\bar{\alpha}) \; X_{n-1}(\ell) + B_n^{1/2}(\ell\bar{\alpha}, \ell\alpha) \; L_n(\ell),\qquad\qquad (3.1)$$

where $\{A_n(\ell\alpha, \ell\bar{\alpha})\}$ is an i.i.d. sequence of Beta$(\ell\alpha, \ell\bar{\alpha})$ r.v.'s; $\{B_n(\ell\bar{\alpha}, \ell\alpha)\}$ is an i.i.d. sequence of Beta$(\ell\bar{\alpha}, \ell\alpha)$ random variables, independent of $\{A_n(\ell\alpha, \ell\bar{\alpha})\}$, and $\{L_n(\ell)\}$ is an i.i.d. sequence, independent of both coefficient sequences, of ℓ-Laplace r.v.'s. Coefficient and innovation sequences are all assumed to be independent of $\{X_k(\ell)\}$ for all k $\leq$ n-1. The process $\{X_n(\ell)\}$ so defined will be called the ℓ-Beta-Laplace AR(1) process, ℓ-BELAR(1).

If it is assumed that $X_{n-1}(\ell)$ has an ℓ-Laplace distribution, then by the theorem in Section 2.3 and the additivity of the ℓ-Laplace family, so does $X_n(\ell)$. The fact that the process is Markovian follows by construction. It is also explicitly autoregressive. To start the process in the stationary distribution, set $X_0(\ell) = L_0(\ell)$. Note also that the parameter space for the process is the set $\{(\ell, \alpha) \mid \ell > 0, 0 \leq \alpha \leq 1\}$.

For the Beta random variables A_n and B_n to be defined properly, each of their parameters must be positive. However, when $\alpha = 0$ or $\alpha = 1$, (3.1) is no longer appropriate. Therefore it is understood that if $\alpha = 0$ then $\{A_n\}$ is identically zero and $\{B_n\}$ is identically one. Thus $\alpha = 0$ implies

$X_n(\ell) = L_n(\ell)$ and $\{X_n(\ell)\}$ is an i.i.d. sequence of ℓ-Laplace variables. For convenience, we denote the innovation process in (3.1) by $\{\epsilon_n\}$, i.e.

$$\epsilon_n = B_n^{1/2}(\ell\bar{\alpha}, \ell\alpha)L_n(\ell). \tag{3.2}$$

Examples of sample path behavior for selected ℓ and α are given in Figure 3.1. Note that although the correlation coefficient is approximately 0.8 for all sets of ℓ and α, there are considerable differences in the sample path behaviors as ℓ changes. In particular when ℓ is small there are runs of values that are very nearly zero in magnitude, so that the process looks very sporadic. Wind velocity data, especially in the tropics, tends to be of this form. Using (3.1) we obtain

$$\rho(1) = \mathrm{Corr}\{X_n(\ell), X_{n-1}(\ell)\} = E\{A_n^{1/2}(\ell\alpha, \ell\bar{\alpha})\} \tag{3.3}$$

$$= \frac{\alpha\Gamma(\ell\alpha+1/2)\Gamma(\ell+1)}{\Gamma(\ell+1/2)\ \Gamma(\ell\alpha+1)}. \tag{3.4}$$

Note that as $\alpha \to 1$, then $\rho(1) \to 1$. Similarly as $\alpha \to 0$, $\rho(1) \to 0$. We can show that (3.4) yields the full range of positive correlations in a one-to-one function of α for any given value of ℓ. Further, it is easily established, using (3.1), that

$$\rho(r) = \mathrm{Corr}(X_n(\ell), X_{n-r}(\ell)) = \rho(1)^{|r|}, \qquad r = 0, \pm 1, \ldots \tag{3.5}$$

Note also that except at $\ell = 0$, $\rho(1) \neq \alpha$. Since we shall frequently refer to expressions of the form of (3.4), we define the notation
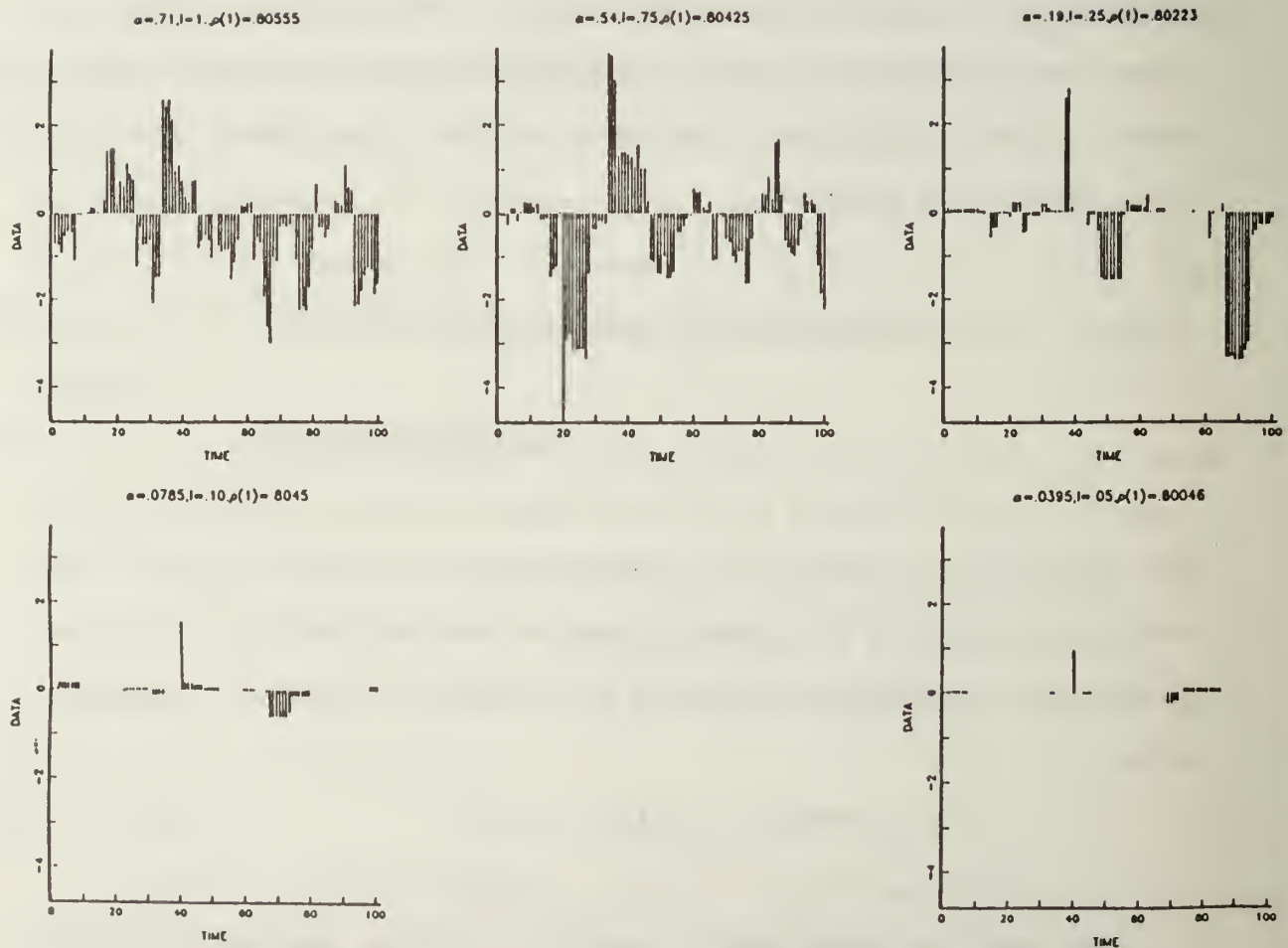
ℓ-BETA-LAPLACE AR(1): SAMPLE PATHS

Figure 3.1   Sample path behaviour for ℓ-Beta-Laplace processes for fixed lag-one serial correlation of 0.8, and different values of ℓ.

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$$\gamma(\ell\alpha) = E\{[\text{Beta}(\ell\alpha, \ell\bar{\alpha})]^{1/2}\}. \tag{3.6}$$

We noted above that the process is explicitly autoregressive. It is autoregressive as well in the sense of expectations (Lawrance and Lewis, 1987) in that $E(X_n(\ell)|X_{n-1}(\ell) = x)$ is a linear function of x. Further, replacing $A_n^{1/2}(.)$ by $-A_n^{1/2}(.)$ in (3.1) yields the full range of negative correlations.

Explicit expressions are very difficult to obtain for joint distributions of consecutive observations in the ℓ–BELAR(1) process for general ℓ. Later, we consider in more detail the very important and more tractable case of the Laplace distribution, i.e. $\ell = 1$. Here, however, we briefly note some aspects of one of the important properties of such joint distributions, viz. time reversibility. A process, $\{X_n\}$, is time-reversible if the joint distributions of $(X_n, X_{n+1}, \ldots, X_{n+r})$ and $(X_{n+r}, X_{n+r-1}, \ldots, X_n)$ are identical for all r and n. This is an important property in practice in the identification problem. However, it is a property of the joint distributions and these are not known. Nevertheless, we can demonstrate partial time reversibility with respect to both directional moments and runs probabilities.

It is straightforward to verify from (3.1) that $E(X_n^2 X_{n-k}) = E(X_n X_{n-k}^2) = 0$ for all n and k. Thus, $\{X_n(\ell)\}$ given by (3.1) is time-reversible with respect to these directional moments. Further, the characteristic function of $X_n - X_{n-1}$ is given by

$$E\{\exp[-i\omega(X_n-X_{n-1})]\} = (1 + \omega^2)^{-\ell\bar{\alpha}} E_A\{[1+(1-A^{1/2})^2\omega^2]^{-\ell\alpha}\} \tag{3.7}$$

and this is clearly real valued. Thus, $X_n - X_{n-1}$ is symmetric about zero and

so $P(X_n > X_{n-1}) = P(X_n < X_{n-1})$, i.e. $\{X_n(\ell)\}$ is time reversible with respect to runs probabilities.

4. THE ℓ-BETA-LAPLACE ARMA PROCESSES

4.1 Introduction

In this section, we introduce time series models that have ℓ-Laplace marginal distributions and the correlation structure of the linear, Gaussian, ARMA processes, although the order of the autoregression is limited to p=1. As before, our constructions are based on the square-root Beta-Laplace transform and the additivity of the ℓ-Laplace family. We begin by considering pure moving average processes.

4.2 The first order moving average process, ℓ-BELMA(1).

Let $\{L_n(\theta)\}$ be an i.i.d. sequence of θ-Laplace random variables, where $\theta = \ell/(1+\beta)$ and $0 < \beta < 1$. Let $\bar{\beta} = 1-\beta$. Also let $\{C_n(\theta\beta, \theta\bar{\beta})\}$ be an i.i.d. sequence of Beta$(\theta\beta, \theta\bar{\beta})$ variates independent of $\{L_n(\theta)\}$. Then the process $\{X_n(\ell)\}$ generated by

$$X_n(\ell) = L_n(\theta) + C_n^{1/2}(\theta\beta, \theta\bar{\beta})L_{n-1}(\theta) \qquad (4.1)$$

has a marginal ℓ-Laplace distribution and an MA(1) structure with $0 \leq \text{Corr}(X_n, X_{n-1}) \leq 0.5$, as shown below.

The marginal distribution of $\{X_n(\ell)\}$ follows directly from the square root Beta-Laplace transform and the additivity of the ℓ-Laplace family. Further, the construction of (4.1) defines $\{X_n(\ell)\}$ as a random coefficient

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moving average of order one. Thus, $X_n(\ell)$ and $X_{n-k}(\ell)$ are, by construction,

independent for $|k| \geq 2$. In addition, we may easily verify from (4.1) that

$$\rho_X(1) = \text{Corr}(X_n, X_{n-1}) = \gamma(\theta\beta)/(1+\beta), \qquad (4.2)$$

where γ is defined by (3.6). Writing out (4.2) explicitly yields

$$\rho_X(1) = \frac{\beta}{1+\beta} \cdot \frac{\Gamma(\beta\theta+1/2)\Gamma(\theta+1)}{\Gamma(\beta\theta+1)\Gamma(\theta+1/2)}. \qquad (4.3)$$

Thus, as $\beta \to 0$ we have $\rho_X(1) \to 0$, and as $\beta \to 1$ we have $\rho_X(1) \to 0.5$. We may show

that $\rho_X(1)$ is a one-to-one function of β extending over the full positive

range of correlation for a moving average process of order one, MA(1), namely

[0,0.5]. In addition, replacing $C_n^{1/2}$ by $-C_n^{1/2}$ yields the full negative range

[-0.5,0]. Thus, the entire range of possible values of $\rho_X(1)$ for an MA(1) is

available for this model. This is important in practice, relating as it does

to the applicability of the model. The NLMA(1) model of Dewald (1985), which

is a MA(1) model with Laplace marginals has, in contrast, $\rho(1)$ bounded by $\pm$

0.4026.

It is straightforward to show that this model, like the autoregressive

model ℓ-BELAR(1), is time reversible with respect to runs probabilities.

4.3 Higher order moving average processes, the ℓ-BELMA(q) process.

The process defined by (4.1) may be extended to order q as follows. Let

$\{L_n(\theta)\}$ be, as before, an i.i.d. sequence of θ-Laplace random variables, where

now $\theta = \ell/(1+\sum\limits_{i=1}^{q} \beta_i)$ and $0 < \beta_i < 1$, for $i = 1,2,\ldots,q$. Also let

$\{C_{n,i}(\theta\beta_i, \theta\bar{\beta}_i), \ i = 1,2,\ldots,q\}$ be independent sequences of i.i.d.

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Beta($\theta\beta_i, \theta\bar{\beta}_i$) random variables independent of $\{L_n(\theta)\}$. Then, the process $\{X_n(\ell)\}$ generated by

$$X_n(\ell) = L_n(\theta) + \sum_{i=1}^{q} C_{n,i}^{1/2} (\theta\beta_i, \theta\bar{\beta}_i) L_{n-i}(\theta) \qquad (4.4)$$

has a marginal ℓ-Laplace distribution and MA(q) structure. These properties follow in the usual way. As regards the autocorrelation function $\{\rho_X(k)\}$, it is clear that $\rho_X(k) = 0$ for $|k| > q$, and we can show that

$$\rho_X(k) = \gamma(\theta\beta_k) + \sum_{i=1}^{q-k} \gamma(\theta\beta_i)\gamma(\theta\beta_{i+k})\}/\{1 + \sum_{i=1}^{q} \beta_i\}, \qquad k=1,2,\ldots,q. \quad (4.5)$$

It is interesting to note that although $\{\rho_X(k)\}$ is similar in form to the autocorrelation function of a Gaussian linear MA(q) process, the correlations are not identical.

This q-th order moving average process, too, is time-reversible with respect to runs probabilities.

    4.4. The first order mixed autoregressive moving average process, ℓ-BELARMA(1,1).

        The first order mixed model is obtained by defining the process $\{X_n(\ell)\}$ by means of two first order difference equations, one corresponding to the autoregressive (AR) component and the other to the moving average (MA) component. These are respectively:

$$Y_n(\theta) = A_n^{1/2} (\theta\alpha, \theta\bar{\alpha}) \; Y_{n-1}(\theta) + B_n^{1/2}(\theta\bar{\alpha}, \theta\alpha) \; L_n(\theta) \qquad (4.6)$$

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and

$$X_n(\ell) = L_n(\theta) + C_n^{1/2} (\theta\beta, \theta\bar{\beta}) \, Y_{n-1}(\theta),$$ (4.7)

where $\theta = \ell/(1+\beta)$; $0 < \alpha$, $\beta < 1$; and $\{L_n(\theta)\}$, $\{A_n(\theta\alpha, \theta\bar{\alpha})\}$, $\{B_n(\theta\bar{\alpha}, \theta\alpha)\}$ and $\{C_n(\theta\beta, \theta\bar{\beta})\}$ are independent sequences of i.i.d. random variates with the usual distributions signified by this notation. As before, we may show that $\{X_n(\ell)\}$ is marginally ℓ-Laplace and has the autocorrelation function of an ARMA(1,1) process. In particular,

$$\rho_X(k) = \gamma(\theta\beta)\frac{[\gamma(\theta\bar{\alpha})+\gamma(\theta\beta)\gamma(\theta\alpha)]}{(1+\beta)} \, [\gamma(\theta\alpha)]^{k-1}, \qquad k = 1,2.\ldots.$$ (4.8)

Note that this process $\{X_n(\ell)\}$ has a structure which is determined in a simple way by the two parameters α and β. If $\beta = 0$, we have the ℓ-BELAR (1) process, and if $\alpha = 0$ the process reduces to the ℓ-BELMA(1) process. Further, if $\alpha = \beta = 0$ then $\{X_n(\ell)\}$ is a sequence of i.i.d. ℓ-Laplace variates.

4.5 Other ℓ-Laplace ARMA processes

We can extend the models described in the previous sections to an ARMA (1,q) form. This is achieved by replacing (4.7) by an expression similar to (4.4) but involving L_n, L_{n-1},..., L_{n-q+1} and Y_{n-q}. However, nothing new emerges here and so we consider instead some other ℓ-Laplace processes generated as the differences between Gamma processes.

(i) Independent Gamma processes.

Suppose that $\{X_n\}$ and $\{Y_n\}$ are independent Gamma processes with Gamma(ℓ,1) marginal distributions and identical ARMA structures. Consider the

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process $\{Z_n\}$ given by $Z_n = X_n - Y_n$. As noted in Section 2, the marginal distribution of $\{Z_n\}$ is ℓ-Laplace and the process shares the same autocorrelation function as $\{X_n\}$ and $\{Y_n\}$. On the other hand, $\{Z_n\}$ is not structurally the same as these two processes. Thus, this approach allows us to construct ℓ-Laplace processes with ARMA autocorrelation functions but not the simple linear additive, random coefficient ARMA structure of the ℓ-BELARMA processes.

(ii) Serial differences in the BGARMA processes.

A feature of the BGARMA processes of Lewis, McKenzie and Hugus (1989) which is relevant here is that the first order AR and MA processes, BGAR(1) and BGMA(1), have the same form of bivariate distribution for two consecutive observations. If the BGAR(1) or BGMA(1) process is $\{X_n\}$ and is marginally Gamma$(\ell,1)$, the joint characteristic function of X_n and X_{n-1} is

$$\phi_{X_n,X_{n-1}}(u,v) = (1+iu)^{-\ell\bar\rho}\,(1+iv)^{-\ell\bar\rho}\,(1+iu+iv)^{-\ell\rho},$$

where $\rho = \mathrm{corr}\,(X_n,X_{n-1})$ in both processes, and $\bar\rho = 1-\rho$. Now, the process of first differences, $\Delta X_n = X_n - X_{n-1}$, has its marginal distribution given by the characteristic function

$$\phi_{\Delta X}(u) = \phi_{X_n,X_{n-1}}(u,-u) = (1+u^2)^{-\ell\bar\rho}.$$

Thus, $\{\Delta X_n\}$ is marginally $(\ell\bar\rho)$-Laplace. Furthermore, if $\{X_n\} \sim$ BGMA(1) originally, then $\{\Delta X_n\}$ is an $(\ell\bar\rho)$-Laplace with MA(2) correlation structure. In fact, $\rho_{\Delta X}(1) = -(1-2\rho)/2(1-\rho)$ and $\rho_{\Delta X}(2) = -\rho/2(1-\rho)$. If $\{X_n\} \sim$ BGAR(1)

originally, then $\{\Delta X_n\}$ is an $(\ell\bar{\rho})$-Laplace process with ARMA(1,1) correlation structure. In this case, $\rho_{\Delta X}(k) = -\frac{1}{2}(1-\rho)\rho^{k-1}$, $k \geq 1$. Again, these first differences processes do not have the simple ARMA structure of the processes presented earlier, although they may share their marginal distributions and autocorrelation functions.

4.6. The ℓ-Laplace ARIMA processes

In the practical application of the Gaussian ARMA processes it has been found extremely useful to extend the basic models to processes $\{X_n\}$ which are non-stationary but whose first (or higher order) differences are an ARMA process. Such processes are called integrated ARMA processes and we refer to such an $\{X_n\}$ as an ARIMA process. This is a trivial extension mathematically, but it would be difficult to overestimate the extra range of applicability it bestows in practice to time series models. The approach is discussed in detail by Box and Jenkins (1976) and has been found enormously useful.

Defining integrated processes presents no problems for Gaussian processes which retain their marginal distributions under simple linear transformations, even of dependent data. However, if a non-stationary time series is thought to be marginally Gamma it is by no means clear just how its differences might be distributed. We have detailed a number of occasions here in which differenced Gammas have the ℓ-Laplace distribution and we would argue that it is not unreasonable to assume that they will often be so in practice, at least approximately. Thus, in attempting to model a non-stationary Gamma process whose differences seem stationary, we would consider modelling the differenced process as an ℓ-Laplace ARMA process. We would thus argue that such ℓ-Laplace ARIMA processes are potentially of great practical importance.

5. THE FIRST-ORDER BETA-LAPLACE AUTOREGRESSIVE PROCESSES, 1-BELAR(1)

5.1 Construction and Correlation Structure

In this section we will study in detail the first-order ℓ-Beta Laplace process with $\ell = 1$ (i.e., 1-BELAR(1)). This process is far more tractable than the general case, and is an interesting alternative to the autoregressive NLAR(1) process introduced by Dewald and Lewis (1985). Thus we set $\ell = 1$ in (3.1) to obtain

$$X_n = A_n^{1/2}(\alpha, 1-\alpha)X_{n-1} + \epsilon_n, \tag{5.1}$$

where $\{\epsilon_n\}$ is i.i.d. sequence with $\epsilon_n \sim (1-\alpha)$-Laplace. Also $\{X_n\}$ has a marginal distribution that is standard Laplace.

The only parameter in the model is α, which describes the dependency structure of the process, as given by (3.4) and (3.5), when $\ell = 1$, as

$$\rho(r) = \left[\frac{\alpha\Gamma(\alpha+1/2)}{\Gamma(3/2)\Gamma(\alpha+1)}\right]^{|r|}, \tag{5.2}$$

for $r = 0, \pm 1, \pm 2, \ldots$.

Now there are infinitely many other first-order AR(1) processes with identical autocorrelation functions and standard Laplace marginals, in particular the NLAR(1) processes by Dewald and Lewis (1985). Dewald (1985) developed fourth-order analyses of the NLAR(1) model based on the linearized residual,

$$R_n = X_n - \rho_X(1)X_{n-1} \qquad\qquad (5.3)$$

to differentiate between different parametric cases of the model. The use of higher-order residual analysis in random coefficient processes was established by Lawrance and Lewis (1987). We use (5.3) again in Section 6.

5.2. The Conditional Density of X_n, given $X_{n-1} = y$.

To find the conditional density of $X_n | X_{n-1} = y$, we begin by deriving the density of $A_n^{1/2}(\alpha, 1-\alpha)$ by differentiating $P(A_n < a^2)$ with respect to a:

$$f_{A^{1/2}}(a;\alpha) = \frac{2}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{a^{2\alpha-1}}{(1-a^2)^\alpha}, \qquad 0 < a < 1. \qquad (5.4)$$

To evaluate $P\{X_n < x \mid X_{n-1} = y\}$ we condition on $A_n^{1/2}(\alpha, 1-\alpha)$ thus:

$$
\begin{aligned}
P\{X_n < x \mid X_{n-1} = y\} &= P\{A_n^{1/2} X_{n-1} + \epsilon_n < x \mid X_{n-1} = y\} \\
&= E_{A_n^{1/2}} \{F_{\epsilon_n}(x-ay)\} \\
&= \int_{L_1(x)}^{L_2(x)} F_{\epsilon_n}(x-ay) \, f_{A_n^{1/2}}(a;\alpha) da, \qquad (5.5)
\end{aligned}
$$

where $F_\epsilon(x-ay)$ is the cummulative distribution function of the $(1-\alpha)$-Laplace random variable, ϵ_n, and $L_i(x)$, for $i = 1,2$, are the limits of integration on a, which may be functions of x.

Since $F_{\epsilon_n}(x-ay)$ changes definition for negative and positive values of $(x-ay)$ and since $0 < a < 1$, differentiating (5.5) with respect to x gives two branches for the conditional density as

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$$f_{X_n|X_{n-1}}(x|y) = \begin{cases} \int_{a=0}^{a=1} f_{\epsilon_n}(x-ay;1-\alpha)f_{A_n^{1/2}}(a;\alpha)da & x/y \geq 1 \text{ or } \\ & x/y \leq 0 \\[2em] \int_{a=0}^{a=x/y} f_{\epsilon_A}(x-ay;1-\alpha)\,f_{A_n^{1/2}}(a;\alpha)da + \\[2em] \int_{a=x/y}^{a=1} f_{\epsilon_n}(x-ay;1-\alpha)f_{A_n^{1/2}}(a;\alpha)da. & 0<x/y<1. \end{cases} \qquad (5.6)$$

Now $f_{\epsilon_n}(.;1-\alpha)$ can be evaluated from (2.6) and $f_{A_n^{1/2}}(.;1-\alpha)$ from (5.4). The equation (5.6) is computed numerically, as given in Dewald (1985).

Note that (5.6), as a function of x, is not absolutely continuous. In fact if $\alpha \leq 1/2$ and $x = y$, the conditional density is not defined.

We use (5.6) in obtaining the maximum likelihood estimates of serial correlations through the parameter α. This is discussed in Section 6.

6. ESTIMATION OF SERIAL CORRELATION IN 1-BELAR(1)

6.1 Least-Squares Estimation of $\rho(1)$

Although the 1-BELAR(1) process is not a linear process (except when $\alpha = 0$) the linearized residual is formed as in (5.3). The process $\{R_n\}$, as shown by Lawrance and Lewis (1987), is uncorrelated but not independent. Also it can be seen that conditional on X_{n-1}, that R_n has zero mean and variance $2\{1-(\rho(1))^2\}$. Therefore a conditional least-squares estimator (Nicholls and Quinn, 1982) of $\rho(1)$, $\hat{\rho}_{LS}$ can be derived by minimizing the sum of squares of

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R_n with respect to $\rho(1)$. This produces the usual product moment estimator

$$\hat{\rho}_{LS} = \sum_{i=2}^{n} x_i x_{i-1} \Big/ \sum_{i=2}^{n} x_{i-1}^2 . \qquad (6.1)$$

Now Dewald (1985) showed that the 1-BELAR(1) process is a Random Coefficient Autoregressive (RCA(1)) process in the sense of Nicholls and Quinn (1982). Therefore from Theorem 3.1 of Nicholls and Quinn, $\hat{\rho}_{LS} \to \rho(1)$, and $\hat{\rho}_{LS}$ is asymptotically unbiased and normally distributed, with asymptotic variance given by

$$n\mathrm{Var}(\hat{\rho}_{LS}) = 1 + 5\alpha - 6[\rho(1)]^2 . \qquad (6.2)$$

Simulations of $\hat{\rho}_{LS}$ were conducted for selected values of $\rho(1)$ in the 1-BELAR(1) process using the Simulation Testbed (SIMTBED) of Lewis et al. (1984). Detailed tabulations of the distribution of $\tilde{\rho}_{LS}$ are given in Dewald (1985) for various values of α and different sample sizes.

Dewald (1985) also showed via simulation that certain robust estimators based on symmetric loss functions of R_n, other than the sum of squares, are biased and apparently asymptotically biased. These robust estimators include the Huber(c), rank, and least absolute deviation (LAD) estimators of $\rho(1)$. Note that these results for robust estimators are different in the 1-BELAR(1) process than in the linear AR(1) processes for which these estimators are reported to be consistent and asymptotically unbiased, e.g. Denby and Martin (1979) and Bloomfield and Steiger (1983).

The least-squares estimator, $\hat{\rho}_{LS}$, is used in the next subsection as a starting value in an iterative procedure to find the maximum likelihood

estimator, $\hat{\rho}_{MLE}$, of $\rho(1)$.

6.2 Maximum Likelihood Estimation

The maximum likelihood estimate of $\rho(1)$ can be obtained from the
conditional density in (5.6). The formula for the joint density of X_n,
$X_{n-1}, \ldots, X_1$ is

$$f(x_n, x_{n-1}, \ldots, x_1) = f_{X_n|X_{n-1}}(x_n|x_{n-1}) f_{X_{n-1}|X_{n-2}}(x_{n-1}|x_{n-2}) \ldots f_{X_1}(x_1), \quad (6.3)$$

where $f_{X_1}(x_1)$ is a standard Laplace density.

Dewald (1985) used (6.3) to obtain estimates for α and then computed $\hat{\rho}_{MLE}$
from the one-to-one function of α in (5.2). The least-squares estimate served
as a good starting point for the numerical evaluations to find $\hat{\rho}_{MLE}$.
Techniques for the numerical integration and search for this estimation
procedure are given in Dewald (1985).

The simulation results in Dewald (1985) indicate that $\hat{\rho}_{MLE}$ has a smaller
standard deviation and bias than does $\tilde{\rho}_{LS}$ at all values of $\rho(1)$. Some
particular results are:

(1) For a sample size of n = 175, and $\alpha = 0.5$, so that $\rho(1) = 0.6366$,
the correlation between the two estimates is estimated to be 0.393.
The estimated standard deviation of $\hat{\rho}_{LS}$ is 0.0728 and that of $\hat{\rho}_{MLE}$
is 0.0289.

(2) For a sample size of n = 125, and $\alpha = .11$, so that
$\rho(1) = .1922$, the correlation between the two estimates is estimated
to be 0.843. The estimated standard deviation of $\hat{\rho}_{LS}$ is 0.099 and
that of $\hat{\rho}_{MLE}$ is 0.069.

(3) For a sample size of n = 250, and $\alpha = 0.844$, so that

$\rho(1) = -0.900$, the correlation between the two estimates is estimated to be 0.101. The estimated standard deviation of $\hat{\rho}_{LS}$ is 0.030 and that of $\hat{\rho}_{MLE}$ is 0.006.

The number of replications in this simulation was only 20, so the results are imprecise. However, it is clear that the maximum likelihood estimator of $\rho(1)$ is better than the least squares estimator, but the computation of the maximum likelihood estimator is not simple.

Dewald (1985) also gives results on joint estimates of $\rho(1)$, $E(X_n)$ and $Var(X_n)$ in the 1-BELAR(1) process.

Also one should note that while the least-squares estimator of α and $\rho(1)$ is available for the general ℓ-BELAR(1) process, maximum likelihood estimation of α and $\rho(1)$ is intractable unless $\ell = 1$.

ACKNOWLEDGMENTS

BIBLIOGRAPHY

Bloomfield, P. and Steiger, W. L. (1983), <u>Least Absolute Deviations Theory,</u>
    <u>Applications, and Algorithms</u>, Birkhaeuser, Boston.

Box, G. E. P. and Jenkins, G. M. (1976), <u>Time Series Analysis, Forecasting,</u>
    <u>and Control</u>, <u>2nd ed.</u>, Holden Day, San Francisco.

Brown, B. G., Katz, R. W. and Murphy, A. H. (1984), "Time Series Models to
    Simulate and Forecast Wind Speed and Wind Powwer," <u>Journal of Climate and</u>
    <u>Applied Meterology</u>, 23, 1184-1195.

Denby, L. and Martin, R. D. (1979), "Robust Estimation of the First-Order
    Autoregressive Parameter," <u>Journal of the American Statistical Assoc.,</u>
    Vol. 74, pp. 140-146.

Dewald, L. S. (1985), "Time Series Models with a Specified Symmetric
    Non-Normal Marginal Distribution," Ph.D. Thesis, Naval Postgraduate
    School, Monterey, CA.

Dewald, L. S. and Lewis, P. A. W. (1985), "A New Laplace Second-Order
    Autoregressive Time Series Model - NLAR(2), "<u>IEEE Trans. On Information</u>
    <u>Theory</u>, Vol. IT-31, pp. 645-652.

Feller, W. (1971), <u>An Introduction to Probability Theory and its Applications</u>,
    2nd ed., Vol. 2, Wiley, New York.

Gastwirth, J. L., and Wolff, S. S. (1965), "A Characterization of the Laplace
    Distribution," John Hopkins University, Department of Statistics
    Report 28.

Gaver, D. P. and Lewis, P. A. W. (1980), "First-Order Autoregressive Gamma
    Sequences and Point Processes," <u>Advances in Applied Probability</u>, Vol. 3,
    pp. 727-745.

Hsu, D. A. (1979). Long-tailed Distributions for Position Errors in
    Navigation. <u>Journal of the Royal Statistical Society</u>, <u>C</u>, <u>28</u> (1), 62-71.

Hugus, D. K. (1982), "Extensions of Some Models for Positive-Valued Time
    Series," Ph.D. Thesis, Naval Postgraduate School, Monterey, CA.

Johnson, N. K. and Kotz, S. (1970), <u>Continuous Univariate Distributions</u>,
    Vol. 2, Wiley, New York.

Lawrance, A. J. (1982), "The Innovation Distribution of a Gamma Distributed
    Autoregressive Process," <u>Scand. J. Statist.</u>, 9, 234-236.

Lawrance, A. J. and Lewis, P. A. W. (1981), "A New Autoregressive Time Series
    Model in Exponential Variables (NEAR(1))," <u>Advances in Applied</u>
    <u>Probability</u>, Vol. 13, pp. 826-845.

Lawrance, A. J. and Lewis, P. A. W. (1985), "Modelling and Residual Analysis of Nonlinear Autoregressive Time Series in Exponential Variables," Journal of the Royal Statistical Society, Vol. B47, pp. 165-202.

Lawrance, A. J. and Lewis, P. A. W. (1987), "Higher-Order Residual Analysis for Nonlinear Time Series with Autoregressive Correlation Structure," International Statistical Review, 55, 1, 21-35.

Lewis, P. A. W. (1981), "Simple Multivariate Time Series for Simulation of Complex Systems," In Proc. Winter Simulation Conference, Ed. T. T. Oren, C. M. Delfosse, and C. M. Shub, pp. 389-390, New York: IEEE Press.

Lewis, P. A. W. (1985). "Some Simple Models for Continuous Variate Time Series," Water Resources Bulletin, 21(4), 635-644.

Lewis, P. A. W., McKenzie, E., and Hugus, D. K. (1989), "Gamma Processes," Naval Postgraduate School Technical Report NPS-55-86-002. Communications in Statistics: Stochastic Models, 3,1, to appear.

Lewis, P. A. W., Orav, E. J. Drueg, H. W., Linnebar, D. G., and Uribe, L. (1985), "An Implementation of Graphical Analysis in Statistical Simulations," International Statistical Review, 53, Vol. 1, pp. 69-90.

Loeve, M. (1963), Probability Theory, 3rd ed., Van Nostrand, New York.

McKenzie, E. (1985), "An Autoregressive Process for Beta Random Variables," Management Science, 31, 8, 988-997.

McKenzie, E. (1987), "Innovation distributions for gamma and negative binomial autoregressions," Scandinavian Journal of Statistics, 14, 1, 645-650.

Nicholls, D. F. and Quinn, B. G. (1982), Random Coefficient Autoregressive Models: An Introduction, Springer Verlag.

Rao, D. S. and Johnson, D. H. (1985). A First Order AR Model for Non-Gaussian Time Series. To appear.

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