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EMPIRICAL BAYES RISK EVALUATION WITH TYPE II CENSORED DATA

Lynn Kuo and Constantin Yiannoutsos

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EMPIRICAL BAYES RISK EVALUATION WITH TYPE II CENSORED DATA

Lynn Kuo[1]
Dept. of Statistics
University of Connecticut
and
Operations Research Department
Naval Postgraduate School
Monterey, CA 93943

and

Constantin Yiannoutsos
Department of Statistics
University of Connecticut
Storrs, CT 06269

Abstract

Empirical Bayes estimators for the scale parameter in a Weibull, Raleigh or an exponential distribution with type II censored data are developed. These estimators are derived by the matching moment method, the maximum likelihood method and by modifying the geometric mean estimators developed by Dey and Kuo (1991). The empirical Bayes risks for these estimators and the Bayes rules are evaluated by extensive simulation. Often, the moment empirical Bayes estimator has the smallest empirical Bayes risk. The cases that the modified geometric mean estimator has the smallest empirical Bayes risk are also identified. We also obtain the risk comparisons for various empirical Bayes estimators when one of the parameters in the hyperprior is known.

Key Words: Type II censored data; Parametric empirical Bayes estimation; ML-II prior; Matching moment method; Geometric mean estimator; EB risk comparison.

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1. INTRODUCTION

In reliability analysis, we often conduct similar tests to assess the reliability parameters. Empirical Bayes (EB) estimators may be employed to reduce the risk by combining all the test data. There are usually several reasonable empirical Bayes estimators. As pointed out by Martz and Waller (1982, p. 633), it is an important question which EB estimator has the smallest EB risk (the expected Bayes risk of the EB rule). We report here some Monte Carlo results that estimate the EB risks. These results will guide us in determining which EB estimator is more desirable.

We assume that we are simultaneously testing p populations. For the i^{th} population, $i = 1, \ldots, p$, we test n_i devices until r_i of them fail. The lifetime for each device tested in the i^{th} population is assumed to be Weibull with known shape parameter ρ_i and unknown scale parameter λ_i,

$$f(t|\lambda_i,\rho_i) = \frac{\rho_i}{\lambda_i} t^{\rho_i - 1} e^{-\frac{t^{\rho_i}}{\lambda_i}}. \tag{1.1}$$

This model includes the exponential ($\rho_i = 1$) and Raleigh ($\rho_i = 2$) distributions.

Let $\boldsymbol{t}_i = (t_{i1} \ldots, t_{ir_i})$ denote the ordered lifetimes of the r_i devices that failed in the i^{th} population, where $t_{i1} < \ldots, < t_{ir_i}$. Then $S_i = (n_i - r_i)t_{ir_i}^{\rho_i} + \sum_{j=1}^{r_i} t_{ij}^{\rho_i}$ is the sufficient statistic for λ_i with gamma distribution $G(r_i,\lambda_i)$.

In this paper, we consider a parametric empirical Bayes formulation. We assume the parameters λ_i are independent and identically distributed with the inverted gamma distribution $IG(\alpha, \beta)$. This conjugate prior is chosen to facilitate a closed form expression for the Bayes estimator λ_i. To obtain empirical Bayes estimators, we are going to estimate α and β from the marginal density of S_i given α and β. We then substitute these estimates $\hat{\alpha}$ and $\hat{\beta}$ for the α and β in the Bayes rule. Usually, we can estimate α and β by a moment method or by an ML-II method. The moment method estimates α and β by equating functions of α and β to the marginal mean and the marginal variance respectively. The ML-II method obtains the maximum likelihood estimates of α and β from the joint density of $S_1, S_2, ..., S_p$ given α and β. Recently, Dey and Kuo (1991) have obtained a different EB estimator when α is known. This estimator expands the usual estimator by a multiple of the geometric mean of the component estimators. They show the geometric mean estimator dominates the best multiple estimator (in frequentists' risk) for a wide class of p, α, and r_i values. In this paper, we propose a modified estimator called the hybrid geometric mean estimator which estimates α by the moment method and estimates β as in Dey and Kuo.

We compare the EB risks of the three EB estimators: moment EB, ML-II EB and hybrid geometric mean EB. The EB risks cannot be expressed in closed form. We employ extensive Monte Carlo simulations to approximate these EB risks. Our simulation results show that all the three EB estimators perform well when compared to the Bayes estimator for $\alpha \geq 5$, $\beta \geq 5$, $r_i > 5$, for all i and $p \geq 5$.

If we fix the number of censored devices, say for example, $r_i = 2$ for all i, our recommendation depends on the different α and β values. A more detailed discussion is given in Section 3. In general, it is safe to recommend the moment EB estimator. Unless we are in the situation with moderate α and moderate β values (around $\alpha = 5$ and $\beta = 3$), where the hybrid geometric mean estimator is recommended. Unfortunately, our recommendation depends on the unknown α and β values. This should not deter us in using the recommended estimators, since we do discuss various methods in estimating α and β in this paper.

If the number of censored devices is moderate, say $r_i = 5$ for all i, then we would recommend using the moment EB estimators for any values of α and β.

We have also evaluated the EB risks for two other cases (1) α known and (2) β known. In case (1), we compare the EB risks among the Dey and Kuo (D/K) estimators, the moment-EB and the ML-II EB estimators. All three estimators have comparable risks independent of α, β, r_i, and p. In case (2), we compare the EB risks between the ML-II and the moment-EB estimators. The ML-II EB is clearly a winner in almost all cases, except the cases where both r_i and α are small, say around $r_i = 2$ and $\alpha = 2$.

Different classes of EB estimators are developed by Lemon and Krutchkoff (1969) and Canavos and Tsokos (1971). Bennett and Martz (1973), Couture and Martz (1972) consider nonparametric EB estimation of $1/\lambda_p$. Martz and Waller (1982) have also provided many relevant developments in this area.

In Section 2, we derive the various estimators for risk comparisons. In Section 3, we discuss the Monte Carlo methods to approximate the EB risk and the results.

2. EB ESTIMATORS

By sufficiency considerations, observe that the S_i have independent gamma distributions $G(r_i, \lambda_i)$. We assume that the λ_i, $i = 1, ..., p$, have independent inverse gamma distributions $IG(\alpha,\beta)$, i.e.,

$$\pi(\lambda_i) = \frac{1}{\Gamma(\alpha)\beta^\alpha \lambda_i^{\alpha+1}} e^{-\frac{1}{\lambda_i \beta}}, \text{ where } \alpha > 0, \beta > 0.$$

The posterior distribution of λ_i given S_i is $IG(\alpha+r_i, \beta/(S_i\beta+1))$. Therefore, the Bayes estimator for λ_i with respect to the squared error loss $\left(L(\boldsymbol{\lambda},\mathbf{a}) = \frac{1}{p}\sum_{i=1}^{p}(\lambda_i - a_i)^2\right)$ is

$$\delta_B(i) = E(\lambda_i|S_i) = \frac{S_i}{\alpha + r_i - 1} + \frac{1}{\beta(\alpha + r_i - 1)}. \tag{2.1}$$

To construct EB estimators, we first obtain the marginal density of S_i,

$$m(S_i|\alpha,\beta) = \int_0^\infty f(S_i|\lambda_i)\pi(\lambda_i)d\lambda_i = \frac{\Gamma(\alpha + r_i)}{\Gamma(\alpha)\Gamma(r_i)} \frac{S_i^{r_i-1}\beta^{r_i}}{(S_i\beta + 1)^{\alpha+r_i}}. \tag{2.2}$$

Therefore, we can find α, β which maximize the joint density of the S_i's,

$$m(\underline{S}|\alpha,\beta) = \prod_{i=1}^{p} \frac{\Gamma(\alpha + r_i)}{\Gamma(\alpha)\Gamma(r_i)} \frac{S_i^{r_i-1}\beta^{r_i}}{(S_i\beta + 1)^{\alpha+r_i}}. \tag{2.3}$$

In Appendix 1, we show there is a unique root of $\frac{\partial}{\partial\alpha}\log m(\underline{S}|\alpha,\beta) = 0 = \frac{\partial}{\partial\beta}\log m(\underline{S}|\alpha,\beta)$. A sufficient condition based on the second derivative test is given to verify that the unique root $\hat{\alpha},\hat{\beta}$ yields the maximum likelihood estimates of (2.3).

In addition to the ML-II method we can also estimate α and β by the moment method. We will assume $r_i = r$ for all i. Using the two equations in Berger (1985, p. 101) which relate the marginal mean and the marginal variance to expectations from the model, we have

$$\mu_m = E^\pi(r\lambda) = \frac{r}{\beta(\alpha-1)}, \text{ and } \sigma_m^2 = E^\pi\left(r\lambda^2 - \frac{r}{\beta(\alpha-1)}\right)^2 = \frac{r}{\beta^2(\alpha-1)^2}\left(1 + \frac{1+r}{\alpha-2}\right). \tag{2.4}$$

Define $\hat{\mu}_m = \Sigma S_i / p, \hat{\sigma}_m^2 = \Sigma(S_i - \hat{\mu}_m)^2 / (p-1)$. We can solve α and β from (2.4) using $\hat{\mu}_m, \hat{\sigma}_m^2$ to substitute for μ_m and σ_m^2. We have

$$\hat{\alpha}_{MM} = \max\left\{\frac{(r-1)\hat{\mu}_m^2 + 2r\hat{\sigma}_m^2}{r\hat{\sigma}_m^2 - \hat{\mu}_m^2}, 2\right\}, \text{ and } \hat{\beta}_{MM} = \max\left\{\frac{r}{(\hat{\alpha}-1)\hat{\mu}_m}, 0\right\}. \quad (2.5)$$

The truncated version is obtained because without the condition $\alpha > 2$ we do not have finite variance in the prior distribution. Moreover, β must be positive. The same truncation methods are also applied to other EB estimators.

Next we will discuss how to obtain the hybrid geometric mean EB estimator. In Dey and Kuo (1991), they show

$$E\left(\prod_{i=1}^{p} S_i^{\frac{1}{p}}\right) = \frac{1}{\beta}\left[\frac{\Gamma\left(\alpha - \frac{1}{p}\right)}{\Gamma(\alpha)}\right]^p \prod_{i=1}^{p} \frac{\Gamma\left(r_i + \frac{1}{p}\right)}{\Gamma(r_i)}.$$

Therefore, they obtain an unbiased estimator of $\frac{1}{\beta}$

$$\hat{\frac{1}{\beta}} = \left[\frac{\Gamma(\alpha)}{\Gamma\left(\alpha - \frac{1}{p}\right)}\right]^p \prod_{i=1}^{p}\left[\frac{\Gamma(r_i)}{\Gamma\left(r_i + \frac{1}{p}\right)} S_i^{\frac{1}{p}}\right]. \quad (2.6)$$

The Dey and Kuo EB geometric mean estimator is obtained from (2.1) with $\hat{\beta}$ estimated from (2.6) and known α. However, the assumption of known α is too restrictive in practice. In this paper, we propose to estimate α by a moment condition. Observe from (2.4),

$$\frac{\mu_m^2}{\sigma_m^2} = \frac{r(\alpha-2)}{\alpha-1+r}. \quad (2.7)$$

Therefore, we can solve for α from (2.7) using $\hat{\mu}_m$ and $\hat{\sigma}_m^2$ for μ_m and σ_m^2,

$$\hat{\alpha}_{\text{ratio}} = \hat{\alpha}_{MM}. \tag{2.8}$$

The hybrid geometric mean EB estimator is obtained from (2.1) using (2.6) and (2.8) to estimate α and β.

We have considered the EB estimators for α and β unknown. We can also develop EB estimators when α is known. In this case, we will consider the following EB estimators: (1) the moment EB, where

$$\hat{\beta} = \max\left\{\frac{r}{(\alpha-1)\hat{\mu}_m}, 0\right\}; \tag{2.9}$$

(2) the ML-II EB, where $\hat{\beta}$ is estimated by maximizing (2.3) as a function β for the given known α; and (3) the Dey and Kuo (D/K) EB, where $\hat{\beta}$ is estimated from (2.6).

For the other case β known, we estimate α by (1) the moment consideration

$$\hat{\alpha} = \max\left\{1 + (\hat{\mu}_m\beta)^{-1} r, 2\right\} \tag{2.10}$$

and (2) the maximum likelihood estimate of α from (2.3), i.e., the root of equation (A.2) in the appendix.

We report the EB risk evaluations of these EB estimators in the next section.

3. EB RISKS

Recall the loss function is defined by

$$L(\lambda, a) = \frac{1}{p}\sum_{i=1}^{p}(\lambda_i - a_i)^2.$$

The Bayes risk of a decision rule $\boldsymbol{\delta}(S) = \left(\delta_1(S), \ldots, \delta_p(S)\right)$ is defined by

$$r\left(\pi, \boldsymbol{\delta}(S)\right) = \int\int L(\boldsymbol{\lambda}, \boldsymbol{\delta}(S)) f(S|\boldsymbol{\lambda}) dS d\pi(\boldsymbol{\lambda}).$$

The EB risk of $\boldsymbol{\delta}(S)$ is the expected Bayes risk, where the expectation is taken over all the samples.

$$\text{EB risk} = Er\left(\pi, \boldsymbol{\delta}(S)\right). \tag{3.1}$$

The EB risk is approximated by the Monte Carlo method. In each iteration, we generate λ_i, i = 1, ..., p from an $IG(\alpha,\beta)$ distribution with α,β fixed. Given the λ_i's, we generate $X_{i\ell}$, ℓ = 1, ..., 10 from the exponential distribution $\text{Exp}(\lambda_i)$. The random variable $t_{i\ell} = X_{i\ell}^{1/\rho}$ is distributed as (1.1). We order the variables $t_{i1} < t_{i2} < \ldots, < t_{ir}$, and compute $S_i = (10-r)t_{ir}^{\rho} + \sum_{\ell=1}^{r} t_{i\ell}^{\rho}$. (Note: we assume $r_i = r$ and $\rho_i = \rho$ for all i.) We compute the Bayes rule δ_B as in (2.1). We also compute each of the EB rules as in Section 2. Then we repeat these steps for 10,000 iterations. The EB risk is approximated by

$$\widehat{\text{EB}}\text{ risk} = \frac{1}{10{,}000} \sum_{j=1}^{10{,}000} \sum_{i=1}^{p} p^{-1}\left(\lambda_{ij} - \delta_{\text{EB},j}(i)\right)^2,$$

where λ_{ij} is the parameter for the i^{th} population simulated in the j^{th} iteration, and $\delta_{EB,j}(i)$ is the EB estimator for the i^{th} population in the j^{th} iteration.

Figure 1 plots the EB risk of the Bayes estimator (2.1), the ML-II EB (from (A.2) and (A.3)), the moment EB (from (2.5)) and the hybrid geometric mean EB (from (2.6 and 2.8)) for five populations with r = 2, 5 and 10. It is expected as r increases, the EB risk decreases because more information on the

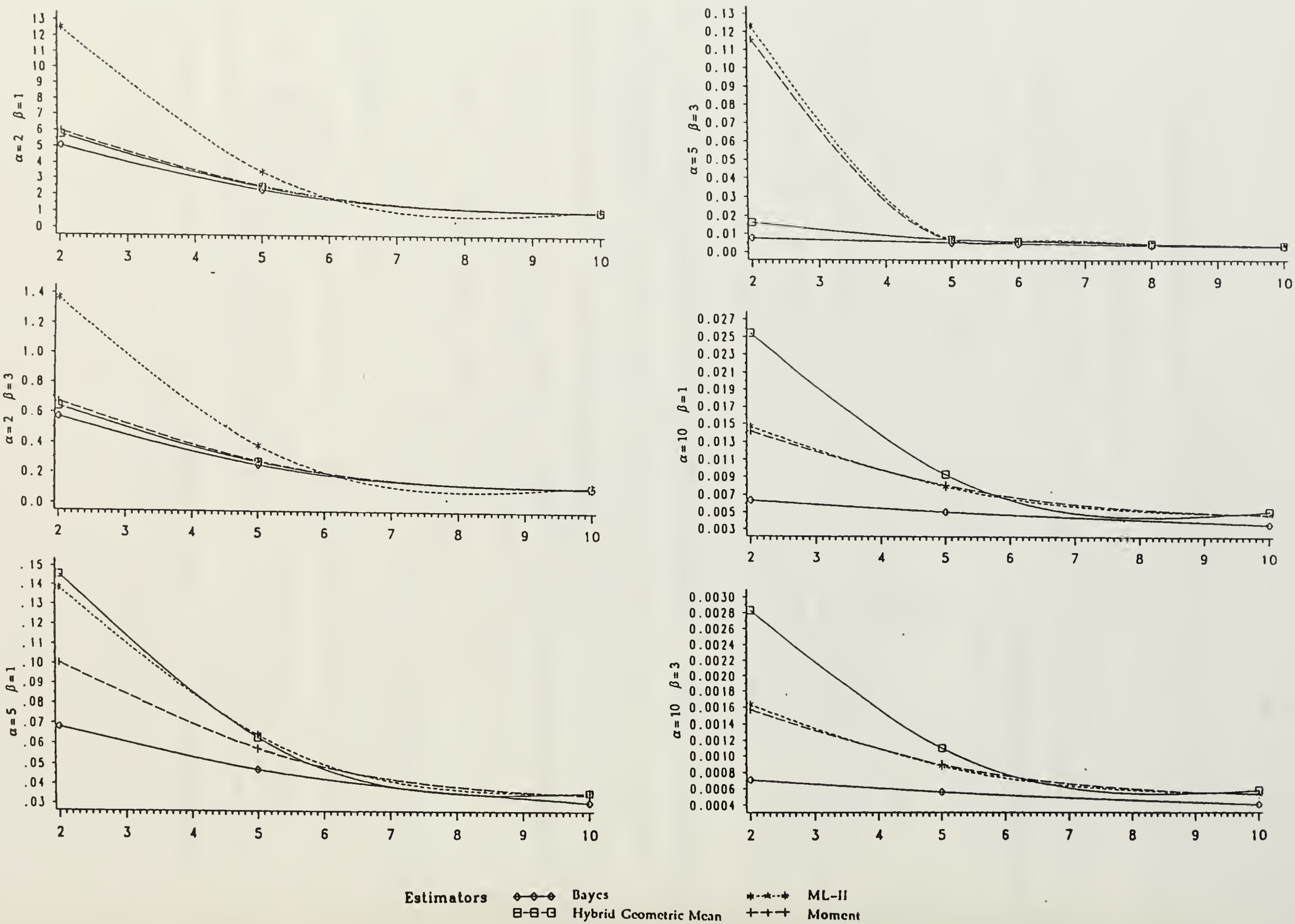

Figure 1. EB risks of the four estimators are plotted versus *r* (the number of failures observed in each population) for a class of configurations when both α and β are unknown

lifetimes is collected. Moreover, the differences of the EB risks of the three estimators and the Bayes rule also decrease. We observe that the Bayes rule has the smallest risk.

If we fix the number of censored devices, say for example, $r_i = 2$ for all i, our recommendation depends on the different α and β values. Case (1): α is small, say around 2. The hybrid geometric mean is the best. It is a bit better than the moment EB. Both outperform the ML-II EB by a substantial amount. Case (2): α is large, say around 10, the moment EB is best. It improves upon the ML-II EB by a small amount. Both dominate the hybrid geometric mean estimator by a substantial amount. Both cases 1 and 2 are somewhat insensitive to the β values. Case (3): α is moderate, say around 5. Then the performances depend on the β values. When β is small around 1, then the moment EB improves the other two by a substantial amount. When β is moderate around 3, then the hybrid geometric mean estimator improves upon the other two estimators by a substantial amount. In summary, it is safe to use the moment EB estimators unless we are in the situation of moderate α and β values (around $\alpha = 5$ and $\beta = 3$).

Next consider the case that number of censored devices is moderate, say r_i 5 for all i. The moment EB consistently performs well among the three estimators independent of the values of α and β.

Figure 2 plots the EB risks of the four estimators for various r values when α is known. The four estimators are the Bayes, moment, ML-II and the one developed in Dey and Kuo (1991). The latter three estimators are computed from (2.1) with β estimated by (2.9), (A.3) and (2.6), respectively.

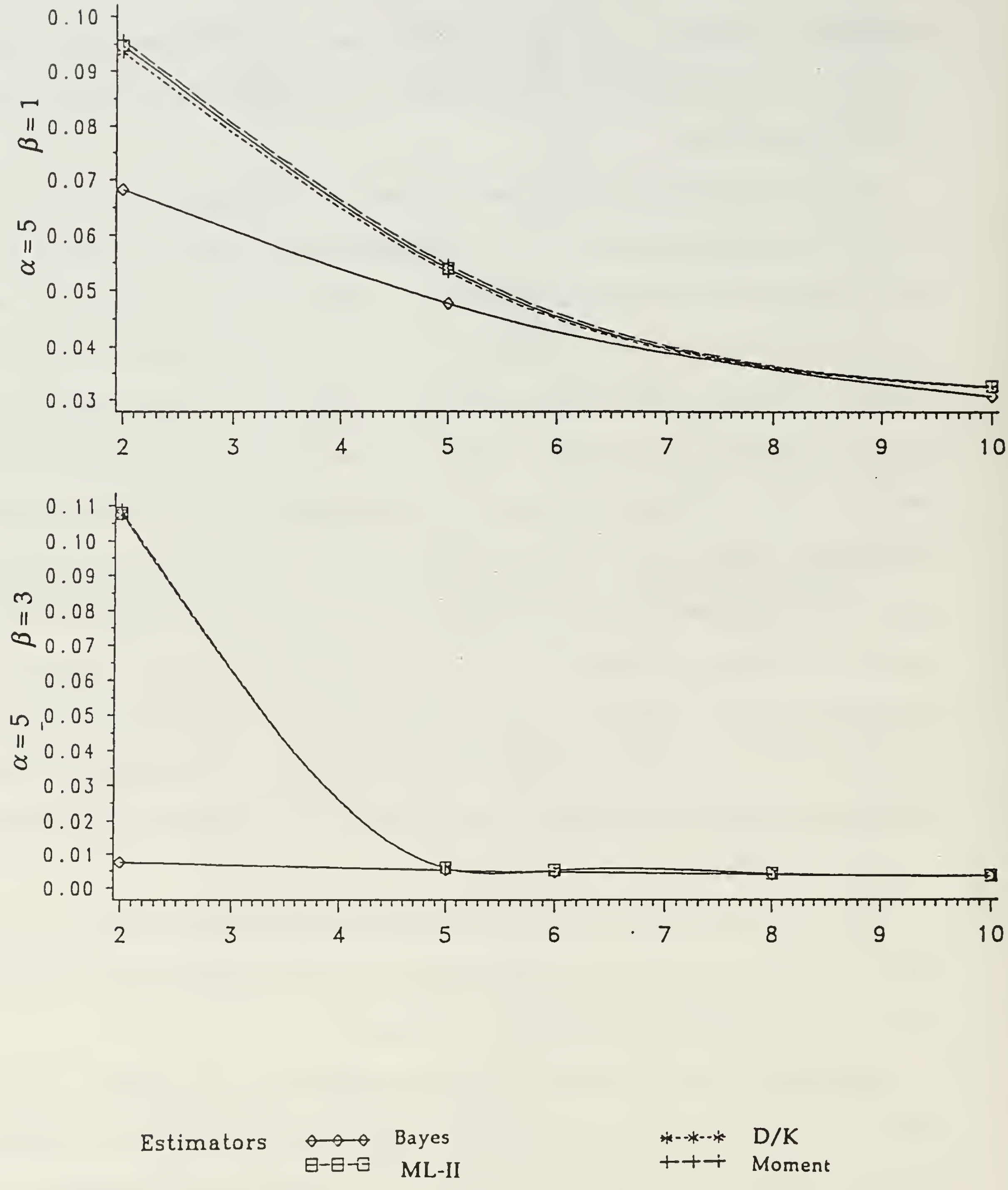

Figure 2. EB risks of the four estimators are plotted versus r for configurations when α is known.

Only the configurations with $\alpha = 5$ and $\beta = 1, 3$ are plotted here. All the other configurations show similar behavior, therefore are omitted. All the results show that the Dey and Kuo estimator dominates the moment and ML-II estimators, sometimes by a very small amount.

Figure 3 compares the EB risks of the three estimators for various r values, where β is known. The three estimators are the Bayes estimators, the moment, and the ML-II estimators. The parameters α in the two latter estimators are computed from (2.10) and (A.2), respectively. Our simulation results show that the ML-II EB dominates the Moment EB usually. The improvement of the ML-II EB over the Moment EB could be substantial in many cases. The exception occurs when both r and α are small, where the moment EB improves upon the ML-II EB by a small amount. Only two cases with $\alpha = 5$ and $\beta = 1, 3$ are presented here for short.

All the figures are produced by the GPLOT procedure of the SAS (Statistical Analysis System).

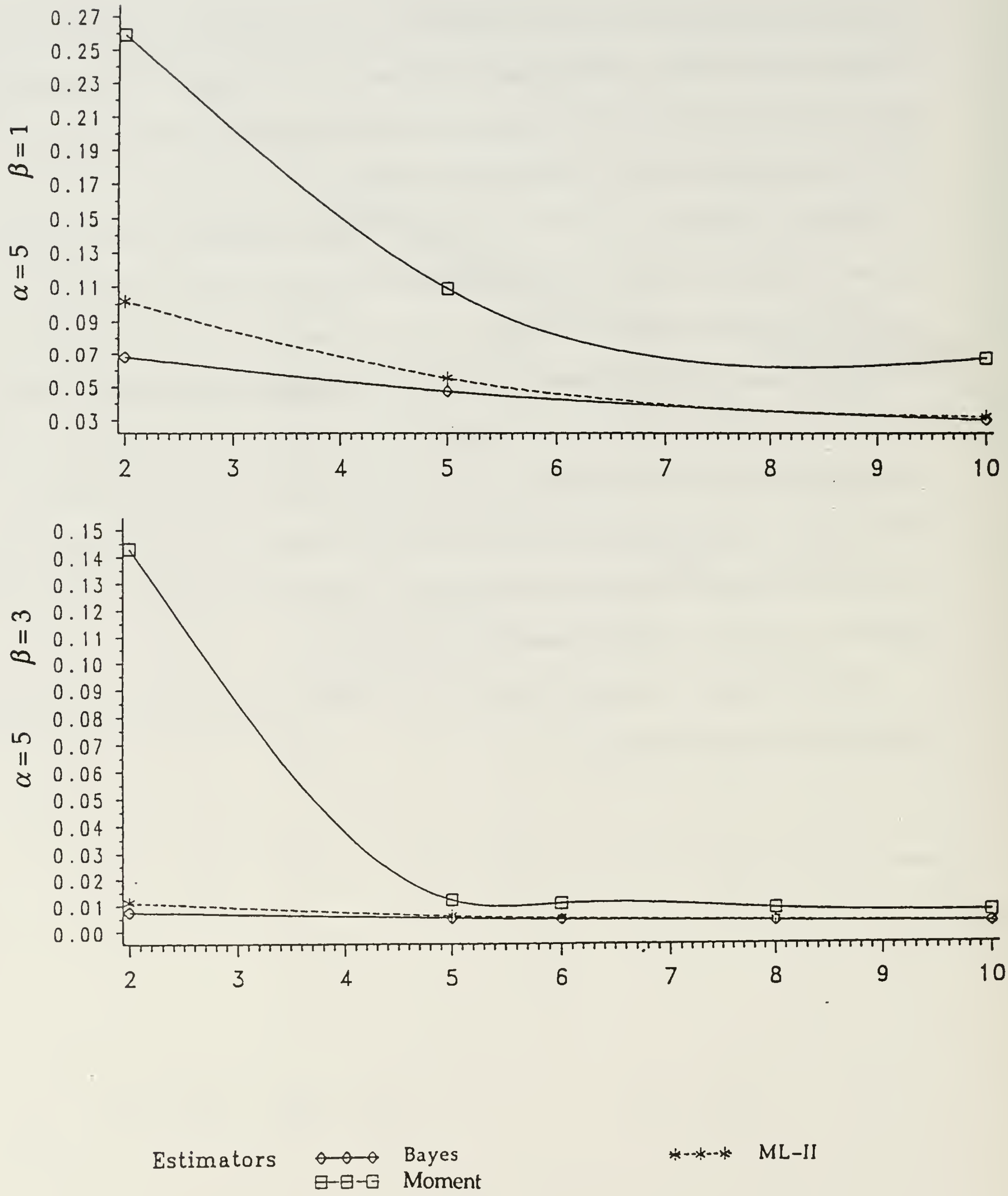

Figure 3. EB risks of the three estimators are plotted versus r for two configurations when β is known.

APPENDIX. ML-II PRIOR APPROACH TO EB ESTIMATION

In this appendix, we derive the maximum likelihood estimates of α and β from the joint density of the S_i's given in (2.3).

To maximize (2.3), we can equivalently maximize the following logarithmic function of (2.3)

$$f(\alpha,\beta)=\sum_{i=1}^{p}\left\{\ln\Gamma(\alpha+r_i)-\ln\Gamma(\alpha)+r_i\ln\beta-(\alpha+r_i)\ln(S_i\beta+1)+g(r_i,S_i)\right\}, \quad (A.1)$$

where g does not depend on α, β.

We first obtain the first derivatives of f and set them to 0:

$$\frac{\partial}{\partial\alpha}f(\alpha,\beta)=\sum_{i=1}^{p}\left\{\Psi(\alpha+r_i)-\Psi(\alpha)-\ln(S_i\beta+1)\right\}=0 \text{ and} \quad (A.2)$$

$$\frac{\partial}{\partial\beta}f(\alpha,\beta)=\sum_{i=1}^{p}\frac{r_i}{\beta}-\frac{(\alpha+r_i)S_i}{S_i\beta+1}=0, \quad (A.3)$$

where $\Psi(\alpha)=\frac{d}{d\alpha}\ln\Gamma(\alpha)$ and $\Psi(\alpha+r_i)=\frac{d}{d\alpha}\ln\Gamma(\alpha+r_i)$.

A computer program using the bisection method has been written to search for the roots of (A.2) and (A.3).

Next we will verify (by the second derivative test) whether the root $(\hat{\alpha},\hat{\beta})$ is a local maximum of f. Let us first evaluate the second derivatives. Using the recurrence formula for $\Psi(\alpha+r_i)$ as in Abramowitz and Stegun (1964, p. 258),

$$\Psi(\alpha+r_i)=\frac{1}{r_i-1+\alpha}+\frac{1}{r_i-2+\alpha}+\ldots+\frac{1}{1+\alpha}+\frac{1}{\alpha}+\Psi(\alpha),$$

we have

$$\frac{\partial^2}{\partial \alpha^2} f(\alpha,\beta) = -\sum_{i=1}^{p} \left\{ \frac{1}{(r_i - 1 + \alpha)^2} + \frac{1}{(r_i - 2 + \alpha)^2} + \ldots + \frac{1}{\alpha^2} \right\} < 0 \qquad (A.4)$$

$$\frac{\partial^2}{\partial \alpha \partial \beta} f(\alpha,\beta) = -\sum_{i=1}^{p} \frac{S_i}{S_i \beta + 1}, \text{ and}$$

$$\left. \frac{\partial^2}{\partial \beta^2} f(\alpha,\beta) \right|_{(\hat{\alpha},\hat{\beta})} = \sum_{i=1}^{p} \left\{ -\frac{r_i}{\hat{\beta}^2} + \frac{(\hat{\alpha} + r_i) S_i^2}{\left(S_i \hat{\beta} + 1\right)^2} \right\}$$

$$= \sum_{i=1}^{p} \left\{ -\frac{(\hat{\alpha} + r_i) S_i}{\left(S_i \hat{\beta} + 1\right)\hat{\beta}} + \frac{(\hat{\alpha} + r_i) S_i^2}{\left(S_i \hat{\beta} + 1\right)^2} \right\} \qquad \text{(using A.3)}$$

$$= \sum_{i=1}^{p} -\frac{(\hat{\alpha} + r_i) S_i}{\hat{\beta}\left(S_i \hat{\beta} + 1\right)^2} < 0. \qquad (A.5)$$

Let $\frac{\partial^2}{\partial \alpha^2} f\left(\hat{\alpha},\hat{\beta}\right)$ denote the 2nd partial derivative with respect to α evaluated at $(\hat{\alpha},\hat{\beta})$. Similar notations are defined for other derivatives. The second derivative test states that if $\frac{\partial^2}{\partial \alpha^2} f\left(\hat{\alpha},\hat{\beta}\right) < 0$ and $D = \frac{\partial^2}{\partial \alpha^2} f\left(\hat{\alpha},\hat{\beta}\right) \cdot \frac{\partial^2}{\partial \beta^2} f\left(\hat{\alpha},\hat{\beta}\right) - \left[\frac{\partial^2}{\partial \alpha \partial \beta} f\left(\hat{\alpha},\hat{\beta}\right) \right]^2 > 0$, then $\left(\hat{\alpha},\hat{\beta}\right)$ is a local maximum of $f(\alpha,\beta)$. Since $\frac{\partial^2}{\partial \alpha^2} f\left(\hat{\alpha},\hat{\beta}\right) < 0$, we need only program the second condition and verify it. That is, we need to show $D = \left\{ \sum_{i=1}^{p} \sum_{j=1}^{r_i} \frac{1}{\left(\hat{\alpha} + r_i - j\right)^2} \right\} \cdot \left\{ \sum_{i=1}^{p} \frac{(\hat{\alpha} + r_i) S_i}{\hat{\beta}\left(S_i \hat{\beta} + 1\right)^2} \right\} - \left\{ \sum_{i=1}^{p} \frac{S_i}{S_i \hat{\beta} + 1} \right\}^2 > 0$, where $(\hat{\alpha},\hat{\beta})$ is the solution to (A.2) and (A.3).

We have discussed how to obtain the maximum likelihood estimates of (A.1) when both α and β are unknown. For the case that β is known, we maximize $f(\alpha,\beta)$ by solving (A.2) as a function of α. There is a unique root of (A.2) which maximizes $f(\alpha,\beta)$ because $f(\alpha,\beta)$ is a concave function of α as shown in (A.4). For the other case that α is known, we solve (A.3) as a function of β. This root maximizes (A.1) because of the condition in (A.5).

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University of California
Los Angeles, CA 90024

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Math Department
University of California
Los Angeles, CA 90024

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Math Department
University of California
Los Angeles, CA 90024

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Statistics Department
University of California
Riverside, CA 92521

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Statistics Department
University of California
Riverside, CA 92521

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Statistics Department
University of California
Riverside, CA 92521

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University of California
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University of California
Santa Barbara, CA 93106

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Statistics Department
University of California
Santa Barbara, CA 931067

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Statistics Department
University of California
Santa Barbara, CA 931067

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Statistics Department
University of California
Santa Barbara, CA 931067

27. Prof. Bradley Efron..1
Statistics Dept.
Sequoia Hall
Stanford University
Stanford, CA 94305

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Statistics Department
Stanford University
Stanford, CA 94305

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Biostatistics Department
Stanford University
Stanford, CA 94305

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Statistics Division
University of California
Davis, CA 95616

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Statistics Division
University of California
Davis, CA 95616

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Statistics Division
University of California
Davis, CA 95616

33. Prof. Jane-Ling Wang..1
Statistics Division
University of California
Davis, CA 95616

34. Dr. Jack C. Lee..1
Bell Communications Research
Morris Research and Engineering Center
435 South Street, 2Q-374
Morristown, NJ 07960

35. Prof. Malay Ghosh 1
Statistics Department
University of Florida
Cainvesville, FL 32611

36. Prof. Glen Meeden 1
Theoretical Statistics Department
University of Minnesota
270 Vincent
206 Church St. SE
Minneapolis, MN 55455

37. Prof. James Zidek 1
Statistics Department
University of British Columbia
Vancouver, V6T 1W5 CANADA

38. Prof. Bruce Turnbull 1
Operations Research and Industrial Engineering
Cornell University
227 FfTC
Ithaca, NY 14853-3801

39. Center for Naval Analyses 1
4401 Ford Avenue
Alexandria, VA 22302-0268

40. Dr. Terry Speed 1
Statistics Department
University of California
Berkeley, CA 94720

41. Dr. P. Welch 1
IBM Research Laboratory
Yorktown Heights, NY 10598

42. Dr. Neil Gerr 1
Office of Naval Research
Arlington, VA 22217

43. Dr. J. Abrahams, Code 1111, Room 607 1
Mathematical Sciences Division, Office of Naval Research
800 North Quincy Street
Arlington, VA 22217-5000

44. Prof. H. Chernoff 1
Department of Statistics
Harvard University
1 Oxford Street
Cambridge, MA 02138

45. Prof. Carl N. Morris 1
Statistics Department
Harvard University
1 Oxford St.
Cambridge, MA 02138

46. Prof. Tom A. Louis 1
School of Public Health
University of Minnesota
Mayo Bldg. A460
Minneapolis, MN 55455

47. Prof. William Jewell 1
Operations Research Department
University of California, Berkeley
Berkeley, CA 94720

48. Dr. S. R. Dalal 1
Bellcore
445 South Street
Morristown, NJ 07962-1910

49. Dr. Michael P. Cohen 1
National Center for Education Statistics
555 New Jersey Ave., NW
Washington, DC 20208-5654

50. Prof. Lynn Kuo 1
Code OR-Ku
Naval Postgraduate School
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