NPS-59BC75111

NAVAL POSTGRADUATE SCHOOL

Monterey, California

DM Approximations for the Gravest Frequency of a Vibrating System

John E. Brock

November 1975

Technical Report for
Period 1 July 1975 - 31 December 1975

Approved for public release; distribution unlimited

Prepared for
Chief of Naval Research, Arlington, Virginia 22217

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral Isham Linder
Superintendent

Jack R. Borsting
Provost

The work reported herein was supported by the Foundation Research Program of the Naval Postgraduate School with funds provided by the Chief of Naval Research.

This report was prepared by:

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 59BC75111	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) DM Approximations for the Gravest Frequency of a Vibrating System		5. TYPE OF REPORT & PERIOD COVERED Technical Report 1 July 75 - 31 December 75 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) John E. Brock		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61152N, RR000-01-10; N0001476WR60052
11. CONTROLLING OFFICE NAME AND ADDRESS Chief of Naval Research Arlington, Virginia 22217		12. REPORT DATE November 1975 13. NUMBER OF PAGES 24
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Approximate Frequency
Dunkerley
Mikhlin
Vibrations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Estimates are made of the smallest nonzero frequency of vibration of undamped linear mechanical systems having lumped and/or distributed mass and permitting rigid body motions. The approximations are smaller than the correct values but remarkable accuracy may be achieved. The procedures are based upon methods of S. Dunkerely and S. G. Mikhlin.

DM Approximations for the Gravest
Frequency of a Vibrating System

by J. E. Brock

Abstract: Estimates are made of the smallest nonzero frequency of vibration of undamped linear mechanical systems having lumped and/or distributed mass and permitting rigid body motions. The approximations are smaller than the correct values but remarkable accuracy may be achieved. The procedures are based upon methods of S. Dunkerley and S. G. Mikhlin.

Note: This report amends, augments, and is intended to replace an earlier report (Dunkerley-Mikhlin Estimates of Gravest Frequency of a Vibrating System, Naval Postgraduate School Report NPS-59BC75101, October 1975). In particular, it corrects errors in an earlier derivation and adds an example which illustrates the equations in the amended derivation.

Acknowledgment: Appreciation is expressed to the Office of Naval Research which has sponsored this work through the Foundation Research Program at the Naval Postgraduate School and also to Professor R. E. Newton who first directed the writer's attention to the merits of the Dunkerley viewpoint.

Contents

Introduction

The purpose of this paper is to develop and to illustrate by a number of examples some interesting and useful extensions of a procedure, generally known as Dunkerley's method, for estimating the gravest frequency of an undamped linear mechanical vibrating system. The theory is developed in the next section hereof. The equivalent of equation 15 was given in the year 1894 by S. Dunkerley (3) who obtained the result empirically based on calculations performed while investigating the vibrations of shaft and disk systems; he certainly did not recognize the generality or the theoretical basis of his formula. Temple and Bickley (12) discussed the procedure in 1933 indicating its applicability to both lumped and continuous mass distributions. However Bickley and Talbot (1) in a later (1961) textbook on vibrations do not mention the method. Southwell (11) treats the method in his well known treatise (1936, 1941) without citing Dunkerely. This may be the reason that the method is sometimes called the Dunkerley-Southwell method. However, although Southwell may have arrived at the result independently (he did indeed discover *another* method of determining lower bounds; cf. Lamb and Southwell (6)), his book does cite Temple and Bickley (12) among the general references.

A formulation in terms of integral equations and their eigenvalues, iterated kernels, etc., which is appropriate for continuous mass distributions, is a logical extrapolation, by analogy, from the formulation in terms of matrices, their eigenvalues, powers of matrices, etc., which is appropriate for systems having a finite number of degrees of freedom, so that, in a sense, all the work reported herein stems from Dunkerley (3) and Temple and Bickley (12). Indeed it was via this path that the writer was led to the results reported here. However, the relationship with the theory of integral equations, a thoroughly developed and explored discipline, is so very close that it seemed likely that the developments at which the writer has arrived had been anticipated, in purely mathematical context, by an earlier writer. Indeed this is the case. Mikhlin (8) definitely attributes the ideas to Mikhlin (7). In neither of these references, however, does Mikhlin refer to Dunkerley (3), Temple and Bickley (12), or Southwell (11). It seems clear that Mikhlin arrived at his results without being aware of Dunkerley's formula or of its relation to his own work. Accordingly, the writer believes that it is appropriate to call the extended procedure by the name Dunkerley-Mikhlin. The present paper may be considered as a brief exposition of this method, and some extensions, and of its application to a variety of problems of engineering vibration.

Derivations

We consider an undamped linear vibrating system characterized by $N \times N$ symmetrical matrices K (stiffness) and M (mass). Suppose that there are p rigid body modes which are known (by inspection). (Thus, if $p > 0$, K has no inverse.) There exists a modal matrix U, not necessarily unique, and a (diagonal) spectral matrix Ω^2 in which frequency-squares are arranged in non-descending order, such that

$$KU = MU\Omega^2; \quad U^T MU = I = N\times N \text{ unit matrix} \tag{1a,b}$$

Now consider any other system which is similar to the original system except that p additional constraints have been incorporated so as to eliminate the rigid body modes. Let the flexibility of this system be C.

Consider the kth mode (k>p) of the original system. There are constants a_{jk} such that

$$v_k = u_k + \sum_{j=1}^{p} a_{jk} u_j \tag{2}$$

does not involve applying loads to the added constraints. Thus, for k>p,

$$v_k = CKv_k = CKu_k + C\sum_{j=1}^{p} a_{jk} Ku_j = CKu_k = CMu_k\omega_k^2 \tag{3}$$

Define the filtering matrix

$$F = I - \sum_{j=1}^{p} u_j u_j^T M \tag{4}$$

which is such that

$$Fu_k = \begin{cases} 0 & \text{if } k\leq p \\ u_k & \text{if } k>p \end{cases} \tag{5}$$

Thus, for k>p,

$$CMFu_k = CMu_k = \omega_k^{-2} v_k = \omega_k^{-2} u_k + \omega_k^{-2} \sum_{j=1}^{p} a_{jk} u_j \tag{6a}$$

while for k≤p,

$$CMFu_k = 0 \tag{6b}$$

Equations 6 may be combined to give

$$U\Lambda + B = CMFU \tag{7}$$

where

$$\Lambda = \text{diag}[0\ 0\ \dots\ 0\ \omega_{p+1}^{-2}\ \omega_{p+2}^{-2}\ \dots\ \omega_N^{-2}] \tag{8}$$

and B is a matrix the first p columns of which are zero and the remaining columns of which are linear combinations of $u_1, u_2, \ldots, u_p$. Now consider the matrix $U^{-1}B = U^T MB$, the kth column of which is

$$\omega_k^{-2} \sum_{j=1}^{p} a_{jk} U^T M u_j \tag{9}$$

and, because of equation 1b, this can have nonzero elements only in the first p positions. Thus $\Lambda + U^{-1}B$ and $(\Lambda + U^{-1}B)^n$ are of the forms

$$\left[\begin{array}{c|c} 0_1 & E \\ \hline 0_2 & D \end{array}\right], \qquad \left[\begin{array}{c|c} 0_1 & ED^{n-1} \\ \hline 0_2 & D^n \end{array}\right] \tag{10a,b}$$

respectively, where 0_1 is a p×p matrix of zeros, 0_2 is a (N-p)×p matrix of zeros, E is the nonzero part of $U^{-1}B$, and D is the nonzero part of Λ. Finally, from the equality

$$(\Lambda + U^{-1}B)^n = (U^{-1}CMFU)^n \tag{11}$$

by taking traces, we get

$$\sum_{k=p+1}^{N} \omega_k^{-2n} = tr[(\Lambda + U^{-1}B)^n] = tr[(U^{-1}CMFU)^n] = tr[(CMF)^n] = tr(Q^n) \tag{12}$$

where Q is defined as the matrix triple product CMF. Note, however, for computational convenience, that

$$tr(Q^n) = tr[(CMF)^n] = tr[(MFC)^n] = tr[(FCM)^n] \tag{13}$$

We have used the fact that if A and B are matrices conformable in either order, then tr(AB) = tr(BA).

We will refer to equation 12 as the nth DM (Dunkerley-Mikhlin) evaluation and to the corresponding relation

$$\omega_{p+1} \gtrapprox [tr(Q^n)]^{-1/2n} \tag{14}$$

as the nth DM approximation. We will repeatedly use the symbol $\gtrapprox$ to mean "greater than but approximately equal to," and a symbol indicating the opposite order.

If M is diagonal, if there are no rigid body modes (i.e., if p=0), and if we take n=1, equation 14 becomes

$$\omega_1^{-2} \lessapprox tr(Q) = tr(CM) = \sum_{k=1}^{N} m_k c_{kk} \tag{15}$$

This is what is generally referred to as Dunkerley's formula. It was (obliquely) stated in 1894 by S. Dunkerley who regarded it as an empirical representation of calculations he had made of shaft and rotor frequencies.

Simple extensions to continuous mass distributions are obvious and have been discussed by many writers, among the first of whom were Temple and Bickley, Reference 6. However, the theoretical basis for the continuous case seems to have been first established by Mikhlin, who, however, did not consider problems of mechanical vibration and who seems to have been unaware of Dunkerley's work. In the continuous case, matrix multiplication is replaced by integration and the compliance matrix C is replaced by a symmetrical function of two variables, z(x,y) which gives deflection at x due to unit loading at y. This notation is appropriate to a one-dimensional field; however, examples given herein ~~in Reference 1~~ illustrate cases involving two dimensional fields. The mass matrix M which heretofore need not have been a diagonal matrix, now becomes the equivalent of an infinite diagonal matrix, namely, a function m(x) specifying mass per unit length. The matrix Q is replaced by the function

$$q(x,y) = m(y)[z(x,y) - \sum_{k=1}^{p} u_k(y) \int_0^L u_k(\xi)\, m(\xi)\, z(x,\xi)\, d\xi] \tag{16}$$

where the functions $u_k(x)$, $k=1,2,\ldots,p$, describe the known orthonormal rigid body modes, satisfying

$$\int_0^L m(\xi)\, u_i(\xi)\, u_j(\xi)\, d\xi = \delta_{ij} \tag{17}$$

(Kronecker delta). In equation 16, $z(x,y)$ is the compliance of any system which is like the given system but with additional constraints so as to eliminate the rigid body modes. The function $z(x,y)$ is sometimes called the Green's function.

The continuous analog of raising the matrix Q to the nth power is the formation of the nth iterated function

$$q_n(x,y) = \int_0^L q_{n-1}(x,\xi)\, q(\xi,y)\, d\xi; \quad q_1(x,y) = q(x,y) \tag{18}$$

and the operation corresponding to taking the trace of Q^n is

$$\sum_{k=p+1}^{\infty} \omega_k^{-2n} = \int_0^L q_n(x,x)\, dx \tag{19}$$

If there is both lumped and distributed mass, we may write

$$m(x) = \tilde{m}(x) + \sum_{k=1}^{r} m_k\, \delta(x-x_k) \tag{20}$$

where $\delta(x-x_k)$ denotes the Dirac "function," $\tilde{m}(x)$ denotes a continuous mass distribution, and the sum represents r distinct point masses m_k located at $x = x_k$, $k=1,2,\ldots,r$.

Illustration

We consider a finite element model of a segment of an Euler-Bernoulli beam. The displacement vector is $u = [x_1\ x_2\ x_3\ x_4]^T$ where x_1 is the (upward lateral) displacement at the left end, x_3 is the similar displacement at the right end, x_2/L is the slope at the left, and x_4/L is the slope at the right. The length of the segment is L. The FEM consistent stiffness and mass matrices are

$$K = (2EI/L^3)\begin{bmatrix} 6 & 3 & -6 & 3 \\ 3 & 2 & -3 & 1 \\ -6 & -3 & 6 & -3 \\ 3 & 1 & -3 & 2 \end{bmatrix}; \quad M = (m/420)\begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}$$

First we exhibit the classical solution. The modal matrix (the first two columns of which are not uniquely determined) is

$$U = \begin{bmatrix} 1 & a & b & c \\ 0 & -2a & -6b & -12c \\ 1 & -a & b & -c \\ 0 & -2a & 6b & -12c \end{bmatrix}/\sqrt{m}, \quad \text{where } a^2 = 3,\ b^2 = 5, \text{ and } c^2 = 7.$$

and the spectral matrix is

$$\Omega^2 = \text{diag}[\omega_2^2\ \ \omega_2^2\ \ \omega_3^2\ \ \omega_4^2] = (840EI/mL^3)\ \text{diag}[0\ \ 0\ \ 6/7\ \ 10]$$

There is no difficulty in verifying equations 1a and 1b.

Next, we employ the DM method and verify the correctness of equation 12. We constrain the system by rigidly fixing the left end so as to form a cantilever. We easily determine

$$C = (L^3/6EI)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 6 \end{bmatrix}; \quad CM = (mL^3/2520EI)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 69 & 17 & 246 & -32 \\ 84 & 21 & 336 & -42 \end{bmatrix}$$

and we also calculate

$$F = \begin{bmatrix} -3 & -4 & 3 & 1 \\ 36 & 33 & 36 & 3 \\ 3 & -1 & -3 & 4 \\ 36 & 3 & -36 & 33 \end{bmatrix}/30$$

Next, we obtain

$$Q = CMF = (mL^3/25200EI)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -19 & 3 & 16 \\ 0 & -35 & 0 & 35 \end{bmatrix},$$

Finally we calculate

$$\mathrm{tr}(Q) = 38mL^3/25200EI = \omega_3^{-2} + \omega_4^{-2}$$

$$\mathrm{tr}(Q^2) = 1234(mL^3/25200EI)^2 = \omega_3^{-4} + \omega_4^{-4}$$

$$\mathrm{tr}(Q^3) = 42902(mL^3/25200EI)^3 = \omega_3^{-6} + \omega_4^{-6}$$

and so on. All of these calculations check out correctly.

One may observe that the two diagonal elements of Q give directly

$$\omega_3^{-2} = 35mL^3/25200EI, \quad \omega_4^{-2} = 3mL^3/25200EI$$

but this is only fortuitous in the present illustration as may be seen in examples 1b and 1c which appear later.

Now we illustrate the equations employed in the derivation of equation 12. First, it is easy to verify that

$$FU = [0 \quad 0 \quad u_3 \ u_4]$$

Next we form

$$CK = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

and obtain

$$[v_3 \ v_4] = CK[u_3 \ u_4] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 6b & 10c \\ 12b & 0 \end{bmatrix} / \sqrt{m}$$

which obviously satisfy the constraints which have been introduced at the left end. We can also calculate the coefficients $a_{13} = 2b$, $a_{23} = -3b/a$, $a_{14} = 5c$, and $a_{24} = -6c/a$, so that, indeed

$$v_3 = u_3 + a_{13}u_1 + a_{23}u_2; \quad v_4 = u_4 + a_{14}u_1 + a_{24}u_2$$

We can also calculate and verify that

$$\omega_3^2 CMu_3 = (6/7)(840EI/mL^3)(bmL^3/2520EI)[0 \ 0 \ 21 \ 42]^T/\sqrt{m} = v_3$$

with a similar calculation relating u_4 and v_4.

Next we calculate

$$CMFU = QU = (\sqrt{m}L^3/840EI)\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 7b & c\\ 0 & 0 & 14b & 0\end{bmatrix}$$

$$\Lambda = (mL^3/25200EI)\ \mathrm{diag}[0\ 0\ 35\ 3]$$

$$U\Lambda = (\sqrt{m}L^3/25200EI)\begin{bmatrix}0 & 0 & 35b & 3c\\ 0 & 0 & -210b & -36c\\ 0 & 0 & 35b & -3c\\ 0 & 0 & 210b & -36c\end{bmatrix}$$

$$B = CMFU - U\Lambda = (\sqrt{m}L^3/25200EI)\begin{bmatrix}0 & 0 & -35b & -3c\\ 0 & 0 & 210b & 36c\\ 0 & 0 & 175b & 33c\\ 0 & 0 & 210b & 36c\end{bmatrix}$$

$$= (mL^3/25200EI)[0\quad 0\quad 35(a_{13}u_1+a_{23}u_2)\quad 3(a_{14}u_1+a_{24}u_2)]$$

Noting that $U^{-1} = (MU)^T$, we finally calculate

$$U^{-1}QU = (mL^3/25200EI)\begin{bmatrix}0 & 0 & 70b & 15c\\ 0 & 0 & -3.5ab & -6ac\\ 0 & 0 & 35 & 0\\ 0 & 0 & 0 & 3\end{bmatrix} = \Lambda + U^{-1}B$$

so that

$$D = (mL^3/25200EI)\begin{bmatrix}35 & 0\\ 0 & 3\end{bmatrix};\quad E = (mL^3/25200EI)\begin{bmatrix}70b & 15c\\ -3.5ab & -6ac\end{bmatrix}$$

We also verify that

$$\mathrm{tr}(U^{-1}QU) = 38mL^3/25200EI,\quad \mathrm{tr}[(U^{-1}QU)^2] = 1234(mL^3/25200EI)^2,\ \text{etc.}$$

Although it is worth remarking that neither K nor C possesses an inverse, this is to be expected in cases for which there are rigid body modes, and the modified system involves perfectly rigid constraints. It is of some interest to note the form of the product CK.

Applications

Example 1, part a. A uniform, massless cantilever beam has concentrated masses m, 9m, and 4m at distances 9L, 21L, and 27L, respectively, from the fixed end. It is desired to estimate the lowest frequency of harmonic oscillation. Taking the lateral deflections of the point masses, namely x_1, x_2, and x_3, as the elements of a vector u, we easily determine

$M = m\ \mathrm{diag}[1\ 9\ 4]$;

$$C = (9L^3/EI)\begin{bmatrix} 27 & 81 & 108 \\ 81 & 343 & 490 \\ 108 & 490 & 729 \end{bmatrix}; \quad Q = CM = (9mL^3/EI)\begin{bmatrix} 27 & 729 & 432 \\ 81 & 3087 & 1960 \\ 108 & 4410 & 2916 \end{bmatrix}$$

From equation 12, taking n=1, we get

$$\omega_1 \gtrsim (EI/54270mL^3)^{1/2} = 4.2926 \cdot 10^{-3}(EI/mL^3)^{1/2}$$

Taking n=2, we get

$$\omega_1 \gtrsim (E^2I^2/287808908 4m^2L^6)^{1/4} = 4.317416 \cdot 10^{-3}(EI/mL^3)^{1/2}$$

A Rayleigh approximation, details of which are not given here, gives the numerical coefficient for an upper bound as $4.317542 \cdot 10^{-3}$, so that the first approximation is in error by less than 0.6% and the second is in error by less than 0.003%.

Example 1, part b. Now suppose that the fixed support at the left of this beam is replaced by a frictionless pivot. The frequency $\omega_1 = 0$ corresponds to the rigid body mode $u_1 = [3\ 7\ 9]^T/(774)^{1/2}$. We obtain

$$F = \begin{bmatrix} 765 & -189 & -108 \\ -21 & 333 & -252 \\ -27 & -567 & 450 \end{bmatrix}/774; \quad Q = CMF = (9mL^3/86EI)\begin{bmatrix} -702 & -810 & 864 \\ -6198 & -10962 & 10952 \\ -9858 & -22806 & 21024 \end{bmatrix}$$

and we have no difficulty in obtaining

$$\omega_2 \gtrsim 0.03195(EI/mL^3)^{1/2}, \quad \omega_2 \gtrsim 0.033494(EI/mL^3)^{1/2}$$

for the first and second DM approximations, respectively. A Rayleigh approximation gives the numerical coefficient as 0.033586 so that the errors are less than 5% and 0.3% respectively.

Example 1, part c. Now suppose that the beam is completely free at the left so that there are two rigid body modes, involving translation and rotation. We use the same matrix C as above although a simpler one could be constructed since the leftmost segment of length 9L obviously does not enter the present problem. We take $u_1 = [1\ 1\ 1]^T/(14)^{1/2}$ and u_2 as a combination of $[1\ 1\ 1]^T$ and $[3\ 7\ 9]^T$ proportioned for orthonormality. Thus we find $u_2 = [15\ 1\ -6]^T/(378)^{1/2}$. We calculate

$$F = \begin{bmatrix} 42 & -126 & 84 \\ -14 & 42 & -28 \\ 21 & -63 & 42 \end{bmatrix}/126; \quad Q = (96mL^3/EI)\begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 2 \\ 3 & -9 & 6 \end{bmatrix}$$

and we get $tr(Q) = 288mL^3/EI$, $tr(Q^2) = 82944(mL^3/EI)^2$ both of which give the exactly correct result

$$\omega_3 = (EL/288mL^3)^{1/2}$$

since there are only three frequencies the smallest two of which are known to be zero.

Example 2. Now we consider a cantilever beam in the form of a truncated right circular cone having base radius b, tip radius a, and length L, and having uniform physical properties. Assuming an Euler-Bernoulli model, we obtain the bending compliance

$$z(x,y) = \{(\xi-\eta)[(1-\alpha\eta)^{-2} - (1+2\alpha\eta)] + 2\alpha^2\eta^3/(1-\alpha\eta)\}/6B\alpha^2$$

where

$$B = \pi Eb^4/4 = (EI)_{root}; \quad \alpha = (b-a)/bL; \quad \xi = Max\{x,y\}; \ \eta = Min\{x,y\}$$

The mass per unit length is

$$m(x) = m_o(1-\alpha x)^2; \quad m_o = \pi b^2\gamma$$

m_o being the value pertaining to the root end and γ denoting mass density. Performing the indicated evaluations gives

$$\omega_1^{-2} \lessapprox \int_0^L m(x)\ z(x,x)\ dx = m_oL^4(1+4q)/60B$$

for the first estimate and, noting the symmetry of z(x,y),

$$\omega_1^{-4} \lessapprox \int_0^L\int_0^L m(x)\, m(y)\, z^2(x,y)\, dx\, dy =$$

$$= m_o{}^2 L^8[3(q^4-1)/8 - q^4\log_e(q)/4 + 3p^2 - 5p^3 + 11p^4/4$$

$$- 3p^5/5 - p^6/20 - p^7/70 + 9p^8/80 - p^9/5 + 2p^{10}/25]/18B^2p^8$$

where $q = a/b$, $p = 1-q$. Although both of these evaluations give results for ω_1 which are smaller than the correct value, the second gives a result which has a maximum error of 0.8% (for $q = 0$) for $0 \leqq q \leqq 1$. The first, and much simpler, evaluation is much less accurate, but it serves as a guide to the following approximation

$$\omega_1 \approx 8.72[(1-0.016q)/(1+5.053q)]^{1/2}(B/m_oL^4)^{1/2}$$

which has a maximum error of 1.4% for $q = 0.1$. The "exact" results, for comparison, were calculated from formulas given by Conway, Becker, and Dubil, Reference 2.

Next, we consider some cases for which the mass distribution is of the form described by equation 20. The first and second DM evaluations, respectively, take the forms

$$\sum_{i=1}^{\infty}\omega_i^{-2} = \int_0^L \tilde{m}(x)\, z(x,x)\, dx + \sum_{k=1}^{r} m_k\, z(x_k,x_k)$$

$$\sum_{i=1}^{\infty}\omega_i^{-4} = \int_0^L\int_0^L \tilde{m}(x)\, \tilde{m}(y)\, z^2(x,y)\, dy\, dx + 2\sum_{k=1}^{r} m_k \int_0^L \tilde{m}(x)\, z^2(x,x_k)\, dx$$

$$+ \sum_{i=1}^{r}\sum_{j=1}^{r} m_i\, m_j\, z^2(x_i,x_j)$$

The first term in these evaluations is what we had previously for continuous distributions and the last term may be identified with the trace of the appropriate matrix.

<u>Example 3.</u> Suppose that the cantilever beam of example 1a itself has a

mass 4m distributed uniformly along its length. (Note that the upper limit of integration is 27L.) We have $m(x) = 4m/27L$ and $z(x,y) = (3xyw-w^3)/6EI$, where $w = \mathrm{Min}\{x,y\}$. We calculate

$$\omega_1 \gtrapprox [(4m/27L)(27L)^4/12EI + 54270mL^3/EI]^{-1/2} = 0.004054(EI/mL^3)^{1/2}$$

for the first DM approximation. Note that we have seen the number 54270 in example 1a. The second DM approximation is more labor; it gives

$$\omega_1 \gtrapprox [11 \cdot 3^{17}/35 + 4759166988/7 + 2878089084]^{-1/4}(EI/mL^3)^{1/2}$$

$$= (81 \cdot 1554923033/35)^{-1/4}(EI/mL^3)^{1/2} = 0.00408289(EI/mL^3)^{1/2}$$

An "exact" evaluation by transfer matrix procedure gives the numerical coefficient 0.00408305, so that the errors for the first and second approximations are 0.7% and 0.004% respectively.

Example 4. Consider the case of a uniform cantilever beam of length L and mass ρm having a concentrated mass $(1-\rho)m$ at its tip. The compliance function $z(x,y)$ is given in example 3 and the mass function is $m(x) = \rho m/L + (1-\rho)\, m\, \delta(x-L)$. The first and second DM approximations give

$$\omega_1 \gtrapprox [(3EI/mL^3)/(1-3\rho/4)]^{1/2}; \quad \{(a):-1.5\% \text{ at } \rho = 1\}$$

$$\omega_1 \gtrapprox [5040(EI/mL^3)^2/(560-856\rho+329\rho^2)]^{1/4}; \quad \{(b):-0.017\% \text{ at } \rho = 1\}$$

(In these and other estimates given in this example 4, the estimate is distinguished by a lower case letter identification, followed by the maximum percent error and the value of ρ for which it obtains.)

This example affords an opportunity to discuss other types of simple but accurate approximations. A Rayleigh approximation based upon the deflection function $y = 3Lx^2-x^3$ gives

$$\omega_1 \lessapprox [(3EI/mL^3)/(1-107\rho/140)]^{1/2}; \quad \{(c):+1.5\% \text{ at } \rho = 1\}$$

This is an upper estimate while (a) is a lower estimate. An obvious average is

$$\omega_1 \approx [(3EI/mL^3)/(1-53\rho/70)]^{1/2}; \quad \{(d):-0.39\% \text{ at } \rho = 0.83\}$$

Approximation (d) gives equal weighting to (a) and (c). This happens to be approximately optimal in this case but this is only fortuitous. The coefficient of ρ should lie somewhere between 105/140 and 107/140. Both are exact for $\rho = 0$. We can choose the coefficient so as to make the result exact for another value of ρ, say $\rho = 1$. This is a uniform cantilever for which

$$\omega_1 = (1.8751040687\ldots)^2 (EI/mL^3)^{1/2}$$

Thus we obtain

$$\omega_1 \approx [(3EI/mL^3)/(1-0.757328\rho)]^{1/2}; \quad \{(e): -.366\% \text{ at } \rho = 0.81\}$$

Exactly the same result is obtained by replacing the first term in the first DM evaluation by the known exact value appropriate for a uniform cantilever with no added mass at the tip. This suggests replacing the coefficient 11/1680 in the first term of the second DM evaluation by $(1.8751\ldots)^{-8}$ and we get

$$\omega_1 \approx [630(EI/mL^3)^2/(70-107\rho+41.1222805\rho^2)]^{1/4}; \quad \{(f): -0.006\% \text{ at } \rho = 0.86\}$$

This accuracy is almost incredible. It goes without saying that the accuracy of these approximations is far greater than that of the physical theory to which they pertain.

<u>Example 5.</u> We next take up an example which combines the features of mixed (i.e., lumped and distributed) mass distribution and rigid body motion. We consider the axial motion of a uniform elastic bar of mass m, length L, and axial stiffness EA, which has a concentrated mass 2m at the left end and a concentrated mass 3m at the right end. This dumb-bell shaped object is not constrained or tied down at any point. The mass function is

$$m(x) = m/L + 2m\delta(x-0) + 3m\delta(x-L)$$

We consider the compliance of a system fixed at the left end.

$$z(x,y) = \mathrm{Min}(x,y)/AE$$

and we obtain

$$q(x,y) = [m(y)/AE][\mathrm{Min}(x,y) - (8Lx-x^2)/12L]$$

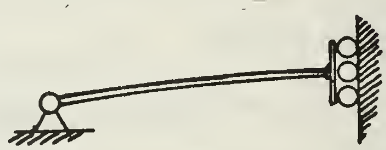
Thus the first DM evaluation gives

$$\omega_2 \gtrapprox (9AE/13mL)^{1/2} = 0.832(AE/mL)^{1/2}$$

and the second gives

$$\omega_2 \gtrapprox (144A^2E^2/242m^2L^2)^{1/4} = 0.87829(AE/mL)^{1/2}$$

The exactly correct coefficient is the smallest positive root of the equation $\tan x = 5x/(6x^2-1)$, which is approximately 0.87935. Thus the errors are 5.4% and 0.12% respectively.



Sketch of uniform hinged-guided beam

Example 6 We determine the fundamental harmonic frequency of a uniform hinged-guided beam of length L. The mass function is $m(x) = m_o = \text{const.}$ The compliance function is

$$z(x,y) = z(y,x) = [3xy(2L-y) - x^3 + \langle x-y\rangle^3]/6EI$$

where

$$\langle x-y\rangle^3 = \begin{cases} (x-y)^3 & \text{if } x \geq y \\ 0 & \text{if } x \leq y \end{cases}$$

The DME1, given by equation 19 with n = 1, is

$$\omega_1^{-2} \lessapprox \sum_{i=1}^{\infty} \omega_i^{-2} = \int_0^L m_o z(x,x)dx = m_o L^4/6EI; \quad \omega_1 \gtrapprox 2.4495(EI/m_o L^4)^{1/2}$$

The DME2 is obtained from equation 19 with n = 2, viz.

$$\omega_1^{-4} \lessapprox \sum_{i=1}^{\infty} \omega_i^{-4} = \int_0^L\int_0^L q(x,\xi)q(\xi,x)d\xi dx =$$

$$= 2\int_0^L\int_0^x m(x)m(\xi)z^2(x,\xi)d\xi dx$$

because of the symmetry of $z(x,y)$. The important consequence of this is that we can take

$$z^2(x,\xi) = [3x\xi(2L-x)-\xi^3]^2/36E^2I^2$$

and not have to concern ourselves with the alternate form applicable for $\xi > x$. Continuing the evaluation we arrive at $(17/630)(m_o L^4/EI)^2$, from which we obtain

$$\omega_1 \gtrapprox 2.4673(EI/m_oL^4)^{1/2}$$

The exact result is

$$\omega_1 = (\pi^2/4)(EI/m_oL^4)^{1/2} = 2.4674(EI/m_oL^4)^{1/2}$$

so that the excellence of the approximations is evident.

Having the excellent approximation to ω_1 given by the DMA2, we can get a reasonable approximation to the *second* frequency, viz.

$$\omega_2^{-2} \approx (\omega_1^{-2})_{DMA1} - [(\omega_1^{-4})_{DMA2}]^{1/2} = [1/6 - (17/630)^{1/2}]m_oL^4/EI$$

giving

$$\omega_2 \approx 20.4(EI/m_oL^4)^{1/2}$$

The correct coefficient is $(3\pi/2)^2 \approx 22.2$.

This example permits us also to exhibit another possibly useful feature of the method. In this case it is known that

$$\omega_k = [(k-\tfrac{1}{2})\pi]^2(EI/m_oL^4)^{1/2}, \; k=1,2,...,$$

so that our precise evaluations of $\Sigma\omega_1^{-2}$ and $\Sigma\omega_1^{-4}$ lead to

$$\sum_{n=1,3,5,...}^{\infty} n^{-4} = \pi^4/96; \quad \sum_{n=1,3,5,...}^{\infty} n^{-8} = 17\pi^8/161280$$

These are well known results; cf. Jolley (5). However, for other cases the corresponding sums may be unknown. For this reason we have looked into all possible cases of a uniform beam having end conditions of the following types: fixed, hinged, guided (see right end of the figure), and free. Letting

$$\eta^4 = \omega^2 m_oL^4/EI$$

the results may be exhibited in the form shown in Table 1.

In Table 1 the physical case described by the specification of end conditions corresponds to the frequency equation indicated. The sum of reciprocal second and fourth powers of the frequencies (except for factors

Table 1 Some results for uniform beams

CASE	EQUATION	END CONDITIONS	$\Sigma\omega_i^{-2}$	$\Sigma\omega_i^{-4}$	η_1
1	$\cos\eta = \operatorname{sech}\eta$	clamped-clamped(*) free-free	1/420	71/17463600	4.7300407449
2	$\tan\eta = \tanh\eta$	clamped-hinged(*) hinged-free	1/210	13/727650	3.9266023120
3	$\tan\eta = -\tanh\eta$	clamped-guided(*) guided-free	1/30	29/2835	2.3650203724
4	$\cos\eta = -\operatorname{sech}\eta$	clamped-free	1/12	11/1680	1.8751040687
5	$\sin\eta = 0$	hinged-hinged(*) guided-guided(*)	1/90	1/9450	3.1415926536
6	$\cos\eta = 0$	hinged-guided	1/6	17/630	1.5707963268

* The root, $\eta = 0$, is to be excluded for cases marked with an asterisk

of EI/m_0L^4 and $E^2I^2/m_0^2L^8$, respecively) are shown. The smallest root of the frequency equation is also given. This information may be useful in augmenting information contained in work by Young et al. (16) and by Gorman (4). It may be remarked that neither of these references treats the end condition $\frac{dy}{dx} = \frac{d^3y}{dx^3} = 0$ which we call "guided;" this condition is certainly not technically important but its inclusion seems to be indicated on the grounds of completeness.

In the table, an asterisk indicates that the obvious root, $\eta = 0$, of the frequency equation is to be excluded. The sums include only non-zero frequencies, and η_1 indicates the smallest positive root. We evidently have such results as

$$\sum_{i=1}^{\infty} [\eta_i^{(1)}]^{-8} = 71/17463600$$

where $\eta_i^{(j)}$ denotes the ith positive root of equation j. From this evaluation we could obtain

$$\eta_1^{(1)} \gtrsim 4.7191$$

Example 7 In order to illustrate a case for which the field of integration is two dimensional, we consider a uniform circular elastic plate which is clamped at its outer radius $r = a$. The influence relationship, $z(P,Q)$, is given by Timoshenko (14) in the form of a series, but for our purposes all that is needed is the "self" influence, i.e., the deflection under the unit lateral load

$$z(P,P) = (a^2-r^2)^2/16\pi Da^2$$

which is also given in (14). Thus, with m_o denoting mass per unit area of plate, we find

$$\omega_1^{-2} \lessapprox \sum_{i=1}^{\infty}\omega_i^{-2} = (m_o/16\pi Da^2)\int_0^{2\pi}\int_0^a (a^2-r^2)^2 r\,dr\,d\theta = m_o a^4/48D$$

whence

$$\omega_1 \approx 6.9(D/m_o a^4)^{1/2}$$

This is actually a poor estimate, the correct coefficient being approximately 10.22 rather than 6.9. The reason lies essentially in the fact that the sum includes a double infinity of terms corresponding to both radial and circumferential nodal lines and the frequencies are not well separated. A better approximation may be obtained with great labor by considering the DME2, or, much more easily, by excluding all modes having radial nodal lines. This is accomplished by considering a unit load uniformly distributed on the circumference of a circle of radius ρ. The deflection is (14)

$$z(r,\rho) = [(r^2+\rho^2)\log_e(\rho/a) + (1+\rho^2/a^2)(a^2-r^2)/2 - (\rho^2-r^2)]/8\pi D$$

for $r \leqq \rho$, and

$$z(r,\rho) = [(r^2+\rho^2)\log_e(r/a) + (1+\rho^2/a^2)(a^2-r^2)/2]/8\pi D$$

for $r \geqq \rho$. The mass function is

$$m(r) = 2\pi r m_o$$

and without difficulty we obtain the DME1 and the DME2

$$\sum_{i=1}^{\infty} \omega_i^{-2} = m_o a^4/96D; \quad \sum_{i=1}^{\infty} \omega_i^{-4} = (17/184320)(m_o a^4/D)^2$$

from which we get the respective estimates

$$\omega_1 \approx 9.8(D/m_o a^4)^{1/2}; \quad \omega_1 \approx 10.2042(D/m_o a^4)^{1/2}$$

It is difficult to assess the accuracy of the latter result. The correct coefficient is the square of the smallest positive root of the equation

$$J_0(x)\, I_1(x) + J_1(x)\, I_0(x) = 0$$

which is given by Rayleigh (9) as $(3.20)^2 = 10.24$. However, an upper bound given by Timoshenko (13) is

$$[19.2(51-2(519)^{\frac{1}{2}}]^{\frac{1}{2}} \approx 10.217$$

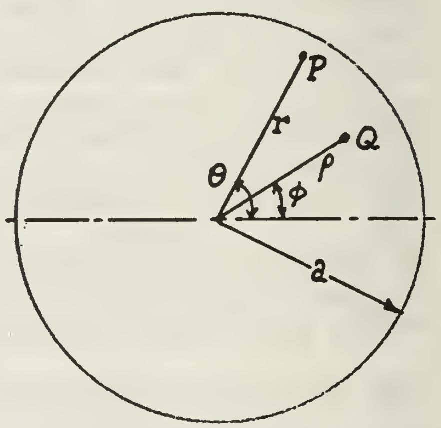
so that our result is definitely in error by no more than 0.13%.

<ins>Example 8</ins> Consider a uniformly tensioned uniform circular elastic membrane of radius a. The influence function is

$$z(P,Q) = z(r,\theta;\rho,\phi) =$$

$$= (1/4\pi T)\log_e\left\{\frac{a^4+r^2\rho^2-2a^2r\rho\cos(\theta-\phi)}{a^2[r^2+\rho^2-2r\rho\cos(\theta-\phi)]}\right\}$$

which is a rearrangement of a form given by Rektorys (10) with the addition of the membrane tension T in the denominator. The DME1 gives



Notation for analysis of circular membrane.

$$\sum_{i=1}^{\infty} \omega_i^{-2} = \int_S z(P,P)dS = \infty$$

Thus, the DME1 is useless except that it demonstrates the divergence of the double sum

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} j_{m,n}^{-2}$$

where $j_{m,n}$ is the nth positive zero of $J_m(x)$. We have been unable to evaluate the integral which gives the DME2. For a successful numerical evaluation we do as was done in the preceding example, namely confine attention to axisymmetric modes. The deflection at radius x of such a circular membrane loaded by unit force uniformly distributed on a circle of radius y is

$$z(x,y) = (1/2\pi T) \log_e[a/\mathrm{Max}(x,y)]$$

and the mass function is

$$m(x) = 2\pi m_o x$$

There is no difficulty in obtaining the DME1 and the DME2, viz.

$$\sum_{i=1}^{\infty} \omega_i^{-2} = m_o a^2/4T \qquad \sum_{i=1}^{\infty} \omega_i^{-4} = m_o^2 a^4/32T^2$$

As in the evaluation in the preceding example, these sums include only frequencies for axisymmetric modes. The exact results in the present case are

$$\omega_i = j_{0,i}(T/m_o a^2)^{1/2}$$

Since $j_{0,1} \approx 2.4048$, the DMA1, which gives the coefficient 2, is in error by 17% and the DMA2, which gives the coefficient $(32)^{1/4}$ is in error by only 1.1%.

Incidentally, we have here established the interesting and possibly useful results

$$\sum_{i=1}^{\infty} j_{0,i}^{-2} = 1/4; \qquad \sum_{i=1}^{\infty} j_{0,i}^{-4} = 1/32$$

Example 9 We now consider the lateral vibrations of a simply supported rectangular elastic plate having dimensions a and b. We assume that the ratio $\rho = b/a$ does not exceed unity. The compliance is (14)

$$z(x,y;\xi,\eta) = (4/\pi^4 abD) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{[\sin(m\pi x/a)\ \sin(m\pi\xi/a)\ \sin(n\pi y/b)\ \sin(n\pi\eta/b)]}{[(m/a)^2 + (n/b)^2]^2}$$

Thus, the DME1 is

$$\sum_{i=1}^{\infty} \omega_i^{-2} = (m_o a^4/D) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m^2\pi^2 + n^2\pi^2\rho^2)^{-2}$$

$$= (m_o a^4/4\pi^4\rho^4 D) \sum_{n=1}^{\infty} [n^2\pi^2\rho^2 \operatorname{csch}^2(n\pi\rho) + n\pi\rho\coth(n\pi\rho) - 2]/n^4$$

which converges fairly rapidly. The DME2 gives

$$\sum_{i=1}^{\infty} \omega_i^{-4} = (m_o a^4/\pi^4 D)^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m^2 + n^2\rho^2)^{-4}$$

$$= (m_o^2 a^8/96\rho^8\pi^8 D^2) \sum_{n=1}^{\infty} [2n^4\pi^4\rho^4(\operatorname{csch}^4 n\pi\rho + 2\operatorname{csch}^2 n\pi\rho\coth n\pi\rho)$$

$$+ 12n^3\pi^3\rho^3 \operatorname{csch}^2 n\pi\rho\coth n\pi\rho$$

$$15n\pi\rho(n\pi\rho\operatorname{csch}^2 n\pi\rho + \coth n\pi\rho - 48]/n^8$$

and this series converges quite rapidly. Thus, for the particular case $a = b$, by taking only one term of the series we get

$$\omega_1^{-4} \approx 6.7556\cdot 10^{-6} m_o^2 a^8/D^2; \omega_1 \approx 19.615(D/m_o a^4)^{1/2}$$

The exactly correct coefficient is $2\pi^2 = 19.739$ so that our estimate is quite good.

For this particular problem, a simple formula (17) gives all of the double infinity of natural frequencies, viz.

$$\omega_{m,n} = \pi^2(m^2/a^2 + n^2/b^2)(D/m_o)^{1/2}$$

and the correctness of the DME1 and the DME2 can be verified term by term. It is merely fortuitous that the exact result for ω_1 $(=\omega_{1,1})$ can be obtained in this case by taking only one term of the double series shown above.

References

1. Bickley, W. G., and Talbot, A., An Introduction to the Theory of Vibrating Systems, Oxford, 1961
2. Conway, H. D., Becker, E. C. H., and Dubil, J. F., Vibration frequencies of tapered bars and circular plates, J. Appl. Mech. 33, June 1964, pp. 329-331.
3. Dunkerley, S., On the whirling and vibration of shafts, Phil. Trans. Roy. Soc. London, A, 185, 1894, pp. 279-360
4. Gorman, D. J., Free Vibration Analysis of Beams and Shafts, Wiley, 1975
5. Jolley, L. B. W., Summation of Series, Dover, 1961
6. Lamb, H., and Southwell, R. V., The vibrations of a spinning disk, Proc. Roy. Soc. London, A, 99 1921, pp. 272-280
7. Mikhlin, S. G., Integral Equations, Pergamon, 1957. (Tr. by A. H. Armstrong.)
8. Mikhlin, S. G., Variational Methods in Mathematical Physics, Macmillan, 1964. (Tr. by T. Boddington.)
9. Rayleigh, Lord (J. W. Strutt), Theory of Sound, Dover, 1945, Vol I, p. 366.
10. Rektorys, K. (Ed.), Survey of Applicable Mathematics, MIT, 1969, p.893
11. Southwell, R. V., An Intro. to the Theory of Elasticity, Oxford, 1936, 1941.
12. Temple, G., and Bickley, W. G., Rayleigh's Principle and Its Applications, Oxford, 1933
13. Timoshenko, S., Vibration Problems in Engineering, 2nd ed., D. Van Nostrand, 1937, p. 429
14. Timoshenko, S., and Woinowsky-Krieger, S., Theory of Plates and Shells, 2nd ed., McGraw-Hill, 1959.
15. Tricomi, F. G., Integral Equations, Interscience, 1957
16. Young, D., and Felgar, R. P., Tables of characteristic functions representing normal modes of vibration of a beam, U. of Texas Pub. No. 4913, 1949
17. Young, D., Continuous Systems, Chap. 61, p.22, in Handbook of Engineering Mechanics, ed. by W. Fluegge, McGraw-Hill, 1962

INITIAL DISTRIBUTION LIST

1.	Professor John E. Brock, Code 59Bc Naval Postgraduate School Monterey, CA 93940	15
2.	Professor Harry D. Conway Thurston Hall Cornell University Ithaca, New York 14850	1
3.	Professor Robert E. Newton, Code 59Ne Naval Postgraduate School Monterey, CA 93940	1
4.	Professor Robert Steidel, Jr. Department of Mechanical Engineering University of California, Berkeley Berkeley, CA 94720	1
5.	Professor W. T. Thomson Department of Mechanical Engineering University of California, Santa Barbara Santa Barbara, CA 93107	1
6.	Dean of Research Naval Postgraduate School Monterey, CA 93940	2
7.	Department of Mechanical Engineering Naval Postgraduate School Monterey, California 93940	2
8.	Director Defense Documentation Center 5010 Duke Street Alexandria, VA 22314	2
9.	Library Naval Postgraduate School Monterey, CA 93940	2
10.	LCDR Richard R. Levin, USN Office of Naval Research 1030 E. Green Street Pasadena, CA 91101	1
11.	Office of Naval Research Arlington, Virginia 22217	2

NPS