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A DIFFUSION APPROXIMATION MODEL

FOR A COMMUNICATION SYSTEM

ALLOWING MESSAGE INTERFERENCE

by

Donald P. Gaver

John P. Lehoczky

February 1977

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Mathematical probability models are presented to describe the service furnished to messages approaching c communications channels, on which messages in progress may be "destroyed" by an attempted access by a new message. Re-tries by destroyed messages are modeled. Numerical results, using the models, are compared to simulations, validating model usefulness.

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A DIFFUSION APPROXIMATION MODEL FOR A COMMUNICATION SYSTEM ALLOWING MESSAGE INTERFERENCE

Donald P. Gaver
Naval Postgraduate School
John P. Lehoczky
Carnegie-Mellon University

1. Introduction and Problem Statement

We study the operating characteristics of an element of a complex communication system, the element consisting of c channels which service an arrival stream of messages. When a message arrives, it selects a channel at random and initiates a transmission (service) time of random duration. If by chance the channel selected is already occupied, i.e. is in the process of transmission, both messages may be "destroyed," i.e. terminated before completion, and the channel reverts to an empty or open condition. The transmitters of the messages are capable of detecting the event of destruction, and following such an event go into a re-try or re-transmission population, from which they later make attempts to occupy a channel and eventually complete message transmission. We develop an approximate probability model to describe the performance of such a system. The approximation tends to become exact as the number of channels, c, becomes large $(c \to \infty)$, but numerical studies indicate that it may be quite adequate for c near ten.

The above problem bears close resemblance to a problem of packet switching of data on the ARPANET and, very likely, on other satellite communications systems, as discussed in Kleinrock (1976), p. 362 ff. We choose to represent the various interacting populations of messages, i.e. those in process, and those in the re-try population, by means of stochastic differential equations as was done by Gaver and Lehoczky (1976) for a related problem.

2. Mathematical Formulations: Model 1

Assume that messages arrive at a system of c channels according to a non-homogeneous Poisson process of rate $c\lambda(t)$; that is, the probability that a message arrives in $(t, t+dt)$ is $c\lambda(t)dt + o(dt)$. Each message then selects a channel at random; any particular channel, whether occupied or not, is selected with probability c^{-1} . If the channel is unoccupied the message begins transmission; the probability that it completes in time $(t, t+dt)$ is $\mu(t)dt + o(dt)$. Note that if $\mu(t) = \mu$, constant, the messages enjoy exponential service times, and if we wish to generate other, say Erlang, service times the device of phases, or extra artificial compartments, may be used, as in Gaver and Lehoczky (1976).

If a message in progress on a channel is interrupted by the arrival of another message, both are assumed instantly destroyed, and the message initiators are transferred to a re-try population, R . A message that has entered R changes its status at time t with probability $\nu(t)dt + o(dt)$: with probability $\alpha(t)$ the change of status implies an attempt to re-transmit on a channel, and with probability $\bar{\alpha}(t) = 1-\alpha(t)$ the change of status implies loss -- the message may no longer be worth transmitting.

Our representation of the above setup is in terms of the following state variables.

$$\underset{\sim}{Q}(t) = \text{the number of messages being transmitted, i.e. occupying channels, at time } t \text{ ; clearly } 0 \le \underset{\sim}{Q}(t) \le c . \tag{2.1}$$

$$\underset{\sim}{R}(t) = \text{the number of messages in } R \text{ at time } t \text{ ; } 0 \le \underset{\sim}{R}(t) < \infty .$$

$$\underset{\sim}{L}(t) = \text{the total number of messages that have been lost by time } t \text{ ; } 0 \le \underset{\sim}{L}(t) < \infty .$$

The state of the system is thus $(\underset{\sim}{Q}(t), \underset{\sim}{R}(t), \underset{\sim}{L}(t))$, a discrete vector-valued Markov process, according to our earlier assumptions. We shall characterize this process approximately when $c\to\infty$, and shall in particular treat $\underset{\sim}{Q}$, $\underset{\sim}{R}$, and $\underset{\sim}{L}$ as continuous stochastic processes, in fact as diffusion processes, see Arnold (1974).

Here is a formalization of the transitions described earlier:

$$
\begin{array}{lll}
\underline{\text{Transition}} & & \underline{\text{Probability}} \\
\text{(in } t,\ t+dt) & & \\
(\underset{\sim}{Q}, \underset{\sim}{R}, \underset{\sim}{L}) \to (\underset{\sim}{Q}+1, \underset{\sim}{R}, \underset{\sim}{L}) & & c\lambda(t)[1-\frac{\underset{\sim}{Q}(t)}{c}]dt \\
\qquad\qquad\ \to (\underset{\sim}{Q}-1, \underset{\sim}{R}+2, \underset{\sim}{L}) & & c\lambda(t)\frac{\underset{\sim}{Q}(t)}{c}\,dt \\
\qquad\qquad\ \to (\underset{\sim}{Q}-1, \underset{\sim}{R}, \underset{\sim}{L}) & & \mu(t)\underset{\sim}{Q}(t)dt \\
\qquad\qquad\ \to (\underset{\sim}{Q}+1, \underset{\sim}{R}-1, \underset{\sim}{L}) & & \nu(t)\alpha(t)[1-\frac{\underset{\sim}{Q}}{c}]\underset{\sim}{R}(t)dt \\
\qquad\qquad\ \to (\underset{\sim}{Q}-1, \underset{\sim}{R}+1, \underset{\sim}{L}) & & \nu(t)\alpha(t)\frac{\underset{\sim}{Q}}{c}\underset{\sim}{R}(t)dt \\
\qquad\qquad\ \to (\underset{\sim}{Q}, \underset{\sim}{R}-1, \underset{\sim}{L}+1) & & \nu(t)\bar{\alpha}(t)\underset{\sim}{R}(t)dt
\end{array}
\tag{2.2}
$$

Other transitions have negligible probability of occurring in $(t, t+dt)$. Now in principle one can write down Kolmogorov equations for the transition probabilities of the $(\underset{\sim}{Q}, \underset{\sim}{R}, \underset{\sim}{L})$ process and solve them. However, such a solution must inevitably be numerical. Here we shall write down an approximate system of Ito stochastic differential equations, see Arnold (1974), to describe process evolution, and from them deduce certain useful information valid when c is large. We write, on the basis of (2.2),

$$
\begin{aligned}
dQ(t) = \{&c\lambda(t)[1-\frac{\underset{\sim}{Q}(t)}{c}] - c\lambda(t)\frac{\underset{\sim}{Q}(t)}{c} - \mu(t)\underset{\sim}{Q}(t) + \nu(t)\alpha(t)[1-\frac{\underset{\sim}{Q}(t)}{c}]\underset{\sim}{R}(t) \\
&- \nu(t)\alpha(t)\frac{\underset{\sim}{Q}(t)}{c}\underset{\sim}{R}(t)\}dt \\
&+ \sqrt{c\lambda(t)[1-\frac{\underset{\sim}{Q}(t)}{c}]}\; d\underset{\sim}{W}_1(t) - \sqrt{c\lambda(t)\frac{\underset{\sim}{Q}(t)}{c}}\; d\underset{\sim}{W}_2(t) \\
&- \sqrt{\mu(t)\underset{\sim}{Q}(t)}\; d\underset{\sim}{W}_3(t) + \sqrt{\nu(t)\alpha(t)[1-\frac{\underset{\sim}{Q}(t)}{c}]\underset{\sim}{R}(t)}\; d\underset{\sim}{W}_4(t) \\
&- \sqrt{\nu(t)\alpha(t)\frac{\underset{\sim}{Q}(t)}{c}\underset{\sim}{R}(t)}\; d\underset{\sim}{W}_5(t) ,
\end{aligned}
\tag{2.3}
$$

$$d\underset{\sim}{R}(t) = \{2c\lambda(t)\frac{\underset{\sim}{Q}(t)}{c} - \nu(t)\alpha(t)[1-\frac{\underset{\sim}{Q}(t)}{c}]\underset{\sim}{R}(t)$$

$$+ \nu(t)\alpha(t)\frac{\underset{\sim}{Q}(t)}{c}\underset{\sim}{R}(t) - \nu(t)\bar{\alpha}(t)\underset{\sim}{R}(t)\}dt$$

$$+ 2\sqrt{c\lambda(t)\frac{\underset{\sim}{Q}(t)}{c}}\; d\underset{\sim}{W}_2(t) - \sqrt{\nu(t)\alpha(t)[1-\frac{\underset{\sim}{Q}(t)}{c}]\underset{\sim}{R}(t)}\; d\underset{\sim}{W}_4(t)$$

$$+ \sqrt{\nu(t)\alpha(t)\frac{\underset{\sim}{Q}(t)}{c}\underset{\sim}{R}(t)}\; d\underset{\sim}{W}_5(t) - \sqrt{\nu(t)\bar{\alpha}(t)\underset{\sim}{R}(t)}\; d\underset{\sim}{W}_6(t) ,$$

$$d\underset{\sim}{L}(t) = \nu(t)\bar{\alpha}(t)\underset{\sim}{R}(t)dt + \sqrt{\nu(t)\alpha(t)\underset{\sim}{R}(t)}\; d\underset{\sim}{W}_6(t)$$

where $(W_i(t), o \leq t, i=1, 2, .. 6)$ is a 6 dimensional standard Wiener process whose components are independent.

The rationale for writing (2.3) is as follows. Each term in the bracketted part of the expression for, say, $d\underset{\sim}{Q}$, is a component of the <u>drift</u> or expected change in $\underset{\sim}{Q}$ between t and $t+dt$; e.g. $c\lambda(t)[1-\frac{\underset{\sim}{Q}(t)}{c}]$ is the expected increase in $\underset{\sim}{Q}$ caused by a newly arrived message immediately reaching an empty channel (line 1 of (2.2)), while $c\lambda(t)\frac{\underset{\sim}{Q}(t)}{c}$ represents the decrease in $\underset{\sim}{Q}$ caused by a new message that arrives and chooses a busy channel, only to be immediately transferred to R , along with the message in progress (see line 2 of (2.2)). The coefficients of the Wiener process terms are proportional to the standard deviations of the motions in $(t, t+dt)$ corresponding to the drift terms.

Next, introduce an expansion of $(\underset{\sim}{Q}, \underset{\sim}{R}, \underset{\sim}{L})$ to terms of order $\sqrt{c}$. Define the stochastic noise processes by

$$\underset{\sim}{X}(t) = \frac{\underset{\sim}{Q}(t) - cq(t)}{\sqrt{c}}$$

$$\underset{\sim}{Y}(t) = \frac{\underset{\sim}{R}(t) - cr(t)}{\sqrt{c}} \qquad (2.4)$$

$$\underset{\sim}{Z}(t) = \frac{\underset{\sim}{L}(t) - c\ell(t)}{\sqrt{c}} \quad ;$$

the vector $(q(t), r(t), \ell(t)) = \lim_{c\to\infty} (\underset{\sim}{Q}(t)/c, \underset{\sim}{R}(t)/c, \underset{\sim}{L}(t)/c)$; the existence of these limiting functions is suggested by results of Kurtz (1971).

An application of Ito's lemma, Arnold (1974), p. 90, provides a stochastic differential equation (s.d.e.) for $(\underset{\sim}{X}, \underset{\sim}{Y}, \underset{\sim}{Z})$ as follows (hereafter we do not explicitly express the t-dependence):

$$\begin{pmatrix} d\underset{\sim}{X} \\ d\underset{\sim}{Y} \\ d\underset{\sim}{Z} \end{pmatrix} = \begin{pmatrix} -(2\lambda + \mu + 2\alpha\nu r) & \alpha\nu(1-2q) \\ 2\lambda + 2\alpha\nu r & -\alpha\nu(1-2q) - \bar{\alpha}\nu \\ 0 & \bar{\alpha}\nu \end{pmatrix} \begin{pmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \\ \underset{\sim}{Z} \end{pmatrix} +$$

$$- \sqrt{c} \begin{pmatrix} q' - \lambda(1-2q) + \mu q - \alpha\nu r(1-2q) \\ r' - 2\lambda q + \alpha\nu r(1-2q) - \bar{\alpha}\nu r \\ \ell' - \bar{\alpha}\nu r \end{pmatrix} + \qquad (2.5)$$

$$\begin{pmatrix} \sqrt{\lambda(1-q)} & -\sqrt{\lambda q} & -\sqrt{\mu q} & \sqrt{\alpha\nu r(1-q)} & -\sqrt{\alpha\nu r q} & 0 \\ 0 & 2\sqrt{\lambda q} & 0 & \sqrt{\alpha\nu(1-q)} & \sqrt{\nu\alpha q r} & -\sqrt{\nu\bar{\alpha} r} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\nu\bar{\alpha} r} \end{pmatrix} \begin{pmatrix} \underset{\sim}{W}_1 \\ \underset{\sim}{W}_2 \\ \underset{\sim}{W}_3 \\ \underset{\sim}{W}_4 \\ \underset{\sim}{W}_5 \\ \underset{\sim}{W}_6 \end{pmatrix}$$

Now the coefficient of $\sqrt{c}$ must be identically zero, for otherwise our system of equations does not converge. This leads to the

Deterministic Equations:

$$
\begin{aligned}
q'(t) &= \lambda(1-2q) - \mu q + \alpha\nu r(1-2q) \\
r'(t) &= 2\lambda q - \alpha\nu r(1-2q) - \bar{\alpha}\nu r \\
\ell'(t) &= \bar{\alpha}\nu r
\end{aligned}
\tag{2.6}
$$

The solutions to these equations, which must be obtained by numerical integration, provide a deterministic approximation to $(\underset{\sim}{Q}, \underset{\sim}{R}, \underset{\sim}{L})$ -- in fact, $(cq, cr, c\ell) \approx (E[\underset{\sim}{Q}], E[\underset{\sim}{R}], E[\underset{\sim}{L}])$.

Stochastic Equations:

In view of (2.6), (2.5) now appears in the form

$$
\begin{pmatrix} d\underset{\sim}{X} \\ d\underset{\sim}{Y} \\ d\underset{\sim}{Z} \end{pmatrix} = \underset{\sim}{A}_t \begin{pmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \\ \underset{\sim}{Z} \end{pmatrix} dt + \underset{\sim}{B}_t d\underset{\sim}{W}' \tag{2.7}
$$

$\underset{\sim}{A}_t$ (a 3 x 2 matrix) being identified as the coefficient of $(\underset{\sim}{X}, \underset{\sim}{Y}, \underset{\sim}{Z})'$, and B_t (a 3 x 6 matrix) identified as the coefficient of the Wiener process term, in (2.5). Next note that the s.d.e. has a special form if the $\underset{\sim}{Z}$ term is omitted; it is reasonable to focus on $\underset{\sim}{X}$ and $\underset{\sim}{Y}$, for clearly $\ell(t)\to\infty$ as $t\to\infty$, since $\underset{\sim}{L}$ represents cumulative losses. Thus we shall be interested in the first two equations in (2.6), and in (2.7) which we now write as

$$
\begin{pmatrix} d\underset{\sim}{X} \\ d\underset{\sim}{Y} \end{pmatrix} = \underset{\sim}{C}_t \begin{pmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{pmatrix} dt + \underset{\sim}{D}_t d\underset{\sim}{W} \tag{2.8}
$$

and $\underset{\sim}{C}_t$ (2 x 2) is $\underset{\sim}{A}_t$ without its last row, while $\underset{\sim}{D}_t$ (2 x 6) is $\underset{\sim}{B}_t$ without its last row. Now the bivariate stochastic process described by (2.8) is nonstationary Ornstein Uhlenbeck, since $\underset{\sim}{C}_t$ has eigenvalues with non-negative real parts; see Arnold (1974), Sec. 8.3, for an account of the scalar case, or see Schach and McNeil (1973). Much that is useful is known about this process. In particular, if $(\underset{\sim}{X}(0), \underset{\sim}{Y}(0)) = \underset{\sim}{0}$, and $q(0)$, $r(0)$ are given, then for all $t>0$ $(\underset{\sim}{X}(t), \underset{\sim}{Y}(t))$ has a bivariate normal distribution with mean $\underset{\sim}{0}$ and covariance matrix $\underset{\sim}{\Sigma}_t$ which satisfies the differential equation

$$\dot{\underset{\sim}{\Sigma}}_t = \underset{\sim}{C}_t\underset{\sim}{\Sigma}_t + \underset{\sim}{\Sigma}_t\underset{\sim}{C}_t' + \underset{\sim}{D}_t\underset{\sim}{D}_t' \tag{2.9}$$

see Arnold (1974), Sec. 8.2.

Combining these facts leads to the

Result: $(\underset{\sim}{Q}, \underset{\sim}{R})$ is approximately bivariate normal (Gaussian), as $c\to\infty$:

$$(\underset{\sim}{Q}(t), \underset{\sim}{R}(t)) \simeq N(c(q,r), c\underset{\sim}{\Sigma}_t) \tag{2.10}$$

From this expression it is possible to estimate the probability that at least a specified number of channels are occupied at time t , and also to estimate the probability that there are no more than any specified number of customers awaiting re-try. Such quantities are useful measures of system performance. Of course both the deterministic equations (2.6) and the equation (2.9) for the covariance matrix $\underset{\sim}{\Sigma}_t$ must be solved numerically, but this should be a less difficult step than is the simulation of such a system.

Steady-State Results:

Suppose $\lambda(t)\to\lambda$, $\mu(t)\to\mu$, $\nu(t)\to\nu$, $\alpha(t)\to\alpha$ all positive constants, as $t\to\infty$. In this case it may happen that $q(t)\to q$ and $r(t)\to r$, the latter satisfying (2.6) with derivatives set equal to zero:

$$0 = \lambda(1-2q) - \mu q + \alpha\nu r(1-2q) \tag{2.11}$$

$$0 = 2\lambda q - \alpha\nu r(1-2q) - \bar{\alpha}\nu r \tag{2.12}$$

and these are immediately solved to give

$$q = \frac{(1+2\rho) - \sqrt{(1+2\rho)^2 - 8\alpha\rho}}{4\alpha} = \frac{2\rho}{(1+2\rho) + \sqrt{(1+2\rho)^2 - 8\rho\alpha}}$$

$$r = \frac{\rho - q}{\bar{\alpha}\,\eta} \tag{2.13}$$

where $\rho = \lambda/\mu$ and $\eta = \nu/\mu$, and provided that the numerical results obtained are feasible: $0 \le q \le 1$, $0 \le r < \infty$. From (2.10), it is necessary that $q < \frac{1}{2}$ in order for a steady state to exist.

When steady state conditions hold $\dot{\underset{\sim}{\Sigma}}_t = 0$, so the steady state covariance matrix, $\underset{\sim}{\Sigma}$, is the unique positive definite solution of

$$-\underset{\sim}{D}\underset{\sim}{D}' = \underset{\sim}{C}\underset{\sim}{\Sigma} + \underset{\sim}{\Sigma}\underset{\sim}{C}' \tag{2.14}$$

where

$$\underset{\sim}{C} = \begin{pmatrix} -(2\lambda + \mu + 2\alpha\nu r) & \alpha\nu(1-2q) \\ 2\lambda + 2\alpha\nu r & -\alpha\nu(1-2q) - \bar{\alpha}\nu \end{pmatrix} \tag{2.15}$$

and

$$\underset{\sim}{D}\underset{\sim}{D}' = \begin{pmatrix} \lambda + \mu q + \alpha\nu r & -(2\lambda q + \alpha\nu r) \\ -(2\lambda q + \alpha\nu r) & 4\lambda q + \nu r \end{pmatrix} \tag{2.16}$$

writing

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \tag{2.17}$$

we may express the solution of (2.14) as follows:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{pmatrix} = \Delta^{-1} \begin{pmatrix} b(a+b-d) & 2b^2 & b^2 \\ bd & 2ab & ab \\ d^2 & 2ad & a(a+b) - bd \end{pmatrix} \begin{pmatrix} \lambda + \mu q + \alpha\nu r \\ -(2\lambda q + \alpha\nu r) \\ 4\lambda q + \nu r \end{pmatrix} \tag{2.18}$$

where

$$a = 2\lambda + \mu + 2\alpha\nu r \ , \quad d = 2\lambda + 2\alpha\nu r$$
$$b = \alpha\nu(1-2q) \ , \qquad \Delta = 2ab(d-a-b) + 2b^2 d \ . \tag{2.19}$$

The values of q and r are available from (2.13). From these results actual numerical expressions for the steady state probability distributions of $\underset{\sim}{Q}$ and $\underset{\sim}{R}$ may be computed. Notice that the long-run loss or defection rate of the final equation of (2.6) i.e. $c\ell'(\infty) = c\bar{\alpha}\nu r$, where r comes from (2.13).

3. Model 2: Intelligent Re-tries

Many variations are possible on the process that leads to Model 1. These are of interest because they represent possibilities for improved performance, or because they describe real system behavior.

In the model of the present section we suppose that re-tries are intelligent: a message in category R that attempts to seize a channel again does not do so at random, with a chance of destroying a message in progress. To represent this change, remove in (2.2) the possibility of the transitions involving $\alpha(t)$: $(\underset{\sim}{Q}, \underset{\sim}{R}, \underset{\sim}{L}) \to (\underset{\sim}{Q}+1, \underset{\sim}{R}-1, \underset{\sim}{L})$, $\to (\underset{\sim}{Q}-1, \underset{\sim}{R}+1, \underset{\sim}{L})$, and $\to (\underset{\sim}{Q}, \underset{\sim}{R}-1, \underset{\sim}{L}+1)$, replacing with the transition rate

$$\underset{\sim}{Q}(t),\ \underset{\sim}{R} \to \underset{\sim}{Q}(t)+1,\ \underset{\sim}{R}(t)-1\ ,\ \text{probability}\ \ \nu(t)\underset{\sim}{R}(t)\left[1-\frac{\underset{\sim}{Q}}{c}\right]dt \tag{3.1}$$

Effectively this change in Model 1 prevents losses, and, when a re-try prepares to access an occupied channel, it senses the presence of the on-going message, and refrains before destruction.

This model can be analyzed by the technique described for Model 1, namely that of setting up approximating Ito-type stochastic differential equations, and then carrying out an expansion for $c \to \infty$. The results are as follows.

Deterministic Component:

We find that for Model 2

$$\begin{aligned} q'(t) &= \lambda(1-2q) - \mu q + \nu r(1-q) \\ r'(t) &= 2\lambda q - \nu r(1-q) \end{aligned} \tag{3.2}$$

Stochastic Component:

Use of the representation (2.4) together with the stochastic differential equations and Ito's lemma yields

$$\begin{pmatrix} d\underset{\sim}{X} \\ d\underset{\sim}{Y} \end{pmatrix} = \underset{\sim}{A}_t \begin{pmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{pmatrix} + \underset{\sim}{B}_t d\underset{\sim}{W}_t\ , \tag{3.3}$$

the s.d.e. for a non-stationary Ornstein-Uhlenbeck process. In this case (suppressing t-dependence),

$$\underset{\sim}{A}_t = \begin{bmatrix} -(2\lambda + \mu + \nu r) & \nu(1-q) \\ 2\lambda + \nu r & -\nu(1-q) \end{bmatrix} \tag{3.4}$$

$$\underset{\sim}{B}_t = \begin{bmatrix} \sqrt{\lambda(1-q)} & -\sqrt{\lambda q} & -\sqrt{\mu q} & \sqrt{\nu r(1-q)} \\ 0 & 2\sqrt{\lambda q} & 0 & -\sqrt{\nu r(1-q)} \end{bmatrix} \tag{3.5}$$

and

$$\underset{\sim}{W}_t = (\underset{\sim}{W}_1(t), \underset{\sim}{W}_2(t), \underset{\sim}{W}_3(t), \underset{\sim}{W}_4(t))' , \tag{3.6}$$

a 4-dimensional standard Wiener process with independent components. Once again the equation (2.9) may be solved for the covariance function $\underset{\sim}{\Sigma}_t$, with $\underset{\sim}{A}_t$ substituted for $\underset{\sim}{C}_t$, and $\underset{\sim}{B}_t$ in place of $\underset{\sim}{D}_t$. And once again we state the

<u>Result</u>: $(\underset{\sim}{Q}, \underset{\sim}{R})$ is approximately bivariate normal as $c \to \infty$:

$$(\underset{\sim}{Q}, \underset{\sim}{R}) \sim N(c(q,r), c\underset{\sim}{\Sigma}_t) \tag{3.7}$$

<u>Steady-State Results</u>:

If $\lambda(t) \to \lambda$, $\mu(t) \to \mu$, $\nu(t) \to \nu$ as $t \to \infty$, then it may happen that $q(t) \to q$, and $r(t) \to r$, these satisfying the equations (3.2) with $q' = r' = 0$. The solutions are easily obtained, and are

$$q = \frac{\lambda}{\mu} \equiv \rho \ , \quad r = \frac{2\lambda^2}{\nu(\mu-\lambda)} \equiv \frac{2\rho^2}{\eta(1-\rho)} \ ; \tag{3.8}$$

apparently it is necessary that $q = \rho < 1$ for steady state to occur. Note that in Model 1 it was necessary that $q < 1/2$ in order that steady state conditions occur. If we compare Models 1 and 2 when $\alpha = 1$ we find

Model 1	Model 2
$q = \rho\ , \quad (0 < \rho < \frac{1}{2})$	$q = \rho\ , \quad (0 < \rho < 1)$
$r = \dfrac{2\rho^2}{\eta(1-2\rho)}$	$r = \dfrac{2\rho^2}{\eta(1-\rho)}$

Clearly the setup of Model 2 results in considerably smaller expected wait per unit time (approximated by $c(q+r)$) than does that of Model 1. The improved service must be purchased in return for investment in the busy channel sensing capacity. The Model 2 can be compared to the model of Gaver and Lehoczky (1976).

The steady state covariance matrix, $\underset{\sim}{\Sigma}$, is obtained from (2.14), replacing $\underset{\sim}{C}$ by $\underset{\sim}{A}$, and $\underset{\sim}{D}$ by $\underset{\sim}{B}$. We find that

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{pmatrix} = \Delta^{-1} \begin{pmatrix} b(a+b-d) & 2b^2 & b^2 \\ bd & 2ab & ab \\ d^2 & 2ad & a(a+b)-bd \end{pmatrix} \begin{pmatrix} \lambda + \mu q + \nu r(1-q) \\ -(2\lambda q + \nu r(1-q)) \\ 4\lambda q + \nu r(1-q) \end{pmatrix} \tag{3.9}$$

where

$$\begin{aligned} a &= 2\lambda + \mu + \nu r\ , \quad & b &= \nu(1-q) \\ d &= 2\lambda + \nu r\ , & \Delta &= 2ab(d-a-b) + 2b^2 d\ . \end{aligned} \tag{3.10}$$

4. Model 3: Transitory Version of Model 1.

Consider next a transitory service system version of the basic Model 1. There are N messages to be transmitted, and each initially is transmitted independently and at a time having absolutely continuous distribution function $F(t)$, with $F(0+) = 0$, and density $f(t)$. Once a given message is transmitted, no more appear from its particular source, so even though some messages enter the re-try population one or more times, the traffic gradually fades away; the problem is fundamentally non-stationary. For analysis of a similar problem see Gaver, Lehoczky, and Perlas (1975).

The following state variables are required.

$\underset{\sim}{I}(t)$ = the number of message arrivals that have occurred by time t;

$$0 \le I(t) \le N .$$

$\underset{\sim}{Q}(t)$ = the number of messages being transmitted at time t;

$$0 \le \underset{\sim}{Q}(t) \le c .$$

$\underset{\sim}{R}(t)$ = the number of messages in the re-try population at time t.

We assume,as we did in formulating Model 1, that if a newly arriving message encounters a channel -- selected at random -- held by a message in progress, then both are instantly "destroyed," and immediately join R. No messages are ever lost. Messages in R re-try at rate $\nu(t)\underset{\sim}{R}(t)$.

The transition scheme is given below. The individual message arrival rate at time t is seen to be $\lambda(t) = f(t)/[1-F(t)]$.

$$\begin{array}{lll}
\underset{(t \text{ to } t+dt)}{\underline{\text{Transition}}} & \underline{\text{Probability}} & \\
(\underset{\sim}{I}, \underset{\sim}{Q}, \underset{\sim}{R}) \to (\underset{\sim}{I}+1, \underset{\sim}{Q}+1, \underset{\sim}{R}) & \lambda(t)[N - \underset{\sim}{I}(t)][1 - \frac{\underset{\sim}{Q}}{c}]dt & \\
\quad \to (\underset{\sim}{I}+1, \underset{\sim}{Q}-1, \underset{\sim}{R}+2) & \lambda(t)[N - \underset{\sim}{I}(t)]\frac{\underset{\sim}{Q}}{c}\,dt & \\
\quad \to (\underset{\sim}{I}, \underset{\sim}{Q}-1, \underset{\sim}{R}) & \mu(t)\underset{\sim}{Q}dt & (4.1) \\
\quad \to (\underset{\sim}{I}, \underset{\sim}{Q}+1, \underset{\sim}{R}-1) & \nu(t)\underset{\sim}{R}[1 - \frac{\underset{\sim}{Q}}{c}]dt & \\
\quad \to (\underset{\sim}{I}, \underset{\sim}{Q}-1, \underset{\sim}{R}+1) & \nu(t)\underset{\sim}{R}\frac{\underset{\sim}{Q}}{c}dt . &
\end{array}$$

From the above we can write down the approximating stochastic differential equations analogous to (2.3). These are, suppressing t-dependence statements,

$$d\underset{\sim}{I} = \lambda[N - \underset{\sim}{I}]dt + \sqrt{\lambda[N - \underset{\sim}{I}][1 - \tfrac{\underset{\sim}{Q}}{c}]}\, d\underset{\sim}{W}_1 + \sqrt{\lambda[N - \underset{\sim}{I}]\tfrac{\underset{\sim}{Q}}{c}}\, d\underset{\sim}{W}_2 \tag{4.2}$$

$$\begin{aligned} d\underset{\sim}{Q} = &\{\lambda[N - \underset{\sim}{I}][1 - \tfrac{\underset{\sim}{Q}}{c}] - \lambda[N - \underset{\sim}{I}]\tfrac{\underset{\sim}{Q}}{c} - \mu\underset{\sim}{Q} - \nu\underset{\sim}{R}\tfrac{\underset{\sim}{Q}}{c} + \nu\underset{\sim}{R}[1 - \tfrac{\underset{\sim}{Q}}{c}]\}dt \\ &+ \sqrt{\lambda[N - \underset{\sim}{I}][1 - \tfrac{\underset{\sim}{Q}}{c}]}\, d\underset{\sim}{W}_1 - \sqrt{\lambda[N - \underset{\sim}{I}]\tfrac{\underset{\sim}{Q}}{c}}\, d\underset{\sim}{W}_2 - \sqrt{\mu\underset{\sim}{Q}}\, d\underset{\sim}{W}_3 \\ &- \sqrt{\nu\underset{\sim}{R}\tfrac{\underset{\sim}{Q}}{c}}\, d\underset{\sim}{W}_4 + \sqrt{\nu\underset{\sim}{R}[1 - \tfrac{\underset{\sim}{Q}}{c}]}\, d\underset{\sim}{W}_5 , \end{aligned}$$

$$\begin{aligned} d\underset{\sim}{R} = &\{2\lambda[N - \underset{\sim}{I}]\tfrac{\underset{\sim}{Q}}{c} + \nu\underset{\sim}{R}\tfrac{\underset{\sim}{Q}}{c} - \nu\underset{\sim}{R}[1 - \tfrac{\underset{\sim}{Q}}{c}]\}dt + \\ &2\sqrt{\lambda\tfrac{\underset{\sim}{Q}}{c}[N - \underset{\sim}{I}]}\, d\underset{\sim}{W}_2 + \sqrt{\nu\underset{\sim}{R}\tfrac{\underset{\sim}{Q}}{c}}\, d\underset{\sim}{W}_4 \\ &- \sqrt{\nu\underset{\sim}{R}[1 - \tfrac{\underset{\sim}{Q}}{c}]}\, d\underset{\sim}{W}_5 \end{aligned}$$

Once again we study the behavior of the system as system parameters, in this case both N , the initial number of messages, and c , the number of channels become large. In fact, we relate these parameters as follows: $c = \beta N$, β being a positive constant. Introduce the noise processes

$$\underset{\sim}{X}(t) = \frac{\underset{\sim}{I}(t) - Ni(t)}{\sqrt{N}} ,$$

$$\underset{\sim}{Y}(t) = \frac{\underset{\sim}{Q}(t) - Nq(t)}{\sqrt{N}} , \tag{4.3}$$

$$\underset{\sim}{Z}(t) = \frac{\underset{\sim}{R}(t) - Nr(t)}{\sqrt{N}} ,$$

where $(i(t), q(t), r(t)) = \lim_{N\to\infty} (\underset{\sim}{I}(t)/c, \underset{\sim}{Q}(t)/c, \underset{\sim}{R}(t)/c)$.

Then an application of Ito's lemma and identification of terms of order $\sqrt{N}$ and N yields the deterministic and stochastic components of the process.

Deterministic Equations:

$$\begin{aligned} i'(t) &= \lambda(1 - i) \\ q'(t) &= \lambda(1 - i)(1 - q/\beta) - \lambda(1 - i)q/\beta - \mu q - \nu rq/\beta + \nu r(1 - q/\beta) \; . \\ r'(t) &= 2\lambda(1 - i)q/\beta + \nu rq/\beta - \nu r(1 - q/\beta) \end{aligned} \tag{4.4}$$

From the first equation and the definition of λ it follows immediately that $\lambda(1-i) = f$, the density of the arrival distribution. Note that when this substitution is made in the second and third equations the latter are precisely of the form of the corresponding equations of Model 1:

Correspondence Between Model 1 and Model 3

Arrivals and Re-tries

Model 1	Model 3
$\lambda(t)$	$f(t)/\beta$
$\nu(t)$	$\nu(t)/\beta$

In the present model $f(\infty) = 0$, and hence the arrival rate is eventually zero, and some time afterwards the system is completely empty. The present model possesses no steady state solution, and to learn about system status at various times it is necessary to solve (4.4) numerically.

Stochastic Equations:

The expansion technique shows that the noise process satisfies

$$\begin{bmatrix} d\underset{\sim}{X} \\ d\underset{\sim}{Y} \\ d\underset{\sim}{Z} \end{bmatrix} = \underset{\sim}{A}_t \begin{bmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \\ \underset{\sim}{Z} \end{bmatrix} + \underset{\sim}{B}_t d\underset{\sim}{W}_t \tag{4.4}$$

where

$$\underset{\sim}{A}_t = \begin{bmatrix} -\lambda & 0 & 0 \\ -\lambda(1-2q/\beta) & -2f/\beta + \mu + 2\nu r/\beta & \nu(1-2q/\beta) \\ -2\lambda q/\beta & 2f/\beta + 2\nu r/\beta & -\nu(1-2q/\beta) \end{bmatrix} \tag{4.5}$$

and

$$\underset{\sim}{B}_t = \begin{bmatrix} \sqrt{f(1-q/\beta)} & \sqrt{fq/\beta} & 0 & 0 & 0 \\ \sqrt{f(1-q/\beta)} & -\sqrt{fq/\beta} & -\sqrt{\mu q} & -\sqrt{\nu rq/\beta} & \sqrt{\nu r(1-q/\beta)} \\ 0 & 2\sqrt{fq/\beta} & 0 & \sqrt{\nu rq/\beta} & -\sqrt{\nu r(1-q/\beta)} \end{bmatrix} \tag{4.6}$$

and $\underset{\sim}{W}_t$ is a 5-dimensional standard Wiener process with independent components.

Result: $(\underset{\sim}{I}, \underset{\sim}{Q}, \underset{\sim}{R})$ is approximately normal as N (hence c) $\to \infty$:

$$(\underset{\sim}{I}, \underset{\sim}{Q}, \underset{\sim}{R}) \sim N(\, c(f, q, r), c\underset{\sim}{\Sigma}_t\,)$$

Here $\dot{\underset{\sim}{\Sigma}}_t$ satisfies the differential equation

$$\dot{\underset{\sim}{\Sigma}}_t = \underset{\sim}{A}_t \underset{\sim}{\Sigma}_t + \underset{\sim}{\Sigma}_t \underset{\sim}{A}_t' + \underset{\sim}{B}_t \underset{\sim}{B}_t' \tag{4.7}$$

and

$$\underset{\sim}{B}_t \underset{\sim}{B}_t' = \begin{bmatrix} f & f(1-2q/\beta) & 2fq/\beta \\ f(1-2q/\beta) & f(1-2q/\beta) + \mu q + \nu r & -(2fq/\beta + \nu r) \\ 2fq/\beta & -(2fq/\beta + \nu r) & 4fq/\beta + \nu r \end{bmatrix} \tag{4.8}$$

As might be anticipated, the fact that

$$\mathrm{Var}[\underset{\sim}{I}(t)] = \sigma_{11}(t) + F(t)[1-F(t)] \tag{4.9}$$

may be deduced from (4.7).

5. Model 4: Transmission to Completion in Model 1.

Suppose now, as may be quite realistic, that a message attempting to transmit on a channel does so until completion before it discovers that it has been "destroyed," i.e. garbled by others--which it also contributes to garbling. The analysis at once becomes much more complex because a channel may contain any number of destroyed messages. Our formulation is that of Model 1, save for the change described above.

The following state variables describe the system.

$\underset{\sim}{Q}(t)$ = the number of channels carrying good, i.e. ungarbled or undestroyed messages at time t.

$\underset{\sim}{S}_k(t)$ = the number of channels carrying exactly k messages that are destroyed at time t; clearly $\underset{\sim}{S}_o(t) \equiv \underset{\sim}{Q}(t)$, and

$$0 \leq \underset{\sim}{Q} + \sum_{k=1}^{\infty} \underset{\sim}{S}_k \leq C .$$

$\underset{\sim}{R}(t)$ = the number of messages in R at time t.

The transition probability scheme is summarized next. We write

$$\underset{\sim}{S}_+(t) = \sum_{k=o}^{\infty} \underset{\sim}{S}_k(t) .$$

Transitions	Probabilities
$(\underset{\sim}{Q}, \underset{\sim}{S}_1, \underset{\sim}{S}_2, \ldots, \underset{\sim}{S}_k, \ldots, \underset{\sim}{R}) \rightarrow$	
$(\underset{\sim}{Q}+1, \underset{\sim}{S}_1, \ldots, \underset{\sim}{S}_k, \ldots, \underset{\sim}{R})$	$c\lambda[1 - \frac{\underset{\sim}{Q}+\underset{\sim}{S}_+}{c}]dt$
$(\underset{\sim}{Q}+1, \underset{\sim}{S}_1, \ldots, \underset{\sim}{S}_k, \ldots, \underset{\sim}{R}-1)$	$\nu\underset{\sim}{R}[1 - \frac{\underset{\sim}{Q}+\underset{\sim}{S}_+}{c}]dt$
$(\underset{\sim}{Q}-1, \underset{\sim}{S}_1, \ldots, \underset{\sim}{S}_k, \ldots, \underset{\sim}{R})$	$\mu\underset{\sim}{Q}dt$
$(\underset{\sim}{Q}-1, \underset{\sim}{S}_1, \underset{\sim}{S}_2+1, \ldots, \underset{\sim}{S}_k, \ldots, \underset{\sim}{R}-1)$	$\nu\underset{\sim}{R}\frac{\underset{\sim}{Q}}{c}dt$
$(\underset{\sim}{Q}-1, \underset{\sim}{S}_1, \underset{\sim}{S}_2+1, \ldots, \underset{\sim}{S}_k, \ldots, \underset{\sim}{R})$	$c\lambda\frac{\underset{\sim}{Q}}{c}$
$(\underset{\sim}{Q}, \underset{\sim}{S}_1, \ldots, \underset{\sim}{S}_k+1, \ldots, \underset{\sim}{R}-1)$ for $k = 1, 2, 3, \ldots$	$\nu\underset{\sim}{R}\frac{\underset{\sim}{S}_k}{c}dt$ (5.1)
$(\underset{\sim}{Q}, \underset{\sim}{S}_1, \ldots, \underset{\sim}{S}_k-1, \underset{\sim}{S}_{k+1}, \ldots, \underset{\sim}{R})$	$c\,\lambda\frac{\underset{\sim}{S}_k}{c}dt$
$(\underset{\sim}{Q}, \underset{\sim}{S}_1-1, \ldots, \underset{\sim}{S}_k, \ldots, \underset{\sim}{R}+1)$	$\mu\frac{\underset{\sim}{S}_1}{c}dt$
. . .	
$(\underset{\sim}{Q}, \underset{\sim}{S}_1, \ldots, \underset{\sim}{S}_k+1, \underset{\sim}{S}_{k+1}-1, \ldots, \underset{\sim}{R}+1)$	$\mu(k+1)\underset{\sim}{S}_{k+1}$

There will be a denumerable infinity of such transitions. While it is in principle possible to carry out the expansion technique, we shall content ourselves with a brief discussion of the deterministic equations analogous to (2.6). Following the example of Section 2, we can write down these differential equations for the limits as $c \to \infty$ of $\underset{\sim}{Q}/c$, $\underset{\sim}{R}/c$, and $\underset{\sim}{S}_k/c$ $(k=1,2,\ldots)$, denoted by q , r , and s_k :

$$q' = -(\lambda + \nu r)(1 - s_+ - q) - (\mu + \lambda + \nu r)q$$

$$\begin{aligned} r' &= -\nu r + \mu \sum_{k=1}^{\infty} k s_k \\ s_k' &= \mu(k+1)s_{k+1} + (\lambda+\nu r)s_{k-1} - (k+\lambda+\nu r)s_k \ , \ k=1,3,4,5,\ldots \\ s_2' &= 3\mu s_3 + (\lambda+\nu r)(s_1+q) - (2\mu+\lambda+\nu r)s_2 \end{aligned} \qquad (5.2)$$

The steady-state values, which exist under circumstances to be derived, are obtained by solving the above system of equations with the derivatives set equal to zero. To simplify, first divide through by μ, putting $\rho = \lambda/\mu$, $\eta = \nu/\mu$. After some algebra it is found that

$$q = \rho \qquad (5.3)$$

$$\rho + \eta r = \theta \ , \quad \text{so} \quad r = (\theta - \rho)\eta^{-1} \qquad (5.4)$$

where θ is the smallest solution of the equation

$$\frac{x}{1+x} e^{-x} = \rho \ , \qquad (5.5)$$

provided one exists.

$$\begin{aligned} s_1 &= \rho\theta \\ s_k &= \frac{e^{-\theta}\theta^k}{k!} \ , \quad k \geq 2 \end{aligned} \qquad (5.6)$$

Further analysis shows that in order for a steady-state to exist,

$$\rho \leq \frac{\sqrt{5} - 1}{\sqrt{5} + 1} e^{-(\sqrt{5} - 1)/2} = 0.20588\ldots$$

Notice that this value is much smaller than the just-intolerable value of 1/2 derived from Model 1. Clearly, transmission to completion of messages provides opportunity for many more transmissions to be destroyed.

6. Model 5: Transmission to Completion with Intelligent Re-tries

A natural variation of the previous model is obtained by insisting on transmission to completion, but also allowing for an intelligent re-try capability, as in Model 2. This means that the transitions $\to(\underset{\sim}{Q}-1, \underset{\sim}{S}_1, \underset{\sim}{S}_2+1, \ldots, \underset{\sim}{S}_k, \ldots, \underset{\sim}{R}-1)$, and $\to(\underset{\sim}{Q}, \underset{\sim}{S}_1, \ldots, \underset{\sim}{S}_{k+1}, \ldots, \underset{\sim}{R}-1)$ are ruled out, i.e. have probability zero in this model. Consequently the transition rates of (5.1) apply, with this change. Confining attention to the deterministic differential equations it may be shown that these have the form given below.

$$
\begin{aligned}
q' &= (\lambda+\nu r)(1 - s_+ - q) - (\lambda+\mu)q \\
r' &= -\nu r(1 - s_+ - q) + \mu \sum_{k=1}^{\infty} ks_k \\
s_k' &= (k+1)\, \mu s_{k+1} + \lambda s_{k-1} - (\lambda+k\mu)s_k \\
s_2' &= 3\mu s_3 + \lambda s_1 - (\lambda+2\mu)s_2 + \lambda q \; .
\end{aligned}
\tag{6.1}
$$

Once again let $\rho=\lambda/\mu$, $\eta=\nu/\mu$, and solve for the steady-state values. These are

$$q = \rho \tag{6.2}$$

$$r = \frac{1}{\eta}\,\frac{\rho[(1+\rho)e^{\rho}-1]}{1-(1+\rho)(e^{\rho}-1)} \tag{6.3}$$

$$s_1 = \rho^2 \tag{6.4}$$

$$s_k = (1+\rho)\frac{\rho^k}{k!} \; , \quad k=2,3,\ldots \tag{6.5}$$

The condition for existence of a steady state is that the denominator of (6.3) be positive, which translates into the requirement that $\rho < 0.50855\ldots$. The latter value may be contrasted to the just-tolerable value of unity derived for Model 2.

7. Numerical Results for Model 1.

In order to check the quality of the diffusion approximations, a simulation program was written that realizes the two-dimensional Markov process describing Model 1. The latter was then exercised through 5×10^6 state changes for several values of the offered load, λ/μ, and for several channel numbers; the parameter $\alpha = 1$, and $\nu/\mu = 0.08$ in this experiment. The results were generally supportive of the approximation, as is seen by examining the following tables. Note, however, that assessments of the mean and variance of the number of active channels, $\underset{\sim}{Q}$, are generally in closer agreement than are those for the size, $\underset{\sim}{R}$, of the re-try population.

TABLE 1.

Means and Variance-Covariance Values by

Simulation and Diffusion Approximation

$c = 10$; $\nu/\mu = 2.0$

	λ/μ: 0.48	0.40	0.32	0.20
$E[\underset{\sim}{Q}]$ simulat.	4.79	3.99	3.19	1.99
diffus.	4.80	4.0	3.2	2.0
$Var[\underset{\sim}{Q}]$	2.52	2.52	2.38	1.76
	2.50	2.52	2.39	1.77
$E[\underset{\sim}{R}]$	65.13	9.31	3.29	0.75
	57.6	8.0	2.84	0.67
$Var[\underset{\sim}{R}]$	1470.	50.26	10.62	1.57
	1466.	42.81	8.80	1.35
$Cov[\underset{\sim}{R},\underset{\sim}{Q}]$	0.034	0.124	0.166	0.149
	0.038	0.143	0.177	0.150

TABLE 2.

Means and Variance-Covariance Values by

Simulation and Diffusion Approximation

$c = 25$; $\nu/\mu = 2.0$

	λ/μ: 0.48	0.40	0.36	0.20
$E[\underset{\sim}{Q}]$ simulat.	11.95	9.97	7.98	4.99
diffus.	12.0	10.0	8.0	5.0
$Var[\underset{\sim}{Q}]$	6.32	6.31	5.98	4.41
	6.26	6.30	5.97	4.42
$E[\underset{\sim}{R}]$	149.5	21.11	7.49	1.73
	144.0	20.0	7.11	1.67
$Var[\underset{\sim}{R}]$	3524.	110.2	23.74	3.56
	3666.	107	22.0	3.38
$Cov[\underset{\sim}{R},\underset{\sim}{Q}]$	0.041	0.132	0.177	0.150
	0.040	0.143	0.177	0.150

8. Summary and Conclusions

In this paper it has been shown that an approximate approach, using stochastic differential equations, is effective for modeling an element of a complex communication system. The approximation improves as c, the number of channels available for transmission, becomes large. The adequacy of the model under such conditions is suggested by the theoretical results of Barbour (1974) and Kurtz (1971); numerical solutions of selected systems also attest to the adequacy of the approximation. The authors wish to gratefully acknowledge the research support of the Office of Naval Research.

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