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# NAVAL POSTGRADUATE SCHOOL

## Monterey, California

ASYMPTOTIC REPRESENTATION OF

STIRLING NUMBERS OF THE SECOND KIND

by

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9 February 1977





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ABSTRACT:

The distribution of the Stirling numbers S(n,k) of the second kind with respect to k has been shown by Harper [Ann. Math. Statist., 38 (1967), 410-414] to be asymptotically normal near the mode. A new single-term asymptotic representation of S(n,k), more effective for large k, is given here. It is based on Hermite's formula for a divided difference and the use of sectional areas normal to the body diagonal of a unit hypercube in k-space. A proof is given that the distribution of these areas is asymptotically normal. A numerical comparison is made with the Harper representation for n=200.

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## 1. Introduction.

Previous asymptotic representations of Stirling numbers $S(n,k)$ of the second kind have been of two types. One type has been a complete infinite series expansion as given by Hsu [1], and by Bleick and Wang [2] and [3]. A second type has been the single-term representation of $S(n,k)$ given by Harper [4] as the normal distribution approximation

$$(1) \qquad S(n,k) \sim \frac{B_n}{\sigma\sqrt{2\pi}} \exp[-(k-\mu)^2/2\sigma^2]$$

where the mean $\mu$ and the variance $\sigma^2$ are expressed in terms of the Bell numbers $B_n$ by

$$(2) \qquad \mu = B_{n+1}/B_n - 1$$

and

$$(3) \qquad \sigma^2 = B_{n+2}/B_n - (B_{n+1}/B_n)^2 - 1 .$$

The purpose of this note is to give a new single-term asymptotic representation based on Hermite's formula for a divided difference, and to compare it with that of Harper.

## 2. Use of Hermite's formula.

A Stirling number $S(n,k)$ of the second kind is defined as the kth difference of $z^n$ at $z=0$ divided by $k!$. By [5,p.10] we find that this divided difference can be represented by a formula of Hermite as the repeated definite integral

$$(4) \qquad S(n,k) = \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{k-1}} (d^k u_1{}^n/du_1{}^k) dt_k$$

where $u_1=t_1+t_2+..+t_k$. We imagine that $t_1$, $t_2$, .., $t_k$ constitute a set of rectangular Cartesian coordiantes and impose an orthogonal transformation of coordinates to $u_1, u_2, .., u_k$. The volume of the space over which the integration in (4) is performed is a portion of a unit hypercube in k-space. If we allow the coordinate $u_1$ to vary along the body diagonal of the hypercube from 0 at one vertex to k at the opposite vertex, the sectional areas normal to the diagonal cut by the hyperplane $u_1=t_1+t_2+..+t_k$ from the domain of integration define a positive function $g(u_1,k)$ even with respect to the argument $u_1-k/2$. We take the integral of $g(u_1,k)$ to be

$$(5)\qquad \int_0^k g(u_1,k)\,du_1 = 1/k!$$

to agree with the volume of the space over which the integration in (1) is performed. We drop the $u_1$ subscript henceforth. Noting that $g(u,k)=0$ for $k<u<0$, we find that

$$(6)\qquad g(u,1) = 1 \quad \text{for } 0 \le u \le 1 \quad ,$$

$$(7)\qquad 2!g(u,2) = (1 - |u-1|) \quad \text{for } 0 \le u \le 2 \ ,$$

and

$$(8)\qquad 3!g(u,3) = \begin{cases} (3/2-|u-3/2|)^2/2 & \text{for } 1/2 \le |u-3/2| \le 3/2 \\ 3/4 - (u-3/2)^2 & \text{for } 1 \le u \le 2 \end{cases} .$$

Consideration of the Laplace transforms of (6), (7) and (8) suggests that we conjecture the Laplace transform of $k!g(u,k)$ to be

$$(9)\qquad (1-e^{-s})^k/s^k = e^{-ks/2}\left(\frac{\sinh s/2}{s/2}\right)^k$$

for all k. We demonstrate the truth of this conjecture later. On performing the integration in (4) over the variables $u_2, u_3, .., u_k$ we find

$$(10)\qquad S(n,k) = k!\binom{n}{k}\int_0^\infty u^{n-k} g(u,k)\,du \ .$$

Using operation 82 of [6,p.10] on the Laplace transform of

(11) $$k! \int_0^u u^m g(u,k)\,du$$

we find the mth moment of the k!g(u,k) distribution about u=0 to be

(12) $$\lim_{s\to 0} (-1)^m (d/ds)^m (1-e^{-s})^k / s^k \ .$$

It is now easy to demonstrate the truth of the conjecture (9) by showing, with the aid of the multinomial theorem, that (12) is the same as the repeated integral

(13) $$\int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 (t_1+t_2+\ldots+t_k)^m dt_k$$

over the volume of the hypercube.

Use of (12) and (5) shows the variance of the k!g(u,k) distribution to be

(14) $$\sigma^2 = k/12 \ .$$

Using (14) the series

(15) $$\exp(\sigma^2 s^2/2) = 1 + \frac{ks^2/24}{1!} + \frac{(ks^2/24)^2}{2!} + \ldots$$

is the bilateral, but not s multiplied, Laplace transform of the normal distribution

(16) $$(1/\sigma\sqrt{2\pi})\exp(-t^2/2\sigma^2)$$

according to [7,p.21]. The corresponding series for (9) multiplied by $e^{ks/2}$, or the bilateral Laplace transform of k!g(u,k) shifted left by k/2, is

(17) $$(2/s)^k \sinh^k s/2 = [1 + \frac{s^2/4}{3!} + \frac{(s^2/4)^2}{5!} + \ldots]^k \ .$$

The dominant k power term in the coefficient of $(s^2/4)^n$ in (15) is $k^n/6^n n!$, and may be shown to be the same in the expansion of (17) by the use of the recurrence formula 6.361 of [8,p.119]. This proves that the k!g(u,k) distribution is asymptotically normal as $k\to\infty$. It is remarkable that the normal distribution should arise in the purely geometrical context of sectional areas normal to the body diagonal of a hypercube of high dimension.

On replacing $k!g(u,k)$ in (10) by its Gaussian normal approximation of mean $\mu=k/2$ and variance $\sigma^2=k/12$ we find

$$(18)\qquad S(n,k) \sim \frac{1}{\sigma\sqrt{2\pi}}\binom{n}{k}\int_0^\infty u^{n-k}\exp[-(u-k/2)^2/2\sigma^2]du$$

$$\sim \frac{1}{\sqrt{2\pi}}\binom{n}{k}\int_{-\infty}^{\sqrt{3k}}(k/2-\sigma t)^{n-k}e^{-t^2/2}dt \ .$$

3. Numerical example.

Table 1 compares the exact values of $S(200,k)$ with the asymptotic approximations computed from the single-term representations (1) and (18). Harper's representation (1), which uses $B_{200}=.62475\ 10^{276}$, $\mu=49.975$ and $\sigma=3.0551$, gives an excellent fit near the mode ($k=50$), but (18) gives a much better fit for large values of $k$.

Table 1. Values of $S(200,k)$

| k | Asymptotic from (1) | Exact | Asymptotic from (18) |
|---|---|---|---|
| 2 | $.23135\ 10^{222}$ | $.80347\ 10^{6}$ | $.69244\ 10^{126}$ |
| 40 | $.39504\ 10^{273}$ | $.24458\ 10^{273}$ | $.42658\ 10^{273}$ |
| 50 | $.81579\ 10^{275}$ | $.81493\ 10^{275}$ | $.15285\ 10^{277}$ |
| 60 | $.37452\ 10^{273}$ | $.53533\ 10^{273}$ | $.29658\ 10^{274}$ |
| 100 | $.49065\ 10^{217}$ | $.22839\ 10^{235}$ | $.27994\ 10^{235}$ |
| 150 | $.13938\ 10^{43}$ | $.30251\ 10^{143}$ | $.30441\ 10^{143}$ |
| 199 | $.16955\ 10^{-241}$ | $.19900\ 10^{5}$ | $.19900\ 10^{5}$ |

## REFERENCES


1. L. C. Hsu, Note on an asymptotic expansion of the nth difference of zero, Ann. Math. Statist. 19, (1948), 273-277. MR9, 578.

2. W. E. Bleick and Peter C. C. Wang, Asymptotics of Stirling numbers of the second kind, Proc. Am. Math. Soc. 42 (1974), 575-580.

3. W. E. Bleick and Peter C. C. Wang, Erratum to 2, Proc. Am. Math. Soc. 48 (1975), 518.

4. L. H. Harper, Stirling behavior is asymptotically normal, Ann. Math. Statist. 38 (1967), 410-414.

5. L. M. Milne-Thomson, The calculus of finite differences, MacMillan and Co., Ltd., London, 1933.

6. G. E. Rober ts and H. Kaufman, Table of Laplace transforms, Saunders, Philadelphia, 1966. MR32 #8050.

7. Balth. van der Pol and H. Bremmer, Operational calculus based on the two-sided Laplace integral, Cambridge University Press, 1955.

8. E. P Adams and R. L. Hippisley, Smithsonian mathematical formulae and tables of elliptic functions, Publication 2672, Smithsonian Institution, Washington, 1922.

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