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AN ANALYTICAL STUDY OF INCOMPRESSIBLE
FREE TURBULENT MIXING IN ADVERSE AND
FAVORABLE PRESSURE GRADIENTS

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ABSTRACT:

The equations governing free turbulent mixing are derived from the Navier-Stokes equations and transformed into a mathematical plane which is explicitly independent of the eddy viscosity model. The coupled momentum and turbulent kinetic energy equations are analytically solved in the transformed plane by a perturbation technique and subsequently retransformed into physical space based on a hypothesized dependence of the eddy viscosity on the turbulent kinetic energy. The adequacy of a given model in reproducing the velocity and turbulent kinetic energy field is assessed by comparing the results of the analysis with some experimental data of planar turbulent wake mixing in constant adverse and favorable pressure gradients.

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NOMENCLATURE

A	=	Cross-Sectional Area of Wind Tunnel Test Section
β	=	Strength of Pressure Gradient, Equations (49) and (51)
C	=	Constant in Equation (32)
D_K	=	Turbulent Dissipation
δ^*	=	Displacement Thickness
ϵ	=	Perturbation Parameter = $1/u_o^2$; DK/ρ in Equation 24
F	=	Arbitrary Function in Equation (73)
F^*	=	$Fe^{-\int f(\varphi)d\varphi}$
G	=	Greens Function
H	=	Shape Factor $\equiv \delta^*/\theta$
I	=	Initial Conditions in Equation (75)
$\vec{i}$	=	Unit Vector in x-direction
J	=	Nonhomogeneous Function in Equation (74) = $g(\varphi,\Psi)e^{-\int f(\varphi)d\varphi}$
$\vec{j}$	=	Unit Vector in y-direction
K	=	Turbulent Kinetic Energy = $\frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$
k	=	Constant in Equation (52)
$\vec{k}$	=	Unit Vector in z-direction
X	=	$u_e^2 - \bar{u}^2$
ℓ	=	Macroscale of Turbulence
μ	=	Coefficient of Viscosity
μ_t	=	Eddy Viscosity
ν	=	Kinematic Viscosity
ν_t	=	Turbulent Kinematic Viscosity
P	=	Pressure
Ψ	=	Stream Function Defined in Equation (25)
φ	=	Transformed Independent Variable in Equation (33)
q	=	Dynamic Pressure

ρ = Density

σ_K = Turbulent Diffusion Coefficient in Equation (23)

S = Velocity Ratio Function = $(u_o/u_e)^2$

t = Time

τ = Apparent Turbulent Shear Stress

θ = Momentum Thickness

u = Axial Velocity (x-direction)

U_e = Average Axial Velocity External to the Wake

v = Lateral Velocity (y-direction)

$\vec{V}$ = Velocity Vector = $u\vec{i} + v\vec{j} + w\vec{k}$

w = Transverse Velocity (z-direction)

x = Axial Coordinate

y = Lateral Coordinate

z = Transverse Coordinate

Subscripts

e = External to the Wake

i,j,k = Three Orthogonal Components of a Vector

L = Exit of Test Section

O = Inlet of Test Section

t = Turbulent Function

0,1 = Zeroth and First Order Solutions of χ and K

Superscripts

$\bar{u}$ = Time Average of u

u' = Fluctuating Part of u

INTRODUCTION

The solution of the Navier-Stokes equations as applied to the problem of turbulent flow has classically been approached either from the point of view of elegant rigor on limitingly simple flows or one of empirically guided analysis of flows of real interest. The Reynolds averaging technique is nearly a pre-requisite to attacking any real turbulent flow and the associated necessity to empirically close the system of equations with a Reynolds stress model makes the mathematics tractable yet often unreliable. Various so-called eddy viscosities which arise from the Boussinesq laminarization of the apparent turbulent stresses are functions of the flow field and as yet there are no known universal functions which adequately model these stresses in all cases.

Possibly the most successful application of eddy viscosity approaches lies in the area of free mixing where velocity differences through the flow field are small and the associated turbulent field is a fairly simple one. However, in the pressure gradient situations which are encountered in ejector and combustor mixing phenomena, classical eddy viscosities fail since they are explicitly independent of the turbulent field which, in these cases, can become complex and exert a dominant effect on the form of the eddy viscosity. Modern approaches have attempted to include the local turbulence structure effect on the eddy viscosity through an explicit dependence on the local turbulent kinetic energy.

In conjunction with a highly idealized wake mixing experiment, the two-dimensional incompressible turbulent wakes in constant adverse and favorable pressure gradients have been studied analytically with a formulation of the equations which allows the coordinated solutions of the velocity and turbulence field in a transformed plane which is explicitly independent of the eddy viscosity model. Subsequently, the retransformation of the equations for a variety of models allows for a comparison of the adequacy of the models and an evaluation as to which most accurately reproduces both the velocity and kinetic energy fields.

EQUATION DEVELOPMENT

The basis for any rigorous analysis of problems involving turbulent fluid flow must, to the best of current knowledge, be founded in the Navier-Stokes equations. For an incompressible, Newtonian fluid these may be written

$$\nabla \cdot \vec{V} = 0 \tag{1}$$

$$\rho \frac{D\vec{V}}{Dt} = - \nabla p + \mu \nabla^2 \vec{V} \tag{2}$$

No analytical solution of the full equations appears possible and, although some success has been shown with numerical approaches to the solution of the equations, as of yet no numerical attack for a genuinely turbulent flow is feasible (Ref. 1). Apart from the numerical approach, only Fourier analysis has shown any progress in the solution of the equations. However, with this method, only limitingly simple turbulent fields have been treated and its relevance to a general turbulent mixing problem has yet to be proven.

Classically, the most fruitful approach in the analysis of turbulence has been to decompose each of the dependent variables in Equations 1 and 2 into a mean term, which is independent of time or has a long characteristic period with respect to the turbulent fluctuations, plus a fluctuating term whose time average is zero. The velocity field

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k} \tag{3}$$

may be decomposed term by term such as the decomposition of u

$$u\ (x,y,z,t) = \overline{u}\ (x,y,z) + u'\ (x,y,z,t) \tag{4}$$

When this decomposition is applied to each term in the continuity equation we obtain

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{5}$$

If we now take the time average of Equation (5), we obtain

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0 \tag{6}$$

which is the continuity equation for the mean flow. Upon subtracting Equation (6) from Equation (5), we obtain

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{7}$$

This is the continuity equation which the velocity fluctuations must satisfy. Prior to applying this decomposition technique to the momentum equations we first reformulate the convective operator in a conservative form, with the aid of the continuity equation. The general convective operator

$$\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + w\,\frac{\partial}{\partial z} \tag{8}$$

becomes

$$\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\,u(\ \ ) + \frac{\partial}{\partial y}\,v(\ \ ) + \frac{\partial}{\partial z}\,w(\ \ ) \tag{9}$$

where the appropriate dependent variable is placed within the parentheses. Applying this formulation to Equations (2) and expanding them in rectangular Cartesian coordinates, we obtain

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(v^2) + \frac{\partial}{\partial z}(wv) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 v \tag{10}$$

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(uw) + \frac{\partial}{\partial y}(vw) + \frac{\partial}{\partial z}(w^2) = \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 w$$

Each of the dependent variables in Equations (10) is now decomposed and the appropriate expression is inserted into the equations. The resulting system of equations is

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t} + \frac{\partial \bar{u}^2}{\partial x} + 2\,\frac{\partial \bar{u}u'}{\partial x} + \frac{\partial u'^2}{\partial x} + \frac{\partial \overline{uv}}{\partial y} + \frac{\partial \bar{u}v'}{\partial y} + \frac{\partial u'\bar{v}}{\partial y} + \frac{\partial u'v'}{\partial y}$$

$$+ \frac{\partial \overline{uw}}{\partial z} + \frac{\partial u'\bar{w}}{\partial z} + \frac{\partial \bar{u}w'}{\partial z} + \frac{\partial u'w'}{\partial z} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} - \frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu \nabla^2 \bar{u} + \nu \nabla^2 u'$$

$$\frac{\partial \bar{v}}{\partial t} + \frac{\partial v'}{\partial t} + \frac{\partial \overline{uv}}{\partial x} + \frac{\partial u'\bar{v}}{\partial x} + \frac{\partial \bar{u}v'}{\partial x} + \frac{\partial u'v'}{\partial x} + \frac{\partial \bar{v}^2}{\partial y} + 2\,\frac{\partial \bar{v}v'}{\partial y} + \frac{\partial v'^2}{\partial y}$$

$$+ \frac{\partial \overline{wv}}{\partial z} + \frac{\partial w'\bar{v}}{\partial z} + \frac{\partial \bar{w}v'}{\partial z} + \frac{\partial w'v'}{\partial z} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} - \frac{1}{\rho}\frac{\partial p'}{\partial y} + \nu \nabla^2 \bar{v} + \nu \nabla^2 v' \tag{11}$$

$$\frac{\partial \bar{w}}{\partial t} + \frac{\partial w'}{\partial t} + \frac{\partial \overline{uw}}{\partial x} + \frac{\partial u'\bar{w}}{\partial x} + \frac{\partial \bar{u}w'}{\partial x} + \frac{\partial u'w'}{\partial x} + \frac{\partial \overline{vw}}{\partial y} + \frac{\partial \bar{v}w'}{\partial y} + \frac{\partial v'\bar{w}}{dy} + \frac{\partial v'w'}{\partial y}$$

$$+ \frac{\partial \bar{w}^2}{\partial z} + 2\,\frac{\partial \bar{w}w'}{\partial z} + \frac{\partial w'^2}{\partial z} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} - \frac{1}{\rho}\frac{\partial p'}{\partial z} + \nu\,\nabla^2 \bar{w} + \nu\,\nabla^2 w'$$

If we now take the time average of each equation, we can simplify the system by dropping out all those terms whose time average is zero (i.e. those which have terms that are linear in the fluctuating properties). Note, however, that we must retain the nonlinear terms containing fluctuations since the products or powers of purely fluctuating terms may generate a steady time averaged value. When the appropriate time averages are taken of each term in Equations (11) we retain the following system of equations

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial}{\partial x}(\bar{u}^2) + \frac{\partial}{\partial y}(\overline{uv}) + \frac{\partial}{\partial z}(\overline{uw}) = -\frac{1}{\rho}\frac{\partial}{\partial x}(\bar{p} + \rho\overline{u'^2}) - \frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial z}\overline{u'w'} + \nu\,\nabla^2\bar{u}$$

$$\frac{\partial \bar{v}}{\partial t} + \frac{\partial}{\partial x}(\overline{uv}) + \frac{\partial}{\partial y}(\bar{v}^2) + \frac{\partial}{\partial z}(\overline{wv}) = -\frac{1}{\rho}\frac{\partial}{\partial y}(\bar{p} + \rho\overline{v'^2}) - \frac{\partial}{\partial x}\overline{u'v'} - \frac{\partial}{\partial z}\overline{w'v'} + \nu\,\nabla^2\bar{v}$$

$$\frac{\partial \bar{w}}{\partial t} + \frac{\partial}{\partial x}(\overline{uw}) + \frac{\partial}{\partial y}(\overline{vw}) + \frac{\partial}{\partial z}(\bar{w}^2) = -\frac{1}{\rho}\frac{\partial}{\partial z}(\bar{p} + \rho\overline{w'^2}) - \frac{\partial}{\partial x}\overline{u'w'} - \frac{\partial}{\partial y}\overline{v'w'} + \nu\,\nabla^2\bar{w}$$

(12)

Note that the price of this "simplification" through the Reynolds averaging has been the introduction of six new unknowns by discarding of all of the phase information of the fluctuations. For a steady, two-dimensional mean flow Equations (12) may be simplified to

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \qquad (13)$$

$$\bar{u}\,\frac{\partial \bar{u}}{\partial x} + v\,\frac{\partial \bar{u}}{\partial y} = -\,\frac{1}{\rho}\,\frac{\partial}{\partial x}\,(\bar{p} + \rho\overline{u'^2}) - \frac{\partial}{\partial y}\,\overline{u'v'} + \nu\,\nabla^2\bar{u} \tag{14}$$

$$\bar{u}\,\frac{\partial \bar{v}}{\partial x} + \bar{v}\,\frac{\partial \bar{v}}{\partial y} = -\,\frac{1}{\rho}\,\frac{\partial}{\partial y}\,(\bar{p} + \rho\overline{v'^2}) - \frac{\partial}{\partial x}\,\overline{u'v'} + \nu\,\nabla^2\bar{v} \tag{15}$$

In general we may make the assumption that the apparent pressures ($\rho\overline{u'^2}$, $\rho\overline{v'^2}$) may be neglected with respect to $\bar{p}$. We should note that when $\bar{p}$ = constant some care must be exercised in applying this approximation since $-\frac{1}{\rho}\frac{\partial}{\partial x}(\rho\overline{u'^2})$ and $-\frac{1}{\rho}\frac{\partial}{\partial y}(\rho\overline{v'^2})$ may be significant terms in the equations for a particular problem. In addition, for a "thin" wake, we may make the standard boundary layer-type parallel flow approximation that

$$\frac{\partial \bar{p}}{\partial x} = \frac{dp}{\partial x} = -\,\rho U_e\,\frac{dU_e}{\partial x} \tag{16}$$

This eliminates the necessity of solving the lateral momentum equation, since the only unknowns are $\bar{u}$ and $\bar{v}$ with $\bar{p}$ being imposed on the mixing region by the external flow. The only remaining term which explicitly involves the turbulent field is $-\overline{u'v'}$ which represents an apparent Reynolds shearing stress (τ).

$$\tau \equiv -\,\overline{\rho u'v'} \tag{17}$$

In general this apparent stress is very large with respect to the average laminar shear stress and we may neglect the laminar component. With these approximations the resulting system of equations to be solved is

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{18}$$

$$\bar{u}\,\frac{\partial \bar{u}}{\partial x} + \bar{v}\,\frac{\partial \bar{u}}{\partial y} = -\,\frac{1}{\rho}\,\frac{dp}{dx} + \frac{1}{\rho}\,\frac{\partial \tau}{\partial y} \tag{19}$$

At this point, some empiricism is necessary since τ is not retrievable from the equation set we have derived. This is clear from Equations (11) when either is multiplied by the appropriate fluctuating velocity and time averaging is performed to obtain an equation for the Reynolds stress, triple correlations appear which are unknowns also. This cascade of unknown higher order correlations continues for all further equations which are derived. Some success (Ref. 2) has been made attacking these problems by making purely heuristic approximations in higher order correlations in an effort to make the least sensitive approximations possible in the equations. However, in lieu of attacking this spiralling set of equations, it has generally proved more effective to introduce some empirically based models of the Reynolds stress into Equations (18) and (19). The most successful models have been based on extensions of Prandtl's mixing length analysis which analogizes turbulent eddy momentum transfer with the molecular manifestation of viscosity. In line with this approach an eddy viscosity, μ_t, is introduced into the problem via the definition

$$\mu_t \equiv - \rho\overline{u'v'}/\frac{\partial \overline{u}}{\partial y} \tag{20}$$

where μ_t is a function of the local mean flow field. Classical models infer that μ_t is purely a function of the local non-turbulent mean flow, however, more modern results (Ref. 3) indicate that some specific dependence on the turbulence field is indicated. Thus we hypothesize that

$$\mu_t = \mu_t \ (\overline{u},\overline{v},K)$$

where

$$K \equiv \tfrac{1}{2} \ (\overline{u'^2} + \overline{v'^2} + \overline{w'^2}) \tag{21}$$

In order to implement a model of this form, we must formulate and solve an equation for K along with the momentum and continuity equations. In order to obtain an equation for K, we multiply Equations (11) by u', v', w' respectively, add them and take the time average. This operation results in the following equation for K which is most conveniently written in tensor notation

$$\frac{DK}{Dt} = - \frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u'_j p'} + \frac{1}{2}\overline{u'_k u'_i u'_j} - \nu \overline{u'_i\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)}\right)$$

$$- \frac{\overline{u'_i u'_j}}{2}\left(\frac{\partial \bar{u}_j}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \nu\overline{\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)^2} \tag{22}$$

The respective convection, diffusion, production, and dissipation terms have been modelled by Patankar and Spalding (Ref. 4) in the following equation

$$\bar{u}\frac{\partial K}{\partial x} + \bar{v}\frac{\partial K}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\nu_t}{\sigma_K}\frac{\partial K}{\partial y}\right] + \nu_t\left(\frac{\partial \bar{u}}{\partial y}\right)^2 - \frac{D_K}{\rho} \tag{23}$$

Where σ_K is the effective Prandtl number for the diffusion of K and D_K is the turbulent dissipation. With appropriate auxiliary expressions or equations for μ_t, D_K, and σ_K we may consider the system of equations to be closed. Clearly the adequacy of the equation system in modelling any particular flow field is dependent on the exact formulation which is used for the unknown coefficients.

Based on dimensional reasoning, we can specify the coefficients μ_t and D_K to be

μ_t = const. x density x velocity x length

D_K = const. x density x velocity^3/length

From experimental results (Ref. 3) we specify σ_K to be a purely empirical constant in the range $0.5 \rightarrow 1.0$.

With the explicit expression which we will test for μ_t, along with a value for σ_K, only the formulation for D_K is left unspecified. Jones (Ref. 5) has hypothesized an equation for $\epsilon = D_K/\rho$ of the following form

$$\frac{D\epsilon}{Dt} = \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_j}\right) + C_1 \frac{\epsilon}{K} \nu_t \frac{\partial \bar{u}_i}{\partial x_j}\left(\frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_j}\right) - \frac{C_2 \epsilon^2}{K} \tag{24}$$

where C_1 and C_2 are empirical constants. In lieu of this complication, we will make some experimentally justified approximations for D_K and its dependence on K and the mean velocity field.

MATHEMATICAL ANALYSIS

I. Equation Reformulation

In order to reduce Equations (18), (19), and (23) to a form more amenable to an analytic approach, we introduce the stream function

$$\Psi_y = \bar{u} \quad , \quad \Psi_x = - \bar{v} \tag{25}$$

which automatically satisfies the continuity equation. With this definition of the stream function, the von Mises transformation is applied to the independent variables (x,y) to transform them into (x,Ψ) via the following equations

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} - \bar{v} \frac{\partial}{\partial \Psi} \tag{26}$$

(26)

$$\frac{\partial}{\partial y} = \bar{u} \frac{\partial}{\partial \Psi}$$

When this transformation is applied to Equations (19) and (23), we obtain

$$\frac{\partial \bar{u}}{\partial x} = - \frac{1}{\rho \bar{\bar{u}}} \frac{dp}{dx} + \frac{\partial}{\partial \Psi}\left(\nu_t \bar{u} \frac{\partial \bar{u}}{\partial \Psi}\right) \tag{27}$$

$$\frac{\partial K}{\partial x} = \frac{\partial}{\partial \Psi}\left(\frac{\nu_t \bar{u}}{\sigma_K} \frac{\partial K}{\partial \Psi}\right) + \nu_t \bar{u} \left(\frac{\partial \bar{u}}{\partial \Psi}\right)^2 - \frac{D_K}{\rho \bar{\bar{u}}} \tag{28}$$

where continuity need not be solved since $\bar{v}$ does not appear explicitly in the equations when $\bar{u}$ and K are expressed in stream function variables.

The dependent variable $\bar{u}$ is now transformed into χ via the equation

$$\chi \equiv u_e^2 - \bar{u}^2 \tag{29}$$

Upon substituting $u = (u_e^2 - \chi)^{\frac{1}{2}}$ into Equations (27) and (28) we can write the equations of momentum and turbulent kinetic energy as

$$\frac{\partial \chi}{\partial x} = \nu_t u_e \left(1 - \frac{\chi}{u_e^2}\right)^{\frac{1}{2}} \frac{\partial^2 \chi}{\partial \Psi^2} \tag{30}$$

$$\frac{\partial K}{\partial x} = \nu_t u_e \left\{\frac{\partial}{\partial \Psi}\left[\frac{1}{\sigma_K}\left(1 - \frac{\chi}{u_e^2}\right)^{\frac{1}{2}} \frac{\partial K}{\partial \Psi}\right] + \frac{1}{4u_e^2}\left(1 - \frac{\chi}{u_e^2}\right)^{-\frac{1}{2}} \left(\frac{\partial \chi}{\partial \Psi}\right)^2 - \frac{D_K}{\mu_t u_e^2}\left(1 - \frac{\chi}{u_e^2}\right)^{-\frac{1}{2}}\right\} \tag{31}$$

To this point, based on the assumptions previously outlined, these equations are as exact as the specifications of D_K, ν_t, and σ_K. To be consistent with our previous dimensional reasoning, the coefficient $\frac{D_K}{\mu_t}$ in the kinetic energy equation may be expressed dimensionally as

$$\frac{D_K}{\mu_t} = \text{constant x (velocity/length)}^2$$

Following experimental indications we hypothesize the exact formulation for this coefficient to be

$$\frac{D_K}{\mu_t} = C \times K/\ell^2 \tag{32}$$

where ℓ is the maxroscale of the turbulent flow, generally accepted to be of the order of the mixing width and a physical measure of those turbulent eddies most intimately involved in momentum transfer. We have not yet explicitly used an independent expression for ν_t in the equations and, as we shall see, the hypothesis for its formulation need not be specified until the equations have been solved in a transformed plane.

With these equations and expressions, we again transform the independent variables, this time from (x,Ψ) to (φ,Ψ) using the following definition of φ

$$\varphi \equiv \int_0^x u_e \, \nu_t \, dx \tag{33}$$

where we have introduced a crucial yet common and experimentally justified assumption that $\nu_t = \nu_t(x)$ alone. However, our interest lies in specifying a useful $\nu_t(K)$ which is implicitly $\nu_t\,[K(\varphi,\Psi)]$ in the transformed plane and we are free to select a particular Ψ_j for optimum results, thus $\nu_t(x)$ is in actuality $\nu_t = \nu_t\,[K(\varphi,\Psi_j)]$.

Applying this transformation to the equations, we can write the momentum and turbulent kinetic energy equations as

$$\frac{\partial X}{\partial \varphi} = \left(1 - \frac{X}{u_e^2}\right)^{\frac{1}{2}} \frac{\partial^2 X}{\partial \Psi^2} \tag{34}$$

$$\frac{\partial K}{\partial \varphi} = \frac{1}{\sigma_K}\frac{\partial}{\partial \Psi}\left[\left(1 - \frac{\chi}{u_e^2}\right)^{\frac{1}{2}}\frac{\partial K}{\partial \Psi}\right] + \frac{1}{4u_e^2}\left(1 - \frac{\chi}{u_e^2}\right)^{-\frac{1}{2}}\left(\frac{\partial \chi}{\partial \Psi}\right)^2 - \frac{C}{\ell^2 u_e^2}\left(1 - \frac{\chi}{u_e^2}\right)^{-\frac{1}{2}} K \qquad (35)$$

Subject to the following boundary and initial conditions:

$$\begin{aligned} &\text{B.C. } \chi \rightarrow 0;\ K \rightarrow 0 \quad \text{as} \quad \Psi \rightarrow = \infty \\ &\text{I.C. } \chi = \chi_i(\Psi) \quad ; \quad K = K_i(\Psi) \text{ at } \varphi = 0 \end{aligned} \qquad (36)$$

II. Equation Simplification--Linearized Case

For many wake or jet-like parallel flows the velocity within the wake differs only slightly from that in the local external flow. In these cases

$$\frac{\chi}{u_e^2} \ll 1$$

and we can write Equations (34) and (35) as

$$\frac{\partial \chi}{\partial \varphi} = \frac{\partial^2 \chi}{\partial \Psi^2} \qquad (37)$$

$$\frac{\partial K}{\partial \varphi} = \frac{1}{\sigma_K}\frac{\partial^2 K}{\partial \Psi^2} - \left(\frac{C}{\ell^2 u_e^2}\right) K + \frac{1}{4u_e^2}\left(\frac{\partial \chi}{\partial \Psi}\right)^2 \qquad (38)$$

subject to the identical boundary and initial conditions specified for Equations (34) and (35).

III. Equation Simplification--Nonlinearized Case

In free mixing cases, although the square of the velocity defect (χ) is small with respect to u_e^2, it is not negligible and some influence of the finiteness of the velocity defect must be included. With the governing equations in the (φ,Ψ) plane it is particularly convenient to include this effect. This is accomplished, first with the momentum equation, by expanding the dependent variables in powers of the parameter which makes χ small, namely some measure of the freestream velocity.

With the following definitions

$$S \equiv \left(\frac{u_0}{u_e}\right)^2 , \quad \epsilon \equiv \frac{1}{u_0^2} \tag{39}$$

the momentum equation becomes

$$\frac{\partial \chi}{\partial \varphi} = (1 - \epsilon\, S\, \chi)^{\frac{1}{2}} \frac{\partial^2 \chi}{\partial \Psi^2} \tag{40}$$

We are seeking here only a first order correction to the linearized set of equations, however, successive higher order approximations may be derived in exactly the same manner. Using the binomial theorem, we expand the diffusive term and, keeping only a first order correction to the linearized case, we can write the momentum equation

$$\frac{\partial \chi}{\partial \varphi} = (1 - \tfrac{1}{2}\, \epsilon\, S\, \chi) \frac{\partial^2 \chi}{\partial \Psi^2} \tag{41}$$

We now expand the dependent variable χ in a power series $\chi = \sum_{i=0}^{\infty} \epsilon^i \chi_i$ and, in line with the objective of obtaining a first order correction to the linearized system, we retain only the first two terms in the series which results in

$$\chi = \chi_0 + \epsilon\, \chi_1$$

Upon substituting this expression into Equation (41) and collecting terms of equal order in ϵ we obtain the following equations for χ_0, χ_1

$$\frac{\partial \chi_0}{\partial \varphi} = \frac{\partial^2 \chi_0}{\partial \Psi^2} \tag{42}$$

$$\frac{\partial \chi_1}{\partial \varphi} = \frac{\partial^2 \chi_1}{\partial \Psi^2} - \tfrac{1}{2} S \chi_0 \frac{\partial^2 \chi_0}{\partial \Psi^2} \tag{43}$$

Clearly the zeroth order equation is the strictly linearized case and the first order solution is the correction term for the finiteness of the velocity defect which tends to force the solution to satisfy the full equation.

Similarly, the linearized solution for K may be corrected for the improved velocity field solution and for the direct effect of a finite velocity defect. Here again we seek a first order correction for the linearized solution by expanding χ and K in Equation (35) in terms of ϵ. The expressions for u are expanded with the binomial theorem with use of the appropriate expressions for $\chi = \chi_0 + \epsilon \chi_1$, the appropriate zeroth and first order solutions to the momentum equation. Clearly the equation for K is now explicitly dependent upon ϵ and we can expand K also in powers of ϵ, again to first order, so that a correction to the linearized case may be obtained. Upon substituting

$$K = K_0 + \epsilon K_1$$

the respective equations for K_0 and K_1 can be obtained by collecting terms of equal order in ϵ. The equations for K_0 and K_1 can be written

$$\frac{\partial K_0}{\partial \varphi} = \frac{1}{\sigma_K} \frac{\partial^2 K_0}{\partial \Psi^2} - \frac{C}{\ell^2 u_e^2} K_0 + \frac{1}{4u_e^2} \left(\frac{\partial \chi_0}{\partial \Psi}\right)^2 \tag{44}$$

$$\frac{\partial K_1}{\partial \varphi} = \frac{1}{\sigma_K} \frac{\partial^2 K_1}{\partial \Psi^2} - \frac{C}{\ell^2 u_e^2} K_1 - \frac{1}{2\sigma_K} S \frac{\partial}{\partial \Psi}\left[\chi_0 \frac{\partial K_0}{\partial \Psi}\right] + \frac{1}{4u_e^2} \left(\frac{S\chi_0}{2}\right) \left(\frac{\partial \chi_0}{\partial \Psi}\right)^2 \tag{45}$$

$$+ \frac{1}{2u_e^2} \frac{\partial \chi_0}{\partial \Psi} \frac{\partial \chi_1}{\partial \Psi} - \frac{C}{\ell^2 u_e^2} \frac{S \chi_0 K_0}{2}$$

IV. Specification of $u_e(\varphi)$, $\ell(\varphi)$

To this point, whether a linearized or nonlinearized approach is taken, the solution can be developed analytically only providing u_e, ℓ are known functions of φ. We want to generate solutions for $\chi(\varphi,\Psi)$, $K(\varphi,\Psi)$, hypothesize a dependence of ν_t upon K and unwind the transformation via the definition of φ in the following manner.

$$d\varphi = \nu_t\, u_e\, dx \tag{46}$$

$$\int \frac{d\varphi}{\nu_t\ (K\,[\varphi,\Psi_j])} = \int u_e\, dx \tag{47}$$

which results implicitly in

$$f(\varphi) = g(x)$$

After performing this integration, we can solve for $\varphi(x)$ based on a given $u_e(x)$ and the hypothesized functional dependence for $\nu_t(K)$. The solutions can then be retransformed into physical variables and the validity of the hypotheses can be checked by comparison with the experimental u and K field.

In order to obtain expressions for u_e and ℓ as functions of φ, we must first specify a general flow field to be studied. The following development serves only as a guide toward the specification of a class of functions which $u_e(\varphi)$ and $\ell(\varphi)$ must be in order to adequately model realistic wake flows in pressure gradients. However, none of the approximations in the following section involves an actual specification of ν_t since u_e and ℓ could be specified as any arbitrary class of functions of φ and the actual physical flow field could later be inferred.

The flow external to the wake in the experimental phase of the investigation was flowing through a channel whose area ratio is

$$\frac{A}{A_0} = \frac{1}{\sqrt{1 + \beta x}} \tag{48}$$

where

$$\beta = \frac{\left(\frac{A_0}{A_L}\right)^2 - 1}{L} \tag{49}$$

For incompressible flow, this area distribution results in a velocity distribution

$$\frac{u_e}{u_0} = \sqrt{1 + \beta x} \tag{50}$$

with a resultant pressure gradient

$$\frac{dp}{dx} = - \beta q_0 \tag{51}$$

In order to map u_e and ℓ into the φ plane, we need a reliable approximate guide as to the shape of the ν_t function so as to specify the transformation from x to φ. From Schlichting (Ref. 6) planar incompressible constant pressure wakes may be approximately solved with a ν_t of the form

$$\nu_t = k\, u_e\, \theta \tag{52}$$

In this expression θ serves as the characteristic length and for weak wakes is a measure of the macroscale (ℓ) as is shown by the following development. From the definition of ψ we can write

$$\ell = \int^{\Psi_e} \frac{1}{\bar{\bar{u}}}\, d\Psi = \int_0^{\Psi_e} \left(1 - \frac{\chi}{u_e^2}\right)^{-\frac{1}{2}} d\Psi \tag{53}$$

which for weak wakes can be approximated by

$$u_e \ell \simeq \int_0^{\Psi_e} \left(1 + \frac{1}{2u_e^2}\chi\right) d\Psi \tag{54}$$

In addition from the definition of momentum thickness

$$\theta \equiv \int_0^{\infty} \frac{\bar{u}}{u_e}\left(1 - \frac{\bar{u}}{u_e}\right) dy = \int_0^{\Psi_e} \frac{1}{u_e}\left(1 - \frac{\bar{u}}{u_e}\right) d\Psi \tag{55}$$

the following approximate expression can be derived

$$u_e \theta \simeq \int_0^{\Psi_e} \frac{1}{2u_e^2}\chi\, d\Psi \tag{56}$$

By comparing Equations 54 and 56

$$u_e \ell = \Psi_e + u_e \theta \tag{57}$$

Since mass flux in the wake and hence Ψ_e will be approximately conserved, θ will reflect the functional dependence of the macroscale on x.

In order to obtain an expression for θ we can return to the momentum integral equation written with the specified external field which generates a constant pressure gradient $\frac{dp}{dx} = - \beta q_0$. The momentum integral equation

$$\frac{d\theta}{dx} + \frac{\theta}{U_e} \frac{dU_e}{dx} [H + 2] = 0 \tag{58}$$

can be integrated for a constant shape factor (H) resulting in

$$\frac{\theta}{\theta_0} = \frac{1}{(1 + \beta x)^{1 + H/2}} \tag{59}$$

With this external velocity field, the Schlichting model for ν_t, and the solution for θ we can write the transformation as

$$\varphi = k \int_0^x u_e^2 \theta \, dx \tag{60}$$

after inserting the expressions for θ and u_e^2 we obtain

$$\varphi = \frac{k\theta_0 u_0^2}{\beta(1 - H/2)} \left[(1 + \beta x)^{1 - H/2} - 1\right] \tag{61}$$

Using this relationship, the expressions for $u_e(x)$ and $\theta(x)$ can be transformed into $u_e(\varphi)$ and $\theta(\varphi)$

$$u_e(\varphi) = u_0 \left(1 - \frac{\beta}{\left(\frac{2}{H - 2}\right) k\theta_0 u_0^2} \varphi\right)^{1/(2 - H)} \tag{62}$$

$$\theta(\varphi) = \theta_0 \left(1 - \frac{\beta}{\left(\frac{2}{H - 2}\right) k\theta_o u_0^2} \varphi\right)^{(H + 2)/(H - 2)} \tag{63}$$

It should now be clear that we can consider the system of governing equations in the problem to be closed and well defined since all functional coefficients have been specified as functions of φ. Namely $S(\varphi)$ and $\ell(\varphi)$

$$S(\varphi) = \left(\frac{u_0}{u_e}\right)^2 = \left(1 - \frac{\beta}{\left(\frac{2}{H - 2}\right) k\theta_0 u_0} \varphi\right)^{2/(H - 2)} \tag{64}$$

and from Equation (57)

$$\ell(\varphi)u_e(\varphi) = \Psi_e + u_0\theta_0 \left(1 - \frac{\beta}{\left(\frac{2}{H - 2}\right) k\theta_0 u_0^2} \varphi\right)^{(H + 1)/(H - 2)} \tag{65}$$

The specification of the value of the shape factor (H) to be used must be consistent with the formulation and the inherent approximations. From the definition of H we can write

$$H \equiv \frac{\delta^*}{\theta} = \frac{\int \left(1 - \frac{\bar{u}}{u_e}\right) dy}{\int \frac{\bar{u}}{u_e} \left(1 - \frac{\bar{u}}{u_e}\right) dy} = \frac{\int \frac{1}{\bar{u}} \left(1 - \frac{\bar{u}}{u_e}\right) d\Psi}{\int \frac{1}{u_e} \left(1 - \frac{\bar{u}}{u_e}\right) d\Psi} \tag{66}$$

$$H = \frac{\int \left(\frac{u_e}{\bar{u}} - 1\right) d\Psi}{\int \left(1 = \frac{\bar{u}}{u_e}\right) d\Psi} \tag{67}$$

$$H = \frac{\int \left[\left(1 - \frac{\chi}{u_e^2}\right)^{-\frac{1}{2}} - 1\right] d\Psi}{\int \left[1 - \left(1 - \frac{\chi}{u_e^2}\right)^{\frac{1}{2}}\right] d\Psi} \tag{68}$$

Expanding the integrand, consistent with the order of the analysis we obtain

$$H \simeq \frac{\int \left[1 + \frac{\chi}{2u_e^2} - 1\right] d\Psi}{\int \left[1 - 1 + \frac{\chi}{2u_e^2}\right] d\Psi} = 1 \tag{69}$$

If we insert this value of H into Equations (61), (64), and (65) we can write

$$\varphi = \frac{2k\theta_0 u_0^2}{\beta} \left[(1 + \beta x)^{\frac{1}{2}} - 1\right] \tag{70}$$

$$S(\varphi) = \left(\frac{U_0}{U_e}\right)^2 = \left(1 + \frac{\beta}{2k\theta_0 u_0^2} \varphi\right)^{-2} \tag{71}$$

$$\ell(\varphi)\ u_e(\varphi) = \Psi_e + u_0\theta_0 \left(1 + \frac{\beta}{2k\theta_0 u_0^2} \varphi\right)^{-2} \tag{72}$$

SOLUTION OF THE EQUATIONS

The general form for each of the linearized and nonlinearized equations for both K, χ can be written

$$\frac{\partial F}{\partial \varphi} = \frac{\partial^2 F}{\partial \Psi^2} + f(\varphi)F + g(\varphi, \Psi) \tag{73}$$

where $f(\varphi) = 0$ for the momentum equations.

If we let $F^* = Fe^{-\int f(\varphi)d\varphi}$, simple differentiation verifies that the solution to the equation

$$\frac{\partial F^*}{\partial \varphi} = \frac{\partial^2 F^*}{\partial \Psi^2} + g\ (\varphi,\ \Psi)e^{-\int f(\varphi)d\varphi} \tag{74}$$

satisfies the general form of Equation (73).

The general solution of Equation (74) can be written in terms of Greens function $G(x,y;\varphi,\Psi)$ as

$$F^*(\varphi,\Psi) = \int_{-\infty}^{+\infty} G(0,y;\varphi,\Psi)\ I\ (y)\ dy + \int_0^{\varphi} \int_{-\infty}^{+\infty} G(x,y;\varphi,\Psi)\ J\ (x,y)\ dx\ dy\ . \tag{75}$$

where the Greens function appropriate to the $\frac{\partial}{\partial \varphi} - \frac{\partial^2}{\partial \Psi^2}$ operator is

$$G(x,y;\varphi,\Psi) = \frac{1}{\sqrt{2\pi(\varphi - x)}} \exp\left[-\frac{(\Psi - y)^2}{4(\varphi - x)}\right] . \tag{76}$$

Once the appropriate initial conditions (I) and nonhomogeneous functions (J) are inserted and integrated, the solution of F is

$$F = F^* e^{\int f(\varphi)d\varphi} \tag{77}$$

COMPARISON WITH EXPERIMENT

In order to test the adequacy with which the solutions presented in Equations (73), (75), and (77) predict a turbulent free mixing flow field, a sample quadrature was performed which could be compared with experiment. Figure 1 presents the initial conditions for χ and K which were used in Equation (75) to solve for χ_0, χ_1, K_0, and K_1 in constant adverse and favorable pressure gradients. Figures 1, 2, and 4 contain the results for a constant favorable pressure gradient with the experimental conditions in indicated on the figures. Figures 1, 3, and 4 contain the results for the same calculations with a constant adverse pressure gradient of the same magnitude. The mathematical retransformation plots in both cases (Figures 4 and 5) were used with the eddy viscosity model

$$\nu_t \sim \sqrt{K}\ \ell \tag{78}$$

to test the ability of this model to reproduce experimental results for wakes in constant pressure gradients. The comparison of the analytical results with experiment shown in Figure 5 verifies that, with some further study of the proper values of the empirical constants, the approach outlined here presents a consistent method for testing wake-type eddy viscosity models with streamwise pressure gradients and predicting untested physical situations with reliable eddy viscosity models.

Although the analytical form of the solution has been obtained, often the initial conditions are discrete and do not satisfactorily fit any known analytic functions. In these cases a numerical solution of the governing equations can be easily obtained and the computer program needed to perform such a computation is listed and explained in the Appendix.

CONCLUSIONS

The equations of momentum and turbulent kinetic energy appropriate to free turbulent mixing have been developed. The resultant equations have been transformed into a plane which is independent of the eddy viscosity model and have been analytically solved by a perturbation technique. The solution depends upon prior knowledge of $u_e(\varphi)$ and $\ell(\varphi)$ and to specify these, an approximation for $\varphi(x)$ must be available in order to study a flow field which is specified a priori. For this analysis, the wake model given by Equation (52) proved to be satisfactory. If, on the other hand, only general classes of flow fields are of interest, $u_e(\varphi)$ and $\ell(\varphi)$ may be specified independent of a specific physical problem with the resultant re-transformation determining the flow field which they implied. The results of the analysis have been compared to experimental wake data in constant adverse and favorable pressure gradients. The results of that comparison indicate that the analysis supplies a satisfactory method for obtaining analytical solutions to wake problems in freestream pressure gradients.

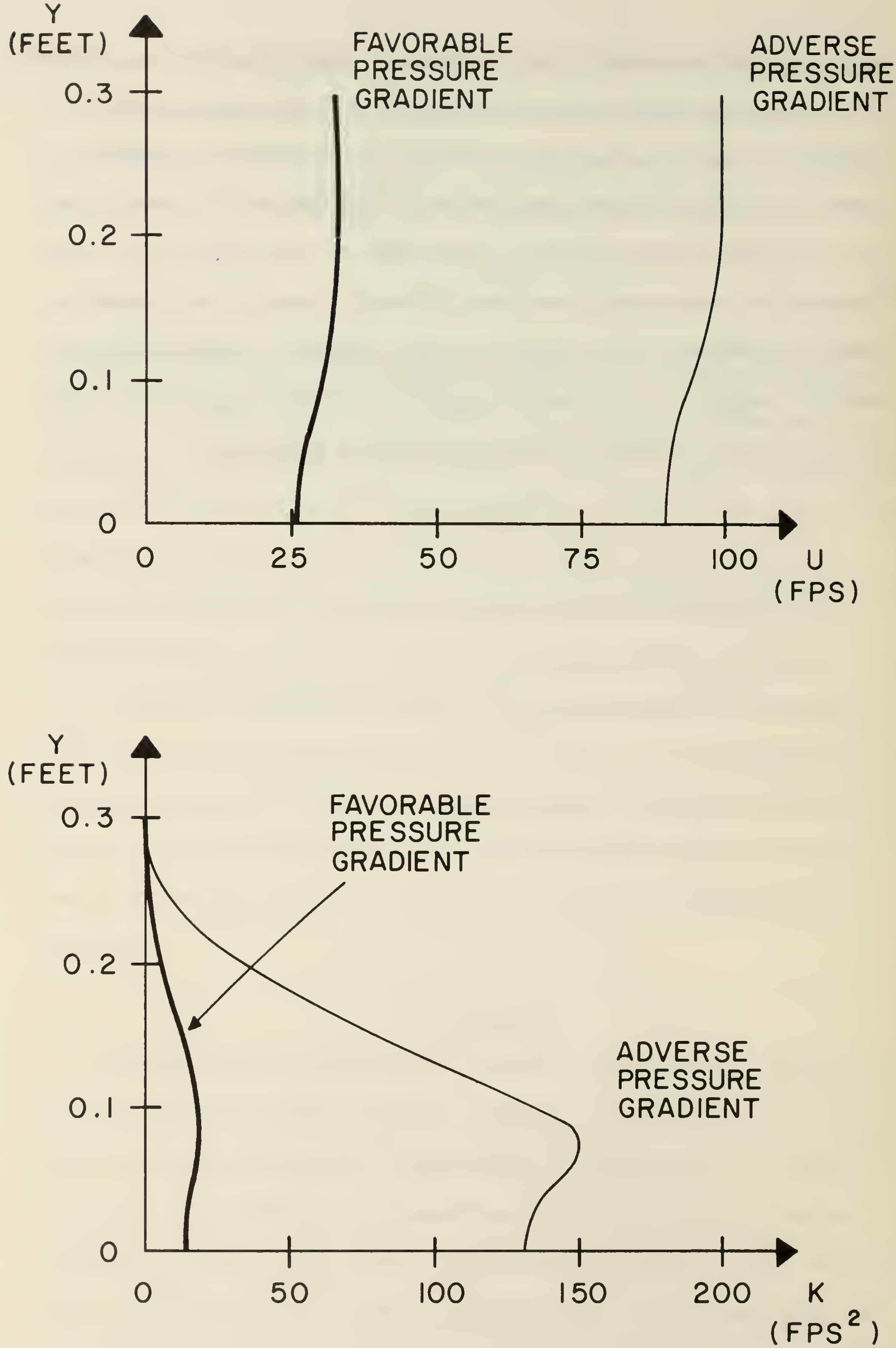

FIG. 1

INITIAL CONDITIONS FOR SAMPLE CALCULATIONS.

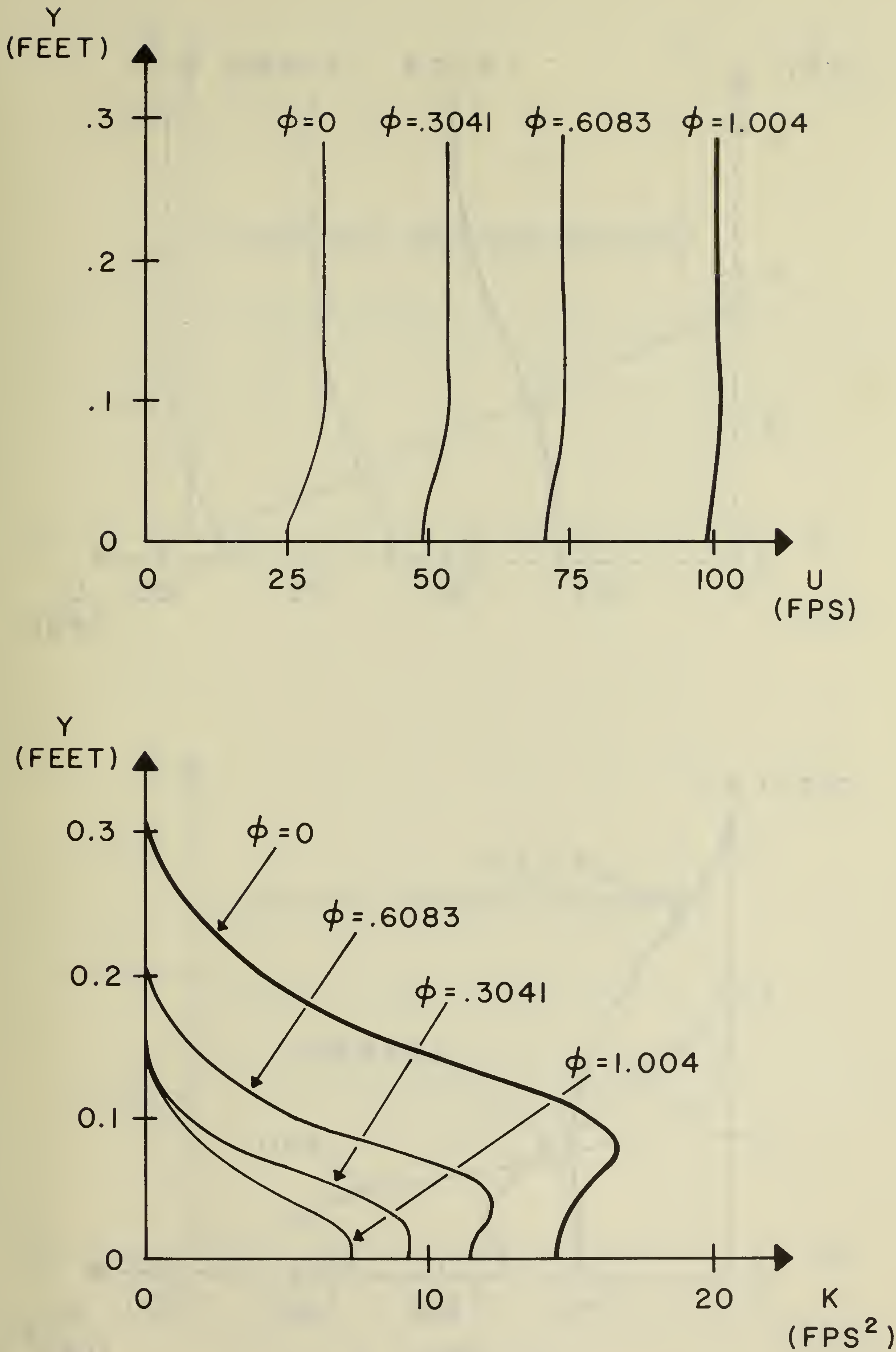

FIG. 2

SOLUTION FIELD FOR FAVORABLE PRESSURE GRADIENT.

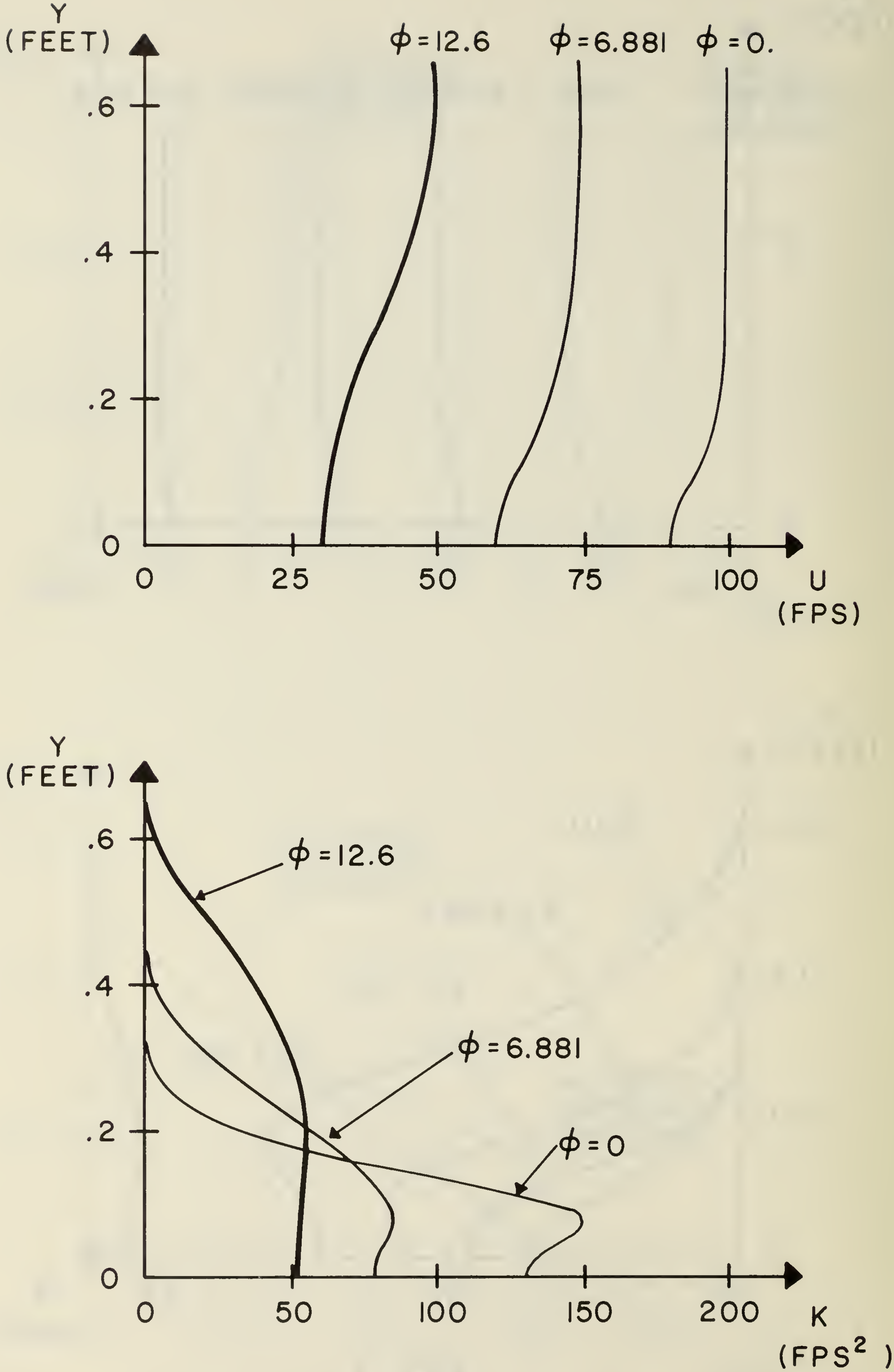

FIG. 3

SOLUTION FIELD FOR ADVERSE PRESSURE GRADIENT.

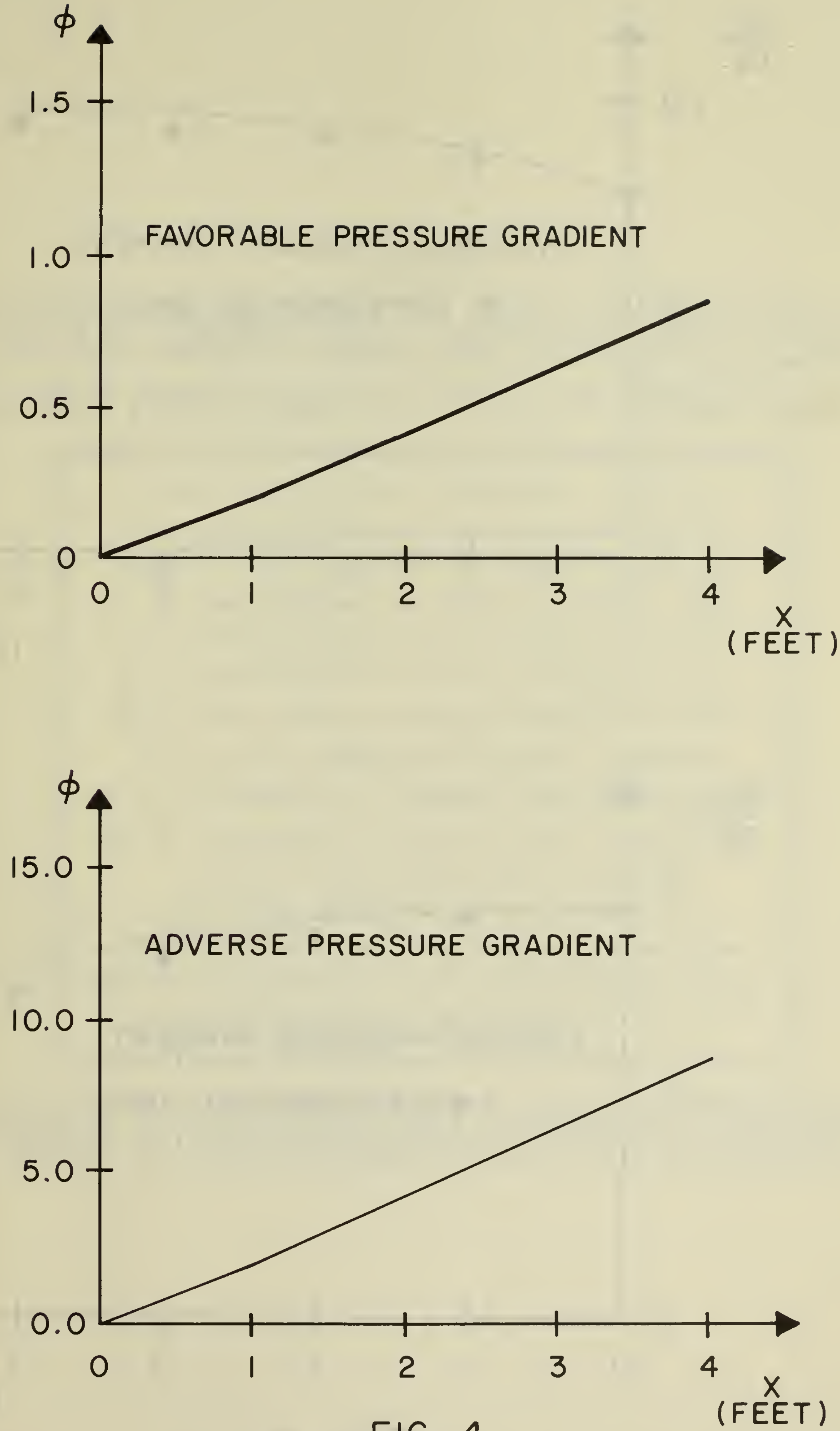

FIG. 4

TRANSFORMATION PLOTS FROM ϕ TO X.

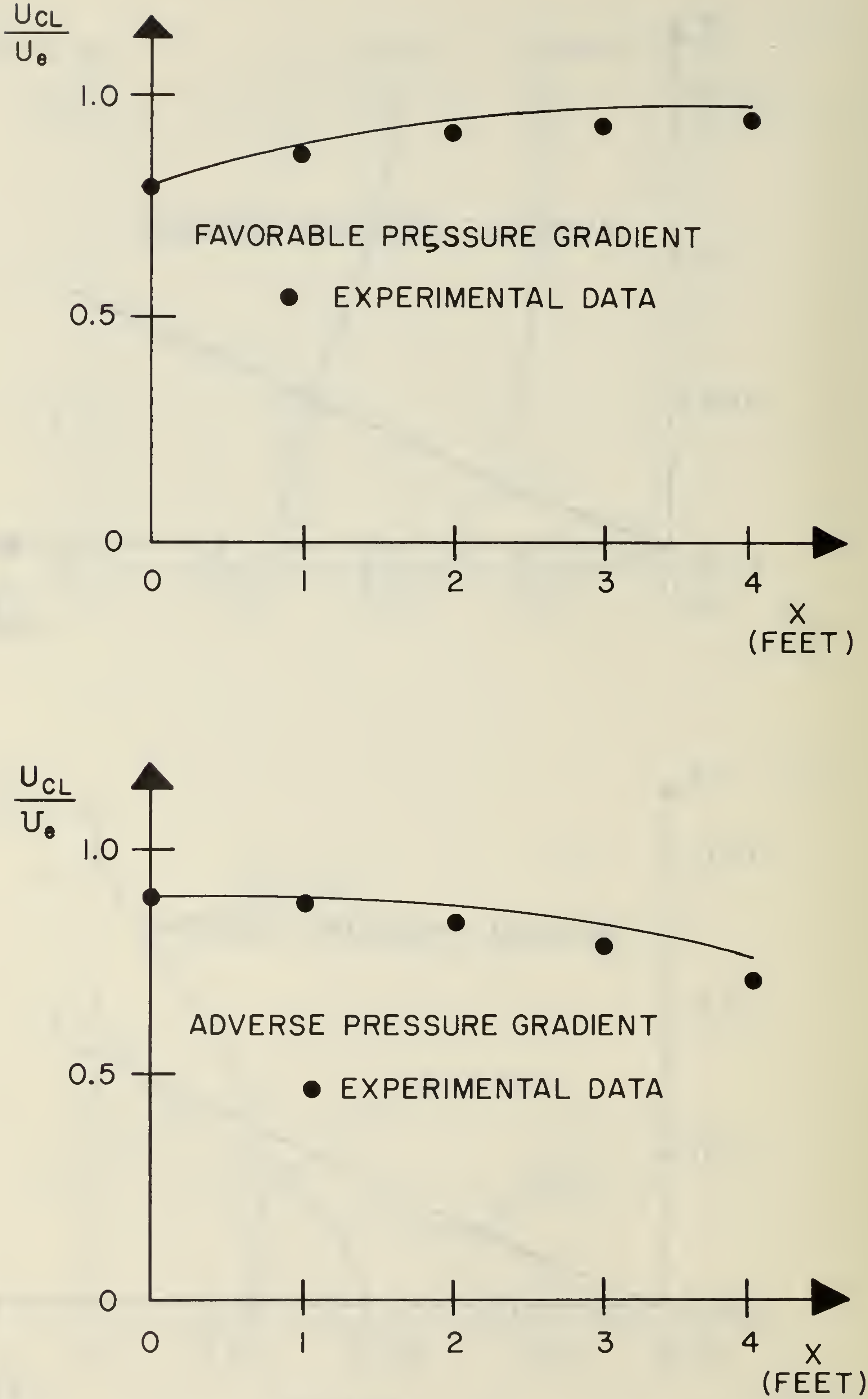

FIG. 5

COMPARISON OF ANALYTICAL RESULTS WITH EXPERIMENT.

```
C                                  APPENDIX
C
C THE FOLLOWING COMPUTER PROGRAM IS A FORTRAN IV CODED FINITE DIFFERENCE
C CALCULATION OF THE ZEROTH AND FIRST ORDER SOLUTIONS TO THE EQUATIONS
C OF MOMENTUM AND TURBULENT KINETIC ENERGY APPLIED TO THE PROBLEM OF
C FREE TURBULENT MIXING IN FAVORABLE AND ADVERSE PRESSURE GRADIENTS
C
C THE FOLLOWING PARAMETERS MUST BE INPUT AT THE BEGINNING OF THE DECK
C    UINF U(1,J) EK1(1,J) SIGMA C C1 AL AO ALL MY MX DELY DELX D1 AKK
C
C          **************************************************
C          *                                                *
C          *      UINF IS THE INLET FREESTREAM VELOCITY     *
C          *                                                *
C          *     UINF ALSO SERVES AS THE PERTURBATION UO    *
C          *                                                *
C          *    SIGMA IS THE CONSTANT IN ENERGY DIFFUSION   *
C          *                                                *
C          *    C IS THE CONSTANT IN THE DISSIPATION TERM   *
C          *                                                *
C          *    C1 IS THE CONSTANT IN THE EDDY VISCOSITY    *
C          *                                                *
C          *    AL IS THE AREA OF THE TEST SECTION AT X=L   *
C          *                                                *
C          *    AO IS THE AREA OF THE TEST SECTION AT X=O   *
C          *                                                *
C          *      ALL IS THE LENGTH OF THE TEST SECTION     *
C          *                                                *
C          *  MY IS THE NUMBER OF LATERAL POINTS CALCULATED *
C          *                                                *
C          *   MX IS THE NUMBER OF AXIAL POINTS CALCULATED  *
C          *                                                *
C          *         DELY IS THE LATERAL GRID SPACING       *
C          *                                                *
C          *       D1 IS THE CONSTANT IN THE STEP SIZE      *
C          *                                                *
C          *     DELX IS THE AXIAL SPACING TO RETRANSFORM   *
C          *                                                *
C          *  AKK IS THE CONSTANT IN THE APPROXIMATE PHI(X) *
C          *                                                *
C          *     MUMMY2 IS AN INDEX FOR A PARTICULAR PSI    *
C          *                                                *
C          **************************************************
C
      DIMENSION CHI1(200,81),CHI2(200,81),AK1(200,81),AK2(200,81)
      DIMENSION THETA(200),DELSTR(200),SHAPE(200),PSIE(200),ELL(200)
      DIMENSION PHI(200),P2(200),P4(200),X(200),FF(200),GG(200)
      DIMENSION APSI(81),AU(81),AK(81),Y(81),U(81),URATIO(81),EK1(81)
```

```
      DIMENSION PSI(81)
C
      DO 1 I=1,MX
      DO 1 J=1,MY
C
      CHI1(I,J)=0.0
      CHI2(I,J)=0.0
      AK1(I,J)=0.0
      AK2(I,J)=0.0
C
      THETA(I)=0.0
      DELSTR(I)=0.0
      SHAPE(I)=0.0
      PSIE(I)=0.0
      ELL(I)=0.0
      PHI(I)=0.0
      PSI(I)=0.0
      P2(I)=0.0
      P4(I)=0.0
      X(I)=0.0
      FF(I)=0.0
      GG(I)=0.0
C
      APSI(J)=0.0
      AU(J)=0.0
      AK(J)=0.0
      Y(J)=0.0
      U(J)=0.0
      URATIO(J)=0.0
      EK1(J)=0.0
C
    1 CONTINUE
C
C THE FOLLOWING METHOD OF INPUTING U AND  EK1 USES A "STANDARD" SHAPE
C FOR BOTH THE VELOCITY AND TURBULENT KINETIC ENERGY PROFILES FOR USE
C IN TEST CASES. IF THIS INPUT IS DESIRED ADDITIONAL INPUTS ARE NEEDED.
C UCL-THE CENTERLINE VELOCITY AT THE INLET, EKCL-THE TURBULENT KINETIC
C ENERGY ON THE CENTERLINE AT THE INLET, AND EKMAX-THE MAXIMUM TURBULENT
C KINETIC ENERGY AT THE INLET. THESE VALUES SCALE THE SHAPE TO PROVIDE
C SUITABLE INITIAL CONDITIONS FOR USE IN EVALUATING THE CONSTANTS AND
C SPECIFYING A STEP SIZE WHICH WILL BE BOTH STABLE AND ACCURATE
C
      U(1)=100.0
      U(2)=99.0
      U(3)=98.0
      U(4)=96.0
      U(5)=92.0
      U(6)=88.0
```

```
      U(7)=81.0
      U(8)=73.0
      U(9)=68.0
      U(10)=59.0
      U(11)=52.0
      U(12)=48.0
      U(13)=41.0
      U(14)=35.0
      U(15)=30.0
      U(16)=26.0
      U(17)=21.0
      U(18)=18.0
      U(19)=14.0
      U(20)=11.0
      U(21)=8.0
      U(22)=5.0
      U(23)=3.0
      U(24)=2.0
      U(25)=1.0
      U(26)=0.5
      U(27)=0.25
      U(28)=0.0
C
      ALPHA=1.0-(UCL/UINF)
C
      DO 2 J=1,51
      U(J)=UINF*(1.0-ALPHA*U(J)/100.0)
    2 CONTINUE
C
      EK1(1)=80.0
      EK1(2)=81.0
      EK1(3)=82.0
      EK1(4)=84.0
      EK1(5)=87.0
      EK1(6)=90.0
      EK1(7)=95.0
      EK1(8)=98.0
      EK1(9)=100.0
      EK1(10)=98.0
      EK1(11)=92.0
      EK1(12)=83.0
      EK1(13)=75.0
      EK1(14)=67.0
      EK1(15)=60.0
      EK1(16)=53.0
      EK1(17)=48.0
      EK1(18)=41.0
      EK1(19)=36.0
```

```
      EK1(20)=32.0
      EK1(21)=26.0
      EK1(22)=21.0
      EK1(23)=17.0
      EK1(24)=13.0
      EK1(25)=10.0
      EK1(26)=8.0
      EK1(27)=6.0
      EK1(28)=4.0
      EK1(29)=2.0
      EK1(30)=1.0
      EK1(31)=0.25
      EK1(32)=0.0
C
      GAMMA=5.0*(1.0-(EKCL/EKMAX))
      DO 344 J=1,9
      EK1(J)=EKMAX*(1.0-(1.0-EK1(J)/100.0)*GAMMA)
  344 CONTINUE
      DO 345 J=10,51
      EK1(J)=EKMAX*EK1(J)/100.0
  345 CONTINUE
C
      UINF2=UINF**2
      C=C/C1
      BETA=((AO/AL)**2-1.0)/ALL
C
C     SET UP INITIAL Y FIELD
      Y(1)=0.0
      DO 50 J=2,MY
      Y(J)=FLOAT(J-1)*DELY
   50 CONTINUE
C
C     SET UP INITIAL PSI FIELD
      PSI(1)=0.0
      PSI(2)=(U(1)+U(2))*0.5*DELY
      DO 51 K=3,MY
      K1=K-1
      A=0.5*(U(1)+U(K))
      B=0.0
      DO 52 L=2,K1
      B=U(L)+B
   52 CONTINUE
      PSI(K)=DELY*(A+B)
   51 CONTINUE
C
      MY1=MY-1
      DELPSI=PSI(MY)/MY1
C
```

```
C      SET UP UNIFORM PSI FIELD
       APSI(1)=0.0
       DO 53 J=2,MY
       APSI(J)=FLOAT(J-1)*DELPSI
   53  CONTINUE
C
       WRITE(6,995)
  995  FORMAT(1H1,40X,'**** TABULATED INPUT VARIABLES IN PHYSICAL COORDIN
      1ATES ****')
C
       WRITE(6,770)
  770  FORMAT(1H0,25X,'VELOCITY',10X,'TURBULENT KINETIC ENERGY',10X,'LATE
      1RAL POSITION',10X,'STREAM FUNCTION')
C
       DO 880 J=1,MY
       WRITE(6,996) U(J),EK1(J),Y(J),APSI(J)
  996  FORMAT(1H0,25X,E15.7,10X,E15.7,10X,E15.7,10X,E15.7)
  880  CONTINUE
C
C      REDISTRIBUTE THE INPUT VALUES IN A UNIFORM PSI FIELD
       AU(1)=U(1)
       AK(1)=EK1(1)
       DO 54 J=2,MY
       POS=APSI(J)
       AU(J)=PIF2(POS,PSI,MY,U)
       AK(J)=PIF2(POS,PSI,MY,EK1)
   54  CONTINUE
C
C      SET THE VALUES FOR THE DEPENDENT VARIABLES OF THE CALCULATION
       DO 55 J=1,MY
       CHI1(1,J)=UINF**2-AU(J)**2
       AK1(1,J)=AK(J)
       PSI(J)=APSI(J)
       U(J)=AU(J)
   55  CONTINUE
C
       DO 40 J=1,MY
       MUMMY=J
       MUMMY1=J-1
       IF(CHI1(1,J).LT.0.1) GO TO 41
   40  CONTINUE
   41  CONTINUE
C
       PSIE(1)=PSI(MUMMY)
       UE =U(MUMMY)
       UE2=UE**2
C
C      SET UP NEW Y FIELD BASED ON UNIFROM PSI FIELD
```

```
      Y(1)=0.0
      Y(2)=((U(1)+U(2))/(2.0*U(1)*U(2)))*DELPSI
      DO 56 K=3,MY
      K1=K-1
      A=0.5*((1.0/U(1))+(1.0/U(K)))
      B=0.0
      DO 57 L=2,K1
      B=(1.0/U(L))+B
   57 CONTINUE
      Y(K)=DELPSI*(A+B)
   56 CONTINUE
C
      ELL(1)=Y(MUMMY)
C
      DO 43 J=1,MY
      URATIO(J)=SQRT(1.0-CHI1(1,J)/UE2)
   43 CONTINUE
C
C     CALCULATE DISPLACEMENT THICKNESS
      A=0.5*((1.0/URATIO(1))+(1.0/URATIO(MUMMY))-2.0)
      B=0.0
      DO 44 J=2,MUMMY1
      B=((1.0/URATIO(J))-1.0)+B
   44 CONTINUE
      DELSTR(1)=DELPSI*(A+B)/UE
C
C     CALCULATE MOMENTUM THICKNESS
      A=0.5*(2.0-URATIO(1)-URATIO(MUMMY))
      B=0.0
      DO 45 J=2,MUMMY1
      B=(1.0-URATIO(J))+B
   45 CONTINUE
      THETA(1)=DELPSI*(A+B)/UE
C
      THETO=THETA(1)
C
      SHAPE(1)=DELSTR(1)/THETA(1)
C
      WRITE(6,997)
  997 FORMAT(1H1,40X,'**** TABULATED INPUT VARIABLES IN STREAM FUNCTION
     1COORDINATES ****')
C
      WRITE(6,666) PHI(1),THETA(1),SHAPE(1),DELSTR(1),ELL(1),PSIE(1)
  666 FORMAT(1H0,2X,'PHI =',E10.4,2X,'THETA =',E10.4,2X,'SHAPE FACTOR ='
     1,E10.4,2X,'DELSTAR =',E10.4,2X,'MACROSCALE =',E10.4,2X,'MASS FLUX
     2=',E10.4)
C
      WRITE(6,771)
```

```
  771 FORMAT(1HO,20X,'CHI',15X,'VELOCITY',10X,'TURBULENT KINETIC ENERGY'
     1,10X,'LATERAL POSITICN',10X,'STREAM FUNCTICN')
C
      DO 881 J=1,MY
      WRITE(6,998) CHI1(1,J),AU(J),AK1(1,J),Y(J),PSI(J)
  998 FORMAT(1HO,10X,E15.7,10X,E15.7,10X,E15.7,10X,E15.7,10X,E15.7)
  881 CONTINUE
C
C     SET STABLE STEP SIZE FOR DELPHI
      DELPHI=D1*(DELPSI**2)
C
      PHI(1)=0.0
      DO 5 I=2,MX
      PHI(I)=FLOAT(I-1)*DELPHI
    5 CONTINUE
C
      AMEGA=0.5*BETA/(AKK*THETO*(UINF**2))
C
      DO 7 I=1,MX
      P2(I)=1.0/((1.0+AMEGA*PHI(I))**2)
      P4(I)=1.0/((1.0+AMEGA*PHI(I))**4)
    7 CONTINUE
C
      MX1=MX-1
C
C     START MAIN CALCULATICN LOOP
      DO 20 I=1,MX1
C
      F=-C/((PSIE(I)+UINF*THETO*P2(I))**2)
C
      DO 10 J=2,MY1
      DERIV=(CHI1(I,J+1)+CHI1(I,J-1)-2.0*CHI1(I,J))/(DELPSI**2)
      CHI1(I+1,J)=CHI1(I,J)+DELPHI*DERIV
   10 CONTINUE
C
      DERIV=2.0*(CHI1(I,2)-CHI1(I,1))/(DELPSI**2)
      CHI1(I+1,1)=CHI1(I,1)+DELPHI*DERIV
C
      DO 11 J=2,MY1
      G=-0.5*P2(I)*CHI1(I,J)*(CHI1(I,J+1)-2.0*CHI1(I,J)+CHI1(I,J-1))/
     1(DELPSI**2)
      DERIV=(CHI2(I,J+1)-2.0*CHI2(I,J)+CHI2(I,J-1))/(DELPSI**2)
      CHI2(I+1,J)=CHI2(I,J)+DELPHI*(DERIV+G)
   11 CONTINUE
C
      G=-0.5*P2(I)*CHI1(I,1)*2.0*(CHI1(I,2)-CHI1(I,1))/(DELPSI**2)
      DERIV=2.0*(CHI2(I,2)-CHI2(I,1))/(DELPSI**2)
      CHI2(I+1,1)=CHI2(I,1)+DELPHI*(DERIV+G)
```

```
C
      DO 12 J=2,MY1
      G=0.25*P2(I)*(((CHI1(I,J+1)-CHI1(I,J-1))/(2.0*DELPSI*UINF))**2)
      DERIV=(AK1(I,J+1)-2.0*AK1(I,J)+AK1(I,J-1))/(DELPSI**2)
      AK1(I+1,J)=AK1(I,J)+DELPHI*((DERIV/SIGMA)+F*AK1(I,J)+G)
   12 CONTINUE
C
      DERIV=2.0*(AK1(I,2)-AK1(I,1))/(DELPSI**2)
      AK1(I+1,1)=AK1(I,1)+DELPHI*((DERIV/SIGMA)+F*AK1(I,1))
C
      DO 13 J=2,MY1
      G1A=CHI1(I,J)*(AK1(I,J+1)-2.0*AK1(I,J)+AK1(I,J-1))/(DELPSI**2)
      G1B=(CHI1(I,J+1)-CHI1(I,J-1))*(AK1(I,J+1)-AK1(I,J-1))/(4.0*
     1(DELPSI**2))
      G1=-0.5*P2(I)*(G1A+G1B)
      G2=0.125*P4(I)*CHI1(I,J)*(((CHI1(I,J+1)-CHI1(I,J-1))/(2.0*DELPSI*
     1UINF))**2)
      G3=0.5*P2(I)*(CHI1(I,J+1)-CHI1(I,J-1))*(CHI2(I,J+1)-CHI2(I,J-1))/
     1(4.0*(DELPSI**2)*(UINF**2))
      G4=-0.5*C*P4(I)*CHI1(I,J)*AK1(I,J)/((PSIE(I)+UINF*THETO*P2(I))**2)
      G=G1+G2+G3+G4
      DERIV=(AK2(I,J+1)-2.0*AK2(I,J)+AK2(I,J-1))/(DELPSI**2)
      AK2(I+1,J)=AK2(I,J)+DELPHI*((DERIV/SIGMA)+F*AK2(I,J)+G)
   13 CONTINUE
C
      G1A=CHI1(I,1)*2.0*(AK1(I,2)-AK1(I,1))/(DELPSI**2)
      G1B=0.0
      G1=-0.5*P2(I)*(G1A+G1B)
      G2=0.0
      G3=0.0
      G4=-0.5*C*P4(I)*CHI1(I,1)*AK1(I,1)/((PSIE(I)+UINF*THETO*P2(I))**2)
      G=G1+G2+G3+G4
      DERIV=2.0*(AK2(I,2)-AK2(I,1))/(DELPSI**2)
      AK2(I+1,1)=AK2(I,1)+DELPHI*((DERIV/SIGMA)+F*AK2(I,1)+G)
C     END OF MAIN CALCULATION LOOP
C
      DO 75 J=1,MY
      U(J)=SQRT(UINF2/P2(I)-CHI1(I+1,J))
      AU(J)=SQRT(UINF2/P2(I)-(CHI1(I+1,J)+CHI2(I+1,J)/UINF2))
   75 CONTINUE
C
      DO 61 J=1,MY
      MUMMY=J
      MUMMY1=J-1
      IF(CHI1(I+1,J).LT.0.1) GO TO 70
   61 CONTINUE
C
   70 CONTINUE
```

```
C
      MUMMY2=MUMMY/2
C
      PSIE(I+1)=PSI(MUMMY)
C
      UE=U(MUMMY)
      UE2=UE**2
C
C     CALCULATE NEW Y FIELD
      Y(1)=0.0
      Y(2)=0.5*DELPSI*(1.0/U(1)+1.0/U(2))
      DO 59 K=3,MY
      K1=K-1
      A=0.5*(1.0/U(1)+1.0/U(K))
      B=0.0
      DO 60 L=2,K1
      B=1.0/U(L)+B
   60 CONTINUE
      Y(K)=DELPSI*(A+B)
   59 CONTINUE
C
      ELL(I+1)=Y(MUMMY)
C
      DO 62 J=1,MY
      URATIO(J)=SQRT(1.0-CHI1(I+1,J)/UE2)
      APSI(J)=SQRT(1.0-(CHI1(I+1,J)+CHI2(I+1,J)/UINF2)/UE2)
      EK1(J)=AK1(I+1,J)+AK2(I+1,J)/UINF2
   62 CONTINUE
C
      A=0.5*((1.0/URATIO(1))+(1.0/URATIO(MUMMY))-2.0)
      B=0.0
      DO 63 J=2,MUMMY1
      B=((1.0/URATIO(J))-1.0)+B
   63 CONTINUE
      DELSTR(I+1)=DELPSI*(A+B)/UE
C
      A=0.5*(2.0-URATIO(1)-URATIO(MUMMY))
      B=0.0
      DO 64 J=2,MUMMY1
      B=(1.0-URATIO(J))+B
   64 CONTINUE
      THETA(I+1)=DELPSI*(A+B)/UE
C
      SHAPE(I+1)=DELSTR(I+1)/THETA(I+1)
C
C
      WRITE(6,991)
  991 FORMAT(1H1,40X,'**** INTEGRATED PROFILE PARAMETERS ****')
```

```
C
      WRITE(6,992) PHI(I+1),THETA(I+1),SHAPE(I+1),DELSTR(I+1),ELL(I+1),
     1PSIE(I+1)
  992 FORMAT(1H0,2X,'PHI =',E10.4,2X,'THETA =',E10.4,2X,'SHAPE FACTOR ='
     1,E10.4,2X,'DELSTAR =',E10.4,2X,'MACROSCALE =',E10.4,2X,'MASS FLUX
     2=',E10.4)
C
      WRITE(6,999)
  999 FORMAT(1H0,40X,'**** CALCULATED OUTPUT VARIABLES ****')
C
      WRITE(6,772)
  772 FORMAT(1H0,'J',8X,'U/UE',8X,'UT/UE',8X,'U',8X,'UT',8X,'Y',8X,'PSI'
     1,8X,'CHI1',8X,'CHI2',8X,'AK1',8X,'AK2',8X,'AKT')
C
      DO 883 J=1,MY
      WRITE(6,990) J,URATIO(J),APSI(J),U(J),AU(J),Y(J),PSI(J),CHI1(I+1,J
     1),CHI2(I+1,J),AK1(I+1,J),AK2(I+1,J),EK1(J)
  990 FORMAT(1H0,I2,2X,F5.3,2X,F5.3,2X,F7.3,2X,F7.3,2X,F5.3,2X,F7.3,2X,E
     112.5,2X,E12.5,2X,E12.5,2X,E12.5,2X,E12.5)
  883 CONTINUE
      ICALC=I
      IF(U(1).LT.5.0) GO TO 21
C
   20 CONTINUE
C
   21 CONTINUE
      WRITE(6,993)
  993 FORMAT(1H1,'**** SUMMARY OF INTEGRATED QUANTITIES ****')
C
      DO 874 I=1,ICALC
      WRITE(6,994) THETA(I),DELSTR(I),SHAPE(I),PSIE(I),ELL(I),I
  994 FORMAT(1H0,5E20.8,I3)
  874 CONTINUE
C
      X(1)=0.0
      DO 869 I=2,MX
      X(I)=FLOAT(I-1)*DELX
  869 CONTINUE
C
      FF(1)=0.0
      DO 870 I=2,MX
      FF(I)=2.0*C1*UINF*((1.0+BETA*X(I))**1.5-1.0)/(3.0*BETA)
  870 CONTINUE
C
      GG(1)=0.0
      GG(2)=0.5*( 1.0/(SQRT(AK1(1,MUMMY2))*ELL(1))+ 1.0/(SQRT(AK1(2,MUMM
     1Y2))*ELL(2)))*DELPHI
      DO 871 I=3,ICALC
```

```
      I1=I-1
      A=0.5*(1.0/(SQRT(AK1(1,MUMMY2))*ELL(1))+1.0/(SQRT(AK1(I,MUMMY2))*
     1ELL(I)))
      B=0.0
      DO 872 K=1,I1
      B=1.0/(SQRT(AK1(I,MUMMY2))*ELL(K))+B
  872 CONTINUE
      GG(I)=DELPHI*(A+B)
  871 CONTINUE
      DO 877 I=ICALC,MX
      GG(I)=0.0
  877 CONTINUE
C
      WRITE(6,850) C1
  850 FORMAT(1H1,30X,'RETRANSFORMATION RESULTS FOR C1 =',E10.3)
C
      WRITE(6,848)
  848 FORMAT(1H0,30X,'F',20X,'G',20X,'PHI',20X,'X')
C
      DO 849 I=1,MX
      WRITE(6,873) FF(I),GG(I),PHI(I),X(I)
  873 FORMAT(1H0,10X,4E20.8)
  849 CONTINUE
C
      END

      FUNCTION PIF2 (X,XLIST,N,FLIST)
C FUNCTION PIF2 IS A SECOND ORDER LOOKUP FUNCTION
      DIMENSION XLIST (100), FLIST (100)
      BLIF (P,Q,R,S,T) = ((Q-P)*(S-T)/(R-Q)+S)
      IF (X-XLIST(N)) 2,1,1
    1 I = N-1
      GO TO 5
    2 IF(X-XLIST(1)) 4,4,6
    4 I = 1
    5 K = 1
      GO TO 30
    6 K = 2
    7 DO 8 I = 1,N
      IF (X-XLIST(I)) 9,9,8
    8 CONTINUE
      I=N
    9 I = I-1
   30 BLIF1 = BLIF(X,XLIST(I),XLIST(I+1),FLIST(I),FLIST(I+1))
   10 IF (K-1) 11,11,12
   11 PIF2 = BLIF1
      RETURN
```

```
12 IF((I+2)-N) 13,13,16
13 IF ((I-1)-1) 15,14,14
14 IF(ABS(XLIST(I-1)-X)-ABS(XLIST(I+2)-X))16,15,15
15 L = I+2
   GO TO 17
16 L = I-1
17 BLIF2 = BLIF (X,XLIST(I),XLIST(L),FLIST(I),FLIST(L))
   PIF2 = BLIF (X,XLIST(I+1),XLIST(L),BLIF1,BLIF2)
18 RETURN
   END
```

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13. ABSTRACT

The equations governing free turbulent mixing are derived from the Navier-Stokes equations and transformed into a mathematical plane which is explicitly independent of the eddy viscosity model. The coupled momentum and turbulent kinetic energy equations are analytically solved in the transformed plane by a perturbation technique and subsequently retransformed into physical space based on a hypothesized dependence of the eddy viscosity on the turbulent kinetic energy. The adequacy of a given model in reporducing the velocity and turbulent kinetic energy field is assessed by comparing the results of the analysis with some experimental data of planar turbulent wake mixing in constant adverse and favorable pressure gradients.

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