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OPTIMAL COMMITMENT OF FORCES

IN SOME LANCHESTER-TYPE COMBAT MODELS

by

James G. Taylor

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This paper shows that one can determine whether or not it is beneficial for the victor to initially commit as many forces as possible to battle in Lanchester-type combat between two homogeneous forces by considering the instantaneous casualty-exchange ratio. It considers the initial-commitment decison as a one-sided static optimization problem and examines this non-linear program for each of three decision criteria (victor's losses, loss ratio, and loss difference) and for each of two different battle-termination conditions (given force-level breakpoint and given force-ratio breakpoint). The paper's main contribution is to show how to determine the sign of the partial derivative of the decision criterion with respect to the victor's initial force level for general combat dynamics without explicitly solving the Lanchester-type combat equations. Consequently, the victor's optimal initial-commitment decision many times may be determined from how the instantaneous casualty-exchange ratio varies with changes in the victor's force level and time. Convexity of the instantaneous casualty-exchange ratio is shown to imply convexity of the decision criterion so that conditions of decreasing marginal returns may be identified also without solving the combat equations. The optimal initial-commitment decision is shown to be sensitive to the decision criterion for fixed force-ratio breakpoint battles.

0. Introduction

This paper[1] analyzes the decision to initially commit forces to battle. The combat is modelled by deterministic Lanchester-type equations with two force-level variables. Our results show that it is not always "best" to initially commit as much as possible to battle but that the optimal decision for the initial commitment of forces depends on a number of factors. The key factor in the victor's optimal commitment of forces is how the trading of casualties depends on the victor's force level and time. In contrast to all previous work, however, our results do not depend on explicitly solving the Lanchester-type differential equations but rather on establishing certain properties for the instantaneous casualty-exchange ratio.[2]

In his now classic 1914 paper F. W. Lanchester[9] (1868-1946) sought to develop a quantitative justification for the principle of concentration[3] with an idealized model of the combat process, and subsequently several other workers (see references 1, 6, 21, 23, and 24) have considered whether or not concentration of forces is "beneficial." The paper at hand extends previous work by Bach, Dolansky, and Stubbs[1] and Taylor and Parry[21]. Bach et al. considered a fight-to-the-finish modelled by Lanchester-type equations of "modern warfare" (see reference 19) with operational losses. Using the overall casualty-exchange ratio as the decision criterion, they showed by explicit computation how the optimal initial commitment of forces depends on a certain parameter k that involves the operational loss rates and the unit effectivenesses. Taylor and Parry[21] pointed out the wider applicability of the model of Bach et al.[1] and simplified their optimal decision rule. Moreover, Taylor and Parry conjectured (but did not prove) that the optimal initial commitment of forces could be determined by considering the instantaneous casualty-exchange ratio and studied the variable-coefficient version of the model of Bach et al.

Thus, the purpose of this paper is to prove the conjecture made by Taylor and Parry[21] that the optimal initial commitment of forces may be determined from how the instantaneous casualty-exchange ratio varies with the victor's force level and time. For general Lanchester-type equations of combat between two homogeneous forces, the victor's decision as to how many forces should be initially committed is analyzed as a one-sided combat optimization problem. The initial-commitment decision is evaluated according to three different criteria (victor's losses, loss ratio, and loss difference) and for two sets of battle-termination conditions (battle with fixed force-level breakpoint and battle with fixed force-ratio breakpoint), and results are compared. In this work, partial derivatives of the decision criteria with respect to the victor's initial force level are calculated without explicitly solving the Lanchester-type equations. Each of the three decision criteria is shown to be a convex function of the victor's initial force level under the appropriate convexity conditions on the instantaneous casualty-exchange ratio so that circumstances of diminishing marginal returns from committing additional forces may be identified.

1. Analysis of Decision to Initially Commit Forces

Let us consider combat between two homogeneous forces described by the following deterministic Lanchester-type equations[4] for $x,y > 0$ [the first equation, for example, becomes $dx/dt = 0$ for $x = 0$]

$$\begin{cases} dx/dt = -F(t,x,y) & \text{with } x(t=0) = x_0, \\ dy/dt = -G(t,x,y) & \text{with } y(t=0) = y_0, \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ denote the X and Y force levels at time t, and F and G denote force-change rates (with a negative force-change rate signifying a net influx of replacements). For simplicity we assume that there are no replacements and withdrawals;[5] and, in this case, F and G are simply casualty rates. To insure the existence of partial derivatives needed in subsequent analysis, we assume that $F(t,x,y)$ and $G(t,x,y)$ are twice continuously differentiable in each of their arguments. The initial force levels at the beginning of battle at $t = 0$ are denoted as x_0 and y_0. Although (1) contains just two force-level variables, this general model does apply to combat between two homogeneous forces with superimposed fire effects of supporting weapons not subject to attrition (see, for example, Taylor and Parry[21]).

Let us now consider the decision by the victor in the above battle as to how many of his available forces he should initially commit to combat. This decision is reflected in the model (1) as the victor's choice (within given force limitations) of the value for his initial force level. Without loss of generality we may take X to be the victor (i.e. assume that he has more than enough forces available to "win" the battle). [In Section 7 below, we will briefly consider X's initial commitment decision in the face of an enemy victory.] Let us consider the initial-commitment decision by X as a one-sided combat optimization problem: we assume that the Y-force commander has adopted a known course of action and consider X's initial commitment decision in this light. This decision is to be made only once, before the battle begins. In other words, we assume that y_0 is given and seek the "best" value of x_0 for X to choose. Thus, the decision variable for X in our combat optimization problem is x_0, the initial number of forces committed to battle.

It is convenient to consider that there are four fundamental aspects of the one-sided combat optimization problem faced by the X-force commander: (1) decision criterion, (2) battle dynamics, (3) battle-termination model, and (4) information structure. In our investigation here let us not consider the inherent uncertainty in the decision problem and assume that X has perfect knowledge about x_0 and y_0, the battle dynamics [i.e. equations (1)] (assumed deterministic), and battle termination (also assumed deterministic).[6] Hence, we will not consider the information structure here further, although it certainly will play a major role in actual real-world military decisions. The purpose of this paper is to show how, in general, the battle dynamics (i.e. the form of equations (1)) influence X's optimal decision.

We assume that the X-force commander bases his decision on a single criterion. Three possible criteria for evaluating his decision are: (C1) friendly losses, $L_X = x_0 - x_f$; (C2) loss ratio, $R_c = (x_0-x_f)/(y_0-y_f)$; and (C3) loss difference, $D_c = (x_0-x_f) - (y_0-y_f)$; where x_f and y_f denote the final force levels at the end of battle at $t = t_f$. The latter two criteria have been suggested by Pugh and

Mayberry[11], who state that the two criteria are "almost equivalent." A major result of this paper is to show that the equivalance of such criteria depends on the battle termination model. Although we are well aware that battle termination is a complex random phenomenon for which it is by no means certain that force levels are the significant variables,[7] in our analysis here we will consider two types of battle-termination conditions: (T1) battle terminated by $y(t)$ reaching its "breakpoint" force level, $y_X^f \geq 0$, while $x(t)$ has always been above its "breakpoint" force level, $x_Y^f \geq 0$; and (T2) battle terminated by $u(t) = x(t)/y(t)$ reaching Y's "breakpoint" force ratio, $u_X^f > u_0 = x_0/y_0$. Analogous conditions may be stated for a Y victory.

We assume that X has limited forces available (but more than enough to win). Since all of the decision criteria are basically costs of engagement, he seeks to minimize his adopted objective function by his choice of the value for the decision variable x_0. Letting C denote one of the above three decision criteria (i.e. either L_X, R_c, or D_c), we may state our combat optimization problem for the initial commitment of X's forces as

$$\underset{x_0}{\text{minimize}}\ C, \qquad \text{subject to: } x_0^{min} \leq x_0 \leq x_0^{max}, \tag{2}$$

where $x_0^{min} = x_0^{draw} + \varepsilon$, $\varepsilon > 0$, and x_0^{draw} denotes the value of the initial X force level which leads to a draw for the given battle-termination conditions (i.e. either (T1) or (T2)).[8] We denote the optimal value of x_0 as x_0^*.

The above nonlinear program (2) is trivial to solve after the partial derivative $\partial C/\partial x_0$ has been calculated. For example, $\partial C/\partial x_0 < 0$ $\forall\, x_0 \in [x_0^{min}, x_0^{max}]$ implies that $x_0^* = x_0^{max}$ so that X should initially commit as much as possible. The determination, however, of $\partial C/\partial x_0$ requires further analysis. Considering (C1) through (C3) above, we see that calculation of $\partial C/\partial x_0$ involves determining $\partial x_f/\partial x_0$, how X's final force level varies with changes in his initial force level.

2. Dependence of Force Level on Initial Conditions

We usually take t as the independent variable or time parameter in (1) and consider $x = x(t)$, $y = y(t)$. For a battle won by X, the final Y force level, y_f, has been driven to satisfy a given battle-termination condition so that the final X force level depends on this y_f. Thus, in general we have

$$x_f = x_f(y_f;x_0,y_0) \quad \text{and} \quad y_f = y_f(x_0,y_0). \tag{3}$$

Hence, for a battle won by X we are motivated to reparameterize the course of battle in terms of y by inverting $y = y(t)$. We must have $dy/dt \neq 0 \ \forall \, t \in [0,t_f]$ to be able to do this, and it seems appropriate to take

$$dy/dt < 0 \qquad \text{for all} \quad t \in [0,t_f]. \tag{4}$$

Accordingly, we have then

$$t = t(y) = t(y;x_0,y_0) \quad \text{and} \quad x = x(y) = x(y;x_0,y_0). \tag{5}$$

We next express x in terms of the <u>instantaneous</u> (or differential) <u>casualty-exchange ratio</u>,[8] dx/dy. It will sometimes be convenient to use the notation

$$\Delta = dx/dy. \tag{6}$$

In general, we have from (1)

$$dx/dy = \Delta = \Delta(t,x,y) = F(t,x,y)/G(t,x,y). \tag{7}$$

When Δ is time-invariant (i.e. $\partial\Delta/\partial t \equiv 0$ for all $t \geq 0$), we will say that the Lanchester-type equations (1) are <u>quasi-autonomous</u>, since they may be transformed to an autonomous system (<u>see</u> p. 163 of Petrovski[10]) by a change of the time scale.[9] When Δ depends on only t and the ratio x/y, we will say that Condition (R) holds. Thus, we have

$$\text{Condition (R):} \quad dx/dy = \Delta = q(t,u), \qquad \text{where} \quad u = x/y. \tag{8}$$

We observe that $(\partial\Delta/\partial x)_{t,y} = (1/y)\partial q/\partial u$ so that $\partial\Delta/\partial x$ and $\partial q/\partial u$ always have the same sign. Let us further observe that on a partial derivative such as $(\partial\Delta/\partial x)_{t,y}$, the subscripts denote the variables being held constant. In terms of our reparameterization (5) in terms of y, we have

$$\Delta = \Delta(t(y;x_0,y_0),x(y;x_0,y_0),y),$$

so that we may write

$$x(y;x_0,y_0) = x_0 - \int_y^{y_0} dx/dy(t(y_1;x_0,y_0),x(y_1;x_0,y_0),y_1)dy_1. \tag{9}$$

Holding y and y_0 constant and differentiating with respect to x_0, we obtain

$$\partial x/\partial x_0 = 1 - \int_y^{y_0} \{(\partial t/\partial x_0)\partial\Delta/\partial t + (\partial x/\partial x_0)\partial\Delta/\partial x\}dy_1, \tag{10}$$

where $\partial x/\partial x_0$ denotes $(\partial x/\partial x_0)_{y,y_0}$, etc.

The Volterra integral equation (10) may be solved by differentiating with respect to y and integrating the resulting first order linear ordinary differential equation. Setting $y = y_f$, we obtain

$$\partial x_f/\partial x_0 = (\partial x_f/\partial x_0)_{y_0,y_f} = \exp[-\int_{y_f}^{y_0} (\partial\Delta/\partial x)dy]$$

$$- \int_{y_f}^{y_0} (\partial t/\partial x_0)\partial\Delta/\partial t \cdot \exp[-\int_{y_f}^{y} (\partial\Delta/\partial x)dy_1]dy. \tag{11}$$

Equation (11) relates changes in the final X force level to variations in X's initial strength. This result (11) is a key one and is used in the development of most subsequent results in this paper.

3. Derivatives of Decision Criteria

For the solution of (2) we need to compute $\partial C/\partial x_0$ for $C = L_X$, R_c, and D_c. As a preliminary step in this computation we recall (3) and observe that

$$(\partial x_f/\partial x_0)_{y_0} = (\partial x_f/\partial x_0)_{y_0,y_f} + (\partial x_f/\partial y_f)_{x_0,y_0} \cdot (\partial y_f/\partial x_0)_{y_0}, \tag{12}$$

where $(\partial x_f/\partial x_0)_{y_0,y_f}$ is given by (11). Setting $y = y_f$ in (9) so that $x = x_f$, holding x_0 and y_0 constant, and differentiating the result with respect to y_f, we obtain

$$(\partial x_f/\partial y_f)_{x_0,y_0} = (dx/dy)_f, \tag{13}$$

where $(dx/dy)_f$ denotes the final instantaneous casualty-exchange ratio for $t = t_f$, $x = x_f$, and $y = y_f$. Sometimes for convenience we will denote $(dx/dy)_f$ as Δ_f [recall equation (6)]. From (12) and (13), we find that

$$(\partial x_f/\partial x_0)_{y_0} = (\partial x_f/\partial x_0)_{y_0,y_f} + (dx/dy)_f \cdot (\partial y_f/\partial x_0)_{y_0}. \tag{14}$$

Henceforth, we will omit denoting which variables are being held constant in such partial derivatives and hope that this will be clear from context. When X wins, we have via (3) that

$$\partial L_X/\partial x_0 = 1 - \partial x_f/\partial x_0 - (dx/dy)_f\ \partial y_f/\partial x_0, \tag{15}$$

$$\partial R_c/\partial x_0 = \{1 - \partial x_f/\partial x_0 + [R_c - (dx/dy)_f]\partial y_f/\partial x_0\}/(y_0 - y_f), \tag{16}$$

$$\partial D_c/\partial x_0 = 1 - \partial x_f/\partial x_0 + [1 - (dx/dy)_f]\partial y_f/\partial x_0. \tag{17}$$

For the case of a fixed final force-level battle in which $y_f = y_X^f$ is fixed beforehand, the above partial derivatives, of course, simplify considerably.

4. Results for Fixed Force-Level Breakpoint Battle

In this case $\partial y_f/\partial x_0 = 0$, and (15) through (17) simplify to

$$\partial L_X/\partial x_0 = \partial D_c/\partial x_0 = 1 - \partial x_f/\partial x_0, \tag{18}$$

and

$$\partial R_c/\partial x_0 = (1 - \partial x_f/\partial x_0)/(y_0 - y_f). \tag{19}$$

Thus, all three decision-criterion partial derivatives have the same sign. Consequently, for a fixed force-level breakpoint battle, the X-force commander makes the same decision regardless of which decision criterion he uses. It suffices, therefore, to consider $\partial L_X/\partial x_0$ in subsequent developments in this section.

By (11) and (18), we have

$$\partial L_X/\partial x_0 = 1 - \exp[-\int_{y_f}^{y_0} (\partial\Delta/\partial x)dy] + \int_{y_f}^{y_0} (\partial t/\partial x_0)\partial\Delta/\partial t \cdot \exp[-\int_{y_f}^{y} (\partial\Delta/\partial x)dy_1]dy. \tag{20}$$

We assume that we always have

$$(\partial t/\partial x_0)_{y,y_0} < 0. \tag{21}$$

This assumption seems reasonable, since we would expect that higher x_0 yields lower y for fixed y_0 and t. Consequently, we may conclude

THEOREM 1: If $\partial(dx/dy)/\partial x < 0$ and $\partial(dx/dy)/\partial t \geq 0$ for all $t \varepsilon [0,t_f]$, then $\partial C/\partial x_0 < 0$ for $C = L_X, R_c, D_c$.

Proof: Immediate by (20) and (21). Q.E.D.

The following theorem shows that under its stated conditions when $q(u)$ is convex and $\partial C/\partial x_0 < 0 \ \forall x_0 \varepsilon [x_0^{min}, x_0^{max}]$, there are decreasing marginal returns from initially committing additional forces to battle. The theorem generalizes results given for a specific model by Bach et al. (see p. 320 and p. 325 of reference 1).

THEOREM 2: Consider a battle with a fixed force-level breakpoint to be won by X. Assume that Condition (R) holds and that the Lanchester-type equations (1) are quasi-autonomous. If $dx/dy = q(u)$ is a strictly convex (concave) function of u on $[0,+\infty)$, then the decision criterion C is a strictly convex (concave) function of x_0 for $C = L_X$, R_c, D_c.

Proof: Computing $\partial^2 x_f/\partial x_0^2 = -\exp[-\int_{y_f}^{y_0} (1/y)(\partial q/\partial u)dy] \int_{y_f}^{y_0} (1/y^2)(\partial^2 q/\partial u^2) \cdot \exp[-\int_{y}^{y_0} (1/y_1)(\partial q/\partial u)dy_1]dy$, we see that $\partial^2 q/\partial u^2 > 0$ implies that $\partial^2 x_f/\partial x_0^2 < 0$, whence the theorem follows from (18) and (19). Q.E.D.

Comment 1: For quasi-autonomous Lanchester-type models we have that $\partial(dx/dy)/\partial x < 0$ for all $t \varepsilon [0,t_f]$ implies $\partial C/\partial x_0 < 0$.

Comment 2: From (20) and (21) we see that, in general, $\partial(dx/dy)/\partial x < 0$ for all $t \in [0, t_f]$ may not always imply $\partial C/\partial x_0 < 0$ when $\partial(dx/dy)/\partial t < 0$. In other words, even though a higher X force level reduces the instantaneous casualty-exchange ratio (i.e. the cost to X of reducing the Y force level a unit amount), it may not be best for X to initially commit as many forces as possible when (for constant force levels) the instantaneous casualty-exchange ratio decreases over time. The reason for this result is that smaller x_0 means that the battle (which X will win by assumption) will last longer, and the longer the battle lasts, the better the instantaneous casualty-exchange ratio becomes for X. Let us give an example of this phenomenon.

Example of $\partial C/\partial X_0 > 0$ even though $\partial(dx/dy)/\partial x < 0$.

Let us consider Helmbold's[8] modification of Lanchester's classic combat formulation to account for inefficiencies of scale for the larger force when force sizes are grossly unequal. We have for time-varying fire effectivenesses (see Taylor and Brown[19] for a discussion of modelling considerations regarding variable coefficients and further references)

$$dx/dt = -a(t)(x/y)^c y, \qquad dy/dt = -b(t)(y/x)^c x, \tag{22}$$

where c is a parameter controlling the relative force-attrition capability. We observe that $c = 0$ corresponds to the usual Lanchester-type equations of modern warfare with variable attrition-rate coefficients.[10] We readily compute that

$$\partial(dx/dy)/\partial x = (1-d)\{a(t)/b(t)\}(x/y)^{-d}/y, \tag{23}$$

where $d = 2(1-c)$. Note that $d \le 2$ for $c \ge 0$. From (23) we see that $\partial(dx/dy)/\partial x < 0$ for $d > 1$. We will show by numerical counterexample that Theorem 1 is in general not true if the assumption that $\partial(dx/dy)/\partial t \ge 0$ is omitted. This result, unfortunately, shows that the conclusion drawn about concentration of forces by Taylor and Parry[21] may not be true in general for variable attrition-rate coefficients. Theorem 1, however, gives sufficient conditions for $\partial C/\partial x_0 < 0$; and for certain battle dynamics, the assumption about $\partial(dx/dy)/\partial t$ may not be absolutely necessary for the theorem to be true. In other cases, it may be possible to weaken this assumption, but we have not investigated this matter further.

Let $a(t)$ and $b(t)$ be piecewise constant[10] and denote

$$a(t)/b(t) = \begin{cases} a/b & \text{for } 0 \leq t < t_c, \\ \tilde{a}/\tilde{b} & \text{for } t \geq t_c, \end{cases} \tag{24}$$

where t_c denotes the time at which the relative effectiveness of combatants changes. We will consider the case in which $a/b > \tilde{a}/\tilde{b}$. This case may be regarded as an approximation to that in which $\partial(dx/dy)/\partial t < 0$. For a battle terminated by a given force-level "breakpoint" being reached, X wins when $t_f \leq t_c$ if and only if

$$(u_0)^d > a(1-f_Y^d)/[b(1-f_X^d)],$$

where, for example, $x_Y^f = f_X x_0$ and x_Y^f denotes the X "breakpoint" for a Y victory. Analytic results that are required for this investigation are given in Table I. Numerical results are shown in Table II. The X force "wins" all three battles. From Table II we see that decreasing u_0 (i.e. decreasing x_0 for fixed y_0) actually leads to a more favorable casualty exchange ratio for X, even though $\partial(dx/dy)/\partial x < 0$. The reason for this result is that reducing the initial force ratio extends the length of battle, and the battle is then fought for $t > t_c$ at greater relative effectiveness per man from X's standpoint (i.e. $\tilde{b}/\tilde{a} > b/a$). For the classic equations of "modern warfare" (i.e. $d = 2$), we have not been able to find any such counterexample. We still feel, however, that for this case with variable attrition-rate coefficients Theorem 1 is probably false without the assumption that $\partial(dx/dy)/\partial t \geq 0$.

5. Results for Fixed Force-Ratio Breakpoint Battle

For a battle terminated by $u(t)$ reaching a given "breakpoint" force ratio[12] we obtain using (3) that when X wins[13]

$$\partial y_f/\partial x_0 = (\partial x_f/\partial x_0)/(u_X^f - \Delta_f), \tag{25}$$

where Δ_f denotes $(dx/dy)_f$ and we recall that u_X^f is a given constant. Observing that (see Taylor[17])

TABLE I.

ANALYTIC RESULTS FOR HELMBOLD'S MODEL USED IN EXAMPLE

1. For[11] $t_f \le t_c$

$$t_f = [2/(d\sqrt{ab})]\ell n\{[\sqrt{(u_0)^d-(a/b)(1-f_Y^d)} - f_Y^{d/2}\sqrt{a/b}]/[(u_0)^{d/2} - \sqrt{a/b}]\}$$

2. For $t_f \le t_c$

$$R_c = \{u_0 - [(u_0)^d - (a/b)(1-f_Y^d)]^{1/d}\}/(1-f_Y)$$

3. For $t_f > t_c$

$$R_c = \{u_0 - [(u_0)^d - (a/b)\{1 - (y_c/y_0)^d\} - (\tilde{a}/\tilde{b})\{(y_c/y_0)^d - f_Y^d\}]^{1/d}\}/(1-f_Y)$$

where

$$(y_c/y_0)^d = \{\cosh(\sqrt{ab}\,t_c\,d/2) - (u_0)^{d/2}\sqrt{b/a}\,\sinh(\sqrt{ab}\,t_c\,d/2)\}^2$$

4. For $t_f > t_c$

$$\partial R_c/\partial x_0 = (1-F_N/F_D)/[(1-f_Y)y_0]$$

where

$$F_N = 1 - (1/u_0^{d/2})\sqrt{(y_c/y_0)^d b/a}\{(a/b) - (\tilde{a}/\tilde{b})\}\sinh(\sqrt{ab}\,t_c\,d/2)$$

$$F_D = \{1 - [(a/b)\{1 - (y_c/y_0)^d\} + (\tilde{a}/\tilde{b})\{(y_c/y_0)^d - f_Y^d\}]/u_0^d\}^{(d-1)/d}$$

Note: The above results hold for $d \ne 0$.

TABLE II.

NUMERICAL RESULTS WHICH SHOW THAT $\partial(dx/dy)/\partial x < 0$

DOES NOT ALWAYS IMPLY THAT $\partial R_c/\partial x_0 < 0$.

Battle	u_0	t_f	t_c	y_c/y_0	R_c	$\partial R_c/\partial x_0$ †
1	4.0	19.5	20.0	----	0.437	----
2	2.0	$t_c < t_f$	20.0	0.69	0.434	$(0.0112)/y_0$
3	1.0	$t_c < t_f$	20.0	0.82	0.420	$(0.0122)/y_0$

Other parameter values (time expressed in the same units for a, t_f, and t_c):

$$d = 1.5, \quad f_Y = 0.5, \quad a = 0.01, \quad a/b = 1.0, \quad \tilde{a}/\tilde{b} = 0.1$$

†Computed using result 4 of Table I.

$$u - \Delta = (du/dt)/\{-(1/y)dy/dt\}, \tag{26}$$

we see by (4) that

$$u_X^f - \Delta_f > 0 \Leftrightarrow X \text{ wins}. \tag{27}$$

By (25) and (27) we see that $\partial x_f/\partial x_0$ and $\partial y_f/\partial x_0$ have the same sign when X wins. Using (25), we find that (15) through (17) become

$$\partial L_X/\partial x_0 = 1 - \{1/(1-\Delta_f/u_X^f)\}\partial x_f/\partial x_0, \tag{28}$$

$$\partial R_c/\partial x_0 = \{1 - [(u_X^f-R_c)/(u_X^f-\Delta_f)]\partial x_f/\partial x_0\}/(y_0-y_f), \tag{29}$$

$$\partial D_c/\partial x_0 = 1 - \{(u_X^f-1)/(u_X^f-\Delta_f)\}\partial x_f/\partial x_0, \tag{30}$$

where we recall that $\partial x_f/\partial x_0$ is given by (11). We observe that (28) through (30) reduce to (18) and (19) for $u_X^f = +\infty$, which is a "fight-to-the-finish." We will now show that the three criteria do not all lead to the same initial-commitment decision.

Let us first consider the criterion of only the friendly losses. If $x_0^{max} \geq y_0 u_X^f$, then clearly $x_0^* = x_0^{max}$ and $L_X = 0$. Let us therefore, assume that $x_0^{max} < y_0 u_X^f$. Recalling (27), we see from (28) that $\partial L_X/\partial x_0 < 1 - \partial x_f/\partial x_0$ so that recalling (11) we have

THEOREM 3: Consider a battle with a fixed force-ratio breakpoint to be won by X. If $\partial(dx/dy)/\partial x \leq 0$ and $\partial(dx/dy)/\partial t \geq 0$ for all $t \in [0,t_f]$, then $\partial L_X/\partial x_0 < 0$.

As seem from Theorem 3, it is advantageous for X to initially commit as many forces as possible even for a quasi-autonomous "linear-law" attrition process for which $\partial\Delta/\partial x \equiv 0 \equiv \partial\Delta/\partial t$. A numerical example of this phenomenon is shown in Table III. We observe that for a fixed force-level breakpoint battle, there is no advantage to X from initially committing additional forces over those required to win for this attrition structure.

TABLE III.

EXAMPLE OF BENEFIT TO X FROM INITIALLY COMMITTING MORE FORCES IN QUASI-AUTONOMOUS LINEAR-LAW BATTLE WITH FIXED FORCE-RATIO BREAKPOINT.

State Equation for Battle: $b(x_0-x) = a(y_0-y)$

Battle	x_0	L_X	R_c
1	150.0	83.33	1.0
2	200.0	66.67	1.0
3	300.0	33.33	1.0

Other parameter values:

$b/a = 1.0$, $u_X^f = 4.0$, $y_0 = 100.0$

When the casualty-exchange ratio R_c is taken as the decision criterion, the decision to initially commit forces is essentially independent of the battle-termination conditions. Before we formally state this result, it is convenient to define the following condition:

$$\text{Condition (G):} \quad R_c = R_c(u_X^f) > (dx/dy)_f = \Delta_f(u_X^f) \quad \text{for all} \quad u_X^f \varepsilon (u_0, +\infty). \tag{31}$$

We have then

THEOREM 4: Assume that Conditions (G) and (R) hold. If $\partial(dx/dy)/\partial x < 0$ and $\partial(dx/dy)/\partial t \geq 0$ for all $t \varepsilon [0, t_f]$, then $\partial R_c/\partial x_0 < 0$.

<u>Proof</u>: Considering $u_X^f - R_c = (u_X^f - u_0)/(1 - y_f/y_0)$, we see that

$$u_X^f > R_c \Leftrightarrow X \text{ wins}. \tag{32}$$

Now consider $N(u_X^f) = N(u_X^f, t_f(u_X^f), y_f(u_X^f))$ for $u_Y^f < u_0 \leq u_X^f < +\infty$, where

$$N(u_X^f) = 1 - \{(u_X^f - R_c)/(u_X^f - \Delta_f)\}\partial x_f/\partial x_0. \tag{33}$$

We then have by (29)

$$\partial R_c/\partial x_0 = N(u_X^f)/(y_0 - y_f). \tag{34}$$

The theorem follows from (34) by showing that $N(u_X^f) < 0$ for $u_0 < u_X^f < +\infty$. The latter result will be proven by showing that (a) $N(u_X^f = u_0) = 0$, and (b) $dN/du_X^f < 0$ for $u_0 < u_X^f < +\infty$.

To show that

$$\lim_{u_X^f \to u_0} N(u_X^f) = 0 = N(u_X^f = u_0), \tag{35}$$

we observe that $\lim_{u_X^f \to u_0} y_f = y_0$ and

$$\lim_{u_X^f \to u_0} N(u_X^f) = 1 - \lim_{u_X^f \to u_0} \{u_X^f - R_c\}/\{u_X^f - (dx/dy)_f\}. \tag{36}$$

Using L'Hospital's rule, we readily compute $\lim_{u_X^f \to u_0} R_c = \lim_{y_f \to y_0} \{\int_{y_f}^{y_0} (dx/dy)dy\}/(y_0 - y_f) = (dx/dy)_f$, whence (35) follows from (36).

We next show that $dN/du_X^f < 0$ for $u_0 < u_X^f < +\infty$. First, we compute dN/du_X^f from (33) to obtain

$$dN/du_X^f = \{1/(u_X^f-\Delta_f)\}\{-1+dR_c/du_X^f+(u_X^f-R_c)(1-d\Delta_f/du_X^f)/(u_X^f-\Delta_f)\}(\partial x_f/\partial x_0)$$

$$- [(u_X^f-R_c)/(u_X^f-\Delta_f)]d(\partial x_f/\partial x_0)/du_X^f. \qquad (37)$$

Considering the definition of R_c, (3), and (26), one may show that[14]

$$dR_c/du_X^f = -y_f(R_c-\Delta_f)/\{(y_0-y_f)(u_X^f-\Delta_f)\}. \qquad (38)$$

Recalling that $y_f = y_f(u_X^f)$, we obtain from (11) that

$$d(\partial x_f/\partial x_0)/du_X^f = \{(\partial\Delta_f/\partial x_f)\partial x_f/\partial x_0 + (\partial\Delta_f/\partial t_f)\partial t_f/\partial x_0\}dy_f/du_X^f, \qquad (39)$$

where $dy_f/du_X^f = -y_f/(u_X^f-\Delta_f) < 0$. By Condition (R) we have $d\Delta_f/du_X^f = \partial\Delta_f/\partial u_X^f + (\partial\Delta_f/\partial t_f)dt_f/du_X^f$. Observing that $y_f \cdot (\partial\Delta_f/\partial x_f)_{t_f,y_f} = (\partial\Delta_f/\partial u_X^f)_{t_f,y_f}$, we may combine (37) and (39) to obtain

$$dN/du_X^f = \{1/(u_X^f-\Delta_f)\}\{-1 + dR_c/du_X^f + (u_X^f-R_c)/(u_X^f-\Delta_f)\}\partial x_f/\partial x_0$$

$$+ \Phi \cdot (u_X^f-R_c)/(u_X^f-\Delta_f)^2, \qquad (40)$$

where $\Phi = \{y_f \cdot \partial t_f/\partial x_0 - (dt/du_X^f)\partial x_f/\partial x_0\}\partial\Delta_f/\partial t_f$. We observe that $\Phi \le 0$, since $\partial x_f/\partial x_0 > 0$, $du_X^f/dt_f > 0$, and by assumption $\partial\Delta_f/\partial t_f \ge 0$ and $\partial t_f/\partial x_0 < 0$ [see (21) above]. It follows by (27), (32), (38), and (40) that for $u_0 < u_X^f < +\infty$

$$dN/du_X^f \le \{-y_0(R_c-\Delta_f)/[(u_X^f-\Delta_f)^2(y_0-y_f)]\}\partial x_f/\partial x_0 < 0,$$

the last inequality being a consequence of Condition (G). Q.E.D

It may be difficult to determine, in general, whether Condition (G) holds. However, it does hold for quasi-autonomous Lanchester-type equations when Condition (R) holds and $\partial\Delta/\partial u < 0$ always. Thus, we have

LEMMA 1: When Condition (R) holds for quasi-autonomous Lanchester-type equations, then $\partial\Delta/\partial u < 0$ for all $t \in [0,t_f]$ implies $R_c > \Delta_f$ (i.e. Condition (G) holds).

Proof: By the assumptions, $u < u_X^f$ implies $\Delta > \Delta_f$ whence follows the lemma by considering $R_c - \Delta_f = \int_{y_f}^{y_0} (\Delta - \Delta_f) dy/(y_0 - y_f)$. Q.E.D.

Then for quasi-autonomous Lanchester-type equations when Condition (R) holds and the casualty-exchange ratio is taken as the decision criterion, the decision to initially commit forces is independent of the battle-termination conditions.

COROLLARY 4.1: Assume that Condition (R) holds for quasi-autonomous Lanchester-type equations. For a battle with either a fixed force-level breakpoint or a fixed force-ratio breakpoint, if $\partial\Delta/\partial x < 0$ for all $t \varepsilon [0,t_f]$, then $\partial R_c/\partial x_0 < 0$.

Proof: By Theorem 1 the result is true for a battle with a fixed force-level breakpoint. For a battle with a fixed force-ratio breakpoint, we know that Condition (G) holds by Lemma 1 so that the corollary follows by observing that all the assumptions of Theorem 4 are satisfied. Q.E.D.

Finally, we consider the loss difference D_c as the decision criterion for initially committing forces. A general result, however, is only available when the final differential casualty-exchange ratio is greater than one.

THEOREM 5: If $\partial(dx/dy)/\partial x < 0$ and $\partial(dx/dy)/\partial t \geq 0$ for all $t \varepsilon [0,t_f]$ and $(dx/dy)_f \geq 1$, then $\partial D_c/\partial x_0 < 0$.

Proof: Recalling (11) and (21), we see that the assumptions of the theorem yield $\partial x_f/\partial x_0 > 0$. Recalling (27) and (30), we then see that $\Delta_f \geq 1$ implies that $\partial D_c/\partial x_0 \leq 1 - \exp\{-\int_{y_f}^{y_0} (\partial\Delta/\partial x) dy\}$ whence follows the theorem. Q.E.D.

If $\Delta_f < 1$, however, it does not follow by the other stated conditions of Theorem 5 that $\partial D_c/\partial x_0 < 0$ so that $x_0^* = x_0^{max}$: it is possible for x_0^* to be an interior point of the interval $[x_0^{min}, x_0^{max}]$ (i.e. D_c has an unconstrained global minimum at x_0^* such that $x_0^{min} < x_0^* < x_0^{max}$). Before we give an example of this occurrence, however, let us give results analogous to those of Theorem 2, which applies for a fixed force-level breakpoint.

THEOREM 6: Consider a battle with a fixed force-ratio breakpoint to be won by X. Assume that Condition (R) holds and that the Lanchester-type equations (1) are quasi-autonomous. If $dx/dy = q(u)$ is a strictly convex (concave) function of u on $[0,+\infty)$ and $(\partial q/\partial u)_f < 0$ (>0), then L_X is a strictly convex (concave) function of x_0. The same is true for R_c if additionally $\partial R_c/\partial x_0 < 0$ $\forall$ $x_0 \varepsilon [x_0^{min}, x_0^{max}]$, while it is true for D_c if $u_X^f > 1$.

<u>Proof</u>: Computing $\partial^2 x_f/\partial x_0^2 = -\left\{\int_{y_f}^{y_0} (1/y^2)(\partial^2 q/\partial u^2)\exp\left[\int_{y_f}^{y_0} (1/y_1)(\partial q/\partial u)dy_1\right]dy\right.$ $\left.-(1/y_f)(\partial q/\partial u)_f/(u_X^f-\Delta_f)\right\} \cdot \exp\left[-2\int_{y_f}^{y_0} (1/y)(\partial q/\partial u)dy\right]$, we see that x_f is a strictly concave (convex) function of x_0 under the stated conditions. The theorem readily follows after we compute $\partial^2 L_X/\partial x_0^2 = 1 - \{1/(1-\Delta_f/u_X^f)\}\partial^2 x_f/\partial x_0^2$, $\partial^2 R_c/\partial x_0^2 =$ $\{2(\partial R_c/\partial x_0)\partial x_f/\partial x_0 - (u_X^f-R_c)\partial^2 x_f/\partial x_0^2\}/\{(y_0-y_f)(u_X^f-\Delta_f)\}$, and $\partial^2 D_c/\partial x_0 =$ $1 - \{(u_X^f-1)/(u_X^f-\Delta_f)\}\partial^2 x_f/\partial x_0^2$. <u>Q.E.D.</u>

We now give an example that an unconstrained optimal initial force level (for fixed y_0, equivalently, an unconstrained optimal initial force ratio u_0^*) can occur when D_c is the decision criterion. Let us first note that it is possible for $\partial D_c/\partial x_0 = 0$ when $\partial\Delta/\partial x < 0$ and $\partial\Delta/\partial t \geq 0$ for all $t \varepsilon [0,t_f]$ and $\Delta_f < 1$. We now assume that Condition (R) holds for quasi-autonomous Lanchester-type equations (1) and that $q(u)$ is convex in u on $[0,+\infty)$ with $(\partial q/\partial u)_f < 0$ and $u_X^f > 1$. Then by Theorem 6 D_c is convex in x_0 and has a global minimum where $\partial D_c/\partial x_0 = 0$. This occurs, for example, for a classic "square-law" battle in which $dx/dt = -ay$ and $dy/dt = -bx$ so that $q(u) = a/(bu)$. Moreover, a direct computation shows that $D_c =$ $y_0\{(u_0-1) - (u_X^f-1)[(u_0^2-a/b)/((u_X^f)^2-a/b)]^{1/2}\}$. For fixed y_0, D_c has a global minimum at $u_0^* = [(a/b)\{(u_X^f)^2-a/b\}/\{2u_X^f - (1+a/b)\}]^{1/2}$. Numerical results are shown in Table IV.

TABLE IV.

NUMERICAL RESULTS WHICH SHOW UNCONSTRAINED MINIMUM OF D_c

FOR "SQUARE-LAW" BATTLE WITH FIXED FORCE-RATIO BREAKPOINT.

Battle	u_0	x_0-x_f	y_0-y_f	D_c	R_c
1	1.2	50.08	77.89	-27.81	0.64
2	$\sqrt{2}$	36.01	66.67	-30.655	0.54
3	$u_0^* = 1.44262$	34.66	65.34	-30.683	0.53
4	1.5	32.15	62.73	-30.583	0.51
5	$\sqrt{3}$	24.13	52.86	-28.73	0.46
6	2	17.43	42.26	-24.84	0.41
7	$\sqrt{5}$	12.79	33.33	-20.54	0.38

Other parameter values:

$b/a = 1.0, \qquad u_X^f = \sqrt{10}, \qquad y_0 = 100.0$

6. Results When the Sign of $\partial(dx/dy)/\partial x$ is Always the Same

Motivated by some Lanchester-type attrition processes that have appeared in the literature for which the sign of $\partial(dx/dy)/\partial x$ is the same for all admissible values of t, x, and y, we state

Condition (P): the sign of $\partial(dx/dy)/\partial x$ is the same for all $t,x,y \geq 0$. (41)

Condition (P) is satisfied, for example, for variable-coefficient Helmbold-type proces (for which $\Delta = \{a(t)/b(t)\}(x/y)^{1-d}$) or constant-coefficient aimed-fire battles with supporting fires not subject to attrition as studied by Taylor and Parry[21] (for whic $\Delta = (a+\beta u)/(\alpha+bu)$). The above results may then be somewhat more strongly stated.

THEOREM 7: Assume that Condition (P) holds and that the Lanchester-type equations (1) are quasi-autonomous. For a battle with a fixed force-level breakpoint to be won by X, $\partial C/\partial x_0 < 0$ for $C = L_X$, R_c, D_c if and only if $\partial(dx/dy)/\partial x < 0$.

THEOREM 8: Assume that Conditions (P) and (R) hold and that the Lanchester-type equations (1) are quasi-autonomous. For a battle with a fixed force-ratio breakpoint to be won by X, $\partial R_c/\partial x_0 < 0$ if and only if $\partial\Delta/\partial u = \partial q/\partial u < 0$. If $dx/dy = q(u)$ is a strictly convex (concave) function of u on $[0,+\infty)$ and $\partial q/\partial u < 0$ (>0), then L_X and R_c are strictly convex (concave) functions of x_0. The same is true for D_c if additionally $u_X^f > 1$.

Theorem 7 follows from (11), (18), (19), and Condition (P), since $\partial(dx/dy)/\partial t \equiv 0$. The statement about $\partial R_c/\partial x_0$ in Theorem 8 follows from (33), (34), and (35), since $dN/du_X^f < 0$ if and only if $\partial\Delta/\partial u < 0$. The latter inequality for dN/du_X^f follows fr $\partial x_f/\partial x_0 > 0$ and

$$dN/du_X^f = \{-y_0(R_c-\Delta_f)/[(u_X^f-\Delta_f)^2(y_0-y_f)]\}\partial x_f/\partial x_0,$$

which holds by (40) with $\partial\Delta/\partial t \equiv 0$, since $R_c > \Delta_f$ if and only if $\partial\Delta/\partial u < 0$. The proof of the last inequality follows along the lines of that for Lemma 1. The proof the convexity statements in Theorem 8 follows along the lines of that for Theorem 6.

7. THE CONCENTRATION DECISION IN THE FACE OF AN ENEMY VICTORY

Let us now briefly consider battles to be won by the enemy (i.e. Y). The above analysis must be entirely redone. In considering the initial-commitment decision, we will assume that X cannot turn the tide of battle (i.e. $0 < x_0^{min} \leq x_0 \leq x_0^{max} = x_0^{draw} - \varepsilon$, where $\varepsilon > 0$).

For a battle won by Y, we assume that $dx/dt < 0$ for all $t \in [0,t_f]$ and parameterize the course of battle in terms of the X force level. Hence, we consider $y_f = y_f(x_f;x_0,y_0)$, $x_f = x_f(x_0,y_0)$, $t = t(x;x_0,y_0)$, and $y = y(x;x_0,y_0)$. Writing $y = y(x;x_0,y_0) = y_0 - \int_x^{x_0} dy/dx(t(x_1;x_0,y_0),x_1,y(x_1;x_0,y_0))dx_1$, we obtain analogous to (10)

$$(\partial y/\partial x_0)_{x,y_0} = \partial y/\partial x_0 = -(dy/dx)_0 - \int_x^{x_0} \{(\partial t/\partial x_0) \cdot \partial(dy/dx)/\partial t + (\partial y/\partial x_0) \cdot \partial(dy/dx)/\partial y\}dx_1. \quad (42)$$

When the equations (1) are quasi-autonomous, $\partial y_f/\partial x_0 = -(1/\Delta_0)\exp\{\int_{x_f}^{x_0} (\partial(dy/dx)/\partial y)dx\}$, which becomes when Condition (R) holds

$$\partial y_f/\partial x_0 = -(1/q_0)\exp\{\int_{x_f}^{x_0}(u/q)^2(1/x)(\partial q/\partial u)dx\}. \quad (43)$$

Taking account of the functional dependencies of x_f and y_f, we see that the partial derivatives (15) through (17) of the decision criteria now take the form

$$\partial L_X/\partial x_0 = 1 - \partial x_f/\partial x_0, \quad (44)$$

$$\partial R_c/\partial x_0 = \{1 + R_c \cdot \partial y_f/\partial x_0 - (1-R_c/\Delta_f)\partial x_f/\partial x_0\}/(y_0-y_f), \quad (45)$$

$$\partial D_c/\partial x_0 = 1 - (1 - 1/\Delta_f)\partial x_f/\partial x_0 + \partial y_f/\partial x_0. \quad (46)$$

For a battle with a fixed force-level breakpoint to be won by Y (i.e. $x_f = x_Y^f$, where x_Y^f is a given quantity[15]), we have that $\partial x_f/\partial x_0 \equiv 0$ so that the above become

$$\partial L_X/\partial x_0 = 1, \quad (47)$$

$$\partial R_c/\partial x_0 = \{1 + R_c \cdot (\partial y_f/\partial x_0)\}/(y_0 - y_f), \tag{48}$$

$$\partial D_c/\partial x_0 = 1 + \partial y_f/\partial x_0, \tag{49}$$

From the above, we see that the initial-commitment decision for X is quite different (at least for a fixed force-level breakpoint) when Y wins. If X conside only his own losses L_X, then $x_0^* = x_0^{min}$. Considering (43) and (49), one can show that $\partial D_c/\partial x_0 > 0$ when Condition (R) holds for quasi-autonomous Lanchester-type equations, $q_0 = (dx/dy)_0 \geq 1$, and $\partial q/\partial u < 0$ for all t $[0,t_f]$, since $q_0 \geq 1$ and $\partial q/\partial u < 0$ for all $t \in [0,t_f]$ imply that $\partial y_f/\partial x_0 > -1$ by (43). Further examination of the initial-commitment decision in the face of an enemy victory is beyond the scope of our current investigation. By the above, however, it should be clear that results differ from those for the case in which X wins.

8. Discussion

In this paper we have shown that under the appropriate conditions Taylor and Parry's[21] conjecture that the consequences from initially committing additional forc to battle may be determined from how the instantaneous casualty-exchange ratio varies with changes in the victor's force level and time is true. This determination does not require that the Lanchester-type combat equations be solved. As the example consi ered in Section 4 showed, temporal variations in the instantaneous casualty-exchange ratio for constant force levels (i.e. $\partial(dx/dy)/\partial t$) must be of a certain nature (<u>see</u>, example, Theorem 1) for our results to hold. This important qualification was not observed by Taylor and Parry[21]. Not only do these results apply to most cases of Lanchester-type combat between two homogeneous forces but also to such cases with superimposed effects of supporting weapons not subject to attrition as treated by Tayl and Parry. Furthermore, our new results may be extended to cases of continuous replac ments and/or withdrawals.[16]

Let us now apply our general results to the constant-coefficient model, dx/dt $-ay - \beta x$, $dy/dt = -bx - \alpha y$, considered by Bach et al.[1] and Taylor and Parry[21]. By Theorems 7 and 8, when the overall casualty-exchange ratio R_c is the decision criter the victor X should initially commit as many forces to battle as possible (i.e. $x_0^* = x_0^{max}$) if and only if $ab > \alpha\beta$, regardless of which of the two battle-termination models is used. For fixed force-level breakpoint battles, the initial-commitment decision does not depend on which of the three criteria is used. Moreover, as first shown by Bach et al.[1], there are diminishing marginal returns from initially committing additional forces to battle when this is the optimal action. Our new results provide an explanation for these diminishing returns: the instantaneous casualty-exchange ratio $dx/dy = q(u) = (a+\beta u)/(\alpha+bu)$ is a convex function of u on $[0,+\infty)$ when $ab > \alpha\beta$ (see Theorem 8).

If the combat between primary systems follows a Helmbold-type[8] attrition process (see Section 4 above and Taylor[17]) in the above example, then the combat dynamics are given by $dx/dt = -a \cdot (x/y)^c y - \beta x$ and $dy/dt = -b \cdot (y/x)^c x - \alpha y$. In this case we have $dx/dy = q(u) = u(au^{-d/2}+\beta)/(\alpha+bu^{d/2})$ and $\partial q/\partial u = \{\alpha\beta-(d-1)ab + (1-d/2)(\alpha a u^{-d/2}+\beta b u^{d/2})\}/(\alpha+bu^{d/2})^2$, where $d = 2(1-c)$. Hence, the victor X should never initially commit as many forces to battle as possible when $d \le 1$. The same conclusion holds for all $d \le 2$ when $\alpha\beta > ab$. For $1 < d < 2$ and $\alpha\beta < ab$, $\partial q/\partial u$ may change sign over the course of battle, and then it is not possible to invoke our theorems. This last example brings to mind an important aspect of our results: our results (in particular, Theorems 1, 3, 4, and 5) provide sufficient conditions for the optimal course of action to be to initially commit as many forces as possible. Since we are dealing with sufficient conditions, it may still be optimal to initially commit as much as possible even when such conditions are not satisfied.

All the above results show that with supporting fires present one should not always commit as many primary forces as possible in aimed-fire battles, but one must trade-off vulnerability to supporting fires against the increased fire effectiveness from massing primary systems. Military interpretations for various quantities such as ab are to be found in Taylor and Parry[21]. Thus, this work shows that in our nuclear age with supporting weapons of great effectiveness, merely commiting large numbers of forces to battle may not always be the "best" thing to do.

Our work here shows the importance of battle-termination conditions for combat evaluations. We saw that different optimal initial-commitment actions were possible in fixed force-level breakpoint battles and fixed force-ratio breakpoint battles. In particular, the loss ratio and the loss difference may yield different initial-commitment decisions for a fixed force-ratio breakpoint battle, although they yield the same decision for a fixed force-level breakpoint battle. Similar results on the sensitivity of optimal time-sequential fire-distribution policies to battle-termination conditions have been pointed out by the author[14,15]. Consequently, we feel that more scientific work is required on modelling conflict termination[17] (see Taylor[14] for references). As is always the case, however, the insights gained into combat dynamics from such Lanchester-type models are no more valid than the models themselves.

Notes

1. It was the author's good fortune to be awarded (jointly with S. Parry) the 1975 MAS Prize by the Military Applications Section of ORSA for the three papers Taylor and Parry[21], Taylor[17], and the paper at hand. The MAS Prize is awarded annually for the best paper on military operations research that is submitted in response to a solicitation.

2. The instantaneous (or differential) casualty-exchange ratio is given by $dx/dy = F(t,x,y)/G(t,x,y)$ for the model (1) with no replacements and withdrawals. We may think of it as the ratio of each side's casualties that occur in a short interval of time dt.

3. One of the half dozen or so principles of war (see references 5, 12, and 22) is the principle of concentration (or mass), which would have a commander concentrate as many men and means for battle as possible at the decisive point. The exact number of principles of war varies from author to author.

4. See references 16, 17, 19, and 21 for further information about such models.

5. Extension to cases with replacements and/or withdrawals is outlined in Note 16 belo

6. As Borch[4] has emphasized, it will not make much sense to study decisions under uncertainty unless we know how to make decisions under full certainty.

7. As pointed out in reference 21, the entire topic of modelling battle termination is a problem area in contemporary defense planning studies, and there is far from universal agreement on this topic. For further references see Taylor[14].

8. For our idealized deterministic model, $\varepsilon > 0$ may be taken to be arbitrarily small. In the real world with its various uncertainties, a larger value would be desirable as a "hedge" against uncertainty (see reference 21 and p. 322 of reference 1).

9. Quasi-autonomous Lanchester-type equations of modern warfare have, for example, been considered by Bonder and Farrell[2] and Taylor[13] (see also Note 4 of reference 19).

10. Piecewise-constant attrition-rate coefficients may be reagrded as a limiting case of twice continuously differentiable coefficients. The former are certainly much more convenient to use for this counterexample.

11. The first equation of Table I may be obtained in the following manner. First, we observe that the substitution $p = x^{1-c}$ and $q = y^{1-c}$ transforms the nonlinear equations (22) into the following linear system

$$dp/dt = -(1-c)a(t)q, \qquad dq/dt = -(1-c)b(t)p.$$

[This important transformation was apparently first noted in Taylor[18] for a more general model.] Next, we consider the case in which the above model has constant attrition-rate coefficients. When X wins, the time for Y to reach his breakpoint, t_f, then follows from well-known constant-coefficient results (see, for example, equation (8) of Taylor and Comstock[20]).

12. See, for instance, Farrell and Freedman[7] for an example of the use of such battle-termination conditions in contemporary defense analysis.

13. Equation (25) is developed in the following manner. From (3) and the definition of u_X^f, we have

$$x_f(y_f(x_0,y_0);x_0,y_0) = u_X^f \cdot y_f(x_0,y_0).$$

Since u_X^f is a given constant, differentiation with respect to x_0 yields

$$(\partial x_f/\partial y_f)_{x_0,y_0} \cdot (\partial y_f/\partial x_0)_{y_0} + (\partial x_f/\partial x_0)_{y_0,y_f} = u_X^f \cdot (\partial y_f/\partial x_0)_{y_0},$$

which yields the desired result via (13).

14. Since x_0 and y_0 are fixed in this development, we have that x_f is a function of only y_f so that

$$R_c = (x_0 - x_f(y_f))/(y_0 - y_f).$$

For a fixed force-ratio breakpoint battle, we may consider that y_f is a function of u_X^f. Differentiation of the above expression for R_c with respect to u_X^f yields the desired result (38) by use of the identity $dy_f/du_X^f = -y_f/(u_X^f - \Delta_f)$, which follows from (26).

15. Thus, one assumes that the X force is effective only for $x > x_Y^f$. In other words one is assuming that by the time the X force level reaches x_Y^f, the unit has suffered so many casualties (and also lost key personnel) that it ceases to be an effective fighting force. One normally writes that $x_Y^f = f_{BP}^X\, x_0$, where f_{BP}^X denotes a given fraction of X's initial strength (for further details, see Section 2 of Taylor and Comstock[20]). The breakpoint fraction f_{BP}^X is usually assumed to depend on the tactical posture of the unit, unit size, its morale and training, etc. A typical value (frequently used in defense analyses) for f_{BP}^X is 0.7 for a company-sized unit in the attack.

16. The extension of these results to cases of continuous replacements and/or withdrawals becomes quite complex, however. We will now briefly examine such an extension. Let $n_X(t)$ denote the net rate of influx of replacements for X, and similarly for $n_Y(t)$. Then, denoting X's casualties as x_c, we have

$$x_c = x_0 - x_f + N_X,$$

where $N_X = \int_0^{t_f} n_X(s)ds$, and similarly for y_c. It follows that

$$\partial x_c/\partial x_0 = 1 - \partial x_f/\partial x_0 + n_X(t_f)\{(dt/dy)_f \cdot \partial y_f/\partial x_0 + \partial t_f/\partial x_0\},$$

and

$$\partial y_c/\partial x_0 = -\partial y_f/\partial x_0 + n_Y(t_f)\{(dt/dy)_f \cdot \partial y_f/\partial x_0 + \partial t_f/\partial x_0\},$$

where $(dt/dy)_f$ denotes the final value for $1/(dy/dt)$. Recalling that $L_X = x_c$, $R_c = x_c/y_c$, and $D_c = x_c - y_c$, we have for a fixed force-level breakpoint battle (in which y_f = constant)

$$\partial L_X/\partial x_0 = 1 - \partial x_f/\partial x_0 + n_X(t_f) \cdot \partial t_f/\partial x_0,$$

$$\partial D_c/\partial x_0 = 1 - \partial x_f/\partial x_0 + \{n_X(t_f)-n_Y(t_f)\}\partial t_f/\partial x_0,$$

and

$$\partial R_c/\partial x_0 = \{1-\partial x_f/\partial x_0+[n_X(t_f)-R_c \cdot n_Y(t_f)]\partial t_f/\partial x_0\}/(y_f-y_0).$$

The above partial derivatives should be compared with the analogous ones (15) through (17) for the case of no replacements and withdrawals. Further examination of such an extension is beyond the scope of our current investigation.

17. Here we mean that more effort should be spent on developing scientifically valid models of conflict termination because of the sensitivity of analysis results to such models.

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