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Integral Identities for Random Variables

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ABSTRACT

Using a general method for deriving identities for random variables, I find a number of new results involving characteristic functions and generating functions. The method is simply to promote a parameter in an integral relation to the status of a random variable and then take expected values of both sides of the equation. Results include formulas for calculating the characteristic functions for x^2, $\sqrt{x}$, $1/x$, $x^2 + x$, $R^2 = x^2 + y^2$, etc. in terms of integral transforms of the characteristic functions for x and (x, y), etc. Generalizations to higher dimensions can be obtained using the same method. Expressions for inverse/fractional moments, $E(n^{1})$, etc. are also presented, demonstrating the method

1. INTRODUCTION

As is well known, it is sometimes easier to study a process using transforms of the relevant probability distributions. Such transforms include: the characteristic function, $C(\omega)$, and the moment generating function, $M(\theta)$, for general random variables; the probability generating function, $G(z)$, for integer valued random variables, and the Laplace transform of the probability density function, $\mathcal{L}(s)$, for non-negative valued random variables. They often allow one to 1) simplify manipulations involving convolutions of probability distributions arising from consideration of sums of random variables and more complicated compound and branching processes, and 2) apply powerful methods from complex analysis and integral transform theory to the solution of differential-difference equations which arise in the study of probability and stochastic processes, and in the analysis of the analytic behavior of those solutions. The value of techniques for manipulating such transforms and of "methods for constructing new characteristic functions out of given ones"[1] is well known. In fact, the theory of probability "depends to a large extent on the method of characteristic functions"[2]. The methods presented here may further aid in the interpretation of complicated characteristic functions and facilitate the identification of independent processes which contribute to the result (see e.g. reference [3]). Apart from their usefulness in probabilistic applications, our results also provide another means of generating new integral identities from old ones.

By promoting a parameter in an integral expression to the status of a random variable (r.v.) and then taking expected values of both sides of the equation, a number of interesting relations involving characteristic functions, generating functions, etc. are found. In general, while there is no guarantee that the resulting integrals can be evaluated in closed form for a particular distribution of interest, the expression may be helpful in numerical work. An analogous technique for generating identities involving operators in Hilbert space, matrices, etc. has been usefully employed in physics (e.g. see the Appendix, below). In the probability context similar methods have long been used to solve problems by averaging conditional results over the conditioning variable.

A number of identities are presented in Sections II - V demonstrating the method of derivation. Some examples of calculations using these identities are then carried out in Section VI. Finally, in Section VII we comment on the generality of the method.

II. Relations Involving the Square of R. V.'s

A. Consider the well-known integral expressing the normalization of a Normal (aka Gaussian) distribution, in which x is an arbitrary constant,

$$\int_{-\infty}^{\infty} \frac{\exp[\,-(\xi - x)^2/(2\sigma^2)\,]\, d\xi}{\sqrt{(2\pi\,\sigma^2)}} = 1$$

Change variables according to, $\xi \;\text{-->}\; \xi/(2\gamma)$, $1/(2\sigma^2) \;\text{-->}\; i\gamma$, and obtain,

$$\int_{-\infty}^{\infty} \frac{\exp[-i\xi^2/(4\gamma) + i\xi x\,]\, d\xi}{\sqrt{(-i4\pi\gamma)}} = \exp[\,i\gamma x^2\,]\,. \qquad (1)$$

Now promote x to be a real random variable and take expected values of both sides of the equation, assuming that the implicit interchange of orders of integration is justified, i.e. that $\int$ and E{ } commute,

$$\sqrt{[\,i/(4\pi\gamma)\,]}\;\int_{-\infty}^{\infty} \exp[-i\xi^2/(4\gamma)]\; C_x(\xi)\, d\xi = C_{x^2}(\gamma)\,. \qquad (2)$$

B. Now multiply Eq. (1) by itself with $x \;\text{-->}\; y$, $\xi \;\text{-->}\; \epsilon$, $\gamma \;\text{-->}\; \delta$, to obtain,

$$\int\int_{-\infty}^{\infty} \frac{\exp[-i\xi^2/(4\gamma) + i\xi x - i\epsilon^2/(4\delta) + i\epsilon y]\, d\xi d\epsilon}{[-i4\pi\sqrt{(\gamma\delta)}]} = \exp[\, i\gamma x^2 + i\delta y^2\,] .$$

Again consider x, y to be r.v.'s and take expected values of both sides,

$$i/[4\pi\sqrt{(\gamma\delta)}\,] \int\int_{-\infty}^{\infty} \exp[-i\xi^2/(4\gamma) - i\epsilon^2/(4\delta)] \;\; C_{x,y}(\xi,\epsilon)\, d\xi d\epsilon \tag{3}$$

$$= \quad C_{x^2,y^2}(\gamma,\delta) .$$

If we now let $\delta = \gamma$ we have,

$$i/[4\pi\gamma] \int\int_{-\infty}^{\infty} \exp[-i(\xi^2 + \epsilon^2)/(4\gamma)] \;\; C_{x,y}(\xi,\epsilon)\, d\xi d\epsilon \tag{4}$$

$$= \quad C_{R^2}(\gamma) ,$$

where $R^2 = x^2 + y^2$. This can be generalized further to 3 or more r.v.'s in an analogous manner.

2. Multiply Eq. (1) by $\exp(\, i\gamma x)$ and take expected values to obtain the characteristic function for $x^2 + x$.

$$\sqrt{[\, i/(4\pi\gamma)\,]} \int_{-\infty}^{\infty} \exp[-i\xi^2/(4\gamma)]\, C_x(\xi + \gamma)\, d\xi \quad = \quad C_{x^2+x}(\gamma) . \tag{5}$$

Again, it is clear that this may be generalized further.

III. Identities for $\sqrt{x}$ and $1/x$

A. Consider the definite integral (e.g. reference [6] p. 341),

$$\int_0^\infty \exp[-a/\xi^2 - b\xi^2]\, d\xi = \sqrt{[\pi/(4b)]}\, \exp[-2\sqrt{(ab)}]\,. \tag{6}$$

Let $a \to x$, $b \to s^2/4$ to obtain the identity,

$$\int_0^\infty \exp[-x/\xi^2 - s^2\xi^2/4]\, d\xi = \sqrt{(\pi)}\,/s\, \exp[-s\sqrt{x}]\,. \tag{7}$$

Now, promote x to be a non-negative r.v. and average over x, to obtain the Laplace transform of the pdf of $\sqrt{x}$,

$$s/\sqrt{\pi} \int_0^\infty \exp[-s^2\xi^2/4]\, \mathcal{L}_x(1/\xi^2)\, d\xi = \mathcal{L}_{\sqrt{x}}(s)\,. \tag{8}$$

Alternatively, a similar integral on p. 399 of reference [6] allows one to express $\mathcal{L}_{\sqrt{x}}(s)$ in terms of the characteristic function, C_x.

B. To obtain the Laplace transform of the pdf for the r.v. $1/x$, i.e. $\mathcal{L}_{1/x}$, given $\mathcal{L}_x$, consider the integral[10],

$$\int_0^\infty \exp[-a\xi]\, J_0(b\sqrt{\xi})\, d\xi = 1/a\, \exp[-b^2/(4a)].$$

Multiply both sides by a and change the parameters $b \to 2\sqrt{s}$, $a \to x$, the latter a non-negative r.v., to obtain,

$$\int_0^\infty E\{\, x \exp[\, -x\xi \,] \,\} \cdot J_0[\, 2\sqrt{(s\xi)} \,]\, d\xi \;=\; E\{\, \exp[\, -s/x \,] \,\}.$$

In terms of the Laplace transform, this is,

$$(-) \int_0^\infty \mathcal{L}'_x(\xi) \cdot J_0[\, 2\sqrt{(s\xi)} \,]\, d\xi \;=\; \mathcal{L}_{1/x}(s). \qquad (9)$$

IV. Identities for Non-Standard Moments and Averages

A. Consider the elementary integral, where x is just a parameter.

$$\int_0^\infty s^{n-1} \exp[\, -x\, s]\, ds \;=\; (n-1)! \,/\, x^n .$$

Now promote x to be a non-negative r.v., whose pdf falls off sufficiently rapidly as x --> 0, (e.g. an Erlang(n+1)) and take expected values wrt x,

$$1/(n-1)! \int_0^\infty s^{n-1}\, \mathcal{L}_x(s)\, ds \;=\; E\{1/x^n\} \qquad (10a)$$

Letting x --> (x + A) leads immediately to the identity,

$$1/(n-1)! \int_0^\infty s^{n-1} \exp(-sA)\, \mathcal{L}_x(s)\, ds \;=\; E\{1/(x+A)^n\} \qquad (10b)$$

Identities for the Laplace transform could also be written in terms of the moment generating function, when it exists. Analogous results for the moment generating function were also derived in references [4], and [5] using methods similar to the above. Those references also contain additional applications of this result.

<u>B.</u> Consider, now, the integral,

$$2 \int_0^\infty \exp[-a\, t^2]\, dt = \sqrt{(\pi/a)}\,, \tag{11}$$

let a --> x, a non-negative r.v., and take expected values,

$$2/\sqrt{\pi} \int_0^\infty C_x(it^2)\, dt = 2/\sqrt{\pi} \int_0^\infty \mathcal{L}_x(t^2)\, dt = E\{1/\sqrt{x}\} \tag{12}$$

Making the change of variable to $y = t^2$ in Eq.(12) results in,

$$1/\sqrt{\pi} \int_0^\infty y^{1/2\,-1}\, \mathcal{L}_x(y)\, dy = E\{1/\sqrt{x}\}\,.$$

This can be recognized as a fractional integration of order 1/2 of the Laplace transform (or MGF). Some of the other moments in this section can also be written as fractional integro-differentiations of moment generating functions or Laplace transforms. This fact, as well as other extensions (and related references) are discussed in references [7 - 9].

C. Multiply Eq. (11) by a, let a --> x and average, to obtain,

$$2/\sqrt{\pi} \int_0^\infty E\{x \exp[-xt^2]\}\, dt = E\{\sqrt{x}\},$$

or, switching to moment generating functions instead of Laplace transforms for this result (either could be used here),

$$2/\sqrt{\pi} \int_0^\infty M'_x(-t^2)\, dt = E\{\sqrt{x}\}. \qquad (13)$$

This can be generalized to obtain a formula for $E\{ x^{m+1/2} \}$, with m an integer, in a straightforward manner.

D. Consider Lipschitz's integral[10] for the ordinary Bessel function of zeroth order, J_0,

$$\int_0^\infty \exp[-as]\, J_0(bs)\, ds = 1/\sqrt{(a^2 + b^2)}.$$

Promote a --> x, a non-negative random variable and take expected values of both sides to obtain,

$$\int_0^\infty \mathfrak{L}_x(s)\, J_0(bs)\, ds = E\{ 1/\sqrt{(x^2 + b^2)} \}. \qquad (14)$$

Successively differentiating this identity wrt the parameter b produces a family of similar identities.

E. Consider one form of Bessel's integral for the n^{th} order ordinary Bessel function[10],

$$J_n(x) = 1/(2\pi) \int_{-\pi}^{\pi} \exp[-n\, i\, \theta + i\, x \sin\theta]\, d\theta,$$

let x be a r.v. and average over all x, to obtain,

$$E\{ J_n(x) \} = 1/(2\pi) \int_{-\pi}^{\pi} \exp[-n\, i\, \theta]\, C_x(\sin\theta)\, d\theta. \qquad (15)$$

Clearly, this result can be generalized in many ways, and is somewhat reminiscent of the well-known formula,

$$E\{ H(x) \} = 1/(2\pi) \int \tilde{H}(\omega)\, C_x(\omega)\, d\omega,$$

where $\tilde{H}(\omega)$ is the Fourier transform of H(x). The latter equation can, in the spirit of this paper, be simply derived by taking expected values of x in the representation of H(x) as the Fourier Transform of $\tilde{H}(\omega)$.

I. Identities for Probability Generating Functions

A. Consider again the well-known integral used to define the gamma function, $\Gamma(n+1)$,

$$\int_0^{\infty} z^n \exp[-s\, z]\, dz = n! / s^{n+1}.$$

This time let n be a non-negative integer valued random variable and average over n,

$$\int_0^\infty G(z) \exp[-sz]\, dz \quad = \quad E\{ n! / s^{n+1} \}. \tag{16}$$

In particular, when s = 1, this yields E{ n! }, when it exists, i.e. the Laplace transform of the probability generating function, evaluated at s = 1 is just E{ n! } (cf. the factorial moments $E\{n(n-1)\cdots(n-k+1)\} = (d/dz)^k G(z)\big|_{z=1}$). For non-integer r.v.'s we can obtain a corresponding expression for $E\{ \Gamma(x) \}$.

B. Consider the integral,

$$\int_0^1 z^n\, dz \quad = \quad 1/(n+1).$$

Now, let n be a non-negative integer valued r.v. and average,

$$\int_0^1 G(z)\, dz \quad = \quad E\{ 1/(n+1) \}, \tag{17}$$

which also follows easily from the power series definition of G(z) and is directly analogous to the usual result for E{n}.

C. Consider the integral expressing the standard result for the even moments of the Normal distribution,

$$\int_{-\infty}^{\infty} z^{2n} \frac{\exp[\,-z^2/(2\sigma^2)\,]\,dz}{\sqrt{(2\pi\sigma^2)}} = (2n-1)!!\,\sigma^{2n},$$

where the double factorial symbol means, e.g., $5!! = 5 \cdot 3 \cdot 1$. Again take averages over n on both sides of the equality,

$$\int_{-\infty}^{\infty} G(z^2) \frac{\exp[\,-z^2/(2\sigma^2)\,]\,dz}{\sqrt{(2\pi\sigma^2)}} = E\{\,(2n-1)!!\,\sigma^{2n}\,\}, \qquad (18)$$

and when $\sigma = 1$ we have $E\{\,(2n-1)!!\,\}$.

D. Consider the two integrals, found on p. 369 of reference [5],

$$\int_0^{\pi/2} [\sin^2\theta]^m\,d\theta = \pi/2 \cdot (2m-1)!!\,/\,(2m)!!\,,$$

and,

$$\int_0^{\pi/2} \sin\theta\,[\sin^2\theta]^m\,d\theta = (2m)!!\,/\,(2m+1)!!\,.$$

Letting m be a r.v. and averaging over all values of m on each side of the above equations, we obtain,

$$2/\pi \int_0^{\pi/2} G(\,\sin^2\theta\,)\,d\theta = E\{\,(2m-1)!!\,/\,(2m)!!\,\}\,, \qquad (19)$$

and,

$$\int_0^{\pi/2} \sin\theta\, G(\sin^2\theta)\, d\theta = E\{ (2m)!! / (2m+1)!! \}, \tag{20}$$

respectively.

VI. Some Applications of the Identities

A. If x has a Normal distribution with zero mean then

$$C_x(\xi) = \exp\{-\xi^2\sigma^2/2\}.$$

Putting this in Eq. (2) and performing the integration, we have,

$$C_{x^2}(\gamma) = \sqrt{[\, 1/(4\pi i\gamma)\,]} \int_{-\infty}^{\infty} \exp[-i\xi^2/(4\gamma) - \xi^2\sigma^2/2\,]\, d\xi .$$

$$= 1/\sqrt{(1 - 2i\sigma^2\gamma)}. \tag{21}$$

This is, indeed, recognized as the characteristic function for the χ^2 distribution with one degree of freedom. (Similarly, if x, y have independent normal distributions with the same value of the variance, that $R^2 = X^2 + Y^2$ has a negative exponential distribution follows trivially from Eq. (4).)

Now, let x have a Normal distribution with non-zero mean, μ, then

$$C_x(\xi) = \exp\{ i\mu\xi - \xi^2\sigma^2/2\}.$$

Substituting this in Eq. (2) and integrating, yields,

$$C_{\chi^2}(\gamma) = \sqrt{[\, i/(4\pi\gamma) \,]} \int_{-\infty}^{\infty} \exp[\, i\mu\xi - i\xi^2/(4\gamma) - \xi^2\sigma^2/2 \,]\, d\xi \,.$$

$$= 1/\sqrt{(1 - 2i\sigma^2\gamma)}\, \exp\{\, i\mu^2\gamma/(1 - 2i\sigma^2\gamma) \,\}, \qquad (22)$$

which is the characteristic function of an offset χ^2 distribution.

B. Calculate $E\{1/\sqrt{x}\}$ where x has an exponential distribution with parameter λ. In this case the characteristic function of x is,

$$C_x(\xi) = \lambda/(\lambda - i\,\xi\,).$$

Hence, substituting this in Eq. (12) we have,

$$2/\sqrt{\pi} \int_0^{\infty} C_x(it^2)\, dt = 2/\sqrt{\pi} \int_0^{\infty} \lambda/(\lambda + t^2)\, dt,$$

and, using a standard integral, we obtain,

$$E\{1/\sqrt{x}\} = \pi\sqrt{\lambda}. \qquad (23)$$

This is easily verified to be correct by a direct calculation. $E\{\sqrt{x}\}$ is also easily verified to be the result produced by Eq. (13).

C. Insert the Laplace transform of the pdf for an exponential distribution, $\lambda/(\lambda + s)$, into Eq. (14), obtaining,

$$E\{1/\sqrt{(x^2 + b^2)}\} = \int_0^{\infty} \lambda/(\lambda + s) \cdot J_0(bs)\, ds$$

This integral is tabulated on p. 685 in reference [6], resulting in,

$$E\{ 1/\sqrt{(x^2 + b^2)} \} = \lambda \pi/2 [H_0(b\lambda) - N_0(b\lambda)], \qquad (24)$$

where H_0 and N_0 are Struve and Neumann functions, respectively, of zeroth order (N_0 can be replaced with Y_0, the Bessel function of the second kind). For example, taking b = 4 and λ = 1 ($H_0(4)$ = .13501and $Y_0(4)$ = -.01694)[6] we find for the exponential distribution,

$$E\{ 1/\sqrt{(x^2 + b^2)} \} = .2387 ,$$

which is easily confirmed by direct Gauss-Laguerre integration of the left-hand-side.

D. We now calculate the average of the n^{th} order Bessel function when x has a N(0, σ) distribution with the use of Eq. (15). After inserting the characteristic function for a normal distribution, using the trig identity $\sin^2\theta = (1 - \cos 2\theta)/2$, and again using Bessel's integral identity, this time for $I_{n/2}$, Eq. (15) leads to,

$$E\{ J_n(x) \} = \exp[-\sigma^2/4] \cdot I_{n/2}(\sigma^2/4), \qquad (25)$$

for n even, and zero when n is odd. This expression can be confirmed by evaluating the expected value directly with the help of an integral tabulated on p. 710 of reference [6].

E. Let G(z) be the generating function for a Poisson distribution,

$$G(z) = \exp [\bar{n} (z - 1)].$$

Putting this in Eq. (16) and integrating yields,

$$E\{ n! / s^{n+1} \} = 1/(s - \bar{n}) \cdot \exp[-\bar{n}], \tag{26}$$

and, in particular, when s = 1,

$$E\{ n! \} = 1/(1 - \bar{n}) \cdot \exp [-\bar{n}] .$$

This is easily verified to be correct, as well as the fact that for a Poisson distribution E{ n! } is only finite for $\bar{n} < 1$.

<u>F.</u> If we substitute the generating function for a Poisson r.v. into Eq.(18) and perform the integration, we easily obtain (letting σ = 1),

$$E\{ (2n - 1)!! \} = 1/\sqrt{(1 - 2\bar{n})} \cdot \exp[-\bar{n}]. \tag{27}$$

Clearly, this is finite only for $\bar{n} < 1/2$.

<u>G.</u> Again using the generating function for a Poisson r.v., Eq. (20) yields, after using a trigonometric identity for $\sin^2\theta$ and Bessel's integral representation for the Bessel function of zeroth order,

$$E\{ (2m)!! / (2m + 1)!! \} = \exp[-\bar{n} / 2] J_0(i\,\bar{n}/2)$$

$$= \exp[-\bar{n} / 2] I_0(\bar{n}/2), \tag{28}$$

where I_0 is the modified Bessel function of zeroth order.

VII. Conclusion

Some of the foregoing integral identities involving characteristic functions and generating functions may be derived or verified using other methods. For example, I originally obtained Eq. (2) by expanding $C_{X^2}(\gamma)$ in a McLaurin series, replacing the derivatives wrt γ by even order derivatives of $C_X(\xi)$ wrt ξ, and re-summing the infinite series. That derivation, which relies on the existence of all moments, is presented in the Appendix. Similarly, the expression for $E\{1/(n + 1)\}$ follows easily from integrating, term by term, the infinite series definition of $G(z)$. In fact, expressions for fractional and/or inverse moments, including some of those derived in Section IV, have been expressed elsewhere[7,8,9] in a unified manner in terms of fractional integro-differentiations of the MGF, generalizing the usual formulas for moments and factorial moments.

However, alternate derivations are not readily identified for all of our integral relations. The point is, that by presenting our unified treatment (containing as a proper subset some of the previously mentioned formalisms) it becomes straightforward to obtain new integral identities for random variables by a judicious search of tables of integrals such as reference [6]. As a final example, a somewhat gratuitous result is obtained by consideration of integral no. 3 on p. 304 in reference [6],

$$\int_{-\infty}^{\infty} \exp(-pt)/[1 + \exp(-qt)]\, dt = \pi/q \operatorname{cosec}[p\pi/q], \qquad q > p > 0, \text{ or } q < p < 0.$$

Let $q = 1$, $p \to x$, a r.v. in the interval [0, 1] and average over x,

$$1/\pi \int_{-\infty}^{\infty} M_x(-t) / [1 + \exp(-t)]\, dt = E\{ \operatorname{cosec} [\pi \cdot x] \}. \tag{29}$$

It is clear that many other integral identities for random variables can be generated in the same manner. The only requirement is that the implicit interchange of orders of integration be justified.

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APPENDIX

In this appendix an alternate derivation of Eq. (2) is presented, patterned after that in reference [3] in which it is applied to help in the interpretation of a complicated characteristic function and identify independent processes which combine to form the final process. The usual definition for the characteristic function yields,

$$C_x(\xi) = E\{ \exp(i\xi x) \}, \qquad \text{and} \qquad C_{x^2}(\gamma) = E\{ \exp(i\gamma x^2) \}. \tag{A1}$$

We solve the problem: given C_x, to find C_{x^2}. First, note that the even moments of x can be expressed alternatively as,

$$[1/i^2 (d/d\xi)^2]^n C_x(\xi)\big|_{\xi=0} = E\{ x^{2n} \},$$

or (A2)

$$[1/i (d/d\gamma)]^n C_{x^2}(\gamma)\big|_{\gamma=0} = E\{ x^{2n} \},$$

i.e. the lhs of these two expressions are equal, assuming the moments exist.

We can now write the ordinary McLaurin series for C_{x^2} as,

$$C_{x^2}(\gamma) = \sum_{m=0}^{\infty} [1/i (d/d\gamma')]^m C_{x^2}(\gamma')\big|_{\gamma'=0} (i\gamma)^m /m! \tag{A3}$$

which can be rewritten using Eqs. (A2) as,

$$C_{x^2}(\gamma) = \sum_{m=0}^{\infty} [-(d/d\xi)^2]^m C_x(\xi)\big|_{\xi=0} (i\gamma)^m /m! . \tag{A4}$$

Next, re-sum this power series in $(d/d\xi)^2$ to obtain,

$$C_{x^2}(\gamma) = \exp\{ -i\gamma\,(d/d\xi)^2 \}\; C_x(\xi)\big|_{\xi=0} \,. \qquad \text{(A5)}$$

Now, note that,

$$\exp\{ \sigma^2(d/dx)^2 \}\, f(x) = 1/\sqrt{(2\pi\sigma^2)} \int \exp\{ -(x - x')^2/(2\sigma^2)\, f(x')\, dx',$$

which can be verified using the convolution theorem of Fourier transforms. This equation could also have been obtained directly from Eq. (1) if we promote x to be the <u>operator</u> d/dx, instead of a r.v., and then post-multiply by f(x). Making the change of variables $x \to \xi$, $\sigma^2 \to -2i\gamma$, and setting $\xi = 0$ we recover Eq. (2).

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