1994-06

A study of the deformation of helical springs under eccentric loading

Leech, Andrew R.

Monterey, California. Naval Postgraduate School

Approved for public release; distribution is unlimited.

A Study of the Deformation of Helical Springs Under Eccentric Loading

by

Andrew R. Leech
Lieutenant, United States Navy
B.S., Virginia Polytechnic Institute and State University, 1986

Submitted in partial fulfillment
of the requirements for the degree of

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
June 1994

REPORT DOCUMENTATION PAGE	Form Approved OMB No. 0704

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE June 1994	3. REPORT TYPE AND DATES COVERED Master's Thesis

4. TITLE AND SUBTITLE A Study of the Deformation of Helical Springs Under Eccentric Loading	FUNDING NUMBERS
6. AUTHOR(S) Andrew Robert Leech	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Naval Postgraduate School Monterey CA 93943-5000	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES The views expressed in this thesis are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.

12a. DISTRIBUTION/AVAILABILITY STATEMENT	12b. DISTRIBUTION CODE
Approved for public release; distribution is unlimited.	A

13. ABSTRACT *(maximum 200 words)*

Much analysis has been done to date on the deformation of helical springs due to normal loading. The aim of this study is to design a helical spring that will deform under eccentric loading a desired amount due to a given force. Under the assumptions of linear stress strain relationships, the spring will be designed in terms of its material properties and its geometry. The deformation of the spring will be made possible utilizing a Shape Memory Alloy (SMA) active element that undergoes phase transformation upon heating above a certain temperature. Two models for spring deformation have been considered. In the first model we study the differential compression of a spring using SMA wire actuators, and in the second model we investigate the bending of an SMA rod placed inside the spring. Our efforts were a first step towards the development of a structural skeleton for a minimally invasive surgical manipulator.

14. SUBJECT TERMS Deformation, design, eccentric, spring.	15. NUMBER OF PAGES 67
	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	UL

NSN 7540-01-280-5500 Standard Form 298 (Rev. 2-89)

Prescribed by ANSI Std. 239-18

ABSTRACT

Much analysis has been done to date on the deformation of helical springs due to normal loading. The aim of this study is to design a helical spring that will deform under eccentric loading a desired amount due to a given force. Under the assumptions of linear stress strain relationships, the spring will be designed in terms of its material properties and its geometry. The deformation of the spring will be made possible utilizing a Shape Memory Alloy (SMA) active element that undergoes phase transformation upon heating above a certain temperature. Two models for spring deformation have been considered. In the first model we study the differential compression of a spring using SMA wire actuators, and in the second model we investigate the bending of an SMA rod placed inside the spring. Our efforts were a first step towards the development of a structural skeleton for a minimally invasive surgical manipulator.

TABLE OF CONTENTS

I. INTRODUCTION 1

II. PRELIMINARIES 3

A. CASTIGLIANO'S THEOREM 3

B. SHAPE MEMORY ALLOYS 6

C. MODEL 1: HELICAL SPRING UNDER ECCENTRIC COMPRESSION . . 8

D. MODEL 2: HELICAL SPRING UNDER BENDING 9

III. DEFORMATION ANALYSIS OF THE FIRST HELICAL SPRING MODEL . . . 12

IV. DEFORMATION ANALYSIS OF THE SECOND HELICAL SPRING MODEL . . . 24

V. SUMMARY AND RECOMMENDATIONS 45

PROGRAMS AND PLOTS 47

LIST OF REFERENCES 61

INITIAL DISTRIBUTION LIST 62

I. INTRODUCTION

The study of the deformation of helical springs has most commonly been limited to those cases due to normal loading [Ref. 1, 4, 5, 6, 7, 8]. In normal loading, a force is applied through the center of a spring, and the displacement of the spring is expressed as a function of the load and the parameters of the spring involving material properties and geometry. This thesis investigates the deformation of helical springs due to eccentric loading, with the aim of designing a helical spring that will deform to a given shape through the application of known forces. The spring design will include the material selection and the selection of helical spring geometry in terms of its length, the diameters of the spring and the spring coil, and the number of coils.

Chapter II provides the background needed to study the deformation of the two helical spring models, such as the development of Castigliano's theorem, and a brief introduction of Shape Memory Alloy's (SMA's). The descriptions of the two kinematic models that have been chosen for this study are also included in this chapter. Chapter III provides the analysis for spring deformation based on the first model, and Chapter IV provides the same for the second model. Chapter V contains computer programs that will be used for simulation based design of the proper helical spring.

The motivation of this study is to design a helical spring that will be used as the primary structural element of a robotic manipulator for minimally invasive surgical applications. In viewing the spring as a skeletal element, it is necessary to analyze the deformation of the skeleton under the action of external forces, so as to control its deformation or motion. The spring deformation will be produced through the use of Shape Memory Alloy (SMA) active elements which undergo a phase transformation upon heating above a certain temperature. When these SMA's undergo their phase transformation, they change their shape to a predetermined form. During this process they will deform the spring they are acting on. By properly selecting the arrangement of the SMA active elements about the spring, and controlling their phase transformation, a properly designed spring can be made to deform to a desired shape.

II. PRELIMINARIES

Certain preliminary topics need to be reviewed before an analysis of the deformation of helical springs due to eccentric loading can be carried out. This chapter includes the development of Castigliano's theorem, an introductory examination of the properties of Shape Memory Alloy's, and descriptions of the spring models we have chosen for analysis.

A. CASTIGLIANO'S THEOREM

Castigliano's theorem is a very useful mathematical tool that can be used to study the deformation of elastic bodies under the application of generalized forces. The deformation of the elastic body is computed from the strain energy of the body. It is an energy based approach and therein lies its simplicity.

Castigliano's theorem, developed by Alberto Castigliano in 1879, is a method by which one can determine the deflection of an elastic body at the point of application of a force. If forces F_A and F_B are exerted on an elastic body at two different points, A and B, there are four associated deflections [Ref 2]

$$\delta_{AA}=f_{AA}F_A \qquad (2.1)$$

$$\delta_{AB}=f_{AB}F_B \qquad (2.2)$$

$$\delta_{BA}=f_{BA}F_A \qquad (2.3)$$

$$\delta_{BB}=f_{BB}F_B \qquad (2.4)$$

where the first subscript implies the point of interest, and the second implies the force of influence. The f's are constants, and are known as influence coefficients. They represent the deflection of one point relative to the other. For example, f_{AB} represents the deflection of point A relative to point B. These influence coefficients are properties of the elastic member.

Maxwell's law of reciprocity states [Ref. 2, p. 637]

$$f_{AB}=f_{BA} \tag{2.5}$$

which implies that the deflection at point A due to a unit force applied at point B is equal to the deflection at point B due to the application of a unit force at A.

Castigliano developed a method where the deflection due to multiple forces acting on an elastic body is obtained as the summation of deflections due to forces applied sequentially, one at a time. The final result is a set of equations like those found in (2.1) through (2.4). One must then superimpose these equations to obtain a series of equations for the displacements, or δ's, at the different points where the forces act on the elastic body. For two forces F_A and F_B we obtain

$$\delta_A=\delta_{AA}+\delta_{AB}=f_{AA}F_A+f_{AB}F_B \tag{2.6}$$

$$\delta_B=\delta_{BA}+\delta_{BB}=f_{BA}F_A+f_{BB}F_B \tag{2.7}$$

Now one needs to determine the work done by each force at each point of the elastic body where it acts. The work done by F_A at point A is

$$W_{AA}=\frac{1}{2}F_A\delta_{AA} \tag{2.8}$$

which, after substituting equation (2.1) becomes

$$W_{AA}=\frac{1}{2}F_A(f_{AA}F_A)=\frac{1}{2}f_{AA}F_A^2 \tag{2.9}$$

Likewise, the work done by F_B at point B is

$$W_{BB}=\frac{1}{2}f_{BB}F_B^2 \tag{2.10}$$

If force F_A continues acting on the body while F_B is gradually applied, we see that there is additional work done on point A due to F_B, namely

$$W_{AB}=F_A\delta_{AB} \tag{2.11}$$

Similarly, if force F_B continues acting on the body as F_A is slowly applied, one can see that there is additional work done on point B due to F_A, namely

$$W_{BA}=F_B\delta_{BA} \tag{2.12}$$

After substituting equations (2.2) and (2.3) into equations (2.11) and (2.12), and by invoking Maxwell's law of reciprocity, equation (2.5), one obtains

$$W_{AB}=f_{AB}F_AF_B \tag{2.13}$$

$$W_{BA}=f_{AB}F_AF_B \tag{2.14}$$

The total work done on the body, or its total strain energy, if F_A is applied before F_B is the summation of equations (2.9), (2.13), and (2.10):

$$W=U=\frac{1}{2}(f_{AA}F_A^2+2f_{AB}F_AF_B+f_{BB}F_B^2) \tag{2.15}$$

If force F_B is applied first, and then F_A, the order of addition becomes equations (2.10), (2.14), and (2.9)

$$W=U=\frac{1}{2}(f_{BB}F_B^2+2f_{AB}F_AF_B+f_{AA}F_A^2) \tag{2.16}$$

Thus, the total work done on the body is irrespective of whether F_A or F_B is applied first.

From Equation (2.16) one finds that the displacement at point A is equal to

$$\frac{\partial U}{\partial F_A}=f_{AA}F_A+f_{AB}F_B=\delta_A \tag{2.17}$$

and the displacement at point B is equal to

$$\frac{\partial U}{\partial F_B}=f_{AB}F_A+f_{BB}F_B=\delta_B \tag{2.18}$$

Castigliano's theorem states that for any force F_i acting on an elastic body, the deformation or deflection at the point of application of the force F_i is

$$\delta_i=\frac{\partial U}{\partial F_i} \tag{2.19}$$

in the direction of F_i, where U is the total strain energy of the elastic body under the application of forces.

B. SHAPE MEMORY ALLOYS

To cause the deformation of a helical spring one needs to apply an external force. Shape Memory Alloy (SMA) active elements were chosen to provide the necessary external forces. An SMA active element has low mass and a very high force to mass ratio; this attractive feature allows the miniaturization of the whole structure. A brief introduction of SMA's is imperative to have a good understanding of how the spring will be deformed.

A Shape Memory Alloy (SMA), is a metallic alloy that is given a certain predetermined shape at a high temperature. Once the alloy is cooled, it can be deformed, and will remain deformed until heated. Once heated above a certain temperature, the alloy "remembers" its undeformed shape and returns to it.

There are many alloys that exhibit this shape memory effect. Among them are Ni-Ti, Ni-Ti-Cu, Cu-Al-Ni-Mn. The alloy is first shaped into its desired "undeformed" shape at a high temperature, when the microstructure is in its austenite phase. These "undeformed" shapes can vary greatly, but the primary shapes we are considering are those of a thin wire of a given length, or a rod with a given circular curve. Once formed, the alloy is quenched to allow the microstructure to come into its martensite phase. It is now ready to be deformed.

The alloy can now be deformed by stretching it, bending it, or reshaping it by any one of a number of means. It will stay deformed from its original shape until it is once again heated up back into the austenite region, where it will return to its original shape. It is beyond the scope of this thesis to present the microscopic analysis of this transformation. It is merely intended to explain what an SMA is and how it will be used as an actuator for spring deformation.

The next two sections provide two spring models that describe the positioning of the SMA actuators relative to the spring. The SMA elements are in their deformed states initially, and they revert back to their undeformed states once heated above a certain temperature. During this process the spring is deformed. The undeformed shape or the memory shape can be given to the alloy by annealing for some time at a fixed temperature and then by rapid cooling back to room temperature. In the discussion to follow it will be assumed that the SMA has already been given its memory shape and attention will be focused on the design of the helical spring for achieving the design goal.

C. MODEL 1: HELICAL SPRING UNDER ECCENTRIC COMPRESSION

This section considers a spring model consisting of a helical spring along with its actuators such that the spring can provide two rotational degrees of freedom besides a single degree of freedom for linear translation. In this model, shown in Figure 2.1, the helical spring is fitted with two end caps. Three SMA wire actuators are attached to the end caps just outside of the spring. All of the SMA wires are to be of the same length, spaced 120° apart. During the process in which the SMA wires are placed, the spring is given a small initial bias compression. This keeps the SMA wires taut and eliminates any slack in the wires.

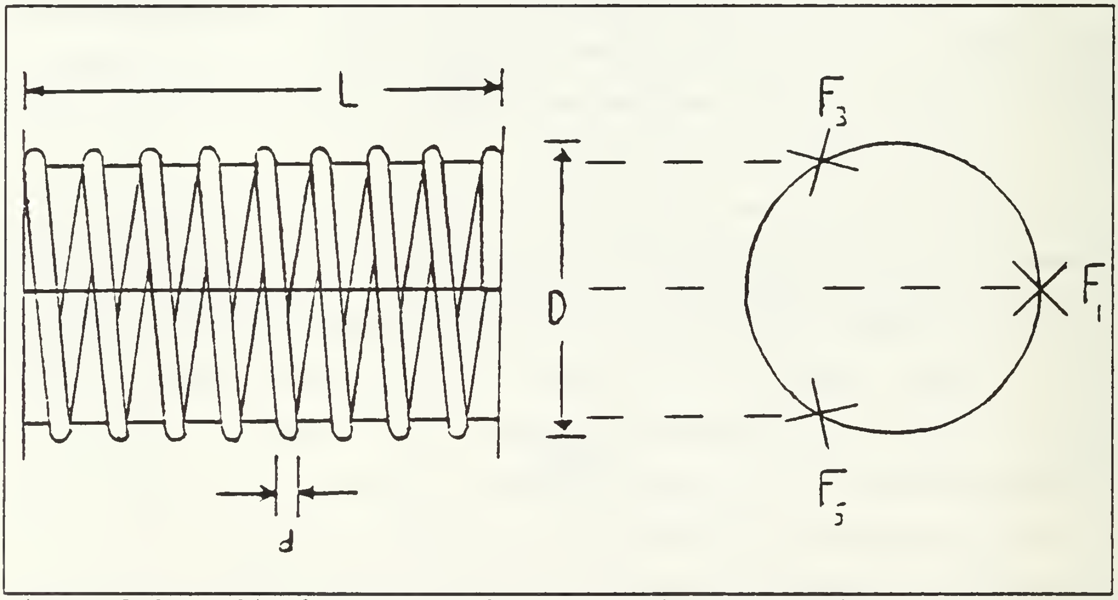


Figure 2.1 Helical spring under eccentric compression.

As current is applied to heat one of the SMA's, the active element shrinks back to its original "undeformed" length. During this process the other two SMA wires remain in their deformed configuration, and the top plane of the spring bends over by virtue of eccentric compression.

F_1, F_3, and F_5 are forces exerted on the spring by the SMA wires, and F_2, F_4, and F_6 are dummy forces. The deflection of the points of application of these forces can be readily obtained using Castigliano's theorem. Position vectors, r_i's, from an arbitrary point A to the points where the forces are applied are constructed. Angle θ is a measurement taken from the point where F_1 is applied, around in a counter-clockwise manner. Using these position vectors, moments and torsions due to forces F_1 through F_6 are summed up at A.

R is the radius of the spring, and L is the length of the spring. E in the modulus of elasticity of the spring material, and G is the shear modulus of that material. I is the area moment of inertia of the cross section of the spring coil, and J is the polar moment of inertia. The number of spring coils is n.

Using these values, the total stain energy of the spring can be calculated. Once this has been done, Castigliano's theorem is invoked, and the displacement of the spring at any one of the six points of application of the forces can be found. Knowing the relative displacements of the different points on the spring coil, the angle of deflection can be computed.

For the spring design problem, the angle of deflection is predetermined. When a helical spring is chosen, R, I, J, E, G, and n are known. From these quantities the force required to deflect the spring is calculated. If this force is one that the SMA wire can exert on the spring, the spring has been properly designed. If not, some of the parameters of the spring, geometry or material, must be changed and the forces recomputed. When the force required to deflect the spring matches the force the SMA can exert, the design problem is completed.

D. MODEL 2: HELICAL SPRING UNDER BENDING

In the second spring model, the assumption is that the spring bends under the action of SMA rods. Three SMA rods are attached to the spring internally along its length and placed 120° apart. Initially the SMA rods are in their deformed shape. When one of the SMA rods is heated it regains its undeformed shape and bends the spring in the process.

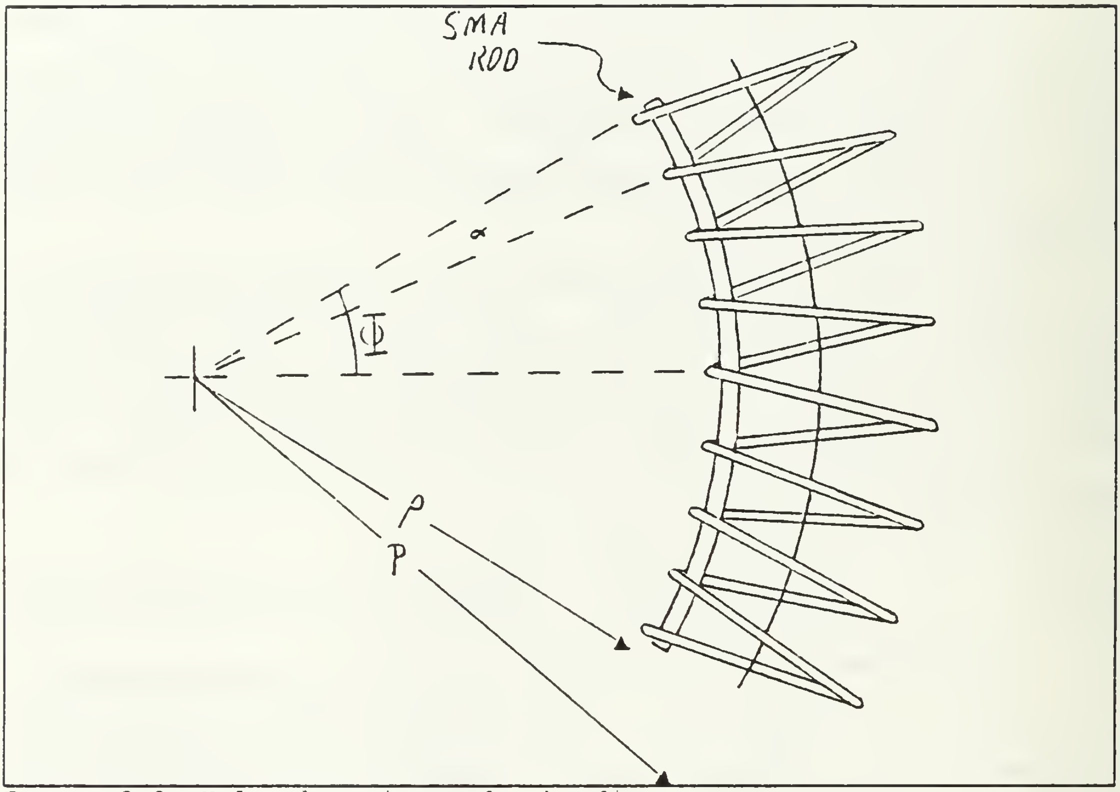


Figure 2.2 Helical spring under bending.

The SMA rod applies a force to each of the spring coils. Each of these forces are assumed to have two components, one normal to the coil directed towards the center of curvature of the spring denoted F, and the other in the tangential direction denoted f. From Figure 2.2, θ is an angular measurement internal to the spring coil measured from the outer most point on the spring, moving in a counter-clockwise direction. R is the radius of the spring. The radius of curvature measured to the center of the spring is P, and ρ is the radius of curvature to the inside point of each of the coils as they are bent. Angle Φ is the angle of curvature of half of the spring. L is the length of the spring, n is the number of coils, E is the modulus of elasticity of the spring material, G is the shear modulus, I is the area moment of inertia, and J is the polar moment of inertia. For an arbitrary point A on the spring, r denotes the position vectors from A to the points of application of the forces, and α is the angle subtended by each coil at the center of curvature of the spring.

Consider now only the top half of the spring, since the spring is symmetric and the bottom half is identical to the top. When an SMA rod is bent through the application of current, each coil of the spring is acted upon by two forces, F and f. Using these forces, and the position vectors from point A to their point of application, a summation of all of the bending moments and torsions at A due to the action of these forces can be obtained. The total strain energy due to these moments and torsions is calculated, and Castigliano's theorem is applied to compute the displacements at each point of application of the forces.

To reiterate, for the helical spring design problem, the spring's displacement is a given. Knowing the equations for the strain energy, and the material and geometry of the spring, one can work backwards to find the force required to bend the spring in this manner. One needs only to iterate using the spring material and geometry to find a force commensurate with the given displacement.

III. DEFORMATION ANALYSIS OF THE FIRST HELICAL SPRING MODEL

Recalling the description of the first spring model described in the preliminaries, one finds that F_1, F_3, and F_5 are forces exerted by the SMA wires on the spring. F_2, F_4, and F_6 are dummy forces on the spring needed to find the displacements at their points of application. Point A is an arbitrary point on the spring, and θ is the angular measurement from F_1 around in a counter-clockwise direction.

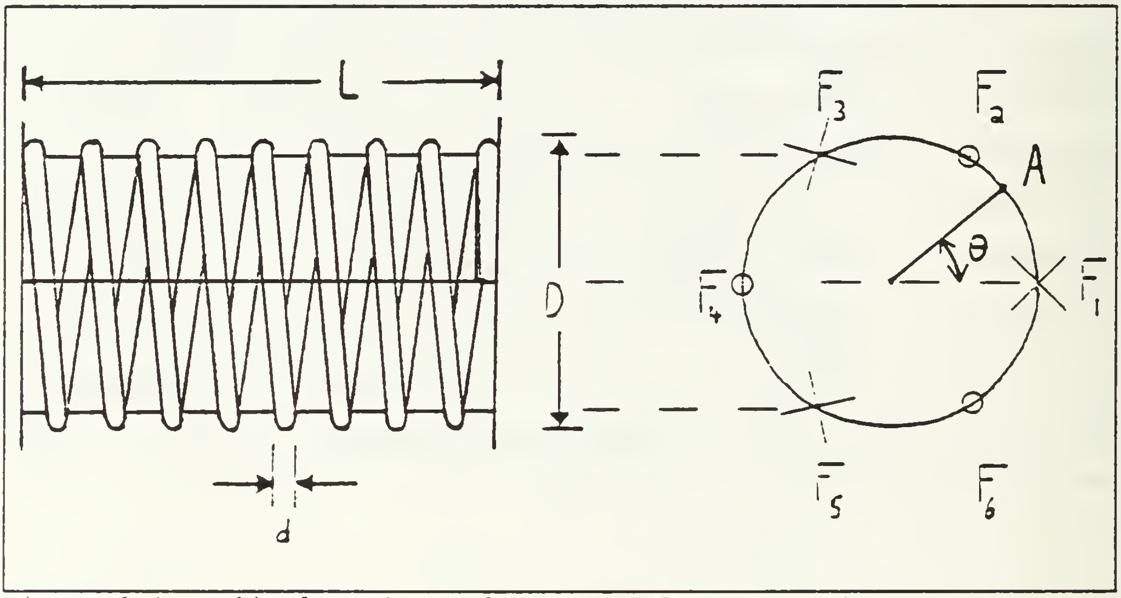


Figure 3.1 Helical spring under eccentric compression.

There are six position vectors from the six points on the spring where the forces are applied to point A.

$$\vec{r}_1=R(1-\cos\theta)\hat{\imath}-R\sin\theta\hat{\jmath} \tag{3.1}$$

$$\vec{r}_2=-R(\cos\theta-\cos60°)\hat{\imath}+R(\sin60°-\sin\theta)\hat{\jmath} \tag{3.2}$$

$$\vec{r}_3=-R(\cos60°+\cos\theta)\hat{\imath}+R(\sin60°-\sin\theta)\hat{\jmath} \tag{3.3}$$

$$\vec{r}_4=-R(1+\cos\theta)\hat{\imath}-R\sin\theta\hat{\jmath} \tag{3.4}$$

$$\vec{r}_5=-R(\cos 60°+\cos\theta)\hat{\imath}-R(\sin 60°+\sin\theta)\hat{\jmath} \tag{3.5}$$

$$\vec{r}_6=-R(\cos\theta-\cos 60°)\hat{\imath}-R(\sin\theta+\sin 60°)\hat{\jmath} \tag{3.6}$$

Summation of moments about point A yields

$$\sum M_A=M_x\hat{\imath}+M_y\hat{\jmath}+\sum_{i=1}^{6}(\vec{r}_i\times\vec{F}_i)=0 \tag{3.7}$$

where,

$$\vec{F}_i=-F_i\hat{k} \quad i=1,2,\ldots,6 \tag{3.8}$$

and where M_x and M_y are the x and y components of the reaction moments at point A.

After computing the cross products of $(r_i \times F_i)$, one can set the like vector components equal to each other and find

$$\begin{aligned} M_x=R[&-(F_1+F_4)\sin\theta+(F_2+F_3)(\sin 60°-\sin\theta)\\ &-(F_5+F_6)(\sin 60°+\sin\theta)] \end{aligned} \tag{3.9}$$

$$\begin{aligned} M_y=R[&-F_1(1-\cos\theta)+F_4(1+\cos\theta)+(F_2+F_6)(\cos\theta-\cos 60°)\\ &+(F_3+F_5)(\cos\theta+\cos 60°)] \end{aligned} \tag{3.10}$$

To compute the strain energy, one needs to find the bending moment and torsion, $\vec{m}$ and $\vec{t}$, on the spring coil. These can be obtained from the reaction moments M_x and M_y through a coordinate transformation in the following manner:

$$\begin{pmatrix} \vec{m} \\ \vec{t} \end{pmatrix} = T(\theta) \begin{pmatrix} M_x \\ M_y \end{pmatrix} \tag{3.11}$$

where

$$T(\theta) \equiv \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{3.12}$$

Following the coordinate conversions, the moment and torsion equations become

$$\begin{aligned} \vec{m} = R[& (-F_1+F_4+(-F_2+F_3+F_5-F_6)\cos 60°)\sin\theta \\ & +(F_2+F_3-F_5-F_6)\sin 60°\cos\theta]\ \vec{\varepsilon}_n \end{aligned} \tag{3.13}$$

$$\begin{aligned} \vec{t} = R[& (F_1+F_2+F_3+F_4+F_5+F_6)+(-F_2-F_3+F_5+F_6)\sin 60°\sin\theta \\ & +(-F_1+F_4+(-F_2+F_3+F_5-F_6)\cos 60°)\cos\theta]\ \vec{\varepsilon}_t \end{aligned} \tag{3.14}$$

The shear force at any point of the spring coil can be obtained from the static equilibrium of forces:

$$\vec{v} = (F_1+F_2+F_3+F_4+F_5+F_6)\,\hat{k} \tag{3.15}$$

The expression for strain energy due to shear is

$$U_V=\int_0^L \frac{fv^2}{2GA}dx=\int_0^{2n\pi} \frac{fv^2}{2GA}Rd\theta \tag{3.16}$$

where U_V is the energy due to shear, *f* is the form factor (10/9 for a solid circle), V is the shear force, L is the length of the spring ($2n\pi$), G is the shear modulus, A is the cross-sectional area of the spring wire, and x is the integration variable denoting the length of the spring coil. After making the substitution of J, the polar moment of inertia, for the circular area A by the expression

$$A=\frac{2J}{r^2}$$

and integrating with respect to θ from 0 to $2n\pi$, the energy due to shear becomes

$$U_V=\frac{10n\pi r^2V^2R}{18GJ} \tag{3.17}$$

One can now see that since r is much smaller that R, the r^2 term will dominate the numerator and make U_V very small. Equation (3.17) will be compared later with the strain energy due to that of torsion to show this difference.

The strain energy due to the bending moment is obtained through the equation

$$U_m=\int_0^L \frac{m^2}{2EI}dx=\int_0^{2n\pi} \frac{m^2}{2EI}Rd\theta \tag{3.18}$$

where U_m is the strain energy due to the bending moment, m is the bending moment, L is the length of the spring ($2n\pi$), and EI is the bending stiffness of the spring. After integration one obtains

$$U_m=\frac{n\pi R^3}{2EI}(F_1^2+F_1F_2-F_1F_3-2F_1F_4-F_1F_5+F_1F_6+F_2^2+F_2F_3-F_2F_4 \\ -2F_2F_5-F_2F_6+F_3^2+F_3F_4-F_3F_5-2F_3F_6+F_4^2 \\ +F_4F_5-F_4F_6+F_5^2+F_5F_6+F_6^2) \tag{3.19}$$

The strain energy due to torsion is obtained through the equation

$$U_t=\int_0^L \frac{t^2}{2JG}dx=\int_0^{2n\pi}\frac{t^2}{2JG}Rd\theta \tag{3.20}$$

where U_t is the strain energy due to torsion, t is the torsion, L is the length of the spring, and JG is the torsional stiffness of the spring. Through integration the strain energy due to torsion becomes

$$U_t=\frac{n\pi R^3}{2JG}(3F_1^2+5F_1F_2+3F_1F_3+2F_1F_4+3F_1F_5+5F_1F_6+3F_2^2 \\ +5F_2F_3+3F_2F_4+2F_2F_5+3F_2F_6+3F_3^2+5F_3F_4+3F_3F_5 \\ +2F_3F_6+3F_4^2+5F_4F_5+3F_4F_6+3F_5^2+5F_5F_6+3F_6^2) \tag{3.21}$$

A comparison of the strain energy due to shear and the strain energy due to the torsional moment U_t demonstrates how much smaller the strain energy due to shear is. Assume that their are eight coils, a spring radius of 5 mm, a spring coil radius of 0.5 mm, that forces F_1 through F_6 are applied uniformly with a unit magnitude of 1 N, and that stainless steel is used as the spring's material. The ratio of strain energy due to torsion, U_t, to the strain energy due to shear, U_v, is

$$\frac{U_t}{U_v}=\frac{9}{10}\left(\frac{R}{r}\right)^2\frac{(3F_1^2+5F_1F_2+\cdots+3F_6^2)}{(F_1+F_2+\cdots+F_6)^2}=180 \tag{3.22}$$

Since this ratio is so large, one can see that strain energy due to shear is very much smaller than the strain energy due to torsion. When the spring deforms by bending, the same can be shown for the ratio of the strain energy due to bending to the strain energy due to shear. This implies that the strain energy due to shear can be neglected.

The total strain energy is simply the addition of equations (3.19) and (3.21),

$$\begin{aligned} U=U_m+U_t=\frac{n\pi R^3}{2EI}&[F_1^2+F_1F_2-F_1F_3-2F_1F_4-F_1F_5+F_1F_6+F_2^2+F_2F_3\\ &-F_2F_4-2F_2F_5-2F_2F_6+F_3^2+F_3F_4-F_3F_5-2F_3F_6\\ &+F_4^2+F_4F_5-F_4F_6+F_5^2+F_5F_6+F_6^2]\\ &+\frac{n\pi R^3}{2JG}[3F_1^2+5F_1F_2+3F_1F_3+2F_1F_4+3F_1F_5\\ &+5F_1F_6+3F_2^2+5F_2F_3+3F_2F_4+2F_2F_5+3F_2F_6\\ &+3F_3^2+5F_3F_4+3F_3F_5+2F_3F_6+3F_4^2\\ &+5F_4F_5+3F_4F_6+3F_5^2+5F_5F_6+3F_6^2] \end{aligned} \tag{3.23}$$

By invoking Castigliano's theorem the six displacements at the points of application of the forces are obtained as:

$$\delta_1 = \frac{n\pi R^3}{2EI}[2F_1 + F_2 - F_3 - 2F_4 - F_5 + F_6] + \frac{n\pi R^3}{2JG}[6F_1 + 5F_2 + 3F_3 + 2F_4 + 3F_5 + 5F_6] \tag{3.24}$$

$$\delta_2 = \frac{n\pi R^3}{2EI}[F_1 + 2F_2 + F_3 - F_4 - 2F_5 - 2F_6] + \frac{n\pi R^3}{2JG}[5F_1 + 6F_2 + 5F_3 + 3F_4 + 2F_5 + 3F_6] \tag{3.25}$$

$$\delta_3 = \frac{n\pi R^3}{2EI}[-F_1 + F_2 + 2F_3 + F_4 - F_5 - 2F_6] + \frac{n\pi R^3}{2JG}[3F_1 + 5F_2 + 6F_3 + 5F_4 + 3F_5 + 2F_6] \tag{3.26}$$

$$\delta_4 = \frac{n\pi R^3}{2EI}[-2F_1 - F_2 + F_3 + 2F_4 + F_5 - F_6] + \frac{n\pi R^3}{2JG}[2F_1 + 3F_2 + 5F_3 + 6F_4 + 5F_5 + 3F_6] \tag{3.27}$$

$$\delta_5 = \frac{n\pi R^3}{2EI}[-F_1 - 2F_2 - F_3 + F_4 + 2F_5 + F_6] + \frac{n\pi R^3}{2JG}[3F_1 + 2F_2 + 3F_3 + 5F_4 + 6F_5 + 5F_6] \tag{3.28}$$

$$\delta_6=\frac{n\pi R^3}{2EI}[F_1-2F_2-2F_3-F_4+F_5+2F_6]$$
$$+\frac{n\pi R^3}{2JG}[5F_1+3F_2+2F_3+3F_4+5F_5+6F_6] \tag{3.29}$$

As stated before, F_2, F_4, and F_6 are dummy forces, which means that they do not really exist, they are used merely to compute the displacements at those points. Setting $F_2 = F_4 = F_6 = 0$ one obtains:

$$\delta_1=\frac{n\pi R^3}{2EI}[2F_1-F_3-F_5]+\frac{n\pi R^3}{2JG}[6F_1+3F_3+3F_5] \tag{3.30}$$

$$\delta_2=\frac{n\pi R^3}{2EI}[F_1+F_3-2F_5]+\frac{n\pi R^3}{2JG}[5F_1+5F_3+2F_5] \tag{3.31}$$

$$\delta_3=\frac{n\pi R^3}{2EI}[-F_1+2F_3-F_5]+\frac{n\pi R^3}{2JG}[3F_1+6F_3+3F_5] \tag{3.32}$$

$$\delta_4=\frac{n\pi R^3}{2EI}[-2F_1+F_3+F_5]+\frac{n\pi R^3}{2JG}[2F_1+5F_3+5F_5] \tag{3.33}$$

$$\delta_5=\frac{n\pi R^3}{2EI}[-F_1-F_3+2F_5]+\frac{n\pi R^3}{2JG}[3F_1+3F_3+6F_5] \tag{3.34}$$

$$\delta_6=\frac{n\pi R^3}{2EI}[F_1-2F_3+F_5]+\frac{n\pi R^3}{2JG}[5F_1+2F_3+5F_5] \tag{3.35}$$

Since all of the SMA actuators will be identical, the force they will exert is expected to be the same. Consequently, if all of the three actuators are activated simultaneously, the deformation at the different points on the spring will be the same. This implies that the spring will undergo compression. The interest here lies in the bending of the top plane of the spring. Therefore, all of the three SMA actuators will not be activated simultaneously. Now define β as the angle by which the top plane of the spring bends through when one or two of the SMA wires are activated simultaneously.

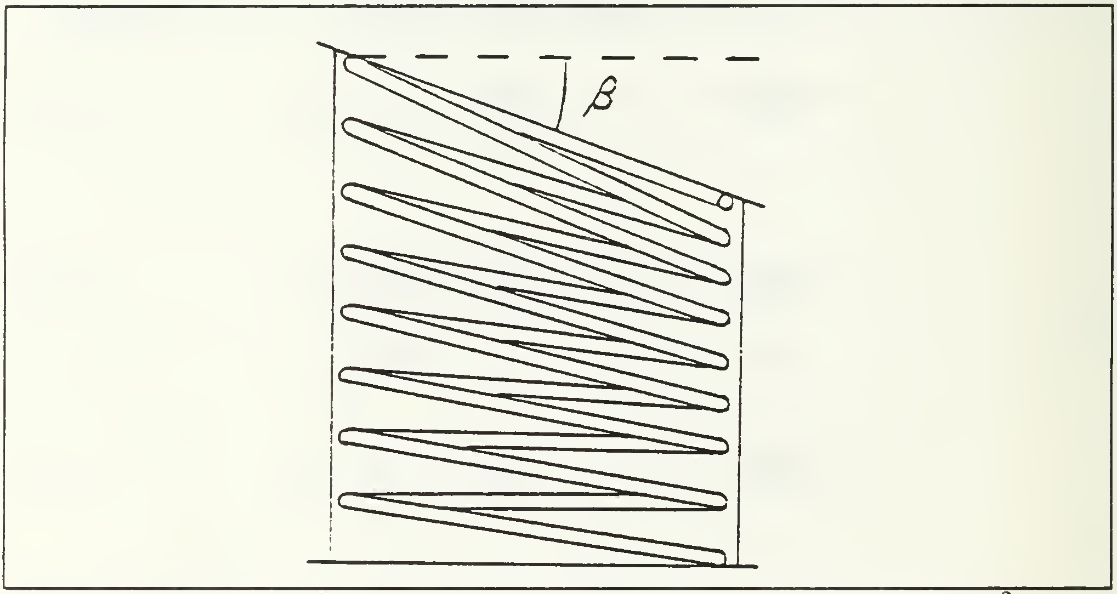


Figure 3.2 Deflection angle of the top plane of the spring, β.

The two possible cases to cause the bending of the top plane of the spring are the application of one force, and the application of two forces. In the first case one force, F_1, is applied. This single force will cause displacements to occur at all six points on the spring. Of interest are the deflections at points 1 and 4 to determine the bending angle of the top plane of the spring. The deflections at points 2, 3, 5, and 6 will be such that the bending will not take place at in any other direction. By knowing the displacements at points 1 and 4, one can calculate β. With F_1 applied one finds

$$\delta_1 = n\pi R^3 F_1\left(\frac{1}{EI} + \frac{3}{JG}\right) \tag{3.36}$$

$$\delta_4 = n\pi R^3 F_1\left(\frac{-1}{EI} + \frac{1}{JG}\right) \tag{3.37}$$

From Figure 3.3 one can see thatsimple geometry one can see that

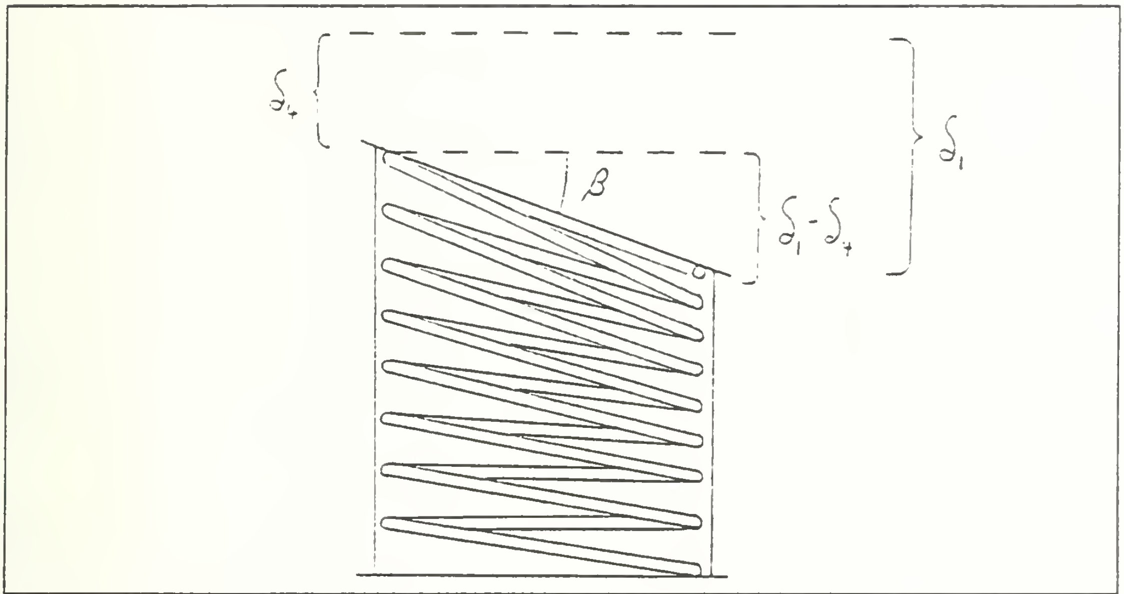


Figure 3.3 Angle of deflection, β.

$$\tan\beta = \left[\frac{\delta_1 - \delta_4}{2R}\right] \tag{3.38}$$

which means

$$\beta = \tan^{-1}\left[n\pi R^2 F_1\left(\frac{1}{EI} + \frac{1}{JG}\right)\right] \tag{3.39}$$

When any one force, say F_1, is applied on the spring, the angle β is given by

$$\beta = \tan^{-1}\left[n\pi R^2 F_1\left(\frac{1}{EI}+\frac{1}{JG}\right)\right] \tag{3.40}$$

The other important case is when two forces are applied simultaneously, say F_1 and F_3. In this case the important displacements are at points 2 and 5, where

$$\delta_2 = n\pi R^3 (F_1+F_3)\left(\frac{1}{2EI}+\frac{5}{2JG}\right) \tag{3.41}$$

$$\delta_5 = n\pi R^3 (F_1+F_3)\left[\frac{-1}{2EI}+\frac{3}{2JG}\right] \tag{3.42}$$

This implies that for the application of the two forces F_1 and F_3

$$\beta = \tan^{-1}\left[\frac{1}{2}n\pi R^2 (F_1+F_3)\left(\frac{1}{EI}+\frac{1}{JG}\right)\right] \tag{3.43}$$

Since each of the SMA wire actuators will be identical, the force that each will exert on the spring will be equal in magnitude. This means, then, if two SMA wires are actuated the resultant angle of deflection will be

$$\beta = \tan^{-1}\left[n\pi R^2 F\left(\frac{1}{EI}+\frac{1}{JG}\right)\right] \tag{3.44}$$

From these results one finds that if one SMA wire is actuated, or two are, the angle of deflection is the same. The difference is the direction in which the helical spring bends over. It is now clear that by activating one or two actuators it will be possible to bend the top coil of the spring in three separate directions by a positive or a negative angle β. The case of actuating all three wires has not been investigated since this would produce the same result as that of applying a single axial force through the center of the spring, or applying a normal load.

Now that the effects of helical spring geometry and material properties are known in computing the deflection angle β, use of these relationships will be instrumental in designing the proper helical spring. SMA wires deform by a constant amount of almost 5% of their original length in the presence or absence of external forces. However, for this property of the SMA to be exhibited repeatedly, the opposing stresses in the SMA wires should not exceed a certain value. The design of the helical spring will be computed from a known value of β that will give the force required to produce that deflection. If these forces produce stresses that are below the stress limit of the SMA wires, then the design is feasible. However, one should try to increase the stresses in the wire as high as possible, within limits, such that the spring to be designed is not too soft. A spring that is too soft will not be strong enough to act as a manipulator for surgical applications.

IV. DEFORMATION ANALYSIS OF THE SECOND HELICAL SPRING MODEL

The second helical spring model considered is one where the spring is bent over directly by an SMA rod. Three rods will actually be placed inside the spring 120° apart to give the spring a full range of motion. Here, consideration is given to only one SMA rod for the development of the model.

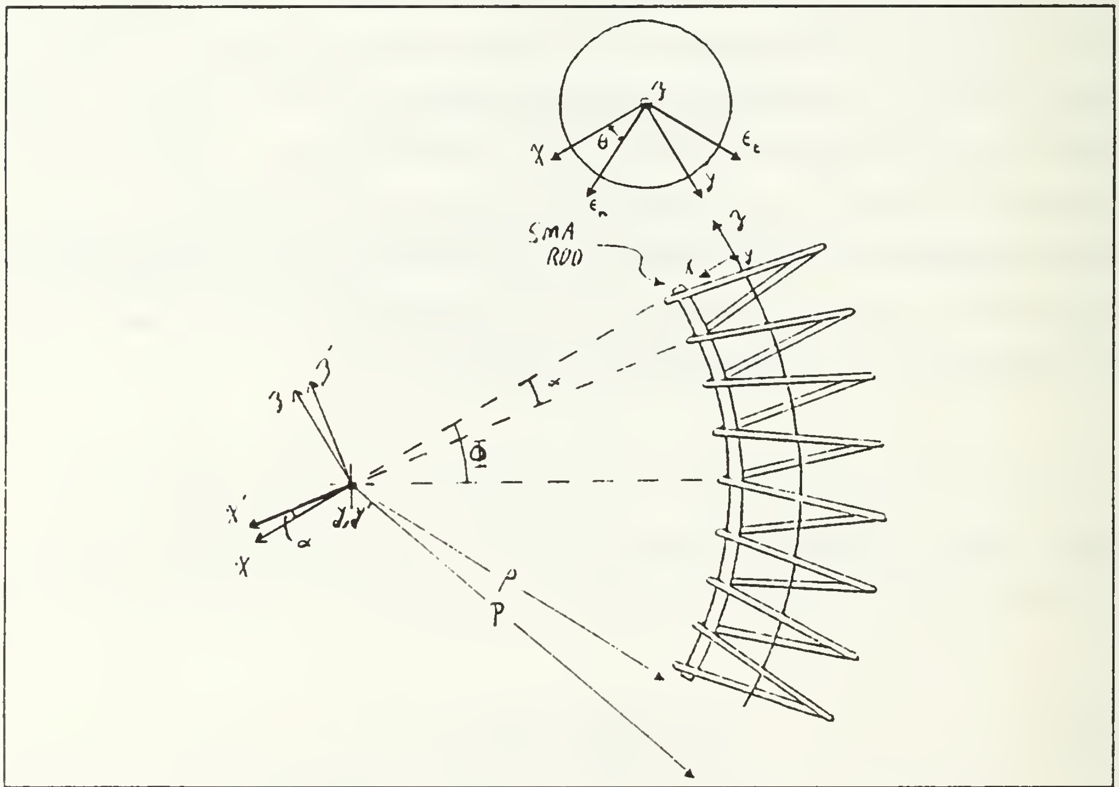


Figure 4.1 Helical spring under bending.

The only major assumption made is that the length of the spring is the same in the bent over configuration as in the undeformed configuration. With this in mind, certain quantities are defined. Let R be the radius of the spring and L be the length of the undeformed spring. In the deformed configuration, it is desired that the spring bends into the shape of a circular arc. Let P be the radius of curvature of the central axis of the spring, and ρ the radius of curvature of the inner part of the spring. Let Φ be the angle between the top coil of the spring and the central coil, ϕ be the angular change of spring coils, θ be the angular measurement along the length of the coils of the spring, α be the angle subtended by two adjacent spring coils, and n be the total number of coils. From these definitions, one sees from Figure 4.1 that the following identities hold:

$$\Phi=\frac{L}{2\mathbf{P}} \tag{4.1}$$

$$\alpha=\frac{2\Phi}{n} \tag{4.2}$$

$$\phi=\left(\frac{\Phi}{n\pi}\right)\theta \tag{4.3}$$

$$\rho=\mathbf{P}-R \tag{4.4}$$

In this model there are three different coordinate frames that are used. The traditional $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ frame is fixed on the top coil of the spring. The $\hat{\imath}'$, $\hat{\jmath}'$, and $\hat{k}'$ frame is relative to each succeeding coil. The ε_n and ε_t frame is a normal and tangential transformation of the $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ frame.

Suppose that as the SMA bends over, it exerts two forces on each spring coil it touches, F in the normal direction, and f in the tangential direction. The line of action of the normal forces passes through the center of curvature. For the ease of computation, it is helpful to translate the point of application of the normal force along its line of action to the center of curvature of the spring. Point A, at the top center of the spring, is the point where position vectors from all other points on the spring will be constructed to. In considering only the first coil, there are two position vectors from each point on that coil, one corresponding to the normal force F, and one to the tangential force f. These position vectors have been defined this way so that they will apply to each coil for these forces. They are

$$\vec{r}_1=(P-R\cos\theta)\hat{i}'-R\sin\theta\hat{j}' \tag{4.5}$$

$$\vec{r}_1{}'=(P-R\cos\theta)\hat{i}'-R\sin\theta\hat{j}'-\rho\hat{i} \tag{4.6}$$

The primes on the i and j vectors indicate that they are in the spring coil's frame of reference, where the unprimed vectors are in the fixed frame of the top coil. These vectors will give the position from any point on the first coil relative to point A. After recognizing that

$$\vec{F}_1= F_1\hat{i} = F_1\cos\phi\hat{i}'-F_1\sin\phi\hat{k}' \tag{4.7}$$

$$\vec{f}_1= f_1\hat{k} = f_1\sin\phi\hat{i}'+f_1\cos\phi\hat{k}' \tag{4.8}$$

the total moment about point A caused by these forces can be calculated by

$$M_{A_1}=(\vec{r}_1\times\vec{F}_1)+(\vec{r}_1{}'\times\vec{f}_1) \tag{4.9}$$

For the second coil the position vectors remain the same, but the forces acting on this coil are F_1, f_1, F_2, and f_2 where

$$\vec{F}_1=F_1\hat{i}=F_1\cos(\alpha+\phi)\hat{i}'-F_1\sin(\alpha+\phi)\hat{k}' \tag{4.10}$$

$$\vec{f}_1=f_1\hat{k}=f_1\sin(\alpha+\phi)\hat{i}'+f_1\cos(\alpha+\phi)\hat{k}' \tag{4.11}$$

$$\vec{F}_2=F_2\cos\phi\hat{\imath}'-F_2\sin\phi\hat{k}' \tag{4.12}$$

$$\vec{f}_2=f_2\sin\phi\hat{\imath}'+f_2\cos\phi\hat{k}' \tag{4.13}$$

Now the total moment about point A is

$$M_{A_2}=(\vec{r}_1\times\vec{F}_1)+(\vec{r}_1{}'\times\vec{f}_1)+(\vec{r}_1\times\vec{F}_2)+(\vec{r}_1{}'\times\vec{f}_2) \tag{4.14}$$

By proceding in a like manner, one arrives at the k-th coil, where this coil is acted upon by the forces F_1, F_2, ..., F_k, and f_1, f_2, ..., f_k. These forces take the form

$$\vec{F}_1=F_1\cos[(k-1)\alpha+\phi]\hat{\imath}'-F_1\sin[(k-1)\alpha+\phi]\hat{k}' \tag{4.15}$$

$$\vec{F}_2=F_2\cos[(k-2)\alpha+\phi]\hat{\imath}'-F_2\sin[(k-2)\alpha+\phi]\hat{k}' \tag{4.16}$$

$$\vdots$$

$$\vec{F}_k=F_k\cos\phi\hat{\imath}'-F_k\sin\phi\hat{k}' \tag{4.17}$$

$$\vec{f}_1=f_1\sin[(k-1)\alpha+\phi]\hat{\imath}'+f_1\cos[(k-1)\alpha+\phi]\hat{k}' \tag{4.18}$$

$$\vec{f}_2=f_2\sin[(k-2)\alpha+\phi]\hat{\imath}'+f_2\cos[(k-2)\alpha+\phi]\hat{k}' \tag{4.19}$$

$$\vdots$$

$$\vec{f}_k=f_k\sin\phi\hat{\imath}'+f_k\cos\phi\hat{k}' \tag{4.20}$$

Now the total moment about point A becomes

$$M_{A_k}=\sum_{i=1}^{k}(\vec{r}_i\times\vec{F}_i)+\sum_{i=1}^{k}(\vec{r}_i{}'\times\vec{f}_i) \tag{4.21}$$

or, after computing the cross products

$$\begin{aligned} M_A=\sum_{i=1}^{k}\{&R\sin\theta(F_i\sin[(k-i)\alpha+\phi]-f_i\cos[(k-i)\alpha+\phi])\hat{i}' \\ &+[(\mathbf{P}-R\cos\theta)(F_i\sin[(k-i)\alpha+\phi] \\ &-f_i\cos[(k-i)\alpha+\phi])+f_i\rho]\hat{j}' \\ &+R\sin\theta(F_i\cos[(k-i)\alpha+\phi]+f_i\sin[(k-i)\alpha+\phi])\hat{k}'\} \end{aligned} \tag{4.22}$$

The easiest way to work with this moment is to break it into parts as follows

$$\begin{aligned} M_{x'}=\sum_{i=1}^{k}\{&F_i(-R\sin\theta\sin[(k-i)\alpha+\phi]) \\ &+f_i(R\sin\theta\cos[(k-i)\alpha+\phi])\} \end{aligned} \tag{4.23}$$

$$\begin{aligned} M_{y'}=\sum_{i=1}^{k}\{&F_i(-(\mathbf{P}-Rcos\theta)\sin[(k-i)\alpha+\phi]) \\ &+f_i((\mathbf{P}-Rsin\theta)\cos[(k-i)\alpha+\phi]-\rho)\} \end{aligned} \tag{4.24}$$

$$\begin{aligned} M_{z'}=\sum_{i=1}^{k}\{&F_i(-R\sin\theta\cos[(k-i)\alpha+\phi]) \\ &+f_i(-R\sin\theta\sin[(k-i)\alpha+\phi])\} \end{aligned} \tag{4.25}$$

A coordinate transformation is now in order to transform the moments from the coil fixed frame to a normal-tangential frame. The transformation matrix T(θ), from equation (3.16), will be used such that

$$\begin{pmatrix} \varepsilon_n \\ \varepsilon_t \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \tag{4.26}$$

The moment equations (4.23) through (4.25) become

$$M_n = \sum_{i=1}^{k} \{F_i(-P\sin\theta\sin[(k-i)\alpha+\phi]) + f_i(P\sin\theta\cos[(k-i)\alpha+\phi] - \rho\sin\theta)\} \tag{4.27}$$

$$M_t = \sum_{i=1}^{k} \{F_i((R-P\cos\theta)\sin[(k-i)\alpha+\phi]) + f_i(-(R-P\cos\theta)\cos[(k-i)\alpha+\phi] - \rho\sin\theta)\} \tag{4.28}$$

$$M_{z'} = \sum_{i=1}^{k} \{F_i(-R\sin\theta\cos[(k-i)\alpha+\phi]) + f_i(-R\sin\theta\sin[(k-i)\alpha+\phi])\} \tag{4.29}$$

With each of these moments in the proper frame of reference, the strain energy due to each of them must be calculated. With the subscript k on each of the energies to remind one that this is the strain energy of the k-th coil, the strain energy due to each of these moments is given by

$$U_{n,k}=\int_0^{2\pi}\frac{M_n^2}{2EI}Rd\theta \tag{4.30}$$

$$U_{t,k}=\int_0^{2\pi}\frac{M_t^2}{2JG}Rd\theta \tag{4.31}$$

$$U_{z',k}=\int_0^{2\pi}\frac{M_{z'}^2}{2EI} \tag{4.32}$$

Since each of the moment terms are expressed as a sum of k quantities, the integrals involving the square of the moments becomes quite cumbersome. To simplify the computation, one needs to use matrix algebra. First define the incremental force vector as follows

$$\Delta F = \begin{pmatrix} \Delta F_1 \\ \Delta F_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta F_k \\ \cdots \\ \Delta f_1 \\ \Delta f_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta f_k \end{pmatrix} \in \mathbb{R}^{2k} \tag{4.33}$$

Then define

$$A_1 = \begin{pmatrix} -P\sin\theta[(k-1)\alpha+\phi] \\ -P\sin\theta\sin[(k-2)\alpha+\phi] \\ \cdot \\ \cdot \\ \cdot \\ -P\sin\theta\sin\phi \end{pmatrix} \in \mathbb{R}^k \tag{4.34}$$

$$A_2 = \begin{pmatrix} P\sin\theta\cos[(k-1)\alpha+\phi]-\rho\sin\theta \\ P\sin\theta\cos[(k-2)\alpha+\phi]-\rho\sin\theta \\ \cdot \\ \cdot \\ \cdot \\ P\sin\theta\cos\phi-\rho\sin\theta \end{pmatrix} \in \mathbb{R}^k \tag{4.35}$$

It can now be shown that

$$M_n = F^T \begin{pmatrix} A_1 \\ \cdots \\ A_2 \end{pmatrix} \tag{4.36}$$

Likewise for the torsional moment, define

$$B_1 = \begin{pmatrix} (R-P\cos\theta)\sin[(k-1)\alpha+\phi] \\ (R-P\cos\theta)\sin[(k-2)\alpha+\phi] \\ \cdot \\ \cdot \\ \cdot \\ (R-P\cos\theta)\sin\phi \end{pmatrix} \tag{4.37}$$

$$B_2 = \begin{pmatrix} -(R-P\cos\theta)\cos[(k-1)\alpha+\phi]-\rho\cos\theta \\ -(R-P\cos\theta)\cos[(k-2)\alpha+\phi]-\rho\cos\theta \\ \cdot \\ \cdot \\ \cdot \\ -(R-P\cos\theta)\cos\phi-\rho\cos\theta \end{pmatrix} \tag{4.38}$$

Then,

$$M_t = F^T \begin{pmatrix} B_1 \\ \cdots \\ B_2 \end{pmatrix} \tag{4.39}$$

For the moment in the z' direction, define

$$C_1=\begin{pmatrix}-R\sin\theta\cos[(k-1)\alpha+\phi]\\-R\sin\theta\cos[(k-2)\alpha+\phi]\\ \cdot\\ \cdot\\ \cdot\\ -R\sin\theta\cos\phi\end{pmatrix} \tag{4.40}$$

$$C_2=\begin{pmatrix}-R\sin\theta\sin[(k-1)\alpha+\phi]\\-R\sin\theta\sin[(k-2)\alpha+\phi]\\ \cdot\\ \cdot\\ \cdot\\ -R\sin\theta\sin\phi\end{pmatrix} \tag{4.41}$$

Then,

$$M_{z'}=F^T\begin{pmatrix}C_1\\ \cdots\\ C_2\end{pmatrix} \tag{4.42}$$

With the moments defined by Equations (4.36), (4.39), and (4.42), one can obtain their squares as follows:

$$M_n^2=\Delta F^T\cdot\begin{pmatrix}A_1\\ \cdots\\ A_2\end{pmatrix}\cdot(A_1\,\vdots\,A_2)\cdot\Delta F \tag{4.43}$$

$$M_t^2=\Delta F^T\cdot\begin{pmatrix}B_1\\ \cdots\\ B_2\end{pmatrix}\cdot(B_1\,\vdots\,B_2)\cdot\Delta F \tag{4.44}$$

$$M_{z'}^2=\Delta F^T \cdot \begin{pmatrix} C_1 \\ \cdots \\ C_2 \end{pmatrix} \cdot (C_1 \vdots C_2) \cdot \Delta F \tag{4.45}$$

Now define

$$\boldsymbol{A}=\begin{pmatrix} A_1 \\ \cdots \\ A_2 \end{pmatrix} \cdot (A_1 \vdots A_2) = \begin{bmatrix} A_1A_1 & \vdots & A_1A_2 \\ \cdots & \cdots & \cdots \\ A_2A_1 & \vdots & A_2A_2 \end{bmatrix} = \begin{bmatrix} A_{11} & \vdots & A_{12} \\ \cdots & \cdots & \cdots \\ A_{21} & \vdots & A_{22} \end{bmatrix} \in \mathbb{R}^{2k\times 2k} \tag{4.46}$$

one arrives at

$$M_n^2=\Delta F^T \cdot \boldsymbol{A} \cdot \Delta F \tag{4.47}$$

where the elements of A_{11} are of the form

$$A_{11}(i,j)=\mathbf{P}^2 \sin^2\theta \sin[(k-i)\alpha+\phi]\sin[(k-j)\alpha+\phi] \tag{4.48}$$

the elements of A_{12} are of the form

$$\begin{aligned} A_{12}(i,j)=-\mathbf{P}^2 \sin^2\theta \sin[(k-i)\alpha+\phi]\cos[(k-j)\alpha+\phi] \\ +\mathbf{P}\rho\sin^2\theta \sin[(k-i)\alpha+\phi] \end{aligned} \tag{4.49}$$

$$A_{21}(i,j)=A_{12}(i,j)^T \tag{4.50}$$

and the elements of A_{22} are of the form

$$A_{22}(i,j)=P^2\sin^2\theta\cos[(k-i)\alpha+\phi]\cos[(k-j)\alpha+\phi]$$
$$-P\rho\sin^2\theta(\cos[(k-i)\alpha+\phi]+\cos[(k-j)\alpha+\phi])$$
$$+\rho^2\sin^2\theta \tag{4.51}$$

Similarly, by defining

$$\boldsymbol{B}=\begin{pmatrix}B_1\\ \cdots\\ B_2\end{pmatrix}\cdot(B_1\vdots B_2)=\begin{bmatrix}B_1B_1 & \vdots & B_1B_2\\ \cdots & \cdots & \cdots\\ B_2B_1 & \vdots & B_2B_2\end{bmatrix}=\begin{bmatrix}B_{11} & \vdots & B_{12}\\ \cdots & \cdots & \cdots\\ B_{21} & \vdots & B_{22}\end{bmatrix} \tag{4.52}$$

one can see that

$$M_t^2=\Delta F^T\cdot\boldsymbol{B}\cdot\Delta F \tag{4.53}$$

where elements of B_{11} take the form

$$B_{11}(i,j)=(R-P\cos\theta)^2\sin[(k-i)\alpha+\phi]\sin[(k-j)\alpha+\phi] \tag{4.54}$$

the elements of B_{12} take the form

$$B_{12}(i,j)=-(R-P\cos\theta)^2\sin[(k-i)\alpha+\phi]\cos[(k-j)\alpha+\phi]$$
$$-(R-P\cos\theta)\rho\cos\theta\sin[(k-i)\alpha+\phi] \tag{4.55}$$

$$B_{21}(i,j)=B_{12}(i,j)^T \tag{4.56}$$

and the elements of B_{22} take the form

$$B_{22}(i,j)=(R-P\cos\theta)^2\cos[(k-i)\alpha+\phi]\cos[(k-j)\alpha+\phi]$$
$$+(R-P\cos\theta)\rho\cos\theta(\cos[(k-i)\alpha+\phi]+\cos[(k-j)\alpha+\phi])\qquad(4.57)$$
$$+\rho^2\cos^2\theta$$

By defining

$$\boldsymbol{C}=\begin{pmatrix}C_1\\ \cdots\\ C_2\end{pmatrix}\cdot(C_1\vdots C_2)=\begin{bmatrix}C_1C_1 & \vdots C_1C_2 & \\ \cdots & \cdots & \cdots\\ C_2C_1 & \vdots C_2C_2 & \end{bmatrix}=\begin{bmatrix}C_{11} & \vdots & C_{12}\\ \cdots & \cdots & \cdots\\ C_{22} & \vdots & C_{22}\end{bmatrix}\qquad(4.58)$$

one can show that

$$M_{x'}^2=\Delta F^T\cdot\boldsymbol{C}\cdot\Delta F\qquad(4.59)$$

where elements of C_{11} take the form

$$C_{11}(i,j)=R^2\sin^2\theta\cos[(k-i)\alpha+\phi]\cos[(k-j)\alpha+\phi]\qquad(4.60)$$

the elements of C_{12} take the form

$$C_{12}(i,j)=R^2\sin^2\theta\sin[(k-i)\alpha+\phi]\cos[(k-j)\alpha+\phi]\qquad(4.61)$$

$$C_{21}(i,j)=C_{12}(i,j)^T\qquad(4.62)$$

and the elements of C_{22} take the form

$$C_{22}(i,j)=R^2\sin^2\theta\sin[(k-i)\alpha+\phi]\sin[(k-j)\alpha+\phi]\qquad(4.63)$$

Now that the squared moments are in compact form, the integrals can be evaluated for the computation of the strain energies. The strain energy in the k-th coil due to the bending moment M_n is

$$U_{n,k} = \int_0^{2\pi} \frac{M_n^2}{2EI} R\,d\theta = \Delta F^T \cdot \left[\int_0^{2\pi} \left(\frac{R}{2EI} \right) A\,d\theta \right] \cdot \Delta F \tag{4.64}$$

where one must evaluate the integral of A term by term. This means that if one defines

$$\begin{aligned} A_{11}'(i,j) &\triangleq \int_0^{2\pi} \left(\frac{R}{2EI} \right) A_{11}(i,j)\,d\theta \\ &= \frac{R}{2EI} \int_0^{2\pi} \mathbf{P}^2 \sin^2\theta \sin[(k-i)\alpha+\phi] \sin[(k-j)\alpha+\phi]\,d\theta \\ &= \left(\frac{\mathbf{P}^2 R}{2EI} \right) \left[\frac{2\pi^3}{\alpha(\alpha^2-4\pi^2)} \sin\alpha \cos[\alpha(2k-i-j+1)] + \frac{\pi}{2} \cos[\alpha(i-j)] \right] \end{aligned} \tag{4.65}$$

$$\begin{aligned} A_{12}'(i,j) &\triangleq \int_0^{2\pi} \left(\frac{R}{2EI} \right) A_{12}(i,j)\,d\theta \\ &= \left(\frac{R}{2EI} \right) \Big\{ \frac{2\pi^3 \mathbf{P}^2}{\alpha(\alpha^2-4\pi^2)} \sin\alpha \sin[\alpha(2k-i-j+1)] \\ &\quad - \frac{\pi \mathbf{P}^2}{2} \sin[\alpha(i-j)] \\ &\quad - \frac{32\pi^3 \mathbf{P}\rho}{\alpha(\alpha^2-16\pi^2)} \sin\left(\frac{\alpha}{2} \right) \sin[\alpha(k-i+\tfrac{1}{2})] \Big\} \end{aligned} \tag{4.66}$$

$$A_{21}'(i,j) \triangleq A_{12}'(i,j)^T \tag{4.67}$$

and

$$\begin{aligned}
A_{22}'(i,j) &\triangleq \int_0^{2\pi} \left(\frac{R}{2EI}\right) A_{22}(i,j)\, d\theta \\
&= \left(\frac{R}{2EI}\right)\Big\{-\frac{2\pi^3 P^2}{\alpha(\alpha^2-4\pi^2)}\sin\alpha\cos[\alpha(2k-i-j+1)] \\
&\quad + \frac{\pi P^2}{2}\cos[\alpha(i-j)] + \pi\rho^2 \\
&\quad + \frac{64\pi^3 P\rho}{\alpha(\alpha^2-16\pi^2)}\sin\left(\frac{\alpha}{2}\right)\cos\left[\frac{\alpha}{2}(2k-i-j+1)\right]\cos\left[\frac{\alpha}{2}(i-j)\right]\Big\}
\end{aligned} \tag{4.68}$$

then, the energy in the k-th coil due to the moment M_n is

$$U_{n,k} = \Delta F^T \cdot A' \cdot \Delta F = \Delta F^T \cdot \begin{bmatrix} A_{11}' & \vdots & A_{12}' \\ \cdots & \cdots & \cdots \\ A_{21}' & \vdots & A_{22}' \end{bmatrix} \cdot \Delta F \tag{4.69}$$

The energy in the k-th coil due to the torsional moment M_t takes the form

$$U_{t,k}=\int_0^{2\pi}\frac{M_t^2}{2JG}R\,d\theta=\Delta F^T\cdot\left[\int_0^{2\pi}\left(\frac{R}{2JG}\right)B\,d\theta\right]\cdot\Delta F \tag{4.70}$$

where a term by term integration, as before, yields Equation (4.75), where the elements of B_{11}', B_{12}', B_{21}', and B_{22}' are defined as follows:

$$\begin{aligned}B_{11}'(i,j)=&\left(\frac{R}{2JG}\right)\Big\{\pi\left(R^2+\frac{P^2}{2}\right)\cos[\alpha(i-j)]\\&-\left(\frac{\pi R^2}{\alpha}-\frac{2\pi\alpha RP}{\alpha^2-\pi^2}+\frac{\pi P^2(\alpha^2-2\pi^2)}{\alpha(\alpha^2-4\pi^2)}\right)\sin\alpha\cos[\alpha(2k-i-j+1)]\Big\}\end{aligned} \tag{4.71}$$

$$\begin{aligned}B_{12}'(i,j)=&\left(\frac{R}{2JG}\right)\Big\{\pi\left(R^2+\frac{P^2}{2}\right)\sin[\alpha(i-j)]\\&-\left(\frac{\pi R^2}{\alpha}-\frac{2\pi\alpha RP}{\alpha^2-\pi^2}+\frac{\pi P^2(\alpha^2-2\pi^2)}{\alpha(\alpha^2-4\pi^2)}\right)\sin\alpha\sin[\alpha(2k-i-j+1)]\\&+\left(\frac{4\pi(\alpha^2-8\pi^2)P\rho}{\alpha(\alpha^2-16\pi^2)}-\frac{4\pi\alpha R\rho}{\alpha^2-4\pi^2}\right)\sin\left(\frac{\alpha}{2}\right)\sin[\alpha(k-i+\tfrac{1}{2})]\Big\}\end{aligned} \tag{4.72}$$

$$B_{21}'(i,j)=B_{12}'(i,j)^T \tag{4.73}$$

and

$$B_{22}{}'(i,j)=\left(\frac{R}{2JG}\right)\{\pi\left(R^2+\frac{P^2}{2}\right)\cos[\alpha(i-j)]+\pi\rho^2$$
$$+\left(\frac{\pi R^2}{\alpha}-\frac{2\pi\alpha RP}{\alpha^2-\pi^2}+\frac{\pi P^2(\alpha^2-2\pi^2)}{(\alpha^2-4\pi^2)}\right)\sin\alpha\cos[\alpha(2k-i-j+1)]$$
$$-\left(\frac{4\pi(\alpha^2-8\pi^2)P\rho}{\alpha(\alpha^2-16\pi^2)}-\frac{4\pi\alpha R\rho}{\alpha^2-4\pi^2}\right)\sin\left(\frac{\alpha}{2}\right)\cos[\alpha(k-i+\tfrac{1}{2})]$$
$$-\left(\frac{4\pi(\alpha^2-8\pi^2)P\rho}{\alpha(\alpha^2-16\pi^2)}-\frac{4\pi\alpha R\rho}{\alpha^2-4\pi^2}\right)\sin\left(\frac{\alpha}{2}\right)\cos[\alpha(k-j+\tfrac{1}{2})]\} \tag{4.74}$$

Thus the strain energy of the k-th coil due to the torsional moment M_t is

$$U_{t,k}=\Delta F^T\cdot B'\cdot\Delta F=\Delta F^T\cdot\begin{bmatrix} B_{11}{}' & \vdots & B_{12}{}' \\ \cdots & \cdots & \cdots \\ B_{21}{}' & \vdots & B_{22}{}' \end{bmatrix}\cdot\Delta F \tag{4.75}$$

The strain energy of the k-th coil due to the moment $M_{z'}$ is

$$U_{z',k}=\int_0^{2\pi}\frac{M_{z'}^2}{2EI}Rd\theta=\Delta F^T\cdot\left[\int_0^{2\pi}\left(\frac{R}{2JG}\right)Cd\theta\right]\cdot\Delta F \tag{4.76}$$

where the elements of the submatricies C_{11}', C_{12}', C_{21}', and C_{22}' are defined in the following manner:

$$C_{11}'(i,j)=\left(\frac{R^3}{2EI}\right)\{-\frac{2\pi^3}{\alpha(\alpha^2-4\pi^2)}\sin\alpha\cos[\alpha(2k-i-j+1)] + \frac{\pi}{2}\cos[\alpha(i-j)]\} \tag{4.77}$$

$$C_{12}'(i,j)=\left(\frac{R^3}{2EI}\right)\{-\frac{2\pi^3}{\alpha(\alpha^2-4\pi^2)}\sin\alpha\sin[\alpha(2k-i-j+1)] + \frac{\pi}{2}\sin[\alpha(i-j)]\} \tag{4.78}$$

$$C_{21}'(i,j)=C_{12}'(i,j)^T \tag{4.79}$$

and

$$C_{22}'(i,j)=\left(\frac{R^3}{2EI}\right)\{\frac{2\pi^3}{\alpha(\alpha^2-4\pi^2)}\sin\alpha\cos[\alpha(2k-i-j+1)] + \frac{\pi}{2}\cos[\alpha(i-j)]\} \tag{4.80}$$

Thus the energy of the k-th coil due to the moment $M_{z'}$ is

$$U_{z',k}=\Delta F^{T}\cdot \boldsymbol{C}'\cdot \Delta F=\Delta F^{T}\cdot \begin{bmatrix} C_{11}{}' & \vdots & C_{12}{}' \\ \cdots & \cdots & \cdots \\ C_{21}{}' & \vdots & C_{22}{}' \end{bmatrix} \cdot \Delta F \tag{4.81}$$

The total energy of the k-th coil of the helical spring is the summation of the energies due to the three moments, or

$$U_k=U_{n,k}+U_{t,k}+U_{z',k}=\Delta F^{T}\cdot(\boldsymbol{A}'+\boldsymbol{B}'+\boldsymbol{C}')\cdot \Delta F \tag{4.82}$$

If one defines

$$D \triangleq \boldsymbol{A}'+\boldsymbol{B}'+\boldsymbol{C}' \tag{4.83}$$

the incremental energy of the helical spring up through the k-th coil now becomes

$$\Delta U=\sum_{i=1}^{k} U_i=\sum_{i=1}^{k} (\Delta F^{T}\cdot D\cdot \Delta F)_i \tag{4.84}$$

Castigliano's theorem is now invoked to determine the incremental displacements of each of the coils in the normal and tangential directions.

$$\delta_1 \triangleq \left(\frac{\Delta U}{\Delta F_1}\right) = D_{11}\Delta F_1 + 2D_{12}\Delta F_2 + 2D_{13}\Delta F_3 + \ldots + 2D_{1k}\Delta F_k \tag{4.85}$$

$$\delta_2 \triangleq \left(\frac{\Delta U}{\Delta F_2}\right) = 2D_{21}\Delta F_1 + D_{22}\Delta F_2 + 2D_{23}\Delta F_3 + \ldots + 2D_{2k}\Delta F_k \tag{4.86}$$

$$\delta_k \triangleq \left(\frac{\Delta U}{\Delta F_k}\right) = 2D_{k1}\Delta F_1 + 2D_{k2}\Delta F_2 + 2D_{k3}\Delta F_3 + \ldots + D_{kk}\Delta F_k \tag{4.87}$$

which simplifies to

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_k \end{pmatrix} = \begin{bmatrix} D_{11} & 2D_{12} & 2D_{13} & \cdots & 2D_{1k} \\ 2D_{21} & D_{22} & 2D_{23} & \cdots & 2D_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ 2D_{k1} & 2D_{k2} & 2D_{k3} & \cdots & D_{kk} \end{bmatrix} \begin{pmatrix} \Delta F_1 \\ \Delta F_2 \\ \vdots \\ \Delta F_k \end{pmatrix} \tag{4.88}$$

Now define

$$\Lambda \triangleq \begin{bmatrix} D_{11} & 2D_{12} & 2D_{13} & \cdots & 2D_{1k} \\ 2D_{21} & D_{22} & 2D_{23} & \cdots & 2D_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ 2D_{k1} & 2D_{k2} & 2D_{k3} & \cdots & D_{kk} \end{bmatrix} \tag{4.89}$$

then the final important relationship becomes

$$\Delta F = \Lambda^{-1}\delta \qquad (4.90)$$

With this relationship, knowing the physical geometry and material properties of the helical spring, and specifing a desired displacement, one can determine the force required by the SMA active element to bend the spring. The matrix Λ contains a conglomeration of information based on all of these known quantities. Once these given quantities have been specified, it is a simple matter to determine the required force matrix F. If the force required cannot be achieved by the SMA, iteration is required for the spring's geometry or material properties. A program is enclosed that will aid in the design of a helical spring under these conditions.

V. SUMMARY AND RECOMMENDATIONS

This thesis investigated the deformation of helical springs under eccentric loading using two different models. The goal was to develop an algorithm for computing the deflection of a helical spring under a known force. The desired final shape of the spring was that of a spring bent over in a circular fashion. The two models used achieved this by two different methods, both utilizing SMA active elements in two different configurations.

The development of this study was to obtain relationships between force and displacement so that one could predict or control the deformation of the spring. A spring under normal loading is governed by an equation such as

$$F=k\delta \tag{5.1}$$

where force F is proportional to displacement δ. This proportionality constant, k, is based on the spring geometry and material properties. In the development of this study, similar relationships were obtained, but with a more complex form of proportionality factor. This factor is dependent on the spring configuration, and is not a constant. The proportionality factor is also a function of the geometry and material properties.

This study has been a first step in the design of a robotic manipulator to be composed of a series of helical springs that utilize SMA actuators to control the manipulator. In the course of future research, SMA actuator design is needed, and in assembling the helical spring with the actuators. Once the actuators are designed, the results acquire by this study can be employed to match a spring's geometry and material to the force that can be generated by the SMA to bend the spring in a desired manner.

When the design process is completed, a helical spring and actuator system will be realized, and can be implemented in a minimally invasive surgical manipulator. The conclusions of this study can also be used in any other situation where an eccentric load is placed on a helical spring causing it to bend over.

PROGRAMS AND PLOTS

The first algorithm delevoped was for the first model, where the spring was deformed by applying an eccentric load axially on the spring. It is a FORTRAN program where material properties, spring geometry, and desired deflection angle are required as inputs, and the force required to deform the spring is the output.

```
    PROGRAM THESIS
*
* This program is for the spring design used in the manipulator skelton
* for minimally invasive surgical applications.  Given the physical
* dimensions and material properties of the spring, and the desired
* deflection where the force will be applied, it will solve for
* the deflection of the spring on the opposite side of where the force
* is applied, the force required, and for the angle of deflection.
*
* Inputs:  Bending stiffness of coil EI, torsional stiffness of the coil

* JG, number of coils n, radius of the spring R, radius of the spring
* coil rc, length of spring, and deflection d where the force will be
* applied.
* Outputs:  deflections d, deflection angle B, and force F.
*
      REAL F, E, I, J, G, R, rc, L, pi, c, EI, JG, dF, dOPPF, B, Z
      INTEGER n, m, k
      CHARACTER*20 MATL
*
      PRINT*
      PRINT*,'THIS PROGRAM WILL AID IN THE DESIGN OF A SPRING FOR USE'
```

```
      PRINT*,' IN THE SKELTON FOR MINIMALLY INVASIVE SURGICAL'
      PRINT*,' APPLICATIONS.'
*
      pi = 3.1415926
*
5     PRINT*
      PRINT*,'SELECT AND ENTER THE TYPE OF MATERIAL YOU WISH TO USE'
      PRINT*,'  (i.e. STAINLESS STEEL): '
      READ*,MATL
      PRINT*
      PRINT*,'ENTER MODULUS OF ELASTICITY (E), IN ENGLISH UNITS'
      PRINT*,'  (i.e. 28.0e6): '
      READ*, E
      PRINT*,'ENTER SHEAR MODULUS (G), IN ENGLISH UNITS'
      PRINT*,'  (i.e. 10.6e6): '
      READ*, G
      PRINT*
      PRINT*,'ENTER "1" IF YOU HAVE DIMENSIONS IN ENGLISH UNITS (in),'
      PRINT*,'  OR "2" IF YOU HAVE DIMENSIONS IN SI UNITS (mm): '
      READ*, k
      PRINT*,'ENTER RADIUS OF SPRING: '
      READ*, R
      PRINT*,'ENTER RADIUS OF SPRING COIL: '
      READ*, rc
      PRINT*,'ENTER LENGTH OF SPRING: '
      READ*, L
      PRINT*,'ENTER DESIRED DEFLECTION: '
      READ*, dF
10    IF (k.EQ.2) THEN
     R = R*(0.001)*39.37
     rc = rc*(0.001)*39.37
```

```
L = L*(0.001)*39.37
dF = dF*(0.001)*39.37
 ENDIF
 PRINT*
 PRINT*,'ENTER NUMBER OF SPRING COILS: '
 READ*, n
 PRINT*
 PRINT*,'ENTER "1" IF ONE FORCE IS APPLIED,'
 PRINT*,'  OR "2" IF TWO FORCES ARE APPLIED: '
 READ*, m
*
 I = 0.25*pi*(rc**4)
 J = 0.5*pi*(rc**4)
 c = n*pi*(R**3)
 EI = E*I
 JG = J*G
 IF (m.EQ.1) THEN
F = dF/(c*(1/(EI) + 3/(JG)))
dOPPF = c*F*(-1/(EI) + 1/(JG))
B = (ATAN((dF - dOPPF)/(2*R)))*(180/pi)
 ELSE
F = dF/(2*c*(1/(2*EI) + 5/(2*JG)))
dOPPF = 2*c*F*(-1/(2*EI) + 3/(2*JG))
B = (ATAN((dF - dOPPF)/(2*R)))*(180/pi)
 ENDIF
*
 PRINT*
 PRINT*,'FOR MATERIAL ',MATL
 PRINT*
 PRINT*,'DEFLECTION AT POINT OF APPLICATION (in): ',dF
 dF = dF/(0.001*39.37)
```

```
      PRINT*,'DEFLECTION AT POINT OF APPLICATION (mm): ',dF
      PRINT*
      PRINT*,'DEFLECTION AT POINT OPPOSITE APPLICATION (in): ',dOPPF
      dOPPF = dOPPF/(0.001*39.37)
      PRINT*,'DEFLECTION AT POINT OPPOSITE APPLICATION (mm): ',dOPFF
      PRINT*
      PRINT*,'DEFLECTION ANGLE (degrees): ',B
      PRINT*,'FORCE NECESSARY (lbs): ',F
      PRINT*
*
      PRINT*,'IF YOU WISH TO BEGIN AGAIN, TYPE ANY NUMBER AND HIT'
      PRINT*,'  RETURN.  ELSE ENTER "99" TO QUIT. '
      READ*, Z
      IF (Z.NE.99) GO TO 5
*
      END
```

The second algorithm developed was for the second model, that of a spring bent over directly by an SMA active element. It is a MATLAB program where one must input the geometry and material properties of the helical spring, as well as the final desired deflection angle of the spring. It generates a proportionality matrix whose size is based on the number of coils of the spring, and computes the force required to bend the spring. It plots force versus angle deflection to obtain a relationship between the magnitude of the force required and the amount of deflection.

```
%  thesis.m
%
% This program is for the spring design used in the manipulator skeleton
% for minimally invasive surgical applications.  Given the physical
% dimensions and material properties of the spring, and the desired
```

```
% deflection of the spring, it will solve for the force required to bend
% the spring using an SMA rod inside the spring.
%
% Inputs:  Spring length (L), radius (R), coil radius (r), Modulus of
%          Elasticity (E), Shear Modulus (G), number of coils (n),
%          desired angle of deflection (phi).
% Outputs: Force required to bend spring (F).
%
clear
%
L = input('Length of spring in millimeters ');
R = input('Radius of spring in millimeters ');
r = input('Radius of spring coil in millimeters ');
E = input('Modulus of elasticity in GPa ');
G = input('Shear modulus in GPa ');
n = input('Number of spring coils (must be even number) ');
phifinal = input('Final desired deflection angle in degrees ');
%
L = L/1000;
R = R/1000;
r = r/1000;
E = E*1e9;
G = G*1e9;
phifinal = phifinal*pi/180;
%
I = 0.25*pi*r^4;
J = 0.5*pi*r^4;
mom = R/(2*E*I);
tor = R/(2*J*G);
%
for i = 1:n
```

```
  initf(i) = 0;
end
f = initf';
%
count = 0;
dradius = 0;
radius(1) = 50;
for z = 1:200
  anglephi = L/(2*radius(z));
  if anglephi <= phifinal, count = count + 1; end
  dradius = -0.05*radius(z);
  radius(z+1) = radius(z) + dradius;
end
%
dP = 0;
P(1) = 50;
for q = 1:count
%
  for i = 1:n
    for j = 1:n
      D(i,j) = 0;
    end
  end
%
  ro = P(q)-R;
  phi(q) = L/(2*P(q));
  a = (2*phi(q))/n;
  c1 = (2*pi^3)/(a*(a^2-4*pi^2));
  c2 = pi/2;
  c3 = P(q)^2;
  c4 = (8*pi^2)/(a^2-4*pi^2);
```

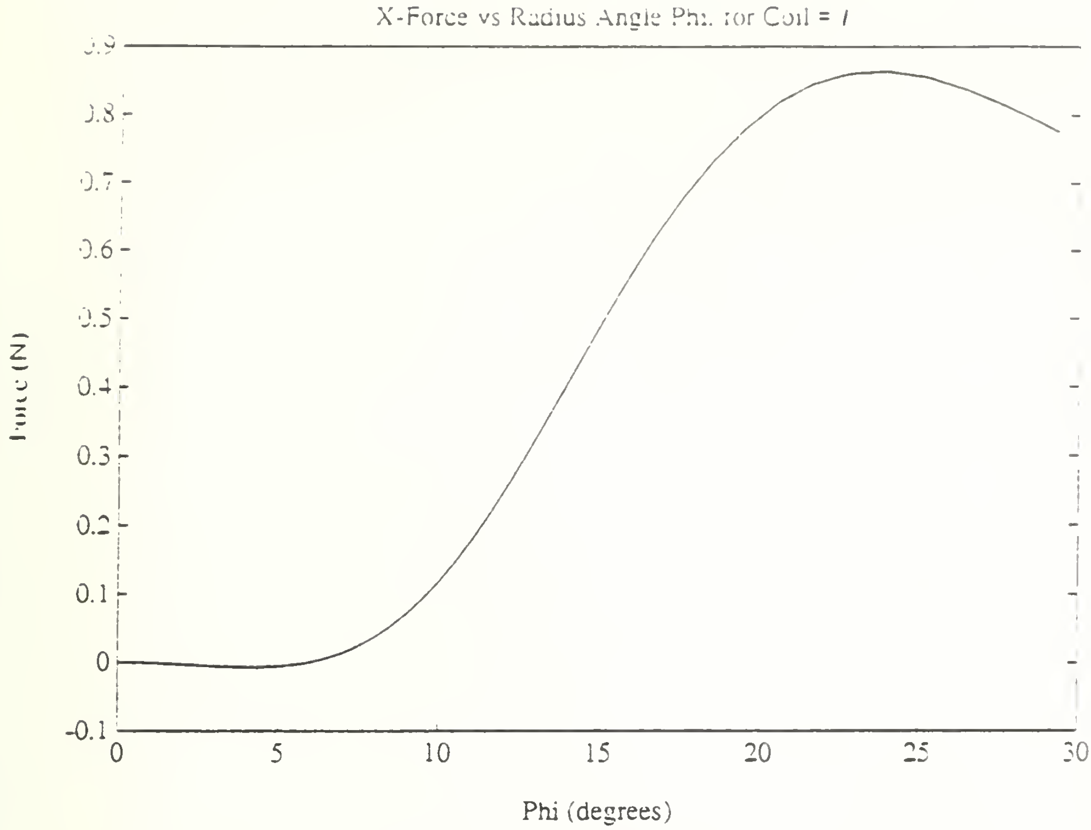
```
  c5 = (64*pi^3)/(a*(a^2-16*pi^2));
  c6 = pi*(R^2+(P(q)^2/2));
  c7 = ((R^2*pi)/a - (2*pi*a*R*P(q))/(a^2-pi^2) ...
        + pi*(a^2-2*pi^2)*P(q)^2/(a*(a^2-4*pi^2)));
  c8 = ((4*pi*(a^2-8*pi^2)*P(q)*ro)/(a*(a^2-16*pi^2)) ...
        - (4*pi*a*R*ro)/(a^2-4*pi^2));
  c9 = (32*pi^3)/(a*(a^2-16*pi^2));
  c10 = sin(a);
  c11 = sin(a/2);
  c12 = mom*R^2;
  c13 = c1*c3*c10;
  c14 = c2*c3;
  c15 = P(q)*ro*c9*c11;
  c16 = c7*c10;
  c17 = c1*c10;
  c18 = P(q)*ro*c5;
%
  for k = 1:n/2
    for j = 1:k
      for i = 1:k
%
        c19 = cos(a*(2*k-i-j+1));
        c20 = sin(a*(2*k-i-j+1));
        c21 = cos(a*(i-j));
        c22 = sin(a*(i-j));
        c23 = sin(a*(k-i+0.5));
        c24 = sin(a*(k-j+0.5));
%
        A11(i,j) = mom*(c13*c19 + c14*c21);
        A12(i,j) = mom*(c13*c20 - c14*c22 - c15*c23);
        A21(i,j) = mom*(c13*c20 + c14*c22 - c15*c24);
```

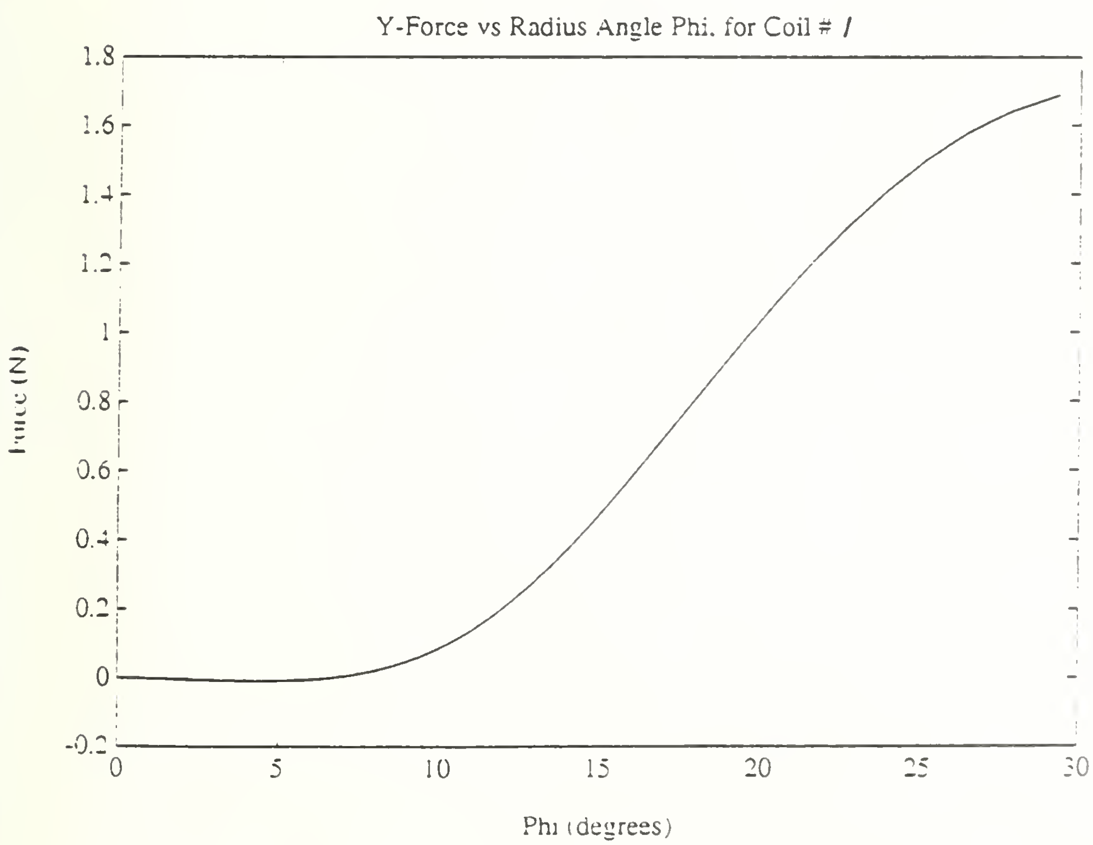
```
        A22(i,j) = mom*(-c13*c19 + c14*c21 + pi*ro^2...
                 + c18*c11*cos((a/2)*(2*k-i-j+1))*cos((a/2)*(i-j)));
%
        B11(i,j) = tor*(c6*c21 - c16*c19);
        B12(i,j) = tor*(c6*c22 - c16*c20 + c8*c11*c23);
        B21(i,j) = tor*(-c6*c22 - c16*c20 + c8*c11*c24);
        B22(i,j) = tor*(c6*c21 + c16*c19 - c8*c11*(cos(a*(k-i+0.5))...
                 + cos(a*(k-j+0.5))) + pi*ro^2);
%
        C11(i,j) = c12*(c2*c21-c17*c19);
        C12(i,j) = c12*(c2*c22-c17*c20);
        C21(i,j) = c12*(-c2*c22-c17*c20);
        C22(i,j) = c12*(c2*c21+c17*c19);
%
      end
    end
%
    d11 = A11+B11+C11;
    d12 = A12+B12+C12;
    d21 = A21+B21+C21;
    d22 = A22+B22+C22;
    for i = 1:k
      for j = 1:k
        D(i,j) = D(i,j) + d11(i,j);
        D(i+k,j) = D(i+k,j) + d21(i,j);
        D(i,j+k) = D(i,j+k) + d12(i,j);
        D(i+k,j+k) = D(i+k,j+k) + d22(i,j);
      end
    end
  end
  D = 2.*D;
```

```
  for i = 1:n
    D(i,i) = 0.5*D(i,i);
  end
%
  for i = 1:n/2
    theta(i) = (n/2 - (i-1))*a;
    deltan(i) = (1 - cos(theta(i)))*dP;
    deltat(i) = (sin(theta(i)) - theta(i))*dP;
  end
%
  delta = [deltan';deltat'];
  F = inv(D)*delta;
  for i = 1:n/2
    fx(i) = F(i)*cos(theta(i)) - F(n/2+i)*sin(theta(i));
    fy(i) = F(i)*sin(theta(i)) + F(n/2+i)*cos(theta(i));
  end
  fxy = [fx';fy'];
  f = f + fxy;
  force(:,q) = f;
  PHI(q) = phi(q)*(180/pi);
  dP = -0.05*P(q);
  P(q+1) = P(q) + dP;
end
%
clg
for j = 1:n/2
  plot(PHI,force(j,:))
  title('X-Force vs Radius Angle Phi, for Coil #')
  xlabel('Phi (degrees)')
  ylabel('Force (N)')
  meta thesplot
```

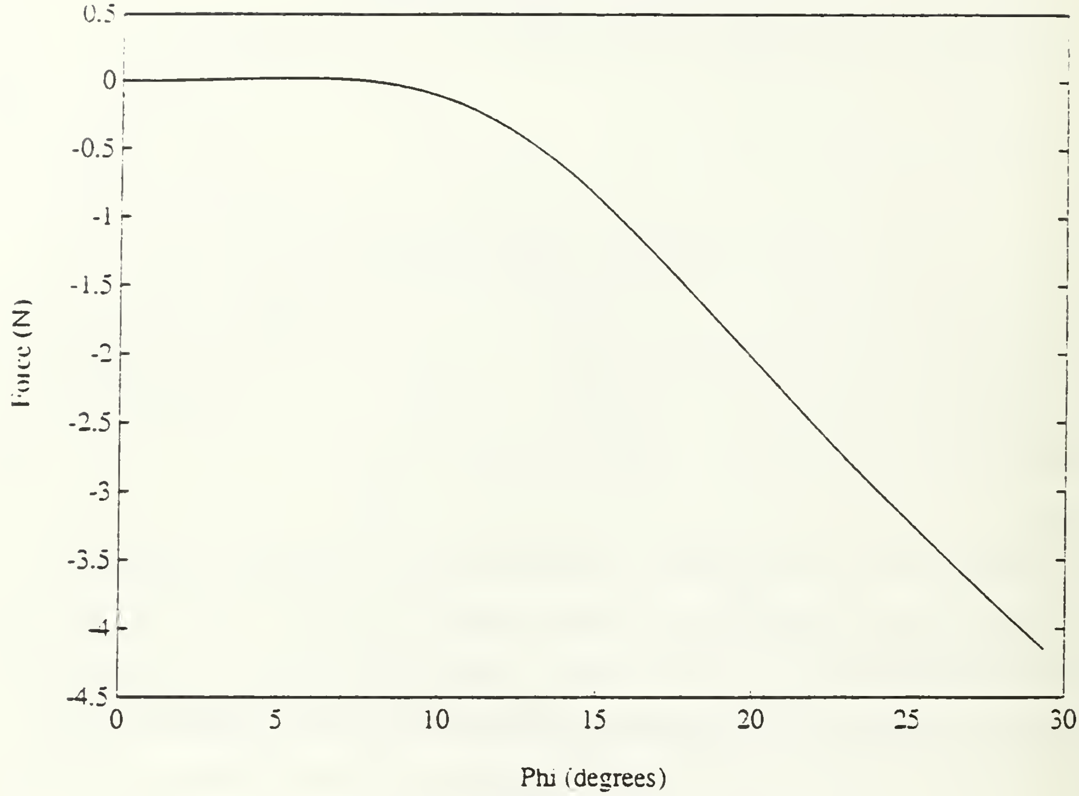
```
end
for j = n/2+1:n
  plot(PHI,force(j,:))
  title('Y-Force vs Radius Angle Phi, for Coil #')
  xlabel('Phi (degrees)')
  ylabel('Force (N)')
  meta thesplot
end
```

The final element is a set of plots using the previous program. They are force versus deflection angle graphs generated by the program using the inputs of a 20 mm long spring, with a radius of 5 mm, coil radius of 0.5 mm, and eight coils. The material chosen was stainless steel with a modulus of elasticity of 190 GPa, and shear modulus of 73 GPa. The final desired deflection angle was 30°. The eight plots are for the top half of the spring. The forces in the bottom half are going to be identical as the spring is symmetrical. Of the eight force/deformation plots, four are for normal forces, and four are for tangential forces.

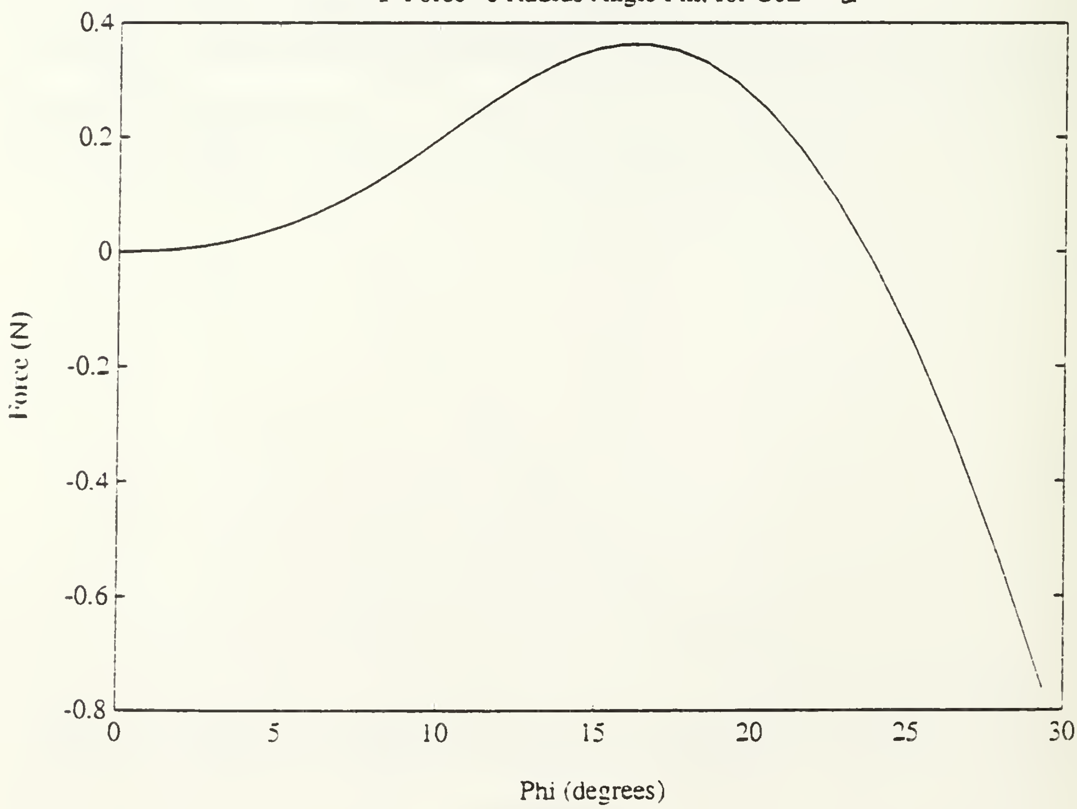

X-Force vs Radius Angle Phi. for Coil = 1
Force (N)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
5
10
15
20
25
30
Phi (degrees)

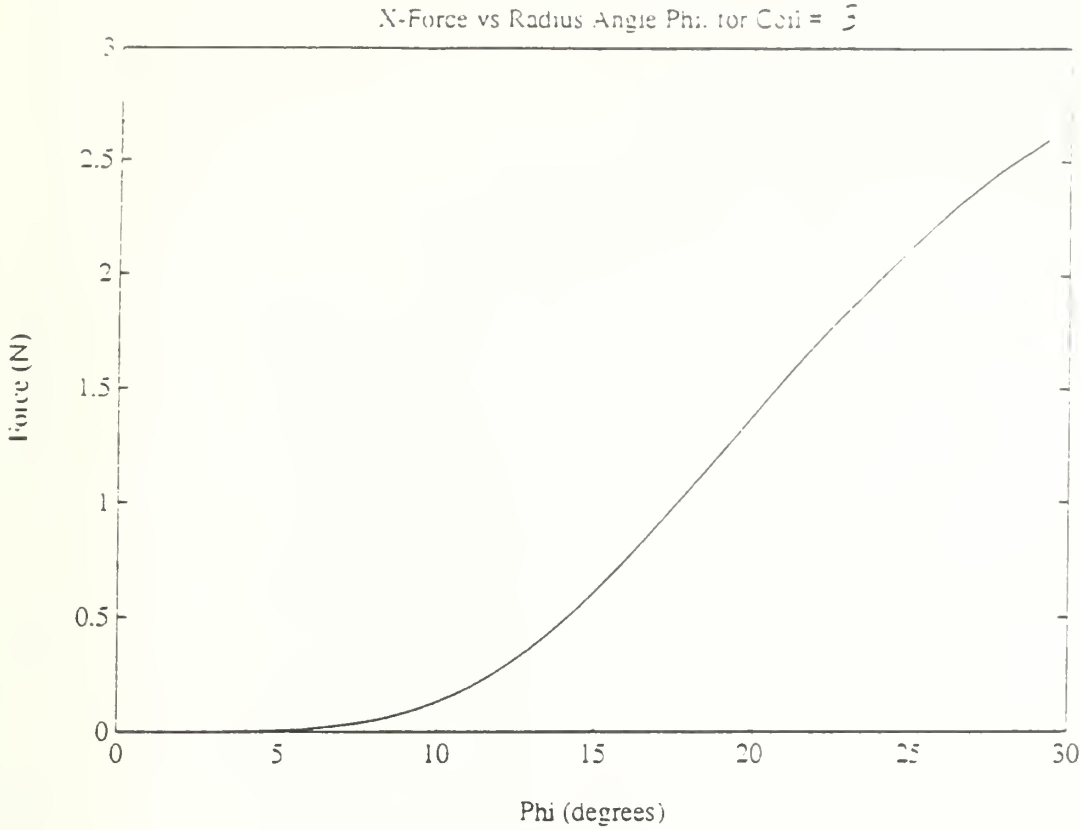

Y-Force vs Radius Angle Phi. for Coil # 1
Force (N)
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
5
10
15
20
25
30
Phi (degrees)

X-Force vs Radius Angle Phi. for Coil = 2

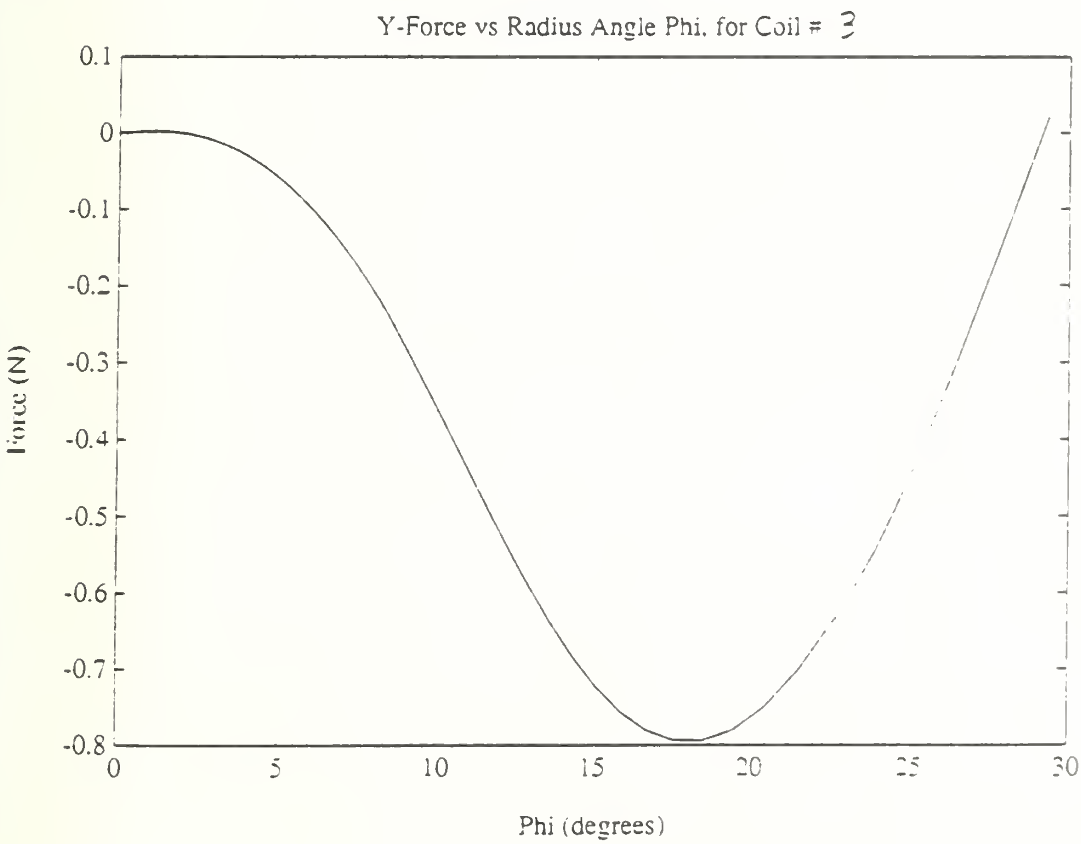


Y-Force vs Radius Angle Phi. for Coil # 2

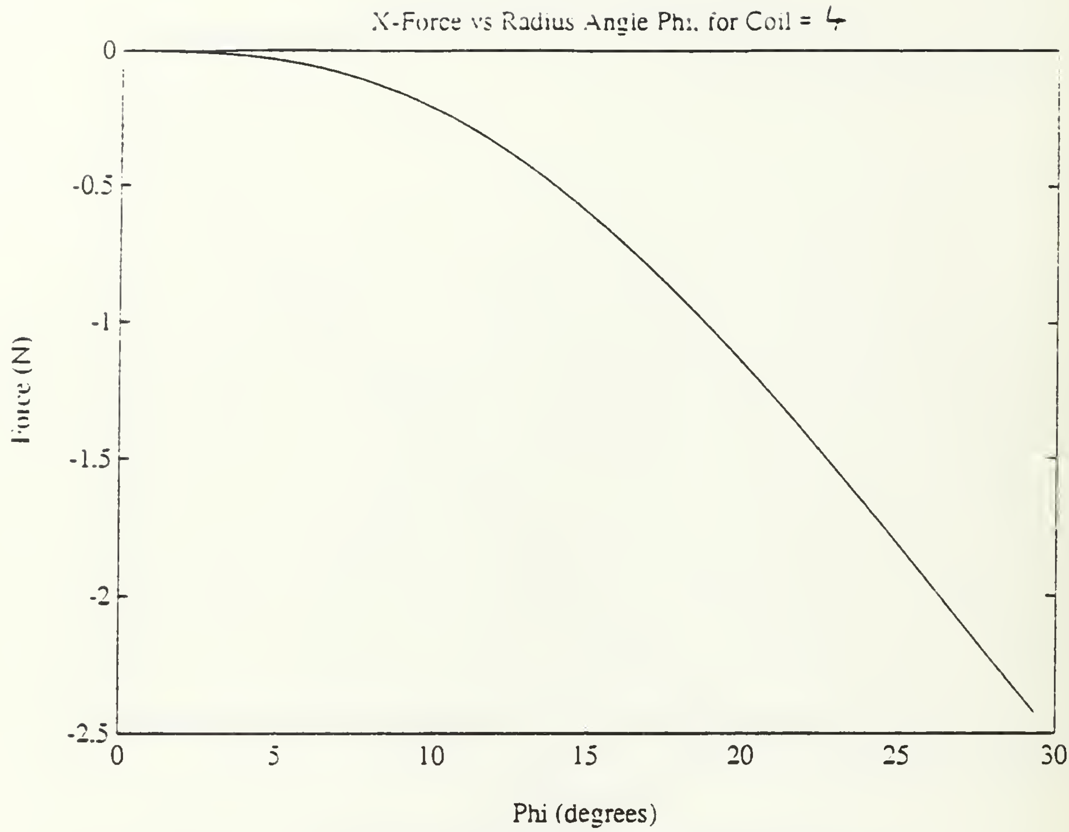


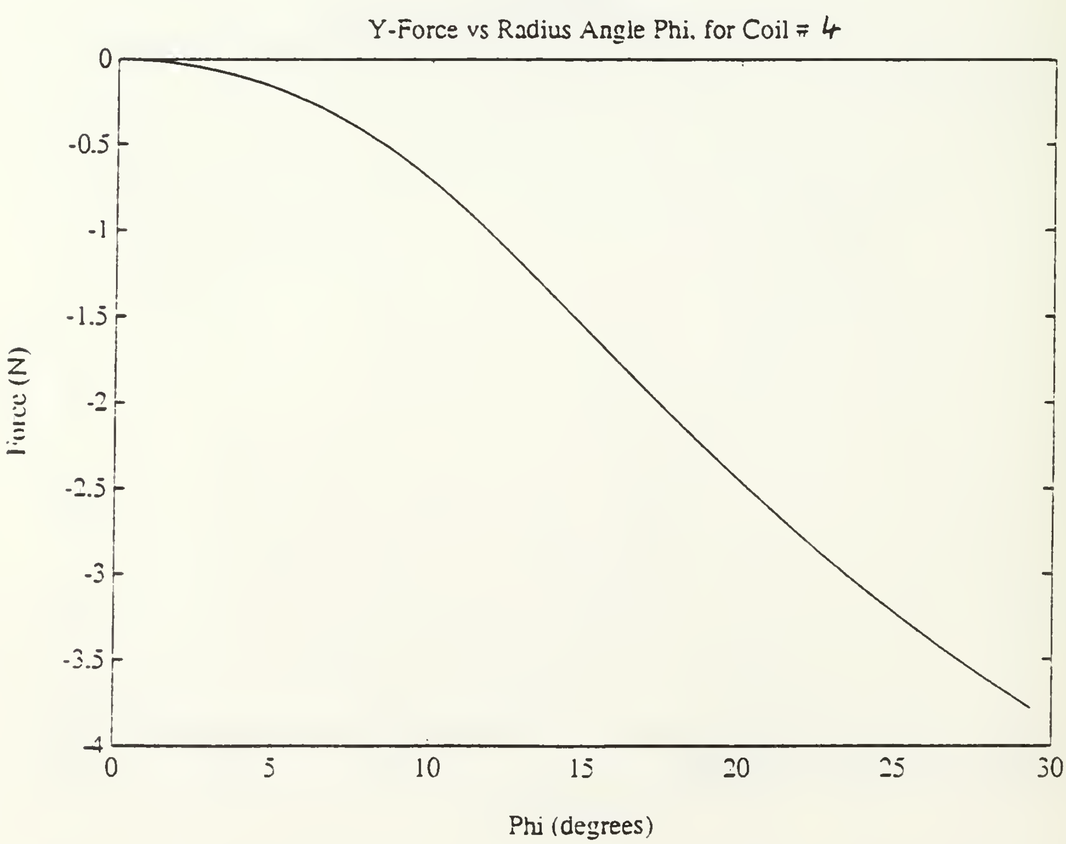


X-Force vs Radius Angle Phi. for Coil = 3
Force (N)
Phi (degrees)



Y-Force vs Radius Angle Phi. for Coil = 3
Force (N)
Phi (degrees)


X-Force vs Radius Angle Phi, for Coil = 4
0
-0.5
-1
-1.5
-2
-2.5
Force (N)
0
5
10
15
20
25
30
Phi (degrees)


Y-Force vs Radius Angle Phi, for Coil = 4
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
Force (N)
0
5
10
15
20
25
30
Phi (degrees)

LIST OF REFERENCES

1. Popov, E.P., *Mechanics of Materials*, 2nd ed., pp. 223-224, Prentice-Hall, Inc., 1976.

2. Logan, D.L., *Mechanics of materials*, pp. 635-638, Harper Collins Publishers, Inc., 1991.

3. Duerig, T.W., and others, *Engineering Aspects of Shape Memory Alloys*, pp. 3-20, Butterworth-Heinemann Ltd., 1990.

4. Associated Spring Corporation, *Handbook of Mechanical Spring Design*, 1956.

5. Associated Spring Company, *Design Handbook*, 1987.

6. Society of Automotive Engineers, Inc., AE-11, *Spring Design Manual*, 1990.

7. Bodenschatz, A., "Conical Springs," *Spring Design and Application*, p. 102, 1990.

8. Hunter Spring Company, *Constant Force Compression Springs*, by E.H. Boerner, pp. 154-160, 1 September 1989.

INITIAL DISTRIBUTION LIST

1.	Defense Technical Information Center Cameron Station Alexandria, Virginia 22304-6145	2
2.	Library, Code 52 Naval Postgraduate School Monterey, CA 93943-5002	2
3.	Department Chairman, Code ME Department of Mechanical Engineering Naval Postgraduate School Monterey, CA 93942-5000	2
4.	Professor R. Mukherjee, Code ME/MK Department of Mechanical Engineering Naval Postgraduate School Monterey, CA 93942-5000	3
5.	Curricular Officer, Code 34 Department of Mechanical Engineering Naval Postgraduate School Monterey, CA 93942-5000	1
6.	LT Andrew R. Leech 5113 Hemlock Avenue Virginia Beach, VA 23464	4