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APPLICATION OF DIFFERENTIAL GAMES TO PROBLEMS

OF MILITARY CONFLICT:

TACTICAL ALLOCATION PROBLEMS - PART I

by

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ABSTRACT:

The mathematical theory of deterministic optimal control/differential games is applied to the study of some tactical allocation problems for combat described by Lanchester-type equations of warfare. A solution procedure is devised for terminal control attrition games. H. K. Weiss' supporting weapon system game is solved and several extensions considered. A sequence of one-sided dynamic allocation problems is considered to study the dependence of optimal allocation policies upon model form. The solution is developed for variable coefficient Lanchester-type equations when the ratio of attrition rates is constant. Several versions of Bellman's continuous stochastic gold-mining problem are solved by the Pontryagin maximum principle, and their relationship to the attrition problems is discussed. A new dynamic kill potential is developed. Several problems from continuous review deterministic inventory theory are solved by the maximum principle.

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I. INTRODUCTION.

This report documents research findings for the time period 30 March 1970 to 19 June 1970 under support of NR 276-027. This report discusses applications of the theory of differential games to tactical allocation problems in the Lanchester theory of combat. We also discuss some extensions for Lanchester-type models of warfare and deterministic inventory theory. A companion report [76] discusses other research findings of the contract period with respect to surveillance-evasion problems of Naval warfare.

The goal of this research is to determine the structure of optimal allocation policies for tactical situations describable by Lanchester-type equations of warfare. We hope to provide insight into such questions as

(1) How should targets be selected?

(2) Do target priorities change with time?

(3) Do battle termination circumstances effect the optimal allocation policies?

(4) How does the nature of the attrition process effect target selection?

(5) What is the effect of ammunition constraints?

(6) How does the uncertainty and confusion of combat effect the optimal selection rules?

We develop our theory of target selection through the examination of a sequence of simplified models. These combat models are too simple to be taken literally but should be interpreted as indicating general principles to serve as hypotheses for subsequent computer simulation studies or field experimentation.

In warfare decisions must be made sequentially over a period of time, and the world is changed as a result of these decisions. The Lanchester theory of combat has been developed to describe such dynamic situations. Of even more interest to defense planners than how to describe combat, is how to optimize the dynamics of combat. Many times the static optimization techniques of linear and non-linear programming are not applicable, so new dynamic optimization techniques were developed in the 1950's.

Actually, many such situations may be formulated as classical constrained calculus of variations problems (technically referred to as the problems of Bolza, Lagrange and Mayer). Because of inequality constraints and non-negative variables in such problems, the classical methods are difficult to apply. Thus, dynamic programming [9] was originally developed as a computational technique for variational problems, although its principles have proven to be of much wider applicability. This was also the impetus for the development of the maximum principle by the Soviet mathematician L. Pontryagin [68]. During this period military problems also rekindled interest in the game theory of J. von Neumann [78] with extensions being made to multi-move discrete games [9], [29] and differential games [50]. It seems appropriate to discuss these techniques briefly.

a. Optimal Control/Differential Games.

These techniques may be used to optimize systems whose behavior is described by a system of differential equations. The same basic concepts are referred to as optimal control when there is one controller and one criterion function and as a differential game with two controllers and two criterion functions (which sum to zero). Recently the term "generalized control theory" has been coined [42], [43] for these dynamic optimization techniques. A common point of such models is that time is treated continuously. Major work has been done by L. Pontryagin and others in the USSR (see survey papers by [13], [71] and references in [8], [33]), and R. Bellman, L. Berkovitz, Y. C. Ho, and others in the US. R. Issacs has independently developed an extensive theory of differential games and has published a book containing numerous examples [50].

However, these techniques apply primarily to deterministic systems. Frequently numerical methods must be used when closed-form analytic solutions can't be obtained. Dynamic programming was developed at RAND by R. Bellman and others [9], [10] for such cases.

b. Dynamic Programming.

Although numerical solution of variational problems was one of the initial reasons for the development of dynamic programming, this technique has proven to be of much wider applicability. It is a dual approach to Lagrange's method of variations, which treats an extremal curve as a sequence of points and develops a differential equation to be satisfied at each such point. On the other hand, dynamic programming generates an optimal trajectory by considering the "direction of best return" working backwards from the problem's end. It bears a close relationship to C. Carathéodory's notion of a geodesic gradient, and this has rekindled interest in much classical work.

Although we haven't explicitly used dynamic programming in the present work, its underlying principle of optimality [9] continues to apply when the assumption required by differential game theory of continuous time no longer holds. Historically (see Chapter X of [9]), multi-move discrete games were considered before differential games, which are a limiting case. For future work in which it may be desirable to closer approximate the real world with less restrictive assumptions (for example, attrition rates which don't lead to closed-form solutions of the corresponding differential equations), it may be necessary to employ numerical procedures, and we have given this consideration.

c. Tactical Allocation Problems.

We think that combining Lanchester-type models of warfare with the theory of differential games/dynamic programming has a great potential for providing insight into the optimization of the dynamics of combat continuing over a period of time with a choice of tactics available to both sides and subject to change with time. In the present work our goal is to determine the factors upon which the optimal allocation depends and also what this dependence is. We have considered the following aspects

(1) combatant objectives (form of criterion function and valuation of surviving forces),

(2) termination conditions of conflict,

(3) type of attrition process,

(4) force strengths,

(5) effect of resource constraints.

Our conclusion is that any or all of the above factors may influence the structure of the optimal allocation policies depending upon the form of the model used. Judgment is required, then, to decide which type of model is most applicable for any specific problem.

Besides the study of problems of land combat, these models have numerous applications to problems of Naval warfare:

(1) optimal allocation of Naval fire support,

(2) allocation of Naval airpower between ground-support and strategic targets,

(3) worth of Naval transport capability for troop build-up in combat zone.

We envision these idealized models as being used to provide insight and to generate hypotheses to be tested in subsequent work under less restrictive assumptions (such as computer Monte Carlo simulation or actual field experimentation).

Our research approach has been to consider a sequence of models of increasing complexity. We have considered models for two types of choice situations

(1) selection of target type,

(2) regulation of firing rate.

We have also found it necessary to develop several extensions to the theory of Lanchester-type models of warfare and also to differential game theory.

In considering more and more complex models, we have started with one-sided models and done some work for the two-sided case. We have learned about the structure of optimal allocation policies by solving numerous specific problems. We have found that the application of existing theory to the prescribed duration battle is straightforward but that (even for the one-sided case) new approaches and concepts had to be developed for battles which terminate by the course of combat being steered to a prescribed state. In these terminal control problems we have considered a "fight to the finish" for mathematical convenience, and our approach, of course, applies to any terminal control game. Our work shows that selection of the appropriate scenario (prescribed duration or terminal control) may be an important decision in a defense planning study. We have also applied the existing theory of differential games to pursuit and evasion problems [76]. We have found that there are numerous mathematical differences between pursuit-evasion and attrition differential games.

These models consider the continual allocation of resources after the battle has started. We could consider models for the initiation and termination of conflict and also the allocation of resources across a broad front before the actual battle begins. Such considerations are beyond the scope of the present work.

We have also looked for other areas of interest to defense planners for the application of the knowledge we have gained through our study of tactical allocation problems. Thus, we consider some models of deterministic, continuous-review inventory processes in Appendix G.

II. REVIEW OF PERTINENT LITERATURE.

We reviewed the literature in two subject areas: Lanchester theory of combat and differential games. We do not attempt an exhaustive review of the literature, since that was not the purpose of this research. However, we try to highlight some major works.

One of the earliest attempts to establish a mathematical model of the dynamics of mass combat was by Lanchester [61] in 1916. He developed several deterministic models that were a system of ordinary differential equations which related the strengths of opposing military forces to length of combat. During World War II B. O. Koopman extended Lanchester's results and also suggested a reformulation of the problem in stochastic form [66]. After World War II the RAND Corporation carried on further studies whose results were summarized by Snow [72]. H. K. Weiss then at Aberdeen Proving Ground and others [7], [22], [28], [37], [38], [80], [81] have subsequently developed deterministic Lanchester models.

R. Brown developed models for the stochastic analysis of combat [23]. The relationship between the above mentioned stochastic and deterministic Lanchester formulations was pointed out relatively early in their development (see [72], for example) but is probably best presented in a recent report by B. O. Koopman [60]. Bonder [21] has done work on the estimation of the Lanchester attrition-rate coefficient (for weapon systems that adjust fire based on results of the previous round fired). A good review of the Lanchester theory of combat is by Dolansky [28], and this includes a comprehensive list of references through 1964.

The study differential games was initiated by R. Isaccs at RAND in the early 1950's [46], [47], [48], [49], but this work has not been available to a wide audience until quite recently [50]. His basic concept, "the tenet of transition," is a generalization of Bellman's [9] "principal of optimality" to a competitive environment, and this is used to develop necessary conditions for optimal strategies. A more recent and more rigorous development of these basic necessary conditions is by Berkovitz [12]. Since the excellent paper by Ho, Bryson and Baron [44] in 1965, there has been a literal explosion of papers on differential games but almost all deal exclusively with pursuit-evasion problems. Excellent survey papers which bear this out are by Simakova (Russian literature) [71] and Berkovitz [13]. A more detailed review of differential game literature for pursuit and evasion applications is to be found in a companion report [76]. At a fairly recent workshop on differential games it was noted that there have been no new significant examples [25] since the publication of Isaacs' book. Other books which treat differential games are by Blaquière et al. [16] (extension of their geometrical approach to optimal control) and Bryson and Ho [24] (Chapter 9).

In 1964 Dolansky [28] noted that the Lanchester theory of combat was insufficiently developed in the area of target selection for combat between heterogeneous forces (optimal control/differential games). Even the two references cited by him, Weiss [82] and Isbell and Marlow [52], have been subsequently extended [74]. Since Dolansky's article, no further examples have been published in the literature except for the ones in Isaacs book [50].

One aspect that has impressed this author has been the diversity of approaches applied to the same problem by the researchers at RAND. Discrete and continuous models, deterministic and stochastic models are used in a complementary manner to help each other and provide insight. We note in this connection the discrete and continuous versions of the strategic bombing problem (Bellman's stochastic gold-mining problem [9]). We also note that the War of Attrition and Attack of Isaacs is the continuous version of other discrete sequential decision-making models of the strategic/tactical deployment of airpower studied at RAND [14], [15], [34].

Differential game theory has also been used to study target selection in combat described by Lanchester-type equations at the University of Michigan. Results are summarized in a report [73], which references working papers for further details. We have not yet reviewed these working papers. However, it appears that this work does not consider the various possible model forms that we do in the present work and, hence, the dependence of optimal allocation policies on model form is not recognized.

III. SOME TACTICAL ALLOCATION PROBLEMS.

In this section we summarize results for the problems we have studied and explain why these problems were studied. A more detailed discussion on many points is to be found in the appendices. The current phase of this work has stressed extension of results in the literature. This has been by necessity both to familiarize ourselves with past work and to extend many partial or incomplete results. The present state of differential game/optimal control theory allows problems, which twenty years ago would be very difficult (if not impossible) to solve by classical variational methods, to be readily solved.

First we review the various tactical allocation problems which we have studied, and then we discuss two extensions we have made to the Lanchester theory of combat. A section is included to summarize some work not included because of its incomplete nature in this report.

a. The Allocation Problems.

In Appendix A we derive a complete solution to the Isbell and Marlow [52] fire programming problem. This is a terminal control problem (the battle terminates when the course of battle has reached some specified state) and such attrition games are not treated in Isaacs' book [50]. We first solved this problem to gain insight into a solution phenomenon of H. K. Weiss' supporting weapon system game [82]. In an optimal control problem one determines extremals and domains of controllability for each terminal state, but in a differential game further investigations are required to verify that one's opponent can't "block" entry to an unfavorable (losing) terminal state against one's extremal strategy. It may be that he can steer the course of battle to an end favorable (winning) to him by use of other than his extremal strategy. This phenomenon has not occurred in any pursuit and evasion differential game in the literature. We discuss the structure of optimal target engagement policies for the Isbell-Marlow problem. Later (in Appendix C) we contrast the same combat model in scenarios of a prescribed duration battle and a "fight to the finish."

In Appendix B we apply the theory of differential games to H. K. Weiss' supporting weapon system game. This problem was originally solved by assuming a special form for the solution [82]. Subsequent work [58] has considered the simpler case of a prescribed duration engagement. We have found the existing framework of differential game theory inadequate for solving the supporting weapon system game and have consequently introduced the concept of a "blockable" terminal state which we have discussed briefly above. Such behavior does not occur in a one-sided problem. The book by Blaquière et al [16] defines a similar concept of a "strongly playable strategy," but there are no concrete examples given to motivate this notion.

In the future we would propose to formalize the notion of a "blockable" terminal state as a contribution to the theory of differential games. We also discuss several extensions of the original supporting weapon system game in Appendix B. It seems appropriate to devise further extensions to study facets like: (a) target priorities for fire support systems, (b) when to engage enemy fire support system instead of fire support for other forces. We have examined some scenarios not included in this report.

In Appendix C we examine a sequence of problems to study the dependence of optimal allocation policies on model form. We consider two types of choice problems: (1) target selection and (2) firing rate. In studying the problem of target selection we re-study the Isbell-Marlow fire programming problem to learn about the structure of best policies through a series of contrasts

(a) prescribed duration versus terminal control battle,

(b) two versus many target types,

(c) square law versus linear law attrition.

We discuss differences in the structure of optimal policies for all these cases. We also find out such things as that if one assigns a worth to targets in proportion to their kill rate against you, then there is never a switch in target priorities. We also are motivated to define the new dynamic kill potential of Appendix F.

We also study the best firing rate in a sequence of models all having resource constraints. We are interested in ascertaining under what circumstances does one "hold his fire." We consider a simplified model for combat between two homogeneous forces in which one side has an ammunition constraint that will be binding in a battle of prescribed duration and the attrition rates are constant. Under these circumstances, the best policy is to fire at one's maximum possible rate until all ammunition has been expended. We see that this model is not too realistic and are led to consider cases where the attrition rates vary with time or force separation. This leads to variable coefficient Lanchester-type equations and has been our impetus for seeking solution methods for such equations. We have, by necessity, had to extend the existing theory of Lanchester-type models, and we discuss this in another appendix (D). We also consider several other scenarios for limited resources.

In Appendix C we have also included a discussion of the usefulness of one-sided models for studying two-sided phenomena. We point out the close relationship between optimal control and differential game theory. Since the Hamiltonian is usually separable in the control variables, i.e., a function independent of ϕ + a function independent of ψ (for a practical example where this isn't true see [11]), we essentially have two "independent" optimal control problems (one a maximization and the other a minimization) and the optimal strategies are pure. We note that this is not true for many important models in game theory (Col. Blotto game, for example [29]).

We also discuss the implications of the idealized models we have considered. Hence, we discuss optimal tactical allocation, intelligence, command and control systems, and human decision making. We have learned that optimal strategies are a function of model form, and there usually will be several possible forms available.

In Appendix E we develop the solution to the continuous version of Bellman's stochastic gold-mining (strategic bombing) problem [9] by optimal control theory. We do so because the solution to this problem has a very similar structure to that for allocation of fire over targets undergoing linear law attrition. We consider two types of models: (1) maximum return for prescribed duration use and (2) maximum return for specified risk. The structures of the optimal allocation policies are slightly different in these two cases. Originally, Bellman used variational methods and knowledge of discrete analogues to solve these problems. The new methods are easier to apply and provide more insight (for example, the distinction between the two problems considered above). Our study of this problem and its similarity to other tactical allocation problems studied in Appendix C suggest that there may be a general structure underlying all such problems. We also are motivated to consider other formulations (for example, a force is only subject to attrition from targets that it engages) of tactical allocation problems with Lanchester-type models of warfare.

b. Extensions of Lanchester-Type Models of Warfare.

We have, by necessity, made two extensions to the Lanchester theory of combat:

(1) solution to Lanchester-type equations with variable coefficients,

(2) development of notion of a dynamic kill potential.

In Appendix D we show how to solve Lanchester-type equations for combat between two homogeneous forces when the attrition rates are variable provided that their quotient is a constant. Solutions are developed for either time or force separation as the independent variable. We also discuss the relationship of our work to that of others [20], [73].

In Appendix F we define the concept of a weapon system firepower potential. We obtained our motivation for this development from our study of tactical allocation problems using optimal control theory. Our approach provides a measure of the firepower capability of a weapon system giving consideration to the dynamics of combat.

When one interprets the maximum principle and dual variables which one is using (or attempts derivations), one sees that the rate of return for engaging a target (as measured by the rate of change of a terminal payoff for the scenario) changes during the course of battle. One is tempted to try to extend the notion of evolution of target worth to cases where there is no allocation problem. By use of the adjoint system to the Lanchester-type equations, one can do this. Our method may be used to study such facets of combat as the worth of mobility in battle, the effect of different range capabilities for weapon systems. This is the end of our guided tour of the appendices.

c. Other Topics Not Included in This Report.

It seems appropriate to note two other areas of work that for one reason or another have not been included in this report: (1) other tactical allocation formulations and (2) target coverage problems. We have done initial work on the formulation of other tactical allocation formulations and (2) target coverage problems. We have done initial work on the formulation of other tactical allocation situations

(a) fire support of several ground units,

(b) weapon system only subject to attrition when engaging a target type.

We also did some work on coverage problems. We obtained a new result for the hit probability against a circular target when the distribution of impact points follows an offset circular bivariate normal distribution. Although this type of problem has been extensively studied (in a recent survey article Eckler [31] gives 60 references; see also Grubbs' [36] brief survey), we have discovered a new representation for the hit probability, and this yields several useful approximations.

Consider a circular target with radius a located at the center of an x-y rectangular coordinate system. Assume that the distribution of impact points follows an offset circular bivariate normal distribution. We let

$\sigma_x = \sigma_y = \sigma$ be standard deviation of impact points,

μ_x, μ_y be average of impact distribution,

and $R = \sqrt{\mu_x^2 + \mu_y^2}$.

Then

for $R < a$

$$P_{hit} = 1 - \exp\{-(a^2 + R^2)/(2\sigma^2)\} \cdot \sum_{k=0}^{\infty} \left(\frac{R}{a}\right)^k I_k\left(\frac{aR}{\sigma^2}\right),$$

where $I_k(Z)$ is the Bessel function with imaginary argument of the first kind, of order k. It may be defined as

$$I_k(Z) = \sum_{m=0}^{\infty} \frac{\left(\frac{Z}{2}\right)^{2m+k}}{m!\,(m+k)!} .$$

Also

for $R > a$

$$P_{hit} = \exp\{-(a^2 + R^2)/(2\sigma^2)\} \sum_{k=1}^{\infty} \left(\frac{a}{R}\right)^k I_k\left(\frac{aR}{\sigma^2}\right) .$$

The above formulas are readily proven through an intermediate result of Gilliland [35]. We may also express the above in closed form through the use of Lommel's functions of two variables (see Watson [79] p. 537).

for $R < a$

$$P_{hit} = 1 + \exp\{-(a^2 + R^2)/(2\sigma^2)\}\{iU_1\left(i \frac{R^2}{\sigma^2}, i \frac{aR}{\sigma^2}\right)$$

$$+ U_2\left(i \frac{R^2}{\sigma^2}, i \frac{aR}{\sigma^2}\right) - I_0\left(\frac{aR}{\sigma^2}\right)\} ,$$

and

for $R > a$

$$P_{hit} = -\exp\{-(a^2 + R^2)/(2\sigma^2)\}\{iU_1\left(i \frac{a^2}{\sigma^2}, i \frac{aR}{\sigma^2}\right)$$

$$+ U_2\left(\frac{ia^2}{\sigma^2}, i \frac{aR}{\sigma^2}\right)\} ,$$

where $i = \sqrt{-1}$ and $U_n(w,z)$ is Lommel's function of two variables defined by

$$U_n(w,z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{n+2m} J_{n+2m}(z).$$

Unfortunately, there exist no tabulations for Lommel's function of two imaginary arguments. Since several problems of physical significance also lead to this type of solution, the creation of such tables seems warranted.

IV. CONCLUSIONS AND FUTURE EXTENSIONS.

Here we summarize what we have done, state some generalizations, and suggest some possible future research. Further amplification of results and conclusions is to be found in the appendices. We have considered the optimization of dynamic systems using the theory of optimal control/differential games. Specifically, we have accomplished the following:

(1) devised method for solving terminal control attrition games,

(2) compared sequence of idealized scenarios to study dependence of optimal allocation policies on model form,

(3) developed solution to Lanchester-type equations with variable coefficients under special circumstances,

(4) developed a new dynamic kill potential,

(5) generalized results in continuous review deterministic inventory theory (optimal inventory policies for linear production costs and effect of budget constraints).

Based on our studies we conclude that

(1) tactics of target selection are dependent on model form and may be sensitive to force strengths, target acquisition processes, attrition processes, and/or termination conditions of combat,

(2) tactics for target selection depend upon "command efficiency,"

(3) for a continuous review deterministic inventory process, when production costs are linear, then the optimal inventory policy is essentially independent of the nature of holding costs except for sometimes operating at the minimum of the shortage/holding cost curve.

We suggest the following as possible future work:

(1) develop in a more mathematical fashion our theory of terminal control attrition games (The examples we have solved suggest several necessary extensions to the existing mathematical theory.),

(2) study extensions of supporting weapon system game (We would examine optimal tactics for various battle termination conditions and attrition processes.),

(3) further study problem of best firing rate when there are ammunition constraints with either time-varying or range-varying attrition rates (This would extend models considered in Appendix C and would use our results developed in Appendix D.),

(4) formulate allocation of forces before the inception of combat problem (It is of interest whether the optimal strategy is mixed for then the element of surprise becomes important in planning a successful attack.),

(5) develop other models of tactical interest and study other extensions in the literature (We would continue to stress the study of the dependence of optimal tactics on model form.).

APPENDIX A. The Isbell-Marlow Fire Programming Problem.

In this appendix we develop a complete solution to the Isbell and Marlow fire programming problem [52]. This is the simplest example of more general tactical allocation problems which are terminated by the system being steered to a specified terminal state. Subsequent work [82] which considered the work of Isbell and Marlow has been heuristic (not using the usual (today's) necessary conditions [12]) possibly because of the incompleteness of this prior work. We originally solved this (the Isbell-Marlow fire programming problem) in order to gain insight into the supporting weapon system game of H. Weiss [82].

In studying simplified models of dynamic tactical allocation problems it is important to understand the dependence of the structure of optimal policies on model form. We have discovered in our researches that the optimal allocation policies may depend on the scenario chosen to study the problem.

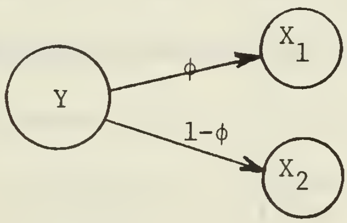
In this appendix we first state fire programming problem before we outline our new solution procedure and indicate its extension to two-sided problems (differential games). Next we present the details of the solution, after which we discuss the structure of the optimal allocation policies. In view of the close connection [12], [41] between optimal control and differential games (Isaacs), the terminology of these two fields is used somewhat interchangeably. We begin by reviewing previous work briefly.

An underdeveloped area [28] of the Lanchester theory of combat is target selection for combat among heterogeneous forces. This type of problem has been studied by Isbell and Marlow, who considered both a truncated stochastic (Lanchester) process by game theoretic means [51] and a terminal control (one-sided) differential game [52]. An attrition differential game is an idealized combat situation described by Lanchester-type equations over a period of time with choices of tactics available to both sides and subject to change with time. Terminal control attrition games only end when the course of combat has been steered to a prescribed state.

In developing a theory of target selection it is important to understand the dependence of allocation rules on the type of model chosen. Tactical allocation problems may be studied in two types of scenarios: (1) the prescribed duration battle and (2) the terminal control battle (a particular case of which is the "fight to the finish"). All the attrition examples in Isaacs' book [50] are of the first type (his "War of Attrition and Attack" is the continuous version of the tactical air war game [14], [15], [34] studied at RAND). Only Isbell and Marlow [52] and Weiss [82] have studied the terminal control problem. Unfortunately, Isbell and Marlow did not obtain a complete solution to their problem. They could not determine when certain terminal states of combat were reached. Weiss studied a problem which may be considered to be a generalization (two-sided version) of their problem. His solution procedure [82] was a heuristic one, not involving the usual (today's) necessary conditions [12], possibly because the simpler problem which he referenced in his paper had not been completely solved.

a. Statement of the Problem.

The situation considered by Isbell and Marlow [52] is the simplest problem of fire distribution: combat between an X-force at two force types (for example, riflemen and grenadiers) and a homogeneous Y-force (for example, riflemen only). This situation is shown diagrammatically below.



It is the objective of the Y-force commander to maximize his survivors at the end of battle and minimize those of his opponent (considering the utilities assigned survivors). This is accomplished through his choice of the fraction of fire, ϕ, directed at X_1. The battle terminates when one side or the other has been annihilated.

Mathematically the problem may be stated as

$$\max_{\phi(t)} \; ry(T) - px_1(T) - qx_2(T) \quad \text{with } T \text{ unspecified}$$

subject to:

$$\frac{dx_1}{dt} = -a_1 y$$

$$\frac{dx_2}{dt} = -(1 - \phi)a_2 y$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2$$

$$x_1, x_2, y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1,$$

where

p, q and r are utilities assigned to surviving forces,

x_1, x_2 and y are average force strengths,

a_1, a_2, b_1 and b_2 are constant attrition rates,

ϕ is fraction of Y-fire directed at x_1,

and with terminal states defined by (1) $x_1(T) = x_2(T) = 0$ and (2) $y(T) = 0$.

The terminal surface of the "realistic" (one-sided) game is seen to consist of five parts:

$$C_1 : x_1(T) = 0, \quad x_2(T) > 0, \quad y(T) = 0,$$

$$C_2 : x_1(T) = \text{ before } \quad x_2(T) = 0, \quad y(T) > 0,$$

$$C_3 : x_1(T) = 0 \text{ after } \quad x_2(T) = 0, \quad y(T) > 0,$$

$$C_4 : x_1(T) > 0, \quad x_2(T) = 0, \quad y(T) = 0,$$

$$C_5 : x_1(T) > 0, \quad x_2(T) > 0, \quad y(T) = 0.$$

b. Solution Procedure and Extensions.

Extremal paths (a path on which the necessary conditions for optimality are almost everywhere satisfied) may be obtained by routine application of Pontryagin's maximum principle [68] (the original authors used equivalent conditions independently developed by Isaacs [48]). However, in a terminal control problem we would like to know the domain of controllability [32] for each terminal state so that tactics are determined in terms of the initial conditions of combat (and also possibly time). We define the domain of controllability for a given terminal state to be that subset of the initial state space from which extremals lead to the terminal state.

The following procedure has been used to solve the above problem:

(a) extremal control is determined by maximizing the Hamiltonian; since the state variables (force strengths) are non-negative, the control depends, in many cases, only on relationships between the dual variables (marginal return from destroying target),

(b) from each separate terminal state, the time history of the dual variables is obtained by a backward integration of the adjoint system of differential equations; for a square law attrition process, the adjoint equations are independent of the state variables,

(c) for each terminal state the domain of controllability is determined by forward integration of the state equations using the time history of extremal control developed in (b); changes in control with time (existence of transition surface) may have to be considered in this step.

It is noted that Isbell and Marlow [52] stopped at step (b) above.

The complete solution to this problem is shown in Table AI. Details are presented below. A significant point to note is that the extremals are unique (non-overlapping of domains of controllability) so that the extremal control turns out to be the optimal control. This solution procedure may be easily extended to terminal control differential games (such as [82] in which the usual necessary conditions [12] were not applied). We do this in Appendix B. However, in two-sided problems this author has noted that domains of controllability may overlap and

Table AI. Solution to Target Selection Problem: Fight to the Finish

Basic assumption: $a_1 b_1 > a_2 b_2$

	Terminal State	Optimal Control	Conditions on Initial Values
Case A: $a_2 q \leq a_1 p$			
C_1	$x_1(t_1) = 0$	$\phi(t) = 1$ for $0 \leq t \leq t_1$	$a_2 b_1 (x_1^\circ)^2 + 2\, a_2 b_2 x_1^\circ x_2^\circ + a_1 b_2 (x_2^\circ)^2 > a_1 a_2 (y^\circ)^2$
	$x_2(T) > 0$	$\phi(t) = 0$ for $t_1 \leq t \leq T$	$b_1 (x_1^\circ)^2 + 2\, b_2 x_1^\circ x_2^\circ \leq a_1 (y^\circ)^2$
	$y(t) = 0$		
C_2	$x_1(t_1) = 0$	$\phi(t) = 1$ for $0 \leq t \leq t_1$	
	$x_2(T) = 0$	$\phi(t) = 0$ for $t_1 \leq t \leq T$	$a_2 b_1 (x_1^\circ)^2 + 2\, a_2 b_2 x_1^\circ x_2^\circ + a_1 b_2 (x_2^\circ)^2 < a_1 a_2 (y^\circ)^2$
	$y(T) > 0$		
C_5	$x_1(T) > 0$		
	$x_2(T) > 0$	$\phi(t) = 1$ for $0 \leq t \leq T$	$a_2 b_1 (x_1^\circ)^2 + 2\, a_2 b_2 x_1^\circ x_2^\circ + a_1 b_2 (x_2^\circ)^2 > a_1 a_2 (y^\circ)^2$
	$y(T) = 0$		$b_1 (x_1^\circ)^2 + 2\, b_2 x_1^\circ x_2^\circ > a_1 (y^\circ)^2$

Terminal State	Optimal Control	Conditions on Initial Values
Case B: $a_2q > a_1p$		
C_1	Same as Case A	$a_2b_1(x_1^\circ)^2+2a_2b_2x_1^\circ x_2^\circ+a_1b_2(x_2^\circ)^2>a_1a_2(y^\circ)^2$ $\frac{a_1b_2(x_2^\circ)^2}{a_2b_1(x_1^\circ)^2+2a_2b_2x_1^\circ x_2^\circ+a_1b_2(x_2^\circ)^2-a_1a_2(y^\circ)^2} \geq \left[\left(\frac{a_1}{q}\right)\left(\frac{qb_1-pb_2}{a_1b_1-a_2b_2}\right)\right]^2$
C_2	Same as Case A	Same as Case A
C_5	See two subcases below	$a_2b_1(x_1^\circ)^2+2a_2b_2x_1^\circ x_2^\circ+a_1b_2(x_2^\circ)^2>a_1a_2(y^\circ)^2$ $\frac{a_1b_2(x_2^\circ)^2}{a_2b_1(x_1^\circ)^2+2a_2b_2x_1^\circ x_2^\circ+a_1b_2(x_2^\circ)^2-a_1a_2(y^\circ)^2} < \left[\left(\frac{a_1}{q}\right)\left(\frac{qb_1-pb_2}{a_1b_1-a_2b_2}\right)\right]^2$
subcase (a):	$\phi(t) = 0$ for $0 \leq t \leq T$	$(b_1x_1^\circ+b_2x_2^\circ)^2\left\{1-\left[\left(\frac{q}{a_1}\right)\left(\frac{a_1b_1-a_2b_2}{qb_1-pb_2}\right)\right]^2\right\} \geq a_2b_2(y^\circ)^2$
subcase (b):	$\phi(t) = 1$ for $0 \leq t \leq T - \tau_1$ $\phi(t) = 0$ for $T - \tau_1 \leq t \leq T$	$(b_1x_1^\circ+b_2x_2^\circ)^2\left\{1-\left[\left(\frac{q}{a_1}\right)\left(\frac{a_1b_1-a_2b_2}{qb_1-pb_2}\right)\right]^2\right\} < a_2b_2(y^\circ)^2$

Note: (a) t_1 is first t such that $x_1(t_1) = 0$, $t_1 < T$

(b) τ_1 is determined by

$$\cosh \sqrt{a_2b_2}\,\tau_1 = \frac{a_1}{q}\,\frac{(qb_1-pb_2)}{(a_1b_1-a_2b_2)}, \quad \tau_1 \leq T$$

(Table AI concluded)

there may be multiple extremals from a given point in the initial state space so that additional considerations must be employed.

c. Some Comments.

We note that the solution to a "fight to the finish" may depend upon the initial strengths of the combatants. This should be contrasted with the optimal allocation which is independent of force strength in the prescribed duration battle. We contrast the solution properties for these two cases in greater detail in Appendix C.

The examining of this solution process provides valuable insight into the corresponding differential (supporting weapon system) game:

(a) devising solution process,

(b) understanding why no transition (switching) surface present in original problem studied by Weiss,

(c) formulating a game which may possess a switching surface (optimal strategies change with time).

It is noted that the supporting weapon system game may be viewed as an extension of this fire programming problem. The following aspects are also noteworthy of these two problems:

(a) both represent simplest allocation problems of their type,

(b) both are terminal control problems (as opposed to tactical war games studied by RAND researchers: [14], [15], [34] it is noted that the continuous version of these is Isaacs' [50] "war of attrition and attack").

It is noteworthy that if the objective function were modified to $ry(T) - px_1(T)$, then the entire solution to the new problem is the same as shown for case A in Table AI, except that the optimal control for entry to C_1 is not unique. Any control which leads to this state is optimal, since the payoff is always zero. Let us note that the deletion of x_2 from the objective function has caused nonuniqueness in the solution and absence of a transition surface under any circumstances. We shall see that these observations are important for understanding the solution of the original version of Weiss' supporting system game.

We note that the approach developed here for solving terminal control attrition games is different than that used to solve pursuit and evasion differential games. Some examples of the latter are worked out in detail in a companion report [76]. In Table AII we summarize some major points of practical difference.

d. Development of Solution.

The solution is actually derived for a "reduced" game (that portion of battle during which Y is faced with a choice problem). We illustrate here for extremals to C_1. It suffices to trace extremals up to t_1 when $x_1(t_1) = 0$, since $\phi = 0$ from then until the end of the game. The determination of the value, denoted by $V(x_1,x_2,y)$ of the reduced game, which is needed to determine the values of the adjoint variables on the terminal surface, and part of the solution originally obtained by Isbell and Marlow will not be repeated here although we shall outline the general steps.

The Hamiltonian is

$$H(t,x,p,\phi) = -\{p_1\phi a_1 y + p_2(1-\phi)a_2 y + p_3(b_1x_1+b_2x_2)\}$$

and the adjoint equations are

Table AII. Some Differences Between Terminal Control Attrition Games and Pursuit and Evasion Games

	Terminal Control Differential Game	Pursuit and Evasion Differential Game
(1) specification of boundary conditions	two point boundary value problem with state variables specified at beginning, $t = 0$, and some state variables and some dual variables being specified at end, $t = T$	values for both state and dual variables given at end $t = T$
(2) concept of useable part	not used in solution process	central position in solution
(3) extremal strategies	may be "blockable" by opponent's non-extremal strategy	always playable
(4) major solution aspect (Taylor)	determination of domain of controllability for each terminal state	determination of barrier (boundary of domain of controllability)

$$\dot{p}_1 = b_1 p_3,$$
$$\dot{p}_2 = b_2 p_3,$$
$$\dot{p}_3 = p_1 a_1 \phi + p_2 (1-\phi) a_2,$$

with

$$p_1(t = t_1) = \text{unspecified}$$

$$p_2(t = t_1) = \frac{\partial V}{\partial x_2} = \frac{-q\sqrt{b_1}\; x_2}{\sqrt{b_2 x_2^2 - a_2 y^2}}$$

$$p_3(t = t_1) = \frac{\partial V}{\partial y} = \frac{q a_2 y}{\sqrt{b_2}\;\sqrt{b_2 x_2^2 - a_2 y^2}}$$

The extremal control is obtained from max $H(t,x,p,\phi)$, and we also have that

$$\max_{\phi} H(t,x,p,\phi) = 0.$$

Obtaining a solution to this problem is simplified by the following considerations. Let $\tau = t_1 - t$ and define

$$v(\tau) = a_2 p_2(\tau) - a_1 p_1(\tau),$$

then we have

$$\frac{dv}{d\tau} = (a_1 b_1 - a_2 b_2) p_3(\tau),$$

with

$$v(\tau = 0) = a_2 p_2(\tau = 0) - a_1 p_1(\tau = 0),$$

and where (up until the first shift of tactics)

$$p_3(\tau) = p_3(\tau = 0)\cosh\{\sqrt{\phi a_1 b_1 + (1-\phi)a_2 b_2}\,\tau\}$$

$$- \frac{\phi a_1 p_1(\tau=0) + (1-\phi)a_2 p_2(\tau=0)}{\sqrt{\phi a_1 b_1 + (1-\phi)a_2 b_2}} \sinh\{\sqrt{\phi a_1 b_1 + (1-\phi)a_2 b_2}\,\tau\}$$

The extremal control is determined by

$$\phi(t) = 0 \quad \text{for} \quad v(\tau) < 0,$$

$$\phi(t) = 1 \quad \text{for} \quad v(\tau) > 0.$$

It is easy to show that it is impossible for $v(\tau) = 0$ over any finite interval of time, and hence the possibility for any singular solution [53] to this problem is excluded. By the symmetry of this problem it suffices to assume that $a_2b_2 < a_1b_1$, and for this case the domains of controllability for C_3 and C_4 are void.

The major contribution of our present research is to show how to determine the domains of controllability. There are two cases to consider.

Case (a) $\quad a_2q \leq a_1p$

This is the easier case and some of these results apply to the other case. The only time when the Y forces win is when terminal state $C_2 : x_1(t_1) = x_2(T) = 0$ and $y(T) > 0$ where T is the time of the end of the battle and $t_1 < T$ is such that $x_1(t_1) = 0$ is entered. We determine the domain of controllability by combining the time history of the extremal control, the non-negativity requirements on the state variables, and the generalized square law

$$Z^2(t_1) - Z^2(t_2) = \{\phi a_1 b_1 + (1-\phi)a_2 b_2\}(y^2(t_1) - y^2(t_2)),$$

where $\phi(t) = \text{const.}$ in $t_1 \leq t \leq t_2$ and $Z(t) = b_1x_1(t) + b_2x_2(t)$. For the case at hand we have

$$(y(t = t_1))^2 = (y^\circ)^2 - \frac{1}{a_1}\{b_1(x_1^\circ)^2 + 2b_2x_1^\circ x_2^\circ\}$$

and

$$-b_2(x_2^\circ)^2 = a_2\{(y(T))^2 - (y(t = t_1))^2\}.$$

The desired condition is found by elimination of $y(t = t_1)$ between the above equations and requiring that $y(T) > 0$.

It remains to distinguish between entry to C_1 and C_5. On entry to C_5, we have that $x_1(T) > 0$, $x_2(T) > 0$, and $y(T) = 0$. The application of our "modified square law" yields,

$$b_1(x_1(T))^2 + 2b_2y^\circ x_1(T) = b_1(x_1^\circ)^2 + 2b_2x_1^\circ x_2^\circ - a_1(y^\circ)^2,$$

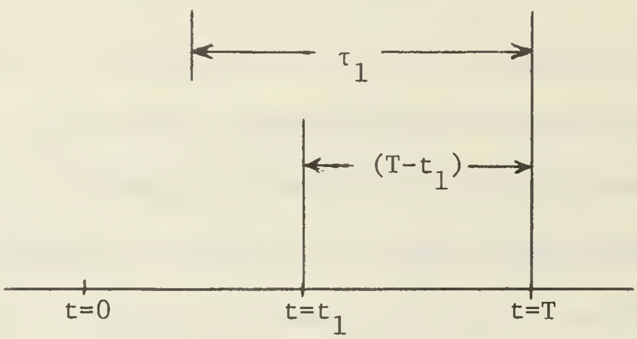
whence our result by requiring that $x_1(T) > 0$.

Case (b) $\quad a_2q > a_1p$

The work of Isbell and Marlow has been extended by showing how to determine the domains of controllability when a switching surface is present in the solution. The conditions for entry to C_2 are as before. We must develop conditions to distinguish between entry to C_1 and C_5 and two subcases for entry to C_5.

C_1 is entered in those cases when the X_1 forces are destroyed before a switch in tactics is required. It is recalled that the latter condition, determined by backward integration of the adjoint differential equations from the terminal surface and the maximum principle, is independent of the initial conditions of the state variables. Entry to

C_1 is determined by the relationship between the proportion of total battle time (forward) to destroy X_1 and the time (backward) of the potential switch. The figure below shows the relationship between these times, where $\tau = T - t$, τ_1 is the time (backward) of the switch, $t = t_1$ is such that $X_1(t_1) = 0$, and T is the time (forward) of the end of the battle. As shown C_5 would be entered.



The condition for entry to C_1 is that $t_2 > \tau_1$ where $T = t_1 + t_2$, i.e., the optimum length of τ-time for engaging X_2 is less than the remaining time for X_2 to destroy Y after Y has annihilated X_1 (battle starts with engagement of X_1). From the "modified square law,"

$$y(t = t_1) = \sqrt{(y°)^2 - \left(\frac{b_1}{a_1}\right)(x_1°)^2 - 2\left(\frac{b_2}{a_1}\right)x_1°x_2°}.$$

After annihilation of X_1, there is another battle of length t_2 remaining. Hence, for this portion where $t_1 \le t \le T$,

$$y(t) = y(t = t_1)\cosh\sqrt{a_2b_2}(t - t_1) - x_2°\sqrt{\frac{b_2}{a_2}}\sinh\sqrt{a_2b_2}(t - t_1).$$

Since $y(t = T) = 0$, we have (using that $T - t_1 = t_2$)

$$\tanh\sqrt{a_2 b_2}\; t_2 = \frac{y(t=t_1)}{x_2^\circ} \sqrt{\frac{a_2}{b_2}}$$

From integration of the adjoint equations and the maximum principle, the τ-time of the switch is given by,

$$\cosh\sqrt{a_2 b_2}\; \tau_1 = \frac{a_1}{q} \frac{(qb_1 - pb_2)}{(a_1 b_1 - a_2 b_2)} .$$

The desired condition is determined by requiring that $t_2 > \tau_1$ (as defined above), use of the identities

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$\tanh^{-1} y = \frac{1}{2} \ln\left(\frac{1 + y}{1 - y}\right) ,$$

and considerable algebraic manipulation.

It finally remains to distinguish between the two cases of entry to C_5. If $\psi(t) = 0$ for $0 \le t \le T$, then

$$y(t) = y^\circ \cosh\sqrt{a_2 b_2}\; t - \frac{(b_1 x_1^\circ + b_2 x_2^\circ)}{\sqrt{a_2 b_2}} \sinh\sqrt{a_2 b_2}\; t.$$

The boundary between the two cases is when $y(T) = 0$ for $T = \tau_1$ and hence,

$$(y^\circ)^2 [\cosh\sqrt{a_2 b_2}\; \tau_1]^2 = \frac{(b_1 x_1^\circ + b_2 x_2^\circ)^2}{a_2 b_2} \{[\cosh\sqrt{a_2 b_2}\; \tau_1]^2 - 1\}$$

where $\cosh\sqrt{a_2 b_2}\,\tau_1$ is given as above. Noting that $\phi = 0$ for the entire battle when $T < \tau_1$ and re-arranging, we obtain the result shown in Table AI.

e. Structure of Optimal Allocation Policies.

For square law attrition it may be shown that the allocation of fraction of fire is always 0 or 1 (see previous section for remark), and fire is concentrated on one target type. This is not surprising, since our model assumes complete and instantaneous information [13] and that fire may be immediately shifted to a new target once the old one has been destroyed [22], [81].

With reference to Table AI, the condition that $a_1 b_1 > a_2 b_2$ may be interpreted to mean that there is more long range return for Y to engage X_1, i.e., more Y's will survive if this is done. Hence, when Y wins, he always engages X_1's while they are available. The condition $a_1 p < a_2 q$ means that at the end of battle there is greater payoff per unit time per Y soldier to engage X_2 not considering X_1's greater attrition effect against Y (short term gain at end of battle).

By the maximum principle and the well-known interpretation of the dual variables [12], Y always allocates his fire entirely to the target type yielding the greatest marginal return. However, marginal return evolves differently in winning or losing causes. When Y loses, he may switch from firing at X_1 entirely to firing at X_2 entirely before the X_1 force has been annihilated. This happens when Y assigns utility to survivors of force type X_2 in excess of their kill rate against Y as compared to force type X_1, and X_1 is abundant enough not to be destroyed before the battle ends.

In this way, we see that tactics may depend on force levels. We also see that Y's target priorities only switch with time in a losing case. This has occurred since a boundary condition at $t = T$ on one of the dual variables is dependent upon values of the state variables by a transversality condition. It may be shown that the structure of optimal allocation policies is different for the prescribed duration battle.

In Appendix F we show how such considerations as those discussed above may be developed into the concept of a dynamic kill potential. However, we do so from the standpoint of the adjoint system for a system of differential equations. (This approach may be used as an alternative to that of Pontryagin for the development of his maximum principle.)

APPENDIX B. H. K. Weiss' Supporting Weapon System Game

In this appendix we develop the solution to the supporting weapon system game of H. K. Weiss [82] by applying the theory of differential games. Previously, this problem had been solved under restrictive assumptions by heuristic means. The solution procedure developed here is general and applies to any terminal control attrition game. A new solution concept is motivated by this development, and solution behavior not previously noted for differential games is encountered.

Our researches on this and similar dynamic tactical allocation problems indicate that there are several significant differences in theory and results between attrition and pursuit-evasion differential games. We have briefly considered such differences in Appendix A. However, much excellent research has been done on generalized control theory applicable to pursuit and evasion problems, and we envision the application of such results to tactical allocation problems as being fruitful future research. For example, the concepts of stochastic control could be applied to a situation in which combatants select targets without knowing precisely what the results of firings will be.

The model considered here is an idealization of a real combat situation. Its value lies in the insight it provides into the relations between system parameters. It should not be expected to produce a numerical answer to a specific problem but rather to indicate general principles to serve as hypotheses for subsequent computer simulation studies or field experimentation. In this manner, the model considered here may be used to study the following facets of supporting weapon systems: performance characteristics, allocation rules, impact of intelligence and command and control factors on the preceding.

There are two types of scenarios in which we may study idealizations of tactical allocation problems: (1) the prescribed duration battle and (2) the terminal control battle, i.e., the game only ends when the course of battle has been steered to a prescribed state. All the attrition problems studied by Isaacs [50] are of the first type. It is noted that his War of Attrition and Attack is the continuous version of other such studies [14], [15], [34]. Only Isbell and Marlow [52] and Weiss have studied the terminal control problem. The former did not obtain a complete solution to their problem but we have in Appendix A and were motivated to the present development. Only by studying several types of models can we begin to understand the dependence of allocation rules on model form.

In this appendix we consider what forms of such dynamic models are available before we review Weiss' problem formulation. We then critique his previous approach before outlining our new solution procedure and presenting details of solution development. We then discuss the structure of optimal allocation policies. We also discuss extensions of the model and a pitfall of model formulation before we contrast some facets of prescribed duration battles to fights to the finish. We finally mention a few implications of the models we have considered. In view of the intimate relationship [12], [41] between optimal control theory and differential games (Isaacs), we use their terminology somewhat interchangeably.

a. <u>Forms of Model Available</u>.

It seems appropriate to discuss the factors affecting the optimal allocation policies. Different assumptions regarding these factors lead to models with different optimal allocation policies. The model for a tactical allocation problem involves three factors:

(1) the payoff,

(2) the description of combat,

(3) the planning horizon.

We will consider a terminal payoff with a linear objective function. The tactical allocation problems studies at RAND [14], [15], [34], [50] all involved an integral payoff. Further comment on the effect of inclusion of only one of the two force types in the payoff by Weiss [82] seems appropriate. What effect does this have on the optimal allocation? From the present work, it seems reasonable to conjecture that for two-on-two combat the optimal strategies for a side will be constant over time (except for the obvious change when a force under attack becomes exhausted) if the payoff only includes one force type. It is further conjectured that this is the reason (only the "men" of each side appearing in the payoff) that the optimal strategies in the reduced supporting weapon system game of H. K. Weiss are constant over time and that optimal strategies may vary over time when all force types are included in the payoff function. It will be seen that optimal strategies only change over time for the loser who engages the force type that does him the most damage in the early stages of the battle and the force included in the payoff on which he has the most effect in the latter stages. We conjecture that the winner's optimal strategy is always constant over time for "fights to the finish."

For our description of the combat attrition process we may consider a generalized Lanchester linear law or a square law (although other mathematical descriptions have been noted as applicable to specific situations). For a square law attrition process the attrition rate is proportional to enemy strength, while for a linear law it is proportional to the product of both enemy and friendly force strengths. With rare exception ([75] or Isaacs' "war of attrition and attack: second version" [50]), previously published work has considered only the square law model. In Appendix C we show that a square-law attrition process leads to a "bang-bang" optimal control while the linear law leads to a singular solution (see p. 481 of [6]). The mathematical development is much more complex in the second case, but we have studied singular problems on numerous occasions (pursuit and evasion [76], inventory theory, the continuous version of Bellman's stochastic gold-mining problem).

It seems appropriate to briefly discuss the physical assumptions which underlie these idealizations of combat attrition. The square law arises under conditions which include that "each unit is informed about the location of the remaining opposing units so that when a target is destroyed, fire may be immediately shifted to a new target" as noted by Weiss [81]. It is noted that differential game theory itself assumes complete information (except that a player does not know the instantaneous strategy of the opposing player). The linear law arises when either target acquisition is subject to diminishing returns [22] or fire is not redirected towards surviving targets after attrition occurs [39], [70], [81].

In the present work a model is formulated for the simplest case of <u>partial information</u>: "area fire" is delivered by the supporting weapon system against the ground troops who use a constant area defense while the perfect information assumption is retained on the state of the supporting weapon system. Again quoting Weiss [81], we assume that the supporting weapon system units are informed about the general areas in which the opposing infantry units are located but are not informed about the consequences of their own fire. Thus, we see that we may account for some changes in the information set by modifying the description of combat. Unfortunately, the mathematics of the resulting problem is much more complex than previously encountered, and a complete solution has not yet been obtained for this case. For this model of incomplete information, one introduces the concept of inferred information (players know more than they can observe directly) based on each player's knowledge of the time history of his control variables and considers the resulting equations in this light.

Another factor having a bearing on the optimal allocation policies is the length of the planning horizon (length of the battle). The following three alternative models are available:

(1) battle of prescribed time duration,

(2) battle of unspecified time duration,

(3) battle until the extermination of one side.

Our researches have subsequently yielded that case (2) is not a properly posed problem in the classical sense [27]. Models applying to the first instance have been extensively studied by RAND researchers [14], [15], [34], [50]. The present work (as an extension of the work of Isbell and Marlow and Weiss) will address the third case, "fights to the finish." The mathematical details of solution and the structure of optimal policies are significantly different for these two cases. Games of prescribed duration are mathematically simpler than "fights to the finish," since the terminal surface consists of one "piece" and many different portions do not have to be considered. Once the adjoint equations have been integrated backward from the terminal surface, the history of the extremal strategies (and hence optimal strategies) becomes uniquely determined unless a state variable goes to zero and a subgame is entered. On the other hand for a terminal control game, extremals to all the distrinct portions of the terminal surface must be considered. Entry to a portion of the terminal surface must be verified by both considerations "in the large" and forward integration of the state equations (after determination of extremal strategies). Many times the potential existence of a transition (switching) surface turns out to be illusory, and the complete solution may turn out to be radically different than was initially anticipated.

b. Problem as Formulated by Weiss

The problem studied by Weiss [82] may be stated as how should the fire support systems of two heterogeneous forces (each consisting of ground forces and its fire support system) optimally engage the opposing combatant. The objective is for each side to minimize its losses in a conflict which terminates when the opposing side is annihilated. The ground forces (infantry) are assumed to have a negligible effect in producing casualties on each other.

Using Weiss' original notation the problem was finally reduced to the payoff:

$$\max_{\phi} \min_{\psi} [y_1(T) - y_2(T)] , \tag{B1}$$

where T is the unspecified terminal time of the battle and ϕ and ψ are decision variables representing the fraction of 'air' of ODD and EVEN which engages the opposing 'infantry'. The average strength of remaining forces are given by the state equations:

$$\begin{aligned} \dot{y}_1 &= -\psi y_4 \ , \\ \dot{y}_2 &= -\phi y_3 \ , \\ \dot{y}_3 &= -(1-\psi) y_4 \ , \\ \dot{y}_4 &= -(1-\phi) y_3 \ , \end{aligned} \tag{B2}$$

with boundary conditions:

$$\begin{aligned} y_1(t=0) &= \overset{\circ}{y}_1 \ , \ y_1(t=T) = 0 \\ y_2(t=0) &= \overset{\circ}{y}_2 \ , \\ y_3(t=0) &= \overset{\circ}{y}_3 \ , \\ y_4(t=0) &= \overset{\circ}{y}_4 \ . \end{aligned} \tag{B3}$$

where $0 \leq \phi$, $\psi \leq 1$, $\dot{y}_i = dy_i/dt$

and

y_1, y_2 = average strength of 'infantry' of ODD and EVEN at time t,

y_3, y_4 = average strength of 'air' of ODD and EVEN at time t.

It is noted that the y_i are transformed variables which include attrition rates. We will also denote terminal values as $y_i(t=T) = y_{is}$, in consonance with Weiss' notation. It is finally noted that the terminal condition on y_1 has been specified as a prelude to the development in a future section.

c. Critique of Previous Solution Procedure.

We should bear in mind that Weiss's excellent paper [82] (it contains much more than the mathematical solution of a differential game) was written over ten years ago. Writing many years before results were known beyond a small number of researchers, he did not employ the usual (today's) necessary conditions [12]. The original solution technique in this pioneering effort used unsupported assumptions which, in general, are not true, although the correct answer was obtained to the particular problem posed. Weiss assumed that optimal strategies would be (a) either 0 or 1 and (b) constant over time and then determined the saddle point of the payoff function. It will be seen that rather laborious computations are required to establish the solution form that Weiss assumed.

Weiss's pioneering effort is especially remarkable when one considers that Isaacs's book [50] had not yet been written and only Isaacs's early RAND memos (see in particular [48], [49]) were available. Also, Isbell and Marlow had failed to obtain a complete solution to a simpler (one-sided) terminal control problem. We note that Weiss's problem (and also Isbell-Marlow fire programming problem) do not appear to be known to the control theorists [5], [13], [24], [71].

Weiss's paper also contains an extension of the attrition model imbedded in an economic model of conflicting systems. It also contains a penetrating analysis of weapon system performance characteristics and concludes with a discussion of insight gained into the optimum design of real world weapon systems.

d. Solution Procedure.

In this section we outline the solution procedure, introduce the concept of the "reduced game," illustrate the determination of extremal strategies, and discuss the concept of a "blockable" terminal state.

Outline of Solution Procedure

In a terminal control problem, we must determine the optimal strategies for each player in terms of the initial conditions of combat (and also possibly time). The solution procedure consists of two phases: (a) determine all extremal strategies and (b) determine optimal strategies from among the extremal strategies. By an extremal, we mean a path on which the necessary conditions [12] for optimality are almost everywhere satisfied.

We must consider each terminal state separately. For each terminal state, there will be one or more extremal paths leading to that state. Extremal paths may be determined by routine application of the well-known necessary conditions. For each extremal path to a terminal state there is a domain of controllability, which we define to be that subset of the initial state space from which a family of extremals leads to the terminal state. The solution procedure may be summarized as:

(1) identify "attainable" terminal states,

(2) determine "domain of controllability" in initial condition space corresponding to each extremal leading to every "attainable" terminal state,

(3) partition the space of initial conditions into exhaustive and mutually exclusive sets, each of which is covered by the "domain(s) of controllability" of one, two, etc., of the extremals to terminal states,

(4) the solution is uniquely determined at this point for regions covered by part of only one domain of controllability,

(5) delete from further consideration those portions of the domain of controllability of any terminal state which is "blockable" from those initial points; again the solution is uniquely determined (extremal is optimal) for those regions reverting to step (4),

(6) if there is still more than one extremal to a given terminal state for a set of points in the initial condition space, compute the value of the game for each extremal; the final solution is determined by comparing these values.

The concept of a "blockable" terminal state is discussed below.

Concept of the "Reduced Game"

The battle is over when either y_1 or y_2 becomes zero. It is convenient to introduce the concept of the "reduced game." Let us henceforth refer to the original problem as the "realistic game." In attrition games (especially "fights to the finish") the allocation problem may disappear before the terminal surface is reached. Let us refer to that part of the game for which the full allocation problem exists as the "reduced game," and we now consider the terminal surface of the reduced game. The value of the reduced game must be backcalculated from the value of the realistic game. To illustrate, the terminal surface for the above problem is defined by three terminal states: (a) $y_1(T) = 0$, (b) $y_2(T) = 0$, and (c) $y_1(T) = 0$ and $y_2(T) = 0$. The terminal surface of the reduced game is seen to consist of five portions and these are shown in Table BI.

It will be seen that the extremal strategies to each of these requires a different development. The payoff on C_3 is $(-y_2(T))$, since ODD has lost all his infantry at the terminal surface of the realistic game. It may be that a portion of the terminal surface is not attainable from any point in the initial state space, and this is

Portions of Terminal Surface

A	EVEN wins	$y_1(T) = 0$
B	EVEN wins	$y_3(T) = 0$
C	ODD wins	$y_2(T) = 0$
D	ODD wins	$y_4(T) = 0$
E	DRAW	

Extremals leading to A

(1) $a_1: \begin{cases} \phi = 1 \\ \psi = 1 \end{cases}$ for $0 \le t \le T$

(2) $a_2: \begin{cases} \begin{cases} \phi = 1 \\ \psi = 0 \end{cases} & \text{for } 0 \le t \le T - \tau_1 \\ \begin{cases} \phi = 1 \\ \psi = 1 \end{cases} & \text{for } T - \tau_1 \le t \le T \end{cases}$

(3) $a_3: \begin{cases} \begin{cases} \phi = 0 \\ \psi = 0 \end{cases} & \text{for } 0 \le t \le T - \tau_2 \\ \begin{cases} \phi = 1 \\ \psi = 0 \end{cases} & \text{for } T - \tau_2 \le t \le T - \tau_1 \\ \begin{cases} \phi = 1 \\ \psi = 1 \end{cases} & \text{for } T - \tau_1 \le t \le T \end{cases}$

Extremals leading to B

(1) $b_1: \begin{cases} \phi = 1 \\ \psi = 0 \end{cases}$ for $0 \le t \le T$

(2) $b_2: \begin{cases} \begin{cases} \phi = 0 \\ \psi = 0 \end{cases} & \text{for } 0 \le t \le T - \tau_1 \\ \begin{cases} \phi = 1 \\ \psi = 0 \end{cases} & \text{for } T - \tau_1 \le t \le T \end{cases}$

Note: Extremals to C and D are symmetric to above.

Table BI. Extremals and Terminal Surface Defined.

what Isaacs refers to as the non-useable portion of the terminal surface [50]. This concept is, however, not particularly useful in the solution of an attrition game. The concept of the domain of controllability for a terminal state is more useful.

Determination of Extremal Strategies

Table BI shows the five terminal states to the ("reduced") supporting weapon system game. Extremal paths are determined for a "reduced game," which is that part of the game for which a full allocation problem exists. For example, after $y_2 = 0$, ODD uses $\phi = 1$ until EVEN's infantry is annihilated, and we only need consider up until that time. Moreover, to determine boundary conditions on the dual variables in the "reduced game," we must consider the payoff of the entire game. We discuss this point further in the next section.

We will now outline the obtaining of extremal strategies when, for example, terminal state A is entered (EVEN wins by destroying ODD's infantry), i.e., $y_1(T) = 0$ and T is unspecified. In this case the objective function becomes:

$$\max_{\phi} \min_{\psi} \{-y_2(T)\}.$$

We introduce "costate" or dual variables, denoted by p_i, one for each state equation and representing rate of change of the game value to the players (here terminal payoff to the game) with respect to the various state variables. We now form the following Hamiltonian:

$$H(t,y,p;\phi,\psi) = \psi y_4(p_3-p_1) + \phi y_3(p_4-p_2) - y_4 p_3 - y_3 p_4.$$

From this Hamiltonian we form the following "adjoint" equations:

$$- \frac{\partial H}{\partial y_1} = \frac{dp_1}{dt} = 0 \Rightarrow p_1(t) = \text{const.},$$

$$- \frac{\partial H}{\partial y_2} = \frac{dp_2}{dt} = 0 \Rightarrow p_2(t) = \text{const.},$$

$$- \frac{\partial H}{\partial y_3} = \frac{dp_3}{dt} = \phi p_2 + (1 - \phi) p_4, \tag{B4}$$

$$- \frac{\partial H}{\partial y_4} = \frac{dp_4}{dt} = \psi p_1 + (1 - \psi) p_3,$$

with boundary conditions

$$p_1(t = T) = \text{unspecified},$$

$$p_2(t = T) = -1,$$

$$p_3(t = T) = 0, \tag{B5}$$

$$p_4(t = T) = 0.$$

Extremal strategies (as a function of time) are determined from $\max_{\phi(t)} \min_{\psi(t)} H(t,y,p;\phi,\psi)$, which is equal to zero, since the terminal time is left unspecified. Thus we have

$$\max_{\phi} \{\phi y_3 (p_4 - p_2)\} + \min_{\psi} \{\psi y_4 (p_3 - p_1)\} - y_4 p_3 - y_3 p_4 = 0, \tag{B6}$$

where it is recalled that we must have $0 \leq \phi$, $\psi \leq 1$.

Extremal strategies are determined by a backward integration of the adjoint equations (B4) with boundary conditions (B5) and considering (B6), since the boundary conditions of the dual variables are at the terminal surface. It is noted that for square law attrition that the adjoint equations are independent of the state variables (except for a boundary condition by a transversality relation) and so are the extremal strategies. The domain of controllability for an extremal so determined is obtained by a forward integration of the state equations. The non-negativity of the state variables plays a central role in these determinations [74]. Details for the case at hand are presented in the next section.

<u>Concept of a "Blockable" Terminal State</u>

It may be shown that for many regions of the initial state space of this problem, there is more than one family of extremals leading to terminal states. The reason for existence of multiple extremals is that the min-max principle is merely necessary and of a local nature (see Athens and Falb [6] for a discussion of the corresponding situation in control theory). The attainable portions of the terminal surface are not "close together" when multiple extremals are present.

A solution aspect unique to terminal control attrition games is that in cases where there are extremals from the same initial point to different terminal states corresponding to the same player both winning and losing, entry to a terminal state may be "blocked" by the "losing" player through use of an admissible strategy other than his extremal strategy. In other words, there is a path determined by the necessary conditions leading from each point in a region of the initial state space to a terminal state, but the "losing" player may use a strategy other than his extremal strategy to actually win. This behavior highlights the local ("in the small") nature of the necessary conditions and the fact that the conditions are, indeed, necessary, i.e., assume that the losing player cannot prevent the terminal state from being reached.

e. Development of Solution.

In this section we determine the optimal strategies from among the extremal strategies as discussed in the previous section. We also present the details of the derivation of extremals and domains of controllability.

Determination of Optimal Strategies

We now apply steps (3) to (6) of our solution procedure. Since the approach developed here may be used to show that Weiss's original solution technique did indeed yield the correct solution to this particular problem, the interested reader is directed to the original paper for the complete solution. We illustrate our procedure for the case when $y_3^\circ = y_4^\circ/\sqrt{2}$.

Application of step (3) yields the regions shown in Figure B1 with further details being provided by Tables BI and BII. It is noted that in region III, EVEN can "block" ODD's steering the course of battle to $y_4(T) = 0$ by countering ODD's strategy of $\phi = 0$ with $\psi = 0$ instead of using his extremal strategy $\psi = 1$. Since EVEN has more air, he would win this strategic war. Hence, ODD would not consider trying to steer the course of combat to state D, since entry to this state is "blockable" for $y_4^\circ > y_3^\circ$. Table BII summarizes such considerations. Discussion is still required on step (6) above for Regions I, II, III, IV, and V as shown in Figure 1. We now show that the "domain of controllability" corresponding to a_1 contains that of a_2 and the payoff to a player 2 for extremal a_1 is always greater than that for a_2 in these regions. Consequently, by applying the principle of optimality [9], extremal a_3 may also be dropped from further consideration. For

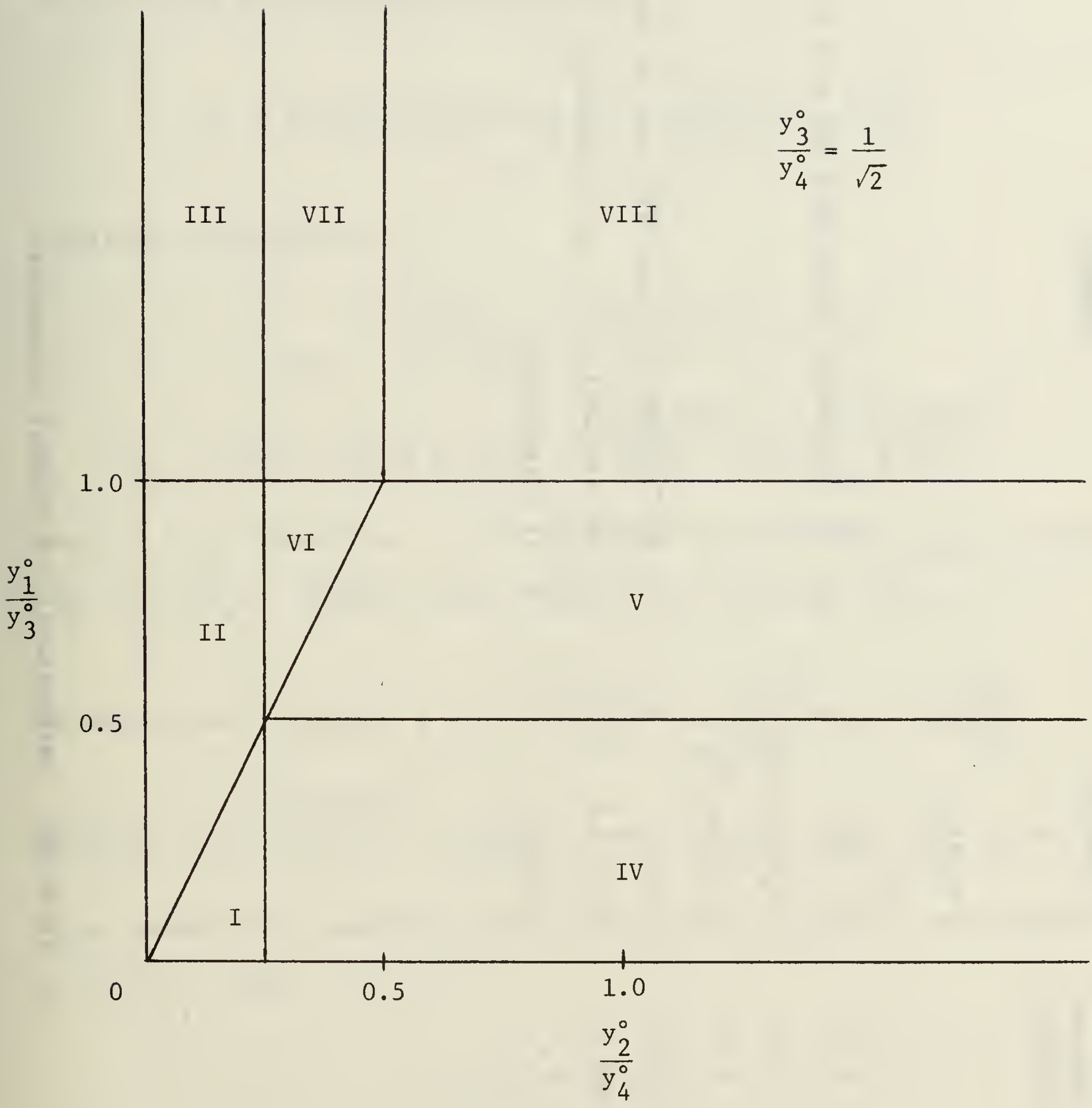


Figure B1. Regions for Determining Optimal Strategies.

Region	Attainable Portion of Terminal Surface	Extremals	Comments
I	A	a_1, a_2, a_3	
II	C	c_1	
III	C,D	c_1, d_1	D is "blockable" since EVEN has more air
IV	A,B	a_1, a_2, a_3, b_1	
V	A,B	a_1, a_2, a_3, b_1	
VI	B,C	b_1, c_1	C is "blockable" since EVEN wins by choosing $\psi = 0$
VII	B,C,D	b_1, c_1, d_1	C is "blockable," D is "blockable"
VIII	B,D	b_1, d_1	D is "blockable"

Table BII. Determination of Optimal Strategies.

extremal a_1, we have that

$$T_{a_1} = y_1^\circ/y_4^\circ \quad \text{and} \quad y_{3s} = y_3^\circ.$$

The domain of controllability is given by:

$$S_{a_1} = \{y^\circ | y_4^\circ > y_3^\circ, y_3^\circ \geq y_1^\circ, y_2^\circ > y_1^\circ\left(\frac{y_3^\circ}{y_4^\circ}\right), y_4^\circ > y_1^\circ\left(\frac{y_3^\circ}{y_4^\circ}\right)\}$$

Similarly, for extremal a_2:

$$\tau_{1(a_2)} = y_1^\circ/y_4^\circ, T_{a_2} = y_3^\circ/y_4^\circ \quad \text{and} \quad y_{3s} = y_1^\circ.$$

$$S_{a_2} = \{y^\circ | y_4^\circ > y_1^\circ, y_3^\circ \geq y_1^\circ, y_2^\circ > \frac{(y_3^\circ)^2+(y_1^\circ)^2}{2\ y_4^\circ}, y_4^\circ \geq \frac{(y_4^\circ)^2+(y_1^\circ)^2}{2\ y_4^\circ}\ \}$$

When $y_4^\circ > y_3^\circ$ (otherwise A is "blockable" for extremal a_2), we have that $S_{a_1} \supset S_{a_2}$. (PROOF: $y^\circ \varepsilon S_{a_2}$ with $y_4^\circ > y_3^\circ$; then $y_3^\circ \geq y_1$ is satisfied; also $(y_3^\circ - y_1^\circ)^2 \geq 0 \Rightarrow \frac{(y_3^\circ)^2+(y_1^\circ)^2}{2\ y_4^\circ} \geq y_1^\circ\left(\frac{y_3^\circ}{y_4^\circ}\right) \Rightarrow y_4^\circ > y_1^\circ\left(\frac{y_3^\circ}{y_4^\circ}\right)$; similarly, $y_2^\circ > \frac{(y_3^\circ)^2+(y_1^\circ)^2}{2\ y_4^\circ} \geq y_1^\circ\left(\frac{y_3^\circ}{y_4^\circ}\right)$; hence $y^\circ \varepsilon S_{a_2}$ with $y_4^\circ > y_3^\circ \Rightarrow y^\circ \varepsilon S_{a_1}$.)

We now consider the payoffs. Denote the payoff to player 2 for extremal a_1 by P_{a_1}. Then

$$P_{a_1} = y_2^\circ - y_1^\circ \frac{y_3^\circ}{y_4^\circ}$$

Similarly, it may be shown that

$$P_{a_2} = y_2^\circ - \frac{(y_3^\circ)^2+(y_1^\circ)^2}{2\ y_4^\circ}$$

It is easy to show that $P_{a_1} > P_{a_2}$ for all $y^\circ \in S_{a_2} \cap \{y^\circ | y_4^\circ > y_3^\circ\}$. Since EVEN determines the choice of these extremals, a_1 will be chosen since it yields the largest payoff for EVEN.

It remains to compare the payoffs to EVEN for a_1 and b_1 in Region IV and V. It may be shown that

$$P_{b_1} = y_2^\circ - \frac{(y_3^\circ)^2}{2\, y_4^\circ}.$$

Hence for $\frac{y_1^\circ}{y_3^\circ} < 1/2$, we have that $P_{a_1} < P_{b_1}$. Thus a_1 is optimal in Region IV, but b_1 is optimal in Region V.

<u>Derivation of Extremals and Domains of Controllability</u>

We provide details for terminal states A and B.

<u>Terminal State A</u>: $y_1(T) = 0$

At $t = T$, it is clear from (B6) that $\phi(t = T) = 1$. Combining this result with (B5), we have at $t = T$:

$$y_{3s} + \min_{\psi} [\psi y_{4s}(-p_1)] = 0$$

Thus $p_1 = \frac{y_{3s}}{y_{4s}}$ and $\psi(t = T) = 1$. Then

$$\phi(t) = \begin{cases} 0 & \text{for } p_4(t) < -1 \\ 1 & \text{for } p_4(t) > -1 \end{cases}$$

and

$$\psi(t) = \begin{cases} 0 & \text{for } p_3(t) > \frac{y_{3s}}{y_{4s}} \\ 1 & \text{for } p_3(t) < \frac{y_{3s}}{y_{4s}} \end{cases}$$

There are now two separate cases which we must consider. We let $\tau = T - t$. The adjoint equations of interest become

$$\frac{dp_3}{d\tau} = \phi - (1 - \phi)p_4, \qquad p_3(\tau = 0) = 0, \qquad \phi(\tau = 0) = 1$$

$$\frac{dp_4}{d\tau} = -\psi \frac{y_{3s}}{y_{4s}} - (1 - \psi)p_3, \qquad p_4(\tau = 0) = 0, \qquad \psi(\tau = 0) = 1$$

Case (a) $\quad 0 < y_{3s} < y_{4s}$

ψ changes first in τ-time, call this τ_1.

For $\tau_1 \leq \tau < \tau_2$, then $p_4(\tau) = -\frac{1}{2}\{\tau^2 + \left(\frac{y_{3s}}{y_{4s}}\right)^2\}$, and for $\tau_2 \leq \tau \leq T$,

$p_3(\tau) = \sqrt{2 - \left(\frac{y_{3s}}{y_{4s}}\right)^2} \cosh(\tau - \tau_2) + \sinh(\tau - \tau_2)$, and

$p_4(\tau) = -\cosh(\tau - \tau_2) - \sqrt{2 - \left(\frac{y_{3s}}{y_{4s}}\right)^2} \sinh(\tau - \tau_2)$.

Hence

(a) for $0 \leq \tau < \tau_1 = \frac{y_{3s}}{y_{4s}}$, $\quad \phi(\tau) = 1$ and $\psi(\tau) = 1$.

(b) for $\tau_1 \leq \tau < \tau_2 = \sqrt{2 - \left(\frac{y_{3s}}{y_{4s}}\right)^2}$, $\quad \phi(\tau) = 1$ and $\psi(\tau) = 0$,

(c) for $\tau_2 \leq \tau \leq T$, $\quad \phi(\tau) = 0$, $\quad \psi(\tau) = 0$.

We now integrate the state equations forward using the above to determine the domains of controllability. When we employ $\phi = 1$ and $\psi = 1$ for $0 \leq t \leq T$, we have that $y_{3s} = y_3^\circ$ and $T = \frac{y_1^\circ}{y_4^\circ}$. Using the facts that $\tau_1 \leq T$ and $y_2(T) > 0$, we find that $y_4^\circ > y_3^\circ, y_3^\circ \geq y_1^\circ, y_2^\circ > y_1^\circ\left(\frac{y_3^\circ}{y_4^\circ}\right)$, and $y_4^\circ > y_1^\circ\left(\frac{y_3^\circ}{y_4^\circ}\right)$.

When we employ $\phi = 1$ and $\psi = 0$ for $0 \leq t \leq T - \frac{y_{3s}}{y_{4s}}$ and $\phi = 1$ and $\psi = 1$ for $T - \frac{y_{3s}}{y_{4s}} \leq t \leq T$, it may be shown that $y_{3s} = y_1^\circ$ and $T = \frac{y_3^\circ}{y_4^\circ}$. Using the facts that $\tau_1 \leq T$, $\tau_2 \geq T$, and $y_2(T) > 0$, we find that $y_4^\circ > y_1^\circ, y_3^\circ \geq y_1^\circ, y_2^\circ > \frac{(y_3^\circ)^2 + (y_1^\circ)^2}{2\, y_4^\circ}, y_4^\circ \geq \frac{(y_3^\circ)^2 + (y_1^\circ)^2}{2\, y_4^\circ}$

Case (b) $0 < y_{4s} < y_{3s}$

As above, we may show that

(a) for $0 \leq \tau < \tau_1 = \dfrac{y_{4s}}{y_{3s}}$, $\phi(\tau) = 1$ and $\psi(\tau) = 1$,

(b) for $\tau_1 \leq \tau < \tau_2 = \sqrt{2 - \left(\dfrac{y_{4s}}{y_{3s}}\right)^2}$, $\phi(\tau) = 1$ and $\psi(\tau) = 0$,

(c) for $\tau_2 \leq \tau \leq T$, $\phi(\tau) = 0$ and $\psi(\tau) = 0$.

Proceeding as before, when we employ $\phi = 1$ and $\psi = 1$ for $0 \leq t \leq T$, we have that $y_{4s} = y_4^\circ$ and $T = \dfrac{y_1^\circ}{y_4^\circ}$. Using the facts that $\tau_1 \geq T$ and $y_2(T) > 0$, we find that $y_4^\circ < y_3^\circ, y_3^\circ > y_1^\circ, y_2^\circ > y_1^\circ\left(\dfrac{y_3^\circ}{y_4^\circ}\right)$, and $y_4^\circ > y_1^\circ\left(\dfrac{y_3^\circ}{y_4^\circ}\right)$.

When we employ $\phi = 1$ and $\psi = 0$ for $0 \leq t \leq T - \dfrac{y_{4s}}{y_3^\circ}$ and $\phi = 1$ and $\psi = 1$ for $T - \dfrac{y_{4s}}{y_3^\circ} \leq t \leq T$, it may be shown that $T = \dfrac{y_4^\circ}{y_3^\circ}$. Using the fact $y_1(T - \dfrac{y_{4s}}{y_3^\circ}) = y_3^\circ$, it may be shown that $y_4^\circ > y_3^\circ, y_1^\circ \geq y_3^\circ, y_2^\circ > y_3^\circ$, and $(y_4^\circ)^2 > 2\{y_1^\circ y_3^\circ - (y_3^\circ)^2\}$.

Terminal State B:

For this case the values of the adjoint variables on the terminal surface are:

$p_1(t = T) = 0$

$p_2(t = T) = -1$

$p_3(t = T) =$ unspecified $\quad y_3(t = T) = 0$

$p_4(t = T) = 0$

It is noted that $p_1(t = T) = 0$ even though $y_1(t = T) = y_1^\circ$. The reason for this is that we must consider the payoff of the entire game to determine boundary conditions for the "reduce game," as noted above.

Thus, we must set $p_1(t = T) = 0$, since ODD must lose all his infantry after his air has been lost and thus has no value for infantry without air.

Subsequent details are similar to those for terminal state A. It may be shown that

(a) for $0 \leq \tau < \tau_1 = \sqrt{2}$, $\phi(\tau) = 1$ and $\psi(\tau) = 0$,

(b) for $\tau_1 \leq \tau \leq T$, $\phi(\tau) = 0$ and $\psi(\tau) = 0$.

When we employ $\phi = 1$ and $\psi = 0$ for $0 \leq \tau \leq T$, we have that $T = \frac{y_3^\circ}{y_4^\circ}$. Using the facts that $\tau_1 > T$ and $y_2(T) > 0$, we find that $y_3^\circ < \sqrt{2}\, y_4^\circ$ and $2\, y_2^\circ y_4^\circ > (y_3^\circ)^2$. The case with the transition surface need not be worked out, since B is "blockable" due to $y_3^\circ \geq \sqrt{2}\, y_4^\circ$. It is noted that terminal states C and D are symmetric with A and B.

f. Structure of Optimal Allocation Policies.

Three characteristics of the solution to the supporting weapon system game are that the optimal strategies are:

(1) either 0 or 1,

(2) constant over time (no transition surfaces),

(3) dependent on initial strengths.

The first characteristic is a consequence of square-law attrition, which makes the existence of a singular control [53] impossible and hence strategies are extreme points in the control variable space. Singular control is, however, possible when there is linear law attrition for the target types over which fire is distributed.

It is conjectured that the absence of transition surfaces in the solution is the consequence of two factors: (a) the problem is a terminal control one and (b) only one target type is in the payoff. In a similar one-sided Problem [52], [74], such a switch in tactics only occurs in a losing cause when both target types are weighted in a terminal payoff. If we were to consider a prescribed duration battle, then it may be shown that transition surfaces may occur for both sides (compare with Isaacs' [50] War of Attrition and Attack). Inclusion of only infantry in the payoff has the effect, in this case, of causing air to always be direct at infantry during the last stages of battle. It is conjectured that there can exist transition surfaces in the solution when all target types are weighted in the payoff. When this is done, however, it may be shown that Weiss's change of variables is inappropriate (payoff must also be transformed), and the original formulation of the state equations with kill rate coefficients must be used.

Finally, it may also be shown that for the prescribed duration battle target selection depends only on the attrition rates of the various force types and relative weights assigned to surviving force types. This should be contrasted with the terminal control case where, as we have just seen, tactics depend on force levels. Thus, we see that tactics depend on the circumstances under which the conflict ends, and Weiss has written a fundamental paper [83] on this topic.

g. Extensions of Model.

It seems appropriate to discuss two extensions of Weiss' original model: one extends the type of payoff and the other modifies the information set available to the players. This second extension is believed to be more descriptive of the deployment of a supporting weapon system against ground forces. Complete solutions haven't yet been developed for either of these. Analytic details of parts of the solution to the first are presented in a section below.

The first extension is the following:

payoff to ODD: $px_1(T) + qx_3(T) - rx_2(T) - sx_4(T)$ with T unspecified

subject to:

$$\dot{x}_1 = - a_1 x_4$$
$$\dot{x}_2 = - b_1 x_3$$
$$\dot{x}_3 = -(1 - \psi) a_2 x_4$$
$$\dot{x}_4 = -(1 - \phi) b_2 x_3$$

with appropriate initial conditions and terminal states as defined before. The reason for the re-introduction of the kill rate coefficients is significant and is discussed in the next section.

It is conjectured that the optimal strategies for this problem may vary with time. The form of the payoff function has modified the marginal advantage of target engagement. This has been caused by the new terms in the payoff. Although the detailed solution has not yet been worked out, extremals so have time varying strategies. By our previous experience with the supporting weapon system game, we see, however, that this is not conclusive proof that the optimal strategies vary with time. One additional factor that we have at our disposal to induce the presence of a switching surface is the value attached to surviving forces. From our earlier experience with the fire programming problem, we would expect the shift in target engagement to apply for the loser (unlike the previous game) of the battle. He would, for example, allocate his air to the force type against which he had the greatest net effect in the early stages of battle and engage the force type for which the payoff (including kill rate) is greatest during the last stage of his losing effort.

The Hamiltonian for this first reformulation is

$$H(t,x,p;\phi,\psi) = \psi x_4(a_2p_3-a_1p_1) + \phi x_3(b_2p_4-b_1p_2) - a_2p_3x_4 - b_2p_4x_3$$

If we were to consider a battle of prescribed duration T, then we would have

$$p_1(t = T) = p$$
$$p_2(t = T) = -r$$
$$p_3(t = T) = q$$
$$p_4(t + T) = -s$$

Optimal strategies (there is only one extremal) are determined from

$$\min_{\psi}[\psi x_4(a_2p_3-a_1p)] + \max_{\phi}[\phi x_3(b_2p_4+b_1r)] - a_2p_3x - b_2p_4x_3$$

Hence

$$\phi = \{\mathrm{sgn}[b_2p_4 + b_1r] + 1\}/2$$

$$\psi = \{\mathrm{sgn}[a_1p - a_2p_3] + 1\}/2$$

where

$$\mathrm{sgn}\, x = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

It may be shown that $\phi(t)$ can only change from 0 to 1 if it does, indeed, change during the course of battle and similarly for $\psi(t)$. Thus an artillery system would never switch from fire support to counter-battery fire in a battle described by this model.

The second extension would replace the state equations by:

$$\dot{x}_1 = -\psi a_1 x_1 x_4$$
$$\dot{x}_2 = -\phi b_1 x_2 x_3$$
$$\dot{x}_3 = -(1 - \psi) a_2 x_4$$
$$\dot{x}_4 = -(1 - \phi) b_2 x_3$$

For this model the Hamiltonian is

$$H(t,x,p;\phi,\psi) = \psi x_4(a_2 p_3 - a_1 x_1 p_1) + \phi x_3(b_2 p_4 - b_1 x_2 p_2) - a_2 p_3 x_4 - b_2 p_4 x_3,$$

and the adjoint equations are:

$$\dot{p}_1 = \psi a_1 x_4 p_1$$
$$\dot{p}_2 = \phi b_1 x_3 p_2$$
$$\dot{p}_3 = \phi b_1 x_2 p_2 + (1 - \phi) b_2 p_4$$
$$\dot{p}_4 = \psi a_1 x_1 p_1 + (1 - \psi) a_2 p_3$$

Since the adjoint equations now depend on the state variables, the resulting two-point boundary value problem does not possess a solution readily obtainable by elementary methods.

The above is believed to be a more realistic model of the deployment of a supporting weapon system against ground forces, since individual soldiers are not engaged as point targets in such combat situations. Weiss [82] has also shown that such a model applies to cases of partial information in the following sense: each supporting unit is informed about the general areas in which opposing infantry are located but is not informed about the consequences of its own fire. This version still maintains the complete information assumption for the supporting weapon systems. It seems more realistic that intelligence efforts would be more intense on a supporting weapon system of large kill potential and that intelligence for ground forces would be primarily concerned with location of troop units (aggregates of troops in specific areas) rather than individual soldiers.

We have also considered other extensions and have done further analytic work on solutions than is presented here, but we do not present this at the present.

h. A Pitfall of Model Formulation.

Weiss [82] transformed his state equations of combat by introducing new variables which "absorbed" the kill rate coefficients. A pitfall of this procedure will now be discussed. It is easy to show that if the state variables are transformed, the payoff must also be appropriately transformed when a tradeoff exists between target types (all target types are present in payoff). This point was not important for the original Weiss formulation, since only one target per side appeared in the payoff. Failure to note this point may lead to failure to identify all significant solution properties for optimal allocation. For example, in the fire programming problem for forces of equal value (payoff: $x_3(T) - x_1(T) - x_2(T)$) if the state equations were to be transformed to:

$$\dot{y}_1 = \psi y_3$$
$$\dot{y}_2 = -(1 - \psi) y_3$$
$$\dot{y}_3 = -y_1 - c y_2,$$

while the original payoffs were retained, then it may be shown that there is no transition surface in the solution under any circumstances.

It is conjectured that in the original version of the supporting weapon system game this aspect of model formulation would have also prevented the existence of time-varying optimal strategies under any circumstances.

i. Battles of Prescribed Duration and Fights to the Finish.

In this section we discuss some differences between the prescribed duration battle and the terminal control battle (a special case of which is the "fight to the finish"). We begin by contrasting various aspects qualitatively and then present some solution details for one of the model extensions mentioned earlier. We do so for both the prescribed duration battle and the fight to the finish.

General Discussion

Of prime interest to the operations research worker who seeks an understanding of complex phenomena, is the extent to which his choice of model influences this perspective. We shall see that what determines the end of a battle is very important to the combatants for their selection of optimal tactics. We shall contrast the battle for a prescribed duration to the battle to a specified terminal state (in particular, the "fight to the finish").

In all cases, target selection depends on the marginal return for engagement. For the supporting weapon system game, marginal return is the rate of change of the value of the game (in terms of forces remaining) per unit of force allocated. It is measured by the product of the rate of change of this value per unit of force type (dual variable) and of the kill rate of this force type by the supporting weapon system. Air or infantry is engaged depending on the difference of such quantities. Similar remarks apply to the fire programming problem. This richness of interpretation of the dual variables is not present in the analysis of multimove discrete games [14], [15], [34]. A very significant point is that the type of model chosen (form of payoff function and planning horizon) may lead to a different evolution of marginal return. This is clear if one only considers the values of the dual variables on the terminal surface. In the terminal control case, such a value of one of the dual variables depends on initial strengths and the history of the battle through the transversality condition $H(t = T, y, p; \phi, \psi) = 0$, whereas for the battle of prescribed duration such values are independent of initial strengths.

In fights to the finish (extension one of section g), a commander must estimate the most vulnerable part of the enemy force (both kill rate and force level) and then concentrate the entire fire of the supporting weapon system on this. The winner continues with his chosen strategy until the desired end is achieved. The loser may shift fire to minimize his losses depending upon the weights he attaches to remaining units of the winner's force types and his effectiveness against each. For the battle of prescribed duration, on the other hand, target selection is independent of initial strengths or tide of the battle. If the battle lasts long enough, the optimal tactic may be to shift fire regardless of whether one is winning or losing.

The fight to the finish is thus strongly dependent upon what are the conditions under which a battle is ended, "the terminal states of combat." It appears that there is more research to be done in this important area, especially in view of the strong dependence of tactics on it as pointed out in this paper. The excellent paper of Weiss' [83] on Richardson's data should be noted. The current development may be readily modified to termination at specified non-zero force levels. There are no mathematical complications from this change.

Thus we conclude that a realistic model for optimal allocation must also consider the conditions under which the battle terminates. We could allow for replacements in such models. In such cases it might be appropriate to consider total losses as defining an additional terminal state. It may be necessary to consider different terminal states for each combatant (not symmetric). For example, we could construct a dynamic allocation model of guerrila warfare in which we might consider the terminal state for the insurgents as reduction to a specified level (possibly zero), while for the counter-insurgents (both sides being allowed replacements) the end of the battle might be determined by the length of the conflict (people get tired of war) and/or total losses.

Of interest to the military tactician is whether target selection rules evolve dynamically with the course of battle. Mathematically, this may be stated as whether there is a transition surface in the solution. For the terminal control problems studied here, such a shift has been conjectured to be present only in a losing cause. For battles of fixed duration, the solution behavior is signigicantly different with the possibility of transition surfaces being present for both sides.

<u>Development of Solution to Prescribed Duration Battle</u>

We consider the following problem (which has been formulated from ODD's standpoint)

$$\max_{\phi} \min_{\psi} \{px_1(T) + qx_3(T) - rx_2(T) - sx_4(T)\} \quad \text{with} \quad T \text{ specified,}$$

subject to: $$\dot{x}_1 = -\psi a_1 x_4,$$
$$\dot{x}_2 = -\phi b_1 x_3,$$
$$\dot{x}_3 = -(1 - \psi) a_2 x_4,$$
$$\dot{x}_4 = -(1 - \phi) b_2 x_3, \quad \text{(B7)}$$

with initial conditions

$$x_1(t = 0) = x_1^\circ, x_2(t = 0) = x_2^\circ, x_3(t = 0) = x_3^\circ, x_4(t = 0) = x_4^\circ.$$

In the subsequent development we assume that all initial strengths are such that a state variable is never reduced to zero so that a "subgame" is entered.

The Hamiltonian, $H(t,x,p;\phi,\psi)$, is given by

$$H(t,x,p;\phi,\psi) = \phi x_3(b_2p_4 - b_1p_2) + \psi x_4(a_2p_3 - a_1p_1) - a_2p_3x_4 - b_2p_4x_3.$$

The adjoint equations are thus given by

$$\dot{p}_1 = 0 \Rightarrow p_1(t) = \text{const} = p,$$
$$\dot{p}_2 = 0 \Rightarrow p_2(t) = \text{const} = -r,$$
$$\dot{p}_3 = -\frac{\partial H}{\partial x_3} = -\phi b_1 r + (1 - \phi) b_2 p_4,$$
$$\dot{p}_4 = -\frac{\partial H}{\partial x_4} = \psi a_1 p + (1 - \psi) a_2 p_3, \quad \text{(B8)}$$

with terminal conditions

$$p_1(t = T) = p, p_2(t = T) = -r, p_3(t = T) = q, p_4(t = T) = -s,$$

so that the Hamiltonian becomes

$$H(t,x,p;\phi,\psi) = \phi x_3(b_2p_4 + b_1r) + \psi x_4(a_2p_3 - a_1p) - a_2p_3x_4 - b_2p_4x_3, \quad \text{(B9)}$$

with the extremal strategies being determined by $\max_{\phi} \min_{\psi} H(t,x,p;\phi,\psi)$. Hence the optimal strategies (there is only one extremal) are given by

$$\phi(t) = \begin{cases} 0 & \text{for } b_2p_4 < -b_1r \\ 1 & \text{for } b_2p_4 > -b_1r, \end{cases}$$

and

$$\psi(t) = \begin{cases} 0 & \text{for } a_2p_3 > a_1p \\ 1 & \text{for } a_2p_3 < a_1p. \end{cases} \qquad \text{(B10)}$$

Let us note that at $t = T$, (B10) becomes

$$\phi(t = T) = \begin{cases} 0 & \text{for } b_1r < b_2s \\ 1 & \text{for } b_1r > b_2s, \end{cases}$$

and

$$\psi(t = T) = \begin{cases} 0 & \text{for } a_2q > a_1p \\ 1 & \text{for } a_2q < a_1p, \end{cases} \qquad \text{(B11)}$$

which conditions the four cases we study below.

We let $\tau = T - t$ in order that we may integrate the adjoint equations backwards from the end of the battle where the boundary condition is given for the dual variables. Then, we have for any τ-time interval over which strategies are constant

$$\frac{dp_3}{d\tau} = \phi b_1 r - (1 - \phi)b_2p_4 \qquad p_3(\tau = 0) = q,$$

$$\frac{dp_4}{d\tau} = -\psi a_1 p - (1 - \psi)a_2p_3 \qquad p_4(\tau = 0) = -s, \qquad \text{(B12)}$$

where $\phi(\tau)$ and $\psi(\tau)$ are given by (B10). From (B11) it is easily seen that there are four cases to consider.

Case I. $b_1r < b_2s$ and $a_2q > a_1p$

We see that $\phi(T) = \psi(T) = 0$, so that near the end of battle (B12) become

$$\frac{dp_3}{d\tau} = -b_2p_4 \qquad p_3(\tau = 0) = q,$$

$$\frac{dp_4}{d\tau} = -a_2p_3 \qquad p_4(\tau = 0) = -s,$$

whose solution is easily seen to be

$$p_3(\tau) = q \cosh\sqrt{a_2b_2}\ \tau + s\sqrt{b_2/a_2} \sinh\sqrt{a_2b_2}\ \tau,$$

$$p_4(\tau) = -s \cosh\sqrt{a_2b_2}\ \tau - q\sqrt{a_2/b_2} \sinh\sqrt{a_2b_2}\ \tau.$$

Noting that $p_3(\tau)a_2 \geq qa_2 > a_1p$ and $-p_4(\tau)b_2 \geq b_2s > b_1r$, we see from (B10) that $\phi(t) = \psi(t) = 0$ for all $t\varepsilon[0,T]$.

Case II. $b_1r > b_2s$ and $a_2q > a_1p$

We see that $\phi(T) = 1$ and $\psi(T) = 0$, so that for $0 \leq \tau \leq \tau_1$ where τ_1 is the time of the first switch (B12) becomes

$$\frac{dp_3}{d\tau} = b_1r \qquad p_3(\tau = 0) = q$$

$$\frac{dp_4}{d\tau} = -a_2p_3 \qquad p_4(\tau = 0) = -s,$$

whose solution is given by

$$p_3(\tau) = b_1r\tau + q,$$

$$p_4(\tau) = -\tau^2a_2b_1r/2 - a_2q\tau - s,$$

from which it is seen that ϕ is the variable which switches at τ_1 which is the solution to

$$-a_2b_1b_2r\tau_1^2/2 - a_2b_2q\tau_1 + (b_1r - b_2s) = 0 \qquad \text{(B13)}$$

It is easily shown that one $\phi(\tau)$ switches to 0 there are no further changes. Hence, we have shown that

for $0 \leq t \leq T - \tau_1$: $\phi(t) = 0$ and $\psi(t) = 0$,

for $T - \tau_1 \leq t \leq T$: $\phi(t) = 1$ and $\psi(t) = 0$,

where τ_1 is determined from (B13).

Case III is similar to Case II.

Case IV. $b_1r > b_2s$ amd $a_2q < a_1p$

We see that $\phi(T) = \psi(T) = 1$, so that for $0 \leq \tau \leq \tau_1$ where τ_1 is the time of the first switch (D12) becomes

$$\frac{dp_3}{d\tau} = b_1r \qquad p_3(\tau = 0) = q$$

$$\frac{dp_4}{d\tau} = -a_1p \qquad p_4(\tau = 0) = -s,$$

whose solution is given by

$$p_3(\tau) = b_1r\tau + q,$$

$$p_4(\tau) = -a_1p\tau - s,$$

whence we see that τ_1 is given by

$$\tau_1 = \min\{ \left(\frac{a_1p - a_2q}{a_2b_1r}\right) , \left(\frac{b_1r - b_2s}{a_1b_2p}\right) \}. \qquad \text{(B14)}$$

We could show that both strategy variables eventually change to 0 (if

T is large enough). For example, if ψ changes first at τ_1, then we may show that for $\tau_1 \le \tau \le \tau_2$

$$p_4(\tau) = -a_2 b_1 r\tau^2/2 - a_2 q\tau - s - (a_1 p - a_2 q)^2/(2a_2 b_1 r),$$

so that $p_4(\tau)$ continues to decrease and ϕ may also change to 0. In this example we have considered we would then have

$$\text{for } 0 \le t \le T - \tau_2 : \phi(t) = 0 \text{ and } \psi(t) = 0,$$
$$\text{for } T - \tau_2 \le t \le T - \tau_1 : \phi(t) = 1 \text{ and } \psi(t) = 0,$$
$$\text{for } T - \tau_1 \le t \le T : \phi(t) = 1 \text{ and } \psi(t) = 1.$$

What we do want to point out from the above development is that the optimum allocation of fire is independent of the force levels and depends only on the attrition rates (and length of battle). We also note that if $q = s = 0$ (only infantry weighted in the payoff), then Case IV above applies and the battle always terminates with the supporting weapon system fires concentrated on the ground forces possibly preceded by a period of counterbattery fire.

Partial Development of Solution to Terminal Control Battle

We consider the following problem (again the payoff is from ODD's standpoint)

$$\max_{\phi} \min_{\psi} \{px_1(T) + qx_3(T) - rx_2(T) - sx_4(T)\} \text{ with } T \text{ unspecified,}$$

subject to:

$$\dot{x}_1 = -\psi a_1 x_4,$$
$$\dot{x}_2 = -\phi b_1 x_3,$$
$$\dot{x}_3 = -(1 - \psi) a_2 x_4,$$
$$\dot{x}_4 = -(1 - \phi) b_2 x_3,$$

with initial conditions

$$x_1(t = 0) = x_1^\circ, x_2(t = 0) = x_2^\circ, x_3(t = 0) = x_3^\circ, x_4(t = 0) = x_4^\circ,$$

and terminal conditions similar to Weiss's original problem (see Figure BI).

We will outline enough (hopefully) of the solution process to show points of difference with the prescribed duration battle. Within the framework of our solution procedure for terminal control attrition games (see Section d above), we have done only the first step (identify terminal states and determine extremal paths).

As before, the Hamiltonian is given by

$$H(t,x,p;\phi,\psi) = \phi x_3(b_2p_4-b_1p_2) + \psi x_4(a_2p_3-a_1p_1) - a_2p_3x_4 - b_2p_4x_3, \qquad (B15)$$

so that the adjoint equations are given by

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = 0 \Rightarrow p_1(t) = \text{const},$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = 0 \Rightarrow p_2(t) = \text{const},$$

$$\dot{p}_3 = -\frac{\partial H}{\partial x_3} = \phi b_1 p_2 + (1 - \phi) b_2 p_4,$$

$$\dot{p}_4 = -\frac{\partial H}{\partial x_4} = \psi a_1 p_1 + (1 - \psi) a_2 p_3. \qquad (B16)$$

From this point on the development is different for each terminal state. We illustrate by considering the case when EVEN wins by destroying ODD's infantry, i.e., $\underline{x_1(T) = 0}$. The boundary conditions at the termination of the battle in this case are

$$p_1(t = T) = \text{unspecified} \quad , \qquad x_1(t = T) = 0,$$

$$p_2(t = T) = -r,$$

$$p_3(= T) = q,$$

$$p_4(t = T) = -s.$$

Extremal strategies are determined by $\max\limits_{\phi} \min\limits_{\psi} H(t,x,p;\phi,\psi)$, which is equivalent to

$$\max_{\phi}\{\phi(b_2p_4 + b_1r)\},$$

and

$$\min_{\psi}\{\psi(a_2p_3 - a_1p_1)\},$$

and, hence, extremal strategies are given by

$$\phi(t) = \begin{cases} 0 & \text{for } b_2p_4 < -b_1r \\ 1 & \text{for } b_2p_4 > -b_1r, \end{cases}$$

and

$$\psi(t) = \begin{cases} 0 & \text{for } a_2p_3 > a_1p_1(T) \\ 1 & \text{for } a_2p_3 < a_1p_1(T). \end{cases} \tag{B17}$$

At $t = T$, we have

$$\phi(t = T) = \begin{cases} 0 & \text{for } b_1r < b_2s \\ 1 & \text{for } b_1r > b_2s, \end{cases}$$

and

$$\psi(t = T) = \begin{cases} 0 & \text{for } a_2q > a_1p_1(T) \\ 1 & \text{for } a_2q < a_1p_1(T), \end{cases} \tag{B18}$$

which gives us various cases to consider.

Since the termination time is unspecified, the following transversality condition must be satisfied at the end of battle

$$H(t=T,x,p;\phi,\psi) = 0. \qquad \text{(B19)}$$

We shall see that this condition has the effect of eliminating $\psi(t) = 0$ as an optimal strategy for EVEN during the closing stages of battle.

We consider two cases of terminating conditions effecting EVEN's strategy variable ψ.

Case A. $a_2q > a_1p_1(T)$ implying $\psi(t = T) = 0$

We show that this case is impossible and drop it from further consideration. We have the following two cases to consider

(a) $b_1r < b_2s$

By (B18), we have $\phi(T) = 0$ so that (B15) and (B19) require that

$$-a_2qx_{4s} + b_2sx_{3s} = 0,$$

where $x_{is} = x_i(t = T)$ as used by Weiss. Since the above will, in general, not be satisfied, this case is impossible.

(b) $b_1r > b_2s$

By (B18), we have $\phi(T) = 1$ so that (B15) and (B19) require that

$$-a_2qx_{4s} + b_1rx_{3s} = 0,$$

which likewise makes this case impossible.

Case B. $a_2q < a_1p_1(T)$ implying $\psi(t = T) = 1$

Again, we have two subcases to consider

(a) $b_1r < b_2s$

By (B18, we have $\phi(T) = 0$ so that (B15) and (B19) require that

$$p_1(T) = (b_2 s x_{3s})/(a_1 x_{4s}), \tag{B20}$$

so that Case B is given by

$$a_2 q x_{4s} < b_2 s x_{3s} \tag{B21}$$

(b) $b_1 r > b_2 s$

By (B18), we have $\phi(T) = 1$ so that (B15) and (B19) require that

$$p_1(T) = (b_1 r x_{3s})/(a_1 x_{4s}), \tag{B22}$$

so that Case B is given by

$$a_2 q x_{4s} < b_1 r x_{3s}. \tag{B23}$$

We will now investigate the above two subcases of Case B more fully. Before we do this, let us rewrite the last two adjoint equations (B16) in terms of the "backwards time" $\tau = T - t$

$$\begin{aligned} \frac{dp_3}{d\tau} &= \phi b_1 r - (1 - \phi) b_2 p_4 & p_3(\tau = 0) &= q, \\ \frac{dp_4}{d\tau} &= -\psi a_1 p_1(T) - (1 - \psi) a_2 p_3 & p_4(\tau = 0) &= -s \end{aligned} \tag{B24}$$

As we have shown above, the terminal state $x_1(T) = 0$ can only be reached ween $a_2 q < a_1 p_1(T)$ so that we have $\psi(t = T) = 1$. We continue with the two subcases above.

(a) $b_1 r < b_2 s$ and $p_1(T) = (b_2 s x_{3s})/(a_1 x_{4s})$ so that $a_2 q x_{4s} < b_2 s x_{3s}$

By (B18), we have $\phi(T) = 0$ so that near the end of battle by (B24) we have

$$\frac{dp_4}{d\tau} = -a_1 p_1(T)$$

and $p_4(\tau) = -a_1 p_1(T)\tau - s < 0$ for all τ.

Hence $\phi(t) = 0$ for $0 \le t \le T$. We may show that $\psi(\tau)$ can switch to 0 at τ_1, so we would have

$$\text{for } 0 \le t \le T - \tau_1 : \phi(t) = 0 \text{ and } \psi(t) = 0,$$

$$\text{for } T - \tau_1 \le t \le T : \phi(t) = 0 \text{ and } \psi(t) = 1.$$

Determination of the domain of controllability is quite messy in this case and we omit it at this time.

(b) $b_1 r > b_2 s$ and $p_1(T) = (b_1 r x_{3s})/(a_1 x_{4s})$ so that $a_2 q x_{4s} < b_1 r x_{3s}$

By (B18), we have $\phi(T) = 1$ so that near the end of battle we have

$$p_4(\tau) = -a_1 p_1(T)\tau - s$$

or

$$p_4(\tau) = -b_1 r x_{3s}\tau/x_{4s} - s$$

$\phi(\tau)$ switches to 0 at τ_1 given by

$$\tau_1 = \frac{(b_1 r - b_2 s)}{b_1 b_2 r}\left(\frac{x_{4s}}{x_{3s}}\right),$$

and to summarize

$$\text{for } 0 \le \tau < \tau_1 : \phi(\tau) = 1$$

$$\text{for } \tau_1 < \tau : \phi(\tau) = 0.$$

Other details are similar to previous case.

j. <u>Implications of Models</u>.

It seems appropriate to discuss briefly the general implications in the following areas:

(1) intelligence,

(2) command and control systems,

(3) human decision making.

Even though the present models assume complete and instantaneous information, their solution does possess certain features capable of being projected to cases where uncertainty is present. The selection of tactics is seen to depend on a knowledge of the enemy's strength and capabilities so that the appropriate target set may be chosen and optimal strategies determined. Previous models [14], [15], [34] (battles of prescribed duration) had not indicated such a conclusion but that tactics depended only on enemy and friendly capabilities and length of combat, not the initial force levels. For such models the estimate of the combat length is critical, since if one were to extend this time, the optimal strategies may have to be determined again from the beginning.

The shifting of tactics with time (instantaneously in the model) indicates requirements for a responsive command structure. For the case studied here, the loser of a battle may receive more benefits from a command structure capable of implementing a change of tactics during the confusion of combat.

Schreiber [70] has proposed "overkill" as a measure of "command efficiency." His idea is to modify the description of combat to reflect differences in command and control capabilities. One uses a linear law (see Section g) when fire is not redirected from killed targets. However, we don't see the full implication of such diminishing returns in combat here. In Appendix C we shall see that when there is a linear law attrition process for the target types over which fire is distributed, the nature of the allocation policy is fundamentally different.

These models may be interpreted to show the value of human judgment in combat. They indicate, as does common sense and experience, that in battle a commander must use his judgment to ascertain to what end can the course of battle be steered so that he may devise his strategy accordingly. The demonstrated sensitivity of these models to many factors shows the importance of human assessment of a situation and value attached to forces remaining after the battle at hand.

A further discussion is to be found in Appendix C.

APPENDIX C. Some One-Sided Dynamic Allocation Problems.

In this appendix we examine a sequence of problems to study the dependence of optimal allocation policies on model form. The problems are for combat over a period of time described by Lanchester-type equations with a choice of tactics available to one side and subject to change with time. We consider two types of choice problems: (1) target-type selection and (2) firing rate.

In 1964 Dolansky [28] noted that the Lanchester theory of combat was insufficiently developed in the area of target selection for combat between heterogeneous forces (optimal control/differential games). This remark was based on consideration of work by Weiss [82] and Isbell and Marlow [52], both of which we have extended in previous appendices. Since that time no further examples have been published in the literature except for the ones in Isaacs' book [50]. This previous work had never systematically investigated the dependence of tactics on model form.

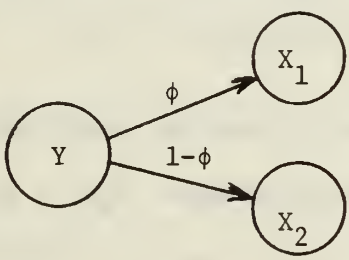
With the first sequence of models our goal is to obtain insight into optimal target selection rules in real combat by gaining a more thorough understanding of some simple models and the solution characteristics of such models. To understand the operations of a complex system, many times the researcher examines a sequence of models of greater and greater complexity to try to see if he can discern a "law of nature." In the first two models we shall see how the objectives of the combatants and the termination conditions of the conflict influence target selection through the evolution of marginal return.

Then we examine the effect of number of target types and type of attrition process.

We then examine a sequence of models to see how ammunition limitations effect firing rates. The results of this section are of a more preliminary nature. Then we discuss two-sided extensions of such problems but point out the value of studying one-sided problems as considered in this paper. Finally, various implications of the models studied are discussed.

a. Target Selection.

The simplest situation of target selection that we could conceive of is one of combat between an X-force of two force types (for example, riflemen and grenadiers) and a homogeneous Y-force (for example, riflemen only). This situation is shown diagrammatically below.



It is the objective of the Y-force commander to maximize his survivors at the end of battle at time T and minimize those of his opponent (considering weighting factors p, q and r). This is accomplished through his choice of the fraction of fire, ϕ, directed at X_1. There are several scenarios that we could apply to the above idealized combat situation: two of these are (1) a battle lasting a specified time, T or (2) a battle lasting until one side or the other was totally annihilated. We will now examine each of these.

1. Battle of Prescribed Duration, T.

Mathematically the problem may be stated as

$$\max_{\phi(t)} \ ry(T) - px_1(T) - qx_2(T) \quad \text{with } T \text{ specified}$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 y$$

$$\frac{dx_2}{dt} = -(1 - \phi)a_2 y$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2$$

$$x_1, x_2, y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1,$$

where

p, q and r are weighting factors assigned to surviving forces,

x_1, x_2 and y are average force strengths,

a_1, a_2, b_1 and b_2 are constant attrition rates, and

ϕ is fraction of Y-fire directed at X_1.

This problem may be solved by routine application of Pontryagin maximum principle [68]. The solution when $a_1b_1 > a_2b_2$ is shown in Table CI. The other case when $a_1b_1 < a_2b_2$ is symmetric to this one. This present analysis ignores those subcases when a state variable is reduced to zero.

The Hamiltonian for this problem is

$$H(t,x,p,\phi) = \phi y(-a_1p_1 + a_2p_2) + \{-a_2p_2y - p_3(b_1x_1 + b_2x_2)\}.$$

The extremal control is determined by $\max_{\phi(t)} H(t,x,p,\phi)$ and hence

$$\phi(t) = \begin{cases} 0 & \text{for } p_2a_2 < p_1a_1 \\ 1 & \text{for } p_2a_2 > p_1a_1 . \end{cases}$$

Table CI. Solution to Target Selection Problem Battle of Prescribed Duration

Basic assumption: $a_1 b_1 > a_2 b_2$

	Optimal Control
Case A: $a_1 p > a_2 q$	$\phi(t) = 1$ for $0 \leq t \leq T$
Case B: $a_1 p < a_2 q$	(a) for $\tau_1 > T$
	$\phi(t) = 0$ for $0 \leq t \leq T$
	(b) for $\tau_1 < T$
	$\phi(t) = 1$ for $0 \leq t \leq T - \tau_1$
	$\phi(t) = 0$ for $T - \tau_1 \leq t \leq T$

Note: τ_1 is determined from the transcendental equation $v(\tau = \tau_1) = 0$, or

$$r\sqrt{\frac{b_2}{a_2}} \sinh\sqrt{a_2 b_2}\, \tau_1 + q \cosh\sqrt{a_2 b_2}\, \tau_1 = \frac{a_1(b_1 q - b_2 p)}{(a_1 b_1 - a_2 b_2)}$$

The adjoint differential equations (note that these are independent of the state variables) are given by

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial x_1} = b_1 p_3 \qquad \text{with} \quad p_1(t = T) = -p,$$

$$\frac{dp_2}{dt} = -\frac{\partial H}{\partial x_2} = b_2 p_3 \qquad \text{with} \quad p_2(t = T) = -q,$$

$$\frac{dp_3}{dt} = -\frac{\partial H}{\partial y} = \phi a_1 p_1 + (1 - \phi) a_2 p_2 \quad \text{with} \quad p_3(t = T) = r.$$

It is convenient to define $v(t) = a_1 p_1(t) - a_2 p_2(t)$. The condition which determines the extremal control is then

$$\phi(t) = \begin{cases} 0 & \text{for} \quad v(t) > 0, \\ 1 & \text{for} \quad v(t) < 0. \end{cases}$$

Introducing the reverse time variable $\tau = T - t$, we consider the following equivalent system of differential equations:

$$\frac{dp_2}{d\tau} = -b_2 p_3 \qquad \text{with} \quad p_2(\tau = 0) = q,$$

$$\frac{dp_3}{d\tau} = -\phi v - a_2 p_2 \qquad \text{with} \quad p_3(\tau = 0) = r,$$

$$\frac{dv}{d\tau} = -(a_1 b_1 - a_2 b_2) p_3 \quad \text{with} \quad v(\tau = 0) = -a_1 p + a_2 q.$$

These equations may be solved to show that up until the first switch in tactics

$$p_3(\tau) = r \cosh\sqrt{\phi a_1 b_1 + (1-\phi) a_2 b_2}\, \tau + \frac{a_1 p + (1-\phi) a_2 q}{\sqrt{\phi a_1 b_1 + (1-\phi) a_2 b_2}} \sinh\sqrt{\phi a_1 b_1 + (1-\phi) a_2 b_2}\, \tau \, .$$

It is easy to show that $p_1(\tau)$, $p_2(\tau) < 0$ and $p_3(\tau) > 0$ for all $\tau \geq 0$.

We see that consideration of the case $a_1b_1 > a_2b_2$ is motivated by the coefficient of $p_3(\tau)$ in the differential equation for $v(\tau)$. There are two further cases to consider.

Case (a) $\quad a_1p > a_2q$

We have that $\phi(\tau = 0) = 1$, since $v(\tau = 0) < 0$. Now since $p_3(\tau) > 0$, we always have $\frac{dv}{d\tau} < 0$ and $v(\tau)$ never can change sign. Thus, we never switch. Hence, for $0 \leq t \leq T$, we have $\phi(t) = 1$.

Case (b) $\quad a_1p < a_2q$

We have that $\phi(\tau = 0) = 0$, since $v(\tau = 0) > 0$. Since $p_3(\tau) > 0$, we always have $\frac{dv}{d\tau} < 0$, and we can have a switch in tactics.

The backward time of this switch in tactics, $\tau = \tau_1$, is determined from the integration of

$$\frac{dv}{d\tau} = -(a_1b_1 - a_2b_2)p_3 \quad \text{for} \quad 0 \leq \tau \leq \tau_1,$$

where it is recalled that $\phi(\tau) = 0$ in this interval. It is easily shown that

$$v(\tau) = -(a_1b_1 - a_2b_2)\left\{\frac{r}{\sqrt{a_2b_2}} \sinh\sqrt{a_2b_2}\,\tau + \frac{q}{b_2}\cosh\sqrt{a_2b_2}\,\tau\right\} - a_1p + \frac{a_1b_1q}{b_2} .$$

Thus, we determine τ_1 from the transcendental equation $v(\tau = \tau_1) = 0$, and the result shown in Table CI is obtained.

It is seen that for the battle of prescribed duration target selection depends only on the attrition rates of the various force types and relative weights assigned to surviving force types. For this model, target selection is independent of force levels. This is not surprising, since the adjoint differential equations are independent of the state variables and the values of the dual variables at the end of battle $t = T$ are independent of force strengths. It is recalled that a dual variable represents the rate of change of the payoff with respect to a particular state variable [12]. Thus, if $V = ry(T) - px_1(T) - qx_2(T)$, then $p_1(T) = \frac{\partial V}{\partial x_1}(t)$, etc. Hence the boundary conditions are given for the dual variables at the end of the battle $t = T$ as $p_1(t = T) = \frac{\partial V}{\partial x_1}(t = T) = -p, p_2(t = T) = -q, p_3(t = T) = r$.

It seems appropriate to discuss further the interpretation of the solution shown in Table CI. From the above definition of the dual variables,

$$a_1 p_1(t) = \begin{pmatrix} \text{return per unit time} \\ \text{for engaging } X_1 \end{pmatrix} = \begin{pmatrix} \text{kill rate of } Y \\ \text{against } X_1 \end{pmatrix} \times \begin{pmatrix} \text{return per unit} \\ \text{of } X_1 \text{ destroyed} \end{pmatrix}.$$

Hence, the condition $a_1 p < a_2 q$ means that at the end of the battle (recall that $p_1(t = T) = -p$, etc.) there is greater payoff per unit time per soldier for Y to engage X_2 (short term gain at the end of battle). The value of the dual variable, for example, $p_1(T)$ also accounts for the effectiveness of X_1 against Y. The condition $a_1 b_1 > a_2 b_2$ may be interpreted to mean that there is more long range return for engaging X_1. Thus, case A of Table CI corresponds to where there is both more long range and also short range return for engaging X_1. Case B corresponds to more short term gain at the end of the battle for engaging X_2, but more long range return for engaging X_1. When remaining forces at $t = T$ are weighted proportional to their kill rates against Y, i.e., $p/q = b_1/b_2$, then case A is the only one possible. A switch in tactics (target priority) is seen to occur for this model when more utility is assigned to survivors of a target-type than in proportion to their destructive capability (kill rate) per unit relative to other target types.

The maximum principle may be interpreted as saying that a target type from several alternatives is engaged when such an engagement yields the greatest marginal return. It turns out, though, that the marginal value of target engagement evolves differently for different model forms. This is clearly seen when we examine the solution for a "fight to the finish."

2. Fight to the Finish.

We consider the similar problem of

$$\underset{\phi(t)}{\text{maximize}}\ ry(T) - px_1(T) - qx_2(T) \quad \text{with } T \text{ unspecified}$$

subject to:

$$\frac{dx_1}{dt} = -\phi a_1 y$$

$$\frac{dx_2}{dt} = -(1 - \phi)a_2 y$$

$$\frac{dy}{dt} = -b_1x_1 - b_2x_2$$

$$x_1, x_2, y \geq 0 \quad , \quad 0 \leq \phi \leq 1 \quad ,$$

and with terminal states defined by (1) $x_1(T) = x_2(T) = 0$ and (2) $y(T) = 0$.

The terminal surface of this problem is seen to consist of five parts:

$$C_1 : x_1(T) = 0, \quad x_2(T) > 0, \quad y(T) = 0,$$

$$C_2 : x_1(T) = 0 \text{ before } x_2(T) = 0, \quad y(T) > 0,$$

$$C_3 : x_1(T) = 0 \text{ after } x_2(T) = 0, \quad y(T) > 0,$$

$$C_4 : x_1(T) > 0, \quad x_2(T) = 0, \quad y(T) = 0,$$

$$C_5 : x_1(T) > 0, \quad x_2(T) > 0, \quad y(T) = 0.$$

The above problem was first studied by Isbell and Marlow [52], and we develop its solution in detail in Appendix A. The solution to this problem when $a_1b_1 > a_2b_2$ is shown in Table AI.

In contrast to the battle of prescribed duration, it is seen that optimal target engagement may depend on initial force levels. When Y wins, he engages X_1 until depletion before X_2. When Y loses, he may switch from firing at X_1 entirely to firing at X_2 entirely before the X_1 force has been annihilated. This happens when survivors of force-type X_2 are assigned utility in excess of their kill rate as compared with force-type X_1, and certain relationships hold between initial force strengths. This dependence of the optimal allocation on initial strengths has been caused by the fact that values of dual variables at $t = T$ are dependent upon values of the state variables. This happens in terminal control attrition problems where a value of a state variable is specified at the terminal surface (and hence the value of the corresponding dual variable is unspecified but may be determined from the transversality condition $H(t = T, x, p, \phi) = 0$).

3. <u>Generalizations to More Target Types</u>.

It is of interest to inquire as to what solution properties generalize to more than two heterogenous force types. For combat described by a generalized Lanchester square law, it turns out that the "bang-bang" allocation, optimal control is an extreme point in the control variable space, will always be true.

Let us consider the following prescribed duration battle model:

$$\underset{\phi_i(t)}{\text{maximize}} \qquad vy(T) - \sum_{i=1}^{n} w_i x_i(T) \quad \text{with } T \text{ specified}$$

$$\text{subject to: } \frac{dx_i}{dt} = -\phi_i a_i y \quad \text{for } i = 1,\ldots,n$$

$$\frac{dy}{dt} = -\sum_{i=1}^{n} b_i x_i$$

$$x_i, y \geq 0 \quad , \quad \phi_i \geq 0 \quad , \quad \text{and} \quad \sum_{i=1}^{n} \phi_i = 1$$

The Hamiltonian, $H(t,x,p,\phi)$, is given by

$$H = -\sum_{i=1}^{n} \phi_i p_i a_i y - p_{n+1} \sum_{i=1}^{n} b_i x_i ,$$

where p_i is the dual variable for the i^{th} state equation. By application of the maximum principle, we are led to

$$\underset{\phi_i}{\text{minimize}} \; \{ \sum_{i=1}^{n} \phi_i p_i a_i \}$$

$$\text{subject to: } \sum_{i=1}^{n} \phi_i = 1, \quad \phi_i \geq 0.$$

Let j be the index such that $a_j p_j$ = minimum $(a_1 p_1, \ldots, a_n p_n)$. Then $\phi_i = \delta_{ij}$, where δ_{ij} is the Kroncecker delta and is equal to 1 for $i = j$ and is equal to 0 otherwise, and all fire is concentrated on one target type.

It is of interest to ask whether the optimal tactic will always be to concentrate fire on only one target type (bang-bang optimal control). The answer to this question turns out to be "no" as the following simple example shows.

4. Linear Law Allocation.

So far the state equations have described combat according to the Lanchester square law in which attrition of a target type is proportional to the number of each force type firing at it. Weiss [81] has given a thorough discussion of the conditions which lead to this. These conditions include that "each unit is informed about the location of the remaining opposing units so that when a target is destroyed, fire may be immediately shifted to a new target." It is noted that the control theory models which we have considered so far have implicitly assumed perfect information.

Another model for attrition is the Lanchester linear law in which the average decrease of a target type is proportional to the product of the average number of targets remaining and the number of each force type firing at it. Such a dependence can arise under two general circumstances: (1) fire is uniformly distributed over a constant target area ("area fire") or (2) the mean time of target acquisition is much larger than target destruction time and is inversely proportional to target density. The first circumstance corresponds to the simplest case of partial information. Again quoting Weiss [81], we assume that units are informed about the general areas in which opposing units are located, but are not informed about the consequences of their own fire. Thus, we see that we may account for some changes in the information set by modifying the description of combat. Brackney [22] has shown that "aimed fire" may lead to a linear law when target acquisition times are considered.

Thus, we consider the following problem in which the X-forces' attrition obeys a linear law and the Y-forces' attrition obeys a square law:

$$\underset{\phi(t)}{\text{minimize}}\ ry(T) - px_1(T) - qx_2(T) \quad \text{with } T \text{ specified}$$

subject to:

$$\frac{dx_1}{dt} = -\phi a_1 x_1 y$$

$$\frac{dx_2}{dt} = -(1 - \phi) a_2 x_2 y$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2$$

$x_1, x_2, y \geq 0$ and $0 \leq \phi \leq 1$.

All analytical details of the solution to the above problem have not been worked out, since the state and adjoint equations do not readily yield an analytic solution. However, it is possible to discuss qualitatively the nature of the optimal control, even though certain quantities have not been explicitly evaluated.

There is a major difference in the solution to this problem from the previous ones. This difference is that the optimal allocation, ϕ, may be other than 0 or 1. The Hamiltonian for this problem is given by

$$H(t,x,p,\phi) = (-p_1a_1x_1y + p_2a_2x_2y)\phi + \{-p_2a_2x_2y - p_3(b_1x_1 + b_2x_2)\} \ , \qquad \text{(C1)}$$

and hence under "normal" circumstances the control is determined by

$$\phi(t) = \begin{cases} 0 & \text{for} \quad p_2a_2x_2 < p_1a_1x_1 \\ 1 & \text{for} \quad p_2a_2x_2 > p_1a_1x_1 \end{cases} \qquad \text{(C2)}$$

The adjoint equations are given by

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = -\{-p_1a_1y\phi - p_3b_1\}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = -\{-p_2a_2y(1 - \phi) - p_3b_2\}$$

$$\dot{p}_3 = -\frac{\partial H}{\partial x_3} = -\{-p_1\phi a_1x_1 - p_2(1 - \phi)a_2x_2\}$$

or

$$\frac{dp_1}{dt} = p_1\phi a_1y + p_3b_1 \qquad p_1(t = T) = -p,$$

$$\frac{dp_2}{dt} = p_2(1 - \phi)a_2y + p_3b_2 \qquad p_2(t = T) = -q,$$

$$\frac{dp_3}{dt} = p_1\phi a_1x_1 + p_2(1 - \phi)a_2x_2 \qquad p_3(t = T) = r, \qquad \text{(C3)}$$

In contrast with the previous problem, it is now possible to have other than a bang-bang optimal control. We may have a singular solution [53] for which the necessary condition that the maximization of the Hamiltonian (with respect to the control variable) does not provide us with a well-defined expression for the extremal control. This occurs when the coefficient of ϕ in the Hamiltonian vanishes for a finite interval of time.

A singular extremal is determined from the conditions [54]

$$\frac{\partial H}{\partial \phi} = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = 0$$

Hence, the following conditions must hold on a singular surface:

$$p_1 a_1 x_1 = p_2 a_2 x_2 \quad \text{and} \quad a_1 b_1 x_1 = a_2 b_2 x_2. \tag{C4}$$

On the singular surface, the extremal control is given by

$$\phi = \frac{a_2}{a_1 + a_2} \tag{C5}$$

It may also be shown that such a singular control is impossible for problems a1 and a2. Thus, singular control (non-concentration of fire on only one target type) is impossible for Lanchester square law attrition but does play a central role in allocation when attrition follows a linear law.

We must test to see if this singular solution can yield the optimal return. A necessary condition for a singular subarc to yield the maximum return [57] is

$$\frac{\partial}{\partial \phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right)\right\} \geq 0.$$

A rather laborious computation shows that

$$\frac{\partial}{\partial \phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right)\right\} = y^2 p_3(t)\{(a_1)^2 b_1 x_1 + (a_2)^2 b_2 x_2\},$$

and hence for $p_3(t) > 0$, we have that $\frac{\partial}{\partial \phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right)\right\} > 0$. Thus, since it may be shown that $p_3(t) > 0$ always, the necessary condition is met for the singular path to be optimal.

In constructing the extremal trajectories and tracing the optimal course of battle (backwards from the end of the prescribed duration battle) it is convenient to introduce

$$v(t) = -a_1 p_1 x_1 + a_2 p_2 x_2, \tag{C6}$$

then

$$\frac{dv}{dt} = -a_1 \frac{dp_1}{dt} x_1 - a_1 p_1 \frac{dx_1}{dt} + a_2 \frac{dp_2}{dt} x_2 + a_2 p_2 \frac{dx_2}{dt}.$$

Using the state equations and the adjoint equations (C3), we obtain from the above

$$\frac{dv}{dt} = -(a_2 b_2 x_2 - a_1 b_1 x_1) p_3,$$

or, in terms of the backwards time $\tau = T - t$, this becomes

$$\frac{dv}{d\tau} = (a_2 b_2 x_2 - a_1 b_1 x_1) p_3 \tag{C7}$$

We may write (C6) as

$$v(\tau) = -\left(\frac{p_2(\tau)}{b_2}\right)\left[\frac{\left(\frac{p_1(\tau)}{p_2(\tau)}\right)}{\left(\frac{b_1}{b_2}\right)} a_1 b_1 x_1 - a_2 b_2 x_2\right]. \tag{C8}$$

We note that (C2) and (C6) may be combined to yield the non-singular control

$$\phi(t) = \begin{cases} 1 & \text{for } v(t) > 0 \\ 0 & \text{for } v(t) < 0, \end{cases} \tag{C9}$$

and the singular control is

$$\phi(t) = \frac{a_2}{a_1 + a_2} \quad \text{for } v(t) = 0, \tag{C10}$$

when the system is in the state described by (C4).

We note that at the end of battle $\tau = 0$, we have

$$v(\tau = 0) = -a_1 p x_1 (t = T) + a_2 q x_2 (t = T). \qquad \text{(C11)}$$

If we were to consider in Figure C1 the line L' defined by $a_1 p x_1 = a_2 q x_2$, then it would appear above, on, or below the line L defined by $a_1 b_1 x_1 = a_2 b_2 x_2$ depending on whether $\frac{p}{q}$ were greater than, equal to, or less than $\frac{b_1}{b_2}$. This is evident from considering the slopes of these two lines

$$\left(\frac{dx_2}{dx_1}\right)_L = \frac{a_1 b_1}{a_2 b_2}, \qquad \left(\frac{dx_2}{dx_1}\right)_{L'} = \frac{a_1 p}{a_2 q},$$

and hence, for example,

$$\left(\frac{dx_2}{dx_1}\right)_{L'} > \left(\frac{dx_2}{dx_1}\right)_L \quad \text{for} \quad \frac{p}{q} > \frac{b_1}{b_2}.$$

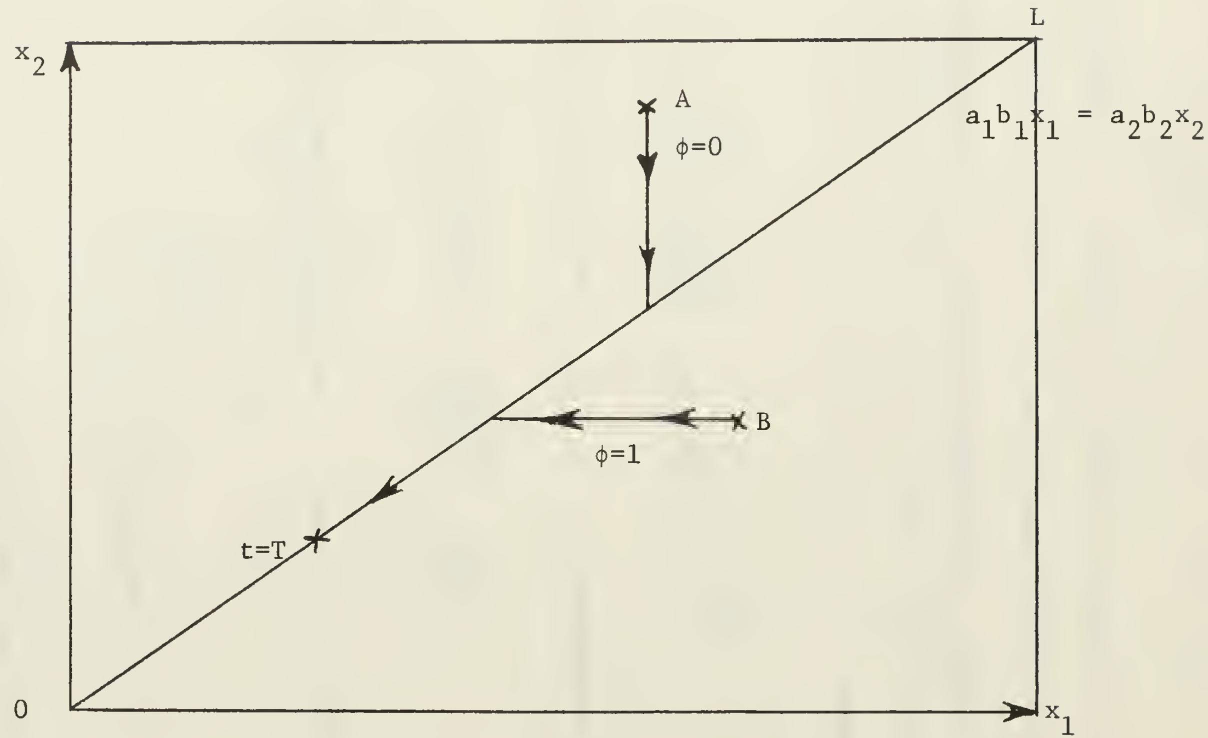
The significance of the line L' and its relationship to the line L is that

$$v(\tau = 0) \begin{cases} > 0 & \text{below } L' \\ < 0 & \text{above } L', \end{cases} \qquad \text{(C12)}$$

and hence by (C9) we find that

$$\phi(t = T) = \begin{cases} 1 & \text{for } P(T) \text{ below } L' \\ 0 & \text{for } P(T) \text{ above } L', \end{cases} \qquad \text{(C13)}$$

Case (a) $\frac{p}{q} = \frac{b_1}{b_2}$



Note: On line L $a_1 b_1 x_1 = a_2 b_2 x_2$, use allocation $\phi = \frac{a_2}{a_1 + a_2}$

Figure C1. Optimal Allocation for Linear Law Attrition where $P(t = T) = (x_1(t = T), x_2(t = T))$. We also note from (C7) that

$$\frac{dv}{d\tau}(\tau) \begin{cases} > 0 & \text{below } L \\ < 0 & \text{above } L. \end{cases} \tag{C14}$$

Thus, (C12) and (C14) give us three cases to consider

$$\text{Case (a)} \quad \frac{p}{q} = \frac{b_1}{b_2},$$

$$\text{Case (b)} \quad \frac{p}{q} > \frac{b_1}{b_2},$$

$$\text{Case (c)} \quad \frac{p}{q} < \frac{b_1}{b_2}.$$

We consider Case (a) first. The solution for this case is shown diagrammatically in Figure C1. Even though explicit expressions have not been obtained for the state and adjoint variables, the dependence of the control on these quantities can still be discussed. It may be shown that the optimal control depends on the state variables x_1 and x_2 (and also attrition coefficients) in each "decision region." Above the line $a_1b_1x_1 = a_2b_2x_2$, denoted by L, the control $\phi = 0$ is used until this line is encountered. When L is reached, the singular control $\phi = \frac{a_2}{a_1 + a_2}$ is used until the end of the battle at $t = T$. The above type of solution holds for arbitrary initial values of x_1 and x_2 : $x_1(t = 0) = x_1^\circ$ and $x_2(t = 0) = x_2^\circ$. The time history of the optimal control is traced for two particular initial force ratios shown as point A and point B. At point B, $\frac{x_1^\circ}{x_2^\circ} > \frac{a_2b_2}{a_1b_1}$ and hence $\phi = 1$ is used until the line L is encountered.

For Case (a) : $\frac{p}{q} = \frac{b_1}{b_2}$, the above statements are proved as follows. At $\tau = 0$ equation (C8) reduces to

$$v(\tau = 0) = \left(\frac{q}{b_2}\right)[a_1b_1x_1(t = T) - a_2b_2x_2(t = T)]. \qquad \text{(C15)}$$

From (C15) we see that there are three cases to consider depending on the sign of the term in square brackets.

Case (1) $a_1b_1x_1(t = T) = a_2b_2x_2(t = T)$

We see that this corresponds to when the system ends up on the singular subarc. In this case $\phi(t = T) = \dfrac{a_2}{a_1 + a_2}$, and we continue (in backwards progression) to use the singular control $\phi(t) = a_2/(a_1+a_2)$ (note that $\frac{dv}{d\tau} = 0$ when this is used and that we had $v(\tau = 0) = 0$) until $x_1(t) = x_1^\circ$ or $x_2(t) = x_2^\circ$. This yields three further subcases.

Subcase (1A) $a_1b_1x_1^\circ < a_2b_2x_2^\circ$

Define t_1 as t such that $x_1(t_1 > 0) = x_1^\circ$. Then we use $\phi = 0$ for $0 \leq t \leq t_1$. This is consistent since $v(\tau = T - t_1) = 0$ and

$$\frac{dv}{d\tau} = p_3(a_1b_1x_1^\circ - a_2b_2x_2) \quad \text{for} \quad T - t_1 \leq \tau \leq T$$

is negative which implies $v(\tau) < 0$ and hence $\phi(\tau) = 0$.

Subcase (1B) $a_1b_1x_1^\circ > a_2b_2x_2^\circ$

Define t_1 as t such that $x_2(t_1 > 0) = x_2^\circ$. Then we use $\phi = 1$ for $0 \leq t \leq t_1$. This is consistent since $v(\tau - T - t_1) = 0$ and

$$\frac{dv}{d\tau} = p_3(a_1b_1x_1 - a_2b_2x_2^\circ) \quad \text{for} \quad T - t_1 \leq \tau \leq T$$

is positive which implies $v(\tau) > 0$ and hence $\phi(\tau) = 1$.

Subcase (1C) $a_1b_1x_1^\circ = a_2b_2x_2^\circ$

We use $\phi(t) = a_2/(a_1 + a_2)$ from the beginning.

Case (2) $a_1b_1x_1(t = T) < a_2b_2x_2(t = T)$

Since $v(\tau = 0) = \left(\frac{q}{x_2}\right)[a_1b_1x_1 - a_2b_2x_2] < 0$, at the end of battle we have $\phi(\tau = 0) = 0$. We work backwards from the end. Since we are above the line L, $\frac{dv}{d\tau} = p_3(a_1b_1x_1 - a_2b_2x_2) < 0$ and hence $v(\tau) < 0$ for all $\tau\varepsilon[0,T]$. Thus we have $\phi(t) = 0$ for $0 \leq t \leq T$.

Case (3) $a_1b_1x_1(t = T) > a_2b_2x_2(t = T)$

Since $v(\tau = 0) = \left(\frac{q}{b_2}\right)[a_1b_1x_1 - a_2b_2x_2] > 0$, at the end of battle we have $\phi(\tau = 0) = 1$. We work backwards from the end. Since we are below the line L, $\frac{dv}{d\tau} = p_3(a_1b_1x_1 - a_2b_2x_2) > 0$ and hence $v(\tau) > 0$ for all $\tau\varepsilon[0,T]$. Thus we have $\phi(t) = 1$ for $0 \leq t \leq T$.

The above cases are shown in Figure C2. It is to be noted that in the above development we have made use of the fact that $p_3(t) > 0$ for all t.

We now consider Case (b) : $\frac{p}{q} > \frac{b_1}{b_2}$. There are two cases to be considered.

Case (1) never on singular subarc for finite interval of time

Again there are two subcases to consider, depending on whether the system winds up above or below L.

Subcase (1a) $a_1b_1x_1(t = T) > a_2b_2x_2(t = T)$

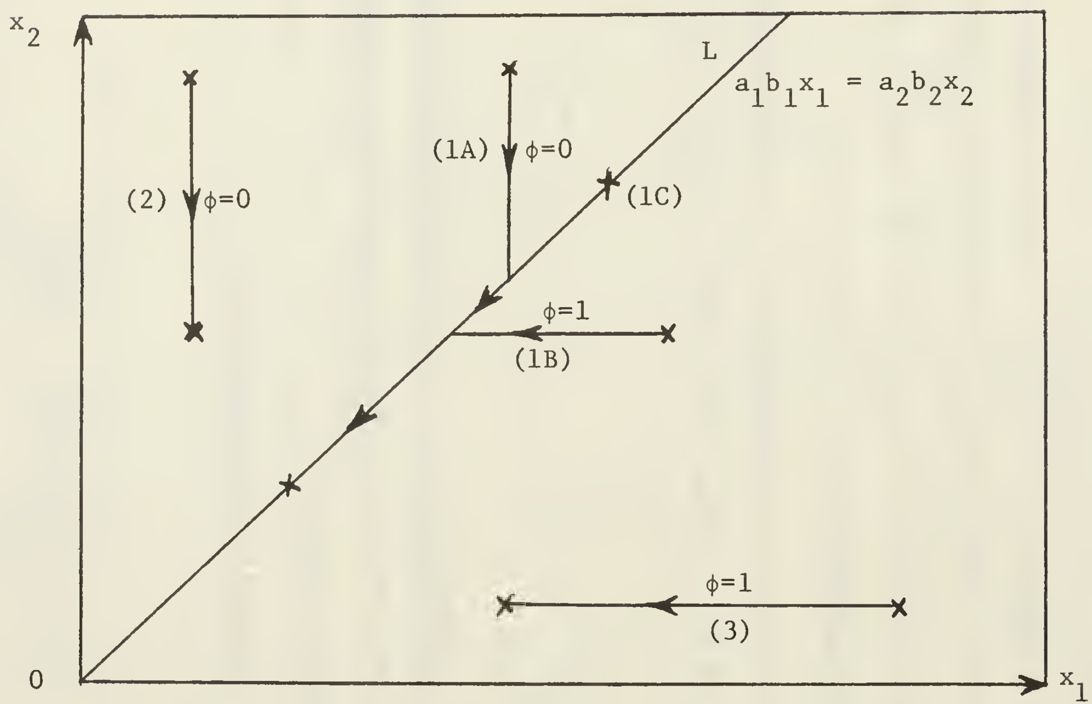
Since

$$v(\tau) = a_1b_1x_1\left(\frac{-p_2}{b_2}\right)\left[\frac{(p_1/p_2)}{(b_1/b_2)} - \frac{a_2b_2x_2}{a_1b_1x_1}\right],$$

we see that $v(\tau = 0) > 0$ and hence by (C9) $\phi(\tau = 0) = 1$. Since

$\frac{dv}{d\tau} = p_3(a_1b_1x_1 - a_2b_2x_2) > 0$ when we are below

Case (a) $\frac{p}{q} = \frac{b_1}{b_2}$



Note: On line L $a_1 b_1 x_1 = a_2 b_2 x_2$, use allocation $\phi = \frac{a_2}{a_1 + a_2}$.

Figure C2. Battle Histories for Prescribed Duration Battle.

L and we stay there by rising $\phi(\tau) = 1$, we have $v(\tau) > 0$ for all $\tau \varepsilon [0,T]$. Thus we have $\phi(t) = 1$ for $0 \leq t \leq T$.

Subcase (1b) $a_1 b_1 x_1(t = T) < a_2 b_2 x_2(t = T)$

Again there are two further subcases to consider, depending on whether the system winds up above or below L'.

Subcase (1bI) $a_1 b_1 x_1(t = T) < a_2 b_2 x_2(t = T)$ and $a_1 p x_1(t = T) < a_2 q x_2(t = T)$

In this case we wind up above L'. Since $v(\tau)$ is given by (C6), we have $v(\tau = 0) < 0$ and hence by (C9) $\phi(\tau = 0) = 0$. Since we are above L, $\frac{dv}{d\tau}$ (given by (C7)) < 0 for all $\tau \varepsilon [0,T]$ and hence $v(\tau) < 0$ for all $\tau \varepsilon [0,T]$. Thus we have $\phi(t) = 0$ for $0 \leq t \leq T$.

Subcase (1bII) $a_1 b_1 x_1(t = T) < a_2 b_2 x_2(t = T)$ and $a_1 p x_1(t = T) > a_2 q x_2(t = T)$

In this case we wind up below L' at the end. Since $v(\tau)$ is given by (C6), we have $v(\tau = 0) > 0$ and hence by (C9) $\phi(\tau = 0) = 1$. We work backwards from the end. Since we are above L, $\frac{dv}{d\tau} < 0$ while we remain above L. Thus $v(\tau)$ decreases for $\tau > 0$. There are two further subcases depending on whether $v(\tau)$ decreases to zero before the line L is encountered. Let τ_1 be such that $v(\tau_1) = 0$. If L has not been reached at τ_1, then $v(\tau)$ for $\tau > \tau_1$ is negative and $\phi(\tau) = 0$ until the beginning of battle. It is also possible to reach L just at $v(\tau_1) = 0$. In this case (assuming we don't remain on singular subarc) $v(\tau) > 0$ for $\tau > \tau_1$, since we pass below L and $\frac{dv}{d\tau} > 0$.

Case (2) on singular subarc for finite interval of time

This can happen only when $a_1b_1x_1(t = T) < a_2b_2x_2(t = T)$ and $a_1px_1(t = T) > a_2qx_2(t = T)$. As usual, we work backwards from the end of battle. We use $\phi(\tau) = 1$ for $0 \leq \tau \leq \tau_1$, and at $\tau = \tau_1$ we must have $a_1b_1x_1(\tau_1) = a_2b_2x_2(\tau_1)$. We use the singular control $\phi(\tau) = a_2/(a_1 + a_2)$ for $\tau_1 \leq \tau \leq \tau_2$. There are three further subcases

(1) $x_1(\tau_2) = x_1^\circ$, $x_2(\tau_2) < x_2^\circ$,

(2) $x_1(\tau_2) < x_1^\circ$, $x_2(\tau_2) = x_2^\circ$,

(3) $x_1(\tau_2) = x_1^\circ$, $x_2(\tau_2) = x_2^\circ$.

We omit the trivial discussion of these cases.

Thus we see from the above that there are six possible cases for the history of combatant force strengths in the battle of prescribed duration:

(1) started below L and never reached L,

(2) always above L',

(3) started above L' and end up above L but below L' without ever reaching L,

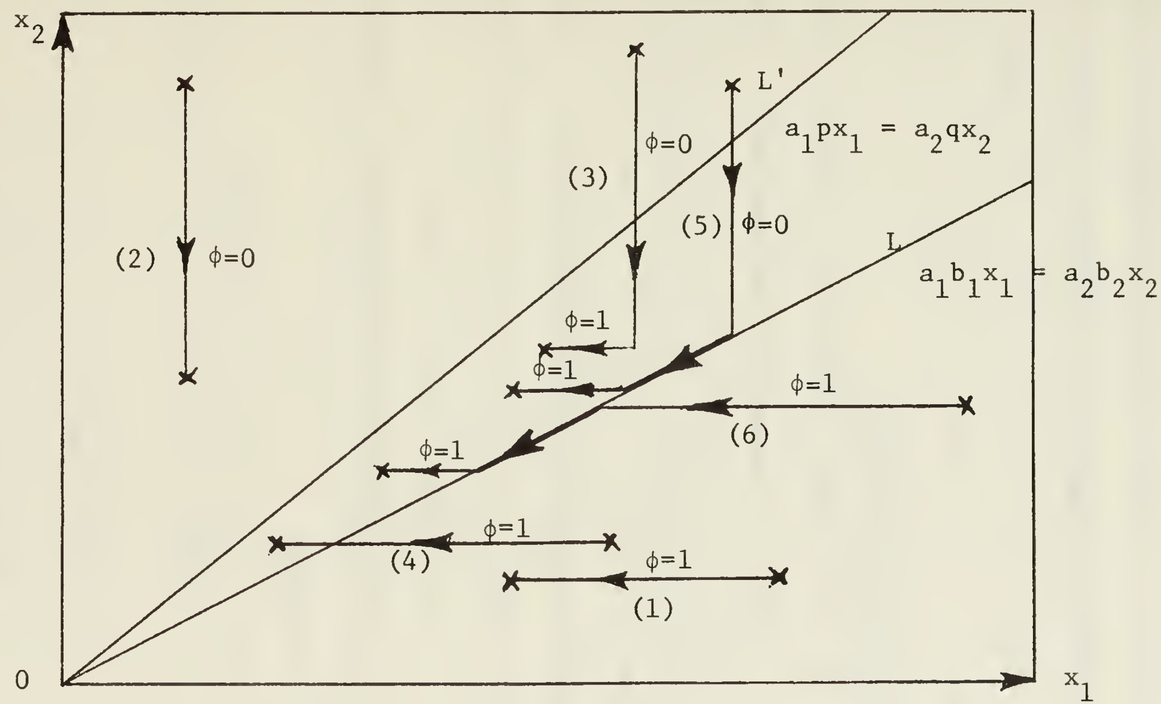
(4) end up above L but started below L and did not remain on L for finite interval of time,

(5) started above (or on) L and were on L for finite interval of time,

(6) started below L and were on L for finite interval of time.

These six cases are shown in Figure C3. The reader should compare the solution we have sketched here with that of Bellman's continuous version of the strategic bombing problem (see [9] pp. 227-233). Case (c) : $\frac{p}{q} < \frac{b_1}{b_2}$ is similar to Case (b).

Case (b) $\frac{p}{q} > \frac{b_1}{b_2}$



Note: On line L $a_1b_1x_1 = a_2b_2x_2$, use allocation $\phi = \frac{a_2}{a_1 + a_2}$.

Figure C3. Optimal Allocation for Linear Law Attrition

The reader's attention is directed to the interpretation of these three cases. Case (a) is when Y assigns utility to surviving X-force types in exact proportion to their destructive capability against Y. Case (b) is when Y assigns a greater utility to surviving X_1's than in proportion to their kill rate against Y relative to that of X_2. It is recalled that similar type remarks were made with respect to the solution of problem a1.

b. Effect of Resource Constraints.

In this section we will examine a sequence of models of increasing complexity for which the effect of ammunition limitations on firing rate (fire discipline) will be explored. In each case, we consider two homogeneous forces engaged in combat described by a square law. The research on these models has not progressed as far as that on the earlier ones. For some of these models the results are of a preliminary nature, the entire solution not having been completely worked out.

1. Battle of Prescribed Duration with Constant Kill Rates.

We consider the situation

$$\underset{\phi(t)}{\text{maximize}}\ px(T) - qy(T) \quad \text{with } T \text{ specified}$$

$$\text{subject to:} \quad \frac{dx}{dt} = -a_1 y$$

$$\frac{dy}{dt} = -\phi \nu a_2 x$$

$$\frac{dz}{dt} = \phi \nu$$

$z, y \geq 0$, $0 \leq \phi \leq 1$, $z(t = 0) = 0$, and $z(t = T) \leq A < \nu T = \nu \int_0^T dt$, where ν is the maximum firing rate of each X unit. It is noted that the nature of the attrition coefficients a_1 and a_2 is different, since a_1 has incorporated in it a constant firing rate.

This corresponds to the case where each X combatant has a limited supply of ammunition, denoted by A. We assume that this supply is such that he could not fire at his maximum firing rate for the prescribed duration of the battle, for when $A \geq \nu T$ it is easily seen that the optimal strategy is to fire at the maximum possible rate, $\phi(t) = 1$ for $0 \leq t \leq T$.

The optimal regulation of firing rate turns out to be

$$\phi(t) = 1 \quad \text{for} \quad 0 \leq t \leq T_1 \quad \text{where} \quad T_1 = \frac{A}{\nu}$$

$$\phi(t) = 0 \quad \text{for} \quad T_1 \leq t \leq T.$$

This was determined as follows. The Hamiltonian is given by

$$H(t,x,p,\phi) = \phi\nu(p_3 - p_2 a_2 x) - p_1 a_1 y,$$

and hence

$$\phi = \begin{cases} 0 & \text{for} \quad p_3 < p_2 a_2 x \\ 1 & \text{for} \quad p_3 > p_2 a_2 x. \end{cases}$$

The adjoint differential equations are given by

$$\dot{p}_1 = -\frac{\partial H}{\partial x} = \phi\nu a_2 p_2 \quad \text{with} \quad p_1(t = T) = p$$

$$\dot{p}_2 = -\frac{\partial H}{\partial y} = a_1 p_1 \quad \text{with} \quad p_2(t = T) = -q$$

$$p_3(t) = \text{const.}$$

We introduce the reverse time variable $\tau = T - t$ and consider a backwards integration of the state and dual variables from the fixed end of the battle, $t = T$. Hence, $\frac{dp_1}{d\tau} = -\phi\nu a_2 p_2$, etc. It is easy to show that $p_1(\tau)$, $x(\tau)$, and $y(\tau)$ are non-decreasing functions of τ (regardless of ϕ) with $p_1(\tau = 0) = p$, $x(\tau = 0) = x_s$, and $y(\tau = 0) = y_s$. Similarly, $p_2(\tau)$ is a strictly decreasing function of τ. Hence, $Q(\tau) = a_2 p_2(\tau)x(\tau)$ is a strictly decreasing function of τ with an initial value of $Q(\tau = 0) = -qa_2 x_s$. Thus, p_3 must be negative, and $\phi(\tau)$ never switches back to 0 once it becomes 1.

This solution is distrubing, since it is not intuitively appealing to fire at one's maximum firing rate until one runs out of ammunition and to spend the final stages of battle without ammunition. Hence, we are led to consider other models for further insight.

2. Battle of Prescribed Duration with Time Varying Kill Rates.

We consider the situation

$$\underset{\phi(t)}{\text{maximize}}\ px(T) - qy(T) \quad \text{with } T \text{ specified}$$

$$\text{subject to:} \quad \frac{dx}{dt} = -a_1(t)y$$

$$\frac{dy}{dt} = -\phi\nu a_2(t)x$$

$$\frac{dz}{dt} = \phi\nu$$

$$x,y \geq 0, \quad 0 \leq \phi \leq 1, \quad z(t = 0) = 0, \quad \text{and} \quad z(t = T) \leq A < \nu T.$$

It seems reasonable to assume that in mnay real world situations $a_1(t)$ and $a_2(t)$ would be monotonically increasing functions of time, e.g., two forces closing with each other. All the previous solution steps remain the same except for the effect of $a_1(t)$ and $a_2(t)$ increasing with time. This may change the solution markedly, although the optimal control is still bang-bang. The quantity $Q(\tau) = a_2(\tau)p_2(\tau)x(\tau)$ is not guaranteed to be a strictly decreasing function of τ, since $a_2(\tau)$

is strictly decreasing (but positive) and $p_2(\tau)$ is negative. This allows the possibility that the optimal tactic may be to hold one's fire and conserve ammunition in the early stages of battle so that $\phi(t = T) = 1$ at the end of battle.

The way in which ammunition is conserved depends on the specific nature of $a_1(t)$ and $a_2(t)$. It seems worthwhile to explore optimal tactics for several simple time dependencies of these quantities, but this hasn't been done as yet. We would recommend that this be a future research task. In Appendix D, we develop the solution to variable coefficient (either force separation or time as the independent variable) Lanchester-type equations when the ratio of attrition rates is a constant. This allows an analytic solution to be obtained for the problem at hand in special instances. It is not unreasonable to expect to encounter cases in which one holds his fire until the kill probability reaches some threshold value. An aspect that is disturbing is that the control has turned out to be bang-bang. One can show, in fact, that a singular solution is impossible for this problem.

R. Isaacs has studied some similar problems in his book Differential Games [50] and has explored some aspects of this problem much deeper than presented here. Isaacs tried to resolve the problem of shooting up all of one's ammunition before the end of the battle by modifying the payoff. Another approach might be to consider a terminal control problem.

3. Fight to the Finish with Limited Ammunition.

Thus we are led to consider

$$\underset{\phi(t)}{\text{maximize}}\ px(T) - qy(T) \quad \text{with } T \text{ unspecified}$$

subject to:

$$\frac{dx}{dt} = -a_1 y$$

$$\frac{dy}{dt} = -\phi \nu a_2 x$$

$$\frac{dz}{dt} = \phi \nu$$

$x, y \geq 0$, $0 \leq \phi \leq 1$, $z(t = 0)$, and $z(t = T) \leq A$,

with terminal states defined by (1) $x(T) = 0$ and (2) $y(T) = 0$. We briefly consider the constant attrition coefficient case, although it is noted that a similar analysis would apply to time dependent attrition coefficients. As with the previous terminal control problem, dual variables (marginal gains) now are related to the final values of the state variables by virtue of $H(t,x,p,\phi) = \text{const.} = 0 = H(t = T,x,p,\phi)$. We might encounter a case where tactics are dependent on enemy force level (in the previous limited ammunition cases, tactics are independent of enemy force level), but this case has not yet been explored very far.

One point worth noting is that for the constant attrition coefficient case the X forces in order to win are required to have enough ammunition to fire at their maximum rate during the entire duration of the battle. Hence, we see that concentration of forces reduces the ammunition requirement per man, since the length of battle is determined by initial numbers of forces committed to battle.

4. Two-Sided Extension.

There appears to be a novel feature in a two-sided version of the above problems. Again, we briefly make a few remarks about the constant attrition coefficient case.

$$\underset{\phi(t)}{\text{maximize}}\ \underset{\psi(t)}{\text{minimize}}\ px(T) - qy(T) \quad \text{with } T \text{ specified}$$

subject to:

$$\frac{dx}{dt} = -\psi a_1 \nu_1 y$$

$$\frac{dy}{dt} = -\phi a_2 \nu_2 x$$

$$\frac{du}{dt} = \phi \nu_2$$

$$\frac{dv}{dt} = \psi \nu_1$$

$$x,y \geq 0, \quad 0 \leq \phi,\psi \leq 1, \quad u(t = 0) = 0, \quad u(t = T) \leq A_2 < \nu_2 T,$$

$$v(t = 0) = 0, \quad v(t = T) \leq A_1 < \nu_1 T.$$

Unlike the previous one-sided version of this problem, it is now possible to have $\phi(t = T) = 1$ with limited ammunition. This possibility has arisen since the Y forces may hold their fire during the early stages of engagement. Questions now arise as to the advantage of delivering the first shot, e.g., is there a time lag before fire is returned?, and we move into the realm of games of timing studied at RAND [55].

c. <u>Extensions to Differential Games</u>.

There is an intimate connection between the mathematical bases of opiimal control theory and differential game theory. It has been stated that optimal control problems may be viewed as one-sided differential games for which the roles of all but one of the competing players have been suppressed [12]. A concise discussion of the inter-relationships between these two subjects is contained in Y. C. Ho's [41] excellent review of Isaacs book [50] (see also Chapter 9 in [24]).

If one takes a Hamilton-Jacobi approach to these variational problems, this relationship becomes particularly evident. In an optimal control problem we are seeking the solution to the following partial differentail equation for the optimal return, S (referred to as Hamilton's characteristic function in the calculus of variations literature [69]),

$$\frac{\partial S}{\partial t} + \underset{\phi(t)}{\text{maximum}}\ H(t,x,\frac{\partial S}{\partial x},\phi) = 0,$$

with appropriate boundary conditions. In a differential game we seek the solution to

$$\frac{\partial S}{\partial t} + \underset{\phi(t)}{\text{maximum}}\ \underset{\psi(t)}{\text{minimum}}\ H(t,x,\frac{\partial S}{\partial x};\phi,\psi) = 0.$$

It also seems appropriate to mention the relationship of dynamic programming to these techniques. Consideration of the equation satisfied by the optimal return points out clearly an important aspect of dynamic programming, its being a discrete approximation technique for solving variational problems [30]. It is, however, a dual approach which generates an optimal trajectory as an envelope of tangents rather than as a sequence of points [10]. The value of the continuous models lies in their ability to exhibit explicitly the dependence of optimal tactics on model parameters rather than any computational ease.

It is noted that the existing theory for differential games assumes that the optimal strategy (during any finite interval of time) is always a pure strategy. Hence, it is necessary that $\max \min H = \min \max H$ almost everywhere in time. There are, however, differential games of practical interest for which pure strategy solutions do not exist [11].

In light of the above discussion, it is easy to see the value of beginning the study of mathematical models of tactical allocation with optimal control. It is true that actual combat is a competitive environment in which the actions of both parties must be considered, but optimal control problems may be used to study most significant aspects of such problems: setting proper boundary conditions, devising solution procedures, study of singular solutions, differences in solutions for different forms of model. Most solution aspects of the one-sided problem are present in the two-sided one. It is assumed that formulation of these two-sided problems is clear from the previous content of this paper.

Of interest to the operations research worker is whether there is any new aspect of solution behavior in a differential game. The answer to this is "yes." In devising a rigorous solution procedure for the supporting weapon system game of H. K. Weiss [82], we have (see Appendix B) encountered solution behavior unique to terminal control attrition games: there may exist a domain of controllability for a given terminal state but entry to this state may be "blockable" by the "losing" player. In other words, there is a path determined by the necessary conditions leading from each point in a region of the initial state space to a terminal state, but the "losing" player may use a strategy other than his extremal strategy for this path to actually win. In the process of solving the supporting weapon system game and trying to understand the many complicated facets of its solution procedure, we gained insight by considering a related optimal control problem (see Appendix A), the Isbell and Marlow fire programming problem [52].

d. Implications of Models.

It seems appropriate to briefly discuss the general implications in the following areas of the models examined in this paper:

(1) optimal tactical allocation,

(2) intelligence,

(3) command and control systems,

(4) human decision making.

The discussion of these areas is not mutually exclusive.

Of interest to the military tactician is whether target selection rules evolve dynamically during the course of battle. Are target priorities static or do they evolve dynamically with the course of battle? With respect to optimal control models, this may be mathematically stated as whether there are transition (switching) surfaces in the solution. We have seen in the idealized and simplified models studied here that target priorities do change. This is related to the evolution of marginal return of target destruction (value of dual variable). We have seen that this evolution depends on the goals of the combatants (utility assigned to surviving force types at the end of the battle) and also the conditions which terminate the battle. In the terminal control problem studied here, a shift in target priorities is present only in a losing case, whereas in a fixed duration battle such a switch is independent of winning or losing but depends only on weapon system capabilities and the prescribed duration of battle.

Even though these models assume complete and instantaneous information, it appears that some inferences may be made for cases where uncertainty is present. In the terminal control case, we saw that selection of tactics depends on a knowledge of the enemy's strength and capabilities, since the terminal state of combat must be determined before optimal strategies can be. For a battle of prescribed duration, e.g., fighting a delaying action in a retrograde movement to protect the withdrawal of troops, tactics depend only on enemy and friendly capabilities and length of combat, not the initial force levels. For such cases the estimate of combat length is critical, since changes in target priorities are determined relative to the end of the engagement.

Schreiber [70] has proposed an idealized and simple, but yet illuminating, way of quantitatively showing the value of intelligence and command control capabilities. He introduces the concept of "command efficiency," which is measured by the fraction of the enemy's destroyed units from which fire has been redirected. The effect of poor intelligence and poor capabilities for redirecting fire from destroyed targets is to produce "overkill." Schreiber's equations for combat involved this fraction called "command efficiency," and they reduce to Lanchester-type equations for area fire when the fraction is 0 and aimed fire for a value of 1. We have seen that the optimal tactics are quite different for these two cases. When intelligence and command control systems are very efficient, the optimal tactic is seen to be concentration of fire on a specific target type. When capability for redirection of fire from destroyed targets is poor (either through damage assessment or constraints on new target acquisition), the optimal tactic may be to allocate fire in a proportional fashion over target types in a way that holds the ratios of target density in each target area to be constant. Another implication is that supporting weapon systems (e.g., artillery) concentrate fire on selected point targets, but that fire is allocated proportionately over various area targets. Thus, these models suggest that the tactics of target engagement may vary with command and control capabilities.

These models also show the importance of intelligence in devising the best tactics in combat. Intelligence on enemy weapon system capabilities (kill rates including target acquisition rates) and potential length of engagement play a central part. We also have seen that for fights to the finish and linear law attrition cases intelligence on enemy force levels is also required. For artillery fire support missions against various troop concentrations, knowledge of troop densities is essential in the assignment of target priorities. Particularly dense concentrations where the initial kill potential is high are seen to be cases where the optimal tactic is to concentrate fire on one target for awhile.

Another argument for the concentration of forces is seen to emerge from the study of these simplified models. When ammunition is limited, a concentration of forces has the effect of counter-balancing this constraint. For example, in a fire fight numerical superiority could mean that the enemy force level would be reduced such that he would disengage in time before the friendly ammunition restriction became critical.

These models may be interpreted to show the value of human judgment in combat. They indicate, as does common sense and experience, that in battle a commander must use his judgment to ascertain to what end can the course of battle be steered so that he may devise his strategy accordingly. The demonstrated sensitivity of these models to many factors shows the importance of human assessment of a situation and the importance of good judgment in assigning utility to forces surviving the battle at hand.

e. Summary.

The results of this appendix may be summarized as follows:

(1) a sequence of one-sided models has been presented which shows that the tactics of target selection may be sensitive to force strengths, target acquisition process, the type of attrition process, and/or the termination conditions of combat,

(2) a sequence of models have been presented which shows some preliminary results on the effect of resource constraints on firing discipline and concentration of forces,

(3) tactics for target selection are heavily dependent upon "command efficiency,"

(4) concentration of fire on one target type among many occurs as an optimal tactic only when target acquisition is not subject to diminishing returns.

APPENDIX D. Solution to Variable Coefficient Lanchester-Type Equations.

In Appendix C, we briefly considered a model involving Lanchester-type equations with variable coefficients. Although such equations have been studied by analysts for over 10 years since H.Weiss' pioneering work [81], analytic solutions for the average force strengths (state variables) as a function of an independent variable (either time or range) have been obtained in only isolated instances [19], [20]. We have discovered a very general method for solving such variable coefficient equations under certain assumptions about the average attrition rates of the combatants. We point out, however, that all previously published results [73] except one are contained in the general results presented here. Additionally, these new results also apply to cases in which the relative velocity of combatant forces is a function of force separation.

We show how to solve Lanchester-type equations for combat between two homogeneous forces when the attrition rates are variable provided that their quotient is a constant. Solutions are developed for either time or force separation as the independent variable. We also investigate under what circumstances each of Bonder's two second order differential equations [20] can be transformed into a constant coefficient equation yielding exponential solutions. We begin by briefly reviewing previous work on this topic.

H. Weiss [81] extended Lanchester-type equations to include the relative movement of two homogeneous forces, allowing time and space to be "traded" for casualties. He considered the two attrition rates to be dependent upon force separation in such a way that their quotient was a constant. S. Bonder [19], [20] and others [73] have used Weiss' extension to study the effects of mobility and various range dependencies of the average attrition rates on the number of surviving forces. For each force type, he developed a second order differential equation which related average force strength to the force separation, r, and obtained solutions for cases of constant relative velocity of forces.

We show that more general results are easily obtainable by considering the original first order system of equations with either time or force separation as the independent variable (as is appropriate for the problem under study). Bonder's results [20] and the constant attrition rate solution are but special instances of our more general results.

a. Range Dependent Attrition Rates.

The case of range dependent attrition rates originally motivated this approach, although it is now seen to be a special case of time dependent attrition rates. We use the same notation as Bonder [20], [73] for the battlefield coordinates.

We consider

$$\frac{dx}{dt} = -\alpha(r)y,$$

$$\frac{dy}{dt} = -\beta(r)x,$$

where

$$\frac{\alpha(r)}{\beta(r)} = \frac{k_\alpha}{k_\beta}$$

and x,y are average force strengths,

$\alpha(r),\beta(r)$ are average (range dependent) attrition rates.

Considering force separation, r, as the independent variable, we have $\frac{dx}{dt} = v\,\frac{dx}{dr}$ and thus the equations become

$$\frac{dx}{dr} = -k_\alpha \frac{g(r)}{v(r)}\, y,$$

$$\frac{dy}{dr} = -k_\beta \frac{g(r)}{v(r)}\, x. \tag{D1}$$

We consider the relative velocity of the forces to be a function of force separation only. As Weiss [81] has pointed out, these equations readily yield a square law relationship between the state variables

$$k_\beta(x^2 - x_0^2) = k_\alpha(y^2 - y_0^2). \tag{D2}$$

Solving equation (D2) for y, substituting the result into the first of equations (D1), and integrating from $r = R_0$ and $x = x_0$ to r and x, we obtain

$$\ell n \left\{ \frac{x + \sqrt{x^2 + (y_0^2 k_\alpha/k_\beta - x_0^2)}}{x_0 + y_0\sqrt{k_\alpha/k_\beta}} \right\} = -\sqrt{k_\alpha k_\beta} \int_{R_0}^{r} \frac{g(u)}{v(u)}\, du \tag{D3}$$

Raising e to the power of each side of equation (D3), we obtain the following result after some algebraic manipulation:

$$x(r) = x_0 \cosh \Theta + y_0\sqrt{k_\alpha/k_\beta}\, \sinh \Theta\ ,$$

where

$$\Theta(r) = -\sqrt{k_\alpha k_\beta} \int_{R_0}^{r} \frac{g(u)}{v(u)}\, du. \tag{D4}$$

A similar expression is readily obtained for y(r). Bonder's [20] results are special cases of equations (D4).

b. Time Dependent Attrition Rates.

More generally, we might be interested in

$$\frac{dx}{dt} = -k_\alpha h(t) y,$$

$$\frac{dy}{dt} = -k_\beta h(t) x.$$

The same approach as above readily yields

$$x(t) = x_0 \cosh \Theta + y_0 \sqrt{k_\alpha / k_\beta} \sinh \Theta$$

where

$$\Theta(t) = -\sqrt{k_\alpha k_\beta} \int_0^t h(u) du. \qquad \text{(D5)}$$

When $h(t) = 1$, equations (D5) reduce to the familiar constant coefficient solution. When $h(t) = g(r(t))$ and $r(t) = R_0 + \int_0^t v(t)dt$, equations (D5) reduce to equations (D4).

c. Some Comments.

We see from the above that the effect of time (range) dependent average attrition rates of the form considered is to transform the time (range)scale of the usual square law attrition process. Thus we see that certain time (range) intervals are weighted more heavily in the transformed time (range) scale than they are in the usual square law attrition process.

Previous analytic work [73] has assumed that the relative velocity between forces to be constant. These results allow this restriction to be relaxed. For example, we may now easily study combat situations in which relative velocity is a decreasing function of force separation.

We would strongly recommend that the results developed here be used in extensions of the allocation models developed in the previous appendix. The approach developed here also applies to the solution of the adjoint equations in the determination of our new dynamic kill potential developed in Appendix F.

d. <u>The Condition for Solution in Terms of Elementary Functions</u>.

We discuss in this section necessary and sufficient conditions for a second order ordinary differential equation which Bonder has derived [20] to be transformed to a constant coefficient equation yielding exponential solutions. This covers all but one of the results obtained by Bonder [73].

We start by considering

$$\frac{dx}{dr} = -\frac{\alpha(r)}{v}\,y,$$

$$\frac{dy}{dr} = -\frac{\beta(r)}{v}\,x, \qquad \text{(D6)}$$

which is implicit in the development of (D1). By differentiation and substitution, we may combine these equations into a single second order equation for x.

$$\frac{d^2x}{dr^2} + y\,\frac{d}{dr}\left\{\frac{\alpha(r)}{v}\right\} + \frac{\alpha(r)}{v}\,\frac{dy}{dr} = 0$$

or

$$\frac{d^2x}{dr^2} - \frac{dx}{dr}\,\frac{d}{dr}\left\{\ln\frac{\alpha(r)}{v}\right\} - \frac{\alpha(r)\beta(r)}{v^2}\,x = 0,$$

which for v = constant (i.e., constant relative velocity of force movement) becomes

$$\frac{d^2x}{dr^2} - \frac{1}{\alpha}\,\frac{d\alpha}{dr}\,\frac{dx}{dr} - \frac{\alpha\beta}{v^2}\,x = 0. \qquad \text{(D7)}$$

A similar equation is similarly obtained for y.

In [40] p. 50 it is stated that a necessary and sufficient condition to be able to transform the equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = h(x)$$

into an equation with constant coefficients is that

$$\frac{a_1 + \frac{1}{2}\frac{a_2'}{a_2}}{\sqrt{a_2}} = \text{constant.}$$

The desired substitution is given by $Z = f(x) = \frac{1}{A}\int^x \sqrt{a_2(x)}\,dx$ (where A is defined on p. 50 of [40]). This reference also gives the transformed second order equation in the new independent variable Z. When the above theorem is applied to (D7), we find out that (D7) can be transformed to an equation with constant coefficients if

$$\frac{1}{\beta}\frac{d\beta}{dr} = \frac{1}{\alpha}\frac{d\alpha}{dr},$$

which is easily seen to be equal to

$$\frac{d}{dr}\left(\frac{\alpha(r)}{\beta(r)}\right) = 0,$$

or $\frac{\alpha(r)}{\beta(r)}$ = constant. It is not surprising in view of our previous development that $\frac{\alpha(r)}{\beta(r)}$ equal to a constant is a sufficient condition for equation (D7) to be transformed into an equation with constant coefficients. The development of necessary conditions in the general case is more complicated.

The above theorem from [40] explains why equation (10) of [73] has not yielded to solution when $R_\alpha \neq R_\beta$. In this case it is seen to be impossible to transform the equation into one yielding exponential solutions. Our work here then confirms the conjecture made in [73] that the condition which facilitated the results obtained at the University of Michigan was that $\frac{\alpha(r)}{\beta(r)}$ = constant.

We also note that the transformations employed by Bonder [20] are readily discovered by p. 50 of [40] but omit the details. We have also briefly tried to solve equation (10) of [73] for $R_\alpha \neq R_\beta$ by classical ordinary differential equation methods (see [45] or pp. 530-576 of [65]). It appears that this equation is not a standard form and series methods must be used. Time has permitted only a very cursory look at this.

APPENDIX E. Connection with Bellman'a Stochastic Gold-Mining Problem.

In this appendix we solve several versions of a continuous stochastic decision process by means of the Pontryagin maximum principle. The basic problem has been called the continuous version of a stochastic gold-mining process (see pp. 227-233 of [9]), but it is really an idealization of an allocation problem for strategic bombers. We consider a decision being made sequentially and continuously over a period of time with the result of the decision not certain. We assume that we know the probabilities associated with each outcome. This type of problem is referred to in the economics literature as decision making under risk.

This is the continuous version of a stochastic decision process. A discrete version has been formulated and solved (see pp. 61-79 of [9]). However, the continuous problem permits certain relationships between model parameters and the structure of the optimal allocation policies to be explicitly exhibited. This is not possible to the degree developed here for a dynamic programming numerical solution procedure. The type of idealization which leads to a simple analytical solution frequently provides insight into the fundamental structure of the optimal allocation policies.

We consider a sequence of models. Two basic cases are allocation in the face of diminishing returns and non-diminishing returns. Two further subcases for each of these are prescribed duration use of a resource and also maximum return for specified risk. Thus we actually consider four models. There is a close relation between these models and their optimal allocation policies and the allocation problems in combat described by Lanchester-type equations of warfare which we considered in Appendix C. This has been our motivation for the current development.

First we give some background on the basic problem and then we develop the solution to each of the four problems. Then we summarize the solutions and discuss the significance of this work.

a. Background.

R. Bellman and R. S. Lehman did the original work on the "continuous gold-mining equation." The problem is actually to maximize the expected damage by a bomber by the proper choice of the bombing sequence of two target areas. The bomber, of course, is subject to being shot down. The problem was originally solved by Bellman and Lehman by use of variational methods (the case of diminishing returns only). In this solution process, they make use of knowledge of the solution to the discrete version of this problem. A significant point to note is that this problem (for the case of diminishing returns) has a singular solution (see [53]). This appears to be the first example in the literature of a problem with a singular control. It was correctly solved ten years before the first publication on singular control problems appeared [54]. We shall use the newer theory to solve it. The current approach provides more insight and also leads to a new interpretation of these problems. The case of non-diminishing returns was not previously solved (it is the less complex case).

The current treatment of these problems by the Pontryagin maximum principle provides further insight. We see that the problem referred to by Bellman as the infinite duration problem is actually the problem of maximizing return for a specified risk. It is not essential that the problem last for an infinite length of time.

We consider the case of non-diminishing returns to contrast its solution with that of diminishing returns. As we have noted previously, there is a close parallel between the solutions of these problems and the solutions to the fire programming problems considered in Appendix C. We may think of a square law attrition process as the case of non-diminishing returns per unit of weapon system, whereas a linear law attrition process corresponds to diminishing returns per unit of weapon system. It appears worthwhile to further study the structure of such allocation problems and to further interpret the various structures of the optimal allocation policies. It also seems worthwhile to consider the interrelationships between such problems in the literature, but time has not permitted this.

The problem is to maximize the expected return for the use of a resource subject to loss (destruction or breakdown) by choice of the operating sequence in two deployment areas. The original motivation for this problem was the allocation of a bomber to strategic targets. Imagine that we had a bomber that we could send to either target A or target B. There is a return (fraction of strategic value destroyed) and a risk (probability of bomber being shot down) for each target area. The problem is to determine the tradeoff between risk and return. The reader is directed to pages 227-228 of [9] for the derivation of the models we consider in the next section.

b. Development of Solution to Problems.

In this section we present the development of the solution to four versions of the continuous gold-mining problem. We consider the following problems

(a) non-diminishing returns - prescribed duration use,

(b) non-diminishing returns - maximum return for specified risk,

(c) diminishing returns - prescribed duration use,

(d) diminishing returns - maximum return for specified risk.

1. Non-diminishing Returns - Prescribed Duration Use.

We consider

$$\underset{\phi(t)}{\text{maximize}} \int_0^T p(t)\,\{\phi r_1 + (1-\phi)r_2\}\,dt \quad \text{with } T \text{ specified,}$$

subject to:

$$\frac{dx}{dt} = -\phi r_1,$$

$$\frac{dy}{dt} = -(1-\phi)r_2,$$

$$\frac{dp}{dt} = -p\{\phi q_1 + (1-\phi)q_2\},$$

$x,y,p \geq 0$ and $0 \leq \phi \leq 1$,

with initial conditions

$$x(t=0) = x_0, \quad y(t=0) = y_0, \quad p(t=0) = 1,$$

where

x,y are strategic values of target areas 1 and 2, respectively, at time t,

p is probability that bomber survives until time t,

r_1, r_2 are rates at which strategic value is destroyed,

q_1, q_2 are rates at which bomber is shot down.

In the present analysis we assume that neither x nor y ever becomes zero.

The Hamiltonian, $H(t,x,p,\phi)$, is given by

$$H(t,x,p,\phi) = p(t)\{\phi r_1+(1-\phi)r_2\}- p_1\phi r_1 - p_2(1-\phi)r_2 - p_3p\{\phi a_1+(1-\phi)q_2\}. \quad \text{(E1)}$$

The adjoint equations are given by

$$\dot{p}_1 = -\frac{\partial H}{\partial x} = 0 \Rightarrow p_1(t) = \text{const}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial y} = 0 \Rightarrow p_2(t) = \text{const}$$

$$\dot{p}_3 = -\frac{\partial H}{\partial p} = -\phi r_1 -(1-\phi)r_2 + p_3\{\phi q_1 +(1-\phi)q_2\}$$

or

$$p_1(t) = 0 \qquad \text{since} \qquad p_1(t = T) = 0$$

$$p_2(t) = 0 \qquad \text{since} \qquad p_2(t = T) = 0$$

$$\frac{dp_3}{dt} = \phi\{-r_1 + p_3q_1\} + (1-\phi)\{-r_2 + p_3q_2\} \qquad p_3(t = T) = 0 \qquad \text{(E2)}$$

Combining (E1) and (E2), we see that the Hamiltonian becomes

$$H(t,x,p,\phi) = p(t)\{\phi r_1 + (1-\phi)r_2\} - p_3p\{\phi q_1 + (1-\phi)q_2\}. \qquad \text{(E3)}$$

The optimal control (there is only one extremal) is determined from $\max_\phi H$, which is the same as $\max_\phi\{\phi[r_1 - p_3q_1] + (1-\phi)[r_2 - p_3q_2]\}$, since $p(t) \geq 0$. Hence, the optimal control is given by

for $q_2 > q_1$

$$\phi(t) = \begin{cases} 1 & \text{for} \quad p_3(t) > \dfrac{r_2 - r_1}{q_2 - q_1} \\ 0 & \text{for} \quad p_3(t) < \dfrac{r_2 - r_1}{q_2 - q_1}, \end{cases}$$

and

for $q_2 < q_1$

$$\phi(t) = \begin{cases} 1 & \text{for } p_3(t) < \dfrac{r_2 - r_1}{q_2 - q_1} \\ 0 & \text{for } p_3(t) > \dfrac{r_2 - r_1}{q_2 - q_1} \end{cases} \tag{E5}$$

We check to see if there is a singular solution [53] to this problem. A more detailed discussion of singular solutions is to be found in Appendix C. A singular extremal is determined by the conditions [54] $\frac{\partial H}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = 0$. Using (E3) for the problem at hand, we obtain

$$p\{r_1 - r_2 - p_3(q_1 - q_2)\} = 0$$

and

$$\frac{dp}{dt}\{r_1 - r_2 - p_3(q_1 - q_2)\} - p(q_1 - q_2)\frac{dp_3}{dt} = 0,$$

which imply (ignoring pathological cases)

$$\frac{dp_3}{dt} = 0 = \phi\{-r_1 + p_3q_1\} + (1 - \phi)\{-r_2 + p_3q_2\}$$

or that $p_3 = r_2/q_2$. The latter condition implies $p_3 = r_1/q_1$ or $\phi = 0$ (which is not a singular control). Thus, we see that unless $\frac{r_1}{q_1} = \frac{r_2}{q_2}$, an unlikely case, there is <u>no singular solution</u>.

We develop the solution by working backwards from the end of the problem at $t = T$. It suffices to consider the case where $q_2 > q_1$. There are two further cases to consider depending on whether $r_2 > r_1$ or $r_1 > r_2$.

Case (a) $r_2 > r_1$ and $q_2 > q_1$

In this case we have $\frac{r_2 - r_1}{q_2 - q_1} > 0$ with $q_2 > q_1$.

Recalling that $p_3(t = T) = 0$ and using (E4), we see that $\phi(t = T) = 0$. We introduce the backwards time $\tau = T - t$ so that the adjoint equation (E2) becomes

$$\frac{dp_3}{d\tau} = \phi\{r_1 - p_3q_1\} + (1 - \phi)\{r_2 - p_3q_2\}.$$

Thus, up until the time of the first switch in tactics, which we denote by τ_1, we have

$$\frac{dp_3}{d\tau} = r_2 - p_3q_2 \quad \text{with} \quad p_3(\tau = 0) = 0.$$

Integration of the above yields

$$p_3(\tau) = \frac{r_2}{q_2}(1 - e^{-q_2\tau}). \qquad \text{(E6)}$$

If $p_3(\tau) < \dfrac{r_2 - r_1}{q_2 - q_1}$ for all $\tau \geq 0$, then we can never switch to $\phi(\tau) = 1$. The above readily yields that we never switch from $\phi(t) = 0$ when $\dfrac{r_2}{q_2} > \dfrac{r_1}{q_1}$. There can be a switch in tactics to $\phi(\tau) = 1$ when $\dfrac{r_2}{q_2} < \dfrac{r_1}{q_1}$, however. The time of this switch, τ_1, is determined from

$$p_3(\tau_1) = \frac{r_2}{q_2}(1 - e^{-q_2\tau_1}) = \frac{r_2 - r_1}{q_2 - q_1}. \qquad \text{(E7)}$$

From (E7) the time of switch is readily computed to be

$$\tau_1 = \ln\left\{\frac{r_2(q_2 - q_1)}{q_2r_1 - q_1r_2}\right\}^{1/q_2}. \qquad \text{(E8)}$$

For this potential switch to actually occur, the planning horizon, T, must be of sufficient length. The condition is that $T - \tau_1 \geq 0$, which implies that for the switch to occur the planning horizon length must satisfy

$$e^{-q_2 T} \le \frac{q_2 r_1 - q_1 r_2}{r_2(q_2 - q_1)} . \tag{E9}$$

Assuming that T satisfies (E9), then for $\frac{r_2}{q_2} < \frac{r_1}{q_1}$ we have

$$\phi(t) = 1 \quad \text{for} \quad 0 \le t \le T - \tau_1,$$

$$\phi(t) = 0 \quad \text{for} \quad T - \tau_1 \le t \le T. \tag{E10}$$

Case (b) $\quad r_2 < r_1$ and $q_2 > q_1$

In this case we have $\frac{r_2 - r_1}{q_2 - q_1} < 0$ with $q_2 > q_1$.

Recalling that $p_3(t = T) = 0$ and using (E4), we see that $\phi(t = T) = 1$. We introduce the backwards time $\tau = T - t$. The adjoint equation (E2) for the dual variable p_3 becomes

$$\frac{dp_3}{d\tau} = \phi\{r_1 - p_3 q_1\} + (1 - \phi)\{r_2 - p_3 q_2\}.$$

Thus, up until the time of the first switch in tactics, which we denote by τ_1, we have

$$\frac{dp_3}{d\tau} = r_1 - p_3 q_1 \quad \text{with} \quad p_3(\tau = 0) = 0.$$

Integration of the above readily yields

$$p_3(\tau) = \frac{r_1}{q_1}(1 - e^{-q_1 \tau}).$$

If $p_3(\tau) > \frac{r_2 - r_1}{q_2 - q_1}$ for all $\tau \ge 0$, then we can never switch to $\phi(\tau) = 0$. The above readily yields that we never switch from $\phi(t) = 0$ when $\frac{r_1}{q_1} > \frac{r_2}{q_2}$, but this is precisely the conditions which define this case. Hence, there is never a switch in tactics and we have

$$\phi(t) = 1 \quad \text{for} \quad 0 \leq t \leq T. \tag{E11}$$

2. Non-diminishing Returns - Maximum Return for Specified Risk.

We consider

$$\underset{\phi(t)}{\text{maximize}} \int_0^T p(t)\{\phi r_1 + (1 - \phi)r_2\}dt \quad \text{with} \quad T \quad \text{unspecified},$$

subject to:

$$\frac{dx}{dt} = -\phi r_1,$$

$$\frac{dy}{dt} = -(1 - \phi)r_2,$$

$$\frac{dp}{dt} = -p\{\phi q_1 + (1 - \phi)q_2\},$$

$x,y,p \geq 0$ and $0 \leq \phi \leq 1$,

with initial conditions

$$x(t = 0) = x_o, \quad y(t = 0) = y_0, \quad p(t = 0) = 1,$$

and terminal condition

$$p(t = T) = \varepsilon > 0 \quad (\text{also} \quad \varepsilon < 1).$$

As before, we assume that neither x nor y ever becomes zero.

As before, the Hamiltonian is given by (E1), but now the adjoint equations have the boundary condition on $p_3(t = T)$ unspecified. Thus

$$p_1(t) = \text{const} = 0,$$

$$p_2(t) = \text{const} = 0,$$

$$\frac{dp_3}{dt} = \phi\{-r_1 + p_3q_1\} + (1 - \phi)\{-r_2 + p_3q_2\} \quad \text{and} \quad p_3(t = T) \quad \text{is} \tag{E12}$$

unspecified.

Since the termination time T is unspecified, we have the following transversality condition (using (E3))

$$H(t,x,p,\phi) = 0 = p(t)\{\phi r_1 + (1 - \phi)r_2\} - p_3 p\{\phi q_1 + (1 - \phi)q_2\}. \quad \text{(E13)}$$

The optimal control is again given by (E4) and (E5). Again, it is impossible to have a singular solution to this problem.

We develop the solution by working backwards from the end of the problem at $t = T$. By the symmetry of the problem, it suffices to consider the case where $q_2 > q_1$. There are two further cases to consider depending on whether $r_2 > r_1$ or $r_1 > r_2$.

Case (a) $r_2 > r_1$ and $q_2 > q_1$

In this case (E13) and $p(t = T) = \varepsilon > 0$ yield

$$\phi[-(r_2 - r_1) + p_3(q_2 - q_1)] + r_2 - p_3 q_2 = 0. \quad \text{(E14)}$$

From the definition of this case, we have $\frac{r_2 - r_1}{q_2 - q_1} > 0$ with $q_2 > q_1$. It is easy to show that we must have $p_3(t) > 0$. We prove this by contradiction. Assume that we had $p_3(t) \leq 0$. Then we would have $p_3(t) \leq 0 < \frac{r_2 - r_1}{q_2 - q_1}$ so that by (E4) we obtain $\phi(t) = 0$. Substituting this in (E14) we obtain

$$p_3(t) = \frac{r_2}{q_2} > 0,$$

which contradicts our assumption. In particular, we must have $p_3(t = T) > 0$. There are two subcases to consider

Subcase (1) $\quad p_3(t = T) > \frac{r_2 - r_1}{q_2 - q_1}$

By (E4) we have $\phi(t = T) = 1$. We combine this with the transversality condition (E14) to obtain

$$p_3(t = T) = \frac{r_1}{q_1} > 0. \tag{E15}$$

This in turn generates further conditions as follows

$$\frac{r_1}{q_1} = p_3(t = T) > \frac{r_2 - r_1}{q_2 - q_1} \Rightarrow \frac{r_1}{q_1} > \frac{r_2}{q_2},$$

which is easily verified to be consistent with Case (a). Using the obtained control and backwards time $\tau = T - t$, we have up until the time of the first switch in tactics, τ_1, from (E2)

$$\frac{dp_3}{d\tau} = r_1 - p_3 q_1 \quad \text{with} \quad p_3(\tau = 0) = \frac{r_1}{q_1}.$$

Integration of the above readily yields

$$p_3(\tau) = \frac{r_1}{q_1} = \text{const.}$$

Thus, we have for $\frac{r_1}{q_1} > \frac{r_2}{q_2}$,

$$\phi(t) = 1 \quad \text{for} \quad 0 \leq t \leq T. \tag{E16}$$

Subcase (2) $\quad p_3(t = T) < \frac{r_2 - r_1}{q_2 - q_1}$

By (E4) we have $\phi(t = T) = 0$. We combine this with the transversality condition (E14) to obtain

$$p_3(t = T) = \frac{r_2}{q_2} > 0. \tag{E17}$$

This in turn generates further conditions as follows

$$\frac{r_2}{q_2} = p_3(t = T) < \frac{r_2 - r_1}{q_2 - q_1} \Rightarrow \frac{r_1}{q_1} < \frac{r_2}{q_2},$$

which is easily verified to be consistent with Case (a). Using the obtained control and backwards time $\tau = T - t$, we have up until the time of the first switch in tactics, τ_1, from (E2)

$$\frac{dp_3}{d\tau} = r_2 - p_3 q_2 \quad \text{with} \quad p_3(\tau = 0) = \frac{r_2}{q_2} .$$

Integration of the above readily yields

$$p_3(\tau) = \frac{r_2}{q_2} = \text{const.}$$

Thus, we have for $\frac{r_2}{q_2} > \frac{r_1}{q_1}$,

$$\phi(t) = 0 \quad \text{for} \quad 0 \leq t \leq T. \tag{E18}$$

Case (b) $r_2 < r_1$ and $q_2 < q_1$

From the definition of this case, we have $\frac{r_2 - r_1}{q_2 - q_1} < 0$ with $q_2 > q_1$. It is easy to show that we must have $p_3(t) > \frac{r_2 - r_1}{q_2 - q_1}$. We prove this by contradiction. Assume that we had $p_3(t) \leq \frac{r_2 - r_1}{q_2 - q_1}$. Then by (E4) we would have $\phi(t)$ so that (E14) would yield

$$p_3(t) = \frac{r_2}{q_2} > 0,$$

which contradicts our assumption. In particular, we must have $p_3(t = T) > \frac{r_2 - r_1}{q_2 - q_1}$ and hence $\phi(t = T) = 1$ by (E4). From (E14) we obtain

$$p_3(t = T) = \frac{r_1}{q_1} > 0.$$

This in turn generates a futher condition as follows

$$\frac{r_1}{q_1} = p_3(t = T) > \frac{r_2 - r_1}{q_2 - q_1} \Rightarrow \frac{r_1}{q_1} > \frac{r_2}{q_2} ,$$

which is easily verified to be consistent with Case (b). It is recognized that this case has turned out to be identical with Subcase (1) of Case (a). Thus, we have for $\frac{r_1}{q_1} > \frac{r_2}{q_2}$,

$$\phi(t) = 1 \quad \text{for} \quad 0 \leq t \leq T. \qquad \text{(E19)}$$

3. Diminishing Returns - Maximum Return for Specified Risk.

We consider

$$\underset{\phi(t)}{\text{maximize}} \int_0^T p(t)\{\phi r_1 x + (1 - \phi) r_2 y\} dt \quad \text{with} \quad T \quad \text{unspecified},$$

subject to:

$$\frac{dx}{dt} = -\phi r_1 x,$$

$$\frac{dy}{dt} = -(1 - \phi) r_2 y,$$

$$\frac{dp}{dt} = -p\{\phi q_1 + (1 - \phi) q_2\},$$

$x, y, p \geq 0$ and $0 \leq \phi \leq 1$,

with initial conditions

$$x(t = 0) = x_0, \quad y(t = 0) = y_0, \quad p(t = 0) = 1,$$

and terminal condition

$$p(t = T) = \varepsilon > 0 \quad (\text{also} \quad \varepsilon < 1).$$

The Hamiltonian, $H(t,x,p,\phi)$, is given by

$$H(t,x,p,\phi) = \phi[p\{r_1 x - r_2 y\} - p_1 r_1 x + p_2 r_2 y - p_3 p(q_1 - q_2)] + pr_2 y - p_2 r_2 y - p_3 p q_2, \qquad \text{(E20)}$$

and the optimal control (there is only one extremal) is determined from $\max_{\phi} H(t,x,p,\phi)$ or

$$\max[\phi\{pr_1x - p_1r_1x - p_3pq_1\} + (1 - \phi)\{pr_2y - p_2r_2y - p_3pq_2\}],$$

which yields the non-singular optimal control to be given by

$$\phi(t) = \begin{cases} 1 & \text{for } pr_1x - p_1r_1x - p_3pq_1 > pr_2y - p_2r_2y - p_3pq_2 \\ 0 & \text{for } pr_1x - p_1r_1x - p_3pq_1 < pr_2y - p_2r_2y - p_3pq_2 \end{cases} \qquad \text{(E21)}$$

From (E20) the adjoint equations for the dual variables are seen to be

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial x} = \phi r_1\{-p(t) + p_1(t)\} \qquad \text{with } p_1(t=T) = 0,$$

$$\frac{dp_2}{dt} = -\frac{\partial H}{\partial y} = (1 - \phi)r_2\{-p(t) + p_2(t)\} \qquad \text{with } p_2(t=T) = 0, \qquad \text{(E22)}$$

$$\frac{dp_3}{dt} = -\frac{\partial H}{\partial p} = -\phi r_1x-(1- \ )r_2y+p_3\{\phi q_1+(1-\phi)q_2\} \quad \text{with } p_3(t=T) \text{ unspecified.}$$

Since the Hamiltonian is a linear function of the control variable ϕ, the maximum principle does not determine the control when the coefficient of ϕ vanishes for a finite interval of time (see p. 481 of [6]). The part of a trajectory for which this happens is called a singular subarc. We determine the conditions for a singular subarc from [54]

$$\frac{\partial H}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = \frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right) = 0. \qquad \text{(E23)}$$

We should also note that since the terminal time is unspecified, we have from a transversality condition

$$H(t,x,p,\phi) = 0. \qquad \text{(E24)}$$

We have from (E20) that

$$\frac{\partial H}{\partial \phi} = p\{r_1x - r_2y\} - p_1r_1x + p_2t_2y - p_3p(q_1 - q_2). \qquad \text{(E25)}$$

A rather lengthy computation, which makes use of both the adjoint equations (E22) and the state equations, yields

$$\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = -p(q_2 r_1 x - q_1 r_2 y). \tag{E26}$$

By (E23) and (E26), we see that a condition for a singular subarc is that

$$\frac{r_1 x}{q_1} = \frac{r_2 y}{q_2} \tag{E27}$$

The singular control is determined from requiring that it keep us on the singular subarc. Thus, (E23) and (E26) yield (note that $\frac{dp}{dt} \neq 0$ and $p \neq 0$)

$$-q_2 r_1 \frac{dx}{dt} + q_1 r_2 \frac{dy}{dt} = 0$$

or using the state equations,

$$q_2 r_1 \, r_1 x - q_1 r_2 (1 - \phi) r_2 y = 0$$

or

$$\frac{r_1 x}{q_1} (\phi r_1) = \frac{r_2 y}{q_2} (1 - \phi) r_2 .$$

Using the fact that we are on a singular subarc so that (E27) holds, we obtain the singular control as

$$\phi = \frac{r_2}{r_1 + r_2} . \tag{E28}$$

A necessary condition for the singular subarc to yield a maximum return is that [57]

$$\frac{\partial}{\partial \phi} \left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right)\right\} \geq 0. \tag{E29}$$

From (E26) we have that

$$\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right) = \frac{d}{dt}(p\{-q_2 r_1 x + q_1 r_2 y\}) = \frac{dp}{dt}\{-q_2 r_1 x + q_1 r_2 y\} + p\{-r_1 q_2 \frac{dx}{dt} + r_2 q_1 \frac{dy}{dt}\},$$

or, using the state equations,

$$\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right) = -p\{\phi q_1 + (1-\phi)q_2\}(-q_2 r_1 x + q_1 r_2 y) + pr_1 q_2\ r_1 x - pr_2 q_1 (1 - \phi) r_2 y.$$

and hence

$$\frac{\partial}{\partial \phi}\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right)\} = p(-q_1 + q_2)(-q_2 r_1 x + q_1 r_2 y) + p(r_1)^2 q_2 x + p(r_2)^2 q_1 y.$$

On the singular subarc we must have (E27), so that the above reduces to

$$\frac{\partial}{\partial \phi}\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right)\} = p\{(r_1)^2 q_2 x + (r_2)^2 q_1 y\} > 0, \qquad \text{(E30)}$$

and the necessary condition is satisfied.

It is convenient to define (where τ is backwards time defined by $\tau = T - t$)

$$A(\tau) = pr_1 x - p_1 r_1 x - p_3 p q_1,$$

and

$$B(\tau) = pr_2 y - p_2 r_2 y - p_3 p q_2. \qquad \text{(E31)}$$

Then (E21) may be written as

$$\phi(\tau) = \begin{cases} 1 & \text{for } A(\tau) > B(\tau) \\ 0 & \text{for } A(\tau) < B(\tau), \end{cases} \qquad \text{(E32)}$$

with the singular control $\phi = \dfrac{r_2}{r_1 + r_2}$ for $A(\tau) = B(\tau)$. (E33)

Also

$$\frac{dA}{d\tau} = -\frac{dA}{dt} = \frac{d}{dt}(-pr_1x + p_1r_1x + p_3pq_1),$$

and a laborious computation, which makes use of both the adjoint equations (E22) and the state equations, yields

$$\frac{dA}{d\tau} = p(1-\phi)q_1q_2\left(\frac{r_1x}{q_1} - \frac{r_2y}{q_2}\right) . \tag{E34}$$

Similarly,

$$\frac{dB}{d\tau} = p\phi\, q_1q_2\left(\frac{r_2y}{q_2} - \frac{r_1x}{q_1}\right) . \tag{E35}$$

We develop the solution by working backwards from the end of the problem at $t = T$. We start by determining the boundary condition on p_3 at the end. There are two cases to be considered: either we are on a singular subarc at $t = T$ or we are not.

If we are on singular subarc, then by transversality condition (E24) and condition of singular subarc $\left(\frac{\partial H}{\partial \phi}\right) = 0$, we have

$$pr_2y - p_2r_2y - p_3pq_2 = 0 ,$$

which yields by use of the boundary conditions on (E22)

$$p_3(t = T) = \frac{r_2y(t=T)}{q_2} . \tag{E36}$$

We also note that on the singular subarc (E27) applies.

If we are not on singular subarc, then there are two further subcases: either $\phi(t = T) = 1$ or $\phi(t = T) = 0$. If $\phi(t = T) = 1$, then (E20), the transversality condition (E24), and the boundary conditions on (E22) yield

$$p_3(t = T) = \frac{r_1 x(t = T)}{q_1}. \tag{E37}$$

Since (t = T) = 1, then by (E21) and fact that $p_1(t = T) = p_2(t = T) = 0$ we have

$$pr_1 x - p_3 p q_1 > pr_2 y - p_3 p q_2,$$

and hence

$$\frac{r_1 x(t = T)}{q_1} > \frac{r_2 y(t = T)}{q_2}. \tag{E38}$$

A similar development shows that for $\phi(t = T) = 0$, we must have

$$\frac{r_1 x(t = T)}{q_1} < \frac{r_2 y(t = T)}{q_2}. \tag{E39}$$

We now trace the optimal trajectories backwards from the end. From the above, we have three cases to consider.

Case (1) at t = T, $\frac{r_1 x}{q_1} > \frac{r_2 y}{q_2}$

In this case by (E38) we have $\phi(t = T) = 1$. From (E21) and boundary conditions we have

$$A(\tau = 0) > B(\tau = 0).$$

Then up until the time τ_1 of the first switch in tactics we have from (E34) and (E35)

$$\frac{dA}{d\tau} = 0,$$

and

$$\frac{dB}{d\tau} = pq_1q_2\left(\frac{r_2y}{q_2} - \frac{r_1x}{q_1}\right) < 0,$$

and hence

$$A(\tau) = A(\tau = 0) > B(\tau = 0) > B(\tau).$$

Thus, we have

$$\phi(t) = 1 \quad \text{for} \quad 0 \le t \le T. \tag{E40}$$

Case (2) at $t = T$, $\dfrac{r_1x}{q_1} < \dfrac{r_2y}{q_2}$

A similar argument shows that

$$\phi(t) = 0 \quad \text{for} \quad 0 \le t \le T. \tag{E41}$$

Case (3) at $t = T$, $\dfrac{r_1x}{q_1} = \dfrac{r_2y}{q_2}$

We see that this corresponds to when the system ends up on the singular subarc at $t = T$. In this case $\phi(t = T) = \dfrac{r_2}{r_1 + r_2}$, and we continue (in backwards progression) to use the singular control $\phi(t) = r_2/(r_1 + r_2)$ (note that $\dfrac{dA}{d\tau} = \dfrac{dB}{d\tau} = 0$ when this is used and that we had $A(\tau = 0) = B(\tau = 0)$) until $x(t) = x_0$ or $y(t) = y_0$. This yields three further subcases.

Subcase (3A) $\dfrac{r_1x_0}{q_1} = \dfrac{r_2y_0}{q_2}$

We use $\phi(t) = r_2/(r_1 + r_2)$ from the beginning.

Subcase (3B) $\dfrac{r_1x_0}{q_1} > \dfrac{r_2y_0}{q_2}$

Define t_1 as t such that $y(t_1 > 0) = y_0$. Then we use $\phi(t) = 1$ for $0 \le t \le t_1$. This is consistent since $A(\tau = T - t_1) = B(\tau = T - t_1)$. Then up until the time τ_2 of the next switch in tactics we have from (E34) and (E35)

$$\frac{dA}{d\tau} = 0,$$

and

$$\frac{dB}{d\tau} = pq_1q_2\left(\frac{r_2y}{q_2} - \frac{r_1x}{q_1}\right) < 0,$$

and hence

$$A(\tau) = A(\tau = T - t_1) = B(\tau = T - t_1) > B(\tau).$$

From (E32) we see that

$$\phi(\tau) = 1 \quad \text{for} \quad T - t_1 \le \tau \le T. \tag{E42}$$

Subcase (3C) $\quad \dfrac{r_1x_0}{q_1} < \dfrac{r_2y_0}{q_2}$

A similar argument as that for Subcase (3B) with the roles of x and y interchanged readily shows that

$$\phi(\tau) = 0 \quad \text{for} \quad T - t_1 \le \tau \le T. \tag{E43}$$

Note that in the above developments we have implicitly made use of the non-negativity of the state variables.

4. Diminishing Returns - Prescribed Duration Use.

We consider

$$\underset{\phi(t)}{\text{maximize}} \int_0^T p(t)\{\phi r_1x + (1 - \phi)r_2y\}dt \quad \text{with } T \text{ specified},$$

subject to: $\frac{dx}{dt} = -\phi r_1 x,$

$$\frac{dy}{dt} = -(1 - \phi) r_2 y,$$

$$\frac{dp}{dt} = -p\{\phi q_1 + (1 - \phi) q_2\},$$

$x,y,p \geq 0$ and $0 \leq \phi \leq 1$,

with initial conditions

$$x(t = 0) = x_0, \quad y(t = 0) = y_0, \quad p(t = 0) = 1.$$

The development of the solution to this problem is similar to that of maximizing return for a specified risk. We have considered the latter problem in Section b3. above. Two main differences between these problems are that (1) the boundary conditions on the dual variables at $t = T$ are slightly different and (2) for the present problem the total time is specified so that the transversality condition $H(t = T,x,p,\phi) = 0$ no longer is applicable. In view of the similarities, we shall frequently summarize results from the previous problem which apply to this one. The interested reader can, of course, refer to the previous problem for full details.

The Hamiltonian, $H(t,x,p,\phi)$, is given by

$$H(t,x,p,\phi) = \phi[p\{r_1 x - r_2 y\} - p_1 r_1 x + p_2 r_2 y - p_3 p(q_1 - q_2)] + pr_2 y - p_2 r_2 y - p_3 p q_2. \quad \text{(E44)}$$

The adjoint equations for the dual variables are the same as (E22) with the exception that the boundary conditions at $t = T$ are now

$$p_1(t = T) = 0, \quad p_2(t = T) = 0, \quad p_3(t = T) = 0. \quad \text{(E45)}$$

The non-singular control obtained by maximizing the Hamiltonian is given by (where, as before, τ is the backwards time defined by $\tau = T - t$)

$$\phi(\tau) = \begin{cases} 1 & \text{for } A(\tau) > B(\tau) \\ 0 & \text{for } A(\tau) < B(\tau), \end{cases} \tag{E46}$$

where

$$A(\tau) = pr_1x - p_1r_1x - p_3pq_1,$$

$$B(\tau) = pr_2y - p_2r_2y - p_3pq_2. \tag{E47}$$

As above, it may also be shown that

$$\frac{dA}{d\tau} = p(1 - \phi)q_1q_2\left(\frac{r_1x}{q_1} - \frac{r_2y}{q_2}\right),$$

and

$$\frac{dB}{d\tau} = p\phi q_1q_2\left(\frac{r_2y}{q_2} - \frac{r_1x}{q_1}\right). \tag{E48}$$

It is convenient for a later development to define

$$D(\tau) = A(\tau) - B(\tau), \tag{E49}$$

so that (E46) becomes

$$\phi(\tau) = \begin{cases} 1 & \text{for } D(\tau) > 0 \\ 0 & \text{for } D(\tau) < 0. \end{cases} \tag{E50}$$

Using (E48) and (E49) we readily obtain

$$\frac{dD}{d\tau} = pq_1q_2\left(\frac{r_1x}{q_1} - \frac{r_2y}{q_2}\right), \tag{E51}$$

with

$$D(\tau = 0) = p(r_1x - r_2y), \tag{E52}$$

where we have made use of (E45) besides obvious definitions.

Since the Hamiltonian is a linear function of the control variable ϕ, the maximum principle does not determine the control when the coefficient of ϕ vanishes for a finite interval of time (see p. 481 of [6]). We recall that the part of an optimal trajectory for which this happens is called a singular aubarc. As in the previous problem on a singular subarc we have

$$\frac{r_1x}{q_1} = \frac{r_2y}{q_2}, \tag{E53}$$

with the singular control to remain on it given by

$$\phi = \frac{r_2}{r_1 + r_2}. \tag{E54}$$

Again, it is readily verified that the necessary condition for the singular subarc to yield a maximum return [57] is met.

Let us now examine the determination of the optimal control at the end of the problem $t = T$ or $\tau = 0$. Substituting the boundary conditions (E45) into (E47), we obtain

$$A(\tau = 0) = pr_1x,$$

and

$$B(\tau = 0) = pr_2y, \tag{E55}$$

and hence (E46) becomes

$$\phi(t = T) = \begin{cases} 1 & \text{for } r_1x(T) > r_2y(T) \\ 0 & \text{for } r_1x(T) < r_2y(T). \end{cases} \tag{E56}$$

In contrasting the optimal trajectories and tracing the optimal course of the bomber utilization (backwards from the end of the prescribed duration period of usage) it is convenient to consider the following. We recall that the optimal control is determined by the sign of $D(\tau)$ (see (E50), (E49), and (E47)). From (E53) a singular subarc must occur on the line L defined by $\frac{r_1 x}{q_1} = \frac{r_2 y}{q_2}$. We recall that at the end of the planning horizon $\tau = 0$, we have

$$D(\tau = 0) = p(t = T)\{r_1 x(t = T) - r_2 y(t = T)\} .$$

Consider now the line L' defined by $r_1 x = r_2 y$. This line will lie above, on, or below the line L defined by $\frac{r_1 x}{q_1} = \frac{r_2 y}{q_2}$ depending on whether q_1 is greater than, equal to, or less than q_2. This is evident from considering the slopes of these two lines which pass through the origin

$$\left(\frac{dy}{dx}\right)_L = \frac{r_1}{q_1} \cdot \frac{q_2}{r_2} , \qquad \left(\frac{dy}{dx}\right)_{L'} = \frac{r_1}{r_2} ,$$

and hence, for example,

$$\left(\frac{dy}{dx}\right)_{L'} > \left(\frac{dy}{dx}\right)_L \quad \text{for } q_1 > q_2.$$

The significance of the line L' and its relationship to the line L is that

$$D(\tau = 0) \begin{cases} > 0 & \text{below } L' \\ < 0 & \text{above } L', \end{cases} \tag{E57}$$

and hence by (E50) we find that

$$\phi(t = T) = \begin{cases} 1 & \text{for } P(T) \text{ below } L' \\ 0 & \text{for } P(T) \text{ above } L', \end{cases} \tag{E58}$$

where $P(t = T) = (x(t = T), y(t = T))$. We also note from (E51) that

$$\frac{dD(\tau)}{d\tau} \begin{cases} > 0 & \text{below } L \\ < 0 & \text{above } L. \end{cases} \tag{E59}$$

Thus, (E59) and (E59) give us three cases to consider

Case (a) $q_1 = q_2 = q$,

Case (b) $q_1 > q_2$,

Case (c) $q_1 < q_2$.

For Case (a): $q_1 = q_2 = q$, equation (E51) and initial condition (E52) are

$$\frac{dD}{d\tau} = pq(r_1x - r_2y)$$

with

$$D(\tau = 0) = p(r_1x - r_2y).$$

There are three cases to consider depending on the sign of $D(\tau = 0)$.

Case (1) $r_1x(t = T) = r_2y(t = T)$

We see that this corresponds to when the system ends up on the singular subarc, i.e., $D(\tau = 0) = 0$. In this case $\phi(t = T) = \dfrac{r_2}{r_1 + r_2}$, and we continue (in backwards progression) to use the singular control $\phi(t) = r_2/(r_1 + r_2)$ to remain on $\dfrac{r_1x}{q_1} = \dfrac{r_2y}{q_2}$ (note that this makes

$\frac{dD}{d\tau} = 0$ and that we had $D(\tau = 0) = 0$) until $x(t) = x_0$ or $y(t) = y_0$. This yields three further subcases.

Subcase (1A) $\quad r_1x_0 < r_2y_0$

Define t_1 as t such that $x(t_1 > 0) = x_0$. Then we use $\phi(t) = 1$ for $0 \leq t \leq t_1$. This is consistent by the following. At $\tau = T - t_1$, we have $D(\tau = T - t_1) = 0$ and up until the time τ_2 of the next switch in tactics we have

$$\frac{dD}{d\tau} = pq(r_1x_0 - r_2y(\tau)) < 0,$$

for $T - t_1 \leq \tau \leq T$ and hence

$$0 = D(\tau = T - t_1) > D(\tau).$$

From (E50) we see that

$$\phi(\tau) = 0 \quad \text{for} \quad T - t_1 \leq \tau \leq T. \tag{E61}$$

Subcase (1B) $\quad r_1x_0 > r_2y_0$

A similar argument as that for Subcase (1A) with the roles of x and y interchanged readily shows that

$$\phi(\tau) = 1 \quad \text{for} \quad T - t_1 \leq \tau \leq T. \tag{E62}$$

Subcase (1C) $\quad r_1x_0 = r_2y_0$

We use $\phi(t) = r_2/(r_1 + r_2)$ from the beginning.

Case (2) $\quad r_1x(t = T) < r_2y(t = T)$

In this case we have $D(\tau = 0) = p\{r_1x(t = T) - r_2y(t = T)\} < 0$, and by (E50) at the end of the planning horizon we have $\phi(\tau = 0) = 0$ so that $y(\tau = 0) < y(\tau)$ for $\tau > 0$. Thus we have until the time τ_1 of the first switch in tactics

$$\frac{dD}{d\tau} = pq\{r_1 x(t = T) - r_2 y(\tau)\} < 0,$$

for $0 \le \tau \le \tau_1$ and hence

$$0 > D(\tau = 0) > D(\tau).$$

From (E50) we see that

$$\phi(t) = 0 \quad \text{for} \quad 0 \le t \le T. \tag{E63}$$

<u>Case(3)</u> $\underline{r_1 x(t = T) > r_2 y(t = T)}$

A similar argument as that for Case (2) with the roles of x and y interchanged readily shows that

$$\phi(t) = 1 \quad \text{for} \quad 0 \le t \le T. \tag{E64}$$

We now consider Case (b): $q_1 > q_2$. There are two cases to be considered.

<u>Case (1) never on singular subarc for finite interval of time</u>

Again there are two subcases to consider, depending on whether the system winds up above or below L.

<u>Subcase (1a)</u> $\dfrac{r_1 x(t = T)}{q_1} > \dfrac{r_2 y(t = T)}{q_2}$

The definitions of Case (b) and Subcase (1a) imply

$$\frac{r_1 x(t = T)}{r_2 y(t = T)} > \frac{q_1}{q_2} > 1,$$

so that we have

$$r_1 x(\tau = 0) > r_2 y(\tau = 0).$$

Thus by (E52) $D(\tau = 0) > 0$ and hence by (E50) $\phi(t = T) = 1$. We consider now the τ-time interval up until the time τ_1 of the first switch in tactics. Use of $\phi(\tau) = 1$ for $\tau\varepsilon[0,\tau_1]$ results in $x(\tau) > x(\tau = 0)$ for $\tau > 0$. Recalling that

$$\frac{dD}{d\tau} = pq_1q_2\left(\frac{r_1x(\tau)}{q_1} - \frac{r_2y(\tau = 0)}{q_2}\right),$$

for $\tau\varepsilon[0,\tau_1]$ and the definition of this case, we easily see that $\frac{dD}{d\tau} > 0$ and hence

$$0 < D(\tau = 0) < D(\tau).$$

From (E50) we see that

$$\phi(t) = 1 \quad \text{for} \quad 0 \leq t \leq T. \tag{E65}$$

<u>Subcase (1b)</u> $\frac{r_1x(t = T)}{q_1} < \frac{r_2y(t = T)}{q_2}$

Again there are two further subcases to consider, depending on whether the system winds up above or below L'.

<u>Subcase (1bI)</u> $\frac{r_1x(t = T)}{q_1} < \frac{r_2y(t = T)}{q_2}$ and $\underline{r_1x(t = T) < r_2y(t = T)}$

In this case we wind up above L'. Since $D(\tau = 0)$ is given by (E52), we have $D(\tau = 0) < 0$ and hence by (E50) $\phi(\tau = 0) = 0$. Since we are initially above L and remain so by use of $\phi(\tau) = 0$, we have by (E59) $\frac{dD}{d\tau} < 0$ for all $\tau\varepsilon[0,T]$ and hence $D(\tau) < 0$ for all τ. Thus we have

$$\phi(t) = 0 \quad \text{for} \quad 0 \leq t \leq T. \tag{E66}$$

Subcase (1bII) $\frac{r_1x(t = T)}{q_1}$ $\frac{r_2y(t = T)}{q_2}$ and $r_1x(t = T)$

$r_2y(t = T)$

In this case we wind up below L' at the end. Since $D(\tau = 0)$ is given by (E52), we have $D(\tau = 0) > 0$ and hence by (E50) $\phi(\tau = 0) = 1$. We work backwards from the end. Since we are above L, $\frac{dD}{d\tau} < 0$ while we remain above L. Thus $D(\tau)$ decreases for $\tau > 0$ while we remain above L. There are two further subcases depending on whether $D(\tau)$ decreases to zero before the line L is encountered. Let τ_1 be such that $D(\tau_1) = 0$. If L has not yet been reached at τ_1, then $D(\tau)$ for $\tau > \tau_1$ is negative and $\phi(\tau) = 0$ until the beginning of battle. It is also possible that the system just reaches L the instant that $D(\tau_1) = 0$. In this case (assuming we don't remain on singular subarc) $D(\tau) > 0$ for $\tau > \tau_1$, since we pass below L and then $\frac{dD}{d\tau} > 0$.

Case (2) on singular subarc for finite interval of time

This can happen only when $\frac{r_1x(t = T)}{q_1} < \frac{r_2y(t = T)}{q_2}$ and $r_1x(t = T) > r_2y(t = T)$. As usual, we work backwards from the end of the planning horizon. We use $\phi(\tau) = 1$ for $0 \leq \tau \leq \tau_1$, and at $\tau = \tau_1$ we must have $\frac{r_1x(\tau_1)}{q_1} = \frac{r_2y(\tau_1)}{q_2}$. We use the singular control $\phi(\tau) = r_2/(r_1 + r_2)$ for $\tau_1 \leq \tau \leq \tau_2$. There are three further subcases

(1) $x(\tau_2) = x_0$, $y(\tau_2) < y_0$,

(2) $x(\tau_2) < x_0$, $y(\tau_2) = y_0$,

(3) $x(\tau_2) = x_0$, $y(\tau_2) = y_0$.

We omit the trivial discussion of these cases.

Thus, to summarize, we see that there are six possible cases for the history of the strategic worth of the two target areas in the use of the bomber for a prescribed length of time:

(1) started below L and never reached L,

(2) always above L',

(3) started above L' and end up above L but below L' without ever reaching L,

(4) end up above L but started below L and did not remain on L for finite interval of time,

(5) started above (or on) L and were on L for finite interval of time,

(6) started below L and were on L for finite interval of time.

Case (c): $q_1 < q_2$ is similar to Case (b).

c. Summary of Solutions.

In this section we summarize the solutions developed in the previous section for the four versions of the continuous stochastic gold-mining problem. We shall summarize the cases of non-diminishing and diminishing returns separately.

The solution for the case of non-diminishing returns is shown in Table EI. We note that for both cases considered the optimal policy is independent of the current strategic values of the two target areas, i.e., the state variables. For the case of maximizing the return for a specified risk, the optimal policy is independent of the risk (cumulative probability of bomber being shot down) and depends only on the ratios of $\frac{r_i}{q_i}$ which we may interpret as the expected gain per unit time divided by the expected loss per unit time.

Table EI. Solution to Continuous Stochastic Gold-Mining Problem
Non-Diminishing Returns

A. Maximum Return for Specified Risk

Case	Optimal Policy
$1 : \frac{r_1}{q_1} > \frac{r_2}{q_2}$	$\phi(t) = 1$ for $0 \le t \le T$
$2 : \frac{r_1}{q_1} < \frac{r_2}{q_2}$	$\phi(t) = 0$ for $0 \le t \le T$

B. Prescribed Duration Use (Basic assumption $q_2 > q_1$)

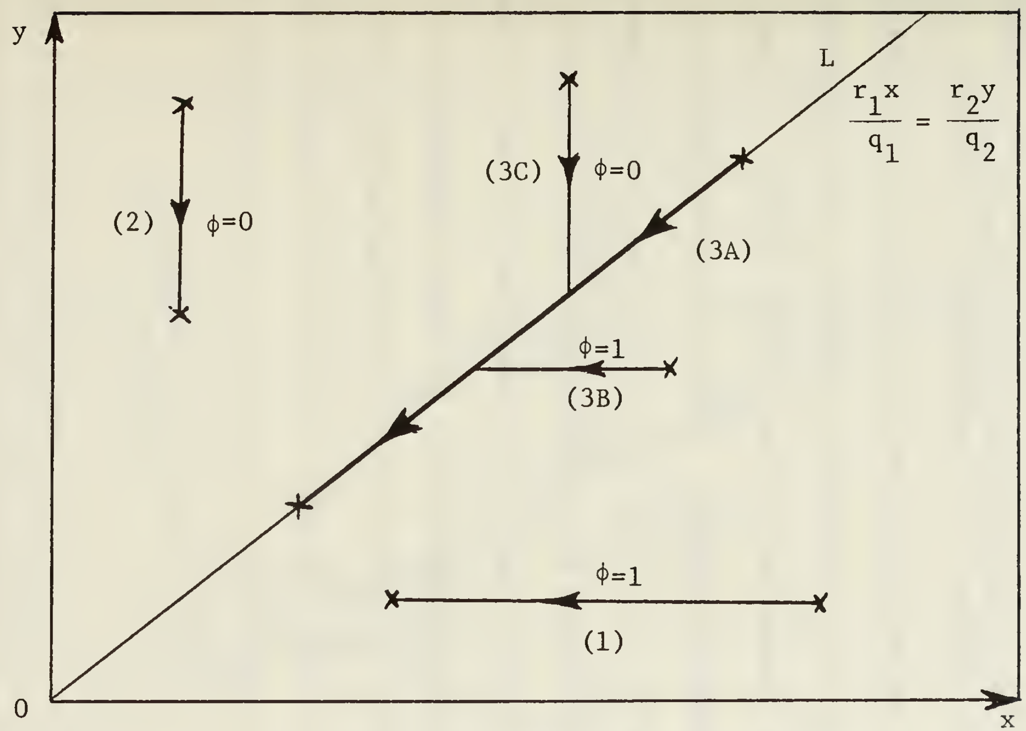
$1 : r_2 > r_1$ and $\frac{r_1}{q_1} < \frac{r_2}{q_2}$	$\phi(t) = 0$ for $0 \le t \le T$
$2 : r_2 < r_1$ and $\frac{r_1}{q_1} > \frac{r_2}{q_2}$	$\phi(t) = 1$ for $0 \le t \le T$
$3 : r_2 > r_1$ and $\frac{r_1}{q_1} > \frac{r_2}{q_2}$	(a) for $\tau_1 \ge T$ $\phi(t) = 0$ for $0 \le t \le T$ (b) for $\tau_1 < T$ $\phi(t) = 1$ for $0 \le t \le T - \tau_1$ $\phi(t) = 0$ for $T - \tau_1 \le t \le T$

Note: τ_1 is given by $\tau_1 = \ln\left\{\frac{r_2(q_2 - q_1)}{q_2 r_1 - q_1 r_2}\right\}^{1/q_2}$

For the case of prescribed duration use with non-diminishing returns, we consider the case of $q_2 > q_1$ with the other case being similar with the roles of x and y interchanged. The condition $q_2 > q_1$ means that there is a larger risk per unit time of the bomber being lost over the second target area. Consider the planning horizon of length T. During the closing stages of length τ_1 of this bombing campaign, we send the bomber to the target area of greater return per unit time regardless of the risk. The length of this interval, τ_1, is, of course, dependent on the risks involved and will be shorter as the chances of the bomber being shot down over target area two become greater. During the initial stages of the bombing campaing, i.e., for $0 \le t \le T - \tau_1$, we allocate the bomber giving consideration to the risks, and the solution is identical to the previous case.

When there are diminishing returns, the solution is seen to depend on the strategic values of the target areas. Consequently, we have chosen to plot the optimal policies as a function of the state variables.

The case of maximizing return for a specified risk with diminishing returns is shown in Figure E1. It is seen that the line L defined by $\frac{r_1 x}{q_1} = \frac{r_2 y}{q_2}$ plays a central role in the solution. We may interpret a quotient like $\frac{r_1 x}{q_1}$ as representing the expected return per unit time divided by the expected loss per unit time for operating in the target area. Another way to do this is return per unit cost per unit time. The optimal policy is to send the bomber to the target area which maximizes the return per unit risk (cost). In this respect this solution is identical to that of non-diminishing returns except now, of course,



Note: On line L $\frac{r_1 x}{q_1} = \frac{r_2 y}{q_2}$, use policy $\phi = \frac{r_2}{r_1 + r_2}$.

Figure E1. Solution to Stochastic Gold-Mining Problem with Diminsihing Returns
Maximum Return for Specified Risk
(also Prescribed Duration Use for $q_1 = q_2$)

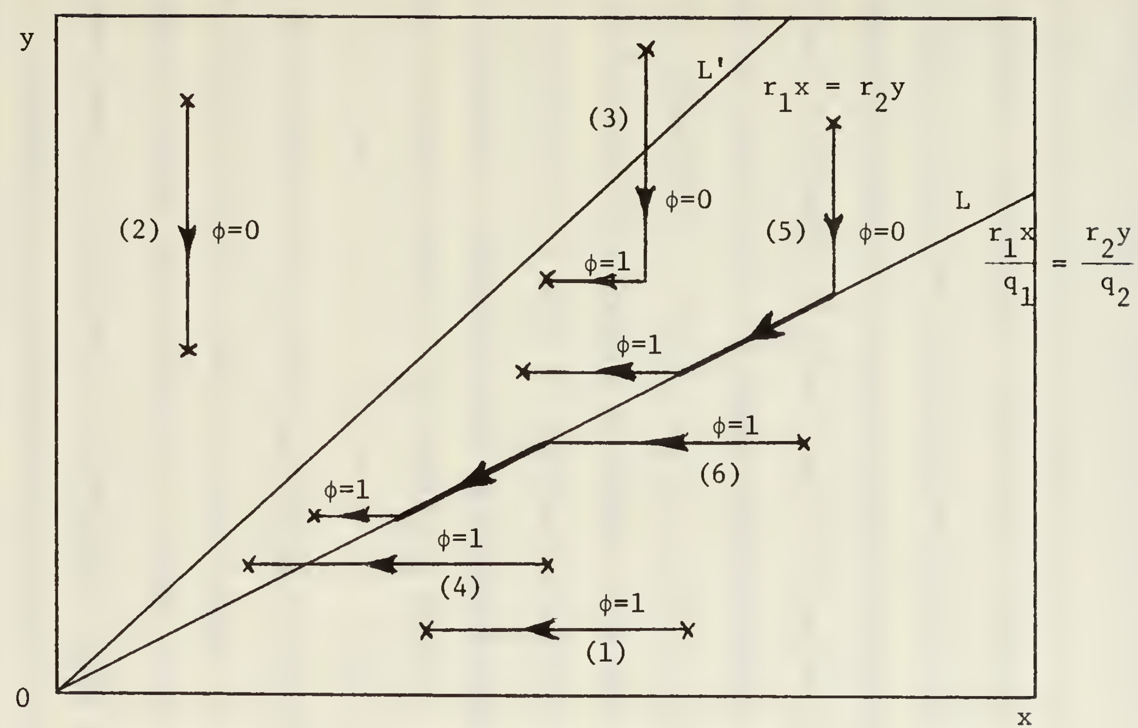
the expected return per unit time depends on the strategic value of the target area. The paths labelled on Figure E1 correspond to the nomenclature of Section b3. above. We note that this solution is the same as that for prescribed duration use when $q_1 = q_2$, i.e., there is equal risk of losing the bomber in the two target areas.

For the case of prescribed duration use with diminishing returns there are three cases to consider. The solution for Case (a): $q_1 = q_2$ is the same as that for maximizing return for specified risk as discussed above. The case when $q_1 > q_2$ is shown in Figure E2. The paths are denoted according to our terminology of Section b4. Again, consider the total time of the bombing campaign. During the early stages we allocate giving consideration to risks, but during the closing stages, the bomber is sent to the target area yielding the greater return per unit time (as measured by r_1x and r_2y) regardless of risk. Although we have not made an explicit determination, it seems reasonable to conjecture by analogy with the case of non-diminishing returns that the greater the risk at target area one, the shorter this interval will be. During the previous period, i.e., $0 \leq t \leq T - \tau_1$, the bomber is allocated on the basis of return per unit cost as before.

d. Discussion.

We have already noted for the non-diminishing returns the allocation is independent of the state variables and effort is concentrated on one alternative, whereas for diminishing returns the values of the state variables must be considered and effort may be split over the alternatives. We shall point out some similarities with the combat allocation models of Appendix C and then attempt some generalizations.

Case (b) $q_1 > q_2$



Note: On L $\frac{r_1 x}{q_1} = \frac{r_2 y}{q_2}$, use policy of $\phi = \frac{r_2}{r_1 + r_2}$.

Figure E2. Solution to Stochastic Gold-Mining Problem with Diminishing Returns Prescribed Duration Use

We should note the similarity of the structure of the optimal allocation policies with that in selection of target type in combat described by Lanchester-type equations. There appears to be an underlying structure for allocation with diminishing returns and allocation with non-diminishing returns. Let us recall that for a square law attrition process, the attrition (return) per unit time per unit of weapon system is a constant; whereas for a linear law attrition process, the attrition (return) per unit time per unit of weapon system is proportional to the number of targets remaining (diminishing returns). This observation has prompted our conclusion in Appendix C that fire is concentrated on a single target type only when the fire is "aimed" and the target acquisition rate is not subject to diminishing returns.

We also note that the termination conditions of the scenario (prescribed time or use until reach given level of risk) has an effect upon the optimal allocation policy. We have noted in Appendix C a similar result for tactical allocation in combat described by Lanchester-type equations.

When we compare the results from the Lanchester attrition models to the stochastic gold-mining problems, the allocation appears to be different when one is not subject to a cost (loss) from the alternative not being used. It seems appropriate to consider in future work this type of attrition model to see what insight may be provided.

We seem to have uncovered a general principle (although we most likely are not the first) that allocation in the face of non-diminishing returns and diminishing returns are two fundamentally different cases. With diminishing returns, we must constantly observe the state of our system.

APPENDIX F. A New Dynamic Kill Potential.

In this appendix we propose a dynamic measure of combat capability by means of the adjoint system of differential equations for Lanchester-type equations of combat. The current results are of a preliminary nature and may be revised in the future.

What is a quantitative measure of effectiveness for a combat unit or weapon system? In many circumstances it appears to be the rate of destruction of the enemy. A more sophisticated approach is to consider the rate of destruction of enemy capability as measured by the rate of destruction of his kill rate against the friendlies.

We have devised a simple way to determine a dynamic kill potential which is the rate of destruction of enemy kill rate giving full consideration to the future course of combat. Consider a weapon system of constant kill rate capability employed in combat against an enemy. The loss of such a weapon is weighted more heavily in the early stages than in later ones. This is because of the "multiplying effect" of the dynamics of combat, i.e., loss of a weapon is also loss of future killing capability of the weapon.

Such a concept has application to force structuring and weapon system analysis. In such work, frequently a large number of alternatives have to be screened. It is infeasible to assess the effectiveness for all the alternate force/weapons mixes by a computer simulation of a standardized scenario. The concept of firepower scores and weapon firepower potential have been developed to screen out unattractive alternatives in preliminary analyses. We have extended these concepts to consider the true dynamics of combat. Originally we were motivated by the interpretation of the adjoint system of differential equations in optimal control theory.

In this appendix we state the problem, give some additional background, and then propose our solution. We then comment on other applications of these ideas before presenting a brief justification of our concept. Finally, we point out the deep relationship of this seemingly simple notion to linear analysis.

This is our initial effort on this problem from a purely mathematical point of view. For the future, we would propose to compare firepower potentials computed by current methods and by our new method and also to improve and expand the exposition. We are currently supervising a student thesis on this topic from a more applied standpoint ("Weapon Firepower Potential" by Major James B. Taylor, USA).

a. Statement of the problem.

To devise a quantitative measure of the combat capability of a unit/weapon system giving consideration to the dyanmics of combat.

b. Some Background.

We could consider a "static" kill potential, the rate of destruction of the enemy kill rate against the friendlies not considering the future course of battle. The concept of firepower scores has evolved into the notion of weapon firepower potential. The latter considers attrition rates as we have indicated but in a "static" fashion. In practice, analysts use operational ammunition consumption rates and operational kill/hit probabilities to estimate attrition rates. Information systems have been designed to make available such information on various systems in numerous circumstances. A high degree of sophistication is not warranted for estimation of kill rates because of the uncertainty in the data.

The current approach to weapon firepower potential does attempt to consider combat dynamics in the following fashion: kill rates are weighted more heavily at the longer ranges. This recognizes the advantage of destroying the enemy at longer ranges before he becomes more effective at killing friendlies at the closer ranges.

What we need is a measure which considers the dynamics of combat: losses early in battle effect the outcome by evolving into more enemy survivors and less friendlies. In the next section we show how to use the concepts of operational definition and adjoint system of differential equations to account for combat dynamics.

c. The Proposed Solution.

We employ the concept of an operational definition (see Chapter 5 in [1]) by defining a dynamic firepower potential of a unit/weapon system under precise circumstances. Numerical measures can only be meaningfully compared under the applicable circumstances.

We consider a standardized scenario of combat between an X-force and a Y-force in a battle lasting a prescribed time T. For illustrative purposes we consider the case of constant attrition rates. Our approach explained in Appendix D allows many variable attrition rate cases to be solved in closed form. This approach applies equally well to the adjoint system of differential equations considered here.

We consider the rate of return of a unit/weapon system (in terms of destruction of enemy kill rate) as measured by the product of a measure of enemy kill-rate worth and the enemy attrition rate by the friendlies. In many circumstances these quantities will have to be properly weighted averages. There is also the problem of combat between heterogeneous forces. Such considerations are beyond the scope of our simple illustrative example.

We define the dynamic firepower potential, F.P., as

$$\text{F.P.} = ap_1, \tag{F1}$$

where

a is the rate of attrition achieved by the unit/weapon system, and

p_1 is the unit worth of enemy forces as measured by the rate of change of the value of engagement in a standardized scenario.

An average firepower potential would be given by

$$\overline{\text{F.P.}} = \frac{1}{T}\int_0^T a(t)p_1(t)dt. \tag{F2}$$

We shall see that $p_1(t)$ is a variable dual to the state variables, x and y, which describe the course of combat as a sequence of points for average force strength.

We consider now a battle lasting from $t = 0$ until $t = T$ with the combat described by

$$\frac{dx}{dt} = -ay,$$

$$\frac{dy}{dt} = -bx, \tag{F3}$$

which we may write as

$$\frac{d\vec{X}}{dt} = \begin{pmatrix} 0 & -a \\ -b & 0 \end{pmatrix}\vec{X}, \tag{F4}$$

where $\vec{X}$ is a column vector of average force strengths, i.e., $\vec{X} = \binom{x}{y}$. The adjoint system of differential equations for (F4) is

$$\frac{d\vec{P}}{dt} = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} \vec{P}, \tag{F5}$$

where $\vec{P} = \binom{p_1}{p_2}$.

What is our motivation for considering the adjoint system of differential equations? The transposed system of equations has long been used to study the consistency (solvability) of a system of linear equations. If we were to use finite differences to approximate the Lanchester-type equations (F3), we would obtain a system of linear equations. Forming the transposed system and passing to the limit, we obtain the adjoint system. Usually, one develops the adjoint system by integrating by parts, but we feel that these considerations here provide more insight.

We may also write (F5) as

$$\frac{dp_1}{dt} = bp_2,$$

$$\frac{dp_2}{dt} = ap_1. \tag{F6}$$

Let us now multiply the first of (F3) by p_1, the second by p_2, and add to obtain

$$p_1 \frac{dx}{dt} + p_2 \frac{dy}{dt} = p_1(-ay) + p_2(-bx).$$

Similarly for (F6)

$$x \frac{dp_1}{dt} + y \frac{dp_2}{dt} = x(bp_2) + y(ap_1).$$

Hence

$$p_1 \frac{dx}{dt} + p_2 \frac{dy}{dt} + x \frac{dp_1}{dt} + y \frac{dp_2}{dt} = 0 = \frac{d}{dt}(xp_1 + yp_2),$$

or

$$\frac{d}{dt}(\vec{X} \cdot \vec{P}) = 0,$$

and hence

$$\vec{X}(t) \cdot \vec{P}(t) = \text{const.} \tag{F7}$$

We may interpret this last condition as a compatability requirement which implies that if initial conditions are given for $\vec{X}$, then the only appropriate boundary condition for $\vec{P}$ is at $t = T$. Hence, we specify the following conditions for (F6)

$$p_1(t = T) = A \ , \ p_2(t = T) = B, \tag{F8}$$

and thus, letting $\tau = T - t$, the solution to (F6) and (F8) is given by

$$p_1(\tau) = A \cosh\sqrt{ab}\ \tau - B\sqrt{\frac{b}{a}} \sinh \sqrt{ab}\ \tau,$$

and

$$p_2(\tau) = B \cosh\sqrt{ab}\ \tau - A\sqrt{\frac{a}{b}} \sinh \sqrt{ab}\ \tau. \tag{F9}$$

Let us call V the value of engagement given by

$$V = x(T)p_1(T) + y(T)p_2(T) = x(t)p_1(t) + y(t)p_2(t). \tag{F10}$$

Hence we see that

$$p_1(t) = \frac{\partial V}{\partial x}(t),$$

and

$$p_2(t) = \frac{\partial V}{\partial y}(t). \tag{F11}$$

We call p_1, p_2 dual variables, and they determine the combat's trajectory in terms of line coordinates, whereas the state variables, x and y, determine it in terms of point coordinates.

We have noted in dynamic tactical allocation models that if surviving forces at $t = T$ are assigned a worth proportional to their kill rate, then target selection depends on the product of kill rates (target and firer). This has influenced our definition of dynamic kill potential.

d. Some Comments.

The above is the same approach used by G. Bliss in developing range tables for correcting artillery fire due to abnormal air densities, weights of projectiles, winds, etc., shortly after World War I [17], [67]. We may think of the p's (dual variables) as the line coordinates of the trajectory (path) of the battle represented by (F3), i.e., $x = x(t)$ and $y = y(t)$ (the solution to (F3)) defines a curve in the x,y space. The duality of Euclidean geometry (after adding the ideal point at infinity) states that we may equally well represent a curve as either a sequence of points (point coordinates) or as an envelope of tangents (line coordinates). When points are transformed by a linear transformation, the line coordinates are transformed by the transposed (or dual) matrix of this transformation. Let us note that we may consider a linear differential equation to be the limit of linear equations.

e. Justification.

We may use the condition $\vec{X} \cdot \vec{P} = \text{const.}$ to develop justification for calling p_1 the rate of change of the value of the engagement with respect to X forces, $\frac{\partial V}{\partial x}$. Consider a battle lasting a specified length of time T. Hence, we have

$$x(t)p_1(t) + y(t)p_2(t) = x(T)p_1(T) + y(T)p_2(T). \qquad \text{(F12)}$$

If at time t the X commander had $\Delta x(t)$ less troops, then this would cause him to have less surviving troops at the end of battle and the enemy (Y) to have more. In fact, the p's tell us how much as we see below

$$(x(t) - \Delta x(t))p_1(t) + y(t)p_2(t) = (x(T) - \Delta x(T))p_1(T) + (y(T) + \Delta y(T))p_2(T). \qquad \text{(F13)}$$

Combining (F12) and (F13), we obtain

$$\Delta x(t)p_1(t) = \Delta x(T)p_1(T) - \Delta y(T)p_2(T).$$

Letting $p_1(T) = 1$ and $p_2(T) = -1$, we see why I have referred to the p's as the value of forces

$$\Delta x(t)p_1(t) = \Delta x(T) + \Delta y(T). \qquad \text{(F14)}$$

From the above, we see that the variable $p_1(t)$ shows what the effect of the loss of one X soldier at time t would have on the outcome of battle. Expressing the value of engagement, V, in terms of survivors, we see that

$$p_1(t) = \frac{\partial V}{\partial x}(t) \quad \text{and} \quad p_2(t) = \frac{\partial V}{\partial y}(t).$$

Bliss's idea for the development of air density corrections for the artillery range tables was similar.

f. Relation to Other Mathematics.

The underlying mathematical structure considered here (duality) manifests itself in many of the modern operations research optimization tools. Let us recall that we showed

for $$\frac{d\vec{X}}{dt} = A\vec{X} \quad \text{and} \quad \frac{d\vec{P}}{dt} = -A^{T}\vec{P} \ ,$$

we must have

$$\vec{X} \cdot \vec{P} = \text{const.} \tag{F15}$$

The finite dimensional analogue of this relationship is

for $$A\vec{x} = \vec{b} \quad \text{and} \quad A^{T}\vec{y} = \vec{c},$$

we must have

$$\vec{y} \cdot \vec{b} = \vec{c} \cdot \vec{x}. \tag{F16}$$

When extended to non-negative variables, this is

for $$\begin{matrix} A\vec{x} = \vec{b} \\ \vec{x} \geq 0 \end{matrix} \quad \text{and} \quad A^{T}\vec{y} \geq \vec{c},$$

we must have

$$\vec{y} \cdot \vec{b} \geq \vec{c} \cdot \vec{x}. \tag{F17}$$

The latter relationship may be used to develop many results in the theory of linear programming. For example, an immediate consequence is that for $\vec{x}$ that maximizes $\vec{c} \cdot \vec{x}$ subject to $A\vec{x} = \vec{b}$ and $\vec{x} \geq 0$, a sufficient condition is given by

$$A^{T}(B^{-1})^{T}c_{B} - c \geq 0$$

where B is non-singular matrix such that $B\vec{x}_B = \vec{b}$ and $\vec{x}_B$ is vector of non-zero components of the solution. The above condition is expressed in the linear programming literature as $Z_j - c_j \geq 0$.

To further indicate the fundamental nature of these concepts, we note that a further generalization of (F15) is

for $Lu(x) = f(x)$ and $L^*v(x) = g(x)$,

we must have

$$\int \{v(x)Lu(x) - u(x)L^*v(x)\}dx = \text{boundary terms}, \qquad \text{(F18)}$$

where L is a linear differential operator and L^* is its adjoint. This is known as Green's identity (p. 183 [62]) and has many important applications to ordinary and partial differential equations. From it one obtains the Green's functions for constructing solutions.

APPENDIX G. Applications to Deterministic Inventory Theory

In this section we consider the optimization of continuous review deterministic inventory models by the Pontryagin Maximum Principle. Several previously published results are extended. For linear production rate costs, we show that when demand is known with certainty and stock may be reordered at any point (continuously) in time, the optimal inventory policy is to only order as needed and only do this after the initial inventory has been depleted. The same type of policy is true when there are budgetary constraints with the constraint being ignored until the budget has been expended. We also have developed an alternate method of analysis to that developed by Arrow and Karlin [3] for the case of convex production rate costs. Our results on this latter topic are not fully documented at this time.

Our reasons for considering inventory problems are twofold: (1) such problems are a major aspect of defense planning and (2) our previous research has considered operations research models with a similar mathematical structure. Our past research has uncovered several facets of formulating and solving such dynamic models. For example, by application of the theory of singular control [53], [54], [57], we have shown that when the production cost rate function is linear, the optimal inventory policy is insensitive to the nature of the shortage (or penalty) cost function (as long as this is not pathological).

Our organization of this section is as follows: we review the general deterministic inventory model and the shortcomings of the classical calculus of variations methods for such a model before we consider our sequence of models. Then, we discuss the insight that we have gained into optimal inventory policies. We begin by surveying some previous work in the field of deterministic inventory theory.

An excellent introduction to elementary inventory theory and inventory theory in general prior to 1957 is to be found in [26]. Dynamic models were not considered prior to 1951. A more advanced introduction to inventory theory is by Arrow, Karlin, and Scarf [4], who summarize work through 1958 and give an extensive bibliography. Variational methods were applied to a deterministic inventory process by Arrow and Karlin [3] in this work. An excellent survey of modelling techniques and results has been written by Karlin [56]. Adiri and Ben-Israel [2] attempted to extend the work of Arrow and Karlin by use of the Pontryagin maximum principle. A comprehensive bibliography of applications of optimal control theory to operations research problems has been published by Tracz [77]. Considering this last reference, it appears as though the above work and references cited therein represents most of the published results on dynamic, deterministic inventory models. Recently McMasters [63] has studied the Arrow and Karlin problem. However, we obtain here different results than McMasters has. Our results are more in consonance with those of Arrow and Karlin [3].

a. The General Model.

We consider a deterministic inventory process subject to continuous review. Karlin has an excellent discussion and classification of inventory models and our present discussion has been based on his [56]. We consider that all processes occur continuously in time. We shall see that this leads to a problem in the calculus of variations. However, two factors that are commonly present in applications preclude the direct application of the classical calculus of variations results: (1) non-negativity of variables and (2) inequality constraints.

Karlin [56] identifies four main factors in the inventory process:

(1) cost factors,

(2) nature of demand for inventory,

(3) nature of supply for inventory,

(4) mechanism of inventory process.

We assume a single item inventory. We consider a production cost, $c(u(t))$, per unit time which only depends upon the rate of production $u(t)$. We also consider storage or holding cost, $h(I(t))$, which depend upon the inventory level $I(t)$. Orginally, $h(I(t))$ is only defined for $I(t) \geq 0$, but we may extend this to $I(t) < 0$ by considering shortage or penalty costs for not meeting inventory demand. We omit considerations of the "time value of money" (discount rate).

The nature of the inventory demand is assumed to be perfectly known and is given by $r(t)$, which is the demand rate. We consider a deterministic supply without setup costs. The production rate is denoted by $u(t)$. We consider an inventory process without lags and continuous in time. Our decision criterion is the minimization of total cost. The basic type of model we consider is the minimization of a cost functional.

$$J[u] = \int_0^T [c(u(t)) + h(I(t))]dt \quad , T \text{ specified,}$$

with the inventory being given by

$$I(t) = I(0) + \int_0^t [u(t) - r(t)]dt.$$

The production rate is, of course, restricted to non-negative, i.e., $u(t) \geq 0$.

b. Shortcomings of the Classical Calculus of Variations.

We have already noted two model factors that prevent direct application of classical calculus of variations results: (1) non-negative variables and (2) inequality constraints. Our own research, however, indicates that these difficulties may overcome by the formulation of an equivalent problem. A similar approach may be used to develope many non-linear programming results by the calculus [59]. For example, when there are non-negative variables in our orginal problem, we may formulate an equivalent problem by replacing x by u^2 . We solve this equivalent problem for u and then recover our orginal variable x. Inequailty constraints are easily converted to equality constraints by the addition of non-negative slack variables.

c. Comments on Previous Work.

Our general comments are than when variational methods were attempted before the advent of the Pontryagin maximum principle, little more than a first variation approach leading to an Euler-Lagrange equation was employed. We should note that the Pontryagin maximum principle involves both the Euler-Lagrange equations and the Weierstrass condition for the Weierstrass excess function. It is not surprising that use of but one calculus of variations' tool from among many (there are four well-known necessary conditions, i.e., Euler equation, Weierstrass Legendre (second order), and Jacobi conditions) has not been able to solve all problems.

F. Morin [64] appears to be one of the first economists to formulate and attempt to solve a deterministic inventory model with continuous time. No backlogging of orders was allowed (no stockouts). It should be noted that Morin tried to apply some theory developed by Bolza (see [18] pp. 41-43) for extremal curves on the boundary of the state space.

Arrow and Karlin [3] have solved Morin's problem. Whereas Morin tried to apply Bolza's results directly to his problem, Arrow and Karlin develop the solution to this specific problem by variational methods. Anyone doubting the complexities of applying variational methods to problems with non-negative variables and inequalities should consult this work. In our notation the Arrow-Karlin problem was

$$\min_{u(t)} \int_0^T [c(u(t)) + h(I(t))]dt \quad \text{with } T \text{ specified},$$

$$\text{subject to: } \frac{dI}{dt} = u(t) - r(t) \quad ,$$

$$\text{and} \quad u(t) \geq 0 \quad , \quad I(t) \geq 0$$

with boundary conditions

$$I(t = 0) = I(0) \text{ and } I(t = T) = 0 \quad . \tag{G1}$$

Arrow and Karlin [3] solve the above model for linear holding rate costs and general convex production rate costs. Their general solution algorithm is applied to linear production rate costs and several other examples, including quadratic production costs. The theoretical foundations of Arrow and Karlin's analysis are not immediately evident from the content of their paper which merely summarizes the results. The central point is that one-sided variations are required when the inventory is at a zero level. Arrow and Karlin apparently developed an extension of the usual variational development for problems where convexity properties can be assumed. Their approach, however, does not seem to be documented in any of the mathematical literature known to this author.

Adiri and Ben-Israel [2] applied to the Pontryagin maximum principle to Arrow and Karlin's problem besides the classical optimal lot size problem. However, because the boundary condition $I(t = T) = 0$, the value of the dual variable $p(t) = (\partial J^* / \partial I)(t)$ is free at $t = T$. Since they never determine the value of the dual variable at $t = T$, i.e., $p(t = T)$, they never do solve this problem. In fact, their conclusion as to the solution for linear production costs is unsupported by their analysis (the conclusion that the partial derivative of the Hamiltonian with respect to the control variable is always negative is unsupported).

We re-examine the solution to the Arrow-Karlin problem given by (G1) above. The constraint on the state variable $I(t) \geq 0$ implies that we must have $dI/dt \geq 0$ when $I(t) = 0$. Hence, we have

$$u(t) \begin{cases} \geq 0 & \text{for } I(t) > 0 \\ \geq r(t) & \text{for } I(t) = 0 . \end{cases} \qquad \text{(G2)}$$

We must further check to see if the state variable constraint has an effect on the adjoint equation (see [24] p. 117), but we see that it does not since $(\partial/\partial I)\ \{dI/dt\} = 0$. The Hamiltonian is given by

$$H(t,I,p,u) = c(u(t)) + h(I(t)) + p(t)\{u(t) - r(t)\},$$

so that the extremal control is given by

$$\min_{u(t)} \{c(u(t)) + p(t)u(t)\}. \tag{G3}$$

We note that $p(t) > 0$ implies that the minimum of (G3) is given by the minimum $u(t)$ given by (G2). The adjoint equation for the dual variable $p(t) = (\partial J^*/\partial I)(t)$ (see [12] for this interpretation) is given by

$$\frac{dp}{dt} = -\frac{\partial H}{\partial I} = -\frac{dh}{dI}.$$

We introduce the backwards time $\tau = T - t$ so that $dp/d\tau = dh/dI$ and hence

$$p(\tau) = \int_0^\tau \frac{dh}{dI}\, d\tau + p(\tau=0).$$

Because of the constraint $I(t) \geq 0$ for all time, it is necessary to consider two separate cases at $\tau = 0$. When $I(t=T) > 0$, then $p(\tau=0) = 0$. This generates a further condition on $I(t=0)$ so that the end state $I(t=T) > 0$ may be reached. When $I(t=T) = 0$, it may be shown that $p(\tau=0)$ must be <0. The precise value of $p(\tau=0)$ is determined by further simultaneous conditions.

McMasters [63] also considers the above models. Unlike Arrow and Karlin [3] who assumed that $I(t=T) = 0$, he makes no assumption about the inventory level at the end of the planning period. He does not distinguish between the two cases that we have above ((1) $I(t=T) > 0$ and (2) $I(t=T) = 0$) and consequently derives different results. He also considered the problem when shortages (stockouts) are allowed. He solves this problem for linear production and holding costs but does not recognize the singular solution [53] in his model. We show in the present work that more general results are possible, i.e., if production costs are linear, then the optimal inventory policy is relatively insensitive to the nature of holding and shortage costs as long as $(dh/dI) > 0$ for $I > 0$ and $(dh/dI) < 0$ for $I < 0$.

d. A Sequence of Models.

In this section we consider a sequence of Arrow-Karlin type models: no stockouts, stockouts allowed with linear production costs, and budget constraints. We have also considered a model where there is a special penalty cost for being out of inventory at the end of the planning period in the stockouts allowed case. This was prompted by the disturbing feature of the developing a shortage at the end of the planning period turning out to be the optimal policy in the stockout model. This is related to future demand being known with certainty. Neither the model nor its policy apply in many real-world circumstances.

No Stockouts

We consider the problem

$$\min_{u(t)} \int_0^T [c(u(t)) + h(I(t))]dt \quad \text{with } T \text{ specified,}$$

$$\text{subject to:} \quad \frac{dI}{dt} = u(t) - r(t),$$

$$\text{and} \quad u(t) \geq 0, \quad I(t) \geq 0$$

with initial condition

$$I(t=0) = I(0). \tag{G4}$$

We assume that holding costs are a non-decreasing function of the inventory level, i.e., $(dh/dI) \geq 0$. As above, the constraint on the state variable $I(t) \geq 0$ implies that we must have $(dI/dt) \geq 0$ when $I(t) = 0$ so that (G2) applies. It is easily checked that this last condition does not modify the adjoint equation (see [24] p. 117). The Hamiltonian is given by

$$H(t,I,p,u) = c(u(t)) + h(I(t)) + p\{u(t) - r(t)\}, \tag{G5}$$

so that the optimal control (there is only one extremal) is given by

$$\min_{u(t)} \{c(u(t)) + p(t)u(t)\}, \tag{G6}$$

where $u(t)$ must satisfy (G2). The adjoint equation for the dual variable is given by

$$\frac{dp}{dt} = -\frac{\partial H}{\partial I} = -\frac{dh}{dI}. \tag{G7}$$

There are two cases to consider for the boundary condition on the dual variable at $t = T$, depending on whether $I(t) > 0$ or $I(t) = 0$.

Case A. $I(T) > 0$.

In this case $p(t{=}T) = 0$, since there is no terminal payoff (we have the problem of Lagrange in the classical literature). We introduce the backward time $\tau = T - t$ so that $(dp/d\tau) = -(dp/dt)$ and hence

$$p(\tau) = \int_0^\tau \frac{dh}{dI}\, d\tau \geq 0 \quad \text{for all} \quad \tau \geq 0. \tag{G8}$$

Since we assume the production costs to be non-decreasing, (G6) immediately yields the optimal inventory policy

$$u^*(t) = \begin{cases} 0 & \text{for } I(t) > 0 \\ r(t) & \text{for } I(t) = 0. \end{cases}$$

Now since $I(T) > 0$, then $u^*(T) = 0$. By a continuity argument, it is easy to show that $u^*(t) = 0$ in a neighborhood of T, i.e., $t \in (T-\delta, T]$ for $\delta > 0$. From the state equation of (G1), we have

$$I(\tau) = \int_0^\tau \{r(s) - u(s)\}ds + I(t=T),$$

and hence

$$I^*(\tau) = \int_0^\tau r(s)ds + I(t=T),$$

so it is easy to see that $I^*(t) > 0$ for all t and hence $u^*(t) = 0$ for all t. Thus, we require that

$$I(0) > \int_0^T r(t)dt.$$

Hence, we see the obvious result that you never produce if you can meet all future demand.

Case B. $I(T) = 0$.

In this case $p(t=T)$ is unspecified. The nature of $c(u(t))$ now effects the structure of the optimal inventory policy. Hence we must consider three further subcases for production rate costs

(1) concave,

(2) linear,

(3) convex.

In the current report we do not carry the analysis any further. We have completed the analysis for a quadratic production-rate cost and constant demand rate. We have obtained the same results in this special case as Arrow and Karlin [3], who used a variational approach which (to the best of this author's knowledge) is found nowhere else in applied mathematics literature. We hope to document our complete results in a future report.

It seems appropriate to indicate the nature of our results. In the cases of concave and linear production rate costs, the optimal inventory policy turns out to be

$$u^*(t) = \begin{cases} 0 & \text{for } I(t) > 0 \\ r(t) & \text{for } I(t) = 0. \end{cases}$$

This is not surprising. In the case of convex production rate costs (this might be due to plant expansion or overtime to attain higher production rates), we have obtained Arrow and Karlin's results. We feel that our approach is more general and hope to explore its capability further in the future.

<u>Stockouts Allowed</u>

We consider the same problem as above only we remove the constraint that $I(t) \geq 0$. We assume that

$$\frac{dh}{dI} \begin{cases} > 0 & \text{for } I(t) > 0 \\ < 0 & \text{for } I(t) < 0. \end{cases}$$

Equations (G5), (G6), and (G7) are readily seen to be still applicable. We can no longer guarantee that $p(\tau) \geq 0$ for all τ and thus (G6) no longer yields the optimal control by inspection. We consider

$$\frac{\partial H}{\partial u} = \frac{dc}{du} + p,$$

and note that $u^*(t) = 0$ for $(\partial H/\partial u) > 0$. To proceed further we must make assumptions on the nature of the production costs $c(u(t))$ (all we had to assume previously was that $c(u(t))$ was a non-decreasing function of u). Since we may also have $(\partial H/\partial u) < 0$, we must further restrict $u(t)$ as follows

$$0 \lesssim u(t) \lesssim b$$

We have not carried the analysis in this most general case further. The details appear to be messy but straightforward. Instead we specialize the problem.

Stockouts Allowed - Linear Production Cost

We consider the problem

$$\min_{u(t)} \int_0^T [au(t) + h(I(t))]dt \quad \text{with } T \text{ specified,}$$

$$\text{subject to:} \quad \frac{dI}{dt} = u(t) - r(t),$$

$$\text{and} \quad 0 \lesssim u(t) \lesssim b \quad (\text{also } a > 0)$$

with initial condition

$$I(t=0) = I(0). \tag{G9}$$

We make the following assumptions on the holding and penalty costs

$$\frac{dh}{dI} \begin{cases} > 0 & \text{for} \quad I(t) > 0 \\ = 0 & \text{for} \quad I(t) = 0 \\ < 0 & \text{for} \quad I(t) < 0\,, \end{cases} \tag{G10}$$

and also $(d^2h/dI^2) > 0$ for $I(t) = 0$. Later we will see that we only require $h(I)$ to have a minimum at $I = 0$ so that $h(I)$ need not be twice differentiable at $I = 0$.

The Hamiltonian is given by

$$H(t,I,p,u) = au + h(I) + p(u-r), \tag{G11}$$

and it is seen that the optimal control (there is only one extremal) is usually given by

$$u^*(t) = \begin{cases} 0 & \text{for} \quad p(t) > -a \\ b & \text{for} \quad p(t) < -a \end{cases} \tag{G12}$$

The adjoint equation for the dual variable (in backwards time $\tau = T - t$) is

$$\frac{dp}{d\tau} = \frac{dh}{dI} \quad \text{with} \quad p(\tau=0) = 0, \tag{G13}$$

and hence

$$p(\tau) = \int_0^\tau \frac{dh}{dI}\, d\tau. \tag{G14}$$

If $I(t=T) \geq 0$, then it is easy to see by (G10), (G12), and (G14) that $u^*(t) = 0$ for $0 \leq t \leq T$. If $I(t=T) < 0$, then we have by (G10) and (G14) that $p(\tau) < 0$ near $\tau = 0$. Also considering (G12), we see that $u^*(\tau) = 0$ for $0 \leq \tau \leq \tau_1$ where τ_1 is determined by

$$\int_0^{\tau_1} \frac{dh}{dI}\, d\tau = -a,$$

and

$$I(\tau) = \int_0^{\tau} r(\tau)d\tau + I(t=T). \tag{G15}$$

Since the Hamiltonian is a linear function of the control variable u, the minimum principle does not determine the control when the coefficient of u vanishes, i.e., $p(\tau) = -a$, for a finite interval of time (see p. 481 of [6]). Part of a trajectory for which this happens is called a singular subarc. We determine the conditions for a singular subarc from [54]

$$\frac{\partial H}{\partial u} = \frac{d}{dt}\left(\frac{\partial H}{\partial u}\right) = 0. \tag{G16}$$

We have from (G11) that

$$\frac{\partial H}{\partial u} = a + p,$$

and

$$\frac{d}{dt}\left(\frac{\partial H}{\partial u}\right) = -\frac{dh}{dI}. \tag{G17}$$

Hence on a singular subarc we have

$$p(\tau) = -a$$

and

$$\frac{dh}{dI} = 0. \tag{G18}$$

The latter of these implies that $I(t) = 0$ on a singular subarc. From (G15) we see that we reach the singular subarc at $\tau = \tau_1$. We stay on it until we have to get off to meet the given initial condition $I(0)$.

We stay on the singular subarc by using $u^*(t) = r(t)$, which keeps $I(t)$ equal to zero.

A necessary condition for a singular subarc to yield a minimum return is that [57]

$$\frac{\partial}{\partial u}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right)\right\} \leq 0. \tag{G19}$$

From (G18) we have that

$$\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) = \frac{d}{dt}\left(-\frac{dh}{dI}\right) = -\frac{d^2h}{dI^2}\frac{dI}{dt} = -\frac{d^2h}{dI^2}(u-r),$$

and hence

$$\frac{\partial}{\partial u}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right)\right\} = -\frac{d^2h}{dI^2}. \tag{G20}$$

Our assumption that $d^2h/dI^2 > 0$ for $I = 0$ guarantees that (G19) is met. Hence, when the holding-shortage cost curve has a minimum at $I = 0$, i.e., $dh/dI = 0$ and $d^2h/dI^2 > 0$, we may have an optimal singular solution holding the inventory at zero. By a limiting argument we may dispense with the condition that $d^2h/dI^2 > 0$ and only require that $h(I)$ has a minimum at $I = 0$.

To summarize, the optimal inventory policy is given by

$$u^*(t) = \begin{cases} 0 & \text{for } I(t) > 0 \\ r(t) & \text{for } I(t) = 0 \\ b & \text{for } I(t) < 0 \end{cases} \quad \text{for } t \in [0, T-\tau_1],$$

and

$$u^*(t) = 0 \quad \text{for } t \in (T-\tau_1, T], \tag{G21}$$

where τ_1 is determined by (G15).

Budget Constraints - Product Costs Only

We consider the same model as immediately above only we assume that there is a budget constraint on production, i.e., we must have

$$\int_0^T c(u(t))dt \leq A,$$

where A is the total production budget. We shall see that the optimal inventory policy is the same as immediately above: only the closing interval of no production begins earlier. Since the problem is the same as above when the budget constraint is not binding, we assume that

$$a\left[\int_0^{T-\tau_1} r(t)dt - I(0)\right] > A, \qquad \text{G22)}$$

where τ_1 is given by (G15). Thus, we consider

$$\min_{u(t)} \int_0^T [au(t) + h(I(t))]dt \quad \text{with } T \text{ specified}, \qquad \text{(G22)}$$

subject to: $\frac{dI}{dt} = u(t) - r(t),$

$$\frac{dM}{dt} = au(t),$$

and $0 \leq u(t) \leq b,$

with boundary conditions

$$I(t=0) = I(0),$$

$$M(t=0) = 0, \quad M(t=T) = A, \qquad \text{(G23)}$$

where $M(t)$ is total expenditures on production through time t. As before we assume (G10) for the holding and penalty costs.

The Hamiltonian is given by

$$H(t,I,p,u) = au + h(I) + p_1(u-r) + p_2 au, \tag{G24}$$

and it is seen that the optimal control on non-singular subarcs is given by

$$u^*(t) = \begin{cases} 0 & \text{for } p_1(t) > -a(1+p_2) \\ b & \text{for } p_1(t) < -a(1+p_2). \end{cases} \tag{G25}$$

The adjoint equations for the dual variables are

$$\begin{aligned} \frac{dp_1}{dt} &= -\frac{\partial H}{\partial I} = -\frac{dh}{dI} \qquad p_1(t=T) = 0 \\ \frac{dp_2}{dt} &= -\frac{\partial H}{\partial M} = 0 \Rightarrow p_2(t) = \text{const and no condition on } p_2(t=T). \end{aligned} \tag{G26}$$

It is easy to see that we must have $p_2 > 0$. Recalling the well-known interpretation of the dual variables [12], we see that $p_2 = \frac{\partial J^*}{\partial M}$. Since increasing total expenditure increases to minimum inventory cost we have $\frac{\partial J^*}{\partial M} > 0$. We could also argue that if p_2 were negative then τ_2 defined by (where $\tau = T - t$)

$$\int_0^{\tau_2} \frac{dh}{dI}\, d\tau = -a(1+p_2)$$

would be less than τ_1 defined by (G15). Thus production would occur for a longer period of time, and this is impossible since we assume that the budget constraint is binding.

Other solution details are similar to the case above, and we omit them. The optimal inventory policy is given by

$$u^*(t) = \begin{cases} 0 & \text{for } I(t) > 0 \\ r(t) & \text{for } I(t) = 0 \\ b & \text{for } I(t) < 0 \end{cases} \quad \text{for } t \in [0, T-\tau_2]$$

and

$$u^*(t) = 0 \quad \text{for} \quad t \in (T-\tau_2, T], \tag{G27}$$

where τ_2 is determined by

$$a \int_0^{T-\tau_2} u^*(t)dt = A,$$

since we assume that (G22) holds.

<u>Budget Constraints - Production and Holding Costs</u>

We extend the above model to the case of a budget constraint on total production plus holding costs, i.e., we must have

$$\int_0^T [c(u(t)) + h_1(I(t))]dt \leq A,$$

where A is the total budget and

$$h_1(I) = \begin{cases} h(I) & \text{for } I \geq 0 \\ 0 & \text{for } I < 0. \end{cases}$$

We shall see that the optimal inventory policy is the same as immediately above only the closing interval of no production begins even earlier.

Since the solution to the problem is the same as (G21) when the constraint is not binding, we assume that

$$a\left[\int_0^{T-\tau_1} \{r(t) + h_1(I(t))\}dt - I(0)\right] > A, \tag{G28}$$

where τ_1 is given by (G15). Thus, we consider

$$\min_{u(t)} \int_0^T [au(t) + h(I(t))]dt \quad \text{with } T \text{ specified,}$$

$$\text{subject to:} \quad \frac{dI}{dt} = u(t) - r(t),$$

$$\frac{dM}{dt} = au(t) + h_1(I(t)),$$

$$\text{and} \quad 0 \leq u(t) \leq b,$$

with boundary conditions

$$I(t=0) = I(0),$$

$$M(t=0) = 0, \quad M(t=T) = A. \tag{G29}$$

As before we assume (G10) for the holding and penalty costs.

The Hamiltonian is given by

$$H(t,I,p,u) = u(a+p_1+p_2a) + h(I) - p_1r + p_2h_1(I), \tag{G30}$$

and the optimal control on non-singular subarcs is given by (G25). The adjoint equations are again given by (G26), and again we must have p_2 = const > 0. The rest is similar to previous isoperimetric problem (integral constraint).

The optimal inventory policy is given again by (G27) with the exception that τ_2 is now determined by

$$\int_0^{T-\tau_2} au^*(t) + h_1(I(t))\, dt = A,$$

since we assume that (G28) holds.

e. Discussion.

In this section we review the structure of optimal inventory policies for the models we have considered in the previous section and attempt some generalizations. We also comment on the nature of deterministic inventory models. As a general comment, we note the similarity of these dynamic inventory models to the (one-sided) attrition games we have considered in previous appendices. This should alert us to the possibility of optimal inventory policies being dependent upon the type of boundary conditions specified.

Considering the sequence of models in the previous section, we observe that when future demand is known with certainty and the production rate costs are concave (a special case which is linear):

(a) never order while you have inventory,

(b) if shortages are allowed, then the best policy is to run out of inventory at the end of the planning period,

(c) budget constraints on production and holding costs are to be ignored (until they become binding).

For convex production rate costs, the situation is more complex. Under certain circumstances it is advantageous to produce at lower rates before inventory is depleted than to hold off production until stocks are entirely depleted after which time higher production rates would be required. This situation arises due to marginal production rate costs which are an increasing function of the production rate. We hope to explore this case more fully in the future.

These models have assumed perfect knowledge of the future. What is the effect of uncertainty? Uncertainty may cause inventory to be backlogged, but we are novices in this field. We have noted previously in the Lanchester theory of combat that if we interpret a linear law attrition process as being the result of uncertainty, then we "split" the allocation of fire among target types as a "hedge" against uncertainty. We should also note that certain aspects of the solution procedure for these dynamic deterministic models extend to the stochastic case. For example, we determine the marginal costs of inventory backwards from the end of the planning horizon.

We should not lose sight that these models are idealizations of a more complex real world process. Therefore, the structure or nature of optimal inventory policies and its dependence on model form is of prime importance. The real world is considerably more uncertain than the perfect knowledge of future demand assumed by these models, but yet there is much that we can learn from deterministic inventory theory. Because of their idealized and simplified nature, it is possible to develop "closed-form" solutions to many deterministic inventory models. We have done this in the current report. In such solutions the interdependence of model parameters is explicitly exhibited. This leads to a better understanding of the structure of trade-off decisions to be made. This should be contrasted to dynamic programming models (both deterministic and probabilistic) for which, in most instances, a solution is developed only for a specific set of parameter values. In this case, it is difficult (if not impossible) to see the structure of optimal inventory policies and its dependence on model form without a parametric analysis of model output.

The intimate connection between variational methods and dynamic programming (their dual relationship in the sense of J. Plücker's principle of duality*) is well known [10], [30]. It is important to understand the Hamilton-Jacobi approach to variational problems. In discrete and stochastic cases, we formulate the analogue of the Hamilton-Jacobi-Bellman equation for the optimal return. Hence, understanding the principles of the solution procedure in the deterministic case provides the insight for extensions.

*Actually first stated in non-algebraic terms by J. Gergonne.

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13. ABSTRACT

The mathematical theory of deterministic optimal control/differential games is applied to the study of some tactical allocation problems for combat described by Lanchester-type equations of warfare. A solution procedure is devised for terminal control attrition games. H. K. Weiss' supporting weapon system game is solved and several extensions considered. A sequence of one-sided dynamic allocation problems is considered to study the dependence of optimal allocation policies on model form. The solution is developed for variable coefficient Lanchester-type equations when the ratio of attrition rates is constant. Several versions of Bellman's continuous stochastic gold-mining problem are solved by the Pontryagin maximum principle, and their relationship to the attrition problems is discussed. A new dynamic kill potential is developed. Several problems from continuous review deterministic inventory theory are solved by the maximum principle.

14 KEY WORDS	LINK A		LINK B		LINK C	
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Tactical Allocation						
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Lanchester Theory of Combat						
Dynamic Kill Potential						
Inventory Theory						