2003

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SIAM J. Control Optim. / Volume 42, Issue 1, 155-177 (electronic)

http://hdl.handle.net/10945/25494

SIAM J. CONTROL OPTIM.
Vol. 42, No. 1, pp. 155–177

LOCALLY CONVERGENT NONLINEAR OBSERVERS*

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Abstract. We introduce a new method for the design of observers for nonlinear systems using backstepping. The method is applicable to a class of nonlinear systems slighter larger than those treated by Gauthier, Hammouri, and Othman [*IEEE Trans. Automat. Control,* 27 (1992), pp. 875–880]. They presented an observer design method that is globally convergent using high gain. In contrast to theirs, our observer is not high gain, but it is only locally convergent. If the initial estimation error is not too large, then the estimation error goes to zero exponentially. A design algorithm is presented.

Key words. nonlinear estimation, nonlinear observers

AMS subject classifications. 93C10, 93B50, 93E11

PII. S0363012900368612

1. Introduction.

The problem of estimating the state of a dynamical system from partial and possibly noisy measurements has a long history. In its nonlinear state space form, one assumes that the dynamics satisfies a known nonlinear differential equation with unknown initial condition and the measurement is a known nonlinear function of the state

$$\begin{aligned} \dot{x} &= f(x), \\ x(0) &= x^0, \\ y &= h(x). \end{aligned} \tag{1.1}$$

The linear form of the problem is

$$\begin{aligned} \dot{x} &= Ax, \\ x(0) &= x^0, \\ y &= Cx. \end{aligned} \tag{1.2}$$

One is given an estimate $\hat{x}^0$ of x^0 and the observations $y(s),\ 0 \le s \le t$, up to time t. The problem is to generate an estimate $\hat{x}(t)$ of $x(t)$ in real time, as the process evolves. The estimate should converge to the true state as $t \to \infty$. Ideally the estimation process should be robust to noise both in the dynamics and in the observations, to the initial state error, and also to modeling errors in the functions $f,\ h$. Furthermore, the error should converge to zero quickly.

One way of approaching this problem is to assume that the dynamics, the initial condition, and the observations are corrupted by noises with known distributions and then to find the conditional density of the state given the past observations. If the dynamics and observations are linear functions of the state and if the noises and

*Received by the editors March 2, 2000; accepted for publication (in revised form) September 6, 2002; published electronically March 19, 2003.
http://www.siam.org/journals/sicon/42-1/36861.html

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the initial condition are independent and Gaussian, then the conditional density is Gaussian and explicitly computable. Wiener [28] solved this problem for stationary Gaussian processes using the method of spectral factorization. Kalman [12], [13] extended this to nonstationary Gaussian processes and reduced the problem to solving an off-line Riccati equation and an on-line linear differential equation driven by the observations.

When the dynamics and/or observations are nonlinear the unnormalized conditional density satisfies the Zakai equation, a parabolic PDE driven by the observations [4]. It is a very difficult task to accurately compute its solution in real time for all but the smallest state dimensions.

The extended Kalman filter [10] is a widely used alternative method for estimating the state of a nonlinear system. It is obtained by linearizing the nonlinear dynamics and the observation along the trajectory of the estimate. It requires that the on-line solution of a Riccati differential equation and a linear differential equation be driven by the observations. The extended Kalman filter is globally defined but it is only a local method. Under certain conditions, the estimate converges to the true state if the initial estimation error is not too large [1], [23].

There are several nonstochastic approaches to state estimation. For linear systems (1.2), Luenberger [19] developed the concept of an observer. This is another linear dynamical system that is driven by the observations in such a way that the error dynamics is asymptotically stable.

Several nonstochastic methods have been proposed for nonlinear estimation. Some of these are surveyed by Misawa and Hedrick [21]. Other methods include linearization [16], [2], [17], H_∞ methods [15], bilinear systems [29], and high gain observers [3], [5], [6], [7], [8], [9], [24], [25], [26], [27].

This paper describes a simple and efficient method for the design of observers for a broad class of nonlinear systems based on backstepping. The backstepping technique has been used extensively to design stabilizing state feedback control laws [18], [20]. The assumptions on the system are that it be smooth and observable in an appropriate sense. The method is applicable to systems whose error dynamics are not necessarily linearizable by a change of coordinates and input/output injection [16], [2], [17]. It is applicable to a slightly larger class of systems than the high gain observer of Gauthier, Hammouri, and Othman [8]. The latter result can be applied to systems that can be globally described in observable form while the backstepping approach requires only a local observable form. Moreover, backstepping is not a high gain design procedure and hence only local convergence is guaranteed. An explicit formula for the observer gain is derived. The gains are functions of the state of the observer. The gains can be derived off-line through an algorithm presented below. The observer is defined on an arbitrarily large compact subset of the state space but is only locally convergent. We shall prove that the estimate converges exponentially to the true state if the state starts in a compact positively invariant set and the initial estimation error is not too large.

The paper is organized as follows. In section 2 the backstepping approach to observer design is illustrated for a scalar output system without inputs in observable form. In section 3 this is generalized to systems in observable form with vector output and no inputs. In section 4, this technique is generalized to systems that locally can be described in observable form. Systems with inputs are discussed in section 5. In section 6, the relative perfomance of the high gain observer and the backstepping observer are discussed. We close with examples in section 7.

2. The backstepping observer. Consider a smooth nonlinear system in observable form:

$$
\begin{aligned}
y &= x_1,\\
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= x_3,\\
&\vdots\\
\dot{x}_{n-1} &= x_n,\\
\dot{x}_n &= f_n(x).
\end{aligned}
\tag{2.1}
$$

The state x is n dimensional and the output y is one dimensional. There is no input. Later we shall relax these assumptions. By smooth we mean C^r for r sufficiently large.

The backstepping observer will be in the following form:

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + \psi_1(\hat{x})(x_1 - \hat{x}_1),\\
\dot{\hat{x}}_2 &= \hat{x}_3 + \psi_2(\hat{x})(x_1 - \hat{x}_1),\\
&\vdots\\
\dot{\hat{x}}_{n-1} &= \hat{x}_n + \psi_{n-1}(\hat{x})(x_1 - \hat{x}_1),\\
\dot{\hat{x}}_n &= f_n(\hat{x}) + \psi_n(\hat{x})(x_1 - \hat{x}_1).
\end{aligned}
\tag{2.2}
$$

The error $e = x - \hat{x}$ dynamics is given by

$$
\begin{aligned}
\dot{e}_1 &= e_2 - \psi_1(\hat{x})e_1,\\
\dot{e}_2 &= e_3 - \psi_2(\hat{x})e_1,\\
&\vdots\\
\dot{e}_{n-1} &= e_n - \psi_{n-1}(\hat{x})e_1,\\
\dot{e}_n &= f_n(x) - f_n(\hat{x}) - \psi_n(\hat{x})e_1.
\end{aligned}
\tag{2.3}
$$

The problem of observer design is to find gains $\psi_i(\hat{x})$, $1 \leq i \leq n$, so that $e(t) \to 0$ as $t \to \infty$.

Notice that the error dynamics (2.3) is dependent on both e and $\hat{x}$. The combined system, consisting of the system and its observer, can be described in $x, \hat{x}$ coordinates (2.1), (2.2), in $e, \hat{x}$ coordinates (2.2), (2.3), or in x, e coordinates (2.1), (2.3).

Suppose that K is a compact subset of x space, which is positively invariant under (2.1), i.e., if a trajectory starts in K, then it remains in K for all future times. The set $K \times \{e = 0\}$ is a positively invariant set of the combined system (2.1), (2.3). Using a backstepping approach [18], we will construct a local Lyapunov function for the combined system to prove local exponential convergence to this positively invariant set. The observer gains ψ_i will be chosen in the course of this construction.

We employ the following notation: an error term $O(e)^k$ is a function of $\hat{x}, e$ such that on any compact subset L of $\hat{x}$ space there exists a constant $N > 0, \delta > 0$ such that

$$
|O(e)^k| \leq N|e|^k
\tag{2.4}
$$

for all $\hat{x} \in L$ and all $|e| < \delta$. We abbreviate $O(e)^1$ as $O(e)$.

We now proceed with the construction of the backstepping observer on a compact, positively invariant set K and show its local convergence.

Define $z_1 = e_1$ and $V_1 = \frac{1}{2}z_1^2$; then

$$\dot{V}_1 = z_1\dot{z}_1 = -c_1 z_1^2 + z_1 z_2 + O(e)^3,$$

where $c_1 > 0$ and z_2 is the linear function of e that satisfies $z_2 = c_1 z_1 + \dot{z}_1 + O(e)^2$.

If $n = 1$, we choose

$$\psi_1(\hat{x}) = c_1 + \frac{df_1}{dx_1}(\hat{x}_1)$$

so that the auxiliary variable $z_2 = 0$ and

$$\dot{V}_1 = z_1\dot{z}_1 = -c_1 z_1^2 + O(e)^3. \tag{2.5}$$

If $n \geq 2$, then z_1, z_2 and e_1, e_2 are linearly related by

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b_{2,1} - \psi_1 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{2.6}$$

where

$$b_{2,1} = c_1. \tag{2.7}$$

Define $V_2 = V_1 + \frac{1}{2}z_2^2$; then

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + O(e)^3,$$

where $c_2 > 0$ and z_3 is the linear function of e that satisfies $z_3 = z_1 + c_2 z_2 + \dot{z}_2 + O(e)^2$. Notice that z_2, z_3 depend on the as yet unspecified observer gains $\psi_1(\hat{x}), \psi_2(\hat{x})$.

If $n = 2$, then we would like to choose the gains so that the auxiliary variable z_3 is 0, for then

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + O(e)^3. \tag{2.8}$$

Now

$$\begin{aligned} z_3 &= z_1 + c_2 z_2 + \dot{z}_2 + O(e)^2 \\ &= (b_{3,1} + f_{2;1} - \psi_2)e_1 + (b_{3,2} + f_{2,2} - \psi_1)e_2, \end{aligned}$$

where

$$f_{n;i} = \frac{\partial f_n}{\partial x_i}(\hat{x}),$$

$$\begin{aligned} b_{3,1} &= 1 + c_2(b_{2,1} - \psi_1) + (b_{2,1} - \psi_1)' - (b_{2,1} - \psi_1)\psi_1, \\ b_{3,2} &= c_1 + c_2. \end{aligned} \tag{2.9}$$

We denote differentiation along the observer dynamics when $e_1 = 0$ by $'$. For an n dimensional observer (2.2), the operation $'$ is defined on functions $\phi(\hat{x})$ by

$$\phi'(\hat{x}) = \sum_{j=1}^{n-1} \frac{\partial \phi}{\partial \hat{x}_j}(\hat{x})\hat{x}_{j+1} + \frac{\partial \phi}{\partial \hat{x}_n}(\hat{x}) f_n(\hat{x}).$$

Notice that $'$ does not involve the gains ψ_i and

$$\phi' = \dot{\phi} + O(e).$$

If $n = 2$, we successively define

$$\psi_1(\hat{x}) = b_{3,2}(\hat{x}) + f_{2;2}(\hat{x}),$$
$$\psi_2(\hat{x}) = b_{3,1}(\hat{x}) + f_{2;1}(\hat{x});$$

then (2.8) holds. Notice that the gains are functions of $\hat{x}$ alone.

If $n \geq 3$, we define z_1, z_2 as before (2.6) and z_3 as a linear function of e so that

$$z_3 = z_1 + c_2 z_2 + \dot{z}_2 + O(e)^2.$$

By a calculation similar to the above, we see that

$$z_3 = (b_{3,1} - \psi_2)e_1 + (b_{3,2} - \psi_1)e_2 + e_3, \tag{2.10}$$

where $b_{3,1},\ b_{3,2}$ are given by (2.9). Define $V_3 = V_2 + \frac{1}{2}z_3^2$; then

$$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_3 z_4 + O(e)^3,$$

where $c_3 > 0$ and z_4 is the linear function of e that satisfies $z_4 = z_2 + c_3 z_3 + \dot{z}_3 + O(e)^2$.

If $n = 3$, we would like to choose the observer gains so that the auxiliary variable z_4 is 0. After a straightforward calculation, one finds that

$$\begin{aligned} z_4 &= (b_{4,1} + f_{3;1} - \psi_3)e_1 + (b_{4,2} + f_{3;2} - \psi_2)e_2 \\ &\quad + (b_{4,3} + f_{3;3} - \psi_1)e_3, \end{aligned} \tag{2.11}$$

where $b_{4,j} = b_{4,j}(\hat{x})$ are functions only of $\hat{x}$ and $b_{4,j}$ depends only on c, ψ_k for $1 \leq k < 3 - j$ and $b_{r,s}$ for $1 < r < 4,\ 1 \leq r - s \leq 4 - j$:

$$\begin{aligned} b_{4,1} &= b_{2,1} - \psi_1 + c_3(b_{3,1} - \psi_2) + (b_{3,1} - \psi_2)' - (b_{3,1} - \psi_2)\psi_1 - (b_{3,2} - \psi_1)\psi_2, \\ b_{4,2} &= 1 + c_3(b_{3,2} - \psi_1) + (b_{3,2} - \psi_1)' + b_{3,1}, \\ b_{4,3} &= c_3 + b_{3,2}. \end{aligned} \tag{2.12}$$

Hence we can successively solve (2.12) for the desired observer gains,

$$\begin{aligned} \psi_1 &= b_{4,3} + f_{3;3}, \\ \psi_2 &= b_{4,2} + f_{3;2}, \\ \psi_3 &= b_{4,1} + f_{3;1}. \end{aligned}$$

Turning to the n dimensional system in observable form, the variables $z_1, \ldots, z_{n+1}$ are defined as follows:

$$\begin{aligned} z_1 &= e_1, \\ z_2 &= c_1 z_1 + \dot{z}_1 + O(e)^2, \\ z_3 &= z_1 + c_2 z_2 + \dot{z}_2 + O(e)^2, \\ &\vdots \\ z_i &= z_{i-2} + c_{i-1} z_{i-1} + \dot{z}_{i-1} + O(e)^2, \\ &\vdots \\ z_{n+1} &= z_{n-1} + c_n z_n + \dot{z}_n + O(e)^2, \end{aligned} \tag{2.13}$$

where $c_i > 0$ and the error terms are chosen so that z is a linear function of e. A straightforward calculation yields

(2.14)
$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ z_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ b_{2,1}-\psi_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1}-\psi_{n-1} & b_{n,2}-\psi_{n-2} & \cdots & 1 \\ b_{n+1,1}+f_{n;1}-\psi_n & b_{n+1,2}+f_{n;2}-\psi_{n-1} & \cdots & b_{n+1,n}+f_{n;n}-\psi_1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix},$$

where $b_{i,j} = b_{i,j}(\hat{x})$ are functions of $\hat{x}$ given by (2.7), (2.9) and for $4 \le i \le n+1$ and $2 \le j \le i-3$,

$$\begin{aligned} b_{i,1} &= b_{i-2,1} - \psi_{i-3} + c_{i-1}(b_{i-1,1} - \psi_{i-2}) + (b_{i-1,1} - \psi_{i-2})' \\ &\quad - \sum_{j=1}^{i-2} (b_{i-1,j} - \psi_{i-j-1})\psi_j, \end{aligned}$$

$$\begin{aligned} b_{i,j} &= b_{i-2,j} - \psi_{i-j-2} + c_{i-1}(b_{i-1,j} - \psi_{i-j-1}) + (b_{i-1,j} - \psi_{i-j-1})' \\ &\quad + b_{i-1,j-1}, \end{aligned} \tag{2.15}$$

$$b_{i,i-2} = 1 + c_{i-1}(b_{i-1,i-2} - \psi_1) + (b_{i-1,i-2} - \psi_1)' + b_{i-1,i-3},$$

$$b_{i,i-1} = c_{i-1} + b_{i-1,i-2}.$$

In the backstepping observer we choose the observer gains to zero the last row of the matrix in (2.14),

$$\begin{aligned} \psi_1 &= b_{n+1,n} + f_{n;n}, \\ &\ \ \vdots \\ \psi_{n-1} &= b_{n+1,2} + f_{n;2}, \\ \psi_n &= b_{n+1,1} + f_{n;1}, \end{aligned} \tag{2.16}$$

so that $z_{n+1} = 0$.

By induction one sees that $b_{i,j}$ depends only on the quantities

$$\begin{aligned} &c_1, \dots, c_{i-1}, \\ &\psi_1, \dots, \psi_{i-j-1}, \\ &b_{r,s}, \quad 1 < r < i, \quad 1 \le r-s < i-j, \end{aligned} \tag{2.17}$$

and so $b_{i,j}$ can be computed down the diagonals of (2.14). We start with the diagonal just below the main one and successively compute $b_{2,1}, b_{3,2}, \dots, b_{n+1,n}$, which yields

$$b_{i,i-1} = c_1 + \cdots + c_{i-1}.$$

Then we define ψ_1 by (2.16). Going down the diagonal two below the main we compute $b_{3,1}, b_{4,2}, \dots, b_{n+1,n-1}$ and then ψ_2, etc.

THEOREM 2.1. *Suppose that K is a compact, positively invariant set for the system* (2.1). *Consider the observer* (2.2) *with backstepping gains* (2.16) *and error dynamics* (2.3). *There exist constants $M > 0$, $\epsilon > 0$, $\gamma > 0$ such that if $x(0) \in K$ and $|e(0)| < \epsilon$, then*

$$|e(t)| < M|e(0)| \exp(-\gamma t).$$

Proof. Define

$$V = \frac{1}{2}\sum_{i=1}^{n} z_i^2; \tag{2.18}$$

then from (2.13)

$$\dot{V} = -\sum_{i=1}^{n} c_i z_i^2 + z_n z_{n+1} + O(e)^3$$

and

$$\dot{V} = -\sum_{i=1}^{n} c_i z_i^2 + O(e)^3. \tag{2.19}$$

Now let U_r be the $r > 0$ neighborhood of K; then its closure $\bar{U}_r$ is a compact subset. Hence there exist constants $N > 0, \epsilon > 0$ such that the error term in (2.19) satisfies

$$|O(e)^3| \leq N|e|^3 \tag{2.20}$$

for all $\hat{x} \in \bar{U}_r, |e| < \epsilon$. Redefine ϵ to be the smaller of r and ϵ.

From (2.14) we know that there exist constants $M_1 > 0, M_2 > 0$ such that for all $\hat{x} \in \bar{U}_r$ and all e, z,

$$M_1|e| \leq |z| \leq M_2|e|. \tag{2.21}$$

Since $c_i > 0$ there exists a constant $\gamma > 0$ such that

$$4\gamma|z|^2 \leq \sum_{i=1}^{n} c_i z_i^2. \tag{2.22}$$

Hence there is an $\epsilon > 0$ sufficiently small so that the error term in (2.19) satisfies

$$|O(e)^3| \leq \frac{1}{2}\sum_{i=1}^{n} c_i z_i^2 \tag{2.23}$$

for all $\hat{x} \in \bar{U}_r$ and all $|e| < \epsilon$. For these $\hat{x}, e$

$$\dot{V} \leq -\frac{1}{2}\sum_{i=1}^{n} c_i z_i^2 \tag{2.24}$$

$$\leq -2\gamma V. \tag{2.25}$$

Consider the set $D = \{(x, e) : x \in K, V(z(e)) < M_1\epsilon/2\}$; this is a neighborhood of $K \times \{0\}$ in x, e space. From (2.21) we see that on D, we have $|e| < \epsilon$, so $\dot{V} < -2\gamma V$, so D is positively invariant, and by Gronwall's inequality

$$V(t) \leq \exp(-2\gamma t)V(0). \tag{2.26}$$

From (2.21) we obtain

$$|e(t)| \leq \frac{M_2}{M_1}\exp(-\gamma t)|e(0)| \qquad \square$$

Remark 1. There are other possible choices of the Lyapunov function (2.18)—this one was chosen to simplify the calculations. The constants c_i appearing in (2.14) can be chosen as functions of $y, \hat{x}$ as long as they are positive and bounded away from zero.

3. Vector output systems in observable form. The above result generalizes immediately to vector output systems with all observability indices the same. These are systems of the form

$$\begin{aligned} y_l &= x_{1,l}, \\ \dot{x}_{1,l} &= x_{2,l}, \\ \dot{x}_{2,l} &= x_{3,l}, \\ &\vdots \\ \dot{x}_{k,l} &= f_{k,l}(x_1, \dots, x_k), \end{aligned} \tag{3.1}$$

where $l = 1, \dots, p$. The output $y = (y_1, \dots, y_p)$ is p dimensional and so is each $x_i = (x_{i,1}, \dots, x_{i,p})$. The state dimension is then $n = pk$ dimensional. The construction of the observer proceeds exactly as before except that the previously scalar quantities $\hat{x}_i, e_i, z_i, \psi_i$ are now p dimensional, z_i^2 is replaced by $|z_i|^2$, $z_i z_j$ is replaced by $z_i \cdot z_j$, and $\psi_i, b_{i,j}, f_{n,j}$ are $p \times p$ dimensional.

More generally we consider systems of the form

$$\begin{aligned} y_l &= x_{1,l}, \\ \dot{x}_{1,l} &= x_{2,l}, \\ \dot{x}_{2,l} &= x_{3,l}, \\ &\vdots \\ \dot{x}_{k_l,l} &= f_{k_l,l}(x_1, \dots, x_{k_l}), \end{aligned} \tag{3.2}$$

where y is p dimensional, x is $n = \sum k_l$ dimensional, and without loss of generality $k_1 \geq k_2 \geq \cdots \geq k_p$. The indices $k_1, \dots, k_p$ are the observability indices of the system [17]. The dual indices are $m_1, \dots, m_{k_1}$, where m_i is the number of k_l's that are greater than or equal to i. The subvectors x_i are defined as $x_i = (x_{i,1}, \dots, x_{i,m_i})$.

The observer is of the form

$$\begin{aligned} \dot{\hat{x}}_{1,l} &= \hat{x}_{2,l} + \psi_{1,l}(\hat{x}_1, \dots, \hat{x}_{k_l})(x_1 - \hat{x}_1), \\ \dot{\hat{x}}_{2,l} &= \hat{x}_{3,l} + \psi_{2,l}(\hat{x}_1, \dots, \hat{x}_{k_l})(x_1 - \hat{x}_1), \\ &\vdots \\ \dot{\hat{x}}_{k_l,l} &= f_{k_l,l}(\hat{x}_1, \dots, \hat{x}_{k_l}) + \psi_{k_l,l}(\hat{x}_1, \dots, \hat{x}_{k_l})(x_1 - \hat{x}_1), \end{aligned} \tag{3.3}$$

where $\psi_{r,l}(\hat{x}_1, \dots, \hat{x}_{k_l})$ is $1 \times p$ dimensional.

The error dynamics is given by

$$\begin{aligned} \dot{e}_{1,l} &= e_{2,l} - \psi_{1,l}(\hat{x}_1, \dots, \hat{x}_{k_l})e_1, \\ \dot{e}_{2,l} &= e_{3,l} - \psi_{2,l}(\hat{x}_1, \dots, \hat{x}_{k_l})e_1, \\ &\vdots \\ \dot{e}_{k_l,l} &= f_{k_l,l}(x_1, \dots, x_{k_l}) - f_{k_l,l}(\hat{x}_1, \dots, \hat{x}_{k_l}) - \psi_{k_l,l}(\hat{x}_1, \dots, \hat{x}_{k_l})e_1. \end{aligned} \tag{3.4}$$

The method is a modification of the previous approach but the notation is cumbersome. The subvector x_j is m_j dimensional and so are the subvectors $\hat{x}_j$ and e_j.

The subvector z_j is m_{j-1} dimensional and is defined by a modification of (2.13),

$$\begin{array}{lcll} z_{1,l} & = & e_{1,l}, & 1 \le l \le p, \\ z_{2,l} & = & c_{1,l} z_{1,l} + \dot{z}_{1,l} + O(e)^2, & 1 \le l \le m_1, \\ z_{3,l} & = & z_{1,l} + c_{2,l} z_{2,l} + \dot{z}_{2,l} + O(e)^2, & 1 \le l \le m_2, \\ & \vdots & & \\ z_{r+1,l} & = & z_{r-1,l} + c_{r,l} z_{r,l} + \dot{z}_{r,l} + O(e)^2, & 1 \le l \le m_r, \\ & \vdots & & \\ z_{k_1+1,l} & = & z_{k_1-1,l} + c_{k_1,l} z_{k_1,l} + \dot{z}_{k_1,l} + O(e)^2, & 1 \le l \le m_{k_1}. \end{array} \tag{3.5}$$

The auxiliary variables are the extra components of z, namely $z_{k_1+1,1}, \dots, z_{k_p+1,p}$, and the observer gains are determined by setting them to zero. If

$$V = \frac{1}{2} \sum_{l=1}^{p} \sum_{r=1}^{k_l} z_{r,l}^2,$$

then

$$\dot{V} = -\sum_{l=1}^{p} \sum_{r=1}^{k_l} c_{r,l} z_{r,l}^2 + O(e)^3$$

and the argument proceeds as before.

We illustrate with an example. Consider a three dimensional system

$$\begin{aligned} y_1 &= x_{1,1}, \\ y_2 &= x_{1,2}, \\ \dot{x}_{1,1} &= x_{2,1}, \\ \dot{x}_{1,2} &= f_{1,2}(x_{1,1}, x_{1,2}), \\ \dot{x}_{2,1} &= f_{2,1}(x_{1,1}, x_{1,2}, x_{2,1}). \end{aligned} \tag{3.6}$$

The indices are $k_1 = 2, k_2 = 1$ and the dual indices are $m_1 = 2, m_2 = 1$. The observer is of the form

$$\begin{aligned} \dot{\hat{x}}_{1,1} &= \hat{x}_{2,1} + \psi_{1,1,1}(\hat{x}_{1,1}, \hat{x}_{1,2}, \hat{x}_{2,1})(x_{1,1} - \hat{x}_{1,1}) + \psi_{1,1,2}(\hat{x}_{1,1}, \hat{x}_{1,2}, \hat{x}_{2,1})(x_{1,2} - \hat{x}_{1,2}), \\ \dot{\hat{x}}_{1,2} &= f_{1,2}(\hat{x}_{1,1}, \hat{x}_{1,2}) + \psi_{1,2,1}(\hat{x}_{1,1}, \hat{x}_{1,2})(x_{1,1} - \hat{x}_{1,1}) + \psi_{1,2,2}(\hat{x}_{1,1}, \hat{x}_{1,2})(x_{1,2} - \hat{x}_{1,2}), \\ \dot{\hat{x}}_{2,1} &= f_{2,1}(\hat{x}_{1,1}, \hat{x}_{1,2}, \hat{x}_{2,1}) + \psi_{2,1,1}(\hat{x}_{1,1}, \hat{x}_{1,2}, \hat{x}_{2,1})(x_{1,1} - \hat{x}_{1,1}) \\ &\quad + \psi_{2,1,2}(\hat{x}_{1,1}, \hat{x}_{1,2}, \hat{x}_{2,1})(x_{1,2} - \hat{x}_{1,2}), \end{aligned} \tag{3.7}$$

and the error dynamics is

$$\begin{aligned} \dot{e}_{1,1} &= e_{2,1} - \psi_{1,1,1}(\hat{x}_{1,1}, \hat{x}_{1,2}, \hat{x}_{2,1}) e_{1,1} - \psi_{1,1,2}(\hat{x}_{1,1}, \hat{x}_{1,2}, \hat{x}_{2,1}) e_{1,2}, \\ \dot{e}_{1,2} &= f_{1,2}(x_{1,1}, x_{1,2}) - f_{1,2}(\hat{x}_{1,1}, \hat{x}_{1,2}) - \psi_{1,2,1}(\hat{x}_{1,1}, \hat{x}_{1,2}) e_{1,1} - \psi_{1,2,2}(\hat{x}_{1,1}, \hat{x}_{1,2}) e_{1,2}, \\ \dot{e}_{2,1} &= f_{2,1}(x_{1,1}, x_{1,2}, x_{2,1}) - f_{2,1}(\hat{x}_{1,1}, \hat{x}_{1,2}, \hat{x}_{2,1}) \\ &\quad - \psi_{2,1,1}(\hat{x}_{1,1}, \hat{x}_{1,2}, \hat{x}_{2,1}) e_{1,1} - \psi_{2,1,2}(\hat{x}_{1,1}, \hat{x}_{1,2}, \hat{x}_{2,1}) e_{1,2}. \end{aligned} \tag{3.8}$$

From (3.5) we obtain

$$
\begin{aligned}
z_{1,1} &= e_{1,1},\\
z_{1,2} &= e_{1,2},\\
z_{2,1} &= (c_{1,1}-\psi_{1,1,1})e_{1,1}-\psi_{1,1,2}e_{1,2}+e_{2,1},\\
z_{2,2} &= \left(\frac{\partial f_{1,2}}{\partial \hat{x}_{1,1}}-\psi_{1,2,1}\right)e_{1,1}+\left(c_{1,2}+\frac{\partial f_{1,2}}{\partial \hat{x}_{1,2}}-\psi_{1,2,2}\right)e_{1,2},\\
z_{3,1} &= \Bigg(1+c_{1,1}c_{2,1}-(c_{1,1}+c_{2,1})\psi_{1,1,1}-\psi'_{1,1,1}+\psi^2_{1,1,1}\\
&\quad -\psi_{1,1,2}\left(\frac{\partial f_{1,2}}{\partial \hat{x}_{1,1}}-\psi_{1,2,1}\right)+\frac{\partial f_{2,1}}{\partial \hat{x}_{1,1}}-\psi_{2,1,1}\Bigg)e_{1,1}\\
&\quad +\Bigg(-(c_{1,1+}c_{2,1})\psi_{1,1,2}-\psi'_{1,1,2}+\psi_{1,1,1}\psi_{1,1,2}\\
&\quad -\psi_{1,1,2}\left(\frac{\partial f_{1,2}}{\partial \hat{x}_{1,2}}-\psi_{1,2,2}\right)+\frac{\partial f_{2,1}}{\partial \hat{x}_{1,2}}-\psi_{2,1,2}\Bigg)e_{1,2}\\
&\quad +\left(c_{1,1}+c_{2,1}+\frac{\partial f_{2,1}}{\partial \hat{x}_{2,1}}-\psi_{1,1,1}\right)e_{2,1}.
\end{aligned} \tag{3.9}
$$

Setting the auxiliary variables $z_{2,2}=z_{3,1}=0$, we obtain a solution

$$
\begin{aligned}
\psi_{1,2,1} &= \frac{\partial f_{1,2}}{\partial \hat{x}_{1,1}},\\
\psi_{1,2,2} &= c_{1,2}+\frac{\partial f_{1,2}}{\partial \hat{x}_{1,2}},\\
\psi_{2,1,1} &= 1+c_{1,1}c_{2,1}-(c_{1,1}+c_{2,1})\psi_{1,1,1}-\psi'_{1,1,1}+\psi^2_{1,1,1}\\
&\qquad -\psi_{1,1,2}\left(\frac{\partial f_{1,2}}{\partial \hat{x}_{1,1}}-\psi_{1,2,1}\right)+\frac{\partial f_{2,1}}{\partial \hat{x}_{1,1}},\\
\psi_{2,1,2} &= -(c_{1,1+}c_{2,1})\psi_{1,1,2}-\psi'_{1,1,2}+\psi_{1,1,1}\psi_{1,1,2}+\frac{\partial f_{2,1}}{\partial \hat{x}_{1,2}},\\
\psi_{1,1,1} &= c_{1,1}+c_{2,1}+\frac{\partial f_{2,1}}{\partial \hat{x}_{2,1}},\\
\psi_{1,1,2} &= 0.
\end{aligned} \tag{3.10}
$$

There are other solutions with $\psi_{1,1,2}\neq 0$.

A computationally simpler approach [22] is to add extra states so as to make all the observability indices the same. We illustrate with the three dimensional system (3.6) above. We imbed it in the four dimensional system

$$
\begin{aligned}
y_1 &= x_{1,1},\\
y_2 &= x_{1,2},\\
\dot{x}_{1,1} &= x_{2,1},\\
\dot{x}_{1,2} &= x_{2,2}+f_{1,2}(x_{1,1},x_{1,2}),\\
\dot{x}_{2,1} &= f_{2,1}(x_{1,1},x_{1,2},x_{2,1}),\\
\dot{x}_{2,2} &= 0.
\end{aligned} \tag{3.11}
$$

In the new coordinates

$$
\begin{aligned}
\bar{x}_{1,1} &= x_{1,1}, \\
\bar{x}_{1,2} &= x_{1,2}, \\
\bar{x}_{2,1} &= x_{2,1}, \\
\bar{x}_{2,2} &= x_{2,2} + f_{1,2}(x_{1,1}, x_{1,2}),
\end{aligned} \tag{3.12}
$$

the system is in observable form

$$
\begin{aligned}
y_1 &= \bar{x}_{1,1}, \\
y_2 &= \bar{x}_{1,2}, \\
\dot{\bar{x}}_{1,1} &= \bar{x}_{2,1}, \\
\dot{\bar{x}}_{1,2} &= \bar{x}_{2,2}, \\
\dot{\bar{x}}_{2,1} &= f_{2,1}(\bar{x}_{1,1}, \bar{x}_{1,2}, \bar{x}_{2,1}), \\
\dot{\bar{x}}_{2,2} &= \frac{\partial f_{1,2}}{\partial \bar{x}_{1,1}}(\bar{x}_{1,1}, \bar{x}_{1,2})\bar{x}_{2,1} + \frac{\partial f_{1,2}}{\partial \bar{x}_{1,2}}(\bar{x}_{1,1}, \bar{x}_{1,2}) f_{1,2}(\bar{x}_{1,1}, \bar{x}_{1,2}).
\end{aligned} \tag{3.13}
$$

This process can be repeated to make all the observability indices identical. An observer can be constructed for the higher dimensional system and since it is convergent it will yield convergent estimates for the original system.

4. Systems locally in observable form. In this section we construct an observer for a nonlinear system with scalar output of the form

$$
\begin{aligned}
\dot{\xi} &= f(\xi), \\
y &= h(\xi),
\end{aligned} \tag{4.1}
$$

where $\xi \in \mathbb{R}^n, y \in \mathbb{R}$.

Following [6] and [8], we say a system is *uniformly observable* if the mapping

$$
\xi \mapsto \begin{bmatrix} h(\xi) \\ L_f h(\xi) \\ \vdots \\ L_f^{n-1} h(\xi) \end{bmatrix} \tag{4.2}
$$

is a global diffeomorphism, where $L_f^j h(\xi)$ is the j-fold Lie derivative of h by f,

$$
\begin{aligned}
L_f h(\xi) &= \frac{\partial h}{\partial \xi}(\xi) f(\xi), \\
L_f^j h(\xi) &= \frac{\partial L_f^{j-1} h}{\partial \xi}(\xi) f(\xi).
\end{aligned} \tag{4.3}
$$

A system can be transformed globally into observable form iff it is uniformly observable. The high gain observer of Gauthier, Hammouri, and Othman [8] requires that the system be uniformly observable while our observer requires only that the system be locally uniformly observable.

A system is *locally uniformly observable* at ξ^0 if the mapping (4.2) is a local diffeomorphism on a neighborhood of ξ^0. A system is *locally uniformly observable* on a set K if it is *locally uniformly observable* at every $\xi^0 \in K$.

If a system is locally uniformly observable at ξ^0, then we can define new local coordinates around ξ^0:

$$x(\xi) = \begin{bmatrix} h(\xi) \\ L_f h(\xi) \\ \vdots \\ L_f^{n-1} h(\xi) \end{bmatrix}. \tag{4.4}$$

In these coordinates the system is in observable form (2.1) with

$$f_n(x) = L_f^n h(\xi(x)). \tag{4.5}$$

LEMMA 4.1. *Suppose that the system* (4.1) *is locally uniformly observable on a compact subset K of ξ space; then there exist an $\epsilon > 0$ and constants $M_1 > 0, M_2 > 0$ such that for all $\xi, \zeta \in K$, $|\xi - \zeta| < \epsilon$*

$$M_1|\xi - \zeta| \le |x(\xi) - x(\zeta)| \le M_2|\xi - \zeta|. \tag{4.6}$$

Proof. The map (4.2) is a local diffeomorphism so at any $\zeta \in K$ there exist $\delta(\zeta) > 0, M_1(\zeta) > 0, M_2(\zeta) > 0$ such that

$$M_1(\zeta)|\xi_1 - \xi_2| \le |x(\xi_1) - x(\xi_2)| \le M_2(\zeta)|\xi_1 - \xi_2| \tag{4.7}$$

for all $|\xi_i - \zeta| < \delta(\zeta)$. Let $B(\zeta)$ denote the open ball around ζ of radius $\delta(\zeta)/2$. These balls form an open cover of the compact set K so there exists a finite subcover $B(\zeta_1), \ldots, B(\zeta_k)$. Define

$$\begin{aligned} \epsilon &= \frac{1}{2} \min\{\delta(\zeta_1), \ldots, \delta(\zeta_k)\}, \\ M_1 &= \min\{M_1(\zeta_1), \ldots, M_1(\zeta_k)\}, \\ M_2 &= \max\{M_2(\zeta_1), \ldots, M_2(\zeta_k)\}. \end{aligned}$$

If $|\xi_1 - \xi_2| < \epsilon$, then there exists a j such that $|\xi_i - \zeta_j| < \delta(\zeta_j)$ for $i = 1, 2$, so the conclusion follows from (4.7). ☐

The observer for (4.1) will be of the form

$$\begin{aligned} \dot{\hat{\xi}} &= f(\hat{\xi}) + \phi(\hat{\xi})\,(y - \hat{y})\,, \\ \hat{y} &= h(\hat{\xi}). \end{aligned} \tag{4.8}$$

THEOREM 4.2. *Suppose the system* (4.1) *is locally uniformly observable on a compact positively invariant set K. There exists an observer* (4.8) *and constants $M > 0$, $\epsilon > 0$, $\gamma > 0$ such that if $\xi(0) \in K$ and $|\xi(0) - \hat{\xi}(0)| < \epsilon$, then*

$$|\xi(t) - \hat{\xi}(t)| < M|\xi(0) - \hat{\xi}(0)| \exp(-\gamma t).$$

Proof. Notice that the mapping (4.4) is globally defined on the compact positively invariant set K. It may fail to define global coordinates on K but it is locally one to one and so defines valid local coordinates. In these local coordinates the system is in observable form (2.1) and so we can proceed as in section 2. In the local x coordinates the observer (4.8) takes the form (2.2) and the local error dynamics is given by (2.3). It is important to note that the x variables are globally defined as are $\hat{x}, e$, although they may be valid coordinates only locally. This allows us to construct the observer in the x variables exactly as before and its local convergence is guaranteed by the above lemma. The observer in the ξ coordinates is given by (4.8), where

$$\phi(\hat{\xi}) = \left[\frac{\partial x}{\partial \xi}(\hat{\xi})\right]^{-1} \psi(x(\hat{\xi})). \tag{4.9}$$

Note that this does not require inverting $x = x(\hat{\xi})$, but it does require inverting the Jacobian matrix $\frac{\partial x}{\partial \xi}(\hat{\xi})$ at each $\hat{\xi}$. □

5. Systems with inputs. In this section we consider systems with inputs

$$\begin{aligned} \dot{\xi} &= f(\xi, u), \\ y &= h(\xi). \end{aligned} \tag{5.1}$$

The state trajectory of a system in observable form (2.1) is completely determined by the output trajectory. The generalization of observable form to a system with inputs is one of the form

$$\begin{aligned} y &= x_1, \\ \dot{x}_1 &= x_2 + g_1(x_1, u), \\ \dot{x}_2 &= x_3 + g_2(x_1, x_2, u), \\ &\vdots \\ \dot{x}_{n-1} &= x_n + g_{n-1}(x_1, \ldots, x_{n-1}, u), \\ \dot{x}_n &= f_n(x) + g_n(x, u). \end{aligned} \tag{5.2}$$

Such a system is said to be *uniformly observable for any input* [9]. Regardless of what input $u = u(t)$ is chosen, the system is observable in the sense that the output and input trajectories uniquely determine the state trajectory,

$$\begin{aligned} x_1 &= y, \\ x_2 &= \dot{x}_1 - g_1(x_1, u), \\ x_3 &= \dot{x}_2 - g_2(x_1, x_2, u), \\ &\vdots \\ x_n &= \dot{x}_{n-1} - g_{n-1}(x_1, \ldots, x_{n-1}, u). \end{aligned}$$

We assume that the state estimate from the observer will be used in a feedback law $u = \kappa(\hat{x})$ to control the system. For a system that is uniformly observable for any input (5.2), the observer will be in the following form:

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + g_1(\hat{x}_1, \kappa(\hat{x})) + \psi_1(\hat{x})(x_1 - \hat{x}_1), \\ \dot{\hat{x}}_2 &= \hat{x}_3 + g_2(\hat{x}_1, \hat{x}_2, \kappa(\hat{x})) + \psi_2(\hat{x})(x_1 - \hat{x}_1), \\ &\vdots \\ \dot{\hat{x}}_{n-1} &= \hat{x}_n + g_{n-1}(\hat{x}_1, \ldots, \hat{x}_{n-1}, \kappa(\hat{x})) + \psi_{n-1}(\hat{x})(x_1 - \hat{x}_1), \\ \dot{\hat{x}}_n &= f_n(\hat{x}) + g_n(\hat{x}, \kappa(\hat{x})) + \psi_n(\hat{x})(x_1 - \hat{x}_1). \end{aligned} \tag{5.3}$$

The error $e = x - \hat{x}$ dynamics is given by

$$
\begin{aligned}
\dot{e}_1 &= e_2 + g_1(x_1, \kappa(\hat{x})) - g_1(\hat{x}_1, \kappa(\hat{x})) - \psi_1(\hat{x})e_1,\\
\dot{e}_2 &= e_3 + g_2(x_1, x_2, \kappa(\hat{x})) - g_2(\hat{x}_1, \hat{x}_2, \kappa(\hat{x})) - \psi_2(\hat{x})e_1,\\
&\vdots\\
\dot{e}_{n-1} &= e_n + g_{n-1}(x_1, \dots, x_{n-1}, \kappa(\hat{x})) - g_{n-1}(\hat{x}_1, \dots, \hat{x}_{n-1}, \kappa(\hat{x})) - \psi_{n-1}(\hat{x})e_1,\\
\dot{e}_n &= f_n(x) - f_n(\hat{x}) + g_n(x, \kappa(\hat{x})) - g_n(\hat{x}, \kappa(\hat{x})) - \psi_n(\hat{x})e_1.
\end{aligned} \tag{5.4}
$$

The observer is constructed as before; define variables $z_1, \dots, z_{n+1}$ by (2.13), where the error terms are chosen so that z is a linear function of e, (2.14). The coefficients $b_{i,j}(\hat{x})$ are given by the generalization of (2.7), (2.9), and (2.15),

$$
\begin{aligned}
b_{2,1} &= c_1 + g_{1,1},\\
b_{3,1} &= 1 + c_2(b_{2,1} - \psi_1) + (b_{2,1} - \psi_1)' + (b_{2,1} - \psi_1)(g_{1,1} - \psi_1) + g_{2,1},\\
b_{3,2} &= c_2 + b_{2,1} + g_{2,2},
\end{aligned} \tag{5.5}
$$

and for $4 \le i \le n+1$ and $2 \le j \le i-3$

$$
\begin{aligned}
b_{i,1} &= b_{i-2,1} - \psi_{i-3} + c_{i-1}(b_{i-1,1} - \psi_{i-2}) + (b_{i-1,1} - \psi_{i-2})'\\
&\quad - \sum_{j=1}^{i-2}(b_{i-1,j} - \psi_{i-j-1})(\psi_j - g_{j;1}) + g_{i-1;1},\\
b_{i,j} &= b_{i-2,j} - \psi_{i-j-2} + c_{i-1}(b_{i-1,j} - \psi_{i-j-1}) + (b_{i-1,j} - \psi_{i-j-1})'\\
&\quad + b_{i-1,j-1} + \sum_{k=j}^{i-2}(b_{i-1,k} - \psi_{i-k-1})g_{k;j} + g_{i-1;j},\\
b_{i,i-2} &= 1 + c_{i-1}(b_{i-1,i-2} - \psi_1) + (b_{i-1,i-2} - \psi_1)' + b_{i-1,i-3}\\
&\quad + (b_{i-1,i-2} - \psi_1)g_{i-2;i-2} + g_{i-1;i-2},
\end{aligned}
$$

$$
b_{i,i-1} = c_{i-1} + b_{i-1,i-2} + g_{i-1;i-1}, \tag{5.6}
$$

where

$$
g_{i;j}(\hat{x}) = \frac{\partial g_i}{\partial x_j}(\hat{x}, \kappa(\hat{x})) \tag{5.7}
$$

and the operation $'$ is defined on functions $\phi(\hat{x})$ by

$$
\phi'(\hat{x}) = \sum_{j=1}^{n-1} \frac{\partial \phi}{\partial \hat{x}_j}(\hat{x})\left(\hat{x}_{j+1} + g_j(\hat{x}, \kappa(\hat{x}))\right) + \frac{\partial \phi}{\partial \hat{x}_n}\left(f_n(\hat{x}) + g_n(\hat{x}, \kappa(\hat{x}))\right). \tag{5.8}
$$

Notice that as before that $'$ does not involve the gains ψ_i and

$$
\phi' = \dot{\phi} + O(e).
$$

Define

$$
V = \frac{1}{2}\sum_{i=1}^{n} z_i^2; \tag{5.9}
$$

then from (2.13)

$$\dot{V} = -\sum_{i=1}^{n} c_i z_i^2 + z_n z_{n+1} + O(e)^3. \tag{5.10}$$

We choose the observer gains

$$\begin{aligned} \psi_1 &= b_{n+1,n} + f_{n;n}, \\ &\vdots \\ \psi_{n-1} &= b_{n+1,2} + f_{n;2}, \\ \psi_n &= b_{n+1,1} + f_{n;1} \end{aligned} \tag{5.11}$$

so that $z_{n+1} = 0$ and

$$\dot{V} = -\sum_{i=1}^{n} c_i z_i^2 + O(e)^3. \tag{5.12}$$

A system (5.1) is said to *locally uniformly observable for any input* if around every ξ^0 the transformation (4.2) locally carries it to the form (5.2). For such systems the above algorithm will yield an observer on any compact positively invariant set.

If the system is not locally uniformly observable for any input, then one can still attempt to design the observer by the above algorithm by defining variables $z_1, \ldots, z_{n+1}$ by (2.13), where the error terms are chosen so that z is a linear function of e, (2.14). The triangular structure of (2.14) will be lost so there is no guarantee that the transformation from e to z is invertible. If it is, then the algorithm yields a locally convergent observer.

Suppose $U(x)$ is a Lyapunov function for the system (5.1) under full state feedback $u = \kappa(x)$,

$$\dot{U}(x) = \frac{\partial U}{\partial x}(x) f(x, \kappa(x)) \leq 0,$$

and suppose that the Lipschitz conditions

$$\begin{aligned} \left| \frac{\partial U}{\partial x}(x) \left(f(x,u) - f(x,\kappa(x)) \right) \right| &\leq M \left| u - \kappa(x) \right|, \\ \left| \kappa(x) - \kappa(\hat{x}) \right| &\leq M \left| x - \hat{x} \right| \end{aligned}$$

hold for some constant M. If we are able to design an observer using the backstepping technique, then we can choose $c_1 = \cdots = c_n = N$ so that $U(x) + V^{\frac{1}{2}}(e, \hat{x})$ is a Lyapunov function for the combined system. For $|e|$ sufficiently small by (2.21) and (2.23),

$$\begin{aligned} \frac{d}{dt}\left(U + V^{\frac{1}{2}} \right) &= \frac{\partial U}{\partial x}(x) f(x, \kappa(\hat{x})) + \frac{1}{2} V^{-\frac{1}{2}} \dot{V} \\ &= \frac{\partial U}{\partial x}(x) f(x, \kappa(x)) + (M^2 - 2^{-\frac{1}{2}} N M_1)|e| \leq 0 \end{aligned}$$

if N is sufficiently large, where M_1 satisfies (2.21). Hence for small initial estimation errors, the output feedback certainty equivalence control inherits the stability of the state feedback control.

6. The high gain observer. In this section, we compare the high gain observer of [8] with high gain, the same observer using low gain, and the backstepping observer. Consider a smooth, scalar output nonlinear system in observable form:

$$
\begin{aligned}
y &= x_1, \\
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
&\vdots \\
\dot{x}_{n-1} &= x_n, \\
\dot{x}_n &= f_n(x).
\end{aligned}
\tag{6.1}
$$

The observer proposed in [8] is of the form

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + L_1(y - \hat{x}_1), \\
\dot{\hat{x}}_2 &= \hat{x}_3 + L_2(y - \hat{x}_1), \\
&\vdots \\
\dot{\hat{x}}_{n-1} &= \hat{x}_n + L_{n-1}(y - \hat{x}_1), \\
\dot{\hat{x}}_n &= f_n(\hat{x}) + L_n(y - \hat{x}_1)
\end{aligned}
\tag{6.2}
$$

with

$$
L_i = \begin{pmatrix} n \\ i \end{pmatrix} \theta^i,
$$

where θ is a parameter that must be chosen "sufficiently large" to insure global convergence—just how large is never explicitly stated. A mathematical proof of the convergence of the high gain observer with sufficiently high gain has been given in [8] for systems without noise. But because of the high gain even relatively small noise can degrade the performance of the high gain observer.

Suppose the variables and functions are of order one and hence one might choose θ to be an order of magnitude bigger, say $\theta = 10$, so as to be "sufficiently large." If the system is three dimensional, then the largest gain is 1000! If the other variables are of order one, then the right side of the observer dynamics (6.2) is completely dominated by its gain times innovation term. The innovation is $y - \hat{x}_1$. If there is any observation noise, then this is magnified by the gains in the error dynamics. For example, suppose there is observation noise of order $\epsilon = 0.01$ so the signal to noise ratio is 100—not a bad situation. However, the noise in the gain times innovation term of the last state is of order 10 while the state is of order 1. When $\hat{x}_1 \approx x_1$, the signal to noise ratio in the observer dynamics is 0.1—hardly conducive to accurate estimation. Even if there is no observation noise, driving noises can have similar but less dramatic effects.

Many successful applications of the high gain observer have been reported in the literature [8], [6], [7], [9]. In most of these applications, a high gain is not actually used. The method of Gauthier, Hammouri, and Othman [8] is used to design an observer but the gain parameter, θ, is chosen to be relatively small. No attempt is made to determine how large θ must be to guarantee global convergence. It appears that the high gain observer with low gain is actually an excellent local observer and this is why it has been successful in applications. It would nice to have a theoretical explanation for why this is so.

Table 6.1
Mean square errors of different observers.

State error	High gain $\theta = 8$	Low gain $\theta = 2$	Backstepping $c_i = 1$
e_1	1.32e-05	3.31e-06	6.27e-07
e_2	6.49e-04	1.22e-05	8.93e-06
e_3	4.18e-03	3.16e-05	1.15e-04

We give a simple example exhibiting this problem. Consider a three dimensional system

$$y = Cx + v, \tag{6.3}$$

$$\dot{x} = Ax + g(x), \tag{6.4}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ 7\sin(x_1^2 + x_2^2 + x_3^2) \end{bmatrix}.$$

There is one equilibrium at $x = 0$ which is asymptotically stable as the eigenvalues of A are $\{-1, -2, -3\}$. The nonlinear term g is bounded. The observation noise v is assumed to be small, band limited, Gaussian noise.

The observer with noise is

$$\dot{\hat{x}} = A\hat{x} + g(\hat{x}) + L(x_1 - \hat{x}_1 + v),$$

where the observer gain is

$$L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} 3\theta \\ 3\theta^2 \\ \theta^3 \end{bmatrix}.$$

To estimate how large the gain parameter θ should be, we started the system at $x(0) = (0, 0, 0)$ and the observer at $\hat{x}(0) = (0, 0, 1)$. The noiseless observer did not converge to the true value when $\theta = 7$ but did converge when $\theta = 8$, so we chose the latter value. We simulated three observers with small observation noise starting from the true state $\hat{x}(0) = x(0) = (0, 0, 0)$. We used white Gaussian noise of covariance 1e$-$04, sampled and held for 0.1 second. The first was the observer [8] with a high gain $\theta = 8$, the second was the observer [8] with a relatively low gain $\theta = 2$ and the third was the backstepping observer that is presented above with all the design parameters $c_i = 1$. As can be seen from Table 6.1, the high gain observer performs poorly as compared with the low gain and backstepping observers, which are comparable in performance. Table 6.2 contains the errors of high and low gain observers relative to the backstepping observer.

It should be noted that in the absence of noise there is no assurance that $\theta = 8$ is high enough so that the high gain observer converges globally or even locally. On

TABLE 6.2
Relative errors of different observers.

State error	High gain $\theta = 8$	Low gain $\theta = 2$	Backstepping $c_i = 1$
e_1	21.0	5.3	1
e_2	72.7	1.4	1
e_3	36.2	0.3	1

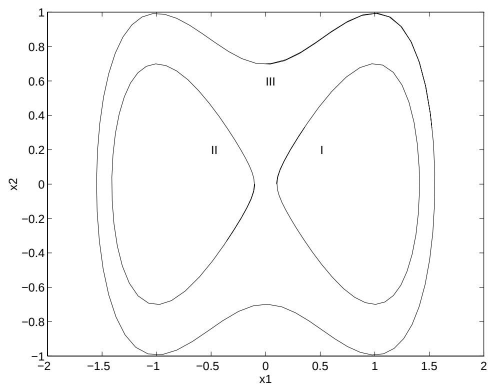

FIG. 1. *The trajectories of Duffing's equation.*

the other hand we have proven that the backstepping observer converges locally for any $c_i > 0$.

One might ask how the example of [8] avoids this high gain difficulty in its the noisy simulations. The answer is very simple: by taking small noise and $\theta = 1,\ n = 2$, it is not really high gain.

7. Examples. The design algorithm described in section 2 has been implemented in MAPLE. This is applied to some examples.

7.1. Duffing's equation. Duffing's equation [11], [14] is

$$\begin{aligned} y &= \xi_1 + \frac{1}{2}\xi_2, \\ \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= \xi_1 - \xi_1^3. \end{aligned} \tag{7.1}$$

It is a conservative system with the energy function

$$E(\xi) = \frac{1}{2}\xi_2^2 - \frac{1}{2}\xi_1^2 + \frac{1}{4}\xi_1^4.$$

It has three equilibrium points, $\xi^0 = (0,0)$, $\xi^I = (1,0)$, and $\xi^{II} = (-1,0)$. There are three typical trajectories (see Figure 1): one is around ξ^I (type I), one is around ξ^{II} (type II), and the third one encloses all three equilibria (type III). We define the compact positively invariant region K to be the area enclosed by a trajectory of type III. The system is locally uniformly observable on $\mathbb{R}^2$. The observation function, $h(\xi) = \xi_1 + \frac{1}{2}\xi_2$, was chosen so that the system is not in observable form. The system can be transformed globally into observable form but cannot be transformed into the

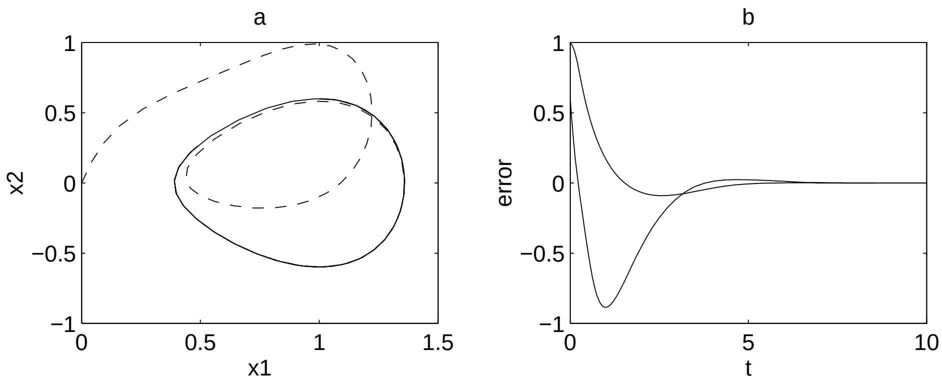

FIG. 2. *Estimation of trajectory of type* I *for Duffing's equation.*

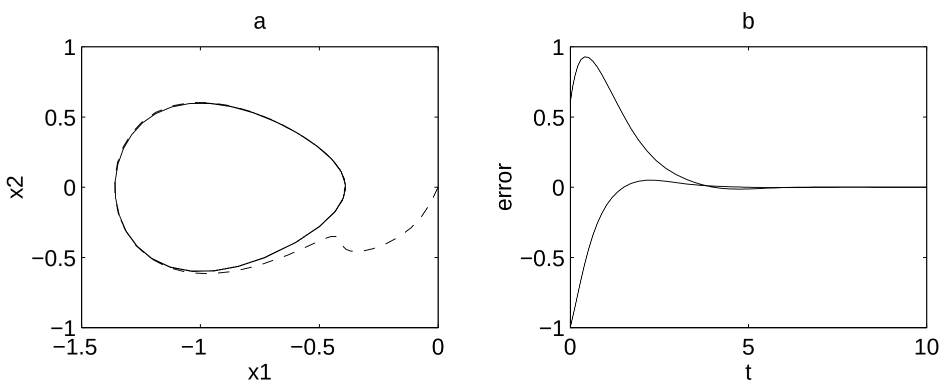

FIG. 3. *Estimation of trajectory of type* II *for Duffing's equation.*

observer form by output injection and change of coordinates [17]. When $c_1 = c_2 = 1$, the observer gain is

$$\phi_1(\hat{\xi}) = -\frac{-2 - 14\,\hat{\xi}_1^2 - 14\,\hat{\xi}_1^4 - 8\,\hat{\xi}_1\hat{\xi}_2 - 8\,\hat{\xi}_1^3\hat{\xi}_2 - 4\,\hat{\xi}_2^2 + 12\,\hat{\xi}_2^2\hat{\xi}_1^2 - 2\,\hat{\xi}_1^6}{3\left(1+\hat{\xi}_1^2\right)^3},$$

$$\phi_2(\hat{\xi}) = \frac{8 + 8\,\hat{\xi}_1^2 + 8\,\hat{\xi}_1^4 - 4\,\hat{\xi}_1\hat{\xi}_2 + 8\,\hat{\xi}_1^3\hat{\xi}_2 + 8\,\hat{\xi}_1^6 + 12\,\hat{\xi}_1^5\hat{\xi}_2 - 8\,\hat{\xi}_2^2 + 24\,\hat{\xi}_2^2\hat{\xi}_1^2}{3\left(1+\hat{\xi}_1^2\right)^3}.$$

Three simulations of the system and the backstepping observer are shown in Figures 2, 3, and 4 for trajectory types I, II, and III. Notice that all the simulations use the same observer with the same gain and the same initial estimate but different initial states. The state trajectories are of different types around different equilibrium points.

The solid and dotted curves in Figures 2a, 3a, and 4a are the graphs of the trajectories of the system and the observer. The curves in Figures 2b, 3b, and 4b show the errors $e_1 = \xi_1 - \hat{\xi}_1$ and $e_2 = \xi_2 - \hat{\xi}_2$.

7.2. Homoclinic bifurcation. The backstepping approach can be used to design observers for systems with parameters. Such systems can undergo bifurcations.

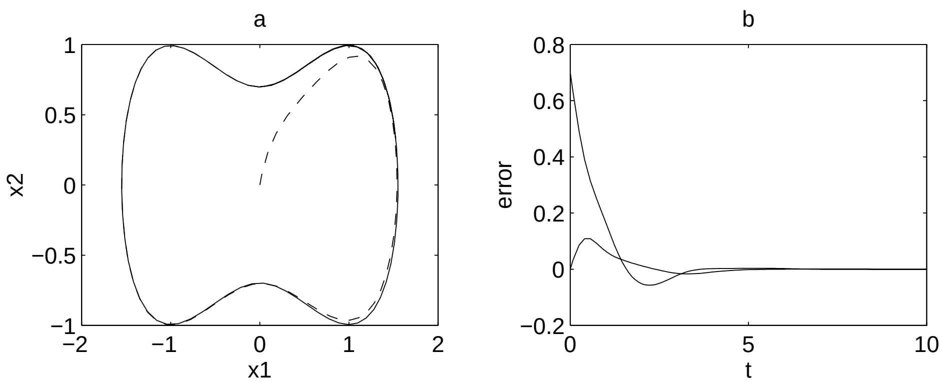

FIG. 4. *Estimation of trajectory of type* III *for Duffing's equation.*

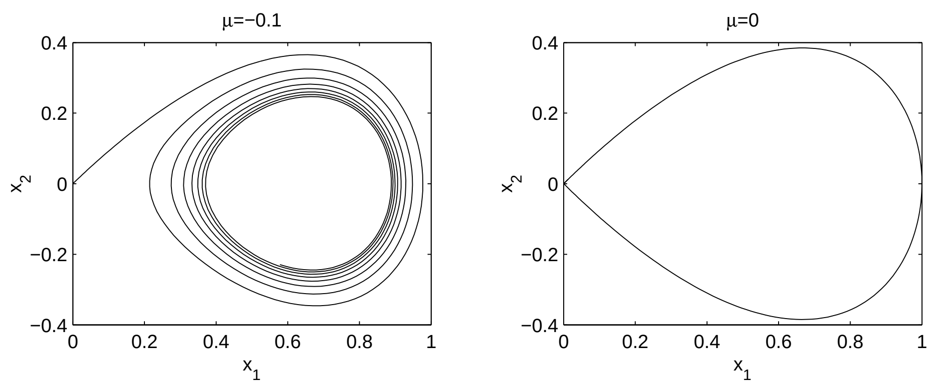

FIG. 5. *The homoclinic bifurcation.*

Consider the following system from [11]:

$$
\begin{aligned}
y &= \xi_1, \\
\dot{\xi}_1 &= 2\xi_2, \\
\dot{\xi}_2 &= 2\xi_1 - 3\xi_1^2 - \xi_2(\xi_1^3 - \xi_1^2 + \xi_2^2 - \mu).
\end{aligned}
\tag{7.2}
$$

The system depends on a parameter c. For all values of the parameter, there is a saddle at $(0, 0)$ with one stable and one unstable direction and an unstable source at $(2/3, 0)$. For $-4/27 < \mu < 0$, there is an asymptotically stable periodic orbit around the unstable source. At $\mu = 0$, the periodic orbit becomes a homoclinic orbit consisting of branches of the stable and unstable manifolds of the saddle. For $\mu > 0$, there are no periodic orbits nearby (see Figure 5). For $\mu < 0$ we can find a compact positively invariant set K containing the attracting limit cycle, and for $\mu = 0$ we can take as K the compact set consisting of the homoclinic orbit and it interior. Because of the parameter μ, (7.2) represents a family of systems. However, the computational algorithm for the observer gain is implemented symbolically and μ can be treated as a parameter in the observer. Notice the construction of the observer does not depend on K.

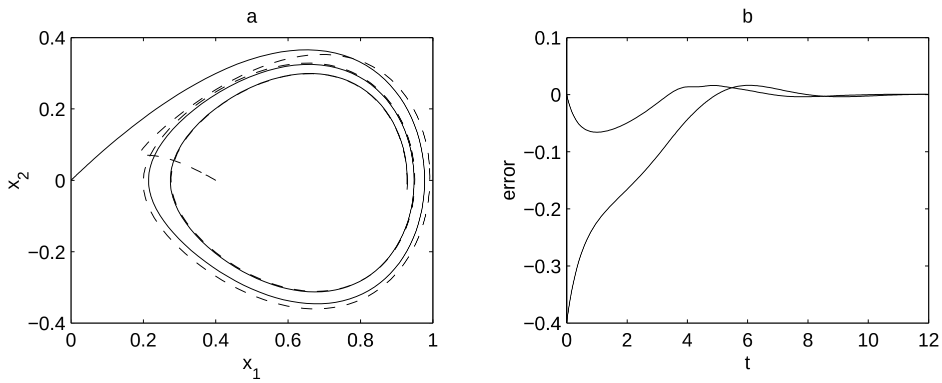

FIG. 6. *Estimation around the periodic solution with* $\mu = -0.1$.

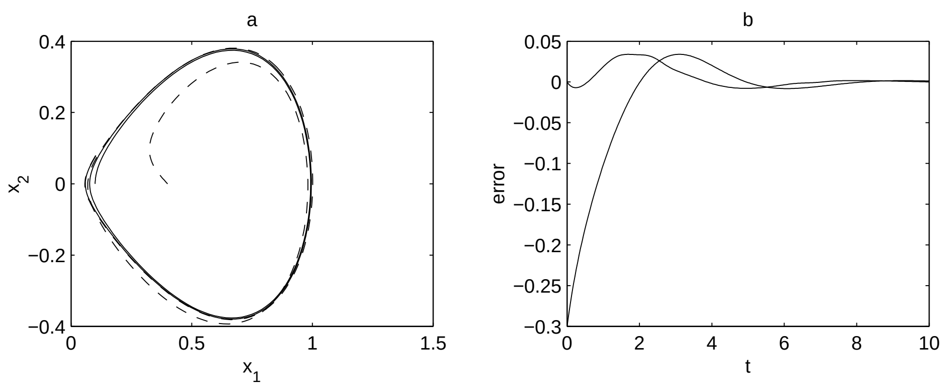

FIG. 7. *Initial state and estimate inside the homoclinic loop with* $\mu = 0$.

If we set $c_1 = c_2 = 1$, the observer gains are

$$\begin{aligned}\phi_1(\hat{\xi}) &= 2 - \hat{\xi}_1^3 + \hat{\xi}_1^2 - 3\,\hat{\xi}_2^2 + \mu,\\ \phi_2(\hat{\xi}) &= 6\,\hat{\xi}_2\hat{\xi}_1 - 6\,\hat{\xi}_1 + \hat{\xi}_1^2 - \hat{\xi}_1^3 - 3\,\hat{\xi}_2^2 + \mu\\ &\quad + \frac{3\,\hat{\xi}_2^4}{2} + \frac{\hat{\xi}_1^6}{2} - \hat{\xi}_1^5 + \frac{\hat{\xi}_1^4}{2} - \hat{\xi}_1^3\mu + \hat{\xi}_1^2\mu + \frac{\mu^2}{2} - 9\,\hat{\xi}_2\hat{\xi}_1^2 + 3.\end{aligned}$$

The performance of the observer for $\mu = -0.1$ and $\mu = 0$ are shown in Figures 6, 7, and 8. In Figures 6a, 7a, and 8a the trajectories of (7.2) (solid curves) and the trajectories of the observer (dotted curves) are shown. The estimation error is plotted in Figures 6b, 7b, and 8b. Notice that, in Figure 8, the state starts inside the homoclinic orbit, the estimate starts outside where the system is unstable, and the observer still converges.

8. Conclusion. We have presented a method for designing observers for nonlinear systems based on the backstepping. The method is broadly applicable and the observer error exponentially converges to zero provided the initial error is not too large. It is applicable to a slightly broader class of systems than the high gain observer of Gauthier, Hammouri, and Othman [8] but differs in that the gain is not high and the convergence is only local. The method is easily implemented in a symbolic

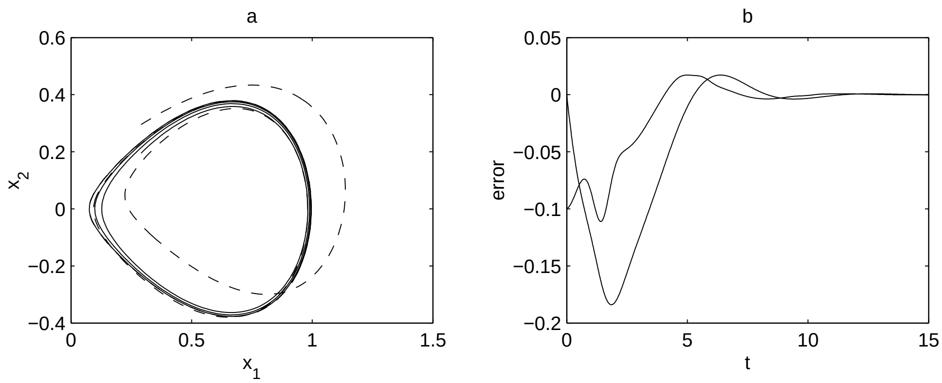

FIG. 8. *Initial state inside the homoclinic loop and initial estimate outside the homoclinic loop with* $\mu = 0$.

computational package such as MAPLE.

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