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On the existence of not necessarily unique solutions of the classical hyperbolic boundary value problems for non-linear second order partial differential equations in two independent variables.

Leehey, Patrick

Brown University



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ON THE EXISTENCE OF NOT NECESSARILY
UNIQUE SOLUTIONS OF THE CLASSICAL HYPER-
BOLIC BOUNDARY VALUE PROBLEMS FOR NON-
LINEAR SECOND ORDER PARTIAL DIFFERENTIAL
EQUATIONS IN TWO INDEPENDENT VARIABLES.

By

Patrick Leehey

B.Sc., United States Naval Academy, 1942

Thesis

submitted in partial fulfillment of the requirements for the
Degree of Doctor of Philosophy in the Graduate Division
of Applied Mathematics at Brown University
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VITA

Patrick Leshey was born at Waterloo, Iowa, October 27, 1921. He attended the College of Engineering, State University of Iowa 1938-1939. Attended the U. S. Naval Academy 1939-1942, receiving the degree of Bachelor of Science in 1942. He was commissioned as Ensign, U. S. Navy, 1942. Served with the U. S. Pacific Fleet 1942-1945. Attended the U. S. Naval Postgraduate School in the course in Naval Engineering Design 1946-1947. Attended Brown University in the Graduate Division of Applied Mathematics 1947-1950. Member of Sigma Xi. He holds the rank of Lieutenant, U.S. Navy.

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NOTATIONS

The following special notations will be used throughout this paper with the meanings as defined below. Other special notations used will be defined at the place where they are introduced.

$$E: \begin{cases} 0 \leq x \leq \lambda \\ 0 \leq y \leq \lambda \end{cases}$$

is a number of; i.e. belongs to.

E is the set of all ordered pairs (x,y) , (points) for which $0 \leq x \leq \lambda$ and $0 \leq y \leq \lambda$.

$$f \in C(B)$$

f is a member of the class of functions continuous on the set B .

$$g \in C^1(H)$$

g is a member of the class of functions continuously differentiable on the set H , (and similarly for higher degrees of differentiability.)

$$u_x$$

$$\frac{\partial u}{\partial x}$$

$$u_{\lambda, x}$$

$$\frac{\partial u_\lambda}{\partial x}$$

$$\dot{x}$$

$\frac{dx}{d\tau}$ where τ is a parameter along a path.

$$x \in [0, \lambda]$$

x belongs to the closed interval, $0 \leq x \leq \lambda$.

$$\implies$$

implies.

$$\iff$$

implies and is implied by; i.e. if and only if.

$$\{g_\lambda\} (x,y; u; p,q)$$

a sequence of functions g_λ , ($\lambda = 1, 2, \dots$), of arguments $(x,y; u; p,q)$.

$$\{g_\lambda\} \rightarrow f \text{ on } E$$

the sequence $\{g_\lambda\}$ converges pointwise on the set E to the function f .

THEOREM

Let $f(x)$ be a function defined on the interval $[a, b]$. If $f(x)$ is continuous on $[a, b]$, then $f(x)$ is integrable on $[a, b]$.

Proof: Let $\epsilon > 0$ be given. Since $f(x)$ is continuous on $[a, b]$, it is uniformly continuous on $[a, b]$. Therefore, there exists a $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

$$\left. \begin{aligned} & \text{Let } \epsilon > 0 \text{ be given.} \\ & \text{Since } f(x) \text{ is continuous on } [a, b], \\ & \text{it is uniformly continuous on } [a, b]. \end{aligned} \right\} \text{Hence, there exists a } \delta > 0 \text{ such that if } |x - y| < \delta, \text{ then } |f(x) - f(y)| < \epsilon.$$

Let P be a partition of $[a, b]$ with mesh δ . Then $|x_i - x_{i-1}| < \delta$ for all i . Therefore, $|f(x_i) - f(x_{i-1})| < \epsilon$ for all i . This implies that the upper and lower Riemann sums differ by less than $\epsilon(b-a)$.

$$\begin{aligned} & \text{Hence, } f(x) \text{ is integrable on } [a, b]. \\ & \text{Q.E.D.} \end{aligned}$$

$$\frac{1}{x^2} = x^{-2}$$

Let $f(x) = \frac{1}{x^2}$. Then $f(x) = x^{-2}$. The derivative of x^{-2} is $-2x^{-3} = -\frac{2}{x^3}$. Therefore, $\int \frac{1}{x^2} dx = -\frac{1}{x} + C$.

$$\begin{aligned} & \int \frac{1}{x^2} dx = \int x^{-2} dx \\ & = \frac{x^{-2+1}}{-2+1} + C \\ & = -\frac{1}{x} + C \end{aligned}$$

Q.E.D.

Let $f(x) = \frac{1}{x^2}$. Then $f(x) = x^{-2}$. The derivative of x^{-2} is $-2x^{-3} = -\frac{2}{x^3}$. Therefore, $\int \frac{1}{x^2} dx = -\frac{1}{x} + C$.

$$\int \frac{1}{x^2} dx = -\frac{1}{x} + C$$

$\{g_\lambda\} \xrightarrow{\text{unif}} f$ on B

the sequence $\{g_\lambda\}$ converges uniformly on the set B to the function f .

$D_\pm y$

the right (+) and left (-) hand derivatives of the function y at the point in question.

CHAPTER I

INTRODUCTION

The purpose of this paper is to present a number of existence theorems pertaining to a class of non-linear second order partial differential equations in two independent variables of the general form

$$(1.1) \quad F(x,y; u; p,q; r,s,t) = 0,$$

where

$$(1.2) \quad p = u_x, \quad q = u_y, \quad r = u_{xx}, \quad s = u_{xy} \quad \text{and} \quad t = u_{yy},$$

in the usual notation. We restrict our attention to those prescriptions of initial conditions for which integral surfaces exist such that the equation is of hyperbolic type thereon, i.e. the inequality

$$(1.3) \quad F_s^2 - 4 F_r F_t > 0$$

must be satisfied on the integral surface in a neighborhood of the initial data.

E. PICARD [1],[7]¹, E. COURSAT [8], E.E. Levi[9], H. LEWY[10], J. HADAMARD[11], M. CINQUINI-CIBRARIO[12],[13], and others have

¹ The number in the bracket [] refers to the reference in the bibliography.

CHAPTER I

DEFINITIONS

The process of this work is to show that the theory of
 the numbers is a branch of the theory of sets, and that
 the theory of sets is a branch of the theory of logic.
 The theory of logic is the study of the principles of
 reasoning and the laws of thought.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad (1.1)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad (1.2)$$

The first of these laws is known as the distributive law,
 and the second as the dual distributive law. These laws
 are fundamental in the theory of sets, and are used
 to prove many other important results. The theory of
 sets is a branch of the theory of logic, and is
 concerned with the study of the principles of
 reasoning and the laws of thought.

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developed existence theorems based on the method of successive approximations. Their concern has been to establish sufficient conditions for the existence of a unique solution. Retaining their restrictions on the initial data, we shall obtain sufficient conditions for the existence of at least one solution. The integrals of the equations we consider will not, in general, be unique.

The concept of characteristic curves in an integral surface plays an important role in all work in this field. We give two definitions of a characteristic curve, the first applicable when the curve is expressed in non-parametric form, the second when expressed in parametric form:

Definition 1

$$\gamma: \begin{cases} a \leq x \leq b \\ y = g(x) \end{cases} \text{ where } g \in C'([a,b]), \text{ or } \gamma: \begin{cases} x = h(y) \\ c \leq y \leq d \end{cases}$$

where $h \in C'([c,d])$, is a characteristic base curve (characteristic projection or, by usage, characteristic) for a particular integral surface $J: u = u(x,y)$ of $F(x,y; u; p,q; r,s,t) = 0 \iff$ for each (x,y)

$$(1.4) \quad F_r dy^2 - F_s dy dx + F_t dx^2 = 0$$

Definition 1a

$$\gamma: \begin{cases} x = x(\tau) \\ y = y(\tau) \end{cases} \text{ for } \tau \in [0,1] \text{ and where } x,y \in C'([0,1]), \text{ is a}$$

characteristic base curve for a particular integral surface $J: u = u(x,y)$ of $F(x,y; u; p,q; r,s,t) = 0 \iff$ for each $\tau \in [0,1]$

$$(1.5) \quad \begin{cases} 1) & F_r \dot{y}^2 - F_s \dot{y} \dot{x} + F_t \dot{x}^2 = 0 \\ 2) & \dot{x}^2 + \dot{y}^2 \neq 0. \end{cases}$$

The first part of the paper is devoted to the study of the
 asymptotic behavior of the solutions of the system

$$\dot{x} = Ax + B u$$
 where A and B are constant matrices. The second part
 is devoted to the study of the asymptotic behavior of the
 solutions of the system

$$\dot{x} = A(x) + B(x) u$$
 where $A(x)$ and $B(x)$ are matrices depending on x .

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Under either definition Υ is rectifiable and possesses a continuously turning tangent (see C. JORDAN[6], p. 100). The two definitions are equivalent in the following sense: We may convert Υ expressed in non-parametric form into its parametric expression by setting $x = \tau$, $y = g(\tau)$, or $x = h(\tau)$, $y = \tau$ as the case may be. That the converse is possible follows directly from condition 2) of Definition 1a and the Implicit Function Theorem. For, suppose at a point $(x(\tau_0), y(\tau_0))$ of Υ that $\dot{x} \neq 0$. Then in a vicinity of $x_0 = x(\tau_0)$ the inverse relation $\tau = \tau(x)$ exists and we may write

$$(1.6) \quad \Upsilon : y = y(\tau(x)) = g(x).$$

Similarly, where $\dot{y} \neq 0$, we may write

$$(1.7) \quad \Upsilon : x = x(\tau(y)) = h(y).$$

By condition 2), one of the two representations (1.6) or (1.7) is always possible in the vicinity of each point of Υ .

Definition 2

$$\Gamma : \begin{cases} x = x(\tau) \\ y = y(\tau) \\ u = u(\tau) \end{cases} \text{ for } \tau \in [0,1] \text{ and where } x, y, u \in C'([0,1]),$$

a space curve lying in a particular integral surface $J: u=u(x,y)$ of $F(x,y; u; p,q; r,s,t) = 0$, is called a characteristic curve in the integral surface $J \iff$ the projection of Γ onto the xy plane is a characteristic projection for the integral surface J .

Let γ be a simple closed curve in \mathbb{R}^n . The orientation of γ is defined to be the direction in which the curve is traversed. If γ is traversed in the counter-clockwise direction, then γ is said to be positively oriented. If γ is traversed in the clockwise direction, then γ is said to be negatively oriented. The orientation of γ is denoted by $\text{or}(\gamma)$. The orientation of γ is important in the definition of the line integral of a vector field over γ . The line integral of a vector field \mathbf{F} over γ is defined to be $\int_{\gamma} \mathbf{F} \cdot d\mathbf{r}$, where $d\mathbf{r}$ is the differential displacement vector along γ . The orientation of γ determines the direction of $d\mathbf{r}$.

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \quad (1.1)$$

where $\mathbf{r}(t)$ is a parametrization of γ from $t=a$ to $t=b$.

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \quad (1.2)$$

where $\mathbf{r}(t)$ is a parametrization of γ from $t=a$ to $t=b$. The orientation of γ is important in the definition of the line integral of a vector field over γ .

The line integral of a vector field \mathbf{F} over a curve γ is defined to be $\int_{\gamma} \mathbf{F} \cdot d\mathbf{r}$. The orientation of γ is important in the definition of the line integral of a vector field over γ . The line integral of a vector field \mathbf{F} over a curve γ is defined to be $\int_{\gamma} \mathbf{F} \cdot d\mathbf{r}$. The orientation of γ is important in the definition of the line integral of a vector field over γ .

Under suitable hypotheses, by virtue of the hyperbolic condition (1.3), for any integral surface $J: u=u(x,y)$ of $F(x,y;u;p,q;r,s,t) = 0$, equations (1.4) or (1.5) determine two one parameter families of characteristic curves lying in the integral surface J . Exactly one characteristic curve from each family passes through any given point $(x_0, y_0, u(x_0, y_0))$ of the integral surface J ; and, moreover, the corresponding two characteristic base curves do not have a common tangent at (x_0, y_0) .

Along any curve, characteristic or otherwise, lying in the integral surface J , the following strip, or band, conditions

$$(1.8) \quad \dot{u} = p\dot{x} + q\dot{y}$$

$$(1.9) \quad \begin{cases} \dot{p} = r\dot{x} + s\dot{y} \\ \dot{q} = s\dot{x} + t\dot{y} \end{cases}$$

must be satisfied.

The modification of Definition 2 and conditions (1.8),(1.9) when the curve Γ is expressed in non-parametric form is obvious.

Definition 3

$$S^1: \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \\ p=p(\tau) \\ q=q(\tau) \end{cases} \text{ for } \tau \in [0,1] \text{ and where } x,y,u,p,q \in C'([0,1]).$$

is called a first order strip \longleftrightarrow for each $\tau \in [0,1]$

$$(1.8) \quad \dot{u} = p\dot{x} + q\dot{y}$$

Suppose a particular integral surface $J: u=u(x,y)$ of

Under certain assumptions of stationarity and ergodicity, the sample means of the process converge to the true means. The central limit theorem states that the distribution of the sample means approaches a normal distribution as the sample size increases. The law of large numbers states that the sample mean converges to the true mean as the sample size increases.

$$\begin{aligned} \mu_1 &= \mu_2 = \mu & (1.1) \\ \sigma_1 &= \sigma_2 = \sigma & (1.2) \\ \rho &= \rho & (1.3) \end{aligned}$$

The joint distribution of the two processes is bivariate normal. The correlation coefficient ρ measures the degree of linear dependence between the two processes. The joint density function is given by:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{x_1-\mu_1}{\sigma_1} - \rho\frac{x_2-\mu_2}{\sigma_2}\right]^2 - \frac{1}{2(1-\rho^2)}\left[\frac{x_2-\mu_2}{\sigma_2} - \rho\frac{x_1-\mu_1}{\sigma_1}\right]^2\right\}$$

The joint distribution is bivariate normal with parameters $\mu_1, \mu_2, \sigma_1, \sigma_2, \rho$.

$F(x,y; u; p,q; r,s,t) = 0$ has a contact of first order with the strip S^1 . Then if $\Gamma : \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \end{cases}$ for $\tau \in [0,1]$ is a characteristic curve in the integral surface J , the strip S^1 is called a characteristic first order strip for the integral surface J .

Definition 4

$$S^2 : \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \\ p=p(\tau) \\ q=q(\tau) \\ r=r(\tau) \\ s=s(\tau) \\ t=t(\tau) \end{cases} \text{ for } \tau \in [0,1] \text{ and where } x,y,u,p,q,r,s,t \in C^1([0,1])$$

is called a second order strip \iff for each $\tau \in [0,1]$

(1.8)
$$\dot{u} = p\dot{x} + q\dot{y}$$

(1.9)
$$\begin{cases} \dot{p} = r\dot{x} + s\dot{y} \\ \dot{q} = s\dot{x} + t\dot{y} \end{cases}$$

If, moreover, equation (1.1) and conditions (1.3) and (1.5) are satisfied for each $\tau \in [0,1]$, then S^1 is called a characteristic second order strip.

Note in Definition 4 that since all the arguments of the functions involved in conditions (1.5) are known upon prescription of the strip S^2 , we may determine whether or not the projection of corresponding space curve $\Gamma : \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \end{cases}$ for $\tau \in [0,1]$ is a characteristic projection without reference to any particular integral surface.

Let \mathcal{L} be a linear operator on a vector space V . We define the matrix A of \mathcal{L} relative to a basis \mathcal{B} of V as follows: $A = [a_{ij}]$ where a_{ij} is the i -th component of $\mathcal{L}(b_j)$ relative to \mathcal{B} .

Let $\mathcal{L}(b_j) = \sum_{i=1}^n a_{ij} b_i$. Then the matrix A is given by

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

It follows that A is a square matrix of order n .

$$\begin{aligned} \mathcal{L}(b_1) &= \sum_{i=1}^n a_{i1} b_i & (1) \\ \mathcal{L}(b_2) &= \sum_{i=1}^n a_{i2} b_i & (2) \\ \mathcal{L}(b_3) &= \sum_{i=1}^n a_{i3} b_i & (3) \end{aligned}$$

Let $\mathcal{L}(b_j) = \sum_{i=1}^n a_{ij} b_i$. Then the matrix A is given by $A = [a_{ij}]$.

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Let $\mathcal{L}(b_j) = \sum_{i=1}^n a_{ij} b_i$. Then the matrix A is given by $A = [a_{ij}]$.

Definitions 3 and 4 can be readily modified to deal with the non-parametric case. See, for example, M. CINQUINI-CIBRARIO[13].

In Chapter 2 we consider the characteristic initial value problem for the equation

$$(1.10) \quad s = f(x, y; u; p, q)$$

and its extension to the system of equations

$$(1.11) \quad s_i = f_i(x, y; u_1, \dots, u_n; p_1, \dots, p_n, q_1, \dots, q_n) \\ (i=1, 2, \dots, n).$$

We modify the customary hypothesis that f be Lipschitzian, i.e. with respect to variables u , p and q , to require that f be partially Lipschitzian, i.e. with respect to variables p and q only. We obtain existence of an integral u over the same closed domain as that obtained in the classical theory. Our integral, however, cannot be shown to be unique. This fact is demonstrated by an example. By further example, we show that the bounds obtained on the domain of existence are maximal bounds.

In Chapter 3 we apply the methods of Chapter 2 to the Cauchy problem for equation (1.10) and the extension to the system (1.11). The conclusions are similar to those obtained in Chapter 2.

The arguments in Chapter 4 serve to establish the equivalence (as defined therein) between the characteristic initial value and the Cauchy problems for the system (1.11) and the corresponding problems for a particular system of first order partial differential equations of the form

The first part of the paper is devoted to the study of the
 asymptotic behavior of the solutions of the system (1.1)
 as $t \rightarrow \infty$. In this part we shall assume that the
 matrix A is constant and the vector f is a
 polynomial in t .

$$(1.1) \quad \dot{x} = Ax + f(t), \quad x(0) = x_0$$

and the solution of the system is denoted by $x(t)$.

$$(1.2) \quad x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} f(s) ds$$

In order to study the asymptotic behavior of the
 solutions of (1.1) as $t \rightarrow \infty$, we shall
 assume that the matrix A is constant and the
 vector f is a polynomial in t . In this part
 we shall assume that the matrix A is constant
 and the vector f is a polynomial in t .

In order to study the asymptotic behavior of the
 solutions of (1.1) as $t \rightarrow \infty$, we shall
 assume that the matrix A is constant and the
 vector f is a polynomial in t .

The asymptotic behavior of the solutions of (1.1)
 as $t \rightarrow \infty$ is studied in this part.

$$(1.12) \quad \begin{cases} \sum_{k=1}^m A_{1k} u_k^x = C_1 & (i = 1, 2, \dots, m < n) \\ \sum_{k=1}^n A_{1k} u_k^y = C_1 & (i = m+1, m+2, \dots, n) \end{cases}$$

where the A_{1k}, C_1 are functions of $x, y, u_1, u_2, \dots, u_n$. The system (1.12) is termed a canonical hyperbolic system.

This equivalence has already been established by M. CINQUINI-CIBRARIO[12]. Under the restriction that the first partial derivatives of the functions A_{1k}, C_1 be Lipschitzian with respect to all their arguments, she obtains her theorems on the existence and uniqueness of the system of functions U_1 as the solution for the canonical hyperbolic system (1.12). We demonstrate that her reasoning establishing the equivalence does not depend upon the uniqueness of the solutions for either system (1.11) or system (1.12). Consequently, from our results in Chapters 2 and 3, we are able to remove the above Lipschitz condition entirely and obtain existence, but not uniqueness, for the solutions of the canonical hyperbolic system for both characteristic and Cauchy initial value prescriptions.

Following the attack of H. LEWY[10], in Chapter 5 we reduce the equation

$$(1.1) \quad F(x, y; u; p, q; r, s, t) = 0$$

to a system of so-called characteristic equations by means of a transformation to the characteristic base curves as coordinates. This system is shown to contain a canonical hyperbolic system.

$$\left. \begin{aligned}
 (1.1) \quad & \sum_{k=1}^n \frac{1}{k} \ln \left| \frac{a_k}{b_k} \right| = 2 \\
 & \sum_{k=1}^n \frac{1}{k} \ln \left| \frac{a_k}{b_k} \right| = 1
 \end{aligned} \right\}$$

where a_k, b_k are integers of \mathbb{Z}^+ , $a_k > b_k$. The system (1.1) is solved in rational numbers.

Some solutions of the system (1.1) are given in [1]. In [2] the problem of the existence of solutions of the system (1.1) is considered. It is shown that the system (1.1) has solutions in rational numbers.

It is known that the system (1.1) has solutions in rational numbers. In [3] it is shown that the system (1.1) has solutions in rational numbers. In [4] it is shown that the system (1.1) has solutions in rational numbers.

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$$\sum_{k=1}^n \frac{1}{k} \ln \left| \frac{a_k}{b_k} \right| = 2$$

The system (1.1) is solved in rational numbers. In [11] it is shown that the system (1.1) has solutions in rational numbers. In [12] it is shown that the system (1.1) has solutions in rational numbers.

We treat the Cauchy problem, i.e. to find an integral surface which has a second order contact with a prescribed second order strip. By virtue of a theorem by M. CINQUINI-CIBRARIO, stated in Chapter 4, LEWY'S work yields immediately the result that for $F \in C'''$ in a suitable region, there exists a unique solution $u \in C'''$ in a sufficiently small neighborhood of the initial curve. We again demonstrate that the equivalence of the problems is not dependent upon uniqueness of their respective solutions; and, hence, by requiring simply that $F \in C''$ we obtain existence but not uniqueness.

In Chapter 6 we treat the characteristic initial value problem for equation (1.1). We follow a modification of H. LEWY'S method introduced by M. CINQUINI-CIBRARIO[13]. Here equation (1.1) is first transformed into the form

$$(1.13) \quad s = f(x, y; u; p, q; r, t).$$

A modified system of characteristic equations is obtained. This system also contains a canonical hyperbolic system. The theorems of Chapter 2 apply and we obtain results similar to those obtained in Chapter 5 for the Cauchy problem.

In Chapter 7 we treat the mixed boundary value problem for the equation

$$(1.10) \quad s = f(x, y; u; p, q),$$

i.e. the problem where any integral surface of (1.10) is required to pass through two space curves issuing from a point, with one of the curves being a characteristic on this surface and the other

The first part of the report deals with the general situation in the country and the progress of the work done during the year. It is followed by a detailed account of the various projects undertaken and the results achieved. The report concludes with a summary of the work done and a list of the names of the staff who have been engaged during the year.

REPORT OF THE DIRECTOR

I have the pleasure to inform you that the work done during the year has been most satisfactory. The various projects undertaken have all been completed and the results achieved are most encouraging. I am sure that the work done during the year will be of great value to the country and I am sure that the staff who have been engaged during the year will be well satisfied with the work they have done.

Yours faithfully,

The Director

1st April 1951

The following is a list of the names of the staff who have been engaged during the year. The names are given in alphabetical order of their surnames.

curve having nowhere a characteristic projection. We show that for equation (1.10) there is no loss in generality if we assume the initial data to be

$$(1.14) \quad u(x,0) = u(x,\pi) = 0.$$

For f continuous, bounded and Lipschitzian, we prove that there exists one and only one integral surface of (1.10) satisfying (1.14) on a domain for which we prescribe explicit bounds. For f continuous, bounded and partially Lipschitzian, we find, by arguments analogous to those used in Chapters 2 and 3, that there exists at least one integral surface of (1.10) satisfying (1.14) on a domain for which we again prescribe the same type of explicit bounds.

In Chapter 8 we consider the characteristic initial value problem for equation (1.10) from a new point of view. Here, in order to extend the theorems of Chapter 2, we introduce the concept of upper and lower bounding functions for the solution (or solutions) of the problem. This idea was first used by O. PERKON [18] to obtain an existence proof for the problem

$$(1.15) \quad y' = f(x,y) \quad , \quad y(x_0) = y_0.$$

His proof is quite independent of the classical proofs.

H. MÜLLER [4] shows that PERKON's method has no direct analogue for a system

$$(1.16) \quad y_i' = f_i(x, y_1, \dots, y_n) \quad , \quad (i = 1, \dots, n).$$

He is able, however, to extend the classical theorem for a system (1.16) to obtain a theorem which reduces to the direct analogue to the PERKON theorem in the case where the f_i are monotonically increasing functions of the arguments y_1, \dots, y_n .

... the ... of ...

$$f(x) = \dots$$

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The extensions to the theorems of Chapter 2 which we obtain are similar to HILLER's conclusions for the system (1.16). Moreover, we demonstrate by example that the FERRON method has no direct analogue for the characteristic initial value problem for equation (1.10). We also give an example illustrating the theorems obtained in this chapter. Finally, we note that the Cauchy problem for equation (1.10) and the Cauchy and characteristic initial value problems for the system

$$(1.11) \quad s_i = f_i(x, y; u_1, \dots, u_n; p_1, \dots, p_n, q_1, \dots, q_n) \\ (i = 1, \dots, n),$$

may also be treated by the methods of this chapter.

the condition of the system is given by the equation

$$L \frac{d^2 \theta}{dt^2} + \frac{1}{2} L \omega^2 \theta = 0$$
 where L is the length of the pendulum and ω is the angular frequency.
 The solution of this equation is given by

$$\theta = A \cos(\omega t + \phi)$$
 where A is the amplitude and ϕ is the phase constant.
 The period of oscillation is given by

$$T = 2\pi \sqrt{\frac{L}{g}}$$
 where g is the acceleration due to gravity.

$$L \frac{d^2 \theta}{dt^2} + \frac{1}{2} L \omega^2 \theta = 0$$

The period of oscillation is given by

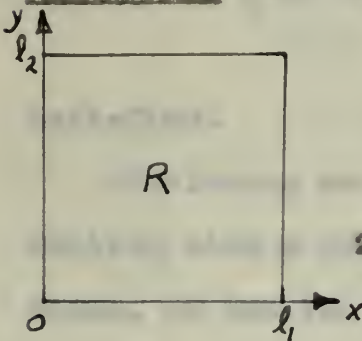
$$T = 2\pi \sqrt{\frac{L}{g}}$$

CHAPTER II

The Characteristic Initial Value Problem for $u_{xy} = f(x,y;u;u_x,u_y)$.

For convenience of reference we first state the following theorem, whose proof is based on the method of successive approximations. The proof of existence was given by É. PICARD [1], while the proof of uniqueness may be found in E. KAMKE [2] p. 410.

Theorem 1.



$$1) f(x,y;u;p,q) \in C(B), B: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \\ -a \leq u \leq a \\ -b_1 \leq p \leq b_1 \\ -b_2 \leq q \leq b_2 \end{cases}$$

2) f is Lipschitzian on B ; i.e. there exists a positive constant K such that for

$$(x,y;u_1;p_1,q_1) \in B, (x,y;u_2;p_2,q_2) \in B,$$

$$|f(x,y;u_1;p_1,q_1) - f(x,y;u_2;p_2,q_2)| \leq K \left\{ |u_1 - u_2| + |p_1 - p_2| + |q_1 - q_2| \right\}$$

3) $M l_1 l_2 \leq a$, $M l_1 \leq b_2$, $M l_2 \leq b_1$, where $M = \max |f|$ on B .

\Rightarrow 4) There exists one and only one function $u(x,y) \in C^1(R)$,

$u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \end{cases}$, such that for each

$(x,y) \in R$ the point $(x,y;u(x,y);u_x(x,y),u_y(x,y)) \in B$, and

$$u_{xy}(x,y) = f(x,y;u(x,y);u_x(x,y),u_y(x,y)), u(x,0) = 0,$$

$$u(0,y) = 0 \quad \text{for each } (x,y) \in R.$$

CHAPTER 11

The characteristic polynomial of the matrix \$A\$ is

$$P(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$$

is given by \$P(\lambda) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)\$.

The eigenvalues are \$\lambda_1 = 3\$ and \$\lambda_2 = -1\$.

For \$\lambda_1 = 3\$, the eigenvector \$v_1\$ satisfies \$(A - 3I)v_1 = 0\$, i.e.,

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies -2x + 2y = 0 \implies x = y.$$

Thus, \$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\$.

$$\left. \begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= -1 \end{aligned} \right\} \begin{aligned} v_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ v_2 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$



For \$\lambda_2 = -1\$, the eigenvector \$v_2\$ satisfies \$(A + I)v_2 = 0\$, i.e.,

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies 2x + 2y = 0 \implies x = -y.$$

Thus, \$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\$.

$$\left\{ \begin{aligned} v_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ v_2 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned} \right\} \text{ are linearly independent vectors in } \mathbb{R}^2.$$

Therefore, the matrix \$P\$ that diagonalizes \$A\$ is \$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\$.

The diagonal matrix \$D\$ is \$D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\$.

$$P^{-1}AP = D \implies A = PDP^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix}.$$

Thus, the matrix \$A\$ is similar to the diagonal matrix \$D\$ via the matrix \$P\$.

\$\square\$

Remarks. a) Suppose we prescribe $u(x,0) = U(x)$, $u(0,y) = V(y)$ where $U(x) \in C^1([0, l_1])$, $V(y) \in C^1([0, l_2])$ and $U(0) = V(0)$. Consider the function $w(x,y) = U(x) + V(y) - U(0)$. Clearly, $w_{xy}(x,y) = 0$ and $w(x,0) = U(x)$, $w(0,y) = V(y)$ hence the function $v = u - w$ must satisfy $v_{xy} = f(x,y; v + w; v_x + w_x, v_y + w_y)$, $v(x,0) = v(0,y) = 0$, a problem of the type covered by Theorem 1.

b) Suppose $f \in C$, bounded and Lipschitzian in the domain B' :

$$\left\{ \begin{array}{l} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \\ -\infty < u < \infty \\ -\infty < p < \infty \\ -\infty < q < \infty \end{array} \right.$$

Then hypothesis 3) is immediately satisfied.

Following an approach used by H. MÜLLER [3] p. 632, in dealing with a system of first order ordinary differential equations, we are led to this improvement of the above theorem:

Theorem 1a. 1)

2)' f is partially Lipschitzian on B ; i.e. there exists a positive constant K such that for $(x,y; u; p_1, q_1) \in B$,

$$(x,y; u; p_2, q_2) \in B, \quad |f(x,y; u; p_1, q_1) - f(x,y; u; p_2, q_2)| \leq K \left\{ |p_1 - p_2| + |q_1 - q_2| \right\}.$$

3)

\Rightarrow 4)' There exists at least one function $u(x,y) \in C^1(B)$, $u_{xy}(x,y) \in C(B)$, where $B: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \end{cases}$ such that for each $(x,y) \in B$

Let \mathcal{L} be the Lie algebra of \mathcal{G} . Then \mathcal{L} is a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. We have $\mathcal{L} = \{ X \in \mathfrak{gl}(n, \mathbb{R}) \mid \text{tr}(X) = 0 \}$. The Lie algebra \mathcal{L} is isomorphic to $\mathfrak{sl}(n, \mathbb{R})$. The Lie algebra \mathcal{L} is a simple Lie algebra of rank $n-1$. The Lie algebra \mathcal{L} is a simple Lie algebra of rank $n-1$.

$$\begin{cases} \mathfrak{L} = \mathfrak{sl}(n, \mathbb{R}) \\ \mathfrak{L} = \mathfrak{sl}(n, \mathbb{R}) \\ \mathfrak{L} = \mathfrak{sl}(n, \mathbb{R}) \\ \mathfrak{L} = \mathfrak{sl}(n, \mathbb{R}) \\ \mathfrak{L} = \mathfrak{sl}(n, \mathbb{R}) \end{cases}$$

The Lie algebra \mathcal{L} is a simple Lie algebra of rank $n-1$.

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REMARKS

(1) The Lie algebra \mathcal{L} is a simple Lie algebra of rank $n-1$.

(2) The Lie algebra \mathcal{L} is a simple Lie algebra of rank $n-1$.

$$\mathfrak{L} = \mathfrak{sl}(n, \mathbb{R}) = \{ X \in \mathfrak{gl}(n, \mathbb{R}) \mid \text{tr}(X) = 0 \}$$

APPENDIX

Let \mathcal{L} be the Lie algebra of \mathcal{G} . Then \mathcal{L} is a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. We have $\mathcal{L} = \{ X \in \mathfrak{gl}(n, \mathbb{R}) \mid \text{tr}(X) = 0 \}$. The Lie algebra \mathcal{L} is isomorphic to $\mathfrak{sl}(n, \mathbb{R})$. The Lie algebra \mathcal{L} is a simple Lie algebra of rank $n-1$.

the point $(x, y; u(x, y); u_x(x, y), u_y(x, y)) \in B$, and $u_{xy}(x, y) = f(x, y; u(x, y); u_x(x, y), u_y(x, y))$, $u(x, 0) = 0$, $u(0, y) = 0$ for each $(x, y) \in B$.

Proof. According to WEIERSTRASS' celebrated theorem [4] p. 1147, on polynomial approximations to a continuous function, there exists a sequence of polynomials, $\{g_\lambda\}(x, y; u; p, q)$, converging uniformly to $f(x, y; u; p, q)$ on B . We designate this uniform convergence by the notation $\{g_\lambda\} \xrightarrow{\text{unif}} f$ on B .

We extend f and the polynomials g_λ , ($\lambda = 1, 2, \dots$), over the domain B to the domain B' , defined in the remark b) above, by the definition

$$f(x, y; u; p, q) = f(x, y; \bar{u}; \bar{p}, \bar{q})$$

$$g_\lambda(x, y; u; p, q) = g_\lambda(x, y; \bar{u}; \bar{p}, \bar{q}), \quad (\lambda = 1, 2, \dots),$$

(2.1) where

$$\begin{aligned} \bar{u} &= u \text{ if } -a \leq u \leq a, & \bar{p} &= p \text{ if } -b_1 \leq p \leq b_1, & \bar{q} &= q \text{ if } -b_2 \leq q \leq b_2. \\ \bar{u} &= a \text{ if } a < u, & \bar{p} &= b_1 \text{ if } b_1 < p, & \bar{q} &= b_2 \text{ if } b_2 < q \\ \bar{u} &= -a \text{ if } u < -a, & \bar{p} &= -b_1 \text{ if } p < -b_1, & \bar{q} &= -b_2 \text{ if } q < -b_2 \end{aligned}$$

From this extended definition we see that $|f| \leq M$ in B' . Since $\{g_\lambda\} \xrightarrow{\text{unif}} f$ in B' , there exists a constant $L > 0$ such that $|g_\lambda| \leq L$ in B' and for all λ . The functions g_λ , ($\lambda = 1, 2, \dots$) are uniformly continuous in B' , moreover they possess bounded difference quotients with respect to the arguments u , p and q everywhere in B' . Hence in B' , for each function g_λ there exists a constant $K_\lambda > 0$ such that

Let \mathcal{A} be a subalgebra of \mathcal{B} . Then \mathcal{A} is a σ -algebra if and only if \mathcal{A} is closed under countable unions and complements. In other words, \mathcal{A} is a σ -algebra if and only if \mathcal{A} is a σ -ring and $\emptyset \in \mathcal{A}$.

Let \mathcal{A} and \mathcal{B} be σ -algebras. Then $\mathcal{A} \cap \mathcal{B}$ is a σ -algebra. Also, $\mathcal{A} \cup \mathcal{B}$ is a σ -algebra if and only if \mathcal{A} and \mathcal{B} are independent. The σ -algebra generated by \mathcal{A} and \mathcal{B} is denoted by $\mathcal{A} \vee \mathcal{B}$. The σ -algebra generated by \mathcal{A} is denoted by $\sigma(\mathcal{A})$.

THEOREM 1.1

Let \mathcal{A} and \mathcal{B} be σ -algebras. Then $\mathcal{A} \cap \mathcal{B}$ is a σ -algebra. Also, $\mathcal{A} \cup \mathcal{B}$ is a σ -algebra if and only if \mathcal{A} and \mathcal{B} are independent. The σ -algebra generated by \mathcal{A} and \mathcal{B} is denoted by $\mathcal{A} \vee \mathcal{B}$. The σ -algebra generated by \mathcal{A} is denoted by $\sigma(\mathcal{A})$.

Let \mathcal{A} and \mathcal{B} be σ -algebras. Then $\mathcal{A} \cap \mathcal{B}$ is a σ -algebra. Also, $\mathcal{A} \cup \mathcal{B}$ is a σ -algebra if and only if \mathcal{A} and \mathcal{B} are independent. The σ -algebra generated by \mathcal{A} and \mathcal{B} is denoted by $\mathcal{A} \vee \mathcal{B}$. The σ -algebra generated by \mathcal{A} is denoted by $\sigma(\mathcal{A})$.

$$(2.2) \left| g_{\lambda}(x, y; u_1; p_1, q_1) - g_{\lambda}(x, y; u_2; p_2, q_2) \right| \leq K_{\lambda} \left\{ |u_1 - u_2| + |p_1 - p_2| + |q_1 - q_2| \right\}.$$

Thus, by Theorem 1, to each g_{λ} there corresponds one and only one function $u_{\lambda}(x, y) \in C^1(R)$, $u_{\lambda, xy}(x, y) \in C(R)$ satisfying

$$(2.3) \begin{cases} u_{\lambda, xy} = g_{\lambda}(x, y; u_{\lambda}(x, y); u_{\lambda, x}(x, y), u_{\lambda, y}(x, y)), \\ u_{\lambda}(x, 0) = 0, \quad u_{\lambda}(0, y) = 0 \quad \text{for each } (x, y) \in R. \end{cases}$$

We may express the characteristic initial value problem for each u_{λ} in the form of an equivalent integral equation

$$(2.4) \quad u_{\lambda}(x, y) = \int_0^x d\xi \int_0^y g_{\lambda}(\xi, \eta; u_{\lambda}(\xi, \eta); u_{\lambda, x}(\xi, \eta), u_{\lambda, y}(\xi, \eta)) d\eta.$$

By differentiation,

$$(2.5) \quad u_{\lambda, x}(x, y) = \int_0^y g_{\lambda}(x, \eta; u_{\lambda}(x, \eta); u_{\lambda, x}(x, \eta), u_{\lambda, y}(x, \eta)) d\eta$$

$$(2.6) \quad u_{\lambda, y}(x, y) = \int_0^x g_{\lambda}(\xi, y; u_{\lambda}(\xi, y); u_{\lambda, x}(\xi, y), u_{\lambda, y}(\xi, y)) d\xi.$$

We now show that the sequences $\{u_{\lambda}\}$, $\{u_{\lambda, x}\}$, $\{u_{\lambda, y}\}$ are each uniformly bounded and equicontinuous on R . For the sequence $\{u_{\lambda}\}$ this follows directly from the integral expression (2.4), for, given $x, x_1, x_2 \in [0, l_1]$ and $y, y_1, y_2 \in [0, l_2]$,

$$(2.7) \quad |u_{\lambda}(x, y)| \leq L l_1 l_2, \quad (\lambda = 1, 2, \dots)$$

$$(2.8) \quad |u_{\lambda}(x_1, y_1) - u_{\lambda}(x_2, y_2)| \leq L |x_1 - x_2| |y_1 - y_2| + L l_2 |x_1 - x_2| + L l_1 |y_1 - y_2|, \quad (\lambda = 1, 2, \dots)$$

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 + 1$. Then $f(x) - g(x) = 2x$.

Since $f(x) - g(x) = 2x$, we have $f(x) = g(x) + 2x$.

$$\begin{aligned} f(x) &= (x^2 + 1) + 2x \\ &= x^2 + 2x + 1 \end{aligned}$$

Therefore, $f(x) = g(x) + 2x$.

$$\begin{aligned} f(x) - g(x) &= (x^2 + 2x + 1) - (x^2 + 1) \\ &= 2x \end{aligned}$$

Thus, $f(x) - g(x) = 2x$.

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 + 1$.

$$f(x) - g(x) = (x^2 + 2x + 1) - (x^2 + 1) = 2x$$

Therefore, $f(x) - g(x) = 2x$.

$$f(x) = g(x) + 2x$$

$$(x^2 + 2x + 1) = (x^2 + 1) + 2x$$

$$x^2 + 2x + 1 = x^2 + 1 + 2x$$

$$2x + 1 = 1 + 2x$$

The uniform boundedness of $\{u_{\lambda,x}\}$ and of $\{u_{\lambda,y}\}$ follow directly from (2.5) and (2.6), respectively, for, given $(x,y) \in R$,

$$(2.9) \quad |u_{\lambda,x}(x,y)| \leq L f_2, \quad (\lambda = 1, 2, \dots)$$

$$(2.10) \quad |u_{\lambda,y}(x,y)| \leq L f_1, \quad (\lambda = 1, 2, \dots).$$

We base the proof of the equicontinuity of the functions of the sequence $\{u_{\lambda,x}\}$ upon the following two lemmas, the first of which is due to T. H. GRONWALL [5].

Lemma 1. 1) $Z(y) \in C([0, l])$

$$(2.11) \quad 2) \quad 0 \leq Z(y) \leq \int_0^y (KZ(\eta) + A) d\eta + B \quad \text{for } y \in [0, l]$$

where K , A and B are constants ≥ 0 .

$$(2.12) \quad 3) \quad 0 \leq Z(y) \leq (Al + B) e^{Kl} \quad \text{for } y \in [0, l].$$

Lemma 2. Given $\mu > 0$, $\zeta > 0$, there exist δ , a positive constant depending upon μ alone, and N , a positive integer depending upon ζ alone, such that whenever $(x_1, y) \in R$, $(x_2, y) \in R$, $|x_1 - x_2| < \delta$ and $\lambda > N$,

$$(2.13) \quad |u_{\lambda,x}(x_2, y) - u_{\lambda,x}(x_1, y)| \leq K \int_0^y |u_{\lambda,x}(x_2, \eta) - u_{\lambda,x}(x_1, \eta)| d\eta + \mu + \zeta$$

where K is the partial Lipschitz constant for $f(x, y; u; p, q)$.

Assume, for the moment, the validity of these two lemmas. Each of the functions $u_{\lambda,x}$ is certainly uniformly continuous on R ; hence, if we let $Z(y) = |u_{\lambda,x}(x_2, y) - u_{\lambda,x}(x_1, y)|$ for any particular $\lambda > N$,

The solution of the system (1) is sought in the form of a power series in ϵ :

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots \quad (1)$$

$$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots \quad (2)$$

Substituting (1) and (2) into the system (1) and equating coefficients of like powers of ϵ , we obtain the following system of equations for the functions $x_i(t)$ and $y_i(t)$:

$$\dot{x}_0 = -x_0, \quad \dot{y}_0 = -y_0 \quad (3)$$

$$\dot{x}_1 = -x_1 + y_0, \quad \dot{y}_1 = -y_1 - x_0 \quad (4)$$

where $x_0(t) = e^{-t}$ and $y_0(t) = e^{-t}$.

$$\dot{x}_2 = -x_2 + y_1, \quad \dot{y}_2 = -y_2 - x_1 \quad (5)$$

It is easy to see that the functions $x_i(t)$ and $y_i(t)$ are bounded for all $t \geq 0$ and $\epsilon < 1$. Therefore, the series (1) and (2) converge uniformly for all $t \geq 0$ and $\epsilon < 1$.

$$\|x(t) - x_0(t)\| \leq \epsilon \|x_1(t)\| + \epsilon^2 \|x_2(t)\| + \dots \leq \epsilon \sum_{k=1}^{\infty} \|x_k(t)\| \leq \epsilon \sum_{k=1}^{\infty} e^{-t} = \epsilon e^{-t} \quad (6)$$

where $\|x(t)\| = \sqrt{x^2(t) + y^2(t)}$ and $\|x_k(t)\| = \sqrt{x_k^2(t) + y_k^2(t)}$.

Thus, the error of the approximation of the solution of the system (1) by the function $x_0(t)$ is of order $O(\epsilon)$ for all $t \geq 0$ and $\epsilon < 1$. The error of the approximation of the solution of the system (1) by the function $y_0(t)$ is of order $O(\epsilon)$ for all $t \geq 0$ and $\epsilon < 1$.

we have immediately that for $|x_2 - x_1| < \delta$,

$$(2.14) \quad |u_{\lambda, x}(x_2, y) - u_{\lambda, x}(x_1, y)| \leq (\mu + \zeta) e^{K\lambda/2}.$$

Suppose $(x_1, y_1) \in R$, $(x_2, y_2) \in R$, then certainly $(x_2, y_1) \in R$ and

$$(2.15) \quad |u_{\lambda, x}(x_2, y_2) - u_{\lambda, x}(x_1, y_1)| \leq |u_{\lambda, x}(x_2, y_2) - u_{\lambda, x}(x_2, y_1)| + |u_{\lambda, x}(x_2, y_1) - u_{\lambda, x}(x_1, y_1)|, \quad (\lambda = 1, 2, \dots).$$

By (2.5) we have that

$$(2.16) \quad |u_{\lambda, x}(x_2, y_2) - u_{\lambda, x}(x_2, y_1)| \leq L |y_2 - y_1|, \quad (\lambda = 1, 2, \dots).$$

Inequalities (2.14), (2.15) and (2.16) yield immediately the equicontinuity on R of the functions of the sequence $\{u_{\lambda, x}\}$; for, given $\epsilon > 0$, we first choose $\mu > 0$ and $\zeta > 0$ such that

$$(2.17) \quad \mu + \zeta < \frac{\epsilon}{2e^{K\lambda/2}}$$

and let δ and N be the corresponding constants given by Lemma 2.

By the uniform continuity on R of each of the functions $u_{\lambda, x}$, there exists a positive constant δ_N , depending on ϵ alone, such that

$$|x_1 - x_2| < \delta_N \quad \text{and} \quad |y_1 - y_2| < \delta_N \quad \Rightarrow$$

$$(2.18) \quad |u_{\lambda, x}(x_2, y_2) - u_{\lambda, x}(x_1, y_1)| < \epsilon, \quad (\lambda = 1, 2, \dots, N).$$

Setting $\delta_0 = \min(\delta, \delta_N, \frac{\epsilon}{2L})$ we obtain

... $\frac{1}{2} \sqrt{2} \dots$

$$\frac{1}{2} \sqrt{2} \dots$$

... $\frac{1}{2} \sqrt{2} \dots$

$$\frac{1}{2} \sqrt{2} \dots$$

... $\frac{1}{2} \sqrt{2} \dots$

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$$\frac{1}{2} \sqrt{2} \dots$$

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... $\frac{1}{2} \sqrt{2} \dots$

$$\frac{1}{2} \sqrt{2} \dots$$

... $\frac{1}{2} \sqrt{2} \dots$

... $\frac{1}{2} \sqrt{2} \dots$

$$|x_1 - x_2| < \delta_0 \quad \text{and} \quad |y_1 - y_2| < \delta_0 \quad \Rightarrow$$

$$(2.19) \quad |u_{\lambda,x}(x_2, y_2) - u_{\lambda,x}(x_1, y_1)| < \epsilon, \quad \text{for } \lambda = 1, 2, \dots, N, N+1, \dots$$

Proof of Lemma 1: Let $Z(y) = e^{My} \cdot w(y)$, without loss for we may always choose $w(y) = e^{-My} \cdot Z(y)$. $w(y) \in C([0, l])$ and hence attains a maximum thereon. Let w_{\max} occur at $y = y_1$, then

$$\begin{aligned} 0 &\leq e^{My_1} w_{\max} \leq \int_0^{y_1} (M e^{M\eta} w(\eta) + A) d\eta + B \\ &\leq w_{\max} \int_0^{y_1} M e^{M\eta} d\eta + A y_1 + B \\ &= w_{\max} (e^{My_1} - 1) + A y_1 + B \end{aligned}$$

Thus $0 \leq w_{\max} \leq A y_1 + B \leq A l + B$ and hence

$$0 \leq Z(y) \leq (A l + B) e^{Ml} \quad \text{for } y \in [0, l].$$

Proof of Lemma 2:

$$\begin{aligned} (2.20) \quad u_{\lambda,x}(x_2, y) - u_{\lambda,x}(x_1, y) &= \int_0^y [\epsilon_{\lambda}(x_2, \eta; u_{\lambda}(x_2, \eta); \\ &\quad u_{\lambda,x}(x_2, \eta), u_{\lambda,y}(x_2, \eta)) \\ &\quad - \epsilon_{\lambda}(x_1, \eta; u_{\lambda}(x_1, \eta); u_{\lambda,x}(x_1, \eta), \\ &\quad u_{\lambda,y}(x_1, \eta))] d\eta \\ &= \int_0^y [\epsilon_{\lambda}(x_2, \eta; u_{\lambda}(x_2, \eta); u_{\lambda,x}(x_2, \eta), \\ &\quad u_{\lambda,y}(x_2, \eta)) \\ &\quad - \epsilon_{\lambda}(x_2, \eta; u_{\lambda}(x_2, \eta); u_{\lambda,x}(x_2, \eta), \\ &\quad u_{\lambda,y}(x_2, \eta))] d\eta \\ &\quad + \int_0^y [\epsilon_{\lambda}(x_2, \eta; u_{\lambda}(x_2, \eta); u_{\lambda,x}(x_2, \eta), \\ &\quad u_{\lambda,y}(x_2, \eta)) \end{aligned}$$

$$\left[\frac{d}{dt} \left(\frac{1}{2} m v^2 \right) = \mathbf{F} \cdot \mathbf{v} \right] \Rightarrow \frac{d}{dt} \left(\frac{1}{2} m v^2 \right) = \mathbf{F} \cdot \mathbf{v}$$

$$\frac{d}{dt} \left(\frac{1}{2} m v^2 \right) = \mathbf{F} \cdot \mathbf{v} \Rightarrow \frac{d}{dt} \left(\frac{1}{2} m v^2 \right) = \mathbf{F} \cdot \mathbf{v} \quad (12.1)$$

Let us now consider the case of a particle moving in a potential field $\mathbf{F} = -\nabla V$. The work done by the force \mathbf{F} in moving the particle from point \mathbf{r}_1 to point \mathbf{r}_2 is given by the line integral $\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}$. This integral is independent of the path taken, and is equal to $V(\mathbf{r}_1) - V(\mathbf{r}_2)$. The work done by the force \mathbf{F} in moving the particle from point \mathbf{r}_1 to point \mathbf{r}_2 is equal to the change in the potential energy of the particle, $\Delta V = V(\mathbf{r}_2) - V(\mathbf{r}_1)$.

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} m v^2 \right) &= \mathbf{F} \cdot \mathbf{v} = -\nabla V \cdot \mathbf{v} \\ &= -\frac{dV}{dt} \end{aligned}$$

$$\frac{d}{dt} \left(\frac{1}{2} m v^2 \right) + \frac{dV}{dt} = 0$$

$$\frac{d}{dt} \left(\frac{1}{2} m v^2 + V \right) = 0$$

Therefore, the total energy of the particle is constant.

$$\left[\frac{1}{2} m v^2 + V \right] = \text{constant} = E$$

$$\frac{1}{2} m v^2 + V = E$$

$$\frac{1}{2} m v^2 = E - V$$

$$v = \sqrt{\frac{2(E - V)}{m}}$$

$$\left[\frac{1}{2} m v^2 + V \right] = E \Rightarrow \frac{1}{2} m v^2 = E - V$$

$$\frac{1}{2} m v^2 = E - V$$

$$v = \sqrt{\frac{2(E - V)}{m}}$$

$$\left[\frac{1}{2} m v^2 + V \right] = E \Rightarrow \frac{1}{2} m v^2 = E - V$$

(2.20)
(Continued)

$$\begin{aligned}
 & - f(x_2, \eta; u_\lambda(x_2, \eta); u_{\lambda, x}(x_1, \eta), \\
 & \quad u_{\lambda, y}(x_2, \eta))] d\eta \\
 & + \int_0^y [f(x_2, \eta; u_\lambda(x_2, \eta); u_{\lambda, x}(x_1, \eta), \\
 & \quad u_{\lambda, y}(x_2, \eta)) \\
 & \quad - f(x_1, \eta; u_\lambda(x_1, \eta); u_{\lambda, x}(x_1, \eta), \\
 & \quad u_{\lambda, y}(x_1, \eta))] d\eta \\
 & + \int_0^y [f(x_1, \eta; u_\lambda(x_1, \eta); u_{\lambda, x}(x_1, \eta), \\
 & \quad u_{\lambda, y}(x_1, \eta)) \\
 & \quad - \varepsilon_\lambda(x_1, \eta; u_\lambda(x_1, \eta); u_{\lambda, x}(x_1, \eta), \\
 & \quad u_{\lambda, y}(x_1, \eta))] d\eta \\
 & \quad (\lambda = 1, 2, \dots).
 \end{aligned}$$

Since $\{\varepsilon_\lambda\} \xrightarrow{\text{unif}} f$ on B' , given $\zeta > 0$, there exists a positive integer N , depending upon ζ alone, such that for $\lambda > N$,

$$\begin{aligned}
 (2.21) \quad & \left| \int_0^y [\varepsilon_\lambda(x_2, \eta; u_\lambda(x_2, \eta); u_{\lambda, x}(x_2, \eta), u_{\lambda, y}(x_2, \eta)) - \right. \\
 & \quad \left. f(x_2, \eta; u_\lambda(x_2, \eta); u_{\lambda, x}(x_2, \eta), u_{\lambda, y}(x_2, \eta))] d\eta \right| \\
 & + \left| \int_0^y [f(x_1, \eta; u_\lambda(x_1, \eta); u_{\lambda, x}(x_1, \eta), u_{\lambda, y}(x_1, \eta)) - \right. \\
 & \quad \left. \varepsilon_\lambda(x_1, \eta; u_\lambda(x_1, \eta); u_{\lambda, x}(x_1, \eta), u_{\lambda, y}(x_1, \eta))] d\eta \right| < \zeta
 \end{aligned}$$

By hypothesis 2)',

$$(2.22) \quad \left| \int_0^y [f(x_2, \eta; u_\lambda(x_2, \eta); u_{\lambda, x}(x_2, \eta), u_{\lambda, y}(x_2, \eta)) - \right.$$

$$- (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 - (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 - (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 -$$

(10.1)

$$+ (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 + (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 + (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 +$$

$$- (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 - (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 - (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 -$$

$$+ (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 + (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 + (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 +$$

$$- (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 - (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 - (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 -$$

(10.2)

... (10.3) ...

... (10.4) ...

$$+ (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 + (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 + (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 +$$

$$- (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 - (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 - (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 -$$

$$+ (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 + (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 + (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 +$$

$$- (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 - (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 - (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 -$$

(10.5)

$$+ (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 + (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 + (\psi_{\alpha}^{\beta} \psi_{\alpha}^{\beta})^2 +$$

$$(2.22) \quad \left| \int_0^{\gamma} \left[f(x_2, \eta; u_{\lambda}(x_2, \eta); u_{\lambda, x}(x_1, \eta), u_{\lambda, y}(x_2, \eta)) \right] d\eta \right| \\ \leq K \int_0^{\gamma} |u_{\lambda, x}(x_2, \eta) - u_{\lambda, x}(x_1, \eta)| d\eta, \quad (\lambda = 1, 2, \dots)$$

Since f is uniformly continuous on E' , the functions of the sequence $\{u_{\lambda}\}$ are equicontinuous on R , and $|u_{\lambda, y}(x_2, \eta) - u_{\lambda, y}(x_1, \eta)| \leq L|x_2 - x_1|$, $(\lambda = 1, 2, \dots)$, it follows that given $\mu > 0$ there exists a positive constant δ , depending upon μ alone, such that for $|x_2 - x_1| < \delta$,

$$(2.23) \quad \left| \int_0^{\gamma} \left[f(x_2, \eta; u_{\lambda}(x_2, \eta); u_{\lambda, x}(x_1, \eta), u_{\lambda, y}(x_2, \eta)) \right. \right. \\ \left. \left. - f(x_1, \eta; u_{\lambda}(x_1, \eta); u_{\lambda, x}(x_1, \eta), u_{\lambda, y}(x_1, \eta)) \right] d\eta \right| < \mu,$$

$(\lambda = 1, 2, \dots)$.

Therefore, from (2.21), (2.22) and (2.23), by virtue of (2.20) we obtain that for $\lambda > N$ and $|x_2 - x_1| < \delta$,

$$(2.13) \quad |u_{\lambda, x}(x_2, y) - u_{\lambda, x}(x_1, y)| < K \int_0^{\gamma} |u_{\lambda, x}(x_2, \eta) - u_{\lambda, x}(x_1, \eta)| d\eta \\ + \mu + \zeta$$

thus verifying Lemma 2.

The proof of the equicontinuity of the functions of the sequence $\{u_{\lambda, y}\}$ follows precisely the same steps as that for the sequence $\{u_{\lambda, x}\}$.

We now invoke the well-known theorem of C. ARZELA [3] p. 1144:

"Given a set F of functions f defined and continuous on the closed bounded set A , then the necessary and sufficient conditions that each sequence of functions contained in F possesses

$$|\beta| = \left[\frac{1}{2} (\beta_{11}^2 + \beta_{22}^2) + \beta_{12}^2 \right]^{1/2}$$

$$|\beta| = \frac{1}{2} \left[(\beta_{11}^2 + \beta_{22}^2) + 4\beta_{12}^2 \right]^{1/2}$$

Let us consider the case of a uniaxial crystal of 2 axes
 - $\beta_{11} = \beta_{22} = \beta$ and $\beta_{12} = 0$
 $\beta = \frac{1}{2} \left[(\beta^2 + \beta^2) + 0 \right]^{1/2} = \beta$
 This shows that the definition of β is correct and we have
 $\beta \geq |\beta_{12}|$ or $\beta \geq \beta_{12}$

$$|\beta| = \frac{1}{2} \left[(\beta_{11}^2 + \beta_{22}^2) + 4\beta_{12}^2 \right]^{1/2}$$

$$|\beta| \geq \frac{1}{2} \left[(\beta_{11}^2 + \beta_{22}^2) + 4\beta_{12}^2 \right]^{1/2}$$

$$|\beta| \geq \beta_{12}$$

Let us consider the case of a uniaxial crystal of 3 axes
 - $\beta_{11} = \beta_{22} = \beta_{33} = \beta$ and $\beta_{12} = \beta_{13} = \beta_{23} = 0$
 $\beta = \frac{1}{2} \left[(\beta^2 + \beta^2 + \beta^2) + 0 \right]^{1/2} = \beta$

$$|\beta| = \frac{1}{2} \left[(\beta_{11}^2 + \beta_{22}^2 + \beta_{33}^2) + 4(\beta_{12}^2 + \beta_{13}^2 + \beta_{23}^2) \right]^{1/2}$$

$$\beta \geq \beta_{12}$$

Let us consider the case of a uniaxial crystal of 3 axes
 - $\beta_{11} = \beta_{22} = \beta_{33} = \beta$ and $\beta_{12} = \beta_{13} = \beta_{23} = 0$
 $\beta = \frac{1}{2} \left[(\beta^2 + \beta^2 + \beta^2) + 0 \right]^{1/2} = \beta$

Let us consider the case of a uniaxial crystal of 3 axes
 - $\beta_{11} = \beta_{22} = \beta_{33} = \beta$ and $\beta_{12} = \beta_{13} = \beta_{23} = 0$
 $\beta = \frac{1}{2} \left[(\beta^2 + \beta^2 + \beta^2) + 0 \right]^{1/2} = \beta$

a subsequence uniformly convergent on A are that \mathcal{F} be uniformly bounded and equicontinuous."

By Theorem 1, there exists a unique triple $(u_\lambda; u_{\lambda,x}; u_{\lambda,y})$ corresponding to g_λ for each λ . Since any subsequence of a uniformly convergent sequence is likewise uniformly convergent; and, since any subsequence of a uniformly bounded and equicontinuous sequence is likewise uniformly bounded and equicontinuous; there exists a subsequence $\{g_\lambda^*\}$ of the sequence $\{g_\lambda\}$ such that the corresponding sequences

$$(2.24) \quad \{u_\lambda^*\} \xrightarrow{\text{unif}} u, \quad \{u_{\lambda,x}^*\} \xrightarrow{\text{unif}} v, \quad \{u_{\lambda,y}^*\} \xrightarrow{\text{unif}} w,$$

where $u, v, w \in C(R)$. This results from the following successive extractions of subsequences:

$\{u_\lambda\}$ is equicontinuous and uniformly bounded on R , hence there exists a subsequence $\{u_\lambda^1\}$ of $\{u_\lambda\}$ uniformly convergent on R . $\{u_{\lambda,x}^1\}$ is equicontinuous and uniformly bounded on R , hence there exists a subsequence $\{u_{\lambda,x}^2\}$ of $\{u_{\lambda,x}^1\}$ uniformly convergent on R . $\{u_{\lambda,y}^2\}$ is equicontinuous and uniformly bounded on R , hence there exists a subsequence $\{u_{\lambda,y}^*\}$ of $\{u_{\lambda,y}^2\}$ uniformly convergent on R . But, by the one-to-one correspondence mentioned above, $\{u_{\lambda,x}^*\}$ is a subsequence of $\{u_{\lambda,x}^2\}$ while $\{u_\lambda^*\}$ is a subsequence of $\{u_\lambda^1\}$. Thus $\{u_{\lambda,x}^*\}$ and $\{u_\lambda^*\}$ are each uniformly convergent on R .

Writing, with the notation $u_0^* = u_{0,x}^* = u_{0,y}^* = 0$,

THEORY OF THE ...

Let $f(x) = x^2 + 2x + 1$...

$$f(x) = x^2 + 2x + 1 = (x+1)^2$$

It is clear that ...

Let $f(x) = x^2 + 2x + 1$...

$$f(x) = x^2 + 2x + 1 = (x+1)^2$$

$$(2.25) \quad u_{\lambda}^* = \sum_{k=1}^{\lambda} (u_k^* - u_{k-1}^*), \quad u_{\lambda,x}^* = \sum_{k=1}^{\lambda} (u_{k,x}^* - u_{k-1,x}^*),$$

$$u_{\lambda,y}^* = \sum_{k=1}^{\lambda} (u_{k,y}^* - u_{k-1,y}^*), \quad (\lambda = 1, 2, \dots),$$

we see that the conditions for differentiation under the summation sign for infinite series are satisfied by (2.24) and the fact that $u_{\lambda}^* \in C^1(R)$, ($\lambda = 1, 2, \dots$). Hence

$$(2.26) \quad v(x,y) = u_x(x,y), \quad w(x,y) = u_y(x,y) \quad \text{for } (x,y) \in R$$

We show that the function u so determined satisfies the integral equation equivalent to the original characteristic initial value problem, i.e.

$$(2.27) \quad u(x,y) = \int_0^x d\xi \int_0^y f(\xi, \eta; u(\xi, \eta); u_x(\xi, \eta), u_y(\xi, \eta)) d\eta$$

for $(x,y) \in R$.

For any λ , by (2.4),

$$(2.28) \quad |u(x,y) - \int_0^x d\xi \int_0^y f(\xi, \eta; u(\xi, \eta); u_x(\xi, \eta), u_y(\xi, \eta)) d\eta|$$

$$\leq |u(x,y) - u_{\lambda}^*(x,y)| + \int_0^x d\xi \int_0^y |f(\xi, \eta; u(\xi, \eta); u_x(\xi, \eta), u_y(\xi, \eta)) - f(\xi, \eta; u_{\lambda}^*(\xi, \eta); u_{\lambda,x}^*(\xi, \eta), u_{\lambda,y}^*(\xi, \eta))| d\eta$$

$$+ \int_0^x d\xi \int_0^y |f(\xi, \eta; u_{\lambda}^*(\xi, \eta); u_{\lambda,x}^*(\xi, \eta), u_{\lambda,y}^*(\xi, \eta)) - f_{\lambda}^*(\xi, \eta; u_{\lambda}^*(\xi, \eta); u_{\lambda,x}^*(\xi, \eta), u_{\lambda,y}^*(\xi, \eta))| d\eta$$

Since $\{g_{\lambda}^*\} \xrightarrow{\text{unif}} f$ on B' , $\{u_{\lambda}^*\} \xrightarrow{\text{unif}} u$ on R , given $\epsilon > 0$ and $(x,y) \in R$, there exists a positive integer N_1 , depending upon ϵ alone, such that for $\lambda > N_1$,

$$\begin{aligned}
 \sum_{k=0}^{\infty} \frac{x^k}{k!} &= e^x \\
 \sum_{k=0}^{\infty} \frac{x^k}{k!} &= e^x
 \end{aligned}$$

The exponential function is defined as the sum of the series above. It is a unique function that is equal to its own derivative.

The exponential function is a continuous function. It is also a smooth function, meaning it has derivatives of all orders.

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d^2}{dx^2} e^x = e^x$$

$$\frac{d^3}{dx^3} e^x = e^x$$

In general, the derivative of the exponential function is the function itself.

$$(2.29) \quad |u(x,y) - u_\lambda^*(x,y)| < \epsilon,$$

$$(2.30) \quad \int_0^x d\xi \int_0^y |f(\xi, \eta; u_\lambda^*(\xi, \eta); u_{\lambda,x}^*(\xi, \eta), u_{\lambda,y}^*(\xi, \eta)) \\ - f(\xi, \eta; u(\xi, \eta); u_x(\xi, \eta), u_y(\xi, \eta))| d\eta \\ < \epsilon / k_1 k_2.$$

Moreover, since f is uniformly continuous in B^1 while $\{u_\lambda^*\}$, $\{u_{\lambda,x}^*\}$, $\{u_{\lambda,y}^*\}$ converge uniformly on R to u , u_x , u_y respectively, there exists a positive integer N_2 , depending on ϵ alone, such that for $\lambda > N_2$,

$$(2.31) \quad \int_0^x d\xi \int_0^y |f(\xi, \eta; u(\xi, \eta); u_x(\xi, \eta), u_y(\xi, \eta)) \\ - f(\xi, \eta; u_\lambda^*(\xi, \eta); u_{\lambda,x}^*(\xi, \eta), u_{\lambda,y}^*(\xi, \eta))| d\eta \\ < \epsilon / k_1 k_2.$$

Introducing (2.29), (2.30) and (2.31) into (2.28), we obtain that for $\lambda > \max(N_1, N_2)$

$$(2.32) \quad |u(x,y) - \int_0^x d\xi \int_0^y f(\xi, \eta; u(\xi, \eta); u_x(\xi, \eta), u_y(\xi, \eta)) \\ < \epsilon(1 + 2k_1 k_2)$$

But ϵ is arbitrary, hence (2.27) is verified for each $(x,y) \in R$. We must verify the one additional fact that for each $(x,y) \in R$, $(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in B$, instead of just belonging to B^1 .

$$|x| \geq -|x| \quad (1)$$

$$((x_1)_{x_1}^2 + (x_2)_{x_2}^2 + (x_3)_{x_3}^2 + \dots + (x_n)_{x_n}^2) \geq 0 \quad (2)$$

$$x^2 + (x_1)_{x_1}^2 + (x_2)_{x_2}^2 + \dots + (x_n)_{x_n}^2 = x^2 + \sum_{i=1}^n (x_i)_{x_i}^2$$

Let $x = (x_1, x_2, \dots, x_n)$ be a vector in \mathbb{R}^n . Then $x^2 = x_1^2 + x_2^2 + \dots + x_n^2$. The expression $x^2 + \sum_{i=1}^n (x_i)_{x_i}^2$ is a sum of squares, and hence it is non-negative. This shows that $x^2 + \sum_{i=1}^n (x_i)_{x_i}^2 \geq 0$.

$$x^2 + \sum_{i=1}^n (x_i)_{x_i}^2 \geq 0$$

$$x^2 + \sum_{i=1}^n (x_i)_{x_i}^2 \geq 0 \quad (3)$$

Let $x = (x_1, x_2, \dots, x_n)$ be a vector in \mathbb{R}^n . Then $x^2 = x_1^2 + x_2^2 + \dots + x_n^2$. The expression $x^2 + \sum_{i=1}^n (x_i)_{x_i}^2$ is a sum of squares, and hence it is non-negative. This shows that $x^2 + \sum_{i=1}^n (x_i)_{x_i}^2 \geq 0$.

$$x^2 + \sum_{i=1}^n (x_i)_{x_i}^2 \geq 0 \quad (4)$$

Let $x = (x_1, x_2, \dots, x_n)$ be a vector in \mathbb{R}^n . Then $x^2 = x_1^2 + x_2^2 + \dots + x_n^2$. The expression $x^2 + \sum_{i=1}^n (x_i)_{x_i}^2$ is a sum of squares, and hence it is non-negative. This shows that $x^2 + \sum_{i=1}^n (x_i)_{x_i}^2 \geq 0$.

By differentiation from (2.37),

$$(2.33) \quad u_x(x,y) = \int_0^y f(x,\eta; u(x,\eta); u_x(x,\eta), u_y(x,\eta)) d\eta$$

$$(2.34) \quad u_y(x,y) = \int_0^x f(\xi,y; u(\xi,y); u_x(\xi,y), u_y(\xi,y)) d\xi.$$

Hence, from the extended definition of f , (2.1), and hypothesis 5),

$$(2.35) \quad |u(x,y)| \leq \int_0^x d\xi \int_0^y |f(\xi,\eta; u(\xi,\eta); u_x(\xi,\eta), u_y(\xi,\eta))| d\eta \\ \leq M'_{12} \leq a$$

$$(2.36) \quad |u_x(x,y)| \leq \int_0^y |f(x,\eta; u(x,\eta); u_x(x,\eta), u_y(x,\eta))| d\eta \\ \leq M'_2 \leq b_1$$

$$(2.37) \quad |u_y(x,y)| \leq \int_0^x |f(\xi,y; u(\xi,y); u_x(\xi,y), u_y(\xi,y))| d\xi \\ \leq M'_1 \leq b_2,$$

thus completing the proof of Theorem 1a.

Remarks a) and b) to Theorem 1 apply as well to Theorem 1a.

By the following example we show that the integral surfaces for Theorem 1a are not necessarily unique:

Example 1 Consider the characteristic initial value problem:

$$(2.38) \quad u_{xy} = |u|^{\frac{1}{2}}; \quad u(x,0) = u(0,y) = 0.$$

Here $f(x,y; u; v,q) = |u|^{\frac{1}{2}}$ is continuous for all u but fails to satisfy a Lipschitz condition on u at $u = 0$. Theorem 1a applies

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$$\int_{\mathbb{R}^n} |f(x)|^p dx = \int_{\mathbb{R}^n} |f(x)|^q dx \quad (1)$$

$$\int_{\mathbb{R}^n} |f(x)|^p dx = \int_{\mathbb{R}^n} |f(x)|^q dx \quad (2)$$

where f is a function on \mathbb{R}^n and p, q are positive real numbers.

(3)

$$\int_{\mathbb{R}^n} |f(x)|^p dx = \int_{\mathbb{R}^n} |f(x)|^q dx \quad (4)$$

$$p > q$$

$$\int_{\mathbb{R}^n} |f(x)|^p dx = \int_{\mathbb{R}^n} |f(x)|^q dx \quad (5)$$

$$p < q$$

$$\int_{\mathbb{R}^n} |f(x)|^p dx = \int_{\mathbb{R}^n} |f(x)|^q dx \quad (6)$$

$$p = q$$

where f is a function on \mathbb{R}^n and p, q are positive real numbers.

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where f is a function on \mathbb{R}^n and p, q are positive real numbers.

$$\int_{\mathbb{R}^n} |f(x)|^p dx = \int_{\mathbb{R}^n} |f(x)|^q dx \quad (7)$$

where f is a function on \mathbb{R}^n and p, q are positive real numbers.

where f is a function on \mathbb{R}^n and p, q are positive real numbers.

to insure existence of a solution in a sufficiently small neighborhood of the origin. However, at least two solutions, valid for all (x,y) in the finite plane, are directly available. First, $u \equiv 0$ obviously satisfies. Second, if we seek a solution u satisfying

- i) $u \geq 0$,
- ii) there exist functions X, Y such that

$$u(x,y) = X(x) \cdot Y(y);$$

that is, by the method of separation of variables, we obtain immediately the solution $u(x,y) = \frac{1}{16} x^2 y^2$.

For purposes of illustrating the various situations that might occur, we give the following:

Example 2. Consider the characteristic initial value problem:

$$(2.39) \quad u_{xy} = |u_x|^{\frac{1}{2}}; \quad u(x,0) = u(0,y); \quad 0.$$

Here $f(x,y; u; p,q) = |p|^{\frac{1}{2}}$ is continuous for all p but fails to satisfy a Lipschitz condition on p at $p = 0$. Since $p(x,0) = u_x(x,0) = 0$ neither Theorem 1 nor Theorem 1a will insure existence of a solution over any domain including a portion of the x axis. Such solutions do exist, however. One is $u \equiv 0$. Under the assumption $p = u_x \geq 0$ we obtain another, for now

$$p_y = p^{\frac{1}{2}} \quad \text{or}$$

$$\frac{dp}{p^{\frac{3}{2}}} = 2p^{\frac{1}{2}} = y + c_1.$$

Since $p(x,0) = 0$, $c_1 = 0$ and

In the case of a function $f(x)$ which is continuous on the interval $[a, b]$ and differentiable on (a, b) , the mean value theorem states that there exists a point ξ in (a, b) such that $f'(\xi) = \frac{f(b) - f(a)}{b - a}$.

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

where ξ is a point in (a, b) such that $f'(\xi) = \frac{f(b) - f(a)}{b - a}$.

Let us now consider the case where $f(x)$ is a function of a complex variable z . If $f(z)$ is analytic in a domain D , then the Cauchy-Riemann conditions must be satisfied in D .

The Cauchy-Riemann conditions are given by $u_x = v_y$ and $u_y = -v_x$, where u and v are the real and imaginary parts of $f(z)$.

Example 1. Let $f(z) = u(x, y) + i v(x, y)$ be a function of a complex variable $z = x + iy$.

$$f(z) = u(x, y) + i v(x, y)$$

Let us assume that $f(z)$ is analytic in a domain D . Then the Cauchy-Riemann conditions must be satisfied in D . Let us assume that $f(z)$ is analytic in a domain D . Then the Cauchy-Riemann conditions must be satisfied in D .

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

$$f'(z) = u_x + i v_x$$

$$f'(z) = u_x + i v_x = \frac{\partial f}{\partial z}$$

where $\frac{\partial f}{\partial z} = u_x + i v_x$.

$$p = u_x = \frac{y^2}{4} \quad \text{or, integrating,}$$

$$u = \frac{xy^2}{4} + c_2.$$

Since $u(0, y) = 0$, $c_2 = 0$; and hence

$$u = \frac{xy^2}{4}$$

is a second solution valid throughout the finite plane.

In Example 2 consider the function

$$u_0(x, y) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{xy^2}{4} & \text{for } x \geq 0. \end{cases}$$

u_0 is continuous for all (x, y) and satisfies the initial value problem (2.39) everywhere except along the y axis, where for $y \neq 0$, $u_{0x}(0, y)$ does not exist. Roughly speaking, u_0 is a continuous integral surface of problem (2.39) having a jump in the normal first derivative across a characteristic.

For equations meeting the continuity, boundedness and partial Lipschitz requirements of Theorem 1a we cannot match integral surfaces in the above fashion to obtain first derivative jumps across characteristics. This follows from the fact that if we prescribe $u(a, y) = V(y) \in C^1([0, \ell_2])$ along the characteristic $x=a$, $a \in [0, \ell_1]$, then

$$(2.40) \quad \begin{cases} p_y(a, y) = f(a, y; V(y); p(a, y), V'(y)) \\ p(a, 0) = 0 \end{cases}$$

represents a first order ordinary differential equation for the

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} (x^2 + y^2) &= x \dot{x} + y \dot{y} \\
 &= -y \dot{x} + x \dot{y}
 \end{aligned}$$

where $\dot{x} = -y$ and $\dot{y} = x$

$$\frac{1}{2} \frac{d}{dt} (x^2 + y^2) = 0$$

It is seen that the quantity $x^2 + y^2$ is constant.

In Example 2 consider the system

$$\begin{cases}
 \dot{x} = -y \\
 \dot{y} = x
 \end{cases}$$

The solution curves are circles centered at the origin in the xy -plane. The direction of motion is counter-clockwise. The trajectories are circles of radius r centered at the origin. The angular velocity is constant and equal to 1. The period of the motion is 2π .

The system can be written in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The eigenvalues of \mathbf{A} are $\pm i$ and the corresponding eigenvectors are $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

$$\mathbf{x}(t) = c_1 e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\begin{cases}
 x(t) = c_1 e^{it} + c_2 e^{-it} \\
 y(t) = i c_1 e^{it} - i c_2 e^{-it}
 \end{cases}$$

where c_1 and c_2 are constants determined by the initial conditions.

unknown function $p = u_x$ under a one point boundary condition. The conditions that f be continuous, bounded and partially Lipschitzian are sufficient to insure the existence of a unique determination of $u_x(a, y)$ for $y \in [0, l_2]$. Note that in Example 2 the function f was continuous only and hence the determination of u_x from the above ordinary differential equation was not unique, thus admitting the possibility of a jump in u_x . The conditions for the determination of u_y along a characteristic $y = \text{const.}$ are similar.

The above remarks serve to permit the extension of the domain of existence of the integral surfaces of Theorems 1 and 1a from R to R^* :

R to $R^* : \begin{cases} -l_1 \leq x \leq l_1 \\ -l_2 \leq y \leq l_2 \end{cases}$. The arguments for the existence may

be made applicable to other quadrants than the first by mere coordinate reflections. Moreover the integrals obtained in the separate quadrants must have first order contacts with each other along the coordinate axes by the above reasoning from ordinary

differential equation theory. Hence we may obtain existence and

uniqueness over the domain R^* by replacing B by B^* :

$$\left. \begin{array}{l} -l_1 \leq x \leq l_1 \\ -l_2 \leq y \leq l_2 \\ -a \leq u \leq a \\ -b_1 \leq p \leq b_1 \\ -b_2 \leq q \leq b_2 \end{array} \right\}$$

in Theorem 1; and we obtain simply existence over R^* by replacing B by B^* in Theorem 1a.

In the classical existence theorem for the ordinary differential equation $\frac{dy}{dx} = f(x, y)$, with $y(0) = 0$, the conditions that f

be continuous on $C: \begin{cases} 0 \leq x \leq a \\ -b \leq y \leq b \end{cases}$, with $M = \max |f|$ on C , were shown to be sufficient to insure existence of at least one integral curve $y = y(x)$ for $x \in [0, \alpha]$ with $\alpha \leq \min(a, \frac{b}{M})$. This bound, $\alpha \leq \min(a, \frac{b}{M})$, was shown by A. WINTNER [15] to be a maximal bound in a certain sense. We apply his method to Theorem 1a in the proof of the following:

Theorem 2.

If, in Theorem 1a, we replace B by B'' :

$$\left\{ \begin{array}{l} 0 \leq x \leq l_1' \\ 0 \leq y \leq l_2' \\ -\infty < u < \infty \\ -b_1 \leq p \leq b_1 \\ -b_2 \leq q \leq b_2 \end{array} \right.$$

and require that f be bounded thereon, then hypothesis 3) in that theorem reduces to

$$3)' \quad l_1 \leq \min(l_1', \frac{b_2}{M}), \quad l_2 \leq \min(l_2', \frac{b_1}{M}),$$

where $M = \max |f|$ on B'' . Moreover, the bounds established by 3)'¹ are maximal bounds in a sense to be explained below.

Proof.

The condition $M l_1 l_2 \leq a$ of hypothesis 3) is immediately satisfied since a approaches $+\infty$. The conditions $M l_1 \leq b_2$, $M l_2 \leq b_1$ may be rewritten as in 3)'¹ and are now the only restrictions on l_1 and l_2 .

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$$\left. \begin{aligned}
 & \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) \\
 & \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) \\
 & \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) \\
 & \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) \\
 & \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right)
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$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right)$$

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If $\ell_1' \leq \frac{b_2}{H}$, ($\ell_2' \leq \frac{b_1}{H}$), then the conclusion is immediate.

For, we may take $f(x, y; u; p, q) = h(x)$, ($g(y)$), which function is not even defined for $x > \ell_1 = \ell_1'$, ($y > \ell_2 = \ell_2'$).

Suppose $\ell_2' > \frac{b_1}{H}$. Then we consider the sequence of problems:

$$(2.41) \quad u_{xy} = (2^{1/m} - u_x)^{1/m+1}, \quad u(x, 0) = u(0, y) = 0, \quad (m=1, 2, \dots).$$

Setting $p = u_x$, (2.41) becomes

$$p_y(x, y) = (2^{1/m} - p(x, y))^{1/m+1}, \quad p(x, 0) = 0.$$

Integrating this ordinary differential equation for p as a function of y , we obtain

$$p(x, y) = 2^{1/m} - \left[2^{1/m+1} - \frac{m}{m+1} y \right]^{m+1/m}.$$

But, since $p = u_x$ and $u(0, y) = 0$ we may integrate again to obtain

$$(2.42) \quad u(x, y) = x \left\{ 2^{1/m} - \left[\frac{m}{m+1} (C_m - y) \right]^{m+1/m} \right\}$$

where

$$(2.43) \quad C_m = \frac{m+1}{m} 2^{\frac{1}{m+1}}.$$

The line $y = C_m$ is a branch line of the solution u . Under the supposition $\ell_2' > \frac{b_1}{H}$, the desired statement is that $\frac{b_1}{H}$ is a maximal bound on ℓ_2' , i.e., for each $\epsilon > 0$, there exists a function $f(x, y; u; p, q)$, depending on ϵ and satisfying hypotheses 1), 2) and 3) on B'' , such that an integral $u(x, y)$ of the problem corresponding to f has a singularity for some $y \in \left(\frac{b_1}{H}, \frac{b_1}{H} + \epsilon \right)$.

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the roots of the characteristic equation $\Delta(\lambda) = 0$. Then the general solution of the system $\dot{x} = Ax$ is given by

$$x(t) = \sum_{i=1}^n c_i e^{\lambda_i t} v_i$$

where v_i are the corresponding eigenvectors. If $\lambda_i = \alpha_i + j\beta_i$, then $e^{\lambda_i t} = e^{\alpha_i t} (\cos \beta_i t + j \sin \beta_i t)$. For real-valued solutions, we take the real and imaginary parts.

Let $\lambda = \alpha + j\beta$ be a complex root of $\Delta(\lambda) = 0$.

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} e^{(\alpha + j\beta)t}$$

Then the real and imaginary parts of $v e^{\lambda t}$ are

Real part: $e^{\alpha t} \cos \beta t$

$$e^{\alpha t} \begin{bmatrix} v_1 \cos \beta t \\ v_2 \cos \beta t \\ \vdots \\ v_n \cos \beta t \end{bmatrix}$$

Imaginary part: $e^{\alpha t} \sin \beta t$

Thus the real solution is

$$e^{\alpha t} \begin{bmatrix} v_1 \cos \beta t \\ v_2 \cos \beta t \\ \vdots \\ v_n \cos \beta t \end{bmatrix} + e^{\alpha t} \begin{bmatrix} v_1 \sin \beta t \\ v_2 \sin \beta t \\ \vdots \\ v_n \sin \beta t \end{bmatrix}$$

and the imaginary solution is

$$e^{\alpha t} \begin{bmatrix} v_1 \sin \beta t \\ v_2 \sin \beta t \\ \vdots \\ v_n \sin \beta t \end{bmatrix} - e^{\alpha t} \begin{bmatrix} v_1 \cos \beta t \\ v_2 \cos \beta t \\ \vdots \\ v_n \cos \beta t \end{bmatrix}$$

Therefore, the general solution of the system $\dot{x} = Ax$ is

$$x(t) = \sum_{i=1}^n c_i e^{\lambda_i t} v_i + \sum_{i=1}^n d_i e^{\alpha_i t} \begin{bmatrix} v_{i1} \cos \beta_i t \\ v_{i2} \cos \beta_i t \\ \vdots \\ v_{in} \cos \beta_i t \end{bmatrix} + \sum_{i=1}^n e_i e^{\alpha_i t} \begin{bmatrix} v_{i1} \sin \beta_i t \\ v_{i2} \sin \beta_i t \\ \vdots \\ v_{in} \sin \beta_i t \end{bmatrix}$$

where c_i, d_i, e_i are arbitrary constants. If $\lambda_i = \alpha_i + j\beta_i$ and $\lambda_j = \alpha_j + j\beta_j$ are complex conjugate roots, then $\alpha_i = \alpha_j = \alpha$ and $\beta_i = -\beta_j = \beta$.

Let $\lambda_1 = \alpha + j\beta$ and $\lambda_2 = \alpha - j\beta$ be a pair of complex conjugate roots of $\Delta(\lambda) = 0$.

Then the corresponding eigenvectors v_1 and v_2 are also complex conjugates.

$$v_1 = \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{bmatrix} e^{(\alpha + j\beta)t}$$

Defining

$$f_m(x, y; u; p, q) = (2^{1/m} - p)^{1/m+1} \text{ for } -2^{1/m+1} \leq p \leq 2^{1/m+1},$$

($m = 1, 2, \dots$), we obtain

$$b_{1m} = 2^{1/m+1}, \quad M_m = (2^{1/m} + 2^{1/m+1})^{1/m+1}; \text{ and, since}$$

$$(2^{1/m} + 2^{1/m+1}) > 2, \quad (m = 1, 2, \dots),$$

$$\lim_{m \rightarrow \infty} \frac{b_{1m}}{M_m} = 1 - .$$

Moreover, by (2.43),

$$\lim_{m \rightarrow \infty} C_m = 1 \quad .$$

Hence, given $\epsilon > 0$, there exists a positive integer M , depending on ϵ alone, such that $m > M \implies$

$$\frac{b_{1m}}{M_m} + \epsilon > C_m > \frac{b_{1m}}{M_m} \quad .$$

Consequently $\frac{b_1}{M}$ is a maximal bound on \mathcal{L}_2 .

To determine that the condition $\mathcal{L}_1 \leq \min(\mathcal{L}_1', \frac{b_2}{M})$ is also a maximal bound we consider the sequence of problems.

$$(2.44) \quad u_{xy} = (2^{1/m} - u_y)^{1/m+1}, \quad u(x, 0) = u(0, y), \quad (m = 1, 2, \dots),$$

and follow the same line of reasoning as before. Thus Theorem 2 is verified.

The close parallelism between our conclusions and the classical theorems for first order ordinary differential equations

giving

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\Omega} \nabla u \cdot \nabla w dx$$

where $v = (u - w)/2$ and $w = (u + w)/2$

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\Omega} \nabla u \cdot \nabla w dx$$

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\Omega} \nabla u \cdot \nabla w dx$$

$$= - \int_{\Omega} \nabla u \cdot \nabla w dx$$

where $v = (u - w)/2$

$$= - \int_{\Omega} \nabla u \cdot \nabla w dx$$

Since $\nabla w \cdot \nabla w \geq 0$, it follows that $\frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx \leq 0$. This implies that $\int_{\Omega} |\nabla u|^2 dx$ is non-increasing.

$$\frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx \leq 0$$

$$\int_{\Omega} |\nabla u|^2 dx \leq \int_{\Omega} |\nabla u_0|^2 dx$$

By the Poincaré inequality, there exists a constant C such that $\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}$. This implies that $\|u\|_{L^2(\Omega)}$ is bounded.

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}$$

Therefore, $\|u\|_{L^2(\Omega)}$ is bounded. This implies that u is bounded in $L^2(\Omega)$.

Finally, since $\frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx \leq 0$, it follows that $\int_{\Omega} |\nabla u|^2 dx$ is bounded. This implies that ∇u is bounded in $L^2(\Omega)$.

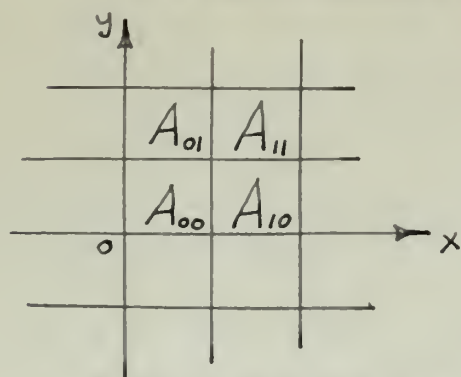
(See F. ZAMES [2]) leads one to suspect that an existence theorem might be proved wherein mere continuity of the function f was demanded. The analogue to the Cauchy polygon method is the attack suggested by the parallelism, and it leads to an existence theorem for the characteristic initial value problem:

$$(2.45) \quad u_{xy} = f(x,y; u) \quad , \quad u(x,0) = u(0,y) = 0.$$

We do not give the proof here; first, because the theorem is a special case of Theorem 1a; and, second, because the steps in the proof are practically identical with those of the Cauchy polygon method for ordinary differential equations.

When $f = f(x,y; u; p,q)$ and f is merely continuous this attack involves difficulties which we have not been able to resolve. We sketch the method to indicate the source of trouble:

In a neighborhood of the origin a partition Π by



characteristics is specified where the subregions A_{ij} in the first quadrant are defined as

$$A_{ij} : \begin{cases} x_j \leq x < x_{j+1} \\ y_j \leq y < y_{j+1} \end{cases} \quad (i,j=0,1,2,\dots)$$

We formulate the approximate integral surface u corresponding to the partition Π as follows:

$$(2.46) \quad u_{\Pi}(x,y) = \int_0^x d\xi \int_0^y F_{\Pi}(\xi,\eta) d\eta,$$

where

Let \mathcal{A} be a subalgebra of \mathcal{B} . Then \mathcal{A} is a σ -algebra if and only if \mathcal{A} is closed under countable unions and complements. In other words, \mathcal{A} is a σ -algebra if and only if \mathcal{A} is a σ -ring and contains the universal set Ω .

$$P(A) = \frac{|A|}{|\Omega|}$$

Let \mathcal{A} and \mathcal{B} be σ -algebras. Then $\mathcal{A} \cap \mathcal{B}$ is a σ -algebra. Also, $\mathcal{A} \cup \mathcal{B}$ is a σ -algebra if and only if \mathcal{A} and \mathcal{B} are independent. In other words, \mathcal{A} and \mathcal{B} are independent if and only if $P(A \cap B) = P(A)P(B)$.

Let \mathcal{A} and \mathcal{B} be σ -algebras. Then \mathcal{A} and \mathcal{B} are independent if and only if $P(A \cap B) = P(A)P(B)$. In other words, \mathcal{A} and \mathcal{B} are independent if and only if $P(A \cap B) = P(A)P(B)$.

Let \mathcal{A} and \mathcal{B} be σ -algebras. Then \mathcal{A} and \mathcal{B} are independent if and only if $P(A \cap B) = P(A)P(B)$.



$$P(A \cap B) = P(A)P(B)$$

Let \mathcal{A} and \mathcal{B} be σ -algebras. Then \mathcal{A} and \mathcal{B} are independent if and only if $P(A \cap B) = P(A)P(B)$.

$$P(A \cap B) = P(A)P(B)$$

$$(2.47) \quad F_{\pi}(x, y) = f(x_1, y_j; u_{\pi}(x_1, y_j); u_{\pi_x}(x_1, y_j), \\ u_{\pi_y}(x_1, y_j))$$

for $(x, y) \in A_{1j}$.

The principal difficulty lies in the fact that the derivatives

$$(2.48) \quad u_{\pi_x} = \int_0^y F_{\pi}(x, \eta) d\eta \quad \text{and}$$

$$(2.49) \quad u_{\pi_y} = \int_0^x F_{\pi}(\xi, y) d\xi$$

are discontinuous across the partition lines $x = \text{constant}$ and $y = \text{constant}$, respectively, thus preventing the direct application of ARZELA's theorem on equicontinuous functions when we consider the sequence of approximate integral surfaces formed by partition refinement.

The equation of (2.45) appears to be more amenable than the more general equation involving the first derivatives p and q . G. FUBINI [16] p. 622, by demanding only that $f(x, y; u)$ be continuous and Lipschitzian with respect to u , has proved the existence of a unique integral of $u_{xy} = f(x, y; u)$ satisfying Dirichlet conditions, i.e. the value of u prescribed on a closed contour. This result, while remarkable, is not contradictory since u is shown to have a discontinuity of the second type at one point of the boundary.

We conclude this chapter with the statement of the extension of Theorems 1 and 1a to a system of equations

$$\frac{1}{2} \frac{d}{dt} (v^2) = \frac{1}{2} \frac{d}{dt} (v_x^2 + v_y^2 + v_z^2) = \frac{1}{2} (2v \cdot \frac{dv}{dt}) = v \cdot \frac{dv}{dt}$$

$$\frac{1}{2} \frac{d}{dt} (v^2) = \frac{1}{2} \frac{d}{dt} (v_x^2 + v_y^2 + v_z^2)$$

$$\frac{1}{2} \frac{d}{dt} (v^2) = \frac{1}{2} \frac{d}{dt} (v_x^2 + v_y^2 + v_z^2)$$

The following is a list of the most important results of the theory of the motion of a particle in a constant field.

$$\frac{1}{2} \frac{d}{dt} (v^2) = \frac{1}{2} \frac{d}{dt} (v_x^2 + v_y^2 + v_z^2) = \frac{1}{2} (2v \cdot \frac{dv}{dt}) = v \cdot \frac{dv}{dt} \quad (10.1)$$

$$\frac{1}{2} \frac{d}{dt} (v^2) = \frac{1}{2} \frac{d}{dt} (v_x^2 + v_y^2 + v_z^2) = \frac{1}{2} (2v \cdot \frac{dv}{dt}) = v \cdot \frac{dv}{dt} \quad (10.2)$$

The following is a list of the most important results of the theory of the motion of a particle in a constant field. The following is a list of the most important results of the theory of the motion of a particle in a constant field. The following is a list of the most important results of the theory of the motion of a particle in a constant field.

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(2.50) $u_i = f_i(x, y; u_1, \dots, u_n; p_1, \dots, p_n, q_1, \dots, q_n), (i=1, 2, \dots, n)$
satisfying the initial conditions

$$(2.51) \quad u_i(x, 0) = u_i(0, y) = 0, \quad (i=1, 2, \dots, n).$$

Theorem 3, below, is a natural extension of Theorem 1. In principle, it was first obtained by O. NICCOLETTI [14] p.7. His statement, however, is not explicit about the bounds on the domain of existence. Moreover, to prove uniqueness he requires the f_i to be of class C^1 . We obtain the improved statement, Theorem 3, by modifying the arguments of E. KANKE [2] p. 402 and p. 403 to apply them to the system (2.50).

Theorem 3)

$$1) \quad f_i(x, y; u_j; p_j, q_j) \in C(B^n), \quad B^n: \begin{cases} 0 \leq x \leq k_1 \\ 0 \leq y \leq k_2 \\ -a \leq u_1 \leq a \\ -b_1 \leq p_1 \leq b_1 \\ -b_2 \leq q_1 \leq b_2 \end{cases}$$

2) The f_i are Lipschitzian on B^n ; i.e. there exists a positive constant K such that for $(x, y; u^1_j; p^1_j, q^1_j) \in B^n$,

$(x, y; u^2_j; p^2_j, q^2_j) \in B^n$, and $i = 1, 2, \dots, n$,

$$\begin{aligned} & |f_i(x, y; u^1_j; p^1_j, q^1_j) - f_i(x, y; u^2_j; p^2_j, q^2_j)| \\ & \leq K \sum_{j=1}^n \left\{ |u^1_j - u^2_j| + |p^1_j - p^2_j| + |q^1_j - q^2_j| \right\}. \end{aligned}$$

3) $M k_1 k_2 \leq a$, $M k_1 \leq b_2$, $M k_2 \leq b_1$ where
 $M = \max \left\{ |f_1|, \dots, |f_n| \right\}$ on B^n .

² Notation: $(x, y; u_j; p_j, q_j) = (x, y; u_1, \dots, u_n; p_1, \dots, p_n, q_1, \dots, q_n)$.

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⇒ 4) There exists one and only one set of functions

$$\{u_1, \dots, u_n\}, u_j(x, y) \in C^1(R), u_{j,xy}(x, y) \in C(R), (j=1, \dots, n),$$

where $R: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \end{cases}$, such that for each $(x, y) \in R$ the point

$$(x, y; u_j(x, y); u_{j,x}(x, y), u_{j,y}(x, y)) \in B^n, \text{ and}$$

$$u_{1,xy}(x, y) = f_1(x, y; u_j(x, y); u_{j,x}(x, y), u_{j,y}(x, y)),$$

$$u_1(x, 0) = u_1(0, y) = 0, \quad (i = 1, \dots, n), \text{ for each } (x, y) \in R.$$

By relaxing hypothesis 2) we obtain the improved theorem below; which stands in the same relation to Theorem 3 that Theorem 1a does to Theorem 1.

Theorem 3a

1)

2)' The f_i are partially Lipschitzian on B^n ; i.e. there exists a positive constant K such that for $(x, y; u_j; p_j^1, q_j^1) \in B^n$,

$(x, y; u_j; p_j^2, q_j^2) \in B^n$, and $i = 1, 2, \dots, n$,

$$\begin{aligned} & |f_i(x, y; u_j; p_j^1, q_j^1) - f_i(x, y; u_j; p_j^2, q_j^2)| \\ & \leq K \sum_{j=1}^n \left\{ |p_j^1 - p_j^2| + |q_j^1 - q_j^2| \right\}. \end{aligned}$$

3)

⇒ 4)' There exists at least one set of functions $\{u_1, \dots, u_n\}$, $u_j(x, y) \in C^1(R)$, $u_{j,xy}(x, y) \in C(R)$, $(j=1, \dots, n)$, where

Let \mathcal{L} be the Lie algebra of the group G . Then \mathcal{L} is a Lie algebra over \mathbb{R} with the following properties:

$$\mathcal{L} = \{ X \in \mathfrak{gl}(n, \mathbb{R}) \mid X^T = -X \}$$

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The Lie algebra \mathcal{L} is isomorphic to the Lie algebra of the group G .

Let \mathcal{L} be the Lie algebra of the group G . Then \mathcal{L} is a Lie algebra over \mathbb{R} with the following properties:

Then in case of $G = SO(n)$

Lemma 1

(1)

Let \mathcal{L} be the Lie algebra of the group G . Then \mathcal{L} is a Lie algebra over \mathbb{R} with the following properties:

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$$\mathcal{L} = \{ X \in \mathfrak{gl}(n, \mathbb{R}) \mid X^T = -X \}$$

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$$\mathcal{L} = \{ X \in \mathfrak{gl}(n, \mathbb{R}) \mid X^T = -X \}$$

(2)

Let \mathcal{L} be the Lie algebra of the group G . Then \mathcal{L} is a Lie algebra over \mathbb{R} with the following properties:

Let \mathcal{L} be the Lie algebra of the group G . Then \mathcal{L} is a Lie algebra over \mathbb{R} with the following properties:

$R: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \end{cases}$, such that for each $(x, y) \in R$ the point

$(x, y; u_j(x, y); u_{j,x}(x, y), u_{j,y}(x, y)) \in B^n$, and

$u_{i,xy}(x, y) = f_i(x, y; u_j(x, y); u_{j,x}(x, y), u_{j,y}(x, y))$,

$u_i(x, 0) = u_i(0, y) = 0$, ($i = 1, \dots, n$), for each $(x, y) \in R$.

The proof of Theorem 3a is essentially a step by step repetition of that for Theorem 1a. WEIERSTRASS' theorem tells us that for each positive integer i there exists a sequence of polynomials $\{g_{i\lambda}\}$ $(x, y; u_j; p_j, q_j)$, ($\lambda = 1, 2, \dots$), converging uniformly on B^n to $f_i(x, y; u_j; p_j, q_j)$. We extend the $g_{i\lambda}$ and the f_i as before and obtain that there exist positive constants L_i such that for each i $|g_{i\lambda}| \leq L_i$ on B^n , extended, and for all λ . We let $L = \max \{L_1, \dots, L_n\}$ and proceed as before, using Theorem 3 in place of Theorem 1 to obtain the integral $u_{i\lambda}$ associated with each $g_{i\lambda}$.

We note only one point of significant difference in the arguments. In place of inequality (2.13) of Lemma 2 we now have the inequalities

$$|u_{i\lambda,x}(x_2, y) - u_{i\lambda,x}(x_1, y)| \leq K \int_0^y \left\{ \sum_{j=1}^n |u_{j\lambda,x}(x_2, \eta) - u_{j\lambda,x}(x_1, \eta)| \right\} d\eta$$

Summing these, and letting

$$Z(\gamma) = \sum_{i=1}^n |u_{i\lambda,x}(x_2, \gamma) - u_{i\lambda,x}(x_1, \gamma)|,$$

we obtain

$$\left. \begin{aligned} & \text{...} \\ & \frac{1}{2} \frac{d^2 x}{dt^2} = \dots \end{aligned} \right\} \dots$$

$$\dots \frac{d^2 x}{dt^2} = \dots$$

$$\dots \frac{d^2 x}{dt^2} = \dots$$

$$\dots \frac{d^2 x}{dt^2} = \dots$$

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$$\begin{aligned} & \dots \\ & \dots \\ & \dots \\ & \dots \end{aligned}$$

$$0 \leq z(y) \leq \epsilon n \int_0^y z(\eta) d\eta + n(\mu + \zeta)$$

to which Lemma 1 applies. Thus the equicontinuity of each of the sequences $\{u_{i\lambda, x}\}$, $(i = 1, \dots, n)$ is assured.

Remarks a) and b) to Theorems 1 and 1a apply, with obvious modifications, to Theorems 3 and 3a. Moreover, as before, we may extend the domain of existence of the integral surfaces of Theorems 3 and 3a from R to R^* : $\begin{cases} -k_1 \leq x \leq k_1 \\ -k_2 \leq y \leq k_2 \end{cases}$.

The set of functions $\{u_1, \dots, u_n\}$ representing the solution to the problem of Theorem 3a cannot be shown to be unique. This is made evident by extending Example 1 to the system

$$\begin{aligned} u_{1,xy} &= |u_1|^{\frac{1}{2}}, & u_1(x,0) &= u_1(0,y) = 0 \\ u_{2,xy} &= 0, & u_2(x,0) &= u_2(0,y) = 0 \\ & \vdots & & \vdots \\ u_{n,xy} &= 0, & u_n(x,0) &= u_n(0,y) = 0 \end{aligned}$$

for which $u_i \equiv 0$ $(i = 2, \dots, n)$

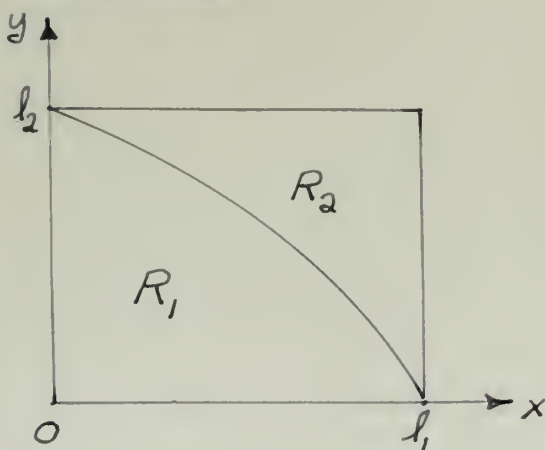
while $u_1 \equiv 0$ or $u_1 = \frac{1}{16} x^2 y^2$. Thus at least two sets of solutions are possible for this system which satisfies the hypotheses of Theorem 3a.

CHAPTER III

The Cauchy Problem for $u_{xy} = f(x, y; u; u_x, u_y)$.

The development of this chapter closely parallels that of Chapter II. Consequently, the notation will be abridged, in particular with respect to the arguments of functions; and the proofs will be merely outlined to show minor variations from the statements in Chapter 2.

For reference, we state the following theorem proved first for systems of equations by O. NICCOLETTI [14] p. 7. Our statement may be easily inferred from that of E. KAMKE [2] p. 405 and p. 410, by a slight modification of his proof.

Theorem 4

$$1) f(x, y; u; p, q) \in C(B),$$

$$B: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \\ -a \leq u \leq a \\ -b_1 \leq p \leq b_1 \\ -b_2 \leq q \leq b_2 \end{cases}$$

2) f is Lipschitzian on B , (as defined in Theorem 1).

3) $M l_1 l_2 \leq a$, $M l_1 \leq b_2$, $M l_2 \leq b_1$, where $M = \max |f|$ on B

4) $\gamma: \begin{cases} 0 \leq x \leq l_1 \\ y = \varphi(x) \end{cases}$ where $\varphi(x) \in C^1([0, l_1])$, $\varphi'(x) \neq 0$ for $x \in [0, l_1]$ and $\varphi(0) = l_2$, $\varphi(l_1) = 0$.

CHAPTER 11

The linear function can be written as $y = kx + b$.

The assignment of this chapter is to study the properties of linear functions and to understand the relationship between the slope and the y-intercept of a linear function. The results will be applied to solve various problems in geometry and algebra.

For reference, we give the following definitions and theorems. The reader is referred to the text for the proofs of these theorems. The reader is also referred to the text for the definitions of the terms used in this chapter.

FIGURE 1



$$\left. \begin{aligned} y &= kx + b \\ k &= \frac{y - b}{x} \\ b &= y - kx \end{aligned} \right\} \text{Equation of a line}$$

11.1 The slope of a line is the ratio of the change in y to the change in x.

$$\left. \begin{aligned} \text{Slope } k &= \frac{\Delta y}{\Delta x} \\ \text{y-intercept } b &= y - kx \end{aligned} \right\} \text{Equation of a line}$$

\Rightarrow 5) There exists one and only one function $u(x,y) \in C^1(R)$, $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq \lambda_1 \\ 0 \leq y \leq \lambda_2 \end{cases}$, such that for each $(x,y) \in R$, the point $(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in E$, and $u_{xy}(x,y) = f(x,y; u(x,y); u_x(x,y), u_y(x,y))$,
 $u(x, \varphi(x)) = u_x(x, \varphi(x)) = u_y(x, \varphi(x)) = 0$

for each $(x,y) \in R$.

Remarks c) Suppose we prescribe $u(x, \varphi(x)) = U(x)$, $u_x(x, \varphi(x)) = P(x)$, $u_y(x, \varphi(x)) = Q(x)$ where $U(x) \in C^1([0, \lambda_1])$ while $P(x), Q(x) \in C([0, \lambda_1])$. Our prescription must satisfy the strip condition $U' = P + Q \cdot \varphi'$ for each $x \in [0, \lambda_1]$. Consider the function $w(x,y) = U(x) + (y - \varphi(x)) Q(x)$. Clearly, $w_{xy} = Q'(x)$ while $w(x, \varphi(x)) = U(x)$, $w_x(x, \varphi(x)) = P(x)$, and $w_y(x, \varphi(x)) = Q(x)$. Hence the function $v = u - w$ must satisfy $v_{xy} = Q'(x) + f(x,y; v+w; v_x+w_x, v_y+w_y)$, with $v(x, \varphi(x)) = v_x(x, \varphi(x)) = v_y(x, \varphi(x)) = 0$, a problem of the type covered by Theorem 4.

d) Hypothesis 4) of Theorem 4 is more restrictive than it need be. At isolated points of γ we may have a horizontal or vertical tangent, provided that γ does not cross the same characteristic more than once. For, under these conditions the inverse function ψ to φ will exist and be continuous for all $y \in [0, \lambda_2]$.

Our improvement of this theorem is as follows:

$\left. \begin{aligned} \text{if } \lambda = 1, \text{ then } \psi(x) = x \\ \text{if } \lambda = -1, \text{ then } \psi(x) = -x \end{aligned} \right\} \text{if } \lambda = \pm 1, \text{ then } \psi(x) = \lambda x$
 and if $\lambda = \pm i$, then $\psi(x) = \pm ix$
 $\psi(x) = \lambda x$

The function $\psi(x)$ is linear and satisfies $\psi(x+y) = \psi(x) + \psi(y)$
 and $\psi(\lambda x) = \lambda \psi(x)$.
 The function $\psi(x)$ is also a homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}, +)$.
 The kernel of ψ is $\{0\}$.
 The image of ψ is \mathbb{R} .
 The function $\psi(x)$ is a linear transformation.

The function $\psi(x)$ is a linear transformation. It is defined by $\psi(x) = \lambda x$.
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 The function $\psi(x)$ is a linear transformation. It is defined by $\psi(x) = \lambda x$.
 The kernel of ψ is $\{0\}$. The image of ψ is \mathbb{R} .

Theorem 4a

1)

2) f is partially Lipschitzian on B , (as defined in Theorem

1a).

3)

4)

\Rightarrow 5) There exists at least one function $u(x,y) \in C^1(R)$,
 $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \end{cases}$, such that for each

$(x,y) \in R$, the point $(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in B$, and

$$u_{xy}(x,y) = f(x,y; u(x,y); u_x(x,y), u_y(x,y)),$$

$$u(x, \varphi(x)) = u_x(x, \varphi(x)) = u_y(x, \varphi(x)) = 0$$

for each $(x,y) \in R$.

Outline of proof.

The path γ may also be expressed as $\gamma: \begin{cases} x = \psi(y) \\ 0 \leq y \leq l_2 \end{cases}$ where

$\psi(y) \in C^1([0, l_2])$, $\psi'(y) \neq 0$ for $y \in [0, l_2]$. ψ is the inverse function to φ .

We may express the problem as the integral equation

$$(3.1) \quad u(x,y) = \int_{\psi(y)}^x d\xi \int_{\varphi(\xi)}^y f(\xi, \eta; u; u_x, u_y) d\eta$$

$$\text{whence} \quad = \int_{\varphi(x)}^y d\eta \int_{\psi(\eta)}^x f(\xi, \eta; u; u_x, u_y) d\xi$$

$$(3.2) \quad u_x(x,y) = \int_{\varphi(x)}^y f(x, \eta; u; u_x, u_y) d\eta$$

$$(3.3) \quad u_y(x,y) = \int_{\psi(y)}^x f(\xi, y; u; u_x, u_y) d\xi.$$

Section 11

11.1. Let \mathcal{H} be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then T is self-adjoint if and only if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in \mathcal{H}$.

11.2. Let \mathcal{H} be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then T is normal if and only if $TT^* = T^*T$.

11.3. Let \mathcal{H} be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then T is unitary if and only if $T^{-1} = T^*$.

11.4. Let \mathcal{H} be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then T is positive if and only if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$.

11.5. Let \mathcal{H} be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then T is self-adjoint and positive if and only if $T = \int_{\sigma(T)} \lambda E_\lambda$ for some spectral measure E_λ .

11.6. Let \mathcal{H} be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then T is self-adjoint and positive if and only if $T = |S|^2$ for some $S \in \mathcal{B}(\mathcal{H})$.

11.7. Let \mathcal{H} be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then T is self-adjoint and positive if and only if $T = \int_{\sigma(T)} \lambda^2 E_\lambda$ for some spectral measure E_λ .

Section 12

12.1. Let \mathcal{H} be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then T is compact if and only if $\|T^n\| \rightarrow 0$ as $n \rightarrow \infty$.

12.2. Let \mathcal{H} be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then T is compact if and only if $T = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n$ for some sequence $\{\lambda_n\}$ and orthonormal basis $\{e_n\}$.

12.3. Let \mathcal{H} be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then T is compact if and only if $T = \int_{\sigma(T)} \lambda E_\lambda$ for some spectral measure E_λ .

12.4. Let \mathcal{H} be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then T is compact if and only if $T = \int_{\sigma(T)} \lambda^2 E_\lambda$ for some spectral measure E_λ .

12.5. Let \mathcal{H} be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then T is compact if and only if $T = \int_{\sigma(T)} \lambda^3 E_\lambda$ for some spectral measure E_λ .

12.6. Let \mathcal{H} be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then T is compact if and only if $T = \int_{\sigma(T)} \lambda^4 E_\lambda$ for some spectral measure E_λ .

12.7. Let \mathcal{H} be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then T is compact if and only if $T = \int_{\sigma(T)} \lambda^5 E_\lambda$ for some spectral measure E_λ .

By WEIERSTRASS' theorem, there exists a sequence of polynomials $\{g_\lambda\}$ $\xrightarrow{\text{unif.}}$ f on B . We extend the domain of definition of f and the polynomials g_λ over B to B' by definition (2.1).

We obtain again the constant $L > 0$ such that $|g_\lambda| \leq L$ in B' for all λ . Moreover, for each g_λ the Lipschitz condition (2.2) is satisfied. Thus, by Theorem 4, for each λ there exists a unique solution u_λ to the problem

$$(3.4) \quad \begin{cases} u_{\lambda,xy} = g_\lambda(x,y; u_\lambda; u_{\lambda,x}, u_{\lambda,y}), \\ u_\lambda(x, \varphi(x)) = u_{\lambda,x}(x, \varphi(x)) = u_{\lambda,y}(x, \varphi(x)) = 0. \end{cases}$$

That the sequences $\{u_\lambda\}$, $\{u_{\lambda,x}\}$, $\{u_{\lambda,y}\}$ are uniformly bounded on R , and that the sequence $\{u_\lambda\}$ is equicontinuous on R is immediately evident from the equivalent integral expressions

$$(3.5) \quad \begin{aligned} u_\lambda(x,y) &= \int_{\varphi(x)}^x d\xi \int_{\varphi(\xi)}^y g_\lambda(\xi, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\eta \\ &= \int_{\varphi(x)}^y d\eta \int_{\varphi(\eta)}^x g_\lambda(\xi, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\xi. \end{aligned}$$

$$(3.6) \quad u_{\lambda,x}(x,y) = \int_{\varphi(x)}^y g_\lambda(x, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\eta,$$

$$(3.7) \quad u_{\lambda,y}(x,y) = \int_{\varphi(y)}^x g_\lambda(\xi, y; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\xi.$$

We now establish the equicontinuity of $\{u_{\lambda,x}\}$ and of $\{u_{\lambda,y}\}$. This done, the same arguments as those for the proof of Theorem 1a will serve to obtain a subsequence $\{u_{\lambda^*}\}$ of $\{u_\lambda\}$ which converges uniformly to the solution u .

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$$\left. \begin{aligned} & \dots \\ & \dots \end{aligned} \right\} (12.1)$$

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There is no loss in generality in restricting ourselves at this point to the consideration of those points $(x, y) \in R_2 : \begin{cases} 0 \leq x \leq \xi_1 \\ \varphi(x) \leq y \leq \xi_2 \end{cases}$.

For we shall see that the arguments developed below will apply as well for $(x, y) \in R_1 : \begin{cases} 0 \leq x \leq \xi_1 \\ 0 \leq y \leq \varphi(x) \end{cases}$ after a simple coordinate

translation and rotation. Thus if we insure existence of a solution on R_2 , existence on R_1 is simultaneously verified. Moreover, the Cauchy initial data insure that such integral surfaces have a first order contact along Υ and hence define an integral surface throughout all of $R = R_1 + R_2$.

Given points $(x_2, y_2) \in R_2$, $(x_1, y_1) \in R_2$, it is always possible to label these points in such a way that $(x_1, y_2) \in R_2$. This being done, we have that

$$(3.8) \quad |u_{\lambda, x}(x_1, y_2) - u_{\lambda, x}(x_1, y_1)| \leq L |y_2 - y_1|,$$

$$(3.9) \quad |u_{\lambda, y}(x_2, y_2) - u_{\lambda, y}(x_1, y_2)| \leq L |x_2 - x_1|.$$

Assuming, without loss, that $y \geq \varphi(x_2) \geq \varphi(x_1)$, we have that

$$(3.10) \quad u_{\lambda, x}(x_2, y) - u_{\lambda, x}(x_1, y) = \int_{\varphi(x_2)}^y [g_{\lambda}(x_2, \eta; u_{\lambda}; u_{\lambda, x}, u_{\lambda, y}) - g_{\lambda}(x_1, \eta; u_{\lambda}; u_{\lambda, x}, u_{\lambda, y})] d\eta + \int_{\varphi(x_1)}^{\varphi(x_2)} g_{\lambda}(x_1, \eta; u_{\lambda}; u_{\lambda, x}, u_{\lambda, y}) d\eta$$

We operate on the first integral on the right hand side of (3.10) in the manner demonstrated in equation (2.20). We obtain a formula identical with (2.20) except that here the lower limit of integration is $y = \varphi(x_2)$ instead of $y = 0$. For brevity, we omit the formula.

There is an issue in connection with the definition of the
norm on the completion of a normed space. It is not clear
if the norm is defined as $\|x\| = \inf\{\|y\| : y \in X, y \sim x\}$

or as $\|x\| = \inf\{\|y\| : y \in X, y \sim x, \|y\| \leq 1\}$.
The first definition is more natural, but the second one
is more convenient for the proof of the Hahn-Banach theorem.

Let X be a normed space and f a linear functional on X .
Let p be a seminorm on X such that $f(x) \leq p(x)$ for all $x \in X$.

Then there exists a linear functional F on the completion \bar{X}
such that $F(x) = f(x)$ for all $x \in X$ and $F(x) \leq p(x)$ for all $x \in \bar{X}$.

The proof of this theorem is based on the fact that the completion
of a normed space is a Banach space. The norm on the completion
is defined as $\|x\| = \inf\{\|y\| : y \in X, y \sim x\}$.

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of a normed space is a Banach space. The norm on the completion
is defined as $\|x\| = \inf\{\|y\| : y \in X, y \sim x\}$.

Since

$$(3.11) \quad \left| \int_{\varphi(x_1)}^{\varphi(x_2)} \varepsilon_{\lambda} (x_1, \eta; u_{\lambda}; u_{\lambda, x}, u_{\lambda, y}) d\eta \right| \leq L |\varphi(x_2) - \varphi(x_1)|, \quad (\lambda = 1, 2, \dots)$$

and since $\varphi(x)$ is uniformly continuous on $[0, \ell_1]$, by the same reasoning as before we arrive at the slight modification to Lemma 2,

$$(3.12) \quad \left| u_{\lambda, x}(x_2, y) - u_{\lambda, x}(x_1, y) \right| \leq K \int_{\varphi(x_2)}^y |u_{\lambda, x}(x_2, \eta) - u_{\lambda, x}(x_1, \eta)| d\eta + \mu + \zeta$$

from which, by Lemma 1,

$$(3.13) \quad \left| u_{\lambda, x}(x_2, y) - u_{\lambda, x}(x_1, y) \right| \leq (\mu + \zeta) e^{k(y - \varphi(x_2))} \leq (\mu + \zeta) e^{k\ell_2}.$$

The equicontinuity of $\{u_{\lambda, x}\}$ is thus assured.

The argument for the equicontinuity of $\{u_{\lambda, y}\}$ is similar, hence Theorem 4a obtains.

Remarks c) and d) to Theorem 4 apply as well to Theorem 4a. Quite obviously, if f is continuous, bounded and Lipschitzian (or partially Lipschitzian) on the infinite cylinder with cross section R , then hypothesis 3) of Theorem 4 (or 4a) is immediately satisfied. In fact, this was the form of Theorem 4 which was utilized in the proof of Theorem 4a.

As previously mentioned, the extension of Theorem 4 to systems of equations was first obtained, in principle, by O. NICCOLETTI [14]. He was not, however, explicit about the domain of existence of the solution. The following statement may be derived

$$\left| \frac{\partial}{\partial x_j} \left(\frac{\partial \phi}{\partial x_i} \right) - \frac{\partial}{\partial x_i} \left(\frac{\partial \phi}{\partial x_j} \right) \right| \leq \epsilon$$

Let $\epsilon > 0$ be arbitrary. Choose $\delta > 0$ such that $\delta < \epsilon/2$. Let ϕ be a function in $C^2(\Omega)$ such that $\|\phi\|_{C^2(\Omega)} < \delta$. Then ϕ is a function in $C^2(\Omega)$ such that $\|\phi\|_{C^2(\Omega)} < \delta$.

$$\left| \frac{\partial}{\partial x_j} \left(\frac{\partial \phi}{\partial x_i} \right) - \frac{\partial}{\partial x_i} \left(\frac{\partial \phi}{\partial x_j} \right) \right| \leq \epsilon$$

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from the same arguments of E. KAMKE [2] p. 405 and p. 410 used as the basis for Theorem 4.

Theorem 5.

1) $f_i(x, y; u_1, \dots, u_n; p_1, \dots, p_n, q_1, \dots, q_n) \in C(B^n)$

$$B^n: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \\ -a \leq u_i \leq a \\ -b_1 \leq p_i \leq b_1 \\ -b_2 \leq q_i \leq b_2 \end{cases} \quad (i = 1, \dots, n).$$

2) The f_i are Lipschitzian on B^n , (as defined in Theorem 3).

3) $M l_1 l_2 \leq a$, $M l_1 \leq b_2$, $M l_2 \leq b_1$, where

$$M = \max \{ |f_1|, \dots, |f_n| \} \text{ on } B^n.$$

4) $\gamma: \begin{cases} 0 \leq x \leq l_1 \\ y = \varphi(x) \end{cases}$ where $\varphi(x) \in C'([0, l_1])$, $\varphi'(x) \neq 0$

$$\text{for } x \in [0, l_1] \text{ and } \varphi(0) = l_2, \varphi(l_1) = 0.$$

\Rightarrow 5) There exists one and only one set of functions $\{u_1, \dots, u_n\}$,

$u_i(x, y) \in C'(R)$, $u_{i,xy}(x, y) \in C(R)$, ($i = 1, \dots, n$), where

$$R: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \end{cases}, \text{ such that for each } (x, y) \in R \text{ the point}$$

$(x, y; u_j(x, y); u_{j,x}(x, y), u_{j,y}(x, y)) \in B$, and

$$u_{i,xy}(x, y) = f_i(x, y; u_j(x, y), u_{j,x}(x, y), u_{j,y}(x, y)),$$

$$u_i(x, \varphi(x)) = u_{i,x}(x, \varphi(x)) = u_{i,y}(x, \varphi(x)) = 0,$$

($i = 1, \dots, n$), for each $(x, y) \in R$.

... ..
... ..

QUESTION

1.

$$\left. \begin{aligned} \lambda^2 - 2\lambda + 1 &= 0 \\ \lambda^2 - 3\lambda + 2 &= 0 \\ \lambda^2 - 4\lambda + 4 &= 0 \\ \lambda^2 - 5\lambda + 6 &= 0 \\ \lambda^2 - 6\lambda + 9 &= 0 \end{aligned} \right\} \text{... ..}$$

... ..

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \Rightarrow \lambda = 1$$

$$\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda = 1, 2$$

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \Rightarrow \lambda = 2$$

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda = 2, 3$$

... ..

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \Rightarrow \lambda = 3$$

$$\lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4) = 0 \Rightarrow \lambda = 3, 4$$

$$\lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0 \Rightarrow \lambda = 4$$

$$\lambda^2 - 9\lambda + 20 = (\lambda - 4)(\lambda - 5) = 0 \Rightarrow \lambda = 4, 5$$

$$\lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 = 0 \Rightarrow \lambda = 5$$

$$\lambda^2 - 11\lambda + 30 = (\lambda - 5)(\lambda - 6) = 0 \Rightarrow \lambda = 5, 6$$

We may extend the arguments in the proof of Theorem 4a to apply to systems of equations. The extension is practically identical with the previous extension of Theorem 1a to Theorem 3a, except that now Theorem 5 is used to establish existence and uniqueness of the solutions of the system

$$u_{i\lambda,xy} = E_{i\lambda}(x,y; u_{j\lambda}; u_{j\lambda,x}, u_{j\lambda,y}), \quad (i=1, \dots, n), \\ (\lambda = 1, 2, \dots),$$

under the Cauchy initial conditions. We obtain the following theorem:

Theorem 5a

1)

2)' the f_i are partially Lipschitzian on B^n , (as defined in Theorem 3a).

3)

4)

⇒ 5)' There exists at least one set of functions $\{u_1, \dots, u_n\}$,

$u_i(x,y) \in C^1(R)$, $u_{i,xy}(x,y) \in C(R)$, ($i = 1, \dots, n$), where

$R: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \end{cases}$, such that for each $(x,y) \in R$ the point

$(x,y; u_j(x,y); u_{j,x}(x,y), u_{j,y}(x,y)) \in B$, and

$u_{i,xy}(x,y) = f_i(x,y; u_j(x,y); u_{j,x}(x,y), u_{j,y}(x,y))$,

$u_i(x, \varphi(x)) = u_{i,x}(x, \varphi(x)) = u_{i,y}(x, \varphi(x)) = 0$,

($i = 1, \dots, n$), for each $(x,y) \in R$.

... the ... of ... in the ... of ...
 ... the ... of ... in the ... of ...
 ... the ... of ... in the ... of ...
 ... the ... of ... in the ... of ...

$$f(x) = \frac{1}{x^2} = x^{-2}$$

$$f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

... the ... of ... in the ... of ...

Example

... the ... of ... in the ... of ...

... the ... of ... in the ... of ...

... the ... of ... in the ... of ...

$$f(x) = \sqrt{x} = x^{1/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

... the ... of ... in the ... of ...

Remark c), with obvious modifications, and Remark d) to Theorem 4 apply as well as to Theorems 5 and 5a. Moreover, in Theorem 5 (or 5a) we may eliminate hypothesis 3) by demanding that the f_1 be continuous, bounded and Lipschitzian (or partially Lipschitzian) on the infinite cylinder with cross section R.

The first part of the paper is devoted to the study of the
 asymptotic behavior of the solutions of the system

$$\dot{x} = Ax + B u$$
 as $t \rightarrow \infty$. It is shown that the solutions
 converge to zero if and only if the matrix A is
 Hurwitz. The second part of the paper is devoted to
 the study of the asymptotic behavior of the solutions
 of the system

$$\dot{x} = Ax + B u + C v$$
 as $t \rightarrow \infty$. It is shown that the solutions
 converge to zero if and only if the matrix A is
 Hurwitz and the matrix C is nonsingular.

The third part of the paper is devoted to the study of
 the asymptotic behavior of the solutions of the system

$$\dot{x} = Ax + B u + C v + D w$$
 as $t \rightarrow \infty$. It is shown that the solutions
 converge to zero if and only if the matrix A is
 Hurwitz and the matrix D is nonsingular.

The fourth part of the paper is devoted to the study of
 the asymptotic behavior of the solutions of the system

$$\dot{x} = Ax + B u + C v + D w + E z$$
 as $t \rightarrow \infty$. It is shown that the solutions
 converge to zero if and only if the matrix A is
 Hurwitz and the matrix E is nonsingular.

The fifth part of the paper is devoted to the study of
 the asymptotic behavior of the solutions of the system

$$\dot{x} = Ax + B u + C v + D w + E z + F y$$
 as $t \rightarrow \infty$. It is shown that the solutions
 converge to zero if and only if the matrix A is
 Hurwitz and the matrix F is nonsingular.

The sixth part of the paper is devoted to the study of
 the asymptotic behavior of the solutions of the system

$$\dot{x} = Ax + B u + C v + D w + E z + F y + G x$$
 as $t \rightarrow \infty$. It is shown that the solutions
 converge to zero if and only if the matrix A is
 Hurwitz and the matrix G is nonsingular.

CHAPTER IV

Existence Theorems for Canonical
Hyperbolic First Order Systems

In this chapter, and in Chapters 5 and 6 as well, we shall not give explicit domains of definition for the functions involved in the differential equations. As a consequence, existence will be shown "in the small" only. We do this because our method of attack will not yield any improvement upon the domains of existence, no matter how large the domains of definition are taken, provided the other hypotheses are not weakened. We shall elaborate on this peculiarity in the course of the exposition.

Theorems 6 and 7 below were given by M. CINQUINI-CIERRARIO [12] p. 180 in the form stated. A statement under somewhat weaker hypotheses, but based on the same proof, may be found in R. COURANT-D. HILBERT [17] p. 324. We examine the proof to show that the arguments therein are independent of the uniqueness of the solutions to the problems involved. As a consequence, our results in Chapters 2 and 3 apply and we arrive at the improved statements given by Theorems 6a and 7a, where hypothesis 2) of Theorems 6 and 7 is dropped altogether and the corresponding conclusions are altered to read "at least one".

Common hypothesis 1) We shall suppose the functions $a_{ik}, c_1,$ ($i, k=1, \dots, n$), of arguments x, y, u_1, \dots, u_n , to be continuously differentiable with bounded derivatives in a certain domain D . Fur-



then, we suppose the determinant

$$(4.1) \quad |a_{ik}| \neq 0 \quad \text{in } D.$$

Under these assumptions, the system

$$(4.2) \quad \begin{cases} A_i(x, y) = \sum_{k=1}^n a_{ik} u_{k,x}(x, y) - c_i = 0, & (i=1, \dots, m < n) \\ B_i(x, y) = \sum_{k=1}^n a_{ik} u_{k,y}(x, y) - c_i = 0, & (i=m+1, \dots, n) \end{cases}$$

is called a canonical hyperbolic first order system.

Theorem 6. (Characteristic initial value problem.)

1)

2) All first derivatives of the functions a_{ik}, c_i , ($i, k=1, \dots, n$) satisfy a Lipschitz condition with respect to arguments u_1, \dots, u_n in D .

$$3) \quad \left. \begin{array}{l} U_i(x) \in C'([0, \ell_1]) \\ V_i(y) \in C'([0, \ell_2]) \\ U_i(0) = V_i(0) \end{array} \right\} \quad (i=1, \dots, n)$$

Moreover, for each $x \in [0, \ell_1]$, the point $(x, 0; U_j(x)) \in D$

and

$$(4.3) \quad \sum_{k=1}^n a_{ik}(x, 0; U_j(x)) U'_k(x) - c_i(x, 0; U_j(x)) = 0, \\ (i=1, \dots, m < n);$$

and, for each $y \in [0, \ell_2]$, the point $(0, y; V_j(y)) \in D$ and

$$(4.4) \quad \sum_{k=1}^n a_{ik}(0, y; V_j(y)) V'_k(y) - c_i(0, y; V_j(y)) = 0, \\ (i=m+1, \dots, n).$$

3. Recall the notation: $(x, y; U_j(x)) = (x, y; U_1(x), \dots, U_n(x))$.

first we suppose the differential

$$y'' + p(x)y' + q(x)y = r(x) \quad (1.1)$$

under some conditions, the system

$$\begin{cases} y_1'(x) = p_1(x)y_1 + q_1(x)y_2 + r_1(x) \\ y_2'(x) = p_2(x)y_1 + q_2(x)y_2 + r_2(x) \end{cases}$$

is called a separated system (first order system).

LEMMA 1.1 (Cauchy's existence theorem)

1)

2) All three members of the functions $p_1, p_2, q_1, q_2, r_1, r_2$ are locally Lipschitz continuous with respect to y_1, y_2 .

3)

$$\begin{cases} y_1(0) = \eta_1 \\ y_2(0) = \eta_2 \end{cases} \quad (1.2)$$

then, for each $\epsilon > 0$, there exists $\delta > 0$ such that

and

$$\|y(x) - \tilde{y}(x)\| \leq \epsilon \quad (1.3)$$

$$\|y(x) - \tilde{y}(x)\| \leq \epsilon$$

and, for each $\epsilon > 0$, there exists $\delta > 0$ such that $\|y(x) - \tilde{y}(x)\| \leq \epsilon$ and $\|y(x) - \tilde{y}(x)\| \leq \epsilon$.

3) If the solution $y(x)$ of (1.1) is bounded on $[0, \infty)$, then

\Rightarrow 4) There exists one and only one set of functions

$$\{u_1, \dots, u_n\}, u_i(x, y) \in C^1(R_\eta), u_{i,xy} \in C(R_\eta), (i = 1, \dots, n),$$

where $R_\eta : \begin{cases} 0 \leq x \leq \eta l_1 \\ 0 \leq y \leq \eta l_2 \end{cases}$, with $0 < \eta \leq 1$ and η sufficiently

small, such that the set of functions satisfies the system (4.2)

for each $(x, y) \in R_\eta$ and satisfies the conditions

$$\left. \begin{aligned} u_i(x, 0) &= U_i(x) \quad \text{for } x \in [0, l_1] \\ u_i(0, y) &= V_i(y) \quad \text{for } y \in [0, l_2] \end{aligned} \right\} (i = 1, \dots, n).$$

Theorem 6a.

1)

3)

\Rightarrow 4)' There exists at least one set of functions, etc. (as in Theorem 6).

Theorem 7. (Cauchy problem.)

1)

2) (as in Theorem 6.)

5) $\gamma : \begin{cases} x = x(\tau) \\ y = y(\tau) \end{cases}$ for $\tau \in [0, 1]$, $x(\tau)$ and $y(\tau) \in C^1([0, 1])$

and strictly monotone, i.e., $\dot{x} \neq 0$, $\dot{y} \neq 0$ on $[0, 1]$.

$U_i(\tau) \in C^1([0, 1])$, $(i = 1, \dots, n)$. For each $\tau \in [0, 1]$, the point $(x(\tau), y(\tau); U_j(\tau)) \in D$.

\Rightarrow 6) There exists one and only one set of functions $\{u_1, \dots, u_n\}$, $u_i(x, y) \in C^1(R_\gamma)$, $u_{i,xy}(x, y) \in C(R_\gamma)$, $(i = 1, \dots, n)$, where R_γ is a sufficiently small neighborhood of the curve γ , such that

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.

$$f(x) + g(x) = (x^2 + 2x + 1) + (x^2 - 2x + 1) = 2x^2 + 2$$

$$f(x) - g(x) = (x^2 + 2x + 1) - (x^2 - 2x + 1) = 4x$$

Therefore, $f(x) + g(x) = 2x^2 + 2$ and $f(x) - g(x) = 4x$.

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.

$$\begin{aligned} f(x) + g(x) &= (x^2 + 2x + 1) + (x^2 - 2x + 1) = 2x^2 + 2 \\ f(x) - g(x) &= (x^2 + 2x + 1) - (x^2 - 2x + 1) = 4x \end{aligned}$$

Problem 11

(1)

(2)

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.

Therefore, $f(x) + g(x) = 2x^2 + 2$ and $f(x) - g(x) = 4x$.

Problem 12

(1)

(2) Let $f(x) = x^2 + 2x + 1$

$$f(x) + g(x) = (x^2 + 2x + 1) + (x^2 - 2x + 1) = 2x^2 + 2$$

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.

$$f(x) + g(x) = (x^2 + 2x + 1) + (x^2 - 2x + 1) = 2x^2 + 2$$

$$f(x) - g(x) = (x^2 + 2x + 1) - (x^2 - 2x + 1) = 4x$$

Therefore, $f(x) + g(x) = 2x^2 + 2$ and $f(x) - g(x) = 4x$.

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.

Therefore, $f(x) + g(x) = 2x^2 + 2$ and $f(x) - g(x) = 4x$.

the set of functions satisfies the system (4.2) for each $(x,y) \in R_\gamma$ and satisfies the conditions

$$u_i(x(\tau), y(\tau)) = u_i(\tau) \quad \text{for } \tau \in [0,1], \quad (i = 1, \dots, n).$$

Theorem 7a

1)

5)

\Rightarrow 6)' There exists at least one set of functions etc. (as in Theorem 7.)

The proofs of these theorems are contained in the following argument:

Suppose we have a set of functions $\{u_1, \dots, u_n\}$, either as a solution to the characteristic initial value problem above on a domain R_η , or as a solution to the Cauchy problem above on a domain R_γ . Then for either case,

$$(4.5) \quad A_{i,y} = \sum_{k=1}^n a_{ik} u_{k,xy} + \sum_{k=1}^n \left[a_{ik,y} + \sum_{r=1}^n \frac{\partial a_{ik}}{\partial u_r} u_{r,y} \right] u_{k,x} - c_{i,y} - \sum_{k=1}^n \frac{\partial c_i}{\partial u_k} u_{k,y} = 0, \quad (i = 1, \dots, m < n),$$

$$(4.6) \quad B_{i,x} = \sum_{k=1}^n a_{ik} u_{k,xy} + \sum_{k=1}^n \left[a_{ik,x} + \sum_{r=1}^n \frac{\partial a_{ik}}{\partial u_r} u_{r,x} \right] u_{k,y} - c_{i,x} - \sum_{k=1}^n \frac{\partial c_i}{\partial u_k} u_{k,x} = 0, \quad (i = m+1, \dots, n).$$

Equations (4.5) and (4.6) are n linear algebraic equations in the

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.
Then $f(x) + g(x) = (x^2 + 2x + 1) + (x^2 - 2x + 1) = 2x^2 + 2$.

Similarly, $f(x) - g(x) = (x^2 + 2x + 1) - (x^2 - 2x + 1) = 4x$.

Problem 2

(1)

(2)

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.

(3)

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.

(4)

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.

$$f(x) + g(x) = (x^2 + 2x + 1) + (x^2 - 2x + 1) = 2x^2 + 2$$

$$f(x) - g(x) = (x^2 + 2x + 1) - (x^2 - 2x + 1) = 4x$$

$$f(x) + g(x) = (x^2 + 2x + 1) + (x^2 - 2x + 1) = 2x^2 + 2$$

$$f(x) - g(x) = (x^2 + 2x + 1) - (x^2 - 2x + 1) = 4x$$

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.

n unknowns $u_{i,xy}$. Since the determinant of this system $|a_{ik}|$, does not vanish over the domain in question, we may solve the system to obtain explicitly

$$(4.7) \quad u_{i,xy} = f_i(x,y; u_j; u_{j,x}, u_{j,y}), \quad (i = 1, \dots, n).$$

Under hypothesis 1) alone, the f_i are continuous and partially Lipschitzian over any bounded domain in the $3n + 2$ dimensional $(x,y; u_j; u_{j,x}, u_{j,y})$ -space where $(x,y; u_j) \in D$. If hypothesis 2) also applies, the f_i are "fully" Lipschitzian over any such domain.

Consider Theorems 6 and 6a. The characteristic initial conditions imposed therein, coupled with the system (4.7), form a problem of the type considered in Theorems 3 and 3a, respectively. (Chapter 2). We have shown above that any solution of a canonical hyperbolic system is also a solution of a particular system of type (4.7). If we now demonstrate the converse for characteristic initial conditions, i.e. that any solution of the derived system (4.7) is also a solution of the original system (4.2), then Theorems 6 and 6a follow directly from Theorems 3 and 3a respectively.

Suppose we have a set of functions $\{u_1, \dots, u_n\}$ as a solution of (4.7) over a certain domain including the initial characteristics. By (4.5) and (4.6), which are merely alternative forms of (4.7), we have

Let \mathcal{L} be a linear operator on a vector space V . Then the characteristic polynomial of \mathcal{L} is defined as $P_{\mathcal{L}}(\lambda) = \det(\lambda I - \mathcal{L})$. The eigenvalues of \mathcal{L} are the roots of this polynomial.

$$P_{\mathcal{L}}(\lambda) = \det(\lambda I - \mathcal{L}) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 \quad (1)$$

Let \mathcal{L} be a linear operator on a vector space V . Then the characteristic polynomial of \mathcal{L} is defined as $P_{\mathcal{L}}(\lambda) = \det(\lambda I - \mathcal{L})$. The eigenvalues of \mathcal{L} are the roots of this polynomial. If λ is an eigenvalue of \mathcal{L} , then $P_{\mathcal{L}}(\lambda) = 0$.

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$$(4.8) \quad \begin{cases} A_{1,y}(x,y) = 0 & , \quad (i = 1, \dots, m < n) \\ B_{1,x}(x,y) = 0 & , \quad (i = m+1, \dots, n) \end{cases}$$

over this domain. But, by (4.3) and (4.4) of hypothesis 3) to both Theorems 6 and 6a, we have that

$$(4.9) \quad \begin{cases} A_1(x,0) = 0 & , \quad (i = 1, \dots, m < n) \\ B_1(0,y) = 0 & , \quad (i = m+1, \dots, n), \end{cases}$$

whence

$$\begin{aligned} A_1(x,y) &\equiv 0 & , \quad (i = 1, \dots, m < n), \\ B_1(x,y) &\equiv 0 & , \quad (i = m+1, \dots, n), \end{aligned}$$

throughout the domain. Hence the converse is shown.

For the Cauchy problem considered in Theorems 7 and 7a, we observe first that we can determine $u_{1,x}(x(\tau), y(\tau))$ and $u_{1,y}(x(\tau), y(\tau))$, ($i = 1, \dots, n$), as functions continuous for each $\tau \in [0,1]$, solely from the prescription of $u_1(x(\tau), y(\tau)) = U_1(\tau)$, ($i = 1, \dots, n$), and the requirement that the canonical hyperbolic system (4.2) must be satisfied at each point of Υ . For, since $\dot{x} + \dot{y}^2 \neq 0$ along Υ , we may write the strip conditions

$$(4.10) \quad \dot{u}_1 = p_1 \dot{x} + q_1 \dot{y}, \quad (i = 1, \dots, n),$$

as one of

$$(4.11) \quad q_1 = \frac{1}{\dot{y}} (\dot{u}_1 - p_1 \dot{x}) \quad \text{or} \quad p_1 = \frac{1}{\dot{x}} (\dot{u}_1 - q_1 \dot{y}), \quad (i = 1, \dots, n).$$

Consider a particular point $P \in \Upsilon$ where $\dot{y} \neq 0$. Here we substitute $q_1 = u_{1,y} = \frac{1}{\dot{y}} (\dot{u}_1 - p_1 \dot{x})$ into equations $B_1(P) = 0$, ($i = m+1, \dots, n$). These, together with the equations $A_1(P) = 0$, ($i = 1, \dots, m < n$),

$$\left. \begin{aligned} (1) \quad & \text{if } x \in \mathbb{R}^n, \text{ then } \|x\| \geq 0 \\ & \|x\| = 0 \iff x = 0 \end{aligned} \right\} (1.1)$$

Let $x, y \in \mathbb{R}^n$. Then $\|x+y\| \leq \|x\| + \|y\|$ and $\|x\| = \|x\|$.
 This property is called the triangle inequality.

$$\left. \begin{aligned} (2) \quad & \|x\| \geq 0 \\ & \|x\| = 0 \iff x = 0 \end{aligned} \right\} (1.2)$$

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

Let $x, y \in \mathbb{R}^n$. Then $\|x+y\| \leq \|x\| + \|y\|$.

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form a linear algebraic system in the $p_1 = u_{1,x}(P)$ with determinant $|a_{ik}| \neq 0$. Thus the p_1 are uniquely determined at P , and, by (4.11), the q_1 as well are uniquely determined at P . If $\dot{y} = 0$ at P , then $\dot{x} \neq 0$ there and a similar argument applies utilizing $p_1 = \frac{1}{\dot{x}} (\dot{d}_1 - q_1 \dot{y})$.

Thus we have, in effect, prescribed all three sets $u_1, u_{1,x}, u_{1,y}$, ($i = 1, \dots, n$), along Υ once the u_1 are prescribed along Υ and the $u_{1,x}$ and the $u_{1,y}$ are merely required to satisfy the strip conditions (4.10) and the canonical hyperbolic system at (4.2) at each point of Υ .

Suppose we have a set of functions $\{u_1, \dots, u_n\}$ as a solution of

(4.7) $u_{1,xy} = f_1(x, y; u_j; u_{j,x}, u_{j,y})$, ($i = 1, \dots, n$) in a neighborhood of the initial curve Υ and taking, with their first derivatives, precisely the above determined values at each point of Υ . Then by (4.5) and (4.6), the fact that these functions and their first derivatives satisfy the canonical hyperbolic system (4.2) at each point of Υ implies further that the system (4.2) corresponding to (4.7) is satisfied everywhere in the neighborhood in question.

With hypothesis 2) imposed, system (4.7) and the initial data on Υ satisfy the hypotheses of Theorem 5, while without hypothesis 2), system (4.7) and the initial data on Υ satisfy the hypotheses of Theorem 5a. But since we have shown above that each solution of (4.7) is a solution of the corresponding canonical

Let \mathcal{L} be a linear differential system in the form $\mathcal{L}(y) = 0$ where $y = (y_1, \dots, y_n)^T$ and \mathcal{L} is a matrix of differential operators. The adjoint system $\mathcal{L}^*(z) = 0$ is defined by $\mathcal{L}^*(z) = (\mathcal{L}(y))^T$.

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hyperbolic system (4.2), we have that Theorem 7 is a consequence of Theorem 5, while Theorem 7a is a consequence of Theorem 5a.

In these four theorems we are unable to prescribe the domain of definition of the functions

$$f_1(x, y; u_j; p_j, q_j), \quad (j = 1, \dots, n),$$

in such a way as to insure existence of a solution throughout

$$R: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \end{cases}. \quad \text{This is because the } f_1 \text{ are continuous for}$$

all p_j and q_j , ($j = 1, \dots, n$), but may turn out to be bounded only when these variables are restricted to finite domains. The following example demonstrates why the existence of solutions can be found only "in the small".

Example 3. Consider the characteristic initial value problem for the system

$$\begin{aligned} u_{1,xy} &= u_{1,x}^2, & u_1(x, -1) &= x, & u_1(0, y) &= 0 \\ u_{2,xy} &= 0, & u_2(x, -1) &= 0, & u_2(0, y) &= 0 \\ & \vdots & & \vdots & & \\ u_{n,xy} &= 0, & u_n(x, -1) &= 0, & u_n(0, y) &= 0. \end{aligned}$$

By quadratures, we obtain the solution $u_1(x, y) = \frac{-x}{y}$, while $u_2 = \dots = u_n = 0$, quite obviously. The f_1 corresponding to this problem possess derivatives of all orders for all values of all variables. However, $f_1 = u_{1,x}^2$ becomes unbounded as the argument $u_{1,x}$ increases indefinitely in absolute value. We note that, despite the specification of initial data everywhere along the

intersecting characteristics $x = 0$ and $y = -1$, the first function in the solution, namely u_1 , has a discontinuity across the line $y = 0$. Hence this example typifies those cases for which solutions exist "in the small" only.

We note that Remark d) of Chapter III applies as well to hypothesis 5) of Theorems 7 and 7a. The statement is that

$\gamma: \begin{cases} x = x(\tau) \\ y = y(\tau) \end{cases}$ for $\tau \in [0,1]$ need only have $x(\tau)$ and

$y(\tau) \in C^1([0,1])$, monotone, and with $\dot{x}^2 + \dot{y}^2 \neq 0$ at each point of γ . In fact, the argument in the proof above applies directly to this statement.

The first part of the paper is devoted to the study of the
 asymptotic behavior of the eigenvalues of the operator
 Δ_{ϵ} as $\epsilon \rightarrow 0$. It is shown that the eigenvalues
 cluster around the eigenvalues of the operator Δ_0 .

In the second part of the paper, we study the asymptotic
 expansion of the eigenfunctions of Δ_{ϵ} . It is shown
 that the eigenfunctions are asymptotically equal to the
 eigenfunctions of Δ_0 plus a boundary layer.

The third part of the paper is devoted to the study of the
 asymptotic expansion of the resolvent of Δ_{ϵ} . It is
 shown that the resolvent is asymptotically equal to the
 resolvent of Δ_0 plus a boundary layer.

The fourth part of the paper is devoted to the study of the
 asymptotic expansion of the heat kernel of Δ_{ϵ} . It is
 shown that the heat kernel is asymptotically equal to the
 heat kernel of Δ_0 plus a boundary layer.

The fifth part of the paper is devoted to the study of the
 asymptotic expansion of the spectral density of Δ_{ϵ} . It
 is shown that the spectral density is asymptotically equal
 to the spectral density of Δ_0 plus a boundary layer.

The sixth part of the paper is devoted to the study of the
 asymptotic expansion of the trace of Δ_{ϵ} . It is shown
 that the trace is asymptotically equal to the trace of
 Δ_0 plus a boundary layer.

The seventh part of the paper is devoted to the study of the
 asymptotic expansion of the determinant of Δ_{ϵ} . It is
 shown that the determinant is asymptotically equal to the
 determinant of Δ_0 plus a boundary layer.

The eighth part of the paper is devoted to the study of the
 asymptotic expansion of the zeta function of Δ_{ϵ} . It
 is shown that the zeta function is asymptotically equal to
 the zeta function of Δ_0 plus a boundary layer.

The ninth part of the paper is devoted to the study of the
 asymptotic expansion of the eta function of Δ_{ϵ} . It
 is shown that the eta function is asymptotically equal to
 the eta function of Δ_0 plus a boundary layer.

The tenth part of the paper is devoted to the study of the
 asymptotic expansion of the spectral zeta function of
 Δ_{ϵ} . It is shown that the spectral zeta function is
 asymptotically equal to the spectral zeta function of
 Δ_0 plus a boundary layer.

CHAPTER V.

The Cauchy Problem for $F(x,y; u; p,q; r,s,t) = 0$.

In this chapter we concern ourselves with the Cauchy problem for the general non-linear second order partial differential equation in the hyperbolic domain. Specifically, the problem is to determine an integral surface of the equation

$$(1.1) \quad F(x,y; u; p,q; r,s,t) = 0$$

such that the hyperbolic condition

$$(1.3) \quad F_{ss}^2 - 4 F_{sr} F_{st} > 0$$

is satisfied thereon; moreover, the integral surface must have a second order contact with a given second order strip which is nowhere a characteristic strip.

In his celebrated paper [10], H. LEWY successfully attacks this problem by reducing equation (1.1) to a system of first order partial differential equations for the unknowns $x,y; u; p,q; r,s,t$ as functions of the parameters λ and μ of the two families of characteristics on the integral surface in question. LEWY's existence proof for the system is based on a finite difference argument. However, the system is of canonical hyperbolic form and the theorem of M. CINQUINI-CIERRARIO, Theorem 7 of Chapter IV, is immediately applicable and insures existence and uniqueness of the solution in a sufficiently small neighborhood of the initial strip. Moreover, as demonstrated below, Theorem 7a may be used to effect an improvement on LEWY's work.

We present simultaneously LEWY's original theorem and our

SECTION 2

The following is a list of the names of the persons who were present at the meeting held on the 15th day of June, 1954.

The names of the persons who were present at the meeting held on the 15th day of June, 1954, are as follows: [List of names]

$$E = \frac{1}{2} (E_1 + E_2) \quad (1.1)$$

where E_1 and E_2 are the values of E at the two ends of the interval.

$$E < \frac{1}{2} (E_1 + E_2) \quad (1.2)$$

It is clear from the above that the value of E at the midpoint of the interval is less than the average of the values at the two ends.

Let us now consider the case where the function E is concave up. In this case, the value of E at the midpoint is greater than the average of the values at the two ends.

Let us now consider the case where the function E is concave down. In this case, the value of E at the midpoint is less than the average of the values at the two ends.

Let us now consider the case where the function E is linear. In this case, the value of E at the midpoint is equal to the average of the values at the two ends.

Let us now consider the case where the function E is constant. In this case, the value of E at the midpoint is equal to the value of E at the two ends.

improvement on it. LEVY's theorem is obtained by omitting the parentheses. Our theorem is obtained by replacing the under-scored statements by the corresponding ones in the parentheses.

Theorem 8 (8a)

$$1) \quad S^2: \begin{cases} x = x(\tau) \\ y = y(\tau) \\ u = u(\tau) \\ p = p(\tau) \\ q = q(\tau) \\ r = r(\tau) \\ s = s(\tau) \\ t = t(\tau) \end{cases} \quad \text{for } \tau \in [0,1] \text{ is a nowhere character-} \\ \text{istic second order strip,}$$

i.e. $x, y; u; p, q; r, s, t(\tau) \in C^1([0,1])$, and for each $\tau \in [0,1]$,

- i) $\dot{x}^2 + \dot{y}^2 \neq 0$,
- ii) $F_r \dot{y}^2 - F_s \dot{y} \dot{x} + F_t \dot{x}^2 \neq 0$,
- iii) $F_s^2 - 4 F_r F_t > 0$,
- iv) $F(x(\tau), y(\tau); u(\tau); p(\tau), q(\tau); r(\tau), s(\tau), t(\tau)) = 0$.

2) $F \in C^{(1)}(\in C^n)$ in a certain neighborhood of S^2 .

3) There exists one and only one (at least one) integral surface $J: u = u(x, y)$ of the equation $F(x, y; u; p, q; r, s, t) = 0$ such that $u(x, y) \in C^{(1)}$ in a sufficiently small neighborhood of the base curve $\gamma: \begin{cases} x = x(\tau) \\ y = y(\tau) \end{cases}$ for $\tau \in [0,1]$, and such that $J: u = u(x, y)$ has a second order contact with the strip S^2 .

Proof

We first demonstrate that any solution of the above problem, together with its derivatives of the first and second orders, represents a solution of a particular canonical hyperbolic system under the same boundary conditions.

We assume that $F_r \neq 0$ and $F_t \neq 0$ in the domains considered in the following argument. This may be done without loss of generality. For, by Definition 1a, a characteristic base curve must satisfy

$$(1.5) \quad \begin{aligned} 1) & \quad F_r \dot{y}^2 - F_s \dot{y} \dot{x} + F_t \dot{x}^2 = 0, \\ 2) & \quad \dot{x}^2 + \dot{y}^2 \neq 0. \end{aligned}$$

Suppose at a point of S^2 that $F_r = 0$. Then $\dot{x} = 0$ represents the vertical tangent taken by one of the characteristic base curves through the projection of this point onto the xy plane. Conversely, if one of the characteristic base curves through a point in the projection of S^2 has a vertical tangent, then $\dot{x} = 0$ there and, consequently, $F_r = 0$ at the corresponding point on S^2 . Likewise, $F_t = 0$ if and only if $\dot{y} = 0$, in the sense above. Thus, by a suitable coordinate rotation in the xy plane, we may insure that $F_r \neq 0$ and $F_t \neq 0$ in a neighborhood of the point in question on S^2 . Granting that this is a local property only and that the particular rotation performed may introduce values of $F_r = 0$ or $F_t = 0$ at some other sufficiently distant points on S^2 , we observe that this local property is sufficient because our proof is ultimately based upon Theorems 4 and 4a of Chapter III. In those

Lemma

Let \mathcal{L} be a linear space over \mathbb{R} with a bilinear form $\langle \cdot, \cdot \rangle$. Let $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$ be a decomposition of \mathcal{L} into two orthogonal subspaces. Then \mathcal{L}_1 and \mathcal{L}_2 are also linear spaces over \mathbb{R} .

Proof. Let $u, v \in \mathcal{L}_1$. Then $u + v \in \mathcal{L}_1$ because \mathcal{L}_1 is a subspace of \mathcal{L} . Similarly, $ku \in \mathcal{L}_1$ for any scalar k . The same holds for \mathcal{L}_2 .

$$\begin{aligned} (1) \quad \langle u + v, u + v \rangle &= \langle u, u \rangle + \langle v, v \rangle + 2\langle u, v \rangle \\ &= \langle u, u \rangle + \langle v, v \rangle \end{aligned} \quad (1.1)$$

Since \mathcal{L}_1 and \mathcal{L}_2 are orthogonal, $\langle u, v \rangle = 0$. Thus $\langle u + v, u + v \rangle = \langle u, u \rangle + \langle v, v \rangle$. This shows that \mathcal{L}_1 and \mathcal{L}_2 are orthogonal subspaces.

Let $u \in \mathcal{L}_1$ and $v \in \mathcal{L}_2$. Then $\langle u, v \rangle = 0$. This implies that \mathcal{L}_1 and \mathcal{L}_2 are orthogonal to each other.

Therefore, \mathcal{L}_1 and \mathcal{L}_2 are linear spaces over \mathbb{R} and are orthogonal to each other.

Q.E.D.

Corollary: If \mathcal{L} is a linear space over \mathbb{R} with a bilinear form $\langle \cdot, \cdot \rangle$, then \mathcal{L} can be decomposed into two orthogonal subspaces.

theorems the integral equation statement of the problem made it plainly evident that the value of the integral at any point P depended only upon the portion of the initial curve cut off by the two characteristics intersecting at P . Consequently, we may consider the arguments below as applying in succession to small overlapping segments of S^2 , with coordinate axes rotated suitably for each segment considered. (See also H. CONRANT - D. HILBERT [17] p. 323 and p. 332.)

Let us assume that we have an integral surface $J: u = u(x, y)$ satisfying the conditions of either Theorem 3 or Theorem 3a. Then by (1.5) we conclude that the related characteristic base curves are the two one-parameter families of curves determined by the equations

$$(5.1) \quad y_\lambda = \rho_1 x_\lambda,$$

$$(5.2) \quad y_\mu = \rho_2 x_\mu,$$

where

$$(5.3) \quad \rho_1 = \frac{p_s + \sqrt{p_s^2 - 4 p_r p_t}}{2 p_r},$$

$$(5.4) \quad \rho_2 = \frac{p_s - \sqrt{p_s^2 - 4 p_r p_t}}{2 p_r}.$$

ρ_1 and ρ_2 are functions of the variables $x, y; u; p, q; r, s, t$ and $\rho_1 \neq \rho_2$ in a neighborhood of S^2 by the hyperbolic condition (1.3).

Consider the coordinate transformation

$$(5.5) \quad \begin{aligned} x &= x(\lambda, \mu) \\ y &= y(\lambda, \mu). \end{aligned}$$

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 ... the ... of the ...
 ... the ... of the ...

$$[1] \quad \dots$$

... the ... of the ...
 ... the ... of the ...
 ... the ... of the ...
 ... the ... of the ...

$$[2] \quad \dots$$

$$[3] \quad \dots$$

$$[4] \quad \dots$$

$$[5] \quad \dots$$

... the ... of the ...
 ... the ... of the ...
 ... the ... of the ...

$$[6] \quad \dots$$

The Jacobian of this transformation,

$$(5.6) \quad y_{\lambda} x_{\mu} - y_{\mu} x_{\lambda} = (\rho_1 - \rho_2) x_{\lambda} x_{\mu},$$

does not vanish in a vicinity of the projection of S^2 . This follows since $\rho_1 \neq \rho_2$; while $x_{\lambda} = 0$ would, by (5.1), imply $y_{\lambda} = 0$, contradicting the requirement $\dot{x}^2 + \dot{y}^2 \neq 0$, (similarly for x_{μ}). Hence the inverse transformation,

$$(5.7) \quad \begin{cases} \lambda = \lambda(x, y) \\ \mu = \mu(x, y) \end{cases},$$

exists in a vicinity of the projection of S^2 .

Along the characteristics on J : $u = u(x, y)$ certain additional equations must be satisfied. These are determined as follows:

Since $F \in C^{III} (\in C^{II})$ and $u \in C^{III}$, we obtain by differentiation

$$(5.8) \quad \begin{cases} F_r r_x + F_s s_x + F_t t_x = - [F]_x \\ x_{\lambda} r_x + y_{\lambda} s_x = r_{\lambda} \\ x_{\lambda} s_x + y_{\lambda} t_x = s_{\lambda}, \end{cases}$$

where

$$(5.9) \quad [F]_x = F_p r + F_q s + F_u p + F_x.$$

similarly,

$$(5.10) \quad \begin{cases} F_r r_y + F_s s_y + F_t t_y = - [F]_y \\ x_{\lambda} r_y + y_{\lambda} s_y = s_{\lambda} \\ x_{\lambda} s_y + y_{\lambda} t_y = t_{\lambda}, \end{cases}$$

where

The solution of this system is

$$x_1 = \frac{1}{\sqrt{2}} \cos t, \quad x_2 = \frac{1}{\sqrt{2}} \sin t, \quad (2.1)$$

and the solution of the homogeneous system is

$$x_1 = c_1 \cos t + c_2 \sin t, \quad x_2 = -c_1 \sin t + c_2 \cos t, \quad (2.2)$$

where c_1 and c_2 are arbitrary constants.

The general solution is

$$\begin{cases} x_1 = \frac{1}{\sqrt{2}} \cos t + c_1 \cos t + c_2 \sin t \\ x_2 = \frac{1}{\sqrt{2}} \sin t - c_1 \sin t + c_2 \cos t \end{cases} \quad (2.3)$$

where c_1 and c_2 are arbitrary constants.

The solution of the system (2.1) is

where c_1 and c_2 are arbitrary constants.

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where c_1 and c_2 are arbitrary constants.

$$\begin{cases} x_1 = \frac{1}{\sqrt{2}} \cos t + c_1 \cos t + c_2 \sin t \\ x_2 = \frac{1}{\sqrt{2}} \sin t - c_1 \sin t + c_2 \cos t \end{cases} \quad (2.4)$$

$$x_1 = \frac{1}{\sqrt{2}} \cos t + c_1 \cos t + c_2 \sin t, \quad (2.5)$$

where c_1 and c_2 are arbitrary constants.

$$\begin{cases} x_1 = \frac{1}{\sqrt{2}} \cos t + c_1 \cos t + c_2 \sin t \\ x_2 = \frac{1}{\sqrt{2}} \sin t - c_1 \sin t + c_2 \cos t \end{cases} \quad (2.6)$$

$$(5.11) \quad [F]_y = F_p s + F_q t + F_u q + F_y \cdot$$

Since λ is the parameter for one family of characteristic curves and, consequently, is the path parameter along each of the curves of the other family, the determinant

$$(5.12) \quad \begin{vmatrix} F_r & F_s & F_t \\ x_\lambda & y_\lambda & 0 \\ 0 & x_\lambda & y_\lambda \end{vmatrix} = F_r y_\lambda^2 - F_s y_\lambda x_\lambda - F_t x_\lambda^2 = 0.$$

Hence the quantities on the right-hand side in each of the systems (5.8) and (5.10) must be linearly dependent, i.e. in each system the augmented matrix of coefficients must be of rank less than three. Consequently,

$$(5.13) \quad \begin{vmatrix} F_r & F_t & [F]_x \\ x_\lambda & 0 & -r_\lambda \\ 0 & y_\lambda & -s_\lambda \end{vmatrix} = F_r r_\lambda y_\lambda + F_t s_\lambda x_\lambda + [F]_x x_\lambda y_\lambda = 0.$$

Recalling the assumption made without loss,

$$x_\lambda = \frac{1}{\rho_1} y_\lambda \quad \text{and} \quad y_\lambda \neq 0, \quad \text{equation (5.13) reduces to}$$

$$(5.14) \quad F_r r_\lambda + \frac{1}{\rho_1} F_t s_\lambda + [F]_x x_\lambda = 0.$$

Likewise, from (5.10) we obtain the linear dependence of the right-hand terms in the form

$$(5.15) \quad \rho_1 F_r s_\lambda + F_t t_\lambda + [F]_y y_\lambda = 0.$$

Along the curves of the other family of characteristics the following relations must be satisfied. These are obtained in a

$$= \sqrt{2} + 2\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} \quad (11.1)$$

From the above we can see that the sum of the square roots of the numbers 2, 2, 2, 2, 2 is equal to 5 times the square root of 2. This is a very interesting result.

$$= \sqrt{2} + 2\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} = \sqrt{2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \end{pmatrix} \quad (11.2)$$

The above equation shows that the sum of the square roots of the numbers 2, 2, 2, 2, 2 is equal to 5 times the square root of 2. This is a very interesting result. The above equation shows that the sum of the square roots of the numbers 2, 2, 2, 2, 2 is equal to 5 times the square root of 2. This is a very interesting result.

$$= \sqrt{2} + 2\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} = \sqrt{2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \end{pmatrix} \quad (11.3)$$

The above equation shows that the sum of the square roots of the numbers 2, 2, 2, 2, 2 is equal to 5 times the square root of 2. This is a very interesting result. The above equation shows that the sum of the square roots of the numbers 2, 2, 2, 2, 2 is equal to 5 times the square root of 2. This is a very interesting result.

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$$= \sqrt{2} + 2\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} = \sqrt{2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \end{pmatrix} \quad (11.4)$$

The above equation shows that the sum of the square roots of the numbers 2, 2, 2, 2, 2 is equal to 5 times the square root of 2. This is a very interesting result. The above equation shows that the sum of the square roots of the numbers 2, 2, 2, 2, 2 is equal to 5 times the square root of 2. This is a very interesting result.

fashion completely analogous to that used in obtaining (5.14) and (5.15):

$$(5.16) \quad p_r r_\mu + \frac{1}{\rho_2} p_t s_\mu + [P]_x x_\mu = 0$$

$$(5.17) \quad \rho_2 p_r s_\mu + p_t t_\mu + [P]_y y_\mu = 0.$$

In addition, the strip conditions

$$(1.8) \quad \dot{u} = p \dot{x} + q \dot{y}$$

$$(1.9) \quad \begin{cases} \dot{p} = r \dot{x} + s \dot{y} \\ \dot{q} = s \dot{x} + t \dot{y} \end{cases}$$

must be satisfied along any curve lying on J : $u = u(x, y)$. In particular, they must be satisfied along any characteristic on J .

From equations (5.1), (5.2), (5.14) through (5.17), (1.8) and (1.9) we obtain the following system of "characteristic equations" i.e. equations which must be satisfied along the characteristics on any integral surface J :

$$(5.18) \quad \left. \begin{aligned} \varphi_1 &= y_\lambda - \rho_1 x_\lambda = 0 \\ \varphi_2 &= p_r r_\lambda + \frac{1}{\rho_1} p_t s_\lambda + [P]_x x_\lambda = 0 \\ \varphi_3 &= \rho_1 p_r s_\lambda + p_t t_\lambda + [P]_y y_\lambda = 0 \\ \varphi_4 &= u_\lambda - p x_\lambda - q y_\lambda = 0 \\ \varphi_5 &= p_\lambda - r x_\lambda - s y_\lambda = 0 \\ \varphi_6 &= q_\lambda - s x_\lambda - t y_\lambda = 0 \\ \psi_1 &= y_\mu - \rho_2 x_\mu = 0 \\ \psi_2 &= p_r r_\mu + \frac{1}{\rho_2} p_t s_\mu + [P]_x x_\mu = 0 \end{aligned} \right\} \text{System A}$$

(5.18)
 (continued)

$$\left. \begin{aligned} \psi_3 &= \rho_2 F_r \mu + F_t \mu + [F]_y y_\mu = 0 \\ \psi_4 &= u_\mu - p x_\mu - q y_\mu = 0 \\ \psi_5 &= p_\mu - r x_\mu - s y_\mu = 0 \\ \psi_6 &= q_\mu - s x_\mu - t y_\mu = 0 \end{aligned} \right\} \text{System B}$$

We observe that System A of (5.18) is of canonical hyperbolic form in $x, y; u; p, q; r, s, t$ as functions of λ and μ . Since for Theorem B, $F \in C^{(3)}$, while for Theorem Ba, $F \in C^{(1)}$, the coefficients of all equations in (5.18) are functions of class $C^{(1)}$ for Theorem B, and of class $C^{(1)}$ for Theorem Ba. Moreover, the determinant of the matrix of coefficients for System A, is, after interchange of rows and columns,

(5.19)

$$\begin{vmatrix} -\rho_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\rho_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & F_r & \frac{1}{\rho_1} F_t & 0 & 0 & 0 & 0 \\ 0 & * & 0 & \rho_1 F_r & F_t & 0 & 0 & 0 \\ * & 0 & F_r & \frac{1}{\rho_2} F_t & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 1 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 1 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= F_r F_t^2 \cdot \frac{(\rho_1 - \rho_2)^2}{\rho_1 \rho_2},$$

where the coefficients designated only by asterisks, *, do not contribute to the value of the determinant. Since $F_r \neq 0, F_t \neq 0$ and $\rho_1 \neq \rho_2$ in a neighborhood of S^2 , the determinant (5.19) does not vanish therein. Hence any solution $J: u = u(x, y)$ of the problem of Theorem B, together with its first and second derivatives,

$$\begin{cases}
 \text{Equation 1} \\
 \text{Equation 2} \\
 \text{Equation 3} \\
 \text{Equation 4}
 \end{cases}$$

The system of equations is solved by the method of elimination. The first equation is multiplied by 2 and then the second equation is subtracted from it. This process is repeated until the system is in row echelon form. The solutions are then found by back substitution.

$$\begin{array}{cccccccc|c}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
 \end{array}$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0, \quad \frac{\partial \mathcal{L}}{\partial y} = 0$$

The partial derivatives are set to zero to find the critical points. The Hessian matrix is then calculated to determine the nature of these points. The final result shows that the function has a local minimum at the point (1, 1).

satisfies the hypotheses for Theorem 7; because the requirement that $F \in C^1$ is certainly sufficient to insure that the first derivatives of the coefficients of System A be Lipschitzian with respect to variables $x, y; u; p, q; r, s, t$. Moreover, the requirement in Theorem 8a that $F \in C^1$ insures that the coefficients of System A are of class C^1 , as demanded by Theorem 7a.

In the $\lambda\mu$, or characteristic, plane, the initial base curve has the parametric form

$$\gamma: \begin{cases} \lambda = \lambda(x(\tau), y(\tau)) & \text{for } \tau \in [0, 1], \\ \mu = \mu(x(\tau), y(\tau)) \end{cases}$$

and is nowhere parallel to either the λ or μ axes. Consequently,

γ may be expressed in the non-parametric form

$$\lambda = \varphi(\mu)$$

where $\varphi(\mu) \in C^1$ and $\varphi'(\mu) \neq 0$. If we introduce $\lambda' = \lambda$ and $\mu' = -\varphi(\mu)$ as new characteristic parameters, we observe that equations (5.18) remain unaltered in form. Hence we may assume, without loss, that the initial base curve γ has the representation

$$(5.20) \quad \lambda + \mu = 0$$

in the $\lambda\mu$ plane.

We now demonstrate that any solution of System A satisfying the given Cauchy initial conditions is also a solution of the problem of Theorems 8 and 8a. This done, Theorems 8 and 8a are immediate consequences of Theorems 7 and 7a, respectively.

Following J. HADAMARD [11] p. 504, we show that for each set of functions satisfying System A and the initial conditions on

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$$\begin{aligned}
 & \frac{1}{2} (2x^2 + 4x + 2) \\
 &= \frac{1}{2} (2x^2 + 4x + 2) \\
 &= x^2 + 2x + 1 \\
 &= (x + 1)^2
 \end{aligned}$$

Additional faint text in the middle section of the page, likely a continuation of the explanation or a separate problem.

$$\frac{1}{2} (2x^2 + 4x + 2) = (x + 1)^2$$

Faint text at the bottom of the page, possibly a conclusion or a final note.

$\lambda + \mu = 0$, the System B is likewise satisfied. Note that in this part of the argument we cannot admit that p, q, r, s and t are derivatives of u . This is now a matter of proof.

Differentiating $F(x, y; u; p, q; r, s, t)$ by λ and observing equations (5.18), we obtain

$$(5.21) \quad \frac{dF}{d\lambda} = \varphi_2 + \varphi_3 + F_u \varphi_4 + F_p \varphi_5 + F_q \varphi_6.$$

Hence $\frac{dF}{d\lambda} = 0$ for each set of functions satisfying System A. However, by hypothesis, $F = 0$ along $\lambda + \mu = 0$. Thus $F \equiv 0$ throughout that region where the set of functions satisfying System A is defined. This in turn implies that

$$(5.22) \quad \frac{dF}{d\mu} = \psi_2 + \psi_3 + F_u \psi_4 + F_p \psi_5 + F_q \psi_6 = 0 \text{ throughout the same region. By hypothesis, } \psi_2 = 0 \text{ in this region, hence}$$

$$(5.23) \quad \psi_3 = -F_u \psi_4 - F_p \psi_5 - F_q \psi_6$$

therein.

Since $\rho_1 \rho_2 = \frac{F_t}{F_r}$, we obtain from (5.18) by simple algebraic

operations

$$(5.24) \quad \frac{\rho_1 y_\mu}{F_t} \varphi_2 = r_\lambda x_\mu + s_\lambda y_\mu + H,$$

$$(5.25) \quad \frac{\rho_2 y_\lambda}{F_t} \psi_2 = r_\mu x_\lambda + s_\mu y_\lambda + H,$$

where

$$(5.26) \quad H = \frac{y_\lambda y_\mu}{F_t} [F]_x = \frac{x_\lambda x_\mu}{F_r} [F]_x;$$

$$(5.27) \quad \frac{y_\mu}{F_t} \varphi_3 = s_\lambda x_\mu + t_\lambda y_\mu + E,$$

Let $\psi_1, \psi_2, \dots, \psi_n$ be a set of linearly independent functions. Then the set $\psi_1, \psi_2, \dots, \psi_n, \psi_{n+1}$ is linearly dependent. For if we assume that they are linearly independent, then we can find constants c_1, c_2, \dots, c_{n+1} , not all zero, such that $c_1\psi_1 + c_2\psi_2 + \dots + c_{n+1}\psi_{n+1} = 0$. But $\psi_1, \psi_2, \dots, \psi_n$ are linearly independent, so $c_1 = c_2 = \dots = c_n = 0$. This implies $c_{n+1}\psi_{n+1} = 0$, and since $\psi_{n+1} \neq 0$, we have $c_{n+1} = 0$. This contradicts our assumption that not all c_i are zero.

$$\psi_1, \psi_2, \dots, \psi_n, \psi_{n+1} \text{ are linearly dependent.}$$

Let $\psi_1, \psi_2, \dots, \psi_n$ be a set of linearly independent functions. Then the set $\psi_1, \psi_2, \dots, \psi_n, \psi_{n+1}$ is linearly dependent. For if we assume that they are linearly independent, then we can find constants c_1, c_2, \dots, c_{n+1} , not all zero, such that $c_1\psi_1 + c_2\psi_2 + \dots + c_{n+1}\psi_{n+1} = 0$. But $\psi_1, \psi_2, \dots, \psi_n$ are linearly independent, so $c_1 = c_2 = \dots = c_n = 0$. This implies $c_{n+1}\psi_{n+1} = 0$, and since $\psi_{n+1} \neq 0$, we have $c_{n+1} = 0$. This contradicts our assumption that not all c_i are zero.

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$$(5.28) \quad \frac{y_\lambda}{F_t} \psi_3 = s_\mu x_\lambda + t_\mu y_\lambda + \kappa,$$

where

$$(5.29) \quad \kappa = \frac{y_\lambda y_\mu}{F_t} [F]_y = \frac{x_\lambda x_\mu}{F_r} [F]_y.$$

By Theorem 7 or Theorem 7a, the functions of the set satisfying System A and the Cauchy initial data are continuously differentiable and possess continuous mixed second derivatives. Thus we may perform the differentiations in the following relations:

$$(5.30) \quad \begin{aligned} \psi_{4,\lambda} - \psi_{4,\mu} &= p_\lambda x_\mu + q_\lambda y_\mu - p_\mu x_\lambda - q_\mu y_\lambda \\ &= \psi_5 x_\mu - \psi_6 y_\mu - \psi_5 x_\lambda - \psi_6 y_\lambda; \end{aligned}$$

$$(5.31) \quad \begin{aligned} \psi_{5,\lambda} - \psi_{5,\mu} &= r_\lambda x_\mu + s_\lambda y_\mu - r_\mu x_\lambda - s_\mu y_\lambda \\ &= \frac{r_1 y_\mu}{F_t} \psi_2 - \frac{r_2 y_\lambda}{F_t} \psi_2, \end{aligned}$$

by (5.24) and (5.25) above;

$$(5.32) \quad \begin{aligned} \psi_{6,\lambda} - \psi_{6,\mu} &= s_\mu x_\lambda + t_\mu y_\lambda - s_\lambda x_\mu - t_\lambda y_\mu \\ &= \frac{y_\lambda}{F_t} \psi_3 - \frac{y_\mu}{F_t} \psi_3, \end{aligned}$$

by (5.27) and (5.28) above. But System A is satisfied, hence (5.30), (5.31) and (5.32), by virtue of (5.23), reduce to

$$(5.33) \quad \begin{cases} \psi_{4,\lambda} &= -\psi_5 x_\lambda - \psi_6 y_\lambda \\ \psi_{5,\lambda} &= 0 \\ \psi_{6,\lambda} &= \frac{-y_\lambda}{F_t} (F_u \psi_4 + F_p \psi_5 + F_q \psi_6). \end{cases}$$

$$x^2 = x^2 \cdot x^2 = x^2 \cdot \frac{x^2}{x^2} \quad (1)$$

$$= x^2 \cdot \frac{x^2}{x^2} = x^2 \cdot \frac{x^2}{x^2} = x^2 \quad (2)$$

The following is a list of the numbers of the first 100 natural numbers. The numbers are listed in order of increasing size. The numbers are listed in order of increasing size. The numbers are listed in order of increasing size.

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad (3)$$

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad (4)$$

The following is a list of the numbers of the first 100 natural numbers.

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad (5)$$

The following is a list of the numbers of the first 100 natural numbers. The numbers are listed in order of increasing size. The numbers are listed in order of increasing size.

$$\left. \begin{aligned} 1^2 + 2^2 + 3^2 + \dots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\ 1^2 + 2^2 + 3^2 + \dots + n^2 &= \frac{n(n+1)(2n+1)}{6} \end{aligned} \right\} (6)$$

In (5.33) all functions are known except ψ_4, ψ_5, ψ_6 and their derivatives with respect to λ . Moreover, along $\lambda = -\mu$ System B is satisfied, i.e. $\psi_4 = \psi_5 = \psi_6 = 0$ for $\lambda = -\mu$. For fixed μ we may consider (5.33) as a homogeneous system of linear first order ordinary differential equations under homogeneous onepoint boundary conditions. This system has the unique solution

$$\psi_4 = \psi_5 = \psi_6 = 0$$

throughout the region of definition of the set of functions satisfying System A. By (5.23), $\psi_3 = 0$ also, and the System B is shown to be dependent upon the System A in the sense above.

From the functions $x = x(\lambda, \mu), y = y(\lambda, \mu)$ of the set satisfying System A, we may form the inverse functions $\lambda = \lambda(x, y), \mu = \mu(x, y)$, since the Jacobian

$$(5.6) \quad y_\lambda x_\mu - y_\mu x_\lambda = (\rho_1 - \rho_2) x_\lambda x_\mu$$

does not vanish. Hence we may express the function $u = u(\lambda, \mu)$ as a function of the independent variables x and y .

We now need to show only that

$$(5.34) \quad p = u_x, \quad q = u_y, \quad r = u_{xx}, \quad s = u_{xy} \quad \text{and} \quad t = u_{yy}$$

throughout the above region to complete the proof.

$$\text{Now} \quad \psi_4 = u_\lambda - px_\lambda - qy_\lambda = 0$$

$$\psi_4 = u_\mu - px_\mu - qy_\mu = 0,$$

while the determinant of this linear system is the Jacobian (5.6) and hence does not vanish. Thus there exists a unique solution.

Let $\psi_1, \psi_2, \dots, \psi_n$ be solutions of the homogeneous system $L\psi = 0$. Then any solution ψ of the inhomogeneous system $L\psi = f$ can be written as $\psi = \psi_h + \psi_p$, where ψ_h is a solution of the homogeneous system and ψ_p is a particular solution of the inhomogeneous system. The general solution of the inhomogeneous system is $\psi = \psi_h + \psi_p$.

$$\psi = \psi_h + \psi_p$$

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90. W. W. WHYTEER, "Over and under functions as related to differential equations," *American Mathematical Monthly*, vol. 47 (1940), pp. 1-10.

THE UNIVERSITY OF CHICAGO
DEPARTMENT OF CHEMISTRY
530 SOUTH EAST ASIAN AVENUE
CHICAGO, ILLINOIS 60607

RECEIVED
MAY 15 1964

TO: DR. J. H. GOLDSTEIN
FROM: DR. R. M. WAYNE
SUBJECT: [Illegible]

Enclosed for you are two copies of a report on the work done during the past few months. I hope you will find it of interest.

Very truly yours,
R. M. Wayne

But $p = u_x$, $q = u_y$ obviously satisfies and hence represents the unique solution.

Similarly,

$$\mathcal{L}\delta = u_{x,\lambda} - rx_\lambda - sy_\lambda = 0$$

$$\Psi\delta = u_{x,\mu} - rx_\mu - sy_\mu = 0,$$

hence $r = u_{xx}$ and $s = u_{xy}$;

$$\mathcal{L}\delta = u_{y,\lambda} - sx_\lambda - ty_\lambda = 0$$

$$\Psi\delta = u_{y,\mu} - sx_\mu - ty_\mu = 0,$$

hence $t = u_{yy}$ and $u_{yx} = u_{xy} = s$. The proof is now complete.

and consequently also for arbitrary values $\mu = 2$, $\mu = 4$ and
 arbitrary values

arbitrary

$$-2 = \sqrt{10} - \sqrt{10} = \sqrt{10^2 - 10^2}$$

$$-2 = -\sqrt{10} + \sqrt{10} = -\sqrt{10^2 - 10^2}$$

$$\sqrt{10^2 - 10^2} = 2 \text{ bzw. } -2 = 2 \text{ bzw. } -2$$

$$2 = \sqrt{10} - \sqrt{10} = \sqrt{10^2 - 10^2}$$

$$2 = -\sqrt{10} + \sqrt{10} = -\sqrt{10^2 - 10^2}$$

und für die Werte $\mu = 2$, $\mu = 4$ und $\mu = 8$ sind die Werte $\mu = 2$ und

CHAPTER VI

The Characteristic Initial Value Problem for

$$F(x,y;u;p,q; r,s,t) = 0.$$

The whole idea of a characteristic initial value problem for the equation

$$(1.1) \quad F(x,y; u; p,q; r,s,t) = 0$$

appears paradoxical at first glance. In the Cauchy problem the prescribed initial data was sufficient to determine whether or not the projection of the initial curve was characteristic. In this problem, however, we merely prescribe two intersecting space curves through which an integral surface of the equation (1.1) must pass. Since the characteristics are, in general, dependent on the integral surface in question, it would appear impossible to determine, a priori, whether or not the prescribed initial curves have characteristic projections.

That such is not the case is demonstrated by M. CINQUINI-CIBRARIO [13]. In this paper she treats the characteristic initial value problem as a special case of the more general Courant problem, i.e. where two arbitrary intersecting space curves are prescribed through which an integral surface of (1.1) must pass. Commencing on p. 220, she gives the necessary and sufficient conditions that these curves be characteristic to any integral surface passing through them. We call curves satisfying these conditions "intrinsically characteristic" curves.

CHAPTER 2

THE DIFFERENTIAL EQUATION

(1.1) $y'' + p(x)y' + q(x)y = r(x)$

Let y_1, y_2, \dots, y_n be solutions of the homogeneous equation (1.1) and let y_p be a particular solution of (1.1). Then the general solution of (1.1) is given by

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n + y_p \quad (1.2)$$

where c_1, c_2, \dots, c_n are arbitrary constants. In the case where the homogeneous equation (1.1) has no solutions other than the trivial solution $y = 0$, the general solution of (1.1) is given by $y = y_p$.

Let y_1, y_2, \dots, y_n be solutions of the homogeneous equation (1.1) and let y_p be a particular solution of (1.1). Then the general solution of (1.1) is given by $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n + y_p$.

Let y_1, y_2, \dots, y_n be solutions of the homogeneous equation (1.1) and let y_p be a particular solution of (1.1). Then the general solution of (1.1) is given by $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n + y_p$.

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Let y_1, y_2, \dots, y_n be solutions of the homogeneous equation (1.1) and let y_p be a particular solution of (1.1). Then the general solution of (1.1) is given by $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n + y_p$.

Let y_1, y_2, \dots, y_n be solutions of the homogeneous equation (1.1) and let y_p be a particular solution of (1.1). Then the general solution of (1.1) is given by $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n + y_p$.

In this chapter we examine her development, for the particular case of the characteristic initial value problem, up to the point where a modified form of the system of characteristic equations (5.18) and the above necessary and sufficient conditions are obtained. There are two important differences between her development and that of H. LEWY given in the preceding chapter. First, she transforms the initial curves into the coordinate axes. Since these curves are characteristic, this implies immediately that $F_x = 0$ and $F_t = 0$ at the origin. Thus many of the divisions performed in Chapter V are now invalidated. Second, she is able to solve (1.1) explicitly for s , obtaining

$$s = f(x, y; u; p, q; r, t)$$

and thus to reduce the number of equations in the system of characteristic equations by two.

We do not follow the remainder of her existence proof, in which she reduces the system of characteristic equations to an integral equation form and then applies successive approximations to obtain the existence of a unique solution to the general Cauchy problem. Instead we deal directly with the special case of the characteristic initial value problem by a method analogous to that of Chapter V. Such an approach is indicated by M. CINTRINI-CIBRARIO, herself, [12] p.190, footnote 8. She states, in effect, that the following Theorem 9 can be shown to be a consequence of Theorem 6, Chapter IV. We present this proof in detail and, in addition, we extend it to apply to the derivation of Theorem 9a as a consequence of Theorem 6a. The improvement obtained corresponds to that of

In this study we consider the asymptotic behavior of the estimator $\hat{\beta}_n$ of the regression coefficients β in the linear model $Y = X\beta + \epsilon$, where Y is the vector of observations, X is the design matrix, and ϵ is the vector of errors. We assume that the errors are independent and normally distributed with mean zero and variance σ^2 . The design matrix X is assumed to be of full rank and the number of observations n tends to infinity as $n \rightarrow \infty$. The asymptotic normality of the least squares estimator is well known, but we provide a detailed proof for completeness.

$$\hat{\beta}_n = (X'X)^{-1}X'Y$$

Under the above assumptions, the least squares estimator $\hat{\beta}_n$ is unbiased and consistent. The asymptotic normality of $\hat{\beta}_n$ can be derived from the central limit theorem and Slutsky's theorem. Specifically, we have $\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{d} N(0, \sigma^2(X'X)^{-1})$. This result is fundamental in many applications of linear regression, particularly in hypothesis testing and confidence interval construction. The asymptotic normality also allows for the use of standard normal distribution tables to approximate probabilities involving the estimator. Furthermore, the asymptotic variance-covariance matrix of $\hat{\beta}_n$ provides a measure of the precision of the estimates. In practice, the asymptotic normality is often used to justify the use of normal distribution-based inference procedures for linear regression models.

Chapter V for the Cauchy problem. Namely, the requirement that $F \in C^{(3)}$ is reduced to require merely that $F \in C^{(1)}$ while the conclusion is altered to read "at least one solution" instead of "one and only one solution".

Theorem 9

$$1) \quad \begin{cases} \Gamma_1: \begin{cases} x_1 - \xi \leq x \leq x_1 + \xi & , f_1(x) \in C^{(1)}([x_1 - \xi, x_1 + \xi]) \\ y = f_1(x) & F_1(x) \in C^{(1)}([x_1 - \xi, x_1 + \xi]). \\ u = F_1(x) \end{cases} \\ \Gamma_2: \begin{cases} x = f_2(y) & , f_2(y) \in C^{(1)}([y_1 - \eta, y_1 + \eta]) \\ y_1 - \eta \leq y \leq y_1 + \eta & F_2(y) \in C^{(1)}([y_1 - \eta, y_1 + \eta]) \\ u = F_2(y) \end{cases} \end{cases}$$

The point (x_1, y_1) is the only point of intersection of Γ_1 and Γ_2 and it is interior to both curves. Moreover, $F_1(x_1) = F_2(y_1)$ and $f_1'(x_1)f_2'(y_1) \neq 1$. (i.e. Γ_1 and Γ_2 do not have a common tangent at the point (x_1, y_1) .)

2) Γ_1 and Γ_2 are "intrinsically characteristic" in a neighborhood of their point of intersection, i.e. they meet the necessary and sufficient conditions, given below, that they be characteristic to any integral surface of

$$(1.1) \quad F(x, y; u; p, q; r, s, t) = 0$$

passing through them. As we shall see below, this hypothesis, together with hypothesis 1), tacitly implies that at the intersection point (x_1, y_1, u_1) of Γ_1 and Γ_2 the values $p_1, q_1, r_1, s_1,$

The following conditions are assumed: the system is linear and time-invariant, the input is a unit step function, and the output is measured at a specific time. The transfer function of the system is given by:

$$\begin{aligned}
 & \left\{ \begin{aligned}
 & (L\ddot{y} + \beta\dot{y} + \gamma y) = \alpha u(t) \\
 & y(0) = 0 \\
 & \dot{y}(0) = 0
 \end{aligned} \right. \quad \left. \begin{aligned}
 & \text{where } \alpha = 1 \\
 & \beta = 2 \\
 & \gamma = 1
 \end{aligned} \right\} \quad \text{Equation (1)} \\
 & \left\{ \begin{aligned}
 & L\ddot{y} + \beta\dot{y} + \gamma y = \alpha u(t) \\
 & y(0) = 0 \\
 & \dot{y}(0) = 0
 \end{aligned} \right. \quad \left. \begin{aligned}
 & \text{where } \alpha = 1 \\
 & \beta = 2 \\
 & \gamma = 1
 \end{aligned} \right\} \quad \text{Equation (2)}
 \end{aligned}$$

The transfer function of the system is given by $G(s) = \frac{\alpha}{Ls^2 + \beta s + \gamma}$. The Laplace transform of the unit step function is $U(s) = \frac{1}{s}$. The Laplace transform of the output is $Y(s) = G(s)U(s) = \frac{\alpha}{s(Ls^2 + \beta s + \gamma)}$. The partial fraction expansion of $Y(s)$ is given by:

$$Y(s) = \frac{A}{s} + \frac{B}{s + \lambda_1} + \frac{C}{s + \lambda_2} \quad (2.1)$$

The constants A , B , and C are determined by the initial conditions and the input function. The inverse Laplace transform of $Y(s)$ gives the time-domain response $y(t)$.

t_1), the hyperbolic condition

$$F_{s_1}^2 - 4 F_{r_1} F_{t_1} > 0,$$

is satisfied, (notation: $F_{s_1} = F_s(x_1, y_1; u_1; p_1, q_1; r_1, s_1, t_1)$, etc.)

3) $F \in C'''$ in a neighborhood of the point

$$(x_1, y_1; u_1; p_1, q_1; r_1, s_1, t_1).$$

\Rightarrow 4) There exists one and only one integral surface $J = u(x, y)$ of $F(x, y; u; p, q; r, s, t) = C$, defined and of class C''' in a sufficiently small neighborhood of the point (x_1, y_1) and passing through subarcs of Γ_1 and Γ_2 intersecting at the point (x_1, y_1, u_1) .

Theorem 9a

1)

2)

3)' $F \in C'''$ in a neighborhood of the point

$$(x_1, y_1; u_1; p_1, q_1; r_1, s_1, t_1).$$

\Rightarrow 4)' There exists at least one integral surface etc.

(as in Theorem 9).

Proof of Theorems 9 and 9a

We first perform the coordinate transformation

$$(6.1) \quad \begin{cases} \bar{x} = x - f_2(y) \\ \bar{y} = y - f_1(x) \end{cases}$$

taking Γ_1 into the \bar{x} axis, Γ_2 into the \bar{y} axis and the point (x_1, y_1) into the origin. This transformation is univalent in a

For the symmetric matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

find the eigenvalues and eigenvectors of A .

1) To find the eigenvalues of A , we solve the characteristic equation $\det(A - \lambda I) = 0$.

2) The characteristic equation is $\det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix} = 0$. This gives $(1-\lambda)(5-\lambda) - 4 = 0$, which simplifies to $\lambda^2 - 6\lambda + 1 = 0$. The eigenvalues are $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$.

Solution

1)

2)

3) For $\lambda_1 = 3 + \sqrt{8}$, the eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ satisfies $(A - \lambda_1 I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

4) For $\lambda_2 = 3 - \sqrt{8}$, the eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ satisfies $(A - \lambda_2 I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Final Answer

The eigenvalues are $3 + \sqrt{8}$ and $3 - \sqrt{8}$.

$$\left[\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} - (3 + \sqrt{8})I \right] \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} - (3 - \sqrt{8})I \right] \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The eigenvectors are $\begin{pmatrix} 1 \\ \frac{1}{2}(3 + \sqrt{8} - 1) \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \frac{1}{2}(3 - \sqrt{8} - 1) \end{pmatrix}$.

neighborhood of (x_1, y_1) since the Jacobian

$$(6.2) \quad 1 - f_1'(x_1)f_2'(y_1) \neq 0$$

by hypothesis 1). Geometrically, this means that γ_1 and γ_2 do not have a common tangent at their point of intersection.

Without loss, we may assume homogeneous initial conditions.

For, suppose we have an integral surface $J: u = u(x, y)$ of equation (1.1) passing through the curves Γ_1 and Γ_2 . Then by the above transformation, considering (6.2),

$$(6.3) \quad u(x, y) = \bar{u}(\bar{x}(x, y), \bar{y}(x, y)),$$

and hence for any such integral surface

$$(6.4) \quad \begin{cases} F_1(x) = u(x, f_1(x)) = u(\bar{x}(x, f_1(x)), 0), \\ F_2(y) = u(f_2(y), y) = \bar{u}(0, \bar{y}(f_2(y), y)). \end{cases}$$

Letting

$$(6.5) \quad w(\bar{x}, \bar{y}) = \bar{u}(\bar{x}, \bar{y}) - \bar{u}(\bar{x}, 0) - \bar{u}(0, \bar{y}) + \bar{u}(0, 0),$$

and since, by hypothesis 1), f_1, f_2, F_1 and $F_2 \in C^1$, we obtain

$$(6.6) \quad \begin{cases} w(\bar{x}, 0) = w_{\bar{x}}(\bar{x}, 0) = w_{\bar{x}\bar{x}}(\bar{x}, 0) = 0, \\ w(0, \bar{y}) = w_{\bar{y}}(0, \bar{y}) = w_{\bar{y}\bar{y}}(0, \bar{y}) = 0. \end{cases}$$

Thus we may reduce the problem to that of finding a function $w = w(\bar{x}, \bar{y})$ which vanishes on the coordinate axes in a vicinity of the origin and satisfies there the transformed form of equation (1.1),

... ..

$$x = \dots \quad (1-2)$$

... ..

... ..

$$\dots \quad (1-3)$$

... ..

$$\dots \quad (1-4)$$

... ..

$$\dots \quad (1-5)$$

... ..

$$\dots \quad (1-6)$$

... ..

... ..

$$(6.7) \quad F(\bar{x}, \bar{y}; [\bar{w} + \delta]; [\bar{w} + \delta], \bar{x}, [\bar{w} + \delta], \bar{y}; [\bar{w} + \delta], \bar{x}\bar{x}, \\ [\bar{w} + \delta], \bar{x}\bar{y}, [\bar{w} + \delta], \bar{y}\bar{y})$$

where

$$(6.8) \quad g(\bar{x}, \bar{y}) = \bar{u}(\bar{x}, 0) + \bar{u}(0, \bar{y}) - \bar{u}(0, 0).$$

The function g is known from the prescribed initial data.

For simplicity, we return to our original notation and state the problem in this way:

To determine the function $u = u(x, y)$ satisfying equation (1.1) and the initial conditions

$$u(x, 0) = u(0, y) = 0,$$

where, in the notation above,

$$u_0 = p_0 = q_0 = r_0 = t_0 = 0$$

and

$$(6.9) \quad F(0, 0; 0; 0, 0; 0, s_0, 0) = 0.$$

By hypothesis 2), there exists a unique value s_0 satisfying (6.9).

The characteristic base curves and, a fortiori, the hyperbolic condition are invariant under the transformation (6.1). (See R. COURANT - D. HILBERT [17] p. 304.) Moreover, the substitution $w = \bar{u} - g$ also preserves the invariance of the equation for the characteristic base curves and the hyperbolic condition as is easily seen by differentiation of (6.7). Hence, by hypothesis 2), we have the hyperbolic condition

$$\frac{1}{\sqrt{2}} \begin{bmatrix} A+1 \\ B+1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} C+1 \\ D+1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (A+1)(C+1) + (B+1)(D+1) \\ (A+1)(C+1) - (B+1)(D+1) \end{bmatrix}$$

where

$$A = \sqrt{2}(\alpha + \beta), \quad B = \sqrt{2}(\alpha - \beta), \quad C = \sqrt{2}(\gamma + \delta), \quad D = \sqrt{2}(\gamma - \delta)$$

The function ψ is found from the previous results and the condition $\psi = 0$ is used to find the values of $\alpha, \beta, \gamma, \delta$ and the results are

By substituting the values of $\alpha, \beta, \gamma, \delta$ in (1) and (2) we get the following equations

$$2\alpha + \sqrt{2}\beta + \sqrt{2}\gamma + \delta = 0$$

where $\alpha, \beta, \gamma, \delta$ are

$$\alpha = \frac{1}{2}(\sqrt{2}\beta + \sqrt{2}\gamma + \delta)$$

and

$$2\alpha - \sqrt{2}\beta + \sqrt{2}\gamma - \delta = 0$$

By substituting the value of α in (1) we get

The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$ and the roots are $\lambda = 1, 1$.

Since the roots are equal, the general solution is $y = (c_1 + c_2x)e^x$. The boundary conditions are $y(0) = 1$ and $y(1) = 0$. Solving these conditions we get $c_1 = 1$ and $c_2 = -2$. Thus the solution is $y = (1 - 2x)e^x$.

$$(6.10) \quad P_{s_0}^2 - 4 P_{r_0} P_{t_0} > 0,$$

while the equation for the characteristic base curve directions at the origin is

$$(6.11) \quad P_{r_0} dy^2 - P_{s_0} dx dy + P_{t_0} dx^2 = 0.$$

Hypothesis 2) implies that the coordinate axes must be characteristic base curves. By (6.11) and (6.10) this in turn implies that $P_{r_0} = P_{t_0} = 0$, and hence that $P_{s_0} \neq 0$. But now the Implicit Function Theorem tells us that in the neighborhood of the point $(0,0; 0; 0,0; 0, s_0, 0)$ equation (1.1) can be solved explicitly in the form

$$(6.12) \quad s = f(x,y; u; p,q; r,t).$$

Under hypothesis 3) or 3)', the function $f \in C'''$ or C'' , respectively, in a neighborhood of this point. Moreover,

$$(6.13) \quad f_{r_0} = f_{t_0} = 0 \quad \text{and} \quad s_0 = f_0$$

while the hyperbolic condition becomes at the origin

$$(6.14) \quad 1 - 4 f_{r_0} f_{t_0} = 1 > 0$$

and the equation for the characteristic base curves becomes

$$(6.15) \quad f_r dy^2 + dx dy + f_t dx^2 = 0.$$

Let us assume that we have a particular integral surface $J: u = u(x,y)$ passing through the coordinate axes in a neighborhood of the origin, with $u(x,y) \in C'''$ in this neighborhood.

We define

$$(12) \quad \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 \right) = \dot{x} \ddot{x} + \dot{y} \ddot{y}$$

the derivative of the kinetic energy is equal to the power of the forces.

$$(13) \quad \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 \right) = \dot{x} \ddot{x} + \dot{y} \ddot{y}$$

Therefore, the kinetic energy is a function of the coordinates and time. The total energy is the sum of the kinetic energy and the potential energy. The total energy is conserved if the forces are conservative.

$$(14) \quad \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 \right) = \dot{x} \ddot{x} + \dot{y} \ddot{y}$$

From equation (13), we have $\dot{x} \ddot{x} + \dot{y} \ddot{y} = \frac{d}{dt} \left(\frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 \right)$. This is a differential equation of the first order.

$$(15) \quad \dot{x} \ddot{x} + \dot{y} \ddot{y} = \frac{d}{dt} \left(\frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 \right)$$

Integrating both sides with respect to time, we get

$$(16) \quad \frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 = \int \left(\dot{x} \ddot{x} + \dot{y} \ddot{y} \right) dt$$

the kinetic energy is equal to the integral of the power of the forces.

$$(17) \quad \frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 = \int \left(\dot{x} \ddot{x} + \dot{y} \ddot{y} \right) dt$$

Let us assume that the forces are conservative. Then the total energy is conserved. The total energy is the sum of the kinetic energy and the potential energy. The total energy is constant in time.

$$(6.16) \quad \delta = \sqrt{1 - 4f_r f_t}, \quad \rho = \frac{-2f_t}{1 + \delta}, \quad \sigma = \frac{-2f_r}{1 + \delta},$$

δ , ρ and σ being of class C^1 by hypothesis 3), or of class C^1 by hypothesis 3)', in the variables $x, y; u; p, q; r, t$ in a neighborhood of the point $(0, 0; 0; 0, 0; 0, 0)$. The two one-parameter families of characteristic base curves corresponding to J are thus represented by the equations

$$(6.17) \quad y_\lambda = \rho x_\lambda$$

$$(6.18) \quad x_\mu = \sigma y_\mu.$$

Note that $\delta_0 = 1$, hence $\delta > 0$ in a neighborhood of the origin, while $\rho_0 = \sigma_0 = 0$.

As in Chapter V, to obtain the system of characteristic equations, we transform to the characteristic base curves as coordinates and consider what relations must be satisfied along these coordinates for any given integral surface J . In particular, we specialize the transformation

$$(6.19) \quad \begin{cases} x = x(\lambda, \mu) \\ y = y(\lambda, \mu) \end{cases}$$

by stipulating that a line $\lambda = \text{constant}$ shall have x -intercept $(\lambda, 0)$ and a line $\mu = \text{constant}$ shall have y -intercept $(0, \mu)$, with $\lambda = \mu = 0$ at the origin. The Jacobian of this transformation, evaluated at the origin, has the value

$$(6.20) \quad x_{\lambda_0} y_{\mu_0} - y_{\lambda_0} x_{\mu_0} = x_{\lambda_0} y_{\mu_0} (1 - \rho_0 \sigma_0) = x_{\lambda_0} y_{\mu_0} \neq 0,$$

since if $x_{\lambda_0} = 0$, then $y_{\lambda_0} = 0$ by (6.17), contradicting the requirement that $\dot{x}^2 + \dot{y}^2 \neq 0$ along any characteristic curve.

$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right)$$

The first part of the proof is to show that the function $f(x)$ is continuous at x_0 . Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$. Since $f(x) = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right)$, we have $f(x) = \frac{1}{2}$ for all x . Therefore, $|f(x) - f(x_0)| = 0 < \epsilon$ for all x . This shows that $f(x)$ is continuous at x_0 .

$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right)$$

$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right)$$

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$$\left. \begin{aligned}
 & \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) \\
 & \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right)
 \end{aligned} \right\}$$

Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$. Since $f(x) = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right)$, we have $f(x) = \frac{1}{2}$ for all x . Therefore, $|f(x) - f(x_0)| = 0 < \epsilon$ for all x . This shows that $f(x)$ is continuous at x_0 .

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Similarly, if $y_{\mu_0} = 0$, then $x_{\mu_0} = 0$ by (6.16) and the contradiction is again obtained.

Paralleling our development in Chapter V, we see that certain determinants must vanish at each point of the integral surface J , yielding equations which must be satisfied along the characteristics on J . We have

$$(6.21) \quad \begin{vmatrix} f_r & -[f]_x & f_t \\ x_\lambda & r_\lambda & 0 \\ 0 & s_\lambda & y_\lambda \end{vmatrix} = f_r r_\lambda y_\lambda + f_t s_\lambda x_\lambda + [f]_x x_\lambda y_\lambda = 0$$

where

$$(6.22) \quad [f]_x = f_p r + f_q s + f_u p + f_x.$$

also

$$(6.23) \quad \begin{vmatrix} f_r & -[f]_y & f_t \\ x_\lambda & s_\lambda & 0 \\ 0 & t_\lambda & y_\lambda \end{vmatrix} = f_r s_\lambda y_\lambda + f_t t_\lambda x_\lambda + [f]_y x_\lambda y_\lambda = 0$$

where

$$(6.24) \quad [f]_y = f_p s + f_q t + f_u q + f_y.$$

Eliminating s_λ between (6.21) and (6.23), we obtain

$$(6.25) \quad f_r^2 r_\lambda y_\lambda^2 - f_t^2 t_\lambda x_\lambda^2 + [f]_x f_r x_\lambda y_\lambda^2 - [f]_y f_t x_\lambda^2 y_\lambda = 0.$$

By virtue of definitions (6.16) and equation (5.17), we may write (6.25) as

$$(6.26) \quad f_t^2 x_\lambda^2 \cdot H(\lambda, \mu) = 0$$

... also in order to ...
 ... in order to ...
 ... in order to ...

$$0 = \dots \begin{vmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} \quad (10.6)$$

$$\dots = \dots + \dots + \dots + \dots \quad (10.7)$$

$$0 = \dots \begin{vmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} \quad (10.8)$$

$$\dots = \dots + \dots + \dots + \dots \quad (10.9)$$

$$\dots = \dots + \dots + \dots + \dots \quad (10.10)$$

$$\dots = \dots + \dots + \dots + \dots \quad (10.11)$$

where

$$(6.27) \quad H(\lambda, \mu) = r_\lambda \sigma^2 - t_\lambda + \frac{2}{1+\delta} \left\{ [f]_y - \sigma [f]_x \right\} x_\lambda.$$

But, as shown above, $x_\lambda \neq 0$ along any of the characteristic base curves of J of the corresponding family, hence (6.26) reduces to

$$(6.28) \quad f_t^2 \cdot H(\lambda, \mu) = 0.$$

Where $f_t \neq 0$ we have immediately that $H(\lambda, \mu) = 0$. Suppose at a particular point of J that $f_t = 0$. Then by (6.16) and (6.17), we have there that

$$(6.29) \quad \rho = 0, \quad \delta = 1, \quad \sigma = -f_r \quad \text{and} \quad y_\lambda = 0.$$

Thus, at this point, by (6.24),

$$(6.30) \quad t_\lambda = s_y x_\lambda = (f_r r_y + [f]_y) x_\lambda;$$

while by (5.22),

$$(6.31) \quad r_\lambda \sigma^2 = f_r^2 r_x x_\lambda = f_r^2 (s_\lambda - [f]_x x_\lambda).$$

Substituting (6.30) and (6.31) into (6.27), we obtain that where $f_t = 0$ on J , $H(\lambda, \mu) = 0$. Hence by (6.28), $H(\lambda, \mu) = 0$ everywhere on J and represents a relation which must be satisfied along each characteristic of the corresponding family on J .

For the other family of characteristics on J , we have determinants corresponding to (6.21) and (6.22) which vanish at each point of J . Eliminating s_μ between these and arguing in a fashion analogous to that above, we arrive at the following rela-

$$r^2 \int_0^\pi [A] \tau - \tau [A] \frac{d}{dt} \tau - \tau^2 \frac{d}{dt} [A] + \tau^2 \frac{d}{dt} [A] \tau \quad (10.1)$$

... the above ...

$$\dots = \dots \quad (10.2)$$

... the above ...

$$\dots = \dots \quad (10.3)$$

... the above ...

$$\dots = \dots \quad (10.4)$$

... the above ...

$$\dots = \dots \quad (10.5)$$

... the above ...

tion which must be satisfied along each characteristic of this family on J :

$$(6.32) \quad K(\lambda, \mu) = \rho^2 t_\mu - r_\mu + \frac{2}{1+\delta} \left\{ [f]_x - \rho [f]_y \right\} y_\mu = 0.$$

We are now in a position to prescribe the necessary and sufficient conditions that the coordinate axes be characteristics for any integral surface of

$$(6.12) \quad z = f(x, y; u; p, q; r, t)$$

passing through them.

Suppose that, in a neighborhood of the origin, the coordinate axes are characteristic to some integral surface $J: u = u(x, y)$ of (6.12) passing through them. Then in terms of the characteristic base curves to J as coordinates, defined by the coordinate transformation (6.13), we have for $\mu = 0$:

$$x = \lambda, \quad y = 0, \quad u = p = r = 0, \quad q = Q(\lambda), \quad t = T(\lambda),$$

where, from (6.12),

$$(6.33) \quad Q'(\lambda) = f(\lambda, 0; 0; 0, Q(\lambda); 0, T(\lambda)),$$

while, from $K(\lambda, \mu) = 0$, since $\rho = f_t = 0$, $\delta = 1$ and $\sigma = -f_p$,

$$(6.34) \quad T'(\lambda) = \left\{ [f]_y + f_r [f]_x \right\} (\lambda, 0; 0; 0, Q(\lambda); 0, T(\lambda)).$$

Moreover,

$$(6.35) \quad Q(0) = T(0) = 0.$$

... ..
... ..

$$u = \frac{1}{\sqrt{2}} \left[\begin{matrix} \cos \theta \\ \sin \theta \end{matrix} \right] = \frac{1}{\sqrt{2}} \left[\begin{matrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{matrix} \right] + \dots = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \dots$$

... ..
... ..

$$(1) \quad \dots \quad (1.1)$$

... ..

... ..

... ..
... ..

$$(2) \quad \dots$$

$$\dots = \dots$$

... ..

$$(3) \quad \dots \quad (3.1)$$

... ..

$$\dots = \dots$$

$$(4) \quad \dots \quad (4.1)$$

... ..

$$\dots = \dots$$

... ..

Equations (6.33) and (6.34) represent a system of first order ordinary differential equations under one point boundary conditions (6.35). The right hand sides of the equations of this system are of class C^1 under hypothesis 3), or of class C^1 under hypothesis 3)', in the variables λ , Q and T . Hence, in either case, the functions Q and T are uniquely determined in a neighborhood of $\lambda = 0$. If the x axis is characteristic, these functions must also satisfy

$$(6.36) \quad f_t(\lambda, 0; 0; 0, Q(\lambda); 0, T(\lambda)) = 0.$$

Similarly, for $\lambda = 0$:

$$x = 0, y = \mu, u = q = t = 0, p = P(\mu), r = R(\mu),$$

where, from (6.12),

$$(6.37) \quad P'(\mu) = f(\mu, \mu; 0; P(\mu), 0; R(\mu), 0),$$

while, from $X(\lambda, \mu) = 0$, since $\sigma = f_p = 0$, $\delta = 1$ and $\rho = -f_t$,

$$(6.38) \quad R'(\mu) = \left\{ [f]_x + f_t [f]_y \right\} (0, \mu; 0; P(\mu), 0; R(\mu), 0).$$

Moreover,

$$(6.39) \quad P(0) = R(0) = 0.$$

Hence, if the y axis is characteristic, the functions P and R , uniquely determined by (6.37), (6.38), and (6.39), must also satisfy

$$(6.40) \quad f_p(0, \mu; 0; P(\mu), 0; R(\mu), 0) = 0.$$

... the ... of ...
 ... the ... of ...
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$$x^2 + y^2 = z^2$$

... the ... of ...

$$x^2 + y^2 = z^2$$

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... the ... of ...

...

$$x^2 + y^2 = z^2$$

To recapitulate, the necessary condition that the x axis be a characteristic of some integral surface is that the functions Q and T determined from the system (6.33) and (6.34), under boundary conditions (6.35), shall satisfy (6.36) for each λ in a neighborhood of $\lambda = 0$. The necessary condition that the y axis be a characteristic of some integral surface is that the functions P and R determined from the system (6.37) and (6.38), under boundary conditions (6.39), shall satisfy (6.40) for each μ in a neighborhood of $\mu = 0$.

We now show that these conditions are also sufficient, i.e. given in the vicinity of the origin, an integral surface $J: u = u(x, y)$ of (6.12) passing through the coordinate axes, with

$$(6.41) \quad P_1(y) = u_x(0, y), \quad R_1(y) = u_{xx}(0, y), \quad Q_1(x) = u_y(x, 0), \\ \text{and } T_1(x) = u_{yy}(x, 0),$$

we show that the requirement

$$(6.40)' \quad f_p(0, y; 0; P_1(y), 0; R_1(y), 0) = 0$$

is sufficient that the y axis be a characteristic on J .

The argument needed to show that the requirement

$$(6.36)' \quad f_t(x, 0; 0; 0, Q_1(x); 0, T_1(x)) = 0$$

is sufficient in order that the x axis be a characteristic on J is analogous to the following and will not be given here.

We need show only that under requirement (6.40)', $P_1(y) = P(y)$ and $R_1(y) = R(y)$, where $P(y)$ and $R(y)$ are those functions obtained

The first part of the proof is devoted to showing that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n^{\text{cl}} \quad (1.1)$$

where Z_n and Z_n^{cl} are the partition functions of the system and of its classical counterpart, respectively. This is done by showing that the two systems are thermodynamically equivalent in the limit of large n .

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n^{\text{cl}} \quad (1.2)$$

and that the two systems are thermodynamically equivalent in the limit of large n . This is done by showing that the two systems are thermodynamically equivalent in the limit of large n .

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n^{\text{cl}} \quad (1.3)$$

and that the two systems are thermodynamically equivalent in the limit of large n . This is done by showing that the two systems are thermodynamically equivalent in the limit of large n .

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n^{\text{cl}} \quad (1.4)$$

and that the two systems are thermodynamically equivalent in the limit of large n . This is done by showing that the two systems are thermodynamically equivalent in the limit of large n .

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n^{\text{cl}} \quad (1.5)$$

and that the two systems are thermodynamically equivalent in the limit of large n . This is done by showing that the two systems are thermodynamically equivalent in the limit of large n .

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n^{\text{cl}} \quad (1.6)$$

and that the two systems are thermodynamically equivalent in the limit of large n . This is done by showing that the two systems are thermodynamically equivalent in the limit of large n .

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n^{\text{cl}} \quad (1.7)$$

and that the two systems are thermodynamically equivalent in the limit of large n . This is done by showing that the two systems are thermodynamically equivalent in the limit of large n .

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n^{\text{cl}} \quad (1.8)$$

and that the two systems are thermodynamically equivalent in the limit of large n . This is done by showing that the two systems are thermodynamically equivalent in the limit of large n .

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n^{\text{cl}} \quad (1.9)$$

previously under the assumption that the y -axis was "intrinsically characteristic".

Now $P_1(0) = R_1(0) = 0$ since $u(x,0) = 0$. Moreover, since u satisfies

$$(6.12) \quad s = f(x, y; u; p, q; r, t),$$

for $x = 0$,

$$(6.37)' \quad P_1'(y) = f(0, y; 0; P_1(y), 0; P_1(y), 0).$$

Now, recalling that $u \in C^{1,1}$,

$$(6.42) \quad s_x = f_r r_x + f_t t_x + [f]_x,$$

$$(6.43) \quad s_y = f_r r_y + f_t t_y + [f]_y.$$

Since $u(0, y) = 0$, we obtain $t_y(0, y) = 0$. Writing $r_x(0, y) = w(y)$ and substituting (6.43) into (6.42) with $x = 0$, we obtain

$$(6.44) \quad \begin{aligned} s_x(0, y) &= r_y(0, y) \\ &= f_r w(y) + f_t f_r r_y + [f]_x + f_t [f]_y \end{aligned}$$

But, $u(0, y) = u_y(0, y) = u_{yy}(0, y) = 0$, hence by (6.44),

$$(6.38)' \quad R_1'(y) = \left[\frac{1}{1-f_r f_t} \left\{ [f]_x + f_t [f]_y + f_r w(y) \right\} \right] (0, y; 0; P_1(y), 0; R_1(y), 0).$$

Now equation (6.37)' is precisely the same as (6.37), while requirement (6.40)' is sufficient to reduce (6.38)' to (6.38). But this implies that $P_1(y) = P(y)$ and $R_1(y) = R(y)$ since the solution of the system of ordinary differential equations in question is unique.

In the foregoing arguments we have developed a procedure for determining whether or not the initial curves are "intrinsically characteristic". By transformation (6.1) and substitution (6.5), we reduce the initial curves Γ_1 and Γ_2 to the coordinate axes. If now s_0 can be uniquely determined from (6.9) we may verify the hyperbolic condition and obtain the characteristic directions at the origin. If these directions coincide with the coordinate axes, then equation (1.1) can be solved explicitly for (6.12). From this, the system (6.37) and (6.38) under boundary condition (6.39) can, in principle at least, be solved for functions P and R. Finally if P and R satisfy (6.40) then the y axis is characteristic to any integral surface of the problem, i.e. "intrinsically characteristic". Likewise, from the system (6.33) and (6.34) under boundary condition (6.35), the functions Q and T can be determined. If these satisfy (6.36) then the x axis is "intrinsically characteristic". Note that P, R, Q and T are evidently of class C^1 .

Having given hypothesis 2) a precise meaning along with a procedure for determining whether or not it is verified for a given problem, we continue with the proof under the assumption that hypothesis 2) is verified.

From equations (6.17), (6.18), (6.27), (6.32) and the strip conditions we obtain the following system of characteristic equations, which must be satisfied along the characteristics on any integral surface J:

is the standard assumption in such cases a procedure for
 representing changes in the field curves and their
 characteristics of transformation (1941) and (1942) (1943)
 we have the field curves η_1 and η_2 in the following form:

$$\eta_1 = \frac{1}{2} \eta_2 \left(\frac{1}{2} \eta_2 \right) \quad (1941)$$
 and $\eta_2 = \frac{1}{2} \eta_1 \left(\frac{1}{2} \eta_1 \right)$ (1942) (1943) (1944)
 The curves η_1 and η_2 are related to the coordinate
 axes, their position (1941) and (1942) when boundary conditions
 are met, the curves (1941) and (1942) when boundary conditions
 (1943) are to be satisfied in detail, in detail for details 1 and
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 (2392) (2393) (2394) (2395) (2396) (2397) (2398) (2399) (2400)

$$\begin{aligned}
 \varphi_1 &= y_\lambda - \rho x_\lambda = 0 \\
 \varphi_2 &= r_\lambda \sigma^2 - t_\lambda + \frac{\rho}{1+\delta} \left\{ [f]_y - \sigma [f]_x \right\} x_\lambda = 0 \\
 \varphi_3 &= u_\lambda - p x_\lambda - q y_\lambda = 0 \\
 \varphi_4 &= p_\lambda - r x_\lambda - t y_\lambda = 0 \\
 \varphi_5 &= q_\lambda - f x_\lambda - t y_\lambda = 0
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \end{aligned}} \right\} \text{System A}$$

$$\begin{aligned}
 \psi_1 &= x_\mu - \sigma y_\mu = 0 \\
 \psi_2 &= r_\mu - \rho^2 t_\mu - \frac{\rho}{1+\delta} \left\{ [f]_x - \rho [f]_y \right\} y_\mu = 0 \\
 \psi_3 &= u_\mu - p x_\mu - q y_\mu = 0 \\
 \psi_4 &= p_\mu - r x_\mu - t y_\mu = 0 \\
 \psi_5 &= q_\mu - f x_\mu - t y_\mu = 0
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{aligned}} \right\} \text{System B}$$

We observe that System A of (6.45) is of canonical hyperbolic form in $x, y; u; p, q; r, t$ as functions of λ and μ . Since for Theorem 9, $F \in C'''$, while for Theorem 9a, $F \in C''$, the coefficients of all equations in (6.45) are functions of class C'' for Theorem 9, and of class C' for Theorem 9a. Moreover, the matrix of coefficients for System A is, after interchange of rows and columns,

$$\begin{aligned}
 (6.45) \quad & \begin{vmatrix} -\rho & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\sigma & 0 & 0 & 0 & 0 & 0 \\ * & 0 & \sigma^2 & -1 & 0 & 0 & 0 \\ 0 & * & 1 & -\rho^2 & 0 & 0 & 0 \\ * & * & 0 & 0 & 1 & 0 & 0 \\ * & * & 0 & 0 & 0 & 1 & 0 \\ * & * & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \\
 & = (1 - \rho\sigma) (\sigma^2 \rho^2 - 1) = \frac{-\rho \delta^2}{(1+\delta)^3}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Ans. } \int_0^1 \frac{1}{x^2} dx = \int_0^1 x^{-2} dx = \left[-x^{-1} \right]_0^1 = \left[-\frac{1}{x} \right]_0^1 \\
 & = \left(-\frac{1}{1} \right) - \left(-\frac{1}{0} \right) = -1 - \left(-\infty \right) = -1 + \infty = \infty
 \end{aligned}$$

The integral $\int_0^1 \frac{1}{x^2} dx$ is an improper integral because the integrand $\frac{1}{x^2}$ is not defined at $x=0$. The limit $\lim_{t \rightarrow 0^+} \int_t^1 \frac{1}{x^2} dx$ does not exist, so the integral diverges to ∞ .

1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0
0	0	0	0	0	0	1

$$\frac{1}{1+1} = \frac{1}{2} = 0.5$$

where the coefficients designated only by asterisks, *, do not contribute to the value of the determinant. But $\delta > 0$ everywhere on J in a neighborhood of the origin, hence the determinant (6.43) does not vanish thereon.

As to the initial conditions, we have, by hypothesis 1) of Theorems 6 and 6a for $\mu = 0$,

$$x = \lambda, \quad y = 0, \quad u = p = r = 0, \quad q = Q(\lambda), \quad t = T(\lambda),$$

and for $\lambda = 0$,

$$x = 0, \quad y = \mu, \quad u = q = t = 0, \quad p = P(\mu), \quad r = R(\mu)$$

where Q, T and P, R are determined from their respective systems and are of class C^1 . Moreover, for $\mu = 0$, by (6.36), $f_t = 0$.

Hence $\rho = 0$, $\delta = 1$, and $\sigma = -f_p$. This together with

$\gamma_\lambda = r_\lambda = u_\lambda = p_\lambda = 0$ and equation (6.34) prove that

$$(6.47) \quad \varphi_1(\lambda, 0) = \varphi_2(\lambda, 0) = \varphi_3(\lambda, 0) = \varphi_4(\lambda, 0) = \varphi_5(\lambda, 0) = 0$$

for all λ in a neighborhood of $\lambda = 0$. Similarly, for $\lambda = 0$,

by (6.40), $f_r = 0$. Hence $\sigma = 0$, $\delta = 1$ and $\rho = -f_t$. This together with

$x_\mu = t_\mu = u_\mu = q_\mu = 0$ and equation (6.38) prove that

$$(6.48) \quad \psi_1(0, \mu) = \psi_2(0, \mu) = \psi_3(0, \mu) = \psi_4(0, \mu) = \psi_5(0, \mu) = 0$$

for all μ in a neighborhood of $\mu = 0$. Thus the initial condition requirements of hypothesis 3) of Theorems 6 and 6a are satisfied.

Since the coefficients in (6.45) are of class C^1 for Theorem 6, hypotheses 1) and 2) of Theorem 6 are satisfied. Also, since the coefficients in (6.45) are of class C^1 for Theorem 6a, the

Let $\delta > 0$ be a constant to be determined later. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$.

Let $\delta > 0$ be a constant to be determined later. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$.

Let $\delta > 0$ be a constant to be determined later. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$.

Let $\delta > 0$ be a constant to be determined later. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$.

Let $\delta > 0$ be a constant to be determined later. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$.

Let $\delta > 0$ be a constant to be determined later. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$.

Let $\delta > 0$ be a constant to be determined later. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$. We choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta < \delta_1$.

common hypothesis 1) of Theorems 6 and 6a is satisfied, but hypothesis 2) of Theorem 6, a hypothesis which does not appear in Theorem 6a, is not satisfied. Thus if we now show that any solution of the canonical hyperbolic system, System A of (6.45), with the given characteristic initial conditions is also a solution of the corresponding problem for the equation

$$(6.12) \quad z = f(x, y; u; p, q; r, t)$$

with the same initial conditions, then Theorem 9 is an immediate consequence of Theorem 6 and Theorem 9a is an immediate consequence of Theorem 6a.

As in the Cauchy problem of Chapter 5, we show that for each solution of System A under the given characteristic initial conditions that System B is likewise satisfied. Note that here we cannot assume that p, q, r and t are derivatives of u ; this is a matter of proof. Recalling from Theorems 6 and 6a that the functions of the solution of System A, x, y, u, p, q, r, t are of class C^1 and that $f \in C^{1,1}$ under hypothesis 3) of Theorem 9, or $f \in C^1$ under hypothesis 3)' of Theorem 9a, we obtain by differentiation and consideration of (6.45) that

$$(6.49) \quad \begin{aligned} \psi_{3,\lambda} - \varphi_{2,\mu} &= p_{\mu} x_{\lambda} + q_{\mu} y_{\lambda} - p_{\lambda} x_{\mu} - q_{\lambda} y_{\mu} \\ &= \psi_{4x_{\lambda}} + \psi_{5y_{\lambda}} - \varphi_{4x_{\mu}} - \varphi_{5y_{\mu}}. \end{aligned}$$

Moreover, since $\varphi_3 = \varphi_4 = \varphi_5 = 0$,

$$(6.50) \quad \begin{aligned} f_{\lambda} &= f_x x_{\lambda} + f_t t_{\lambda} + f_p p_{\lambda} + f_q q_{\lambda} + f_u u_{\lambda} + f_x x_{\lambda} + f_y y_{\lambda} \\ &= f_x x_{\lambda} + f_t t_{\lambda} + [f]_x x_{\lambda} + [f]_y y_{\lambda}, \end{aligned}$$

while

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$$x^2 - 2x + 1 = (x-1)^2$$

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$$x^2 - 2x + 1 = (x-1)^2$$

$$x^2 - 2x + 1 = (x-1)^2$$

$$x^2 - 2x + 1 = (x-1)^2$$

$$\begin{aligned}
 (6.51) \quad f_{\mu} &= f_r x_{\mu} + f_t t_{\mu} + f_p p_{\mu} + f_q q_{\mu} + f_u u_{\mu} + f_x x_{\mu} + f_y y_{\mu} \\
 &= f_r x_{\mu} + f_t t_{\mu} + [f]_x x_{\mu} + [f]_y y_{\mu} \\
 &\quad + f_p \psi_4 + f_q \psi_5 + f_u \psi_3.
 \end{aligned}$$

Thus by (6.45), (6.50) and (6.51),

$$\begin{aligned}
 (6.52) \quad \psi_{4,\lambda} - \varphi_{4,\mu} &= f_{\mu} x_{\lambda} + f_{\mu} y_{\lambda} - f_{\lambda} x_{\mu} - f_{\lambda} y_{\mu} \\
 &= y_{\lambda} \{ f_p \psi_4 + f_q \psi_5 + f_u \psi_3 \} \\
 &\quad + \left(\frac{1+\delta}{2}\right) x_{\lambda} \psi_2 - \left(\frac{1+\delta}{2}\right) p y_{\mu} \varphi_2,
 \end{aligned}$$

and

$$\begin{aligned}
 (6.53) \quad \psi_{5,\lambda} - \varphi_{5,\mu} &= f_{\mu} x_{\lambda} + f_{\mu} y_{\lambda} - f_{\lambda} x_{\mu} - f_{\lambda} y_{\mu} \\
 &= x_{\lambda} \{ f_p \psi_4 + f_q \psi_5 + f_u \psi_3 \} \\
 &\quad - \left(\frac{1+\delta}{2}\right) \sigma x_{\lambda} \psi_2 + \left(\frac{1+\delta}{2}\right) y_{\mu} \varphi_2.
 \end{aligned}$$

Taking into account the fact that System A is satisfied, we reduce (6.49), (6.52) and (6.53) to the system

$$\begin{aligned}
 (6.54) \quad \psi_{3,\lambda} &= \psi_4 x_{\lambda} + \psi_5 y_{\lambda} \\
 \psi_{4,\lambda} &= y_{\lambda} \{ f_u \psi_3 + f_p \psi_4 + f_q \psi_5 \} \\
 \psi_{5,\lambda} &= x_{\lambda} \{ f_u \psi_3 + f_p \psi_4 + f_q \psi_5 \}
 \end{aligned}$$

For fixed μ , (6.54) represents a system of linear, homogeneous, first order ordinary differential equations for the functions ψ_3 , ψ_4 and ψ_5 of the variable λ . Moreover, by (6.43),

$$x^2 + 2x + 1 = (x+1)^2$$

$$x^2 + 2x + 1 = (x+1)^2$$

$$x^2 + 2x + 1 = (x+1)^2$$

(This is the form of the square)

$$x^2 + 2x + 1 = (x+1)^2$$

$$x^2 + 2x + 1 = (x+1)^2$$

$$x^2 + 2x + 1 = (x+1)^2$$

and

$$x^2 + 2x + 1 = (x+1)^2$$

$$x^2 + 2x + 1 = (x+1)^2$$

$$x^2 + 2x + 1 = (x+1)^2$$

Using this method we can solve a 2nd degree equation

and we can find the roots of the equation

$$x^2 + 2x + 1 = (x+1)^2$$

$$x^2 + 2x + 1 = (x+1)^2$$

$$x^2 + 2x + 1 = (x+1)^2$$

The first is a special case of the general one

because there are some special circumstances which are the same

as the general one (the method is the same)

the homogeneous one point boundary conditions

$$\psi_3(0, \mu) = \psi_4(0, \mu) = \psi_5(0, \mu) = 0$$

must be satisfied. Hence, the unique solution for the system (6.54) is

$$\psi_3 = \psi_4 = \psi_5 = 0$$

wherever the solution of system A is defined.

Consider the linear algebraic system,

$$(6.55) \quad \begin{cases} \psi_3 = u_\lambda - px_\lambda - qy_\lambda = 0 \\ \psi_3 = u_\mu - px_\mu - qy_\mu = 0. \end{cases}$$

The determinant of this system, by (6.20), does not vanish in a neighborhood of the origin, hence in this neighborhood there exists a unique solution for p and q . Since $p = u_x$ and $q = u_y$ satisfy (6.55) they are the solution of (6.55)

Similarly, from

$$(6.56) \quad \begin{cases} \psi_4 = p_\lambda - rx_\lambda - fy_\lambda \\ \psi_4 = p_\mu - rx_\mu - fy_\mu, \end{cases}$$

we obtain $r = u_{xx}$ and $f = u_{xy}$,

while from

$$(6.57) \quad \begin{cases} \psi_5 = q_\lambda - fx_\lambda - ty_\lambda \\ \psi_5 = q_\mu - fx_\mu - ty_\mu, \end{cases}$$

we obtain the additional information that $t = u_{yy}$. Consequently, any solution of system A under the given characteristic initial conditions satisfies the equation

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$$\frac{1}{2}(\sqrt{2} + \sqrt{2}) = \sqrt{2} = \sqrt{2}$$

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THE PROPERTIES OF THE...

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$$u_{xy} = f(x, y; u; u_x, u_y; u_{xx}, u_{yy})$$

in a neighborhood of the point $(0,0; 0; 0,0; 0,0)$ and the proof of Theorems 9 and 9a is now complete.

Let us designate the problem considered in Theorems 9 and 9a as Problem I. By virtue of the exposition of Chapter IV and this present chapter, we may associate to this problem a particular Problem II, of the type considered in Theorems 3 and 3a of Chapter II. As we have shown, any solution of I is a solution of II, and, conversely, any solution of II is a solution of I. Where for I, $P \in C'''$, Theorem 3 tells us that the solution of the related Problem II is unique. Hence, as is stated in Theorem 9, the solution for I is likewise unique. If, however, for Problem I, $P \in C''$ only, then Theorem 3a tells us merely that the related Problem II has at least one solution. Moreover, Example 1, Chapter II, tells us that this solution cannot be shown to be unique.

We must not conclude merely from the above that for $P \in C''$ the solution to Problem I cannot be shown to be unique. We can say, though, that any proof for uniqueness, if such can be made at all, will apparently have to be based upon arguments independent of those of this paper.

THE UNIVERSITY OF CHICAGO

IN THE DEPARTMENT OF CHEMISTRY
BY

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Chapter VII

The Mixed Boundary Value Problem

$$\text{for } u_{xy} = f(x, y; u; u_x, u_y).$$

In the terminology of J. HADAMARD [11], appendix II, p. 456, the mixed hyperbolic boundary value problem is one in which we prescribe the values of the integral surface along two lines issuing from a point, one of which is characteristic to the surface in question, while the other is nowhere characteristic.

J. HADAMARD, in the reference above, and E. PICARD [7], p.136, prove the existence of a unique solution to the linear equation

$$(7.1) \quad u_{xy} = a u_x + b u_y + c u,$$

a , b and c continuous functions of x and y alone, satisfying the initial conditions

$$(7.2) \quad u(x, 0) = u(x, x) = 0.$$

In Theorem 10, below, we extend their conclusions to the equation

$$(7.3) \quad u_{xy} = f(x, y; u; u_x, u_y)$$

maintaining initial conditions (7.2). The result is well known, but does not appear in the literature in the precise form stated. We require this precise statement because we wish to proceed from Theorem 10 by the methods of Chapters II and III in which we relax the Lipschitz condition on the function f to require merely

Section 11

The first condition is that

$$\frac{d^2x}{dt^2} = -\frac{g}{l}x \quad (1.1)$$

is the equation of a simple harmonic oscillator with angular frequency $\omega = \sqrt{g/l}$. The general solution of this equation is $x = A \cos(\omega t + \phi)$, where A and ϕ are constants determined by the initial conditions. The initial conditions are $x(0) = x_0$ and $\dot{x}(0) = v_0$. Substituting these into the general solution gives $x_0 = A \cos \phi$ and $v_0 = -A\omega \sin \phi$. Solving these two equations for A and ϕ gives $A = \sqrt{x_0^2 + (v_0/\omega)^2}$ and $\phi = \arctan(-v_0/\omega x_0)$. Therefore the solution of the differential equation is $x = \sqrt{x_0^2 + (v_0/\omega)^2} \cos(\omega t + \arctan(-v_0/\omega x_0))$.

$$\frac{d^2x}{dt^2} + \frac{g}{l}x = 0 \quad (1.2)$$

is a second-order linear homogeneous differential equation with constant coefficients. The characteristic equation is $\lambda^2 + g/l = 0$, which has roots $\lambda = \pm i\sqrt{g/l}$. Therefore the general solution is $x = C_1 \cos(\sqrt{g/l} t) + C_2 \sin(\sqrt{g/l} t)$.

$$x = C_1 \cos(\omega t) + C_2 \sin(\omega t) \quad (1.3)$$

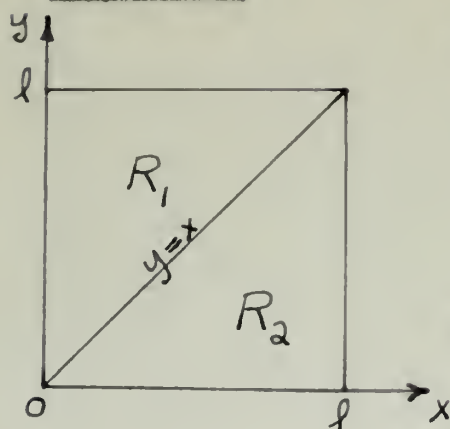
is the general solution. The initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$ give $x_0 = C_1$ and $v_0 = \omega C_2$. Therefore $C_1 = x_0$ and $C_2 = v_0/\omega$. The solution is $x = x_0 \cos(\omega t) + (v_0/\omega) \sin(\omega t)$.

$$x = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t) \quad (1.4)$$

is the solution of the differential equation. The period of the motion is $T = 2\pi/\omega = 2\pi\sqrt{l/g}$. The amplitude of the motion is $A = \sqrt{x_0^2 + (v_0/\omega)^2}$. The phase constant is $\phi = \arctan(-v_0/\omega x_0)$. The total energy of the oscillator is $E = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2 = \frac{1}{2}m\omega^2(x_0^2 + (v_0/\omega)^2) = \frac{1}{2}m\omega^2 A^2$.

that f be partially Lipschitzian. Thus we obtain the improved statement of Theorem 10a.

Theorem 10



$$1) f(x,y; u; p,q) \in C(B), B: \begin{cases} 0 \leq x \leq l \\ 0 \leq y \leq l \\ -a \leq u \leq a \\ -b \leq p \leq b \\ -b \leq q \leq b \end{cases}$$

2) f is Lipschitzian on B (as defined in Theorem 1.)

3) $M l^2 \leq a, M l \leq b$, where

$$M = \max |f| \text{ on } B$$

4) There exists one and only one function $u(x,y) \in C^1(B)$, $u_{xy}(x,y) \in C(B)$, where $B: \begin{cases} 0 \leq x \leq l \\ 0 \leq y \leq l \end{cases}$, such that for each

$(x,y) \in B$, the point $(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in B$, and

$$u_{xy}(x,y) = f(x,y; u(x,y); u_x(x,y), u_y(x,y)),$$

$$u(x,0) = u(x,x) = 0 \quad \text{for each } (x,y) \in B.$$

Proof

This proof is based upon FIGARD's variation of the method of successive approximations, [1] p. 359 or [7] p. 117. Here the uniform convergence of the approximating functions to the solution is verified by means of a majorant series. The majorant series used is that obtained from the approximating functions converging uniformly to the solution for the particular linear equation

... ..

$$\begin{aligned} \lambda &= 1 = 0 \\ \lambda &= 1 = 0 \\ \lambda &= 1 = 0 \\ \lambda &= 1 = 0 \\ \lambda &= 1 = 0 \end{aligned}$$

$$f(x,y) = \dots$$



... ..

... ..

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Example

... ..

$$(7.4) \quad u_{xy} = K (u + u_x + u_y)$$

with the same initial conditions. K is the Lipschitz constant for the function f of (7.3). PICARD applied this technique to the characteristic initial value problem, obtaining Theorem 1 of Chapter II. He thus obtained the theorem for the characteristic initial value problem for the non-linear equation (7.3) from the theorem for the characteristic initial value problem for the linear equation (7.1).

For the mixed boundary value problem under consideration, a curious situation arises. We do not obtain a majorant series from equation (7.4) under mixed initial conditions. However, we do find that PICARD's majorant series for the characteristic initial value problem serves as well for this problem. Thus Theorem 10 follows not from the theorem for the mixed boundary value problem for the linear equation (7.1) but from the theorem for the characteristic initial value problem for equation (7.1).

It is sufficient, as we shall demonstrate later, to show existence of a unique solution in region $R_2: \begin{cases} 0 \leq x \leq l \\ 0 \leq y \leq x \end{cases}$. Assuming $(x, y) \in R_2$, we may express the problem as the integral equation

$$(7.5) \quad u(x, y) = \int_y^x d\xi \int_0^y f(\xi, \eta; u; u_x, u_y) d\eta.$$

By differentiation,

$$(7.6) \quad u_x(x, y) = \int_0^y f(x, \eta; u; u_x, u_y) d\eta,$$

and

$$\int_{-\infty}^{\infty} \delta(x) dx = 1 \quad (1.1)$$

The Dirac delta function is a distribution that is zero everywhere except at the origin, where it is infinite. It is defined by the property that its integral over the entire real line is equal to one. This function is used to model point charges, impulses, and other phenomena that are localized in space or time.

The Dirac delta function is a linear functional on the space of test functions. It is defined by the equation $\int_{-\infty}^{\infty} \delta(x) \phi(x) dx = \phi(0)$, where $\phi(x)$ is a test function. This property is used to derive the sifting property of the delta function, which states that $\int_{-\infty}^{\infty} \delta(x-a) \phi(x) dx = \phi(a)$.

The Dirac delta function is also used to define the Dirac delta distribution, which is a linear functional on the space of test functions. It is defined by the equation $\int_{-\infty}^{\infty} \delta(x-a) \phi(x) dx = \phi(a)$. This distribution is used to model point charges, impulses, and other phenomena that are localized in space or time.

$$\int_{-\infty}^{\infty} \delta(x-a) \phi(x) dx = \phi(a) \quad (1.2)$$

This equation shows that the Dirac delta function acts as a sifting operator, picking out the value of the test function at the point $x=a$.

$$\int_{-\infty}^{\infty} \delta(x-a) \phi(x) dx = \phi(a) \quad (1.3)$$

This equation is a restatement of the sifting property of the Dirac delta function.

$$(7.7) \quad u_y(x, y) = \int_y^x f(\xi, y; u; u_x, u_y) d\xi - \int_0^y f(y, \eta; u; u_x, u_y) d\eta.$$

We form the successive approximations

$$(7.8) \quad \left\{ \begin{array}{l} u_1(x, y) = \int_y^x d\xi \int_0^y f(\xi, \eta; 0; 0, 0) d\eta \\ u_2(x, y) = \int_y^x d\xi \int_0^y f(\xi, \eta; u_1; u_{1,x}, u_{1,y}) d\eta \\ \vdots \\ u_n(x, y) = \int_y^x d\xi \int_0^y f(\xi, \eta; u_{n-1}; u_{n-1,x}, u_{n-1,y}) d\eta \\ \vdots \end{array} \right.$$

where, by differentiation,

$$(7.9) \quad u_{n,x}(x, y) = \int_0^y f(x, \eta; u_{n-1}; u_{n-1,x}, u_{n-1,y}) d\eta, \\ (n = 1, 2, \dots).$$

$$(7.10) \quad u_{n,y}(x, y) = \int_y^x f(\xi, y; u_{n-1}; u_{n-1,x}, u_{n-1,y}) d\xi \\ - \int_0^y f(y, \eta; u_{n-1}; u_{n-1,x}, u_{n-1,y}) d\eta, \\ (n = 1, 2, \dots).$$

Since the point $(x, y; 0; 0, 0) \in B$ for $(x, y) \in R_2$, by hypothesis 3),

$$|u_1(x, y)| \leq M |x-y| \cdot |y| \leq M l^2 \leq a,$$

$$|u_{1,x}(x, y)| \leq M |y| \leq M l \leq b,$$

$$|u_{1,y}(x, y)| \leq M \{|x-y| + |y|\} \\ = M|x| \leq M l \leq b$$

Thus, by induction, for all n and for any $(x, y) \in R_2$

$$(7.11) \quad \left\{ \begin{array}{l} |u_n(x, y)| \leq M l^2 \leq a, \\ |u_{n,x}(x, y)| \leq M l \leq b, \\ |u_{n,y}(x, y)| \leq M l \leq b. \end{array} \right.$$

Our purpose is to show that on R_2

$$(7.12) \quad \{u_n\} \xrightarrow{\text{unif}} u, \quad \{u_{n,x}\} \xrightarrow{\text{unif}} u_x \text{ and } \{u_{n,y}\} \xrightarrow{\text{unif}} u_y$$

such that the function u and its derivatives satisfy conclusion 4) for $(x,y) \in R_2$. To accomplish this we consider the successive approximations

$$(7.13) \quad \begin{aligned} w_1(x,y) &= \int_0^x d\xi \int_0^y M d\eta \\ w_2(x,y) &= \int_0^x d\xi \int_0^y K(w_1 + w_{1,x} + w_{1,y}) d\eta \\ &\vdots \\ w_n(x,y) &= \int_0^x d\xi \int_0^y K(w_{n-1} + w_{n-1,x} + w_{n-1,y}) d\eta \\ &\vdots \end{aligned}$$

where, by differentiation,

$$(7.14) \quad w_{n,x}(x,y) = \int_0^y K [w_{n-1} + w_{n-1,x} + w_{n-1,y}] (x, \eta) d\eta, \quad (n = 1, 2, \dots),$$

$$(7.15) \quad w_{n,y}(x,y) = \int_0^x K [w_{n-1} + w_{n-1,x} + w_{n-1,y}] (\xi, y) d\xi, \quad (n = 1, 2, \dots).$$

Here $M = \max |f|$ on E while K is the Lipschitz constant of hypothesis 2).

Now $w_1(x,y) = Kxy$, hence $w_1(x,y) = w_1(y,x)$. Moreover, $w_{1,x}(x,y) = Ky$, $w_{1,y}(x,y) = Kx$, hence $w_{1,x}(x,y) = w_{1,y}(y,x)$.

Let us make the inductive hypothesis that for some fixed positive integer n ,

$$(7.16) \quad w_n(x,y) = w_n(y,x), \quad w_{n,x}(x,y) = w_{n,y}(y,x).$$

The function is in the form of

$$y = \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} \quad (1)$$

Let us find the derivative of the function with respect to x.

$$\frac{d}{dx} \left(x^{-2} + x^{-3} + x^{-4} \right) = -2x^{-3} - 3x^{-4} - 4x^{-5}$$

$$= -\frac{2}{x^3} - \frac{3}{x^4} - \frac{4}{x^5} \quad (2)$$

Therefore, the derivative is

$$\frac{d}{dx} \left(\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} \right) = -\frac{2}{x^3} - \frac{3}{x^4} - \frac{4}{x^5} \quad (3)$$

$$\frac{d}{dx} \left(x^{-2} + x^{-3} + x^{-4} \right) = -2x^{-3} - 3x^{-4} - 4x^{-5} \quad (4)$$

The derivative of the function is

$$-2x^{-3} - 3x^{-4} - 4x^{-5}$$

$$= -\frac{2}{x^3} - \frac{3}{x^4} - \frac{4}{x^5}$$

Thus, the derivative of the function is

$$-\frac{2}{x^3} - \frac{3}{x^4} - \frac{4}{x^5} \quad (5)$$

But this implies that

$$(7.17) \quad [w_n + w_{n,x} + w_{n,y}](x,y) = [w_n + w_{n,x} + w_{n,y}](y,x)$$

and thus, by (7.13),

$$w_{n+1}(x,y) = w_{n+1}(y,x).$$

Also, by (7.14) and (7.15), (7.17) implies that

$$\begin{aligned} w_{n+1,x}(x,y) &= \int_0^y K [w_n + w_{n,x} + w_{n,y}](x,\eta) d\eta \\ &= \int_0^y K [w_n + w_{n,x} + w_{n,y}](\xi,x) d\xi \\ &= w_{n+1,y}(y,x). \end{aligned}$$

Hence, by induction, (7.16) holds for $n = 1, 2, \dots$.

PICARD, in the references quoted above, shows that

$$(7.18) \quad \sum_{n=1}^{\infty} w_n = w, \quad \sum_{n=1}^{\infty} w_{n,x} = w_x, \quad \sum_{n=1}^{\infty} w_{n,y} = w_y,$$

each uniformly convergent on R , where the function w and its derivatives satisfy

$$(7.19) \quad \begin{cases} w_{xy} = K(w + w_x + w_y), \\ w(x,0) = w(0,y) = 0. \end{cases}$$

We now show that these series are majorant to the series

$$(7.20) \quad \sum_{n=1}^{\infty} (u_n - u_{n-1}), \quad \sum_{n=1}^{\infty} (u_{n,x} - u_{n-1,x}), \quad \sum_{n=1}^{\infty} (u_{n,y} - u_{n-1,y}),$$

respectively, for each $(x,y) \in R_2$, (with $u_0 = 0$).

Now, for $(x,y) \in R_2$,

$$\begin{aligned} |u_1(x,y)| &\leq \int_0^x d\xi \int_0^y |f(\xi,\eta; 0; 0,0)| d\eta \leq \int_0^x d\xi \int_0^y M d\eta = w_1(x,y) \\ |u_{1,x}(x,y)| &\leq \int_0^y |f(x,\eta; 0; 0,0)| d\eta \leq \int_0^y M d\eta = w_{1,x}(x,y) \end{aligned}$$

Let $x = \cos \theta$

$$T_n(x) = \cos(n\theta) = \cos(n \arccos x)$$

$$T_0(x) = 1$$

Let $T_1(x) = x$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

Let $T_5(x) = 16x^5 - 20x^3 + 5x$

Let $T_6(x) = 32x^6 - 48x^4 + 24x^2 - 1$

$$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$

Let $T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$

Let $T_9(x) = 256x^9 - 576x^7 + 432x^5 - 144x^3 + 16x$

$$T_{10}(x) = 512x^{10} - 1280x^8 + 1152x^6 - 512x^4 + 80x^2 - 1$$

Let $T_{11}(x) = 1024x^{11} - 2816x^9 + 2816x^7 - 1536x^5 + 448x^3 - 32x$

$$T_{12}(x) = 2048x^{12} - 6144x^{10} + 6912x^8 - 4096x^6 + 1344x^4 - 192x^2 + 1$$

Let $T_{13}(x) = 4096x^{13} - 13824x^{11} + 17920x^9 - 11264x^7 + 4736x^5 - 896x^3 + 64x$

Let $T_{14}(x) = 8192x^{14} - 28672x^{12} + 42240x^{10} - 32768x^8 + 17920x^6 - 5376x^4 + 640x^2 - 1$

$$T_{15}(x) = 16384x^{15} - 57344x^{13} + 82944x^{11} - 57344x^9 + 28672x^7 - 10240x^5 + 1792x^3 - 128x$$

$$T_{16}(x) = 32768x^{16} - 117760x^{14} + 184320x^{12} - 122880x^{10} + 57344x^8 - 20480x^6 + 4480x^4 - 320x^2 + 1$$

$$\begin{aligned}
|u_{1,y}(x,y)| &\leq \int_y^x |f(\xi,y;0;0,0)| d\xi + \int_0^y |f(y,\eta;0;0,0)| d\eta \\
&\leq \int_y^x M d\xi + \int_0^y M d\eta \\
&= \int_0^x M d\xi = w_{1,y}(x,y).
\end{aligned}$$

Also, abbreviating our notation somewhat,

$$\begin{aligned}
|u_2 - u_1| &\leq \int_y^x d\xi \int_0^y |f(\xi,\eta;u_1;u_{1,x},u_{1,y}) \\
&\quad - f(\xi,\eta;0;0,0)| d\eta \\
&\leq \int_y^x d\xi \int_0^y K [|u_1| + |u_{1,x}| + |u_{1,y}|] (\xi,\eta) d\eta \\
&\leq \int_0^x d\xi \int_0^y K [w_1 + w_{1,x} + w_{1,y}] (\xi,\eta) d\eta
\end{aligned}$$

= w_2 ,

$$|u_{2,x} - u_{1,x}| \leq \int_0^y K [w_1 + w_{1,x} + w_{1,y}] (x,\eta) d\eta = w_{2,x}$$

$$|u_{2,y} - u_{1,y}| \leq \int_y^x K [w_1 + w_{1,x} + w_{1,y}] (\xi,y) d\xi$$

$$+ \int_0^y K [w_1 + w_{1,x} + w_{1,y}] (y,\eta) d\eta$$

$$= \int_y^x K [w_1 + w_{1,x} + w_{1,y}] (\xi,y) d\xi$$

$$+ \int_0^y K [w_1 + w_{1,x} + w_{1,y}] (\xi,y) d\xi$$

$$= \int_0^x K [w_1 + w_{1,x} + w_{1,y}] (\xi,y) d\xi$$

$$= w_{2,y}.$$

Hence, by induction, we obtain for $n = 1, 2, \dots$

$$|u_n - u_{n-1}| \leq w_n, \quad |u_{n,x} - u_{n-1,x}| \leq w_{n,x},$$

$$(7.21) \quad |u_{n,y} - u_{n-1,y}| \leq w_{n,y} \quad \text{for each } (x,y) \in R_2.$$

Thus the series of (7.18) are majorant to the corresponding series of (7.20). Moreover, the requirements for termwise differentiation of an infinite sum are satisfied since each of the series of (7.20) is now known to be uniformly convergent on R_2 . Hence, for $(x, y) \in R_2$,

$$(7.22) \quad \begin{cases} \sum_{n=1}^{\infty} (u_n - u_{n-1}) = u \\ \sum_{n=1}^{\infty} (u_{n,x} - u_{n-1,x}) = u_x \\ \sum_{n=1}^{\infty} (u_{n,y} - u_{n-1,y}) = u_y \end{cases}$$

or, in other terms, since each of these series telescopes,

$$(7.22) : \{u_n\} \xrightarrow{\text{unif}} u, \quad \{u_{n,x}\} \xrightarrow{\text{unif}} u_x, \quad \{u_{n,y}\} \xrightarrow{\text{unif}} u_y$$

on R_2 .

We now verify that the function u and its derivatives u_x and u_y satisfy the integral equation statement of the problem (7.3):

$$(7.23) \quad \begin{aligned} & \left| u(x, y) - \int_y^x d\xi \int_0^y f(\xi, \eta; u; u_x, u_y) d\eta \right| \\ & \leq |u(x, y) - u_n(x, y)| + \int_y^x d\xi \int_0^y |f(\xi, \eta; u; u_x, u_y) \\ & \quad - f(\xi, \eta; u_{n-1}; u_{n-1,x}, u_{n-1,y})| d\eta \\ & \leq |u(x, y) - u_n(x, y)| \\ & \quad + \int_y^x d\xi \int_0^y K [|u - u_{n-1}| + |u_x - u_{n-1,x}| + |u_y - \\ & \quad u_{n-1,y}|] (\xi, \eta) d\eta \end{aligned}$$

... the ... of ... and ... of ... in ... of ...

$$\left. \begin{aligned}
 \frac{d}{dt} \left(\frac{1}{2} m v^2 \right) &= \dots \\
 \frac{d}{dt} \left(\frac{1}{2} m v^2 \right) &= \dots \\
 \frac{d}{dt} \left(\frac{1}{2} m v^2 \right) &= \dots
 \end{aligned} \right\} \text{Equation 1}$$

... the ... of ... and ... of ... in ... of ...

$$\left[\frac{d}{dt} \left(\frac{1}{2} m v^2 \right) \right] = \dots$$

... the ... of ... and ... of ... in ... of ...

$$\left[\frac{d}{dt} \left(\frac{1}{2} m v^2 \right) \right] = \dots$$

$$\left[\frac{d}{dt} \left(\frac{1}{2} m v^2 \right) \right] = \dots$$

$$\left[\frac{d}{dt} \left(\frac{1}{2} m v^2 \right) \right] = \dots \tag{Equation 2}$$

$$\left[\frac{d}{dt} \left(\frac{1}{2} m v^2 \right) \right] = \dots$$

... the ... of ... and ... of ... in ... of ...

Thus, by (7.22)', given $\epsilon > 0$, there exists a positive integer N , depending on ϵ alone, such that $n > N \Rightarrow$

$$|u(x,y) - \int_y^x d\xi \int_0^y f(\xi, \eta; u; u_x, u_y) d\eta| < \epsilon(1+3K\epsilon^2),$$

for $(x,y) \in R_2$. But ϵ is arbitrary, hence the integral equation is satisfied.

By (7.11) and (7.22)' we see that for any $(x,y) \in R_2$, the point $(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in B$. Thus existence of a solution on R_2 is now proved.

To prove uniqueness, let us suppose that u_1 and u_2 are two solutions on R_2 , then

$$(7.24) \quad |u_1(x,y) - u_2(x,y)| \leq \int_y^x d\xi \int_0^y |f(\xi, \eta; u_1; u_{1,x}, u_{1,y}) - f(\xi, \eta; u_2; u_{2,x}, u_{2,y})| d\eta \\ \leq \int_y^x d\xi \int_0^y K[|u_1 - u_2| + |u_{1,x} - u_{2,x}| + |u_{1,y} - u_{2,y}|] (\xi, \eta) d\eta,$$

$$(7.25) \quad |u_{1,x}(x,y) - u_{2,x}(x,y)| \leq \int_0^y |f(x, \eta; u_1; u_{1,x}, u_{1,y}) - f(x, \eta; u_2; u_{2,x}, u_{2,y})| d\eta \\ \leq \int_0^y K[|u_1 - u_2| + |u_{1,x} - u_{2,x}| + |u_{1,y} - u_{2,y}|](x, \eta) d\eta,$$

$$(7.26) \quad |u_{1,y}(x,y) - u_{2,y}(x,y)| \leq \int_y^x |f(\xi, y; u_1; u_{1,x}, u_{1,y}) - f(\xi, y; u_2; u_{2,x}, u_{2,y})| d\xi \\ + \int_0^y |f(y, \eta; u_1; u_{1,x}, u_{1,y}) - f(y, \eta; u_2; u_{2,x}, u_{2,y})| d\eta.$$

Let \mathcal{L} be the Lie algebra of G . Then \mathcal{L} is a vector space over \mathbb{R} .

Let $X, Y \in \mathcal{L}$. Then $[X, Y] \in \mathcal{L}$.

$$[X, Y] = \lim_{t \rightarrow 0} \frac{1}{t} (e^{-tY} e^{tX} e^{tY} e^{-tX} - 1)$$

The map $\exp: \mathcal{L} \rightarrow G$ is a diffeomorphism near the identity element.

Let $X \in \mathcal{L}$. Then $\frac{d}{dt} \exp(tX) = X \exp(tX)$. To compute $\frac{d}{dt} \exp(tX)$, we use the definition of the exponential map.

Let $X \in \mathcal{L}$. Then $\frac{d}{dt} \exp(tX) = X \exp(tX)$.

$$\frac{d}{dt} \exp(tX) = \lim_{h \rightarrow 0} \frac{\exp(tX + hX) - \exp(tX)}{h} = X \exp(tX)$$

$$\frac{d}{dt} \exp(tX) = X \exp(tX)$$

$$\frac{d}{dt} \exp(tX) = X \exp(tX)$$

$$\frac{d}{dt} \exp(tX) = X \exp(tX)$$

$$\frac{d}{dt} \exp(tX) = X \exp(tX)$$

$$\frac{d}{dt} \exp(tX) = X \exp(tX)$$

Let $\Psi(x, y) = (|u_1 - u_2| + |u_{1,x} - u_{2,x}| + |u_{1,y} - u_{2,y}|)(x, y)$.

With $R^* = \begin{cases} 0 \leq x \leq l^* \\ 0 \leq y \leq x \end{cases}$, $l^* = \min(l, l, \frac{1}{6K})$, we have

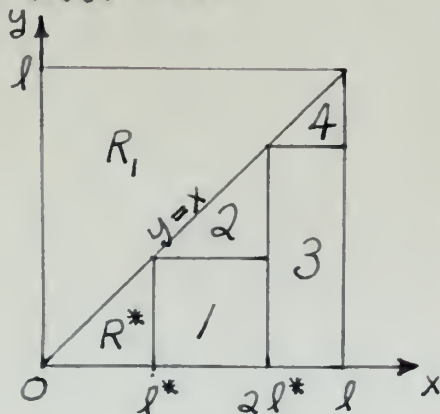
$\Psi(x, y) \in C(R^*)$. Moreover, there exists a point $(x^*, y^*) \in R^*$ such that $\Psi(x^*, y^*) = \mu$ where $\mu = \max \Psi(x, y)$ on R^* . But, adding (7.24), (7.25) and (7.26) we obtain

$$\begin{aligned} \Psi(x, y) &\leq K \mu \{ (x-y)y + y + (x-y) + y \} \\ &\leq K \mu (xy + x + y) \\ &\leq K \mu \cdot \frac{3}{6K} = \frac{\mu}{2}, \end{aligned}$$

hence $\Psi(x^*, y^*) = \mu \leq \frac{\mu}{2}$, which implies $\mu = 0$ and thus

$$(7.27) \quad u_1(x, y) = u_2(x, y)$$

for $(x, y) \in R^*$



To extend this uniqueness proof to the domain R_2 , we subdivide R_2 as shown in the diagram. We know that the solution u is unique on R^* and hence determines $u(l^*, y)$ for $0 \leq y \leq l^*$.

But $u(x, 0) = 0$ by hypothesis, consequently, by Theorem 1, Chapter II, we have a unique solution u_1 to the characteristic initial value problem on sub-region 1. Since $u_x(l^*, 0) = u_{1,x}(l^*, 0)$, we have from the differential equation that $u_x(l^*, y) = u_{1,x}(l^*, y)$ for $0 \leq y \leq l^*$, i.e. u and u_1 have a first order contact across the line $x = l^*$ and hence together represent a unique solution for the region $R^* + 1$. Analogously, by the preceding "in the

Let $f(x) = \sqrt{x}$ and $g(x) = x^2$. Then $f(x) = g(x)$ when $\sqrt{x} = x^2$. Squaring both sides, we get $x = x^4$, which implies $x^4 - x = 0$. Factoring, we have $x(x^3 - 1) = 0$. This gives $x = 0$ or $x^3 = 1$. The real solution is $x = 1$. Thus, the curves intersect at $(0, 0)$ and $(1, 1)$.

To find the area between the curves from $x = 0$ to $x = 1$, we integrate the absolute difference of the functions. Since $x^2 \geq \sqrt{x}$ for $x > 1$ and $\sqrt{x} \geq x^2$ for $0 < x < 1$, the area is given by $\int_0^1 (\sqrt{x} - x^2) dx$.

$$\int_0^1 (\sqrt{x} - x^2) dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Therefore, the area between the curves $y = \sqrt{x}$ and $y = x^2$ from $x = 0$ to $x = 1$ is $\frac{1}{3}$.

$$f(x) = \sqrt{x}, g(x) = x^2$$

For $x > 1$, $x^2 > \sqrt{x}$

The area between the curves is found by integrating the absolute difference of the functions. The curves intersect at $(0, 0)$ and $(1, 1)$. For $0 < x < 1$, $\sqrt{x} > x^2$, and for $x > 1$, $x^2 > \sqrt{x}$. The area is $\int_0^1 (\sqrt{x} - x^2) dx + \int_1^{\infty} (x^2 - \sqrt{x}) dx$.



The area between the curves $y = \sqrt{x}$ and $y = x^2$ from $x = 0$ to $x = 1$ is $\frac{1}{3}$. For $x > 1$, the area between the curves is $\int_1^{\infty} (x^2 - \sqrt{x}) dx$, which diverges to infinity. Thus, the total area between the curves is infinite.

small" uniqueness proof for the mixed boundary value problem, the solution u_2 is unique in sub-region 2 and has a first order contact with u_1 across the line $y = x$. We continue obtaining unique solutions for characteristic initial value and mixed initial value problems, alternatively as indicated by the numerical sequence in the diagram. These solutions have first order contacts with each other across the characteristics forming the boundaries of the sub-regions, hence we have extended our uniqueness proof from the region R_1 to the region R_2 .

Having thus determined the existence of a unique solution satisfying conclusion 4) throughout R_2 , we now consider the Cauchy problem for region R_1 with the same equation and hypotheses thereon and with the initial conditions

$$(7.28) \begin{cases} u^0(x, x) = 0, \quad u_x^0(x, x) = u_{x+}(x, x), \text{ and} \\ u_y^0(x, x) = u_{y-}(x, x) \quad \text{for } x \in [0, \ell]. \end{cases}$$

In (7.28) u_{x+} and u_{y-} are the right-hand x and lower y derivatives, respectively, determined at each point of the line $y = x$ by the known solution u on R_2 . By Theorem 4, Chapter III, there exists a unique solution u^0 to this Cauchy problem for each $(x, y) \in R_1$, hence

$$u_1(x, y) = \begin{cases} u_0(x, y) & \text{for } (x, y) \in R_1 \\ u(x, y) & \text{for } (x, y) \in R_2 \end{cases}$$

is the unique solution valid for each $(x, y) \in R = R_1 + R_2$, since u_0 and u have, by prescription, a first order contact across the line $y = x$. This completes the proof of Theorem 10.

Let \mathcal{A} be a subalgebra of \mathcal{B} . Then \mathcal{A} is a σ -algebra if and only if it is closed under countable unions and complements. In other words, if $\{A_n\}_{n \in \mathbb{N}}$ is a sequence of sets in \mathcal{A} , then $\bigcup_{n \in \mathbb{N}} A_n$ and A^c are also in \mathcal{A} . This property is essential for the definition of a probability measure on \mathcal{A} .

$$P(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} P(A_n)$$

Given two σ -algebras \mathcal{A} and \mathcal{B} , their intersection $\mathcal{A} \cap \mathcal{B}$ is also a σ -algebra. Similarly, the union $\mathcal{A} \cup \mathcal{B}$ is a σ -algebra if and only if \mathcal{A} and \mathcal{B} are independent. This concept is crucial in understanding the relationship between different sources of information in a probability space.

$$\left. \begin{aligned} P(A \cap B) &= P(A)P(B) \\ P(A \cup B) &= P(A) + P(B) - P(A \cap B) \end{aligned} \right\} \text{Independence}$$

Let \mathcal{A} and \mathcal{B} be independent σ -algebras. Then for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the events A and B are independent. This means that the occurrence of one event does not affect the probability of the other. This property is fundamental in the study of random variables and stochastic processes.

$$P(A \cap B) = P(A)P(B) \quad \text{if } A \in \mathcal{A} \text{ and } B \in \mathcal{B}$$

The concept of independence is closely related to the notion of a product probability space. If $(\Omega_1, \mathcal{A}_1, P_1)$ and $(\Omega_2, \mathcal{A}_2, P_2)$ are two probability spaces, then their product space $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, P_1 \otimes P_2)$ is a natural way to combine them. This construction is used extensively in the theory of multivariate random variables.

Relaxing only hypothesis 2) of Theorem 10, we obtain the following improvement:

Theorem 10a

- 1)
- 2)' f is partially Lipschitzian on B (as defined in Theorem 1a.)
- 3)
- ⇒ 4)' There exists at least one function, etc. (as in Theorem 10.)

Outline of the proof:

As in the proof of Theorem 10, we may, without loss, prove existence on R_2 only. For, prescribing Cauchy conditions on $y = x$ as before, we may extend the solution from R_2 to R_1 , by use of Theorem 4a, Chapter III.

In this proof we follow very closely the derivation of Theorem 1a, Chapter II; hence only the differences between the two proofs will be noted.

WEIERSTRASS' theorem tells us that there exists a sequence of polynomials, $\{g_\lambda\}$, converging uniformly to f on B . We extend the g_λ , ($\lambda = 1, 2, \dots$), and f from B to

$$B': \begin{cases} 0 \leq x \leq 1 \\ -\infty < u < \infty \\ -\infty < p < \infty \\ -\infty < q < \infty \end{cases}$$

by definitions analogous to (2.1). There

exists a constant $L > 0$ such that $|g_\lambda| \leq L$ in B' and for all λ . More-

Let \mathcal{A} be a collection of subsets of X .

Define $\sigma(\mathcal{A})$ to be the σ -algebra generated by \mathcal{A} .

Lemma 1.1

- (a) $\mathcal{A} \subseteq \sigma(\mathcal{A})$.
- (b) $\sigma(\mathcal{A})$ is the smallest σ -algebra containing \mathcal{A} .
- (c) If \mathcal{A} is a π -system, then $\sigma(\mathcal{A})$ is a σ -algebra.
- (d) If \mathcal{A} is a λ -system, then $\sigma(\mathcal{A})$ is a λ -system.

Proof of Lemma 1.1

(a) Let $A \in \mathcal{A}$. Then $A \in \sigma(\mathcal{A})$ because $\sigma(\mathcal{A})$ is the smallest σ -algebra containing \mathcal{A} .

(b) Let \mathcal{B} be a σ -algebra containing \mathcal{A} . Then $\sigma(\mathcal{A}) \subseteq \mathcal{B}$ because $\sigma(\mathcal{A})$ is the smallest σ -algebra containing \mathcal{A} .

(c) Let \mathcal{A} be a π -system. Then $\sigma(\mathcal{A})$ is a σ -algebra because \mathcal{A} is a π -system and $\sigma(\mathcal{A})$ is the smallest σ -algebra containing \mathcal{A} .

(d) Let \mathcal{A} be a λ -system. Then $\sigma(\mathcal{A})$ is a λ -system because \mathcal{A} is a λ -system and $\sigma(\mathcal{A})$ is the smallest σ -algebra containing \mathcal{A} .

$$\begin{aligned} \sigma(\mathcal{A}) &= \bigcap \{ \mathcal{B} : \mathcal{A} \subseteq \mathcal{B}, \mathcal{B} \text{ is a } \sigma\text{-algebra} \} \\ &= \bigcap \{ \mathcal{B} : \mathcal{A} \subseteq \mathcal{B}, \mathcal{B} \text{ is a } \lambda\text{-system} \} \end{aligned}$$

Let \mathcal{A} be a π -system. Then $\sigma(\mathcal{A})$ is a σ -algebra.

Let \mathcal{A} be a λ -system. Then $\sigma(\mathcal{A})$ is a λ -system.

over, the g_λ are "fully" Lipschitzian in B' . Hence by Theorem 10, (with $a \rightarrow \infty$, $b \rightarrow \infty$), for each g_λ there exists a unique function u_λ such that for $(x, y) \in R_2$

$$(7.29) \quad u_\lambda = \int_y^x d\xi \int_0^y g_\lambda(\xi, \eta; u_\lambda; u_{\lambda, x}, u_{\lambda, y}) d\eta,$$

and thus

$$(7.30) \quad u_{\lambda, x} = \int_0^y g_\lambda(x, \eta; u_\lambda; u_{\lambda, x}, u_{\lambda, y}) d\eta,$$

$$(7.31) \quad u_{\lambda, y} = \int_y^x g_\lambda(\xi, y; u_\lambda; u_{\lambda, x}, u_{\lambda, y}) d\xi \\ - \int_0^y g_\lambda(y, \eta; u_\lambda; u_{\lambda, x}, u_{\lambda, y}) d\eta.$$

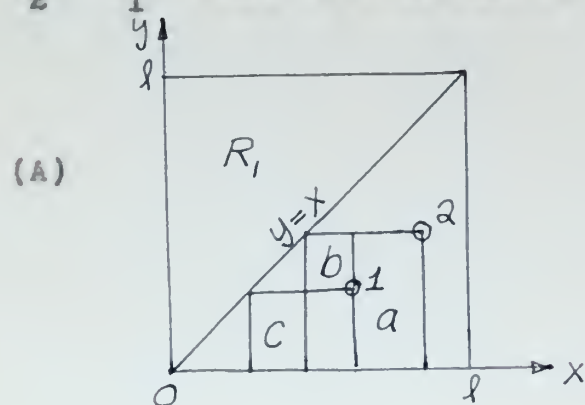
For $(x, y) \in R_2$, by (7.29), (7.30) and (7.31),

$$(7.32) \quad \left. \begin{aligned} |u_\lambda(x, y)| &\leq Lx^2 \\ |u_{\lambda, x}(x, y)| &\leq Lx \\ |u_{\lambda, y}(x, y)| &\leq L\{(x-y) + y\} \\ &\leq Lx \end{aligned} \right\} (\lambda = 1, 2, \dots)$$

i.e. the sequences $\{u_\lambda\}$, $\{u_{\lambda, x}\}$ and $\{u_{\lambda, y}\}$ are uniformly bounded on R_2 .

Given two points, $(x_1, y_1) \in R_2$, $(x_2, y_2) \in R_2$, we may assume, without loss, that $x_1 \leq x_2$. Then, if $y_1 \leq y_2$, let us assume that $y_2 < x_1$. Then by integrating over the regions a, b and c in

diagram (A) we obtain



Consider the region R in the first quadrant bounded by the lines $x=1$, $y=1$, and the curve $y=x^2$. The region is shown in the figure below.

$$\text{Area of } R = \int_0^1 (1 - x^2) dx = \left[x - \frac{x^3}{3} \right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

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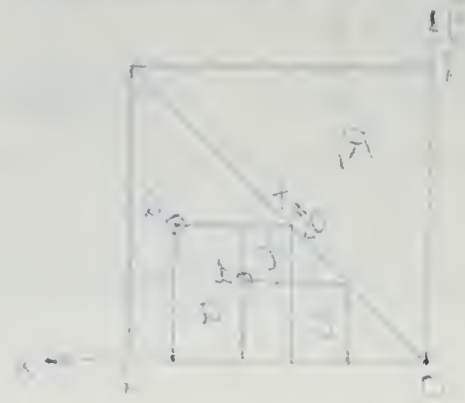
$$\text{Area of } R = \int_0^1 (1 - x^2) dx = \left[x - \frac{x^3}{3} \right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

The area of the region R is $\frac{2}{3}$.

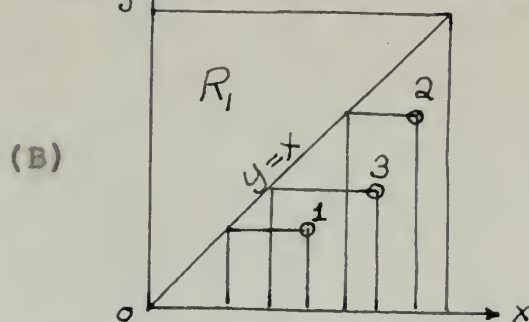
$$\begin{cases} \int_0^1 (1 - x^2) dx = \frac{2}{3} \\ \int_0^1 (1 - x^2) dx = \frac{2}{3} \\ \int_0^1 (1 - x^2) dx = \frac{2}{3} \\ \int_0^1 (1 - x^2) dx = \frac{2}{3} \end{cases}$$

The area of the region R is $\frac{2}{3}$.

The area of the region R is $\frac{2}{3}$.



$$(7.33) \quad |u_\lambda(x_2, y_2) - u_\lambda(x_1, y_1)| \leq L \{ \lambda(x_2 - x_1) + 2\lambda(y_2 - y_1) \}.$$

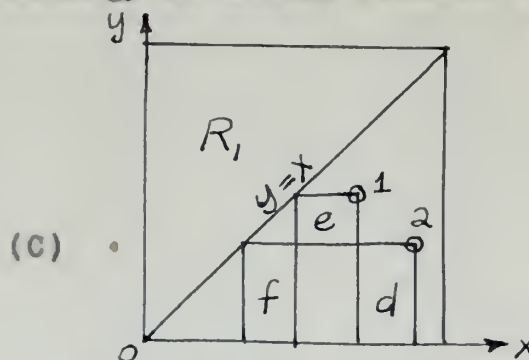


If $y_2 \geq x_1$ we may always choose a point (x_3, y_3) with $y_2 < x_3 < x_2$ and $y_1 < y_3 < x_1$ (as in diagram (B)). Then, as above,

$$|u_\lambda(x_2, y_2) - u_\lambda(x_3, y_3)| \leq L \{ \lambda(x_2 - x_3) + 2\lambda(y_2 - y_3) \}$$

$$|u_\lambda(x_3, y_3) - u_\lambda(x_1, y_1)| \leq L \{ \lambda(x_3 - x_1) + 2\lambda(y_3 - y_1) \}.$$

Adding, we obtain (7.33). Further if $y_1 \geq y_2$, we have the case



shown in diagram (C). Here by integrating over the regions d , e and f we again obtain (7.33). Hence the sequence $\{u_\lambda\}$ is equicontinuous on R_2 .

Now, for $(x, y_2) \in R_2$, $(x, y_1) \in R_2$, by (7.30)

$$(7.34) \quad |u_{\lambda, x}(x, y_2) - u_{\lambda, x}(x, y_1)| \leq L|y_2 - y_1|.$$

Likewise, for $(x_2, y) \in R_2$, $(x_1, y) \in R_2$, by (7.31)

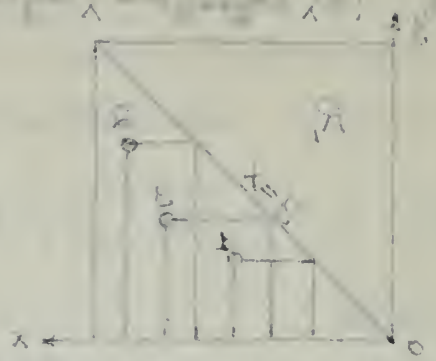
$$(7.35) \quad |u_{\lambda, y}(x_2, y) - u_{\lambda, y}(x_1, y)| \leq L|x_2 - x_1|.$$

Moreover, by precisely the same argument as that used to prove Lemma 2 of Chapter II, given $\mu > 0$, $\zeta > 0$, there exist $\delta > 0$, $N > 0$, depending only on μ and ζ , respectively, such that for

$$(x_2, y) \in R_2, \quad (x_1, y) \in R_2,$$

$$\lambda > N \text{ and } |x_2 - x_1| < \delta$$

$\int_0^1 (1-x)^2 dx = \int_0^1 (1 - 2x + x^2) dx = [x - x^2 + \frac{x^3}{3}]_0^1 = 1 - 1 + \frac{1}{3} = \frac{1}{3}$
 $\int_0^1 (1-x)^3 dx = \int_0^1 (1 - 3x + 3x^2 - x^3) dx = [x - \frac{3x^2}{2} + x^3 - \frac{x^4}{4}]_0^1 = 1 - \frac{3}{2} + 1 - \frac{1}{4} = \frac{1}{4}$
 $\int_0^1 (1-x)^4 dx = \int_0^1 (1 - 4x + 6x^2 - 4x^3 + x^4) dx = [x - 2x^2 + 2x^3 - x^4 + \frac{x^5}{5}]_0^1 = 1 - 2 + 2 - 1 + \frac{1}{5} = \frac{1}{5}$

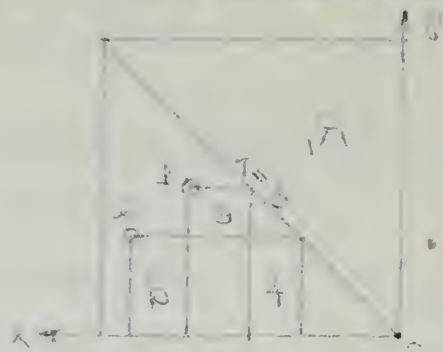


(10)

$$\int_0^1 (1-x)^2 dx = \int_0^1 (1 - 2x + x^2) dx = [x - x^2 + \frac{x^3}{3}]_0^1 = \frac{1}{3}$$

$$\int_0^1 (1-x)^3 dx = \int_0^1 (1 - 3x + 3x^2 - x^3) dx = [x - \frac{3x^2}{2} + x^3 - \frac{x^4}{4}]_0^1 = \frac{1}{4}$$

The area under the curve $y=1-x$ from $x=0$ to $x=1$ is $\frac{1}{2}$.
 The area under the curve $y=1-x^2$ from $x=0$ to $x=1$ is $\frac{2}{3}$.
 The area under the curve $y=1-x^3$ from $x=0$ to $x=1$ is $\frac{3}{4}$.
 The area under the curve $y=1-x^4$ from $x=0$ to $x=1$ is $\frac{4}{5}$.



(11)

$$\int_0^1 (1-x)^2 dx = \int_0^1 (1 - 2x + x^2) dx = [x - x^2 + \frac{x^3}{3}]_0^1 = \frac{1}{3}$$

$$\int_0^1 (1-x)^3 dx = \int_0^1 (1 - 3x + 3x^2 - x^3) dx = [x - \frac{3x^2}{2} + x^3 - \frac{x^4}{4}]_0^1 = \frac{1}{4}$$

$$\int_0^1 (1-x)^4 dx = \int_0^1 (1 - 4x + 6x^2 - 4x^3 + x^4) dx = [x - 2x^2 + 2x^3 - x^4 + \frac{x^5}{5}]_0^1 = \frac{1}{5}$$

$$\int_0^1 (1-x)^5 dx = \int_0^1 (1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5) dx = [x - \frac{5x^2}{2} + \frac{5x^3}{3} - \frac{5x^4}{4} + \frac{x^5}{5}]_0^1 = \frac{1}{6}$$

The area under the curve $y=1-x$ from $x=0$ to $x=1$ is $\frac{1}{2}$.
 The area under the curve $y=1-x^2$ from $x=0$ to $x=1$ is $\frac{2}{3}$.
 The area under the curve $y=1-x^3$ from $x=0$ to $x=1$ is $\frac{3}{4}$.
 The area under the curve $y=1-x^4$ from $x=0$ to $x=1$ is $\frac{4}{5}$.
 The area under the curve $y=1-x^5$ from $x=0$ to $x=1$ is $\frac{5}{6}$.

$$\int_0^1 (1-x)^6 dx = \int_0^1 (1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6) dx = [x - 3x^2 + \frac{5x^3}{2} - \frac{5x^4}{2} + \frac{3x^5}{2} - \frac{x^6}{6}]_0^1 = \frac{1}{7}$$

$$\int_0^1 (1-x)^7 dx = \int_0^1 (1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7) dx = [x - \frac{7x^2}{2} + \frac{7x^3}{2} - \frac{7x^4}{4} + \frac{7x^5}{5} - \frac{7x^6}{6} + \frac{x^7}{7}]_0^1 = \frac{1}{8}$$

$$\Rightarrow$$

$$(7.36) \quad |u_{\lambda,x}(x_2,y) - u_{\lambda,x}(x_1,y)| \\ \leq K \int_0^y |u_{\lambda,x}(x_2,\eta) - u_{\lambda,x}(x_1,\eta)| d\eta + \mu + \xi.$$

Thus by (7.34), (7.36) and Lemma 1, Chapter II, the sequence

$\{u_{\lambda,x}\}$ is equicontinuous on R_2 .

We need the following refinement of the argument in order to show that the sequence $\{u_{\lambda,y}\}$ is equicontinuous on R_2 :

Let us suppose $(x,y_2) \in R_2$, $(x,y_1) \in R_2$. Without loss, we may assume that $x \geq y_2 \geq y_1$. Then

$$u_{\lambda,y}(x,y_2) - u_{\lambda,y}(x,y_1) \\ = \int_{y_2}^x [g_\lambda(\xi, y_2; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) - g_\lambda(\xi, y_1; u_\lambda; u_{\lambda,x}, u_{\lambda,y})] d\xi \\ (7.37) \quad - \int_{y_1}^{y_2} g_\lambda(\xi, y_1; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\xi \\ - \int_0^{y_1} [g_\lambda(y_2, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) - g_\lambda(y_1, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y})] d\eta \\ - \int_{y_1}^{y_2} g_\lambda(y_2, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\eta$$

We have just proved that the sequences $\{u_\lambda\}$ and $\{u_{\lambda,x}\}$ are equicontinuous on R_2 . The sequence $\{g_\lambda\}$ is certainly equicontinuous on E' . Hence, considering (7.35), given $\mu > 0$, there exists $\delta > 0$, depending upon μ alone, such that $|y_2 - y_1| < \delta$

$$\Rightarrow$$

$$(7.38) \quad \left| \int_0^{y_1} [g_\lambda(y_2, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) - g_\lambda(y_1, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y})] d\eta \right| < \mu,$$

$$(7.39) \quad \left| \int_{y_2}^x [g_\lambda(\xi, y_2; u_\lambda(\xi, y_2); u_{\lambda,x}(\xi, y_2), \underline{u_{\lambda,y}(\xi, y_2)}) \right. \\ \left. - g_\lambda(\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda,x}(\xi, y_1), \underline{u_{\lambda,y}(\xi, y_2)})] d\xi \right| < \mu,$$

$$|f(x)| = |f(x) - f(x)| = 0$$

$$f(x) = \frac{1}{x^2} = x^{-2}$$

Let us find the derivative of $f(x) = x^{-2}$ using the power rule.

$$f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

Therefore, the derivative of $f(x) = \frac{1}{x^2}$ is $f'(x) = -\frac{2}{x^3}$.

$$f'(x) = -\frac{2}{x^3}$$

$$f'(x) = -\frac{2}{x^3} = -2x^{-3}$$

$$f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

$$f'(x) = -\frac{2}{x^3}$$

$$f'(x) = -\frac{2}{x^3}$$

Let us verify this result using the definition of the derivative.

The derivative of $f(x)$ is defined as:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{x^2 - (x+h)^2}{x^2(x+h)^2}}{h}$$

for $\lambda = 1, 2, \dots$.

Also, since $\{g_\lambda\} \xrightarrow{\text{unif}} f$ on B' , given $\zeta > 0$, there exists $N > 0$, depending upon ζ alone, such that $\lambda > N$

\Rightarrow

$$(7.40) \left| \int_{y_2}^x [g_\lambda - f](\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda, x}(\xi, y_1), \underline{u_{\lambda, y}(\xi, y_2)}) d\xi \right| < \zeta,$$

$$\left| \int_{y_2}^x [f - g_\lambda](\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda, x}(\xi, y_1), \underline{u_{\lambda, y}(\xi, y_1)}) d\xi \right| < \zeta.$$

By hypothesis 2)',

$$(7.41) \left| \int_{y_2}^x [f(\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda, x}(\xi, y_1), \underline{u_{\lambda, y}(\xi, y_2)}) - f(\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda, x}(\xi, y_1), \underline{u_{\lambda, y}(\xi, y_1)})] d\xi \right| \\ \leq \int_{y_2}^x K |u_{\lambda, y}(\xi, y_2) - u_{\lambda, y}(\xi, y_1)| d\xi.$$

Moreover, since $|g_\lambda| \leq L$, ($\lambda = 1, 2, \dots$),

$$(7.42) \left| \int_{y_1}^{y_2} g_\lambda(\xi, y_1; u_\lambda; u_{\lambda, x}, u_{\lambda, y}) d\xi \right| \leq L |y_2 - y_1|$$

$$\left| \int_{y_1}^{y_2} g_\lambda(y_2, \eta; u_\lambda; u_{\lambda, x}, u_{\lambda, y}) d\eta \right| \leq L |y_2 - y_1|.$$

Thus by equations (7.37) through (7.41), given $\mu > 0$, $\zeta > 0$, there exists $\delta > 0$, $N > 0$, depending only upon μ and ζ , respectively, such that $|y_2 - y_1| < \delta$ and $\lambda > N$

$$2000 \times 10^3 = 2 \times 10^6$$

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$$\dots \int \dots \int \dots$$

... ..

$$\begin{aligned}
 \Rightarrow \\
 (7.43) \quad & |u_{\lambda,y}(x,y_2) - u_{\lambda,y}(x,y_1)| \\
 & \leq K \int_{y_2}^x |u_{\lambda,y}(\xi,y_2) - u_{\lambda,y}(\xi,y_1)| d\xi \\
 & \quad + 4\mu + 2\zeta.
 \end{aligned}$$

By Lemma 1, Chapter II, inequalities (7.35) and (7.43) imply that the sequence $\{u_{\lambda,y}\}$ is equicontinuous on R_2 .

From this point on the proof is practically identical with that for Theorem 1a. Since the sequences $\{u_\lambda\}$, $\{u_{\lambda,x}\}$ and $\{u_{\lambda,y}\}$ are uniformly bounded and equicontinuous on R_2 , we may apply ARZELA's theorem to obtain a subsequence of each, uniformly convergent on R_2 . Hence, as for Theorem 1a, by successive extractions of subsequences we obtain a subsequence $\{u_\lambda^*\}$ of $\{u_\lambda\}$ converging uniformly on R_2 to a solution u of the integral equation

$$u(x,y) = \int_y^x d\xi \int_0^y f(\xi,\eta; u; u_x, u_y) d\eta,$$

and such that for $(x,y) \in R_2$

$(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in E$. The proof for Theorem 10a is now complete.

Following E. PICARD [7] p. 135 and p. 139, we show that the general statement of the mixed boundary conditions, (i.e. where u is prescribed along two intersecting curves, one characteristic and the other nowhere characteristic), can be reduced to the statement found in Theorems 10 and 10a, (i.e. where $u(x,0) = u(x,x) = 0$ for $x \in [0,1]$).

First, let us suppose that we prescribe

$$(7.44) \quad \begin{cases} u(x,0) = \varphi(x) \\ u(x,x) = \psi(x) \end{cases}$$

for $x \in [0, l]$, $\varphi(x)$ and $\psi(x) \in C^1[0, l]$ and $\varphi(0) = \psi(0)$.

Consider

$$(7.45) \quad w(x,y) = \varphi(x) + \psi(y) - \varphi(y).$$

We have $w_{xy} = 0$ on R while

$$(7.46) \quad \begin{cases} w(x,0) = \varphi(x) \\ w(x,x) = \psi(x) \end{cases}$$

for $x \in [0, l]$. Hence, instead of the problem with non-homogeneous boundary conditions (7.44), by setting

$$(7.47) \quad v = u - w$$

we may consider the problem

$$(7.48) \quad \begin{cases} v_{xy} = f(x,y; v+w; v_x + w_x, v_y + w_y) \\ v(x,0) = 0 \\ v(x,x) = 0, \end{cases}$$

a problem of the type covered by Theorems 10 and 10a.

Second, suppose we prescribe u along the characteristic $y = 0$ and the nowhere characteristic curve $y = F(x)$, where $F(x) \in C^1([0, l_1])$, $F'(x) \neq 0$ for $x \in [0, l_1]$ and $F(0) = 0$.

The coordinate transformation

$$(7.49) \quad \begin{cases} \bar{x} = F(x) \\ \bar{y} = y \end{cases}$$

reduces the curve $y = F(x)$ to the diagonal $\bar{y} = \bar{x}$ since the inverse F^{-1} exists and is of class C^1 on $[0, F(l_1)]$. Moreover,

$$(7.50) \quad u_{xy} = F'(x) u_{\bar{x}\bar{y}}.$$

$$\left. \begin{aligned} (1) \psi &= (0, 1, 0) \\ (2) \psi &= (1, 0, 0) \end{aligned} \right\} \text{ (17.27)}$$

For $\psi = (0, 1, 0)$ and $\psi = (1, 0, 0)$ we have $\psi = (0, 1, 0)$ and $\psi = (1, 0, 0)$ respectively.

$$(17.28) \quad \psi = (0, 1, 0) \quad \psi = (1, 0, 0)$$

$$\left. \begin{aligned} (1) \psi &= (0, 1, 0) \\ (2) \psi &= (1, 0, 0) \end{aligned} \right\} \text{ (17.29)}$$

For $\psi = (0, 1, 0)$ and $\psi = (1, 0, 0)$ we have $\psi = (0, 1, 0)$ and $\psi = (1, 0, 0)$ respectively.

$$(17.30) \quad \psi = (0, 1, 0) \quad \psi = (1, 0, 0)$$

we have $\psi = (0, 1, 0)$ and $\psi = (1, 0, 0)$.

$$\left. \begin{aligned} \psi &= (0, 1, 0) \\ \psi &= (1, 0, 0) \end{aligned} \right\} \text{ (17.31)}$$

we have $\psi = (0, 1, 0)$ and $\psi = (1, 0, 0)$.

For $\psi = (0, 1, 0)$ and $\psi = (1, 0, 0)$ we have $\psi = (0, 1, 0)$ and $\psi = (1, 0, 0)$ respectively.

$$\left. \begin{aligned} \psi &= (0, 1, 0) \\ \psi &= (1, 0, 0) \end{aligned} \right\} \text{ (17.32)}$$

For $\psi = (0, 1, 0)$ and $\psi = (1, 0, 0)$ we have $\psi = (0, 1, 0)$ and $\psi = (1, 0, 0)$ respectively.

$$\psi = (0, 1, 0) \quad \psi = (1, 0, 0) \text{ (17.33)}$$

Since $F'(x) \neq 0$, the form of the differential equation remains unchanged and we reduce the problem to one with initial conditions in the form (7.44).

Thus the general statement of the mixed boundary value problem for

$$(7.3) \quad u_{xy} = f(x, y; u; u_x, u_y)$$

can be reduced to the form treated in Theorems 10 and 10a. We note that whatever continuity and Lipschitz conditions are satisfied by (7.3) before transformation (7.49) and substitution (7.47) are satisfied as well after these operations are performed.

CHAPTER VIII

EXISTENCE THEOREMS BASED ON THE
CONCEPT OF UPPER AND LOWER BOUNDING FUNCTIONS

For the ordinary differential equation $y' = f(x, y)$ with $y(x_0) = y_0$, O. PERRON [18], assuming f merely continuous, gives an existence proof that is entirely independent of the classical proofs and contains them as special cases. He bases his proof on the concept of under and over functions, defining $\varphi(x)$ to be an under function if $\varphi(x_0) = y_0$ and

$$(8.1) \quad D_{\pm} \varphi(x) < f(x, \varphi(x))$$

and defining $\psi(x)$ to be an over function if $\psi(x_0) = y_0$ and

$$(8.2) \quad D_{\pm} \psi(x) > f(x, \psi(x)).$$

The solutions are found to lie between the upper limit function g of the set of underfunctions and the lower limit function G of the set of overfunctions, g and G themselves being solutions.

M. MÜLLER [4] shows that PERRON's proof will not carry over directly to apply to a system.

$$(8.3) \quad y_i' = f_i(x, y_1, \dots, y_n) \quad (i = 1, \dots, n).$$

However, he is able to extend the classical theorem, obtaining a statement which is similar to that of PERRON and which reduces to the direct analogue of PERRON's theorem in the particular case where the functions f_i are monotonically increasing in the arguments y_1, \dots, y_n .

THE PROBLEM

Let \mathcal{H} be a Hilbert space and $\mathcal{L}(\mathcal{H})$ the space of bounded linear operators on \mathcal{H} . Let $T \in \mathcal{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$.

The resolvent set $\rho(T)$ of T is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is invertible. The spectrum $\sigma(T)$ of T is the complement of $\rho(T)$ in \mathbb{C} . The point spectrum $\sigma_p(T)$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not injective. The continuous spectrum $\sigma_c(T)$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is injective but not surjective. The residual spectrum $\sigma_r(T)$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not surjective.

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

$$\sigma_p(T) \cup \sigma_r(T) \subseteq \sigma(T)$$

Let $\lambda \in \sigma(T)$. Then $\lambda I - T$ is not invertible. If $\lambda \in \sigma_p(T)$, then $\lambda I - T$ is not injective. If $\lambda \in \sigma_c(T)$, then $\lambda I - T$ is injective but not surjective. If $\lambda \in \sigma_r(T)$, then $\lambda I - T$ is not surjective.

$$\sigma_p(T) \cup \sigma_r(T) \subseteq \sigma(T)$$

The spectrum $\sigma(T)$ is a non-empty compact subset of \mathbb{C} . The point spectrum $\sigma_p(T)$ is a subset of $\sigma(T)$. The continuous spectrum $\sigma_c(T)$ and the residual spectrum $\sigma_r(T)$ are subsets of $\sigma(T)$.

Let $\lambda \in \sigma(T)$. Then $\lambda I - T$ is not invertible. If $\lambda \in \sigma_p(T)$, then $\lambda I - T$ is not injective. If $\lambda \in \sigma_c(T)$, then $\lambda I - T$ is injective but not surjective. If $\lambda \in \sigma_r(T)$, then $\lambda I - T$ is not surjective.

Let $\lambda \in \sigma(T)$. Then $\lambda I - T$ is not invertible. If $\lambda \in \sigma_p(T)$, then $\lambda I - T$ is not injective. If $\lambda \in \sigma_c(T)$, then $\lambda I - T$ is injective but not surjective. If $\lambda \in \sigma_r(T)$, then $\lambda I - T$ is not surjective.

Let $\lambda \in \sigma(T)$. Then $\lambda I - T$ is not invertible. If $\lambda \in \sigma_p(T)$, then $\lambda I - T$ is not injective. If $\lambda \in \sigma_c(T)$, then $\lambda I - T$ is injective but not surjective. If $\lambda \in \sigma_r(T)$, then $\lambda I - T$ is not surjective.

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

The spectrum $\sigma(T)$ is a non-empty compact subset of \mathbb{C} . The point spectrum $\sigma_p(T)$ is a subset of $\sigma(T)$. The continuous spectrum $\sigma_c(T)$ and the residual spectrum $\sigma_r(T)$ are subsets of $\sigma(T)$.

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

In this chapter we return to the characteristic initial value problem for

$$(8.4) \quad u_{xy} = f(x, y; u; u_x, u_y).$$

We obtain results similar to those of MULLER above. In the following Theorems 11 and 11a we improve the statements of Theorems 1 and 1a, Chapter II, by the introduction of upper and lower bounding functions Ω and ω .

Theorem 11 (11a)

$$1) \quad f(x, y; u; p, q) \in C(T), \quad T: \begin{cases} 0 \leq x \leq l \\ 0 \leq y \leq l \\ \omega(x, y) \leq u \leq \Omega(x, y) \\ \omega_x(x, y) \leq p \leq \Omega_x(x, y) \\ \omega_y(x, y) \leq q \leq \Omega_y(x, y) \end{cases}$$

2) (2') f is Lipschitzian (partially Lipschitzian) on T (as defined in Theorems 1 and 1a).

3) The functions $\omega(x, y)$ and $\Omega(x, y) \in C^1(R)$, $R: \begin{cases} 0 \leq x \leq l \\ 0 \leq y \leq l \end{cases}$ with $\omega_{xy}(x, y)$ and $\Omega_{xy}(x, y) \in C(R)$. Moreover,

$$\omega(x, 0) = \Omega(x, 0) = 0 \quad \text{for } x \in [0, l],$$

$$\omega(0, y) = \Omega(0, y) = 0 \quad \text{for } y \in [0, l],$$

and, for each $(x, y) \in R$,

$$(8.5) \quad \omega_{xy}(x, y) \leq \min_{S(x, y)} [f(x, y; u; p, q)],$$

$$(8.6) \quad \Omega_{xy}(x, y) \geq \max_{S(x, y)} [f(x, y; u; p, q)]$$

where

Let \mathcal{L} be the Lie algebra of G . Then \mathcal{L} is a vector space over \mathbb{R} or \mathbb{C} with a bilinear operation $[\cdot, \cdot]$ satisfying the Jacobi identity. The Lie bracket is defined by $[X, Y] = XY - YX$ for $X, Y \in \mathcal{L}$.

$$[X, Y] = XY - YX \tag{1}$$

The Lie algebra \mathcal{L} is said to be solvable if there exists a chain of subalgebras $\mathcal{L} = \mathcal{L}_0 \supset \mathcal{L}_1 \supset \dots \supset \mathcal{L}_n = \{0\}$ such that $[\mathcal{L}_i, \mathcal{L}_i] \subset \mathcal{L}_{i+1}$ for all i . The derived series of \mathcal{L} is defined by $\mathcal{L}^{(0)} = \mathcal{L}$ and $\mathcal{L}^{(i)} = [\mathcal{L}^{(i-1)}, \mathcal{L}^{(i-1)}]$ for $i \geq 1$. \mathcal{L} is solvable if and only if $\mathcal{L}^{(n)} = \{0\}$ for some n .

THEOREM 1

$$\mathcal{L} \text{ is solvable} \iff \text{there exists a basis } \{X_1, \dots, X_n\} \text{ of } \mathcal{L} \text{ such that } [X_i, X_j] \in \text{span}\{X_1, \dots, X_{i+j-1}\} \text{ for all } i, j.$$

Proof. Suppose \mathcal{L} is solvable. Then there exists a chain of subalgebras $\mathcal{L} = \mathcal{L}_0 \supset \mathcal{L}_1 \supset \dots \supset \mathcal{L}_n = \{0\}$ such that $[\mathcal{L}_i, \mathcal{L}_i] \subset \mathcal{L}_{i+1}$ for all i . Let $\{X_1, \dots, X_n\}$ be a basis of \mathcal{L} such that $X_i \in \mathcal{L}_i \setminus \mathcal{L}_{i+1}$ for $i = 1, \dots, n$. Then $[X_i, X_j] \in \mathcal{L}_{i+j-1}$ for all i, j .

Conversely, suppose there exists a basis $\{X_1, \dots, X_n\}$ of \mathcal{L} such that $[X_i, X_j] \in \text{span}\{X_1, \dots, X_{i+j-1}\}$ for all i, j . Let $\mathcal{L}_i = \text{span}\{X_1, \dots, X_i\}$ for $i = 0, \dots, n$. Then \mathcal{L}_i is a subalgebra of \mathcal{L} and $[\mathcal{L}_i, \mathcal{L}_i] \subset \mathcal{L}_{i+1}$ for all i . Thus \mathcal{L} is solvable.

$$\mathcal{L} \text{ is solvable} \iff \text{there exists a basis } \{X_1, \dots, X_n\} \text{ of } \mathcal{L} \text{ such that } [X_i, X_j] \in \text{span}\{X_1, \dots, X_{i+j-1}\} \text{ for all } i, j. \tag{2}$$

$$\mathcal{L} \text{ is solvable} \iff \text{there exists a basis } \{X_1, \dots, X_n\} \text{ of } \mathcal{L} \text{ such that } [X_i, X_j] \in \text{span}\{X_1, \dots, X_{i+j-1}\} \text{ for all } i, j. \tag{3}$$

$$(8.7) \quad \mathcal{B}(x,y): \begin{cases} x = x \\ y = y \\ \omega(x,y) \leq u \leq \Omega(x,y) \\ \omega_x(x,y) \leq p \leq \Omega_x(x,y) \\ \omega_y(x,y) \leq q \leq \Omega_y(x,y) \end{cases}$$

\Rightarrow 4) (4)' There exists one and only one (at least one) function $u(x,y) \in C^1(R)$, $u_{xy} \in C(R)$ such that for each $(x,y) \in R$ the point $(x,y; u(x,y); u_x(x,y); u_y(x,y)) \in T$, and

$$u_{xy}(x,y) = f(x,y; u(x,y); u_x(x,y), u_y(x,y)),$$

$$u(x,0) = u(0,y) = 0 \quad \text{for each } (x,y) \in R.$$

Proof

We extend the domain of definition of the function f over T

$$\text{to } B': \begin{cases} 0 \leq x \leq l \\ 0 \leq y \leq l \\ -\infty < u < \infty \\ -\infty < p < \infty \\ -\infty < q < \infty \end{cases} \quad \text{by defining } f(x,y; u; p, q)$$

$$= f(x,y; \bar{u}; \bar{p}, \bar{q}), \text{ where}$$

$$\bar{u} = u \text{ if } \omega(x,y) \leq u \leq \Omega(x,y), \quad \bar{p} = p \text{ if } \omega_x(x,y) \leq p \leq \Omega_x(x,y),$$

$$(8.8) \quad \bar{u} = \omega(x,y) \text{ if } u < \omega(x,y) \quad \bar{p} = \omega_x(x,y) \text{ if } p < \omega_x(x,y)$$

$$\bar{u} = \Omega(x,y) \text{ if } \Omega(x,y) < u \quad \bar{p} = \Omega_x(x,y) \text{ if } \Omega_x(x,y) < p$$

$$\text{and} \quad \bar{q} = q \text{ if } \omega_y(x,y) \leq q \leq \Omega_y(x,y)$$

$$\bar{q} = \omega_y(x,y) \text{ if } q < \omega_y(x,y)$$

$$\bar{q} = \Omega_y(x,y) \text{ if } \Omega_y(x,y) < q.$$

By definition (8.8), f is uniformly continuous and uniformly bounded in B' . Moreover, by hypothesis 2)(2)' and (8.8) f satisfies a Lipschitz (partial Lipschitz) condition in B' .

Hence, by Theorem 1 (1a) (Chapter II, there exists one and only one (at least one) function satisfying conclusion 4)(4)' except that for $(x,y) \in R$ we are assured only that the point $(x,y;u(x,y);u_x(x,y),u_y(x,y)) \in S'$. To complete the proof we must show that this point actually lies in T ; i.e. we must show that for each $(x,y) \in R$,

$$(8.9) \quad \begin{cases} \omega(x,y) \leq u(x,y) \leq \Omega(x,y) \\ \omega_x(x,y) \leq u_x(x,y) \leq \Omega_x(x,y) \\ \omega_y(x,y) \leq u_y(x,y) \leq \Omega_y(x,y) \end{cases} .$$

To accomplish this, we first prove the following lemma:

Lemma 3 i) $\omega_{xy}(x,y) \leq u_{xy}(x,y)$ for all $(x,y) \in R$

$$\Rightarrow \quad \omega(x,y) \leq u(x,y) \quad "$$

$$\omega_x(x,y) \leq u_x(x,y) \quad "$$

$$\omega_y(x,y) \leq u_y(x,y) \quad "$$

ii) $\Omega_{xy}(x,y) \geq u_{xy}(x,y)$ for all $(x,y) \in R$

$$\Rightarrow \quad \Omega(x,y) \geq u(x,y) \quad "$$

$$\Omega_x(x,y) \geq u_x(x,y) \quad "$$

$$\Omega_y(x,y) \geq u_y(x,y) \quad "$$

Proof: For i),

$$\omega(x,y) = \int_0^x dx \int_0^y \omega_{xy} dy \leq \int_0^x dx \int_0^y u_{xy} dy = u(x,y)$$

$$\omega_x(x,y) = \int_0^y \omega_{xy} dy \leq \int_0^y u_{xy} dy = u_x(x,y)$$

$$\omega_y(x,y) = \int_0^x \omega_{xy} dx \leq \int_0^x u_{xy} dx = u_y(x,y).$$

The proof for ii) is analogous.

To prove (3.9) it only remains to verify that hypothesis i) and ii) of Lemma 3 are satisfied by u . By hypothesis 3) and definition (3.8), for each $(x,y) \in R$,

$$\begin{aligned}\omega_{xy}(x,y) &\leq \min_{S(x,y)} [f(x,y; u; p, q)] \\ &\leq f(x,y; u(x,y); u_x(x,y), u_y(x,y)) \\ &= u_{xy}(x,y)\end{aligned}$$

and

$$\begin{aligned}\Omega_{xy}(x,y) &\geq \max_{S(x,y)} [f(x,y; u; p, q)] \\ &\geq f(x,y; u(x,y); u_x(x,y), u_y(x,y)) \\ &= u_{xy}(x,y).\end{aligned}$$

Thus, by Lemma 3, requirement (2.9) is satisfied for each $(x,y) \in R$ and the proof of Theorems 11 and 11a is complete.

It is evident upon inspection of Theorems 11 and 11a that if, instead of homogeneous initial conditions, we prescribe

$$u(x,0) = U(x) \quad \text{with } U(x) \in C^1([0, l]),$$

$$u(0,y) = V(y) \quad \text{with } V(y) \in C^1([0, l]),$$

where $U(0) = V(0)$, then we must require

$$\omega(x,0) = \Omega(x,0) = U(x),$$

$$\omega(0,y) = \Omega(0,y) = V(y).$$

The proof then goes through as before.

The following example is an illustration of Theorem 11:

Example 4

For the problem

Let f be a function from X to Y . Let A and B be subsets of X . Then $f(A \cup B) = f(A) \cup f(B)$.

$$f(A \cup B) = \{y \in Y \mid \exists x \in A \cup B, f(x) = y\}$$
$$= \{y \in Y \mid \exists x \in A, f(x) = y\} \cup \{y \in Y \mid \exists x \in B, f(x) = y\}$$
$$= f(A) \cup f(B)$$

Let f be a function from X to Y . Let A and B be subsets of X . Then $f(A \cap B) \subseteq f(A) \cap f(B)$.

Let f be a function from X to Y . Let A and B be subsets of X . Then $f(A \setminus B) \subseteq f(A) \setminus f(B)$.

Let f be a function from X to Y . Let A and B be subsets of X . Then $f(A \setminus B) \subseteq f(A) \setminus f(B)$.

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Let f be a function from X to Y . Let A and B be subsets of X . Then $f(A \setminus B) \subseteq f(A) \setminus f(B)$.

Q.E.D.

$$(8.10) \quad u_{xy} = (2^{1/m} - u_x)^{1/m+1}, \quad u(x,0) = u(0,y) = 0,$$

we may readily verify that

$$(8.11) \quad \omega(x,y) = \left(\frac{1}{m+1}\right)^{1/m+1} \cdot 2^{1/m(m+1)} xy$$

and

$$(8.12) \quad \Omega(x,y) = 2^{1/m(m+1)} xy$$

satisfy the hypotheses of Theorem 11 for all $x \geq 0$ and

$$0 \leq y \leq C_m^* = \frac{m}{m+1} 2^{1/m+1}$$

In Chapter 11 we obtained the exact solution

$$(8.42) \quad u(x,y) = x \left\{ 2^{1/m} - \left[\frac{m}{m+1} (C_m^* - y) \right]^{m+1/m} \right\}$$

where

$$(8.43) \quad C_m^* = \frac{m+1}{m} 2^{1/m+1}$$

is a branch point of the solution. We observe that as m increases indefinitely ω and Ω approach u from below and above, respectively, while C_m^* approaches C_m from below.

We see from this example that it is possible to obtain approximate solutions, with known limits of error, and to locate singularities in the actual solution by use of Theorem 11, provided that suitable functions ω and Ω can be obtained. For problems where explicit solutions cannot be obtained in "closed form", the procedure is to alter the right-hand side of the equation

$$u_{xy} = f(x,y; u; u_x, u_y)$$

so that an explicit solution of the altered equation can be ob-

$$\frac{1}{2} \frac{d}{dt} (v^2) = \frac{1}{2} \frac{d}{dt} (v_x^2 + v_y^2 + v_z^2) = \frac{1}{2} \frac{d}{dt} (v^2)$$

Let us now consider the case

$$\frac{1}{2} \frac{d}{dt} (v^2) = \frac{1}{2} \frac{d}{dt} (v_x^2 + v_y^2 + v_z^2) = \frac{1}{2} \frac{d}{dt} (v^2)$$

and

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and

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Let us now consider the case

tained satisfying the boundary conditions. This may lead to functions ω and Ω satisfying the hypotheses of Theorem 11. (See W. W. WHITBURN [19] and [20].) The motivation for equations (3.11) and (3.12) of Example 4 is now evident.

When we consider the possibility of applying, as explained below, the PERRON method using under and over functions to the characteristic initial value problem under consideration, we find the situation much the same as that in the case of a system of first order ordinary differential equations. We arrive at the unsatisfactory state of affairs wherein there is no assurance that the under functions remain below the over functions throughout the entire region on which a solution is known to exist. In fact, we shall presently give an example where an under function exceeds an over function within the domain of existence of a solution.

Recalling inequalities (3.1) and (3.9), we may express the application of the PERRON method as follows: We require both the under and over functions to satisfy the given characteristic initial conditions and to be continuously differentiable and to possess a mixed second derivative at each point of the domain $R: \begin{cases} 0 \leq x \leq X \\ 0 \leq y \leq Y \end{cases}$. We further stipulate that each under function, φ , shall satisfy

$$(3.13) \quad \varphi_{xy}(x,y) < f(x,y; \varphi(x,y); \varphi_x(x,y), \varphi_y(x,y)),$$

and that each over function, ψ , shall satisfy

$$(3.14) \quad \psi_{xy}(x,y) > f(x,y; \psi(x,y); \psi_x(x,y), \psi_y(x,y))$$

for each $(x,y) \in R$.

Analogous arguments to those used by PERMON for the ordinary differential equation $y' = f(x,y)$ lead to the inequalities

$$\begin{aligned} \varphi_x(0,y) < \psi_x(0,y) & \quad \text{for} \quad 0 < y \leq l, \\ \varphi_y(x,0) < \psi_y(x,0) & \quad \text{for} \quad 0 < x \leq l, \end{aligned}$$

for any under function φ and any over function ψ . These inequalities, together with the requirement that φ and ψ satisfy the characteristic initial data on the positive x and y axes, insure that $\psi > \varphi$ in a sufficiently small "L" shaped strip in the first quadrant adjacent to the initial characteristics. Unfortunately, this is inadequate as the following example demonstrates.

Example 5

Consider the problem

$$(8.15) \quad u_{xy} = 0, \quad u(x,0) = u(0,y) = 0.$$

This problem has the unique solution $u \equiv 0$ throughout the finite plane. Let

$$(8.16) \quad \begin{cases} \psi_{xy} = Ax - By^2 + C \\ \varphi_{xy} = -D, \end{cases}$$

where A , B , C and D are positive constants. By integration in (8.16) we may obtain functions ψ and φ satisfying the initial conditions of (8.15). Obviously, φ is an under function for all (x,y) . Moreover, $\psi_{xy} > 0$ for all (x,y) lying in the portion of the first quadrant below the parabolic arc

$$y = +\sqrt{\frac{A}{B}x + \frac{C}{B}};$$

and hence ψ meets the requirements for an over function on a domain R_ℓ : $\begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \sqrt{\frac{C}{B}} \end{cases}$ where ℓ is arbitrarily large but finite.

Defining $h = \psi - \varphi$ we have

$$h_{xy}(x,y) = Ax - By^2 + C + D.$$

Since $h(x,0) = h(0,y) = 0$, we obtain by integration

$$h(x,y) = \frac{A}{2} x^2 y - \frac{B}{2} x^2 y^2 + (C+D) xy.$$

We note that $h > 0$ in that portion of the first quadrant below the hyperbola branch

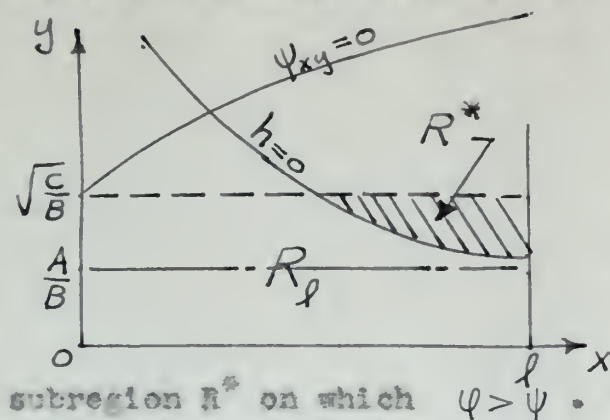
$$y = \frac{A}{B} + \frac{2(C+D)}{Bx}$$

while $h < 0$ above this branch. From the diagram it is evident

that if we require

$$\frac{A}{B} < \sqrt{\frac{C}{B}}$$

then there exists a positive constant ℓ such that within the corresponding domain R_ℓ we have a



subregion R^* on which $\varphi > \psi$. Hence the PERRON method is not directly applicable to this class of problems.

Returning to Theorems 11 and 11a, we observe that if, for fixed (x,y) , f is a monotonically increasing function for the arguments u , p and q , then

$$\begin{aligned} f(x,y; \omega(x,y); \omega_x(x,y), \omega_y(x,y)) \\ = \min_{S(x,y)} [f(x,y; u; p,q)], \end{aligned}$$

and

$$f(x,y; \Omega(x,y); \Omega_x(x,y), \Omega_y(x,y)) = \max_{S(x,y)} [f(x,y; u; p,q)].$$

Let $y = \sqrt{x}$ and $z = \sqrt{y}$. Then $z = \sqrt{\sqrt{x}} = x^{1/4}$.
 The derivative of z with respect to x is $\frac{dz}{dx} = \frac{1}{4}x^{-3/4}$.

$$\frac{dz}{dx} = \frac{1}{4}x^{-3/4}$$

At $x = 16$, $\frac{dz}{dx} = \frac{1}{4}(16)^{-3/4} = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$.

$$dz = \frac{1}{32} dx$$

When x increases from 16 to 17, $dx = 1$.
 The corresponding change in z is $dz = \frac{1}{32}$.

$$\frac{dz}{dx} = \frac{1}{32}$$

Thus, the change in z is $\frac{1}{32}$ when x increases from 16 to 17.

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 The derivative of z with respect to x is $\frac{dz}{dx} = \frac{1}{4}x^{-3/4}$.



At $x = 16$, $\frac{dz}{dx} = \frac{1}{4}(16)^{-3/4} = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$.
 When x increases from 16 to 17, $dx = 1$.
 The corresponding change in z is $dz = \frac{1}{32}$.

$$\frac{dz}{dx} = \frac{1}{4}x^{-3/4}$$

Thus, the change in z is $\frac{1}{32}$ when x increases from 16 to 17.

In this case we may alter hypothesis 3) to require merely that

$$\begin{aligned}\omega_{xy}(x,y) &\leq f(x,y; \omega(x,y); \omega_x(x,y), \omega_y(x,y)) \\ \Omega_{xy}(x,y) &\geq f(x,y; \Omega(x,y); \Omega_x(x,y), \Omega_y(x,y))\end{aligned}$$

for each $(x,y) \in R$. This is the direct analogue to PERSON'S theorem (see [13]) and corresponds to the previously mentioned result of MULLER for a system (8.3).

We close this chapter with the remark that Theorems 11 and 11a can be extended immediately in two ways. First, the method is directly applicable to the Cauchy problem. We require the functions ω and Ω to satisfy the Cauchy initial data and observe that the proof of Lemma 3 is essentially unchanged. Second, the method extends to apply to a system

$$u_{i,xy} = f_i(x,y; u_j; u_{j,x}, u_{j,y}), \quad (i = 1, \dots, n)$$

for both characteristic and Cauchy initial value prescriptions. The modifications in the hypotheses and proof for Theorems 11 and 11a are obvious.

Let \mathcal{A} be a σ -algebra on Ω . Let \mathcal{B} be a σ -algebra on Ω .

$$\mathcal{A} \cap \mathcal{B} = \{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$$

$$\mathcal{A} \cup \mathcal{B} = \{A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$$

Let \mathcal{C} be a σ -algebra on Ω . Let \mathcal{D} be a σ -algebra on Ω . Let \mathcal{E} be a σ -algebra on Ω . Let \mathcal{F} be a σ -algebra on Ω .

Let \mathcal{G} be a σ -algebra on Ω . Let \mathcal{H} be a σ -algebra on Ω . Let \mathcal{I} be a σ -algebra on Ω . Let \mathcal{J} be a σ -algebra on Ω . Let \mathcal{K} be a σ -algebra on Ω .

$$\mathcal{A} \cap \mathcal{B} = \{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$$

Let \mathcal{L} be a σ -algebra on Ω . Let \mathcal{M} be a σ -algebra on Ω . Let \mathcal{N} be a σ -algebra on Ω . Let \mathcal{O} be a σ -algebra on Ω .

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ON THE EXISTENCE OF NOT NECESSARILY
UNIQUE SOLUTIONS OF THE CLASSICAL HYPER-
BOLIC BOUNDARY VALUE PROBLEMS FOR NON-
LINEAR SECOND ORDER PARTIAL DIFFERENTIAL
EQUATIONS IN TWO INDEPENDENT VARIABLES.

By

Patrick Leehy

B.Sc., United States Naval Academy, 1942

Thesis

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THE UNIVERSITY OF CHICAGO
DEPARTMENT OF CHEMISTRY
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VITA

Patrick Leehey was born at Waterloo, Iowa, October 27, 1921. He attended the College of Engineering, State University of Iowa 1938-1939. Attended the U. S. Naval Academy 1939-1942, receiving the degree of Bachelor of Science in 1942. He was commissioned as Ensign, U. S. Navy, 1942. Served with the U. S. Pacific Fleet 1942-1945. Attended the U. S. Naval Postgraduate School in the course in Naval Engineering Design 1946-1947. Attended Brown University in the Graduate Division of Applied Mathematics 1947-1950. Member of Sigma Xi. He holds the rank of Lieutenant, U.S. Navy.

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ANSWERS TO QUESTIONS

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31. $1 + 2 + 3 + \dots + 100 = 5050$	5050
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50. $1 + 2 + 3 + \dots + 100 = 5050$	5050

NOTATIONS

The following special notations will be used throughout this paper with the meanings as defined below. Other special notations used will be defined at the place where they are introduced.

$$B: \begin{cases} 0 \leq x \leq l \\ 0 \leq y \leq l \end{cases}$$

is a member of; i.e. belongs to.

B is the set of all ordered pairs (x,y) , (points) for which $0 \leq x \leq l$ and $0 \leq y \leq l$.

$$f \in C(B)$$

f is a member of the class of functions continuous on the set B .

$$g \in C^1(H)$$

g is a member of the class of functions continuously differentiable on the set H , (and similarly for higher degrees of differentiability.)

$$u_x$$

$$\frac{\partial u}{\partial x}$$

$$u_{\lambda, x}$$

$$\frac{\partial u_\lambda}{\partial x}$$

$$\dot{x}$$

$\frac{dx}{dz}$ where z is a parameter along a path.

$$x \in [0, l]$$

x belongs to the closed interval, $0 \leq x \leq l$.



implies.



implies and is implied by; i.e. if and only if.

$$\{g_\lambda\} (x,y; u; p,q)$$

a sequence of functions g_λ , ($\lambda = 1, 2, \dots$), of arguments $(x,y; u; p,q)$.

$$\{g_\lambda\} \rightarrow f \text{ on } B$$

the sequence $\{g_\lambda\}$ converges pointwise on the set B to the function f .

$\{g_\lambda\} \xrightarrow{\text{unif}} f$ on B

the sequence $\{g_\lambda\}$ converges uniformly on the set B to the function f .

$D_\pm y$

the right(+) and left (-) hand derivatives of the function y at the point in question.

The following is a list of the
 names of the persons who
 were present at the meeting
 held on the 1st day of
 the month of ...

...

...

The following is a list of the names of the persons who were present at the meeting held on the 1st day of the month of ...

CHAPTER I

INTRODUCTION

The purpose of this paper is to present a number of existence theorems pertaining to a class of non-linear second order partial differential equations in two independent variables of the general form

$$(1.1) \quad F(x,y; u; p,q; r,s,t) = 0,$$

where

$$(1.2) \quad p = u_x, \quad q = u_y, \quad r = u_{xx}, \quad s = u_{xy} \quad \text{and} \quad t = u_{yy},$$

in the usual notation. We restrict our attention to those prescriptions of initial conditions for which integral surfaces exist such that the equation is of hyperbolic type thereon, i.e. the inequality

$$(1.3) \quad F_s^2 - 4 F_r F_t > 0$$

must be satisfied on the integral surface in a neighborhood of the initial data.

E. PICARD [1],[7]¹, E. COURSAT [8], E.E.Levi[9], H.LEWY[10], J. HADAMARD[11], M. CINQUINI-CIARRARIO[12],[13], and others have

¹ The number in the bracket [] refers to the reference in the bibliography.

CHAPTER I

DEFINITIONS

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¹ The subject of this book is the theory of the...
and the theory of the...

developed existence theorems based on the method of successive approximations. Their concern has been to establish sufficient conditions for the existence of a unique solution. Retaining their restrictions on the initial data, we shall obtain sufficient conditions for the existence of at least one solution. The integrals of the equations we consider will not, in general, be unique.

The concept of characteristic curves in an integral surface plays an important role in all work in this field. We give two definitions of a characteristic curve, the first applicable when the curve is expressed in non-parametric form, the second when expressed in parametric form:

Definition 1

$$\gamma : \begin{cases} a \leq x \leq b \\ y = g(x) \end{cases} \text{ where } g \in C'([a,b]), \text{ or } \gamma : \begin{cases} x = h(y) \\ c \leq y \leq d \end{cases}$$

where $h \in C'([c,d])$, is a characteristic base curve (characteristic projection or, by usage, characteristic) for a particular integral surface $J: u = u(x,y)$ of $F(x,y; u; p,q; r,s,t) = 0 \iff$ for each (x,y)

$$(1.4) \quad F_r dy^2 - F_s dy dx + F_t dx^2 = 0$$

Definition 1a

$$\gamma : \begin{cases} x = x(\tau) \\ y = y(\tau) \end{cases} \text{ for } \tau \in [0,1] \text{ and where } x,y \in C'([0,1]), \text{ is a}$$

characteristic base curve for a particular integral surface

$J: u = u(x,y)$ of $F(x,y; u; p,q; r,s,t) = 0 \iff$ for each $\tau \in [0,1]$

$$(1.5) \quad \begin{cases} 1) & F_r \dot{y}^2 - F_s \dot{y} \dot{x} + F_t \dot{x}^2 = 0 \\ 2) & \dot{x}^2 + \dot{y}^2 \neq 0. \end{cases}$$

The first part of the proof is devoted to showing that the function f is continuous at the origin. Let (x, y) be a point in the domain of f . We consider the function $g(t) = f(x + ty, y)$. Then $g(0) = f(x, y)$ and $g'(0) = f_x(x, y)$. By the Mean Value Theorem, there exists a θ between 0 and 1 such that $g(1) - g(0) = g'(\theta)$. This implies that $f(x + y, y) - f(x, y) = f_x(x + \theta y, y)$. Since f_x is continuous at (x, y) , it follows that $f(x + y, y) - f(x, y) \rightarrow 0$ as $y \rightarrow 0$. A similar argument shows that $f(x, y + h) - f(x, y) \rightarrow 0$ as $h \rightarrow 0$. Therefore, f is continuous at (x, y) .

Lemma 2

$$\left. \begin{aligned}
 & f(x, y) = f(x, 0) + f_y(x, 0)y + \frac{1}{2} f_{yy}(x, 0)y^2 + o(y^2) \\
 & f(x, y) = f(x, 0) + f_y(x, 0)y + \frac{1}{2} f_{yy}(x, 0)y^2 + o(y^2)
 \end{aligned} \right\} (1.1)$$

The proof of Lemma 2 is straightforward. It follows from the Taylor expansion of $f(x, y)$ around $y = 0$. The remainder term $o(y^2)$ indicates that the error term goes to zero faster than y^2 as $y \rightarrow 0$.

$$f(x, y) = f(x, 0) + f_y(x, 0)y + \frac{1}{2} f_{yy}(x, 0)y^2 + o(y^2) \quad (1.2)$$

Lemma 3

$$\left. \begin{aligned}
 & f(x, y) = f(x, 0) + f_y(x, 0)y + \frac{1}{2} f_{yy}(x, 0)y^2 + o(y^2) \\
 & f(x, y) = f(x, 0) + f_y(x, 0)y + \frac{1}{2} f_{yy}(x, 0)y^2 + o(y^2)
 \end{aligned} \right\} (1.3)$$

The proof of Lemma 3 is similar to Lemma 2. It follows from the Taylor expansion of $f(x, y)$ around $y = 0$. The remainder term $o(y^2)$ indicates that the error term goes to zero faster than y^2 as $y \rightarrow 0$.

Under either definition γ is rectifiable and possesses a continuously turning tangent (see C. JORDAN[6], p. 100). The two definitions are equivalent in the following sense: We may convert γ expressed in non-parametric form into its parametric expression by setting $x = \tau$, $y = g(\tau)$, or $x = h(\tau)$, $y = \tau$ as the case may be. That the converse is possible follows directly from condition 2) of Definition 1a and the Implicit Function Theorem. For, suppose at a point $(x(\tau_0), y(\tau_0))$ of γ that $\dot{x} \neq 0$. Then in a vicinity of $x_0 = x(\tau_0)$ the inverse relation $\tau = \tau(x)$ exists and we may write

$$(1.6) \quad \gamma : y = y(\tau(x)) = g(x).$$

Similarly, where $\dot{y} \neq 0$, we may write

$$(1.7) \quad \gamma : x = x(\tau(y)) = h(y).$$

By condition 2), one of the two representations (1.6) or (1.7) is always possible in the vicinity of each point of γ .

Definition 2

$$\Gamma : \begin{cases} x = x(\tau) \\ y = y(\tau) \\ u = u(\tau) \end{cases} \text{ for } \tau \in [0,1] \text{ and where } x, y, u \in C'([0,1]),$$

a space curve lying in a particular integral surface $J: u=u(x,y)$ of $F(x,y; u; p,q; r,s,t) = 0$, is called a characteristic curve in the integral surface $J \iff$ the projection of Γ onto the xy plane is a characteristic projection for the integral surface J .

Under suitable hypotheses, by virtue of the hyperbolic condition (1.3), for any integral surface $J: u=u(x,y)$ of $F(x,y,u;p,q,r,s,t) = 0$, equations (1.4) or (1.5) determine two one parameter families of characteristic curves lying in the integral surface J . Exactly one characteristic curve from each family passes through any given point $(x_0, y_0, u(x_0, y_0))$ of the integral surface J ; and, moreover, the corresponding two characteristic base curves do not have a common tangent at (x_0, y_0) .

Along any curve, characteristic or otherwise, lying in the integral surface J , the following strip, or band, conditions

$$(1.8) \quad \dot{u} = p\dot{x} + q\dot{y}$$

$$(1.9) \quad \begin{cases} \dot{p} = r\dot{x} + s\dot{y} \\ \dot{q} = s\dot{x} + t\dot{y} \end{cases}$$

must be satisfied.

The modification of Definition 2 and conditions (1.8), (1.9) when the curve Γ is expressed in non-parametric form is obvious.

Definition 3

$$S^1: \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \\ p=p(\tau) \\ q=q(\tau) \end{cases} \text{ for } \tau \in [0,1] \text{ and where } x,y,u,p,q \in C'([0,1]).$$

is called a first order strip \iff for each $\tau \in [0,1]$

$$(1.8) \quad \dot{u} = p\dot{x} + q\dot{y}$$

Suppose a particular integral surface $J: u=u(x,y)$ of

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$$\begin{aligned}
 (1) \quad & x + y = 10 \\
 (2) \quad & 2x + 3y = 25 \\
 (3) \quad & 3x + 4y = 35
 \end{aligned}$$

...the ... of ...
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 ...the ... of ...
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$$\begin{bmatrix}
 1 & 1 & 0 & 0 \\
 2 & 3 & 0 & 0 \\
 3 & 4 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
 \end{bmatrix}$$

...the ... of ...

$$x + y = 10 \quad (1)$$

...the ... of ...

$F(x,y; u; p,q; r,s,t) = 0$ has a contact of first order with the strip S^1 . Then if $\Gamma : \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \end{cases}$ for $\tau \in [0,1]$ is a characteristic curve in the integral surface J , the strip S^1 is called a characteristic first order strip for the integral surface J .

Definition 4

$$S^2 : \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \\ p=p(\tau) \\ q=q(\tau) \\ r=r(\tau) \\ s=s(\tau) \\ t=t(\tau) \end{cases} \text{ for } \tau \in [0,1] \text{ and where } x,y,u,p,q,r,s,t \in C^1([0,1])$$

is called a second order strip \iff for each $\tau \in [0,1]$

(1.8) $\dot{u} = p\dot{x} + q\dot{y}$

(1.9) $\begin{cases} \dot{p} = r\dot{x} + s\dot{y} \\ \dot{q} = s\dot{x} + t\dot{y} \end{cases}$

If, moreover, equation (1.1) and conditions (1.3) and (1.5) are satisfied for each $\tau \in [0,1]$, then S^1 is called a characteristic second order strip.

Note in Definition 4 that since all the arguments of the functions involved in conditions (1.5) are known upon prescription of the strip S^2 , we may determine whether or not the projection of corresponding space curve $\Gamma : \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \end{cases}$ for $\tau \in [0,1]$ is a characteristic projection without reference to any particular integral surface.

... of the ... $\left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right]$...

Exercise 1

The ...

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The ...

$$\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \quad (1.1)$$

$$\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \quad (1.2)$$

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... of the ...

Definitions 3 and 4 can be readily modified to deal with the non-parametric case. See, for example, M. CINQUINI-CIBRARIO[13].

In Chapter 2 we consider the characteristic initial value problem for the equation

$$(1.10) \quad s = f(x, y; u; p, q)$$

and its extension to the system of equations

$$(1.11) \quad s_i = f_i(x, y; u_1, \dots, u_n; p_1, \dots, p_n, q_1, \dots, q_n) \\ (i=1, 2, \dots, n).$$

We modify the customary hypothesis that f be Lipschitzian, i.e. with respect to variables u , p and q , to require that f be partially Lipschitzian, i.e. with respect to variables p and q only. We obtain existence of an integral u over the same closed domain as that obtained in the classical theory. Our integral, however, cannot be shown to be unique. This fact is demonstrated by an example. By further example, we show that the bounds obtained on the domain of existence are maximal bounds.

In Chapter 3 we apply the methods of Chapter 2 to the Cauchy problem for equation (1.10) and the extension to the system (1.11). The conclusions are similar to those obtained in Chapter 2.

The arguments in Chapter 4 serve to establish the equivalence (as defined therein) between the characteristic initial value and the Cauchy problems for the system (1.11) and the corresponding problems for a particular system of first order partial differential equations of the form

$\hat{y}_1 = \hat{y}_2 = \dots = \hat{y}_n = \hat{y}$ (11.11)

and the variance of the errors

$$\text{var}(\hat{y}_i) = \sigma^2 \frac{1}{n} \left(1 - \frac{1}{n} \right) \quad (11.12)$$

as easily as the variance-covariance matrix Σ of the observations y . This matrix is positive definite if and only if the matrix A is positive definite. In this case, the matrix A is symmetric and positive definite, and the matrix Σ is symmetric and positive definite. The matrix Σ is positive definite if and only if the matrix A is positive definite. The matrix Σ is positive definite if and only if the matrix A is positive definite.

In Chapter 10, we saw that the matrix of Lagrange multipliers λ can be found by solving the system of equations (10.11) and (10.12). The solution λ can be found by solving the system of equations (10.11) and (10.12). The solution λ can be found by solving the system of equations (10.11) and (10.12). The solution λ can be found by solving the system of equations (10.11) and (10.12).

$$(1.12) \quad \begin{cases} \sum_{k=1}^n A_{ik} U_{uk}, x = C_i & (i = 1, 2, \dots, m < n) \\ \sum_{k=1}^n A_{ik} U_{uk}, y = C_i & (i = m+1, m+2, \dots, n) \end{cases}$$

where the A_{ik}, C_i are functions of $x, y, u_1, u_2, \dots, u_n$. The system (1.12) is termed a canonical hyperbolic system.

This equivalence has already been established by M. CINQUINI-CIBRARIO[12]. Under the restriction that the first partial derivatives of the functions A_{ik}, C_i be Lipschitzian with respect to all their arguments, she obtains her theorems on the existence and uniqueness of the system of functions U_i as the solution for the canonical hyperbolic system (1.12). We demonstrate that her reasoning establishing the equivalence does not depend upon the uniqueness of the solutions for either system (1.11) or system (1.12). Consequently, from our results in Chapters 2 and 3, we are able to remove the above Lipschitz condition entirely and obtain existence, but not uniqueness, for the solutions of the canonical hyperbolic system for both characteristic and Cauchy initial value prescriptions.

Following the attack of H. LEWY[10], in Chapter 5 we reduce the equation

$$(1.1) \quad F(x, y; u; p, q; r, s, t) = 0$$

to a system of so-called characteristic equations by means of a transformation to the characteristic base curves as coordinates. This system is shown to contain a canonical hyperbolic system.

$$\left. \begin{aligned} (1) \quad & \mathcal{L} \{ \delta(x) \} = 1 \\ (2) \quad & \mathcal{L} \{ \delta(x-a) \} = e^{-as} \end{aligned} \right\} \text{Table 1}$$

where \mathcal{L} is the Laplace transform of $f(x)$ and s is the complex frequency variable.

The Laplace transform of a function $f(x)$ is defined by

$$\mathcal{L} \{ f(x) \} = \int_0^{\infty} f(x) e^{-sx} dx \quad (3)$$

where s is a complex number with a positive real part. The inverse Laplace transform is given by

$$f(x) = \mathcal{L}^{-1} \{ F(s) \} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s) e^{sx} ds \quad (4)$$

where γ is a real number greater than the real part of all the poles of $F(s)$.

The Laplace transform is particularly useful for solving differential equations.

For example, the Laplace transform of the differential equation

$$y'' + y = \delta(x) \quad (5)$$

is given by

$$s^2 Y(s) + Y(s) = 1 \quad (6)$$

where $Y(s)$ is the Laplace transform of $y(x)$. Solving for $Y(s)$ gives

$$Y(s) = \frac{1}{s^2 + 1} \quad (7)$$

which can be inverted to give the solution

$$y(x) = \sin(x) \quad (8)$$

for $x > 0$ and $y(x) = 0$ for $x < 0$.

The Laplace transform is also useful for solving integral equations.

For example, the Laplace transform of the integral equation

$$y(x) = \int_0^x y(t) dt + \delta(x) \quad (9)$$

is given by

$$Y(s) = \frac{1}{s^2 + 1} \quad (10)$$

We treat the Cauchy problem, i.e. to find an integral surface which has a second order contact with a prescribed second order strip. By virtue of a theorem by M. CINQUINI-CIARRARIO, stated in Chapter 4, LEWY'S work yields immediately the result that for $F \in C'''$ in a suitable region, there exists a unique solution $u \in C'''$ in a sufficiently small neighborhood of the initial curve. We again demonstrate that the equivalence of the problems is not dependent upon uniqueness of their respective solutions; and, hence, by requiring simply that $F \in C''$ we obtain existence but not uniqueness.

In Chapter 6 we treat the characteristic initial value problem for equation (1.1). We follow a modification of H. LEWY'S method introduced by M. CINQUINI-CIARRARIO[13]. Here equation (1.1) is first transformed into the form

$$(1.13) \quad s = f(x,y; u; p,q; r,t).$$

A modified system of characteristic equations is obtained. This system also contains a canonical hyperbolic system. The theorems of Chapter 2 apply and we obtain results similar to those obtained in Chapter 5 for the Cauchy problem.

In Chapter 7 we treat the mixed boundary value problem for the equation

$$(1.10) \quad s = f(x,y; u; p,q),$$

i.e. the problem where any integral surface of (1.10) is required to pass through two space curves issuing from a point, with one of the curves being a characteristic on this surface and the other

In 1871 the census showed that in 1870 the population of the United States was 39,000,000. This was a very small number compared with the population of the United States in 1900, which was 76,000,000. The increase in population was due to a number of causes, the most important of which were immigration and a high birth rate. The population of the United States in 1900 was 76,000,000, which was an increase of 37,000,000 over the population in 1870. This increase was due to a number of causes, the most important of which were immigration and a high birth rate.

The population of the United States in 1900 was 76,000,000, which was an increase of 37,000,000 over the population in 1870. This increase was due to a number of causes, the most important of which were immigration and a high birth rate. The population of the United States in 1900 was 76,000,000, which was an increase of 37,000,000 over the population in 1870.

The population of the United States in 1900 was 76,000,000, which was an increase of 37,000,000 over the population in 1870. This increase was due to a number of causes, the most important of which were immigration and a high birth rate. The population of the United States in 1900 was 76,000,000, which was an increase of 37,000,000 over the population in 1870.

curve having curve having nowhere a characteristic projection. We show that for equation (1.10) there is no loss in generality if we assume the initial data to be

$$(1.14) \quad u(y,0) = u(x,x) = 0.$$

For f continuous, bounded and Lipschitzian, we prove that there exists one and only one integral surface of (1.10) satisfying (1.14) on a domain for which we prescribe explicit bounds. For f continuous, bounded and partially Lipschitzian, we find, by arguments analogous to those used in Chapters 2 and 3, that there exists at least one integral surface of (1.10) satisfying (1.14) on a domain for which we again prescribe the same type of explicit bounds.

In Chapter 8 we consider the characteristic initial value problem for equation (1.10) from a new point of view. Here, in order to extend the theorems of Chapter 2, we introduce the concept of upper and lower bounding functions for the solution (or solutions) of the problem. This idea was first used by O. PERRON [18] to obtain an existence proof for the problem

$$(1.15) \quad y' = f(x,y) \quad , \quad y(x_0) = y_0.$$

His proof is quite independent of the classical proofs.

H. WILMER [4] shows that PERRON's method has no direct analogue for a system

$$(1.16) \quad y_i' = f_i(x, y_1, \dots, y_n) \quad , \quad (i = 1, \dots, n).$$

He is able, however, to extend the classical theorem for a system (1.16) to obtain a theorem which reduces to the direct analogue to the PERRON theorem in the case where the f_i are monotonically increasing functions of the arguments y_1, \dots, y_n .

... ..

$$... .. (17.1)$$

... ..

... ..

$$... .. (17.2)$$

... ..

$$... .. (17.3)$$

... ..

The extensions to the theorems of Chapter 2 which we obtain are similar to MULLER's conclusions for the system (1.16). Moreover, we demonstrate by example that the PERKON method has no direct analogue for the characteristic initial value problem for equation (1.16). We also give an example illustrating the theorems obtained in this chapter. Finally, we note that the Cauchy problem for equation (1.10) and the Cauchy and characteristic initial value problems for the system

$$(1.11) \quad \begin{aligned} \dot{x}_i &= f_i(x, y; u_1, \dots, u_n; p_1, \dots, p_n, q_1, \dots, q_n) \\ &\quad (i = 1, \dots, n), \end{aligned}$$

may also be treated by the methods of this chapter.

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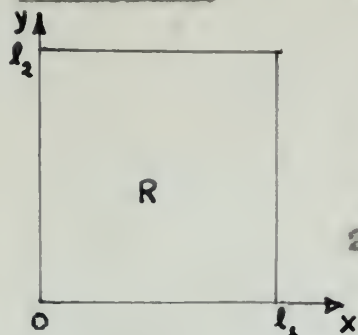
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CHAPTER II

The Characteristic Initial Value Problem for $u_{xy} = f(x,y;u;u_x,u_y)$.

For convenience of reference we first state the following theorem, whose proof is based on the method of successive approximations. The proof of existence was given by É. PICARD [1], while the proof of uniqueness may be found in E. KAMKE [2] p. 410.

Theorem 1.

$$1) \quad f(x,y;u;p,q) \in C(B), B: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \\ -a \leq u \leq a \\ -b_1 \leq p \leq b_1 \\ -b_2 \leq q \leq b_2 \end{cases}$$

2) f is Lipschitzian on B ; i.e. there exists a positive constant K such that for

$$(x,y;u_1;p_1,q_1) \in B, (x,y;u_2;p_2,q_2) \in B,$$

$$|f(x,y;u_1;p_1,q_1) - f(x,y;u_2;p_2,q_2)| \leq K \{ |u_1 - u_2| + |p_1 - p_2| + |q_1 - q_2| \}$$

3) $M l_1 l_2 \leq a$, $M l_1 \leq b_2$, $M l_2 \leq b_1$, where $M = \max |f|$ on B .

\Rightarrow 4) There exists one and only one function $u(x,y) \in C^1(R)$, $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \end{cases}$, such that for each $(x,y) \in R$ the point $(x,y;u(x,y);u_x(x,y),u_y(x,y)) \in B$, and $u_{xy}(x,y) = f(x,y;u(x,y);u_x(x,y),u_y(x,y))$, $u(x,0) = 0$, $u(0,y) = 0$ for each $(x,y) \in R$.

CHAPTER 12

The following is a list of the exercises in this chapter.

12.1.1. 1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13. 14. 15. 16. 17. 18. 19. 20. 21. 22. 23. 24. 25. 26. 27. 28. 29. 30. 31. 32. 33. 34. 35. 36. 37. 38. 39. 40. 41. 42. 43. 44. 45. 46. 47. 48. 49. 50. 51. 52. 53. 54. 55. 56. 57. 58. 59. 60. 61. 62. 63. 64. 65. 66. 67. 68. 69. 70. 71. 72. 73. 74. 75. 76. 77. 78. 79. 80. 81. 82. 83. 84. 85. 86. 87. 88. 89. 90. 91. 92. 93. 94. 95. 96. 97. 98. 99. 100.

The following is a list of the exercises in this chapter. The exercises are arranged in order of increasing difficulty. The first few exercises are relatively easy, while the last few are more challenging. The exercises are numbered 1 through 100.

$$\begin{cases} x^2 + y^2 = 1 \\ x^2 - y^2 = 1 \\ x + y = 1 \\ x - y = 1 \end{cases}$$

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The following is a list of the exercises in this chapter. The exercises are arranged in order of increasing difficulty. The first few exercises are relatively easy, while the last few are more challenging. The exercises are numbered 1 through 100.



$$\begin{aligned} & (x^2 + y^2)^2 = (x^2 - y^2)^2 \\ & (x^2 + y^2)^2 - (x^2 - y^2)^2 = 0 \\ & (x^2 + y^2 + x^2 - y^2)(x^2 + y^2 - x^2 + y^2) = 0 \\ & (2x^2)(2y^2) = 0 \end{aligned}$$

The following is a list of the exercises in this chapter. The exercises are arranged in order of increasing difficulty. The first few exercises are relatively easy, while the last few are more challenging. The exercises are numbered 1 through 100.

Remarks. a) Suppose we prescribe $u(x,0) = U(x)$, $u(0,y) = V(y)$ where $U(x) \in C'([0, \lambda_1])$, $V(y) \in C'([0, \lambda_2])$ and $U(0) = V(0)$. Consider the function $w(x,y) = U(x) + V(y) - U(0)$. Clearly, $w_{xy}(x,y) = 0$ and $w(x,0) = U(x)$, $w(0,y) = V(y)$ hence the function $v = u - w$ must satisfy $v_{xy} = f(x,y; v + w; v_x + w_x, v_y + w_y)$, $v(x,0) = v(0,y) = 0$, a problem of the type covered by Theorem 1.

b) Suppose $f \in C$, bounded and Lipschitzian in the domain B' :

$$\left\{ \begin{array}{l} 0 \leq x \leq \lambda_1 \\ 0 \leq y \leq \lambda_2 \\ -\infty < u < \infty \\ -\infty < p < \infty \\ -\infty < q < \infty \end{array} \right.$$

Then hypothesis 3) is immediately satisfied.

Following an approach used by M. MULLER [3] p. 632, in dealing with a system of first order ordinary differential equations, we are led to this improvement of the above theorem:

Theorem 1a. 1)

2)' f is partially Lipschitzian on B ; i.e. there exists a positive constant K such that for $(x,y; u; p_1, q_1) \in B$, $(x,y; u; p_2, q_2) \in B$, $|f(x,y; u; p_1, q_1) - f(x,y; u; p_2, q_2)| \leq K \{ |p_1 - p_2| + |q_1 - q_2| \}$.

3)

\Rightarrow 4)' There exists at least one function $u(x,y) \in C'(R)$, $u_{xy}(x,y) \in C(R)$, where $B: \begin{cases} 0 \leq x \leq \lambda_1 \\ 0 \leq y \leq \lambda_2 \end{cases}$ such that for each $(x,y) \in R$

$(1) \frac{1}{2} \frac{d}{dt} (x^2 + y^2) = x \dot{x} + y \dot{y} = x(-y) + y(x) = 0$
 $\Rightarrow x^2 + y^2 = C$
 This is the equation of a circle centered at the origin with radius \sqrt{C} .

Let $x = r \cos \theta$ and $y = r \sin \theta$

$$\begin{cases}
 x^2 + y^2 = r^2 \\
 x = r \cos \theta \\
 y = r \sin \theta
 \end{cases}$$

Then $\frac{d}{dt} (x^2 + y^2) = 2x \dot{x} + 2y \dot{y}$

which is

which is zero. This means that the quantity $x^2 + y^2$ is constant.

Therefore

$$\begin{aligned}
 x^2 + y^2 &= C \\
 r^2 &= C \\
 r &= \sqrt{C}
 \end{aligned}$$

is

\Rightarrow The trajectory is a circle centered at the origin with radius \sqrt{C} .

the point $(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in B$, and $u_{xy}(x,y) = f(x,y; u(x,y); u_x(x,y), u_y(x,y))$, $u(x,0) = 0$, $u(0,y) = 0$ for each $(x,y) \in R$.

Proof. According to WEIERSTRASS' celebrated theorem [4] p. 1147, on polynomial approximations to a continuous function, there exists a sequence of polynomials, $\{g_\lambda\}(x,y;u;p,q)$, converging uniformly to $f(x,y; u; p,q)$ on B . We designate this uniform convergence by the notation $\{g_\lambda\} \xrightarrow{\text{unif}} f$ on B .

We extend f and the polynomials g_λ , $(\lambda = 1, 2, \dots)$, over the domain B to the domain B' , defined in the remark b) above, by the definition

$$f(x,y; u; p,q) = f(x,y; \bar{u}; \bar{p}, \bar{q})$$

$$g_\lambda(x,y; u; p,q) = g_\lambda(x,y; \bar{u}; \bar{p}, \bar{q}), \quad (\lambda = 1, 2, \dots),$$

(2.1) where

$$\begin{aligned} \bar{u} &= u \text{ if } -a \leq u \leq a, & \bar{p} &= p \text{ if } -b_1 \leq p \leq b_1, & \bar{q} &= q \text{ if } -b_2 \leq q \leq b_2. \\ \bar{u} &= a \text{ if } a < u & \bar{p} &= b_1 \text{ if } b_1 < p & \bar{q} &= b_2 \text{ if } b_2 < q \\ \bar{u} &= -a \text{ if } u < -a & \bar{p} &= -b_1 \text{ if } p < -b_1 & \bar{q} &= -b_2 \text{ if } q < -b_2 \end{aligned}$$

From this extended definition we see that $|f| \leq M$ in B' . Since $\{g_\lambda\} \xrightarrow{\text{unif}} f$ in B' , there exists a constant $L > 0$ such that $|g_\lambda| \leq L$ in B' and for all λ . The functions g_λ , $(\lambda = 1, 2, \dots)$ are uniformly continuous in B' , moreover they possess bounded difference quotients with respect to the arguments u , p and q everywhere in B' . Hence in B' , for each function g_λ there exists a constant $K_\lambda > 0$ such that

the same that shows $\int_{\mathbb{R}^n} \chi_{\Omega} \delta(x) dx = 1$ and $\int_{\mathbb{R}^n} \chi_{\Omega} \delta(x-a) dx = \chi_{\Omega}(a)$

Lemma Let $\Omega \subset \mathbb{R}^n$ be an open set and χ_{Ω} its characteristic function. Then $\chi_{\Omega} \in L^1(\mathbb{R}^n)$ if and only if Ω has finite volume. In this case $\int_{\mathbb{R}^n} \chi_{\Omega} dx = \text{Vol}(\Omega)$.

Proof If Ω has finite volume, then $\chi_{\Omega} \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \chi_{\Omega} dx = \text{Vol}(\Omega)$. Conversely, if $\chi_{\Omega} \in L^1(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} \chi_{\Omega} dx < \infty$, which implies that Ω has finite volume.

Let $\Omega \subset \mathbb{R}^n$ be an open set with finite volume. Then $\chi_{\Omega} \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \chi_{\Omega} dx = \text{Vol}(\Omega)$. Moreover, $\chi_{\Omega} \in L^p(\mathbb{R}^n)$ for all $p \geq 1$.

$$(2.2) \quad |g_\lambda(x, y; u_1; p_1, q_1) - g_\lambda(x, y; u_2; p_2, q_2)| \leq K_\lambda \left\{ |u_1 - u_2| + |p_1 - p_2| + |q_1 - q_2| \right\}.$$

Thus, by Theorem 1, to each g_λ there corresponds one and only one function $u_\lambda(x, y) \in C^1(R)$, $u_{\lambda, xy}(x, y) \in C(R)$ satisfying

$$(2.3) \quad \begin{cases} u_{\lambda, xy} = g_\lambda(x, y; u_\lambda(x, y); u_{\lambda, x}(x, y), u_{\lambda, y}(x, y)), \\ u_\lambda(x, 0) = 0, \quad u_\lambda(0, y) = 0 \quad \text{for each } (x, y) \in R. \end{cases}$$

We may express the characteristic initial value problem for each u_λ in the form of an equivalent integral equation

$$(2.4) \quad u_\lambda(x, y) = \int_0^x d\xi \int_0^y g_\lambda(\xi, \eta; u_\lambda(\xi, \eta); u_{\lambda, x}(\xi, \eta), u_{\lambda, y}(\xi, \eta)) d\eta.$$

By differentiation,

$$(2.5) \quad u_{\lambda, x}(x, y) = \int_0^y g_\lambda(x, \eta; u_\lambda(x, \eta); u_{\lambda, x}(x, \eta), u_{\lambda, y}(x, \eta)) d\eta$$

$$(2.6) \quad u_{\lambda, y}(x, y) = \int_0^x g_\lambda(\xi, y; u_\lambda(\xi, y); u_{\lambda, x}(\xi, y), u_{\lambda, y}(\xi, y)) d\xi.$$

We now show that the sequences $\{u_\lambda\}$, $\{u_{\lambda, x}\}$, $\{u_{\lambda, y}\}$ are each uniformly bounded and equicontinuous on R . For the sequence $\{u_\lambda\}$ this follows directly from the integral expression

$$(2.4), \text{ for, given } x, x_1, x_2 \in [0, \ell_1] \text{ and } y, y_1, y_2 \in [0, \ell_2],$$

$$(2.7) \quad |u_\lambda(x, y)| \leq L \ell_1 \ell_2, \quad (\lambda = 1, 2, \dots)$$

$$(2.8) \quad |u_\lambda(x_1, y_1) - u_\lambda(x_2, y_2)| \leq L |x_1 - x_2| |y_1 - y_2| + L \ell_2 |x_1 - x_2| + L \ell_1 |y_1 - y_2|, \quad (\lambda = 1, 2, \dots)$$

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.
Then $f(x) + g(x) = 2x^2 + 2$ and $f(x) - g(x) = 4x$.

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.
Then $f(x) + g(x) = 2x^2 + 2$ and $f(x) - g(x) = 4x$.

$$\begin{aligned} & (f(x) + g(x))^2 = (2x^2 + 2)^2 = 4x^4 + 8x^2 + 4 \\ & (f(x) - g(x))^2 = (4x)^2 = 16x^2 \end{aligned}$$

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.
Then $f(x) + g(x) = 2x^2 + 2$ and $f(x) - g(x) = 4x$.

$$\begin{aligned} & (f(x) + g(x))^2 - (f(x) - g(x))^2 = (2x^2 + 2)^2 - (4x)^2 \\ & = 4x^4 + 8x^2 + 4 - 16x^2 = 4x^4 - 8x^2 + 4 \\ & = (2x^2 - 2)^2 \end{aligned}$$

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$$(f(x) + g(x))^2 - (f(x) - g(x))^2 = (2x^2 + 2)^2 - (4x)^2$$

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$.
Then $f(x) + g(x) = 2x^2 + 2$ and $f(x) - g(x) = 4x$.

The uniform boundedness of $\{u_{\lambda,x}\}$ and of $\{u_{\lambda,y}\}$ follow directly from (2.5) and (2.6), respectively, for, given $(x,y) \in R$,

$$(2.9) \quad |u_{\lambda,x}(x,y)| \leq L f_2, \quad (\lambda = 1, 2, \dots)$$

$$(2.10) \quad |u_{\lambda,y}(x,y)| \leq L f_1, \quad (\lambda = 1, 2, \dots).$$

We base the proof of the equicontinuity of the functions of the sequence $\{u_{\lambda,x}\}$ upon the following two lemmas, the first of which is due to T. H. GRONWALL [5].

Lemma 1. 1) $Z(y) \in C([0, l])$

$$(2.11) \quad 2) \quad 0 \leq Z(y) \leq \int_0^y (MZ(\eta) + A) d\eta + B \quad \text{for } y \in [0, l]$$

where M , A and B are constants ≥ 0 .

$$(2.12) \quad 3) \quad 0 \leq Z(y) \leq (Al + B) e^{My} \quad \text{for } y \in [0, l].$$

Lemma 2. Given $\mu > 0$, $\zeta > 0$, there exist δ , a positive constant depending upon μ alone, and N , a positive integer depending upon ζ alone, such that whenever $(x_1, y) \in R$, $(x_2, y) \in R$, $|x_1 - x_2| < \delta$ and $\lambda > N$,

$$(2.13) \quad |u_{\lambda,x}(x_2, y) - u_{\lambda,x}(x_1, y)| \leq K \int_0^y |u_{\lambda,x}(x_2, \eta) - u_{\lambda,x}(x_1, \eta)| d\eta + \mu + \zeta$$

where K is the partial Lipschitz constant for $f(x, y; u; p, q)$.

Assume, for the moment, the validity of these two lemmas. Each of the functions $u_{\lambda,x}$ is certainly uniformly continuous on R ; hence, if we let $Z(y) = |u_{\lambda,x}(x_2, y) - u_{\lambda,x}(x_1, y)|$ for any particular $\lambda > N$,

the region bounded by $x^2 + y^2 = 4$ and $y = x^2$ in the first quadrant. The region is bounded by the x-axis, the y-axis, the curve $y = x^2$, and the circle $x^2 + y^2 = 4$.

$$V = \int_0^2 \int_{x^2}^{\sqrt{4-x^2}} dy dx = \int_0^2 [y]_{x^2}^{\sqrt{4-x^2}} dx = \int_0^2 (\sqrt{4-x^2} - x^2) dx$$

$$= \left[\frac{x}{2} \sqrt{4-x^2} + \frac{2}{3} x^3 \right]_0^2 = \left[\frac{2}{2} \sqrt{4-4} + \frac{2}{3} (8) \right] - 0 = \frac{16}{3}$$

The volume of the solid is $\frac{16}{3}$ cubic units. The region is bounded by the x-axis, the y-axis, the curve $y = x^2$, and the circle $x^2 + y^2 = 4$.

Example 2 Find the volume of the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

$$V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz dy dx = \int_0^1 \int_0^{1-x} (1-x-y) dy dx = \int_0^1 \left[(1-x)y - \frac{1}{2}y^2 \right]_0^{1-x} dx$$

$$= \int_0^1 \left((1-x)(1-x) - \frac{1}{2}(1-x)^2 \right) dx = \int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2 \right) dx = \int_0^1 \frac{1}{2}(1-x)^2 dx$$

$$= \frac{1}{2} \left[-\frac{1}{3}(1-x)^3 \right]_0^1 = \frac{1}{2} \left(0 - \left(-\frac{1}{3} \right) \right) = \frac{1}{6}$$

$$\therefore V = \frac{1}{6} \text{ cubic units}$$

The volume of the solid is $\frac{1}{6}$ cubic units. The solid is bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

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we have immediately that for $|x_2 - x_1| < \delta$,

$$(2.14) \quad |u_{\lambda, x}(x_2, y) - u_{\lambda, x}(x_1, y)| \leq (\mu + \zeta) e^{K/2}.$$

Suppose $(x_1, y) \in R$, $(x_2, y) \in R$, then certainly $(x_2, y_1) \in R$ and

$$(2.15) \quad |u_{\lambda, x}(x_2, y_2) - u_{\lambda, x}(x_1, y_1)| \leq |u_{\lambda, x}(x_2, y_2) - u_{\lambda, x}(x_2, y_1)| \\ + |u_{\lambda, x}(x_2, y_1) - u_{\lambda, x}(x_1, y_1)|, \quad (\lambda = 1, 2, \dots).$$

By (2.5) we have that

$$(2.16) \quad |u_{\lambda, x}(x_2, y_2) - u_{\lambda, x}(x_2, y_1)| \leq L |y_2 - y_1|, \quad (\lambda = 1, 2, \dots).$$

Inequalities (2.14), (2.15) and (2.16) yield immediately the equicontinuity on R of the functions of the sequence $\{u_{\lambda, x}\}$; for, given $\epsilon > 0$, we first choose $\mu > 0$ and $\zeta > 0$ such that

$$(2.17) \quad \mu + \zeta < \frac{\epsilon}{2e^{K/2}}$$

and let δ and N be the corresponding constants given by Lemma 2.

By the uniform continuity on R of each of the functions $u_{\lambda, x}$, there exists a positive constant δ_N , depending on ϵ alone, such that

$$|x_1 - x_2| < \delta_N \quad \text{and} \quad |y_1 - y_2| < \delta_N \implies$$

$$(2.18) \quad |u_{\lambda, x}(x_2, y_2) - u_{\lambda, x}(x_1, y_1)| < \epsilon, \quad (\lambda = 1, 2, \dots, N).$$

Setting $\delta_0 = \min(\delta, \delta_N, \frac{\epsilon}{2L})$ we obtain

we have $\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}^n} f(x) dx$

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$$|x_1 - x_2| < \delta_0 \quad \text{and} \quad |y_1 - y_2| < \delta_0 \quad \Rightarrow$$

$$(2.19) \quad |u_{\lambda, x}(x_2, y_2) - u_{\lambda, x}(x_1, y_1)| < \epsilon, \quad \text{for } \lambda = 1, 2, \dots, N, N+1, \dots$$

Proof of Lemma 1: Let $Z(y) = e^{My} \cdot w(y)$, without loss, for we may always choose $w(y) = e^{-My} \cdot Z(y)$. $w(y) \in C([0, \ell])$ and hence attains a maximum thereon. Let w_{\max} occur at $y = y_1$, then

$$\begin{aligned} 0 &\leq e^{My_1} w_{\max} \leq \int_0^{y_1} (M e^{M\eta} w(\eta) + A) d\eta + B \\ &\leq w_{\max} \int_0^{y_1} M e^{M\eta} d\eta + A y_1 + B \\ &= w_{\max} (e^{My_1} - 1) + A y_1 + B \end{aligned}$$

Thus $0 \leq w_{\max} \leq A y_1 + B \leq A\ell + B$ and hence

$$0 \leq Z(y) \leq (A\ell + B) e^{M\ell} \quad \text{for } y \in [0, \ell].$$

Proof of Lemma 2:

$$\begin{aligned} (2.20) \quad u_{\lambda, x}(x_2, y) - u_{\lambda, x}(x_1, y) &= \int_0^y \left[g_{\lambda}(x_2, \eta; u_{\lambda}(x_2, \eta); \right. \\ &\quad \left. u_{\lambda, x}(x_2, \eta), u_{\lambda, y}(x_2, \eta)) \right. \\ &\quad \left. - g_{\lambda}(x_1, \eta; u_{\lambda}(x_1, \eta); u_{\lambda, x}(x_1, \eta), \right. \\ &\quad \left. u_{\lambda, y}(x_1, \eta)) \right] d\eta \\ &= \int_0^y \left[g_{\lambda}(x_2, \eta; u_{\lambda}(x_2, \eta); u_{\lambda, x}(x_2, \eta), \right. \\ &\quad \left. u_{\lambda, y}(x_2, \eta)) \right. \\ &\quad \left. - f(x_2, \eta; u_{\lambda}(x_2, \eta); u_{\lambda, x}(x_2, \eta), \right. \\ &\quad \left. u_{\lambda, y}(x_2, \eta)) \right] d\eta \\ &\quad + \int_0^y \left[f(x_2, \eta; u_{\lambda}(x_2, \eta); u_{\lambda, x}(x_2, \eta), \right. \\ &\quad \left. u_{\lambda, y}(x_2, \eta)) \right] d\eta \end{aligned}$$

$$\Rightarrow \frac{1}{2} > |a_2 - \frac{1}{2}| \Rightarrow \frac{1}{2} > |a_2 - \frac{1}{2}|$$

$$x_1, x_2, x_3, x_4, x_5 = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \quad (10.20)$$

Let us consider the function $f(x) = x^2 + 1$ and $f'(x) = 2x$. The function $f(x)$ is convex and $f'(x)$ is increasing. The function $f(x)$ is also concave and $f'(x)$ is decreasing. The function $f(x)$ is also linear and $f'(x)$ is constant.

$$f(x) = x^2 + 1 = f'(x) \cdot x + f(0) = 2x \cdot x + 1 = 2x^2 + 1$$

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(10.21)

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(2.20)
(Continued)

$$\begin{aligned}
 & - f(x_2, \eta; u_\lambda(x_2, \eta); u_{\lambda, x}(x_1, \eta), \\
 & \quad u_{\lambda, y}(x_2, \eta))] d\eta \\
 & + \int_0^y [f(x_2, \eta; u_\lambda(x_2, \eta); u_{\lambda, x}(x_1, \eta), \\
 & \quad u_{\lambda, y}(x_2, \eta)) \\
 & - f(x_1, \eta; u_\lambda(x_1, \eta); u_{\lambda, x}(x_1, \eta), \\
 & \quad u_{\lambda, y}(x_1, \eta))] d\eta \\
 & + \int_0^y [f(x_1, \eta; u_\lambda(x_1, \eta); u_{\lambda, x}(x_1, \eta), \\
 & \quad u_{\lambda, y}(x_1, \eta)) \\
 & - \varepsilon_\lambda(x_1, \eta; u_\lambda(x_1, \eta); u_{\lambda, x}(x_1, \eta), \\
 & \quad u_{\lambda, y}(x_1, \eta))] d\eta \\
 & \quad (\lambda = 1, 2, \dots).
 \end{aligned}$$

Since $\{\varepsilon_\lambda\} \xrightarrow{\text{unif}} f$ on E' , given $\zeta > 0$, there exists a positive integer N , depending upon ζ alone, such that for $\lambda > N$,

$$\begin{aligned}
 (2.21) \quad & \left| \int_0^y [\varepsilon_\lambda(x_2, \eta; u_\lambda(x_2, \eta); u_{\lambda, x}(x_2, \eta), u_{\lambda, y}(x_2, \eta)) - \right. \\
 & \quad \left. f(x_2, \eta; u_\lambda(x_2, \eta); u_{\lambda, x}(x_2, \eta), u_{\lambda, y}(x_2, \eta))] d\eta \right| \\
 & + \left| \int_0^y [f(x_1, \eta; u_\lambda(x_1, \eta); u_{\lambda, x}(x_1, \eta), u_{\lambda, y}(x_1, \eta)) - \right. \\
 & \quad \left. \varepsilon_\lambda(x_1, \eta; u_\lambda(x_1, \eta); u_{\lambda, x}(x_1, \eta), u_{\lambda, y}(x_1, \eta))] d\eta \right| < \zeta
 \end{aligned}$$

By hypothesis 2)',

$$(2.22) \quad \left| \int_0^y [f(x_2, \eta; u_\lambda(x_2, \eta); u_{\lambda, x}(x_2, \eta), u_{\lambda, y}(x_2, \eta)) - \right.$$

$$y = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d}{dx} \right]^k f(x) \Big|_{x=0}$$

(Taylor's series)

$$y = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d}{dx} \right]^k f(x) \Big|_{x=0} + \dots$$

$$y = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d}{dx} \right]^k f(x) \Big|_{x=0}$$

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$$y = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d}{dx} \right]^k f(x) \Big|_{x=0}$$

is a series

Let $f(x) = e^{ax}$, then $f'(x) = ae^{ax}$, $f''(x) = a^2 e^{ax}$, $f'''(x) = a^3 e^{ax}$, $f^{(k)}(x) = a^k e^{ax}$.
 At $x=0$, $f(0) = 1$, $f'(0) = a$, $f''(0) = a^2$, $f'''(0) = a^3$, $f^{(k)}(0) = a^k$.

$$y = \sum_{k=0}^{\infty} \frac{1}{k!} a^k e^{ax} \Big|_{x=0} = \sum_{k=0}^{\infty} \frac{a^k}{k!}$$

$$y = \sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$$

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is a series

$$y = \sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$$

$$(2.22) \quad \begin{aligned} & \text{(Continued)} \quad -f(x_2, \eta; u_{\lambda}(x_2, \eta); u_{\lambda, x}(x_1, \eta), u_{\lambda, y}(x_2, \eta))] \, d\eta | \\ & \leq K \int_0^y |u_{\lambda, x}(x_2, \eta) - u_{\lambda, x}(x_1, \eta)| \, d\eta, \quad (\lambda = 1, 2, \dots) \end{aligned}$$

Since f is uniformly continuous on B' , the functions of the sequence $\{u_{\lambda}\}$ are equicontinuous on R , and $|u_{\lambda, y}(x_2, \eta) - u_{\lambda, y}(x_1, \eta)| \leq L |x_2 - x_1|$, $(\lambda = 1, 2, \dots)$, it follows that given $\mu > 0$ there exists a positive constant δ , depending upon μ alone, such that for $|x_2 - x_1| < \delta$,

$$(2.23) \quad \begin{aligned} & \left| \int_0^y [f(x_2, \eta; u_{\lambda}(x_2, \eta); u_{\lambda, x}(x_1, \eta), u_{\lambda, y}(x_2, \eta)) \right. \\ & \quad \left. - f(x_1, \eta; u_{\lambda}(x_1, \eta); u_{\lambda, x}(x_1, \eta), u_{\lambda, y}(x_1, \eta))] \, d\eta \right| < \mu, \\ & (\lambda = 1, 2, \dots). \end{aligned}$$

Therefore, from (2.21), (2.22) and (2.23), by virtue of (2.20) we obtain that for $\lambda > N$ and $|x_2 - x_1| < \delta$,

$$(2.13) \quad |u_{\lambda, x}(x_2, y) - u_{\lambda, x}(x_1, y)| < K \int_0^y |u_{\lambda, x}(x_2, \eta) - u_{\lambda, x}(x_1, \eta)| \, d\eta + \mu + \zeta$$

thus verifying Lemma 2.

The proof of the equicontinuity of the functions of the sequence $\{u_{\lambda, y}\}$ follows precisely the same steps as that for the sequence $\{u_{\lambda, x}\}$.

We now invoke the well-known theorem of C. ARZELA [3] p. 1144:

"Given a set F of functions f defined and continuous on the closed bounded set A , then the necessary and sufficient conditions that each sequence of functions contained in F possesses

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi$$

$$\int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi$$

... and the conditions for the convergence of the integral ...
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a subsequence uniformly convergent on A are that \mathcal{P} be uniformly bounded and equicontinuous."

By Theorem 1, there exists a unique triple $(u_\lambda; u_{\lambda,x}; u_{\lambda,y})$ corresponding to g_λ for each λ . Since any subsequence of a uniformly convergent sequence is likewise uniformly convergent; and, since any subsequence of a uniformly bounded and equicontinuous sequence is likewise uniformly bounded and equicontinuous; there exists a subsequence $\{g_\lambda^*\}$ of the sequence $\{g_\lambda\}$ such that the corresponding sequences

$$(2.24) \quad \{u_\lambda^*\} \xrightarrow{\text{unif}} u, \quad \{u_{\lambda,x}^*\} \xrightarrow{\text{unif}} v, \quad \{u_{\lambda,y}^*\} \xrightarrow{\text{unif}} w,$$

where $u, v, w \in C(\mathbb{R})$. This results from the following successive extractions of subsequences:

$\{u_\lambda\}$ is equicontinuous and uniformly bounded on \mathbb{R} , hence there exists a subsequence $\{u_\lambda^1\}$ of $\{u_\lambda\}$ uniformly convergent on \mathbb{R} . $\{u_{\lambda,x}^1\}$ is equicontinuous and uniformly bounded on \mathbb{R} , hence there exists a subsequence $\{u_{\lambda,x}^2\}$ of $\{u_{\lambda,x}^1\}$ uniformly convergent on \mathbb{R} . $\{u_{\lambda,y}^2\}$ is equicontinuous and uniformly bounded on \mathbb{R} , hence there exists a subsequence $\{u_{\lambda,y}^*\}$ of $\{u_{\lambda,y}^2\}$ uniformly convergent on \mathbb{R} . But, by the one-to-one correspondence mentioned above, $\{u_{\lambda,x}^*\}$ is a subsequence of $\{u_{\lambda,x}^2\}$ while $\{u_\lambda^*\}$ is a subsequence of $\{u_\lambda^1\}$. Thus $\{u_{\lambda,x}^*\}$ and $\{u_\lambda^*\}$ are each uniformly convergent on \mathbb{R} .

Writing, with the notation $u_0^* = u_{0,x}^* = u_{0,y}^* = 0$,

Let $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ be three vectors in space. Then the volume of the parallelepiped formed by these vectors is given by the scalar triple product $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$. This product is equal to the determinant of the matrix whose rows are the components of the vectors.

$$(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

The volume of the parallelepiped is zero if the vectors are coplanar. This is the case when the determinant is zero. The volume is positive or negative depending on the orientation of the vectors.

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The volume of the parallelepiped is zero if the vectors are coplanar. This is the case when the determinant is zero. The volume is positive or negative depending on the orientation of the vectors.

$$(2.25) \quad u_{\lambda}^* = \sum_{k=1}^{\lambda} (u_k^* - u_{k-1}^*), \quad u_{\lambda,x}^* = \sum_{k=1}^{\lambda} (u_{k,x}^* - u_{k-1,x}^*),$$

$$u_{\lambda,y}^* = \sum_{k=1}^{\lambda} (u_{k,y}^* - u_{k-1,y}^*), \quad (\lambda = 1, 2, \dots),$$

we see that the conditions for differentiation under the summation sign for infinite series are satisfied by (2.24) and the fact that $u_{\lambda}^* \in C^1(R)$, $(\lambda = 1, 2, \dots)$. Hence

$$(2.26) \quad v(x,y) = u_x(x,y), \quad w(x,y) = u_y(x,y) \quad \text{for } (x,y) \in R$$

We show that the function u so determined satisfies the integral equation equivalent to the original characteristic initial value problem, i.e.

$$(2.27) \quad u(x,y) = \int_0^x d\xi \int_0^y f(\xi, \eta; u(\xi, \eta); u_x(\xi, \eta), u_y(\xi, \eta)) d\eta$$

for $(x,y) \in R$.

For any λ , by (2.4),

$$(2.28) \quad |u(x,y) - \int_0^x d\xi \int_0^y f(\xi, \eta; u(\xi, \eta); u_x(\xi, \eta), u_y(\xi, \eta)) d\eta| \\ \leq |u(x,y) - u_{\lambda}^*(x,y)| + \int_0^x d\xi \int_0^y |f(\xi, \eta; u(\xi, \eta); u_x(\xi, \eta), u_y(\xi, \eta)) - f(\xi, \eta; u_{\lambda}^*(\xi, \eta); u_{\lambda,x}^*(\xi, \eta), u_{\lambda,y}^*(\xi, \eta))| d\eta \\ + \int_0^x d\xi \int_0^y |f(\xi, \eta; u_{\lambda}^*(\xi, \eta); u_{\lambda,x}^*(\xi, \eta), u_{\lambda,y}^*(\xi, \eta)) - g_{\lambda}^*(\xi, \eta; u_{\lambda}^*(\xi, \eta); u_{\lambda,x}^*(\xi, \eta), u_{\lambda,y}^*(\xi, \eta))| d\eta$$

Since $\{g_{\lambda}^*\} \xrightarrow{\text{unif}} f$ on B' , $\{u_{\lambda}^*\} \xrightarrow{\text{unif}} u$ on R , given $\epsilon > 0$ and $(x,y) \in R$, there exists a positive integer N_1 , depending upon ϵ alone, such that for $\lambda > N_1$,

$$\frac{d}{dt} \left(\frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 + \frac{1}{2} m \dot{z}^2 \right) = \sum_i \mathbf{F}_i \cdot \mathbf{v}_i$$

$$= \sum_i \left(\frac{\partial L}{\partial \mathbf{r}_i} \cdot \mathbf{v}_i + \frac{\partial L}{\partial \dot{\mathbf{r}}_i} \cdot \dot{\mathbf{v}}_i \right)$$

For the case of a particle in a potential field, the Lagrangian is given by $L = T - V$. The total energy is conserved if the potential is time-independent, i.e., $\frac{\partial L}{\partial t} = 0$.

The Hamiltonian function is defined as $H = T + V$. For a system with a time-independent potential, the Hamiltonian is constant in time, representing the total mechanical energy of the system.

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_i \left(\frac{\partial H}{\partial \mathbf{r}_i} \cdot \dot{\mathbf{r}}_i + \frac{\partial H}{\partial \dot{\mathbf{r}}_i} \cdot \ddot{\mathbf{r}}_i \right)$$

$$= 0 + \sum_i \left(\mathbf{F}_i \cdot \dot{\mathbf{r}}_i + \mathbf{v}_i \cdot m \ddot{\mathbf{r}}_i \right)$$

$$= \sum_i \left(\mathbf{F}_i \cdot \dot{\mathbf{r}}_i + \dot{\mathbf{r}}_i \cdot \mathbf{F}_i \right) = 2 \sum_i \mathbf{F}_i \cdot \dot{\mathbf{r}}_i$$

Since $\mathbf{F}_i = -\nabla V$, the expression for $\frac{dH}{dt}$ becomes $\frac{dH}{dt} = -2 \sum_i \nabla V \cdot \dot{\mathbf{r}}_i$. This is zero if the potential is time-independent, confirming energy conservation.

$$\frac{dH}{dt} = -2 \sum_i \nabla V \cdot \dot{\mathbf{r}}_i$$

$$= 0 \quad \text{if } \frac{\partial V}{\partial t} = 0$$

Therefore, the total energy of a system is conserved when the potential energy does not depend explicitly on time.

$$(2.29) \quad |u(x,y) - u_{\lambda}^*(x,y)| < \epsilon,$$

$$(2.30) \quad \int_0^x d\xi \int_0^y |f(\xi, \eta; u_{\lambda}^*(\xi, \eta); u_{\lambda, x}^*(\xi, \eta), u_{\lambda, y}^*(\xi, \eta)) \\ - f(\xi, \eta; u_{\lambda}^*(\xi, \eta); u_{\lambda, x}^*(\xi, \eta), u_{\lambda, y}^*(\xi, \eta))| d\eta \\ < \epsilon l_1 l_2.$$

Moreover, since f is uniformly continuous in B' while $\{u_{\lambda}^*\}$, $\{u_{\lambda, x}^*\}$, $\{u_{\lambda, y}^*\}$ converge uniformly on R to u , u_x , u_y respectively, there exists a positive integer N_2 , depending on ϵ alone, such that for $\lambda > N_2$,

$$(2.31) \quad \int_0^x d\xi \int_0^y |f(\xi, \eta; u(\xi, \eta); u_x(\xi, \eta), u_y(\xi, \eta)) \\ - f(\xi, \eta; u_{\lambda}^*(\xi, \eta); u_{\lambda, x}^*(\xi, \eta), u_{\lambda, y}^*(\xi, \eta))| d\eta \\ < \epsilon l_1 l_2.$$

Introducing (2.29), (2.30) and (2.31) into (2.28), we obtain that for $\lambda > \max(N_1, N_2)$

$$(2.32) \quad |u(x,y) - \int_0^x d\xi \int_0^y f(\xi, \eta; u(\xi, \eta); u_x(\xi, \eta), u_y(\xi, \eta)) \\ < \epsilon(1 + 2l_1 l_2)$$

But ϵ is arbitrary, hence (2.27) is verified for each $(x,y) \in R$. We must verify the one additional fact that for each $(x,y) \in R$, $(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in B$, instead of just belonging to B' .

$$f(x) = \frac{1}{x^2} = x^{-2} \Rightarrow f'(x) = -2x^{-3} = -\frac{2}{x^3} \quad (2.2)$$

$$f(x) = \frac{1}{x^3} = x^{-3} \Rightarrow f'(x) = -3x^{-4} = -\frac{3}{x^4}$$

$$f(x) = \frac{1}{x^4} = x^{-4} \Rightarrow f'(x) = -4x^{-5} = -\frac{4}{x^5}$$

$$f'(x) = -\frac{4}{x^5}$$

Example 3: Find the derivative of $f(x) = \frac{1}{x^2} + \frac{1}{x^3}$.
 Solution: $f(x) = x^{-2} + x^{-3}$
 $f'(x) = -2x^{-3} - 3x^{-4} = -\frac{2}{x^3} - \frac{3}{x^4}$

$$f(x) = \frac{1}{x^2} + \frac{1}{x^3} \Rightarrow f'(x) = -\frac{2}{x^3} - \frac{3}{x^4}$$

$$f(x) = \frac{1}{x^5} = x^{-5} \Rightarrow f'(x) = -5x^{-6} = -\frac{5}{x^6}$$

$$f'(x) = -\frac{5}{x^6}$$

Example 4: Find the derivative of $f(x) = \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4}$.
 Solution: $f(x) = x^{-2} + x^{-3} + x^{-4}$
 $f'(x) = -2x^{-3} - 3x^{-4} - 4x^{-5} = -\frac{2}{x^3} - \frac{3}{x^4} - \frac{4}{x^5}$

$$f(x) = \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} \Rightarrow f'(x) = -\frac{2}{x^3} - \frac{3}{x^4} - \frac{4}{x^5}$$

$$f'(x) = -\frac{2}{x^3} - \frac{3}{x^4} - \frac{4}{x^5}$$

Example 5: Find the derivative of $f(x) = \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5}$.
 Solution: $f(x) = x^{-2} + x^{-3} + x^{-4} + x^{-5}$
 $f'(x) = -2x^{-3} - 3x^{-4} - 4x^{-5} - 5x^{-6} = -\frac{2}{x^3} - \frac{3}{x^4} - \frac{4}{x^5} - \frac{5}{x^6}$

By differentiation from (2.27),

$$(2.33) \quad u_x(x,y) = \int_0^y f(x,\eta; u(x,\eta); u_x(x,\eta), u_y(x,\eta)) d\eta$$

$$(2.34) \quad u_y(x,y) = \int_0^x f(\xi,y; u(\xi,y); u_x(\xi,y), u_y(\xi,y)) d\xi.$$

Hence, from the extended definition of f , (2.1), and hypothesis 3),

$$(2.35) \quad |u(x,y)| \leq \int_0^x d\xi \int_0^y |f(\xi,\eta; u(\xi,\eta); u_x(\xi,\eta), u_y(\xi,\eta))| d\eta \\ \leq M'_{12} \leq a$$

$$(2.36) \quad |u_x(x,y)| \leq \int_0^y |f(x,\eta; u(x,\eta); u_x(x,\eta), u_y(x,\eta))| d\eta \\ \leq M'_2 \leq b_1$$

$$(2.37) \quad |u_y(x,y)| \leq \int_0^x |f(\xi,y; u(\xi,y); u_x(\xi,y), u_y(\xi,y))| d\xi \\ \leq M'_1 \leq b_2,$$

thus completing the proof of Theorem 1a.

Remarks a) and b) to Theorem 1 apply as well to Theorem 1a.

By the following example we show that the integral surfaces for Theorem 1a are not necessarily unique:

Example 1 Consider the characteristic initial value problem:

$$(2.38) \quad u_{xy} = |u|^{\frac{1}{2}}; \quad u(x,0) = u(0,y) = 0.$$

Here $f(x,y; u; p,q) = |u|^{\frac{1}{2}}$ is continuous for all u but fails to satisfy a Lipschitz condition on u at $u = 0$. Theorem 1a applies

to insure existence of a solution in a sufficiently small neighborhood of the origin. However, at least two solutions, valid for all (x,y) in the finite plane, are directly available. First, $u = 0$ obviously satisfies. Second, if we seek a solution u satisfying

- 1) $u \geq 0$,
- ii) there exist functions X, Y such that

$$u(x,y) = X(x) \cdot Y(y);$$

that is, by the method of separation of variables, we obtain immediately the solution $u(x,y) = \frac{1}{16} x^2 y^2$.

For purposes of illustrating the various situations that might occur, we give the following:

Example 2. Consider the characteristic initial value problem:

$$(2.39) \quad u_{xy} = |u_x|^{\frac{1}{2}}; \quad u(x,0) = u(0,y) = 0.$$

Here $f(x,y; u; p,q) = |p|^{\frac{1}{2}}$ is continuous for all p but fails to satisfy a Lipschitz condition on p at $p = 0$. Since $p(x,0) = u_x(x,0) = 0$ neither Theorem 1 nor Theorem 1a will insure existence of a solution over any domain including a portion of the x axis. Such solutions do exist, however. One is $u = 0$. Under the assumption $p = u_x \geq 0$ we obtain another, for now

$$p_y = p^{\frac{1}{2}} \quad \text{or}$$

$$\frac{dp}{p^{\frac{3}{2}}} = 2p^{-\frac{1}{2}} = y + c_1.$$

Since $p(x,0) = 0$, $c_1 = 0$ and

$$p = u_x = \frac{y^2}{4} \quad \text{or, integrating,}$$

$$u = \frac{xy^2}{4} + c_2.$$

Since $u(0,y) = 0$, $c_2 = 0$; and hence

$$u = \frac{xy^2}{4}$$

is a second solution valid throughout the finite plane.

In Example 2 consider the function

$$u_0(x,y) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{xy^2}{4} & \text{for } x \geq 0. \end{cases}$$

u_0 is continuous for all (x,y) and satisfies the initial value problem (2.39) everywhere except along the y axis, where for $y \neq 0$, $u_{0x}(0,y)$ does not exist. Roughly speaking, u_0 is a continuous integral surface of problem (2.39) having a jump in the normal first derivative across a characteristic.

For equations meeting the continuity, boundedness and partial Lipschitz requirements of Theorem 1a we cannot match integral surfaces in the above fashion to obtain first derivative jumps across characteristics. This follows from the fact that if we prescribe $u(a,y) = v(y) \in C'([0, \ell_2])$ along the characteristic $x=a$, $a \in [0, \ell_1]$, then

$$(2.40) \quad \begin{cases} p_y(a,y) = f(a,y; v(y); p(a,y), v'(y)) \\ p(a,0) = 0 \end{cases}$$

represents a first order ordinary differential equation for the

unknown function $p = u_x$ under a one point boundary condition. The conditions that f be continuous, bounded and partially Lipschitzian are sufficient to insure the existence of a unique determination of $u_x(a,y)$ for $y \in [0, l_2]$. Note that in Example 2 the function f was continuous only and hence the determination of u_x from the above ordinary differential equation was not unique, thus admitting the possibility of a jump in u_x . The conditions for the determination of u_y along a characteristic $y = \text{const.}$ are similar.

The above remarks serve to permit the extension of the domain of existence of the integral surfaces of Theorems 1 and 1a from R to R^* : $\begin{cases} -l_1 \leq x \leq l_1 \\ -l_2 \leq y \leq l_2 \end{cases}$. The arguments for the existence may be made applicable to other quadrants than the first by mere coordinate reflections. Moreover the integrals obtained in the separate quadrants must have first order contacts with each other along the coordinate axes by the above reasoning from ordinary differential equation theory. Hence we may obtain existence and uniqueness over the domain R^* by replacing B by B^* : $\begin{cases} -l_1 \leq x \leq l_1 \\ -l_2 \leq y \leq l_2 \\ -a \leq u \leq a \\ -b_1 \leq p \leq b_1 \\ -b_2 \leq q \leq b_2 \end{cases}$

in Theorem 1; and we obtain simply existence over R^* by replacing B by B^* in Theorem 1a.

In the classical existence theorem for the ordinary differential equation $\frac{dy}{dx} = f(x,y)$, with $y(0) = 0$, the conditions that f

be continuous on $C: \begin{cases} 0 \leq x \leq a \\ -b \leq y \leq b \end{cases}$, with $M = \max |f|$ on C , were shown to be sufficient to insure existence of at least one integral curve $y = y(x)$ for $x \in [0, \alpha]$ with $\alpha \leq \min(a, \frac{b}{M})$. This bound, $\alpha \leq \min(a, \frac{b}{M})$, was shown by A. WINTNER [15] to be a maximal bound in a certain sense. We apply his method to Theorem 1a in the proof of the following:

Theorem 2.

If, in Theorem 1a, we replace B by B'' :

$$B'': \begin{cases} 0 \leq x \leq l_1' \\ 0 \leq y \leq l_2' \\ -\infty < u < \infty \\ -b_1 \leq p \leq b_1 \\ -b_2 \leq q \leq b_2 \end{cases}$$

and require that f be bounded thereon, then hypothesis 3) in that theorem reduces to

$$3)' \quad l_1 \leq \min(l_1', \frac{b_2}{M}), \quad l_2 \leq \min(l_2', \frac{b_1}{M}),$$

where $M = \max |f|$ on B'' . Moreover, the bounds established by 3) are maximal bounds in a sense to be explained below.

Proof.

The condition $M l_1 l_2 \leq a$ of hypothesis 3) is immediately satisfied since a approaches $+\infty$. The conditions $M l_1 \leq b_2$, $M l_2 \leq b_1$ may be rewritten as in 3) and are now the only restrictions on l_1 and l_2 .

If $x'_2 \leq \frac{b_2}{M}$, ($x'_1 \leq \frac{b_1}{M}$), then the conclusion is immediate.

For, we may take $f(x,y; u; p,q) = h(x), (g(y))$, which function is not even defined for $x > x'_1 = x'_1$, ($y > x'_2 = x'_2$).

Suppose $x'_2 > \frac{b_1}{M}$. Then we consider the sequence of problems:

$$(2.41) \quad u_{xy} = (2^{1/m} - u_x)^{1/m+1}, \quad u(x,0) = u(0,y) = 0, \quad (m=1,2,\dots).$$

Setting $p = u_x$, (2.41) becomes

$$p_y(x,y) = (2^{1/m} - p(x,y))^{1/m+1}, \quad p(x,0) = 0.$$

Integrating this ordinary differential equation for p as a function of y , we obtain

$$p(x,y) = 2^{1/m} - \left[2^{1/m+1} - \frac{m}{m+1} y \right]^{m+1/m}.$$

But, since $p = u_x$ and $u(0,y) = 0$ we may integrate again to obtain

$$(2.42) \quad u(x,y) = x \left\{ 2^{1/m} - \left[\frac{m}{m+1} (C_m - y) \right]^{m+1/m} \right\}$$

where

$$(2.43) \quad C_m = \frac{m+1}{m} 2^{\frac{1}{m+1}}.$$

The line $y = C_m$ is a branch line of the solution u . Under the supposition $x'_2 > \frac{b_1}{M}$, the desired statement is that $\frac{b_1}{M}$ is a maximal bound on x'_2 , i.e., for each $\epsilon > 0$, there exists a function $f(x,y; u; p,q)$, depending on ϵ and satisfying hypotheses 1), 2)' and 3)' on E^n , such that an integral $u(x,y)$ of the problem corresponding to f has a singularity for some $y \in (\frac{b_1}{M}, \frac{b_1}{M} + \epsilon)$.

Defining

$$f_m(x, y; u; p, q) = (2^{1/m} - p)^{1/m+1} \text{ for } -2^{1/m+1} \leq p \leq 2^{1/m+1},$$

($m = 1, 2, \dots$), we obtain

$$b_{1m} = 2^{1/m+1}, \quad M_m = (2^{1/m} + 2^{1/m+1})^{1/m+1}; \text{ and, since}$$

$$(2^{1/m} + 2^{1/m+1}) > 2, \quad (m = 1, 2, \dots),$$

$$\lim_{m \rightarrow \infty} \frac{b_{1m}}{M_m} = 1 \dots$$

Moreover, by (2.43),

$$\lim_{m \rightarrow \infty} C_m = 1 \dots$$

Hence, given $\epsilon > 0$, there exists a positive integer N , depending on ϵ alone, such that $m > N \implies$

$$\frac{b_{1m}}{M_m} + \epsilon > C_m > \frac{b_{1m}}{M_m}.$$

Consequently $\frac{b_1}{M}$ is a maximal bound on λ_2 .

To determine that the condition $\lambda_1 \leq \min(\lambda_1', \frac{b_2}{M})$ is also a maximal bound we consider the sequence of problems.

$$(2.44) \quad u_{xy} = (2^{1/m} - u_y)^{1/m+1}, \quad u(x, 0) = u(0, y), \quad (m = 1, 2, \dots),$$

and follow the same line of reasoning as before. Thus Theorem 2 is verified.

The close parallelism between our conclusions and the classical theorems for first order ordinary differential equations

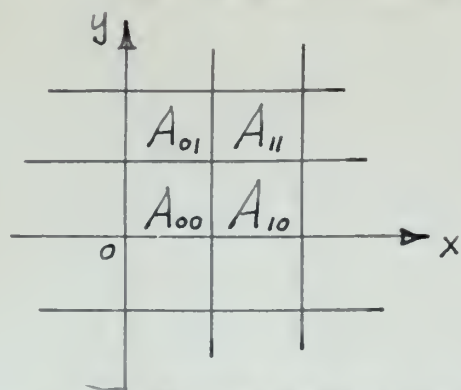
(See F. ZAMKE [2]) leads one to suspect that an existence theorem might be proved wherein mere continuity of the function f was demanded. The analogue to the Cauchy polygon method is the attack suggested by the parallelism, and it leads to an existence theorem for the characteristic initial value problem:

$$(2.45) \quad u_{xy} = f(x,y; u) \quad , \quad u(x,0) = u(0,y) = 0.$$

We do not give the proof here; first, because the theorem is a special case of Theorem 1a; and, second, because the steps in the proof are practically identical with those of the Cauchy polygon method for ordinary differential equations.

When $f = f(x,y; u; p,q)$ and f is merely continuous this attack involves difficulties which we have not been able to resolve. We sketch the method to indicate the source of trouble:

In a neighborhood of the origin a partition Π by



characteristics is specified where the subregions A_{1j} in the first quadrant are defined as

$$A_{1j} : \begin{cases} x_1 \leq x < x_{11} \\ y_j \leq y < y_{j1} \end{cases} \quad (1, j=0, 1, 2, \dots)$$

We formulate the approximate integral surface u corresponding to the partition Π as follows:

$$(2.46) \quad u_{\Pi}(x,y) = \int_0^x d\xi \int_0^y F_{\Pi}(\xi, \eta) d\eta$$

where

$$(2.47) \quad F_{\pi}(x, y) = f(x_1, y_1; u_{\pi}(x_1, y_1); u_{\pi x}(x_1, y_1), \\ u_{\pi y}(x_1, y_1))$$

for $(x, y) \in A_{1j}$.

The principal difficulty lies in the fact that the derivatives

$$(2.48) \quad u_{\pi x} = \int_0^y F_{\pi}(x, \eta) d\eta \quad \text{and}$$

$$(2.49) \quad u_{\pi y} = \int_0^x F_{\pi}(\xi, y) d\xi$$

are discontinuous across the partition lines $x = \text{constant}$ and $y = \text{constant}$, respectively, thus preventing the direct application of ARKELA's theorem on equicontinuous functions when we consider the sequence of approximate integral surfaces formed by partition refinement.

The equation of (2.45) appears to be more amenable than the more general equation involving the first derivatives p and q . G. FURUKI [16] p. 622, by demanding only that $f(x, y; u)$ be continuous and Lipschitzian with respect to u , has proved the existence of a unique integral of $u_{xy} = f(x, y; u)$ satisfying Dirichlet conditions, i.e. the value of u prescribed on a closed contour. This result, while remarkable, is not contradictory since u is shown to have a discontinuity of the second type at one point of the boundary.

We conclude this chapter with the statement of the extension of Theorems 1 and 1a to a system of equations

(2.50) $u_i = f_i(x, y; u_1, \dots, u_n; p_1, \dots, p_n, q_1, \dots, q_n), (i=1, 2, \dots, n)$
satisfying the initial conditions

$$(2.51) \quad u_i(x, 0) = u_i(0, y) = 0, \quad (i=1, 2, \dots, n).$$

Theorem 3, below, is a natural extension of Theorem 1. In principle, it was first obtained by O. NICCOLETTI [14] p.7. His statement, however, is not explicit about the bounds on the domain of existence. Moreover, to prove uniqueness he requires the f_i to be of class C^1 . We obtain the improved statement, Theorem 3, by modifying the arguments of E. KAMKE [2] p. 402 and p. 403 to apply them to the system (2.50).

Theorem 3)

$$1) \quad f_i(x, y; u_j; p_j, q_j)^2 \in C(B^n), \quad B^n: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \\ -a \leq u_1 \leq a \\ -b_1 \leq p_1 \leq b_1 \\ -b_2 \leq q_1 \leq b_2 \end{cases}$$

2) The f_i are Lipschitzian on B^n ; i.e. there exists a positive constant K such that for $(x, y; u^1_j; p^1_j, q^1_j) \in B^n$,

$(x, y; u^2_j; p^2_j, q^2_j) \in B^n$, and $i = 1, 2, \dots, n$,

$$\begin{aligned} & |f_i(x, y; u^1_j; p^1_j, q^1_j) - f_i(x, y; u^2_j; p^2_j, q^2_j)| \\ & \leq K \sum_{j=1}^n \left\{ |u^1_j - u^2_j| + |p^1_j - p^2_j| + |q^1_j - q^2_j| \right\}. \end{aligned}$$

3) $K l_1 l_2 \leq a$, $K l_1 \leq b_2$, $K l_2 \leq b_1$ where

$$K = \max \left\{ |f_1|, \dots, |f_n| \right\} \text{ on } B^n.$$

² Notation: $(x, y; u_j; p_j, q_j) = (x, y; u_1, \dots, u_n; p_1, \dots, p_n, q_1, \dots, q_n)$.

\Rightarrow 4) There exists one and only one set of functions

$\{u_1, \dots, u_n\}$, $u_j(x, y) \in C^1(R)$, $u_{j,xy}(x, y) \in C(R)$, ($j=1, \dots, n$),
 where $R: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \end{cases}$, such that for each $(x, y) \in R$ the point

$(x, y; u_j(x, y); u_{j,x}(x, y), u_{j,y}(x, y)) \in B^n$, and

$u_{1,xy}(x, y) = f_1(x, y; u_j(x, y); u_{j,x}(x, y), u_{j,y}(x, y))$,

$u_1(x, 0) = u_1(0, y) = 0$, ($i = 1, \dots, n$), for each $(x, y) \in R$.

By relaxing hypothesis 2) we obtain the improved theorem below; which stands in the same relation to Theorem 3 that Theorem 1a does to Theorem 1.

Theorem 3a

1)

2)' The f_1 are partially Lipschitzian on B^n ; i.e. there exists a positive constant K such that for $(x, y; u_j; p_j^1, q_j^1) \in B^n$,
 $(x, y; u_j; p_j^2, q_j^2) \in B^n$, and $i = 1, 2, \dots, n$,

$$\begin{aligned} & |f_1(x, y; u_j; p_j^1, q_j^1) - f_1(x, y; u_j; p_j^2, q_j^2)| \\ & \leq K \sum_{j=1}^n \left\{ |p_j^1 - p_j^2| + |q_j^1 - q_j^2| \right\}. \end{aligned}$$

3)

\Rightarrow 4)' There exists at least one set of functions $\{u_1, \dots, u_n\}$,
 $u_j(x, y) \in C^1(R)$, $u_{j,xy}(x, y) \in C(R)$, ($j=1, \dots, n$), where

$R: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \end{cases}$, such that for each $(x, y) \in R$ the point

$(x, y; u_1(x, y); u_{j,x}(x, y), u_{j,y}(x, y)) \in B^n$, and

$u_{i,xy}(x, y) = f_i(x, y; u_j(x, y); u_{j,x}(x, y), u_{j,y}(x, y))$,

$u_i(x, 0) = u_i(0, y) = 0$, ($i = 1, \dots, n$), for each $(x, y) \in R$.

The proof of Theorem 3a is essentially a step by step repetition of that for Theorem 1a. WEIERSTRASS' theorem tells us that for each positive integer i there exists a sequence of polynomials $\{g_{i\lambda}\}$ ($x, y; u_j; p_j, q_j$), ($\lambda = 1, 2, \dots$), converging uniformly on B^n to $f_i(x, y; u_j; p_j, q_j)$. We extend the $g_{i\lambda}$ and the f_i as before and obtain that there exist positive constants L_i such that for each i $|g_{i\lambda}| \leq L_i$ on B^n , extended, and for all λ . We let $L = \max \{L_1, \dots, L_n\}$ and proceed as before, using Theorem 3 in place of Theorem 1 to obtain the integral $u_{i\lambda}$ associated with each $g_{i\lambda}$.

We note only one point of significant difference in the arguments. In place of inequality (2.13) of Lemma 2 we now have the inequalities

$$\begin{aligned} & |u_{i\lambda, x}(x_2, y) - u_{i\lambda, x}(x_1, y)| \\ & \leq K \int_0^y \sum_{j=1}^n |u_{j\lambda, x}(x_2, \eta) - u_{j\lambda, x}(x_1, \eta)| d\eta \\ & \qquad \qquad \qquad (i = 1, \dots, n). \end{aligned}$$

Summing these, and letting

$$Z(y) = \sum_{i=1}^n |u_{i\lambda, x}(x_2, y) - u_{i\lambda, x}(x_1, y)|,$$

we obtain

$$0 \leq Z(y) \leq Kn \int_0^y z(\eta) d\eta + n(\mu + \zeta)$$

to which Lemma 1 applies. Thus the equicontinuity of each of the sequences $\{u_{i\lambda, x}\}$, $(i = 1, \dots, n)$ is assured.

Remarks a) and b) to Theorems 1 and 1a apply, with obvious modifications, to Theorems 3 and 3a. Moreover, as before, we may extend the domain of existence of the integral surfaces of Theorems 3 and 3a from R to R^* : $\begin{cases} -k_1 \leq x \leq k_1 \\ -k_2 \leq y \leq k_2 \end{cases}$.

The set of functions $\{u_1, \dots, u_n\}$ representing the solution to the problem of Theorem 3a cannot be shown to be unique. This is made evident by extending Example 1 to the system

$$\begin{aligned} u_{1,xy} &= |u_1|^{\frac{1}{2}}, & u_1(x,0) &= u_1(0,y) = 0 \\ u_{2,xy} &= 0, & u_2(x,0) &= u_2(0,y) = 0 \\ & \vdots & & \vdots \\ u_{n,xy} &= 0, & u_n(x,0) &= u_n(0,y) = 0 \end{aligned}$$

for which $u_i \equiv 0$ $(i = 2, \dots, n)$

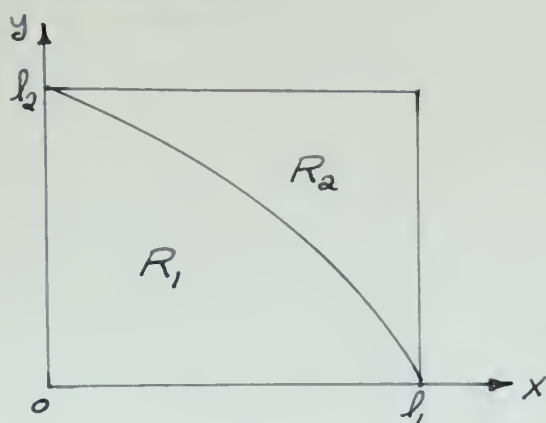
while $u_1 \equiv 0$ or $u_1 = \frac{1}{16} x^2 y^2$. Thus at least two sets of solutions are possible for this system which satisfies the hypotheses of Theorem 3a.

CHAPTER III

The Cauchy Problem for $u_{xy} = f(x, y; u; u_x, u_y)$.

The development of this chapter closely parallels that of Chapter II. Consequently, the notation will be abridged, in particular with respect to the arguments of functions; and the proofs will be merely outlined to show minor variations from the statements in Chapter 2.

For reference, we state the following theorem proved first for systems of equations by O. NICCOLETTI [14] p. 7. Our statement may be easily inferred from that of E. KAMKE [2] p. 405 and p. 410, by a slight modification of his proof.

Theorem 4

$$1) f(x, y; u; p, q) \in C(B),$$

$$B: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \\ -a \leq u \leq a \\ -b_1 \leq p \leq b_1 \\ -b_2 \leq q \leq b_2 \end{cases}$$

2) f is Lipschitzian on B , (as defined in Theorem 1).

3) $M l_1 l_2 \leq a$, $M l_1 \leq b_2$, $M l_2 \leq b_1$, where $M = \max |f|$ on B

4) $\gamma: \begin{cases} 0 \leq x \leq l_1 \\ y = \varphi(x) \end{cases}$ where $\varphi(x) \in C^1([0, l_1])$, $\varphi'(x) \neq 0$ for $x \in [0, l_1]$ and $\varphi(0) = l_2$, $\varphi(l_1) = 0$.

\Rightarrow 5) There exists one and only one function $u(x,y) \in C^1(R)$,
 $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq \lambda_1 \\ 0 \leq y \leq \lambda_2 \end{cases}$, such that for each
 $(x,y) \in R$, the point $(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in R$, and
 $u_{xy}(x,y) = f(x,y; u(x,y); u_x(x,y), u_y(x,y))$,
 $u(x, \varphi(x)) = u_x(x, \varphi(x)) = u_y(x, \varphi(x)) = 0$

for each $(x,y) \in R$.

Remarks c) Suppose we prescribe $u(x, \varphi(x)) = U(x)$,
 $u_x(x, \varphi(x)) = P(x)$, $u_y(x, \varphi(x)) = Q(x)$ where $U(x) \in C^1([0, \lambda_1])$
while $P(x), Q(x) \in C([0, \lambda_1])$. Our prescription must satisfy
the strip condition $U' = P + Q \cdot \varphi'$ for each $x \in [0, \lambda_1]$.
Consider the function $w(x,y) = U(x) + (y - \varphi(x)) Q(x)$. Clearly,
 $w_{xy} = Q'(x)$ while $w(x, \varphi(x)) = U(x)$, $w_x(x, \varphi(x)) = P(x)$, and
 $w_y(x, \varphi(x)) = Q(x)$. Hence the function $v = u - w$ must satisfy
 $v_{xy} = Q'(x) + f(x,y; v + w; v_x + w_x, v_y + w_y)$, with $v(x, \varphi(x))$
 $= v_x(x, \varphi(x)) = v_y(x, \varphi(x)) = 0$, a problem of the type covered
by Theorem 4.

d) Hypothesis 4) of Theorem 4 is more restrictive than
it need be. At isolated points of γ we may have a horizontal or
vertical tangent, provided that γ does not cross the same char-
acteristic more than once. For, under these conditions the in-
verse function ψ to φ will exist and be continuous for all
 $y \in [0, \lambda_2]$.

Our improvement of this theorem is as follows:

Theorem 4a

1)

2) f is partially Lipschitzian on B , (as defined in Theorem 1a).

3)

4)

\Rightarrow 5) There exists at least one function $u(x,y) \in C^1(R)$,
 $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \end{cases}$, such that for each

$(x,y) \in B$, the point $(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in B$, and

$$u_{xy}(x,y) = f(x,y; u(x,y); u_x(x,y), u_y(x,y)),$$

$$u(x, \varphi(x)) = u_x(x, \varphi(x)) = u_y(x, \varphi(x)) = 0$$

for each $(x,y) \in R$.

Outline of proof.

The path γ may also be expressed as $\gamma: \begin{cases} x = \psi(y) \\ 0 \leq y \leq l_2 \end{cases}$ where

$\psi(y) \in C^1([0, l_2])$, $\psi'(y) \neq 0$ for $y \in [0, l_2]$. ψ is the inverse function to φ .

We may express the problem as the integral equation

$$(3.1) \quad u(x,y) = \int_{\psi(y)}^x \frac{d\xi}{\psi'(y)} \int_{\varphi(\xi)}^y f(\xi, \eta; u; u_x, u_y) d\eta$$

whence

$$(3.2) \quad u_x(x,y) = \int_{\varphi(x)}^y f(x, \eta; u; u_x, u_y) d\eta$$

$$(3.3) \quad u_y(x,y) = \int_{\psi(y)}^x f(\xi, y; u; u_x, u_y) d\xi.$$

By WEIERSTRASS' theorem, there exists a sequence of polynomials $\{g_\lambda\} \xrightarrow{\text{unif.}} f$ on B . We extend the domain of definition of f and the polynomials g_λ over B to B' by definition (2.1).

We obtain again the constant $L > 0$ such that $|g_\lambda| \leq L$ in B' for all λ . Moreover, for each g_λ the Lipschitz condition (2.2) is satisfied. Thus, by Theorem 4, for each λ there exists a unique solution u_λ to the problem

$$(3.4) \quad \begin{cases} u_{\lambda,xy} = g_\lambda(x,y; u_\lambda; u_{\lambda,x}, u_{\lambda,y}), \\ u_\lambda(x, \varphi(x)) = u_{\lambda,x}(x, \varphi(x)) = u_{\lambda,y}(x, \varphi(x)) = 0. \end{cases}$$

That the sequences $\{u_\lambda\}$, $\{u_{\lambda,x}\}$, $\{u_{\lambda,y}\}$ are uniformly bounded on R , and that the sequence $\{u_\lambda\}$ is equicontinuous on R is immediately evident from the equivalent integral expressions

$$(3.5) \quad \begin{aligned} u_\lambda(x,y) &= \int_{\psi(y)}^x d\xi \int_{\varphi(\xi)}^y g_\lambda(\xi, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\eta \\ &= \int_{\varphi(x)}^y d\eta \int_{\psi(\eta)}^x g_\lambda(\xi, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\xi, \end{aligned}$$

$$(3.6) \quad u_{\lambda,x}(x,y) = \int_{\varphi(x)}^y g_\lambda(x, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\eta,$$

$$(3.7) \quad u_{\lambda,y}(x,y) = \int_{\psi(y)}^x g_\lambda(\xi, y; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\xi.$$

We now establish the equicontinuity of $\{u_{\lambda,x}\}$ and of $\{u_{\lambda,y}\}$. This done, the same arguments as those for the proof of Theorem 1a will serve to obtain a subsequence $\{u_{\lambda^*}\}$ of $\{u_\lambda\}$ which converges uniformly to the solution u .

There is no loss in generality in restricting ourselves at this point to the consideration of those points $(x, y) \in R_2$: $\begin{cases} 0 \leq x \leq \lambda_1 \\ \varphi(x) \leq y \leq \lambda_2 \end{cases}$.

For we shall see that the arguments developed below will apply as well for $(x, y) \in R_1$: $\begin{cases} 0 \leq x \leq \lambda_1 \\ 0 \leq y \leq \varphi(x) \end{cases}$ after a simple coordinate

translation and rotation. Thus if we insure existence of a solution on R_2 , existence on R_1 is simultaneously verified. Moreover, the Cauchy initial data insure that such integral surfaces have a first order contact along Υ and hence define an integral surface throughout all of $R = R_1 + R_2$.

Given points $(x_2, y_2) \in R_2$, $(x_1, y_1) \in R_2$, it is always possible to label these points in such a way that $(x_1, y_1) \in R_2$. This being done, we have that

$$(3.8) \quad |u_{\lambda, x}(x_2, y_2) - u_{\lambda, x}(x_1, y_1)| \leq L |y_2 - y_1|,$$

$$(3.9) \quad |u_{\lambda, y}(x_2, y_2) - u_{\lambda, y}(x_1, y_1)| \leq L |x_2 - x_1|.$$

Assuming, without loss, that $y \geq \varphi(x_2) \geq \varphi(x_1)$, we have that

$$(3.10) \quad u_{\lambda, x}(x_2, y) - u_{\lambda, x}(x_1, y) = \int_{\varphi(x_2)}^y \left[\varepsilon_{\lambda}(x_2, \eta; u_{\lambda, x}, u_{\lambda, y}) - \varepsilon_{\lambda}(x_1, \eta; u_{\lambda, x}, u_{\lambda, y}) \right] d\eta + \int_{\varphi(x_1)}^{\varphi(x_2)} \varepsilon_{\lambda}(x_1, \eta; u_{\lambda, x}, u_{\lambda, y}) d\eta$$

We operate on the first integral on the right hand side of (3.10) in the manner demonstrated in equation (2.90). We obtain a formula identical with (2.90) except that here the lower limit of integration is $y = \varphi(x_2)$ instead of $y = 0$. For brevity, we omit the formula.

Since

$$(3.11) \quad \left| \int_{\varphi(x_1)}^{\varphi(x_2)} g_{\lambda}(x_1, \eta; u_{\lambda}, u_{\lambda, x}, u_{\lambda, y}) d\eta \right| \leq L |\varphi(x_2) - \varphi(x_1)|, \quad (\lambda = 1, 2, \dots)$$

and since $\varphi(x)$ is uniformly continuous on $[0, \ell_1]$, by the same reasoning as before we arrive at the slight modification to Lemma 2,

$$(3.12) \quad \begin{aligned} & |u_{\lambda, x}(x_2, y) - u_{\lambda, x}(x_1, y)| \\ & \leq K \int_{\varphi(x_2)}^y |u_{\lambda, x}(x_2, \eta) - u_{\lambda, x}(x_1, \eta)| d\eta + \mu + \zeta \end{aligned}$$

from which, by Lemma 1,

$$(3.13) \quad \begin{aligned} |u_{\lambda, x}(x_2, y) - u_{\lambda, x}(x_1, y)| & \leq (\mu + \zeta) e^{k(y - \varphi(x_2))} \\ & \leq (\mu + \zeta) e^{k\ell_2}. \end{aligned}$$

The equicontinuity of $\{u_{\lambda, x}\}$ is thus assured.

The argument for the equicontinuity of $\{u_{\lambda, y}\}$ is similar, hence Theorem 4a obtains.

Remarks c) and d) to Theorem 4 apply as well to Theorem 4a. Quite obviously, if f is continuous, bounded and Lipschitzian (or partially Lipschitzian) on the infinite cylinder with cross section R , then hypothesis 3) of Theorem 4 (or 4a) is immediately satisfied. In fact, this was the form of Theorem 4 which was utilized in the proof of Theorem 4a.

As previously mentioned, the extension of Theorem 4 to systems of equations was first obtained, in principle, by O. NICCOLETTI [14]. He was not, however, explicit about the domain of existence of the solution. The following statement may be derived

from the same arguments of E. KAMKE [2] p. 405 and p. 410 used as the basis for Theorem 4.

Theorem 5.

$$1) f_i(x, y; u_1, \dots, u_n; p_1, \dots, p_n, q_1, \dots, q_n) \in C(B^n)$$

$$B^n: \begin{cases} 0 \leq x \leq \lambda_1 \\ 0 \leq y \leq \lambda_2 \\ -a \leq u_i \leq a \\ -b_1 \leq p_i \leq b_1 \\ -b_2 \leq q_i \leq b_2 \end{cases} \quad (i = 1, \dots, n).$$

2) The f_i are Lipschitzian on B^n , (as defined in Theorem 3).

3) $M \lambda_1 \lambda_2 \leq a$, $M \lambda_1 \leq b_2$, $M \lambda_2 \leq b_1$, where

$$M = \max \{ |f_1|, \dots, |f_n| \} \text{ on } B^n.$$

$$4) \gamma: \begin{cases} 0 \leq x \leq \lambda_1 \\ y = \varphi(x) \end{cases} \text{ where } \varphi(x) \in C^1([0, \lambda_1]), \varphi'(x) \neq 0$$

$$\text{for } x \in [0, \lambda_1] \text{ and } \varphi(0) = \lambda_2, \varphi(\lambda_1) = 0.$$

\Rightarrow 5) There exists one and only one set of functions $\{u_1, \dots, u_n\}$,

$u_i(x, y) \in C^1(R)$, $u_{i,xy}(x, y) \in C(R)$, ($i = 1, \dots, n$), where

$$R: \begin{cases} 0 \leq x \leq \lambda_1 \\ 0 \leq y \leq \lambda_2 \end{cases}, \text{ such that for each } (x, y) \in R \text{ the point}$$

$(x, y; u_j(x, y); u_{j,x}(x, y), u_{j,y}(x, y)) \in B$, and

$$u_{i,xy}(x, y) = f_i(x, y; u_j(x, y), u_{j,x}(x, y), u_{j,y}(x, y)),$$

$$u_i(x, \varphi(x)) = u_{i,x}(x, \varphi(x)) = u_{i,y}(x, \varphi(x)) = 0,$$

($i = 1, \dots, n$), for each $(x, y) \in R$.

We may extend the arguments in the proof of Theorem 4a to apply to systems of equations. The extension is practically identical with the previous extension of Theorem 1a to Theorem 3a, except that now Theorem 5 is used to establish existence and uniqueness of the solutions of the system

$$u_{i\lambda,xy} = g_{i\lambda}(x,y; u_{j\lambda}; u_{j\lambda,x}, u_{j\lambda,y}), \quad (i=1, \dots, n),$$

$$(\lambda = 1, 2, \dots),$$

under the Cauchy initial conditions. We obtain the following theorem:

Theorem 5a

1)

2)' the f_i are partially Lipschitzian on E^n , (as defined in Theorem 3a).

3)

4)

\Rightarrow 5)' There exists at least one set of functions $\{u_1, \dots, u_n\}$,

$u_i(x,y) \in C^1(R)$, $u_{i,xy}(x,y) \in C(R)$, ($i = 1, \dots, n$), where

$R: \begin{cases} 0 \leq x \leq k_1 \\ 0 \leq y \leq k_2 \end{cases}$, such that for each $(x,y) \in R$ the point

$(x,y; u_j(x,y); u_{j,x}(x,y), u_{j,y}(x,y)) \in E$, and

$u_{i,xy}(x,y) = f_i(x,y; u_j(x,y); u_{j,x}(x,y), u_{j,y}(x,y))$,

$u_{i,x}(x, \varphi(x)) = u_{i,x}(x, \varphi(x)) = u_{i,y}(x, \varphi(x)) = 0$,

($i = 1, \dots, n$), for each $(x,y) \in R$.

Remark c), with obvious modifications, and Remark d) to Theorem 4 apply as well as to Theorems 5 and 5a. Moreover, in Theorem 5 (or 5a) we may eliminate hypothesis 3) by demanding that the f_i be continuous, bounded and Lipschitzian (or partially Lipschitzian) on the infinite cylinder with cross section R .

CHAPTER IV

Existence Theorems for Canonical
Hyperbolic First Order Systems

In this chapter, and in Chapters 5 and 6 as well, we shall not give explicit domains of definition for the functions involved in the differential equations. As a consequence, existence will be shown "in the small" only. We do this because our method of attack will not yield any improvement upon the domains of existence, no matter how large the domains of definition are taken, provided the other hypotheses are not weakened. We shall elaborate on this peculiarity in the course of the exposition.

Theorems 6 and 7 below were given by M. CINQUINI-CIERRARIO [12] p. 180 in the form stated. A statement under somewhat weaker hypotheses, but based on the same proof, may be found in R. COURANT-D. HILBERT [17] p. 324. We examine the proof to show that the arguments therein are independent of the uniqueness of the solutions to the problems involved. As a consequence, our results in Chapters 2 and 3 apply and we arrive at the improved statements given by Theorems 6a and 7a, where hypothesis 2) of Theorems 6 and 7 is dropped altogether and the corresponding conclusions are altered to read "at least one".

Common hypothesis 1) We shall suppose the functions a_{ik}, c_i , ($i, k=1, \dots, n$), of arguments x, y, u_1, \dots, u_n , to be continuously differentiable with bounded derivatives in a certain domain D . Fur-

then, we suppose the determinant

$$(4.1) \quad |a_{ik}| \neq 0 \quad \text{in } D.$$

Under these assumptions, the system

$$(4.2) \quad \begin{cases} A_i(x, y) = \sum_{k=1}^n a_{ik} u_{k,x}(x, y) - c_i = 0, & (i=1, \dots, m < n) \\ B_i(x, y) = \sum_{k=1}^n a_{ik} u_{k,y}(x, y) - c_i = 0, & (i=m+1, \dots, n) \end{cases}$$

is called a canonical hyperbolic first order system.

Theorem 6. (Characteristic initial value problem.)

1)

2) All first derivatives of the functions a_{ik}, c_i , ($i, k=1, \dots, n$) satisfy a Lipschitz condition with respect to arguments u_1, \dots, u_n in D .

$$3) \quad \left. \begin{aligned} U_1(x) &\in C'([0, \lambda_1]) \\ V_1(y) &\in C'([0, \lambda_2]) \\ U_1(0) &= V_1(0) \end{aligned} \right\} \quad (i=1, \dots, n)$$

Moreover, for each $x \in [0, \lambda_1]$, the point $(x, 0; U_j(x)) \in D$

and

$$(4.3) \quad \sum_{k=1}^n a_{ik}(x, 0; U_j(x)) U'_k(x) - c_i(x, 0; U_j(x)) = 0, \\ (i=1, \dots, m < n);$$

and, for each $y \in [0, \lambda_2]$, the point $(0, y; V_j(y)) \in D$ and

$$(4.4) \quad \sum_{k=1}^n a_{ik}(0, y; V_j(y)) V'_k(y) - c_i(0, y; V_j(y)) = 0, \\ (i=m+1, \dots, n).$$

3. Recall the notation: $(x, y; U_j(x)) = (x, y; U_1(x), \dots, U_n(x))$.

\Rightarrow 4) There exists one and only one set of functions

$$\{u_1, \dots, u_n\}, u_i(x, y) \in C^1(R_\eta), u_{i,xy} \in C(R_\eta), (i = 1, \dots, n),$$

where $R_\eta : \begin{cases} 0 \leq x \leq \eta \lambda_1 \\ 0 \leq y \leq \eta \lambda_2 \end{cases}$, with $0 < \eta \leq 1$ and η sufficiently

small, such that the set of functions satisfies the system (4.2)

for each $(x, y) \in R_\eta$ and satisfies the conditions

$$\left. \begin{aligned} u_i(x, 0) &= U_i(x) \quad \text{for } x \in [0, \lambda_1] \\ u_i(0, y) &= V_i(y) \quad \text{for } y \in [0, \lambda_2] \end{aligned} \right\} (i = 1, \dots, n).$$

Theorem 6a.

1)

3)

\Rightarrow 4)' There exists at least one set of functions, etc. (as in Theorem 6).

Theorem 7. (Cauchy problem.)

1)

2) (as in Theorem 6.)

$$5) \gamma : \begin{cases} x = x(\tau) \\ y = y(\tau) \end{cases} \text{ for } \tau \in [0, 1], x(\tau) \text{ and } y(\tau) \in C^1([0, 1])$$

and strictly monotone, i.e., $\dot{x} \neq 0, \dot{y} \neq 0$ on $[0, 1]$.

$U_i(\tau) \in C^1([0, 1]), (i = 1, \dots, n)$. For each $\tau \in [0, 1]$, the point $(x(\tau), y(\tau); U_j(\tau)) \in D$.

\Rightarrow 6) There exists one and only one set of functions $\{u_1, \dots, u_n\}$, $u_i(x, y) \in C^1(R_\gamma), u_{i,xy}(x, y) \in C(R_\gamma), (i = 1, \dots, n)$, where R_γ is a sufficiently small neighborhood of the curve γ , such that



the set of functions satisfies the system (4.2) for each $(x,y) \in R_\gamma$ and satisfies the conditions

$$u_i(x(\tau), y(\tau)) = U_i(\tau) \quad \text{for } \tau \in [0,1], \quad (i = 1, \dots, n).$$

Theorem 7a

1)

5)

\Rightarrow 6) There exists at least one set of functions etc. (as in Theorem 7.)

The proofs of these theorems are contained in the following argument:

Suppose we have a set of functions $\{u_1, \dots, u_n\}$, either as a solution to the characteristic initial value problem above on a domain R_η , or as a solution to the Cauchy problem above on a domain R_γ . Then for either case,

$$(4.5) \quad A_{i,y} = \sum_{k=1}^n a_{ik} u_{k,xy} + \sum_{k=1}^n \left[a_{ik,y} + \sum_{r=1}^n \frac{\partial a_{ik}}{\partial u_r} u_{r,y} \right] u_{k,x} - c_{i,y} - \sum_{k=1}^n \frac{\partial c_i}{\partial u_k} u_{k,y} = 0, \quad (i = 1, \dots, m < n),$$

$$(4.6) \quad B_{i,x} = \sum_{k=1}^n a_{ik} u_{k,xy} + \sum_{k=1}^n \left[a_{ik,x} + \sum_{r=1}^n \frac{\partial a_{ik}}{\partial u_r} u_{r,x} \right] u_{k,y} - c_{i,x} - \sum_{k=1}^n \frac{\partial c_i}{\partial u_k} u_{k,x} = 0, \quad (i = m+1, \dots, n).$$

Equations (4.5) and (4.6) are n linear algebraic equations in the



n unknowns $u_{i,xy}$. Since the determinant of this system, $|a_{ik}|$, does not vanish over the domain in question, we may solve the system to obtain explicitly

$$(4.7) \quad u_{i,xy} = f_i(x,y; u_j; u_{j,x}, u_{j,y}), \quad (i = 1, \dots, n).$$

Under hypothesis 1) alone, the f_i are continuous and partially Lipschitzian over any bounded domain in the $3n + 2$ dimensional $(x,y; u_j; u_{j,x}, u_{j,y})$ -space where $(x,y; u_j) \in D$. If hypothesis 2) also applies, the f_i are "fully" Lipschitzian over any such domain.

Consider Theorems 6 and 6a. The characteristic initial conditions imposed therein, coupled with the system (4.7), form a problem of the type considered in Theorems 3 and 3a, respectively. (Chapter 2). We have shown above that any solution of a canonical hyperbolic system is also a solution of a particular system of type (4.7). If we now demonstrate the converse for characteristic initial conditions, i.e. that any solution of the derived system (4.7) is also a solution of the original system (4.2), then Theorems 6 and 6a follow directly from Theorems 3 and 3a respectively.

Suppose we have a set of functions $\{u_1, \dots, u_n\}$ as a solution of (4.7) over a certain domain including the initial characteristics. By (4.5) and (4.6), which are merely alternative forms of (4.7), we have

$$(4.8) \quad \begin{cases} A_{1,y}(x,y) = 0 & , \quad (i = 1, \dots, m < n) \\ B_{1,x}(x,y) = 0 & , \quad (i = m+1, \dots, n) \end{cases}$$

over this domain. But, by (4.3) and (4.4) of hypothesis 3) to both Theorems 6 and 6a, we have that

$$(4.9) \quad \begin{cases} A_1(x,0) = 0 & , \quad (i = 1, \dots, m < n) \\ B_1(0,y) = 0 & , \quad (i = m+1, \dots, n), \end{cases}$$

whence

$$\begin{aligned} A_i(x,y) &\equiv 0 & , \quad (i = 1, \dots, m < n), \\ B_i(x,y) &\equiv 0 & , \quad (i = m+1, \dots, n), \end{aligned}$$

throughout the domain. Hence the converse is shown.

For the Cauchy problem considered in Theorems 7 and 7a, we observe first that we can determine $u_{1,x}(x(\tau), y(\tau))$ and $u_{1,y}(x(\tau), y(\tau))$, ($i = 1, \dots, n$), as functions continuous for each $\tau \in [0,1]$, solely from the prescription of $u_1(x(\tau), y(\tau)) = U_1(\tau)$, ($i = 1, \dots, n$), and the requirement that the canonical hyperbolic system (4.2) must be satisfied at each point of Υ . For, since $\dot{x} + \dot{y}^2 \neq 0$ along Υ , we may write the strip conditions

$$(4.10) \quad \dot{u}_1 = p_1 \dot{x} + q_1 \dot{y}, \quad (i = 1, \dots, n),$$

as one of

$$(4.11) \quad q_1 = \frac{1}{\dot{y}} (\dot{u}_1 - p_1 \dot{x}) \quad \text{or} \quad p_1 = \frac{1}{\dot{x}} (\dot{u}_1 - q_1 \dot{y}), \quad (i = 1, \dots, n).$$

Consider a particular point $P \in \Upsilon$ where $\dot{y} \neq 0$. Here we substitute $q_1 = u_{1,y} = \frac{1}{\dot{y}} (\dot{u}_1 - p_1 \dot{x})$ into equations $E_1(P) = 0$, ($i = m+1, \dots, n$). These, together with the equations $A_1(P) = 0$, ($i = 1, \dots, m < n$),

form a linear algebraic system in the $p_i = u_{i,x}(P)$ with determinant $|a_{ik}| \neq 0$. Thus the p_i are uniquely determined at P , and, by (4.11), the q_i as well are uniquely determined at P . If $\dot{y} = 0$ at P , then $\dot{x} \neq 0$ there and a similar argument applies utilizing $p_i = \frac{1}{\dot{x}} (\dot{a}_i - q_i \dot{y})$.

Thus we have, in effect, prescribed all three sets $u_i, u_{i,x}, u_{i,y}$, ($i = 1, \dots, n$), along γ once the u_i are prescribed along γ and the $u_{i,x}$ and the $u_{i,y}$ are merely required to satisfy the strip conditions (4.10) and the canonical hyperbolic system at (4.2) at each point of γ .

Suppose we have a set of functions $\{u_1, \dots, u_n\}$ as a solution of

(4.7) $u_{i,xy} = f_i(x, y; u_j; u_{j,x}, u_{j,y})$, ($i = 1, \dots, n$) in a neighborhood of the initial curve γ and taking, with their first derivatives, precisely the above determined values at each point of γ . Then by (4.5) and (4.6), the fact that these functions and their first derivatives satisfy the canonical hyperbolic system (4.2) at each point of γ implies further that the system (4.2) corresponding to (4.7) is satisfied everywhere in the neighborhood in question.

With hypothesis 2) imposed, system (4.7) and the initial data on γ satisfy the hypotheses of Theorem 5, while without hypothesis 2), system (4.7) and the initial data on γ satisfy the hypotheses of Theorem 5a. But since we have shown above that each solution of (4.7) is a solution of the corresponding canonical

hyperbolic system (4.2), we have that Theorem 7 is a consequence of Theorem 5, while Theorem 7a is a consequence of Theorem 5a.

In these four theorems we are unable to prescribe the domain of definition of the functions

$$f_i(x, y; u_j; p_j, q_j), \quad (i = 1, \dots, n),$$

in such a way as to insure existence of a solution throughout

$$R: \begin{cases} 0 \leq x \leq l_1 \\ 0 \leq y \leq l_2 \end{cases}. \quad \text{This is because the } f_i \text{ are continuous for}$$

all p_j and q_j , ($j = 1, \dots, n$), but may turn out to be bounded only when these variables are restricted to finite domains. The following example demonstrates why the existence of solutions can be found only "in the small".

Example 3. Consider the characteristic initial value problem for the system

$$u_{1,xy} = u_{1,x}^2, \quad u_1(x, -1) = x, \quad u_1(0, y) = 0$$

$$u_{2,xy} = 0, \quad u_2(x, -1) = 0, \quad u_2(0, y) = 0$$

!

$$u_{n,xy} = 0, \quad u_n(x, -1) = 0, \quad u_n(0, y) = 0.$$

By quadratures, we obtain the solution $u_1(x, y) = \frac{-x}{y}$, while $u_2 = \dots = u_n = 0$, quite obviously. The f_i corresponding to this problem possess derivatives of all orders for all values of all variables. However, $f_1 = u_{1,x}^2$ becomes unbounded as the argument $u_{1,x}$ increases indefinitely in absolute value. We note that, despite the specification of initial data everywhere along the

intersecting characteristics $x = 0$ and $y = -1$, the first function in the solution, namely u_1 , has a discontinuity across the line $y = 0$. Hence this example typifies those cases for which solutions exist "in the small" only.

We note that Remark d) of Chapter III applies as well to hypothesis 5) of Theorems 7 and 7a. The statement is that

$$\gamma : \begin{cases} x = x(\tau) \\ y = y(\tau) \end{cases} \text{ for } \tau \in [0,1] \text{ need only have } x(\tau) \text{ and}$$

$y(\tau) \in C^1([0,1])$, monotone, and with $\dot{x}^2 + \dot{y}^2 \neq 0$ at each point of γ . In fact, the argument in the proof above applies directly to this statement.

CHAPTER V.

The Cauchy Problem for $F(x,y; u; p,q; r,s,t) = 0$.

In this chapter we concern ourselves with the Cauchy problem for the general non-linear second order partial differential equation in the hyperbolic domain. Specifically, the problem is to determine an integral surface of the equation

$$(1.1) \quad F(x,y; u; p,q; r,s,t) = 0$$

such that the hyperbolic condition

$$(1.3) \quad F_s^2 - 4 F_r F_t > 0$$

is satisfied thereon; moreover, the integral surface must have a second order contact with a given second order strip which is nowhere a characteristic strip.

In his celebrated paper [10], H. LEWY successfully attacks this problem by reducing equation (1.1) to a system of first order partial differential equations for the unknowns $x,y; u; p,q; r,s,t$ as functions of the parameters λ and μ of the two families of characteristics on the integral surface in question. LEWY's existence proof for the system is based on a finite difference argument. However, the system is of canonical hyperbolic form and the theorem of M. CINQUINI-CIERRARIO, Theorem 7 of Chapter IV, is immediately applicable and insures existence and uniqueness of the solution in a sufficiently small neighborhood of the initial strip. Moreover, as demonstrated below, Theorem 7a may be used to effect an improvement on LEWY's work.

We present simultaneously LEWY's original theorem and our

improvement on it. LEVY's theorem is obtained by omitting the parentheses. Our theorem is obtained by replacing the under-scored statements by the corresponding ones in the parentheses.

Theorem 8 (8a)

$$1) \quad S^2: \begin{cases} x = x(\tau) \\ y = y(\tau) \\ u = u(\tau) \\ p = p(\tau) \\ q = q(\tau) \\ r = r(\tau) \\ s = s(\tau) \\ t = t(\tau) \end{cases} \quad \text{for } \tau \in [0,1] \text{ is a nowhere character-} \\ \text{istic second order strip,}$$

i.e. $x, y; u; p, q; r, s, t(\tau) \in C^1([0,1])$, and for each $\tau \in [0,1]$,

- i) $\dot{x}^2 + \dot{y}^2 \neq 0$,
- ii) $F_r \dot{y}^2 - F_s \dot{y} \dot{x} + F_t \dot{x}^2 \neq 0$,
- iii) $F_s^2 - 4 F_r F_t > 0$,
- iv) $F(x(\tau), y(\tau); u(\tau); p(\tau), q(\tau); r(\tau), s(\tau), t(\tau)) = 0$.

2) $F \in C^{(4)}$ ($\in C^4$) in a certain neighborhood of S^2 .

3) There exists one and only one (at least one) integral surface $J: u = u(x, y)$ of the equation $F(x, y; u; p, q; r, s, t) = 0$ such that $u(x, y) \in C^{(4)}$ in a sufficiently small neighborhood of the base curve $\tau: \begin{cases} x = x(\tau) \\ y = y(\tau) \end{cases}$ for $\tau \in [0,1]$, and such that $J: u = u(x, y)$ has a second order contact with the strip S^2 .

Proof

We first demonstrate that any solution of the above problem, together with its derivatives of the first and second orders, represents a solution of a particular canonical hyperbolic system under the same boundary conditions.

We assume that $F_r \neq 0$ and $F_t \neq 0$ in the domains considered in the following argument. This may be done without loss of generality. For, by Definition 1a, a characteristic base curve must satisfy

$$(1.5) \quad \begin{aligned} 1) & \quad F_r \dot{y}^2 - F_x \dot{y} \dot{x} + F_t \dot{x}^2 = 0, \\ 2) & \quad \dot{x}^2 + \dot{y}^2 \neq 0. \end{aligned}$$

Suppose at a point of S^2 that $F_r = 0$. Then $\dot{x} = 0$ represents the vertical tangent taken by one of the characteristic base curves through the projection of this point onto the xy plane. Conversely, if one of the characteristic base curves through a point in the projection of S^2 has a vertical tangent, then $\dot{x} = 0$ there and, consequently, $F_r = 0$ at the corresponding point on S^2 . Likewise, $F_t = 0$ if and only if $\dot{y} = 0$, in the sense above. Thus, by a suitable coordinate rotation in the xy plane, we may insure that $F_r \neq 0$ and $F_t \neq 0$ in a neighborhood of the point in question on S^2 . Granting that this is a local property only and that the particular rotation performed may introduce values of $F_r = 0$ or $F_t = 0$ at some other sufficiently distant points on S^2 , we observe that this local property is sufficient because our proof is ultimately based upon Theorems 4 and 4a of Chapter III. In those

theorems the integral equation statement of the problem made it plainly evident that the value of the integral at any point P depended only upon the portion of the initial curve cut off by the two characteristics intersecting at P . Consequently, we may consider the arguments below as applying in succession to small overlapping segments of S^2 , with coordinate axes rotated suitably for each segment considered. (See also R. COURANT - D. HILBERT [17] p. 323 and p. 332.)

Let us assume that we have an integral surface $J: uz(x,y)$ satisfying the conditions of either Theorem 8 or Theorem 8a. Then by (1.5) we conclude that the related characteristic base curves are the two one-parameter families of curves determined by the equations

$$(5.1) \quad y_\lambda = \rho_1 x_\lambda,$$

$$(5.2) \quad y_\mu = \rho_2 x_\mu,$$

where

$$(5.3) \quad \rho_1 = \frac{F_s + \sqrt{F_s^2 - 4F_r F_t}}{2F_r},$$

$$(5.4) \quad \rho_2 = \frac{F_s - \sqrt{F_s^2 - 4F_r F_t}}{2F_r}.$$

ρ_1 and ρ_2 are functions of the variables $x, y; u; p, q; r, s, t$ and $\rho_1 \neq \rho_2$ in a neighborhood of S^2 by the hyperbolic condition (1.3).

Consider the coordinate transformation

$$(5.5) \quad \begin{cases} x = x(\lambda, \mu) \\ y = y(\lambda, \mu) . \end{cases}$$

The Jacobian of this transformation,

$$(5.6) \quad y_{\lambda} x_{\mu} - y_{\mu} x_{\lambda} = (\rho_1 - \rho_2) x_{\lambda} x_{\mu},$$

does not vanish in a vicinity of the projection of S^2 . This follows since $\rho_1 \neq \rho_2$; while $x_{\lambda} = 0$ would, by (5.1), imply $y_{\lambda} = 0$, contradicting the requirement $\dot{x}^2 + \dot{y}^2 \neq 0$, (similarly for x_{μ}). Hence the inverse transformation,

$$(5.7) \quad \begin{cases} \lambda = \lambda(x, y) \\ \mu = \mu(x, y) \end{cases},$$

exists in a vicinity of the projection of S^2 .

Along the characteristics on $J: u = u(x, y)$ certain additional equations must be satisfied. These are determined as follows:

Since $F \in C'''$ ($\in C''$) and $u \in C'''$, we obtain by differentiation

$$(5.8) \quad \begin{cases} F_r r_x + F_s s_x + F_t t_x = - [F]_x \\ x_{\lambda} r_x + y_{\lambda} s_x = r_{\lambda} \\ x_{\lambda} s_x + y_{\lambda} t_x = s_{\lambda}, \end{cases}$$

where

$$(5.9) \quad [F]_x = F_p r + F_q s + F_u u + F_x.$$

Similarly,

$$(5.10) \quad \begin{cases} F_r r_y + F_s s_y + F_t t_y = - [F]_y \\ x_{\lambda} r_y + y_{\lambda} s_y = s_{\lambda} \\ x_{\lambda} s_y + y_{\lambda} t_y = t_{\lambda}, \end{cases}$$

where

$$(5.11) \quad [F]_y = F_p s + F_q t + F_u q + F_v y.$$

Since λ is the parameter for one family of characteristic curves and, consequently, is the path parameter along each of the curves of the other family, the determinant

$$(5.12) \quad \begin{vmatrix} F_r & F_s & F_t \\ x_\lambda & y_\lambda & 0 \\ 0 & x_\lambda & y_\lambda \end{vmatrix} = F_r y_\lambda^2 - F_s y_\lambda x_\lambda + F_t x_\lambda^2 = 0.$$

Hence the quantities on the right-hand side in each of the systems (5.8) and (5.10) must be linearly dependent, i.e. in each system the augmented matrix of coefficients must be of rank less than three. Consequently,

$$(5.13) \quad \begin{vmatrix} F_r & F_t & [F]_x \\ x_\lambda & 0 & -r_\lambda \\ 0 & y_\lambda & -s_\lambda \end{vmatrix} = F_r r_\lambda y_\lambda + F_t s_\lambda x_\lambda + [F]_x x_\lambda y_\lambda = 0.$$

Recalling the assumption made without loss,

$$x_\lambda = \frac{1}{\rho_1} y_\lambda \quad \text{and} \quad y_\lambda \neq 0, \quad \text{equation (5.13) reduces to}$$

$$(5.14) \quad F_r r_\lambda + \frac{1}{\rho_1} F_t s_\lambda + [F]_x x_\lambda = 0.$$

Likewise, from (5.10) we obtain the linear dependence of the right-hand terms in the form

$$(5.15) \quad \rho_1 F_r s_\lambda + F_t t_\lambda + [F]_y y_\lambda = 0.$$

Along the curves of the other family of characteristics the following relations must be satisfied. These are obtained in a

fashion completely analogous to that used in obtaining (5.14) and (5.15):

$$(5.16) \quad P_r r_\mu + \frac{1}{P_2} P_{ts} s_\mu + [P]_x x_\mu = 0$$

$$(5.17) \quad P_2 P_{rs} s_\mu + P_{tt} t_\mu + [P]_y y_\mu = 0.$$

In addition, the strip conditions

$$(1.8) \quad \dot{u} = p \dot{x} + q \dot{y}$$

$$(1.9) \quad \begin{cases} \dot{p} = r \dot{x} + s \dot{y} \\ \dot{q} = s \dot{x} + t \dot{y} \end{cases}$$

must be satisfied along any curve lying on $J: u = u(x, y)$. In particular, they must be satisfied along any characteristic on J .

From equations (5.1), (5.2), (5.14) through (5.17), (1.8) and (1.9) we obtain the following system of "characteristic equations" i.e. equations which must be satisfied along the characteristics on any integral surface J :

$$(5.18) \quad \left. \begin{aligned} \varphi_1 &= y_\lambda - P_1 x_\lambda = 0 \\ \varphi_2 &= P_r r_\lambda + \frac{1}{P_1} P_{ts} s_\lambda + [P]_x x_\lambda = 0 \\ \varphi_3 &= P_1 P_{rs} s_\lambda + P_{tt} t_\lambda + [P]_y y_\lambda = 0 \\ \varphi_4 &= u_\lambda - p x_\lambda - q y_\lambda = 0 \\ \varphi_5 &= p_\lambda - r x_\lambda - s y_\lambda = 0 \\ \varphi_6 &= q_\lambda - s x_\lambda - t y_\lambda = 0 \\ \psi_1 &= y_\mu - P_2 x_\mu = 0 \\ \psi_2 &= P_r r_\mu + \frac{1}{P_2} P_{ts} s_\mu + [P]_x x_\mu = 0 \end{aligned} \right\} \text{System A}$$

(5.18)
 (continued)

$$\left. \begin{aligned} \psi_3 &= \rho_2 F_r s \mu + F_t t \mu + [F]_y y \mu = 0 \\ \psi_4 &= u \mu - p x \mu - q y \mu = 0 \\ \psi_5 &= p \mu - r x \mu - s y \mu = 0 \\ \psi_6 &= q \mu - s x \mu - t y \mu = 0 \end{aligned} \right\} \text{System B}$$

We observe that System A of (5.18) is of canonical hyperbolic form in x, y ; u ; p, q ; r, s, t as functions of λ and μ . Since for Theorem B, $F \in C'''$, while for Theorem Ba, $F \in C''$, the coefficients of all equations in (5.18) are functions of class C'' for Theorem B, and of class C' for Theorem Ba. Moreover, the determinant of the matrix of coefficients for System A, is, after interchange of rows and columns,

(5.19)

$$\begin{vmatrix} -\rho_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\rho_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & F_r & \frac{1}{\rho_1} F_t & 0 & 0 & 0 & 0 \\ 0 & * & 0 & \rho_1 F_r & F_t & 0 & 0 & 0 \\ * & 0 & F_r & \frac{1}{\rho_2} F_t & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 1 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 1 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= F_r F_t^2 \cdot \frac{(\rho_1 - \rho_2)^2}{\rho_1 \rho_2},$$

where the coefficients designated only by asterisks, *, do not contribute to the value of the determinant. Since $F_r \neq 0$, $F_t \neq 0$ and $\rho_1 \neq \rho_2$ in a neighborhood of S^E , the determinant (5.19) does not vanish therein. Hence any solution $J: u(x, y)$ of the problem of Theorem B, together with its first and second derivatives,

satisfies the hypotheses for Theorem 7; because the requirement that $F \in C'''$ is certainly sufficient to insure that the first derivatives of the coefficients of System A be Lipschitzian with respect to variables $x, y; u; p, q; r, s, t$. Moreover, the requirement in Theorem 8a that $F \in C''$ insures that the coefficients of System A are of class C' , as demanded by Theorem 7a.

In the $\lambda\mu$, or characteristic, plane, the initial base curve has the parametric form

$$\Upsilon : \begin{cases} \lambda = \lambda(x(\tau), y(\tau)) & \text{for } \tau \in [0, 1], \\ \mu = \mu(x(\tau), y(\tau)) \end{cases}$$

and is nowhere parallel to either the λ or μ axes. Consequently, Υ may be expressed in the non-parametric form

$$\lambda = \varphi(\mu)$$

where $\varphi(\mu) \in C'$ and $\varphi'(\mu) \neq 0$. If we introduce $\lambda' = \lambda$ and $\mu' = -\varphi(\mu)$ as new characteristic parameters, we observe that equations (5.18) remain unaltered in form. Hence we may assume, without loss, that the initial base curve Υ has the representation

$$(5.20) \quad \lambda + \mu = 0$$

in the $\lambda\mu$ plane.

We now demonstrate that any solution of System A satisfying the given Cauchy initial conditions is also a solution of the problem of Theorems 8 and 8a. This done, Theorems 8 and 8a are immediate consequences of Theorems 7 and 7a, respectively.

Following J. HADAMARD [11] p. 504, we show that for each set of functions satisfying System A and the initial conditions on

$\lambda + \mu = 0$, the System B is likewise satisfied. Note that in this part of the argument we cannot admit that p, q, r, s and t are derivatives of u . This is now a matter of proof.

Differentiating $F(x, y; u; p, q; r, s, t)$ by λ and observing equations (5.18), we obtain

$$(5.21) \quad \frac{dF}{d\lambda} = \varphi_2 + \varphi_3 + F_u \varphi_4 + F_p \varphi_5 + F_q \varphi_6.$$

Hence $\frac{dF}{d\lambda} = 0$ for each set of functions satisfying System A. However, by hypothesis, $F = 0$ along $\lambda + \mu = 0$. Thus $F \equiv 0$ throughout that region where the set of functions satisfying System A is defined. This in turn implies that

$$(5.22) \quad \frac{dF}{d\mu} = \psi_2 + \psi_3 + F_u \psi_4 + F_p \psi_5 + F_q \psi_6 = 0 \text{ throughout the same region. By hypothesis, } \psi_2 = 0 \text{ in this region, hence}$$

$$(5.23) \quad \psi_3 = -F_u \psi_4 - F_p \psi_5 - F_q \psi_6$$

therein.

Since $\rho_1 \rho_2 = \frac{F_t}{F_r}$, we obtain from (5.18) by simple algebraic

operations

$$(5.24) \quad \frac{\rho_1 y_\mu}{F_t} \varphi_2 = r_\lambda x_\mu + s_\lambda y_\mu + H,$$

$$(5.25) \quad \frac{\rho_2 y_\lambda}{F_t} \psi_2 = r_\mu x_\lambda + s_\mu y_\lambda + H,$$

where

$$(5.26) \quad H = \frac{y_\lambda y_\mu}{F_t} [F]_x = \frac{x_\lambda x_\mu}{F_r} [F]_x ;$$

$$(5.27) \quad \frac{y_\mu}{F_t} \varphi_3 = s_\lambda x_\mu + t_\lambda y_\mu + K,$$

$$(5.28) \quad \frac{y_\lambda}{F_t} \psi_3 = s_\mu x_\lambda + t_\mu y_\lambda + K,$$

where

$$(5.29) \quad K = \frac{y_\lambda y_\mu}{F_t} [F]_y = \frac{x_\lambda x_\mu}{F_r} [F]_y.$$

By Theorem 7 or Theorem 7a, the functions of the set satisfying System A and the Cauchy initial data are continuously differentiable and possess continuous mixed second derivatives. Thus we may perform the differentiations in the following relations:

$$(5.30) \quad \begin{aligned} \psi_{4,\lambda} - \varphi_{4,\mu} &= p_\lambda x_\mu + q_\lambda y_\mu - p_\mu x_\lambda - q_\mu y_\lambda \\ &= \varphi_5 x_\mu - \varphi_6 y_\mu - \psi_5 x_\lambda - \psi_6 y_\lambda, \end{aligned}$$

$$(5.31) \quad \begin{aligned} \psi_{5,\lambda} - \varphi_{5,\mu} &= r_\lambda x_\mu + s_\lambda y_\mu - r_\mu x_\lambda - s_\mu y_\lambda \\ &= \frac{r_1 y_\mu}{F_t} \varphi_2 - \frac{r_2 y_\lambda}{F_t} \psi_2, \end{aligned}$$

by (5.24) and (5.25) above;

$$(5.32) \quad \begin{aligned} \psi_{6,\lambda} - \varphi_{6,\mu} &= s_\mu x_\lambda + t_\mu y_\lambda - s_\lambda x_\mu - t_\lambda y_\mu \\ &= \frac{y_\lambda}{F_t} \psi_3 - \frac{y_\mu}{F_t} \varphi_3. \end{aligned}$$

by (5.27) and (5.28) above. But System A is satisfied, hence (5.30), (5.31) and (5.32), by virtue of (5.23), reduce to

$$(5.33) \quad \begin{cases} \psi_{4,\lambda} &= -\psi_5 x_\lambda - \psi_6 y_\lambda \\ \psi_{5,\lambda} &= 0 \\ \psi_{6,\lambda} &= \frac{-y_\lambda}{F_t} (F_u \psi_4 + F_p \psi_5 + F_q \psi_6). \end{cases}$$

In (5.33) all functions are known except ψ_4, ψ_5, ψ_6 and their derivatives with respect to λ . Moreover, along $\lambda = -\mu$ System B is satisfied, i.e. $\psi_4 = \psi_5 = \psi_6 = 0$ for $\lambda = -\mu$. For fixed μ we may consider (5.33) as a homogeneous system of linear first order ordinary differential equations under homogeneous onepoint boundary conditions. This system has the unique solution

$$\psi_4 = \psi_5 = \psi_6 = 0$$

throughout the region of definition of the set of functions satisfying System A. By (5.23), $\psi_3 = 0$ also, and the System B is shown to be dependent upon the System A in the sense above.

From the functions $x = x(\lambda, \mu), y = y(\lambda, \mu)$ of the set satisfying System A, we may form the inverse functions $\lambda = \lambda(x, y), \mu = \mu(x, y)$, since the Jacobian

$$(5.6) \quad y_\lambda x_\mu - y_\mu x_\lambda = (\rho_1 - \rho_2) x_\lambda x_\mu$$

does not vanish. Hence we may express the function $u = u(\lambda, \mu)$ as a function of the independent variables x and y .

We now need to show only that

$$(5.34) \quad p = u_x, \quad q = u_y, \quad r = u_{xx}, \quad s = u_{xy} \quad \text{and} \quad t = u_{yy}$$

throughout the above region to complete the proof.

$$\text{Now} \quad \varphi_4 = u_\lambda - px_\lambda - qy_\lambda = 0$$

$$\psi_4 = u_\mu - px_\mu - qy_\mu = 0,$$

while the determinant of this linear system is the Jacobian (5.6) and hence does not vanish. Thus there exists a unique solution.

But $p = u_x$, $q = u_y$ obviously satisfies and hence represents the unique solution.

Similarly,

$$\varphi_5 = u_{x,\lambda} - rx_\lambda - sy_\lambda = 0$$

$$\psi_5 = u_{x,\mu} - rx_\mu - sy_\mu = 0,$$

hence $r = u_{xx}$ and $s = u_{xy}$;

$$\varphi_6 = u_{y,\lambda} - ax_\lambda - ty_\lambda = 0$$

$$\psi_6 = u_{y,\mu} - ax_\mu - ty_\mu = 0,$$

hence $t = u_{yy}$ and $u_{yx} = u_{xy} = s$. The proof is now complete.

CHAPTER VI

The Characteristic Initial Value Problem for

$$F(x,y;u;p,q;r,s,t) = 0.$$

The whole idea of a characteristic initial value problem for the equation

$$(1.1) \quad F(x,y;u;p,q;r,s,t) = 0$$

appears paradoxical at first glance. In the Cauchy problem the prescribed initial data was sufficient to determine whether or not the projection of the initial curve was characteristic. In this problem, however, we merely prescribe two intersecting space curves through which an integral surface of the equation (1.1) must pass. Since the characteristics are, in general, dependent on the integral surface in question, it would appear impossible to determine, a priori, whether or not the prescribed initial curves have characteristic projections.

That such is not the case is demonstrated by M. CINQUINI-CISERARIO [13]. In this paper she treats the characteristic initial value problem as a special case of the more general Goursat problem, i.e. where two arbitrary intersecting space curves are prescribed through which an integral surface of (1.1) must pass. Commencing on p. 220, she gives the necessary and sufficient conditions that these curves be characteristic to any integral surface passing through them. We call curves satisfying these conditions "intrinsically characteristic" curves.

In this chapter we examine her development, for the particular case of the characteristic initial value problem, up to the point where a modified form of the system of characteristic equations (5.18) and the above necessary and sufficient conditions are obtained. There are two important differences between her development and that of H. LEWY given in the preceding chapter. First, she transforms the initial curves into the coordinate axes. Since these curves are characteristic, this implies immediately that $P_r = 0$ and $P_t = 0$ at the origin. Thus many of the divisions performed in Chapter V are now invalidated. Second, she is able to solve (1.1) explicitly for s , obtaining

$$s = f(x, y; u; p, q; r, t)$$

and thus to reduce the number of equations in the system of characteristic equations by two.

We do not follow the remainder of her existence proof, in which she reduces the system of characteristic equations to an integral equation form and then applies successive approximations to obtain the existence of a unique solution to the general Goursat problem. Instead we deal directly with the special case of the characteristic initial value problem by a method analogous to that of Chapter V. Such an approach is indicated by M. CINQUINI-CIBRARIO, herself, [12] p.180, footnote 8. She states, in effect, that the following Theorem 9 can be shown to be a consequence of Theorem 6, Chapter IV. We present this proof in detail and, in addition, we extend it to apply to the derivation of Theorem 9a as a consequence of Theorem 6a. The improvement obtained corresponds to that of

Chapter V for the Cauchy problem. Namely, the requirement that $F \in C'''$ is reduced to require merely that $F \in C''$ while the conclusion is altered to read "at least one solution" instead of "one and only one solution".

Theorem 9

$$1) \quad \Pi_1: \begin{cases} \Gamma_1: \begin{cases} x_1 - \xi \leq x \leq x_1 + \xi, & f_1(x) \in C''([x_1 - \xi, x_1 + \xi]) \\ y = f_1(x) & F_1(x) \in C''([x_1 - \xi, x_1 + \xi]). \\ u = F_1(x) \end{cases} \\ \\ \Gamma_2: \begin{cases} x = f_2(y), & f_2(y) \in C''([y_1 - \eta, y_1 + \eta]) \\ y_1 - \eta \leq y \leq y_1 + \eta & F_2(y) \in C''([y_1 - \eta, y_1 + \eta]) \\ u = F_2(y) \end{cases} \end{cases}$$

The point (x_1, y_1) is the only point of intersection of Γ_1 and Γ_2 and it is interior to both curves. Moreover, $F_1(x_1) = F_2(y_1)$ and $f_1'(x_1)f_2'(y_1) \neq 1$. (i.e. Γ_1 and Γ_2 do not have a common tangent at the point (x_1, y_1) .)

2) Π_1 and Π_2 are "intrinsically characteristic" in a neighborhood of their point of intersection, i.e. they meet the necessary and sufficient conditions, given below, that they be characteristic to any integral surface of

$$(1.1) \quad F(x, y; u; p, q; r, s, t) = 0$$

passing through them. As we shall see below, this hypothesis, together with hypothesis 1), tacitly implies that at the intersection point (x_1, y_1, u_1) of Π_1 and Π_2 the values $p_1, q_1, r_1, s_1,$

t_1), the hyperbolic condition

$$F_{s_1}^2 - 4 F_{r_1} F_{t_1} > 0,$$

is satisfied, (notation: $F_{s_1} = F_s(x_1, y_1; u_1; p_1, q_1; r_1, s_1, t_1)$, etc.)

3) $F \in C'''$ in a neighborhood of the point

$$(x_1, y_1; u_1; p_1, q_1; r_1, s_1, t_1).$$

\Rightarrow 4) There exists one and only one integral surface $J_{\text{mod}}(x, y)$ of $F(x, y; u; p, q; r, s, t) = 0$, defined and of class C''' in a sufficiently small neighborhood of the point (x_1, y_1) and passing through subarcs of Γ_1 and Γ_2 intersecting at the point (x_1, y_1, u_1) .

Theorem 9a

1)

2)

3)' $F \in C''$ in a neighborhood of the point

$$(x_1, y_1; u_1; p_1, q_1; r_1, s_1, t_1).$$

\Rightarrow 4)' There exists at least one integral surface etc.

(as in Theorem 9).

Proof of Theorems 9 and 9a

We first perform the coordinate transformation

$$(6.1) \quad \begin{cases} \bar{x} = x - f_2(y) \\ \bar{y} = y - f_1(x) \end{cases}$$

taking γ_1 into the \bar{x} axis, γ_2 into the \bar{y} axis and the point (x_1, y_1) into the origin. This transformation is univalent in a

neighborhood of (x_1, y_1) since the Jacobian

$$(6.2) \quad 1 - f_1'(x_1)f_2'(y_1) \neq 0$$

by hypothesis 1). Geometrically, this means that Γ_1 and Γ_2 do not have a common tangent at their point of intersection.

Without loss, we may assume homogeneous initial conditions. For, suppose we have an integral surface $J: u=u(x,y)$ of equation (1.1) passing through the curves Γ_1 and Γ_2 . Then by the above transformation, considering (6.2),

$$(6.3) \quad u(x,y) = \bar{u}(\bar{x}(x,y), \bar{y}(x,y)),$$

and hence for any such integral surface

$$(6.4) \quad \begin{cases} P_1(x) = u(x, f_1(x)) = \bar{u}(\bar{x}(x, f_1(x)), 0), \\ P_2(y) = u(f_2(y), y) = \bar{u}(0, \bar{y}(f_2(y), y)). \end{cases}$$

Letting

$$(6.5) \quad w(\bar{x}, \bar{y}) = \bar{u}(\bar{x}, \bar{y}) - \bar{u}(\bar{x}, 0) - \bar{u}(0, \bar{y}) + \bar{u}(0, 0),$$

and since, by hypothesis 1), f_1, f_2, P_1 and $P_2 \in C^1$, we obtain

$$(6.6) \quad \begin{aligned} w(\bar{x}, 0) &= w_{\bar{x}}(\bar{x}, 0) = w_{\bar{x}\bar{x}}(\bar{x}, 0) = 0, \\ w(0, \bar{y}) &= w_{\bar{y}}(0, \bar{y}) = w_{\bar{y}\bar{y}}(0, \bar{y}) = 0. \end{aligned}$$

Thus we may reduce the problem to that of finding a function $w = w(\bar{x}, \bar{y})$ which vanishes on the coordinate axes in a vicinity of the origin and satisfies there the transformed form of equation (1.1),

$$(6.7) \quad F(\bar{x}, \bar{y}; [w + \varepsilon]; [w + \varepsilon], \bar{x}, [w + \varepsilon], \bar{y}; [w + \varepsilon], \bar{x}, \\ [w + \varepsilon], \bar{y}, [w + \varepsilon], \bar{x}, [w + \varepsilon], \bar{y})$$

where

$$(6.8) \quad g(\bar{x}, \bar{y}) = \bar{u}(\bar{x}, 0) + \bar{u}(0, \bar{y}) - \bar{u}(0, 0).$$

The function g is known from the prescribed initial data.

For simplicity, we return to our original notation and state the problem in this way:

To determine the function $u = u(x, y)$ satisfying equation (1.1) and the initial conditions

$$u(x, 0) = u(0, y) = 0,$$

where, in the notation above,

$$u_0 = p_0 = q_0 = r_0 = t_0 = 0$$

and

$$(6.9) \quad F(0, 0; 0; 0, 0; 0, s_0, 0) = 0.$$

By hypothesis 2), there exists a unique value s_0 satisfying (6.9).

The characteristic base curves and, a fortiori, the hyperbolic condition are invariant under the transformation (6.1). (See R. COURANT - D. HILBERT [17] p. 304.) Moreover, the substitution $w = \bar{u} - g$ also preserves the invariance of the equation for the characteristic base curves and the hyperbolic condition as is easily seen by differentiation of (6.7). Hence, by hypothesis 2), we have the hyperbolic condition

$$(6.10) \quad F_{s_0}^2 - 4 F_{r_0} F_{t_0} > 0,$$

while the equation for the characteristic base curve directions at the origin is

$$(6.11) \quad F_{r_0} dy^2 - F_{s_0} dx dy + F_{t_0} dx^2 = 0.$$

Hypothesis 2) implies that the coordinate axes must be characteristic base curves. By (6.11) and (6.10) this in turn implies that $F_{r_0} = F_{t_0} = 0$, and hence that $F_{s_0} \neq 0$. But now the Implicit Function Theorem tells us that in the neighborhood of the point $(0,0; 0; 0,0; 0, s_0, 0)$ equation (1.1) can be solved explicitly in the form

$$(6.12) \quad s = f(x,y; u; p,q; r,t).$$

Under hypothesis 3) or 3)', the function $f \in C'''$ or C'' , respectively, in a neighborhood of this point. Moreover,

$$(6.13) \quad f_{r_0} = f_{t_0} = 0 \quad \text{and} \quad s_0 = f_0$$

while the hyperbolic condition becomes at the origin

$$(6.14) \quad 1 - 4 f_{r_0} f_{t_0} = 1 > 0$$

and the equation for the characteristic base curves becomes

$$(6.15) \quad f_r dy^2 + dx dy + f_t dx^2 = 0.$$

Let us assume that we have a particular integral surface $J: u = u(x,y)$ passing through the coordinate axes in a neighborhood of the origin, with $u(x,y) \in C'''$ in this neighborhood..

We define

$$(6.16) \quad \delta = \sqrt{1 - 4 f_r f_t}, \quad \rho = \frac{-2f_t}{1+\delta}, \quad \sigma = \frac{-2f_r}{1+\delta},$$

δ , ρ and σ being of class C^1 by hypothesis 3), or of class C^1 by hypothesis 3)', in the variables $x, y; u; p, q; r, t$ in a neighborhood of the point $(0, 0; 0; 0, 0; 0, 0)$. The two one-parameter families of characteristic base curves corresponding to J are thus represented by the equations

$$(6.17) \quad y_\lambda = \rho x_\lambda$$

$$(6.18) \quad x_\mu = \sigma y_\mu.$$

Note that $\delta_0 = 1$, hence $\delta > 0$ in a neighborhood of the origin, while $\rho_0 = \sigma_0 = 0$.

As in Chapter V, to obtain the system of characteristic equations, we transform to the characteristic base curves as coordinates and consider what relations must be satisfied along these coordinates for any given integral surface J . In particular, we specialize the transformation

$$(6.19) \quad \begin{cases} x = x(\lambda, \mu) \\ y = y(\lambda, \mu) \end{cases}$$

by stipulating that a line $\lambda = \text{constant}$ shall have x -intercept $(\lambda, 0)$ and a line $\mu = \text{constant}$ shall have y -intercept $(0, \mu)$, with $\lambda = \mu = 0$ at the origin. The Jacobian of this transformation, evaluated at the origin, has the value

$$(6.20) \quad x_{\lambda_0} y_{\mu_0} - y_{\lambda_0} x_{\mu_0} = x_{\lambda_0} y_{\mu_0} (1 - \rho_0 \sigma_0) = x_{\lambda_0} y_{\mu_0} \neq 0,$$

since if $x_{\lambda_0} = 0$, then $y_{\lambda_0} = 0$ by (6.17), contradicting the requirement that $\dot{x}^2 + \dot{y}^2 \neq 0$ along any characteristic curve.

Similarly, if $y_{\mu_0} = 0$, then $x_{\mu_0} = 0$ by (6.18) and the contradiction is again obtained.

Paralleling our development in Chapter V, we see that certain determinants must vanish at each point of the integral surface J , yielding equations which must be satisfied along the characteristics on J . We have

$$(6.21) \quad \begin{vmatrix} f_r & -[f]_x & f_t \\ x_\lambda & r_\lambda & 0 \\ 0 & s_\lambda & y_\lambda \end{vmatrix} = f_r r_\lambda y_\lambda + f_t s_\lambda x_\lambda + [f]_x x_\lambda y_\lambda = 0$$

where

$$(6.22) \quad [f]_x = f_p r + f_q f + f_u p + f_x.$$

also

$$(6.23) \quad \begin{vmatrix} f_r & -[f]_y & f_t \\ x_\lambda & s_\lambda & 0 \\ 0 & t_\lambda & y_\lambda \end{vmatrix} = f_r s_\lambda y_\lambda + f_t t_\lambda x_\lambda + [f]_y x_\lambda y_\lambda = 0$$

where

$$(6.24) \quad [f]_y = f_f f + f_t t + f_q q + f_y.$$

Eliminating s_λ between (6.21) and (6.23), we obtain

$$(6.25) \quad f_r^2 r_\lambda y_\lambda^2 - f_t^2 t_\lambda x_\lambda^2 + [f]_x f_r x_\lambda y_\lambda^2 -$$

$$[f]_y f_t x_\lambda^2 y_\lambda = 0.$$

By virtue of definitions (6.16) and equation (5.17), we may write (6.25) as

$$(6.26) \quad f_t^2 x_\lambda^2 \cdot H(\lambda, \mu) = 0$$

where

$$(6.27) \quad H(\lambda, \mu) = r_\lambda \sigma^2 - t_{\lambda+} \frac{2}{1+\delta} \left\{ [f]_y - \sigma [f]_x \right\} x_\lambda.$$

But, as shown above, $x_\lambda \neq 0$ along any of the characteristic base curves of J of the corresponding family, hence (6.26) reduces to

$$(6.28) \quad f_t^2 \cdot H(\lambda, \mu) = 0.$$

Where $f_t = 0$ we have immediately that $H(\lambda, \mu) = 0$. Suppose at a particular point of J that $f_t = 0$. Then by (6.16) and (6.17), we have there that

$$(6.29) \quad \rho = 0, \quad \delta = 1, \quad \sigma = -f_r \quad \text{and} \quad y_\lambda = 0.$$

Thus, at this point, by (6.24),

$$(6.30) \quad t_\lambda = s_y x_\lambda = (f_r r_y + [f]_y) x_\lambda;$$

while by (6.22),

$$(6.31) \quad r_\lambda \sigma^2 = f_r^2 r_x x_\lambda = f_r^2 (s_\lambda - [f]_x x_\lambda).$$

Substituting (6.30) and (6.31) into (6.27), we obtain that where $f_t = 0$ on J , $H(\lambda, \mu) = 0$. Hence by (6.28), $H(\lambda, \mu) = 0$ everywhere on J and represents a relation which must be satisfied along each characteristic of the corresponding family on J .

For the other family of characteristics on J , we have determinants corresponding to (6.21) and (6.22) which vanish at each point of J . Eliminating s_μ between these and arguing in a fashion analogous to that above, we arrive at the following rela-



tion which must be satisfied along each characteristic of this family on J :

$$(6.32) \quad K(\lambda, \mu) = \rho^2 \mu - r\mu + \frac{2}{1+\delta} \left\{ [f]_x - \rho [f]_y \right\} y\mu = 0.$$

We are now in a position to prescribe the necessary and sufficient conditions that the coordinate axes be characteristics for any integral surface of

$$(6.12) \quad z = f(x, y; u; p, q; r, t)$$

passing through them.

Suppose that, in a neighborhood of the origin, the coordinate axes are characteristic to some integral surface $J: z = u(x, y)$ of (6.12) passing through them. Then in terms of the characteristic base curves to J as coordinates, defined by the coordinate transformation (6.19), we have for $\mu = 0$:

$$x = \lambda, \quad y = 0, \quad u = p = r = 0, \quad q = Q(\lambda), \quad t = T(\lambda),$$

where, from (6.12),

$$(6.33) \quad Q'(\lambda) = f(\lambda, 0; 0; 0, Q(\lambda); 0, T(\lambda)),$$

while, from $H(\lambda, \mu) = 0$, since $\rho = f_t = 0$, $\delta = 1$ and

$$\sigma = -f_r,$$

$$(6.34) \quad T'(\lambda) = \left\{ [f]_y + f_r [f]_x \right\} (\lambda, 0; 0; 0, Q(\lambda); 0, T(\lambda)).$$

Moreover,

$$(6.35) \quad Q(0) = T(0) = 0.$$



Equations (6.33) and (6.34) represent a system of first order ordinary differential equations under one point boundary conditions (6.35). The right hand sides of the equations of this system are of class C'' under hypothesis 3), or of class C' under hypothesis 3)', in the variables λ , Q and T . Hence, in either case, the functions Q and T are uniquely determined in a neighborhood of $\lambda = 0$. If the x axis is characteristic, these functions must also satisfy

$$(6.36) \quad f_t(\lambda, 0; 0; 0, Q(\lambda); 0, T(\lambda)) = 0.$$

Similarly, for $\lambda = 0$:

$$x = 0, y = \mu, u = q = t = 0, p = P(\mu), r = R(\mu),$$

where, from (6.12),

$$(6.37) \quad P'(\mu) = f(\mu, \mu; 0; P(\mu), 0; R(\mu), 0),$$

while, from $K(\lambda, \mu) = 0$, since $\sigma = f_p = 0$, $\delta = 1$ and $\rho = -f_t$,

$$(6.38) \quad R'(\mu) = \left\{ [f]_x + f_t [f]_y \right\} (0, \mu; 0; P(\mu), 0; R(\mu), 0).$$

Moreover,

$$(6.39) \quad P(0) = R(0) = 0.$$

Hence, if the y axis is characteristic, the functions P and R , uniquely determined by (6.37), (6.38), and (6.39), must also satisfy

$$(6.40) \quad f_r(0, \mu; 0; P(\mu), 0; R(\mu), 0) = 0.$$

To recapitulate, the necessary condition that the x axis be a characteristic of some integral surface is that the functions Q and T determined from the system (6.33) and (6.34), under boundary conditions (6.35), shall satisfy (6.36) for each λ in a neighborhood of $\lambda = 0$. The necessary condition that the y axis be a characteristic of some integral surface is that the functions P and R determined from the system (6.37) and (6.38), under boundary conditions (6.36), shall satisfy (6.40) for each μ in a neighborhood of $\mu = 0$.

We now show that these conditions are also sufficient, i.e. given in the vicinity of the origin, an integral surface $J: u = u(x, y)$ of (6.12) passing through the coordinate axes, with

$$(6.41) \quad P_1(y) = u_x(0, y), \quad R_1(y) = u_{xx}(0, y), \quad Q_1(x) = u_y(x, 0), \\ \text{and } T_1(x) = u_{yy}(x, 0),$$

we show that the requirement

$$(6.40)' \quad f_T(0, y; 0; P_1(y), 0; R_1(y), 0) = 0$$

is sufficient that the y axis be a characteristic on J .

The argument needed to show that the requirement

$$(6.36)' \quad f_Q(x, 0; 0; 0, Q_1(x); 0, T_1(x)) = 0$$

is sufficient in order that the x axis be a characteristic on J is analogous to the following and will not be given here.

We need show only that under requirement (6.40)', $P_1(y) = P(y)$ and $R_1(y) = R(y)$, where $P(y)$ and $R(y)$ are those functions obtained

previously under the assumption that the y-axis was "intrinsically characteristic".

Now $P_1(0) = R_1(0) = 0$ since $u(x,0) = 0$. Moreover, since u satisfies

$$(6.12) \quad s = f(x,y; u; p,q; r,t),$$

for $x = 0$,

$$(6.37)' \quad P_1'(y) = f(0,y; 0; P_1(y), 0; R_1(y), 0).$$

Now, recalling that $u \in C'''$,

$$(6.42) \quad s_x = f_r r_x + f_t t_x + [f]_x,$$

$$(6.43) \quad s_y = f_r r_y + f_t t_y + [f]_y.$$

Since $u(0,y) = 0$, we obtain $t_y(0,y) = 0$. Writing $r_x(0,y) = w(y)$ and substituting (6.43) into (6.42) with $x = 0$, we obtain

$$(6.44) \quad \begin{aligned} s_x(0,y) &= r_y(0,y) \\ &= f_r w(y) + f_t f_r r_y + [f]_x + f_t [f]_y \end{aligned}$$

But, $u(0,y) = u_y(0,y) = u_{yy}(0,y) = 0$, hence by (6.44),

$$(6.38)' \quad R_1'(y) = \left[\frac{1}{1-f_r f_t} \left\{ [f]_x + f_t [f]_y + f_r w(y) \right\} \right] (0,y; 0; P_1(y), 0; R_1(y), 0).$$

Now equation (6.37)' is precisely the same as (6.37), while requirement (6.40)' is sufficient to reduce (6.38)' to (6.38).

But this implies that $P_1(y) = P(y)$ and $R_1(y) = R(y)$ since the solution of the system of ordinary differential equations in question is unique.

In the foregoing arguments we have developed a procedure for determining whether or not the initial curves are "intrinsically characteristic". By transformation (6.1) and substitution (6.5), we reduce the initial curves Γ_1 and Γ_2 to the coordinate axes. If now s_0 can be uniquely determined from (6.9) we may verify the hyperbolic condition and obtain the characteristic directions at the origin. If these directions coincide with the coordinate axes, then equation (1.1) can be solved explicitly for (6.12). From this, the system (6.37) and (6.38) under boundary condition (6.39) can, in principle at least, be solved for functions P and R. Finally if P and R satisfy (6.40) then the y axis is characteristic to any integral surface of the problem, i.e. "intrinsically characteristic". Likewise, from the system (6.33) and (6.34) under boundary condition (6.35), the functions Q and T can be determined. If these satisfy (6.36) then the x axis is "intrinsically characteristic". Note that P, R, Q and T are evidently of class C^1 .

Having given hypothesis 2) a precise meaning along with a procedure for determining whether or not it is verified for a given problem, we continue with the proof under the assumption that hypothesis 2) is verified.

From equations (6.17), (6.18), (6.27), (6.32) and the strip conditions we obtain the following system of characteristic equations, which must be satisfied along the characteristics on any integral surface J:

$$\begin{aligned}
 \varphi_1 &= y_\lambda - \rho x_\lambda = 0 \\
 \varphi_2 &= r_\lambda \sigma^2 - t_\lambda + \frac{2}{1+\delta} \left\{ [f]_y - \sigma [f]_x \right\} x_\lambda = 0 \\
 \varphi_3 &= u_\lambda - p x_\lambda - q y_\lambda = 0 \\
 \varphi_4 &= p_\lambda - r x_\lambda - f y_\lambda = 0 \\
 (6.45) \quad \varphi_5 &= q_\lambda - f x_\lambda - t y_\lambda = 0
 \end{aligned}
 \left. \vphantom{\begin{aligned} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \end{aligned}} \right\} \text{System A}$$

$$\begin{aligned}
 \psi_1 &= x_\mu - \sigma y_\mu = 0 \\
 \psi_2 &= r_\mu - \rho^2 t_\mu - \frac{2}{1+\delta} \left\{ [f]_x - \rho [f]_y \right\} y_\mu = 0 \\
 \psi_3 &= u_\mu - p x_\mu - q y_\mu = 0 \\
 \psi_4 &= p_\mu - r x_\mu - f y_\mu = 0 \\
 \psi_5 &= q_\mu - f x_\mu - t y_\mu = 0
 \end{aligned}
 \left. \vphantom{\begin{aligned} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{aligned}} \right\} \text{System B}$$

We observe that System A of (6.45) is of canonical hyperbolic form in $x, y; u; p, q; r, t$ as functions of λ and μ . Since for Theorem 9, $F \in C'''$, while for Theorem 9a, $F \in C''$, the coefficients of all equations in (6.45) are functions of class C'' for Theorem 9, and of class C' for Theorem 9a. Moreover, the matrix of coefficients for System A is, after interchange of rows and columns,

$$(6.46) \quad \begin{vmatrix} -\rho & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\sigma & 0 & 0 & 0 & 0 & 0 \\ * & 0 & \sigma^2 & -1 & 0 & 0 & 0 \\ 0 & * & 1 & -\rho^2 & 0 & 0 & 0 \\ * & * & 0 & 0 & 1 & 0 & 0 \\ * & * & 0 & 0 & 0 & 1 & 0 \\ * & * & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (1 - \rho\sigma) (\rho^2 \sigma^2 - 1) = \frac{-8\delta^2}{(1+\delta)^3}$$

where the coefficients designated only by asterisks, *, do not contribute to the value of the determinant. But $\delta > 0$ everywhere on J in a neighborhood of the origin, hence the determinant (6.46) does not vanish thereon.

As to the initial conditions, we have, by hypothesis 1) of Theorems 9 and 9a for $\mu = 0$,

$$x = \lambda, \quad y = 0, \quad u = p = r = 0, \quad q = Q(\lambda), \quad t = T(\lambda),$$

and for $\lambda = 0$,

$$x = 0, \quad y = \mu, \quad u = q = t = 0, \quad p = P(\mu), \quad r = R(\mu)$$

where Q , T and P , R are determined from their respective systems and are of class C^1 . Moreover, for $\mu = 0$, by (6.36), $f_t = 0$.

Hence $\rho = 0$, $\delta = 1$, and $\sigma = -f_x$. This together with $y_\lambda = r_\lambda = u_\lambda = p_\lambda = 0$ and equation (6.34) prove that

$$(6.47) \quad \varphi_1(\lambda, 0) = \varphi_2(\lambda, 0) = \varphi_3(\lambda, 0) = \varphi_4(\lambda, 0) = \varphi_5(\lambda, 0) = 0$$

for all λ in a neighborhood of $\lambda = 0$. Similarly, for $\lambda = 0$, by (6.40), $f_x = 0$. Hence $\sigma = 0$, $\delta = 1$ and $\rho = -f_t$. This together with $x_\mu = t_\mu = u_\mu = q_\mu = 0$ and equation (6.33) prove that

$$(6.48) \quad \psi_1(0, \mu) = \psi_2(0, \mu) = \psi_3(0, \mu) = \psi_4(0, \mu) = \psi_5(0, \mu) = 0$$

for all μ in a neighborhood of $\mu = 0$. Thus the initial condition requirements of hypothesis 3) of Theorems 6 and 6a are satisfied.

Since the coefficients in (6.45) are of class C^1 for Theorem 9, hypotheses 1) and 2) of Theorem 6 are satisfied. Also, since the coefficients in (6.45) are of class C^1 for Theorem 9a, the

common hypothesis 1) of Theorems 6 and 6a is satisfied, but hypothesis 2) of Theorem 6, a hypothesis which does not appear in Theorem 6a, is not satisfied. Thus if we now show that any solution of the canonical hyperbolic system, System A of (6.45), with the given characteristic initial conditions is also a solution of the corresponding problem for the equation

$$(6.12) \quad s = f(x, y; u; p, q; r, t)$$

with the same initial conditions, then Theorem 9 is an immediate consequence of Theorem 6 and Theorem 9a is an immediate consequence of Theorem 6a.

As in the Cauchy problem of Chapter 5, we show that for each solution of System A under the given characteristic initial conditions that System B is likewise satisfied. Note that here we cannot assume that p, q, r and t are derivatives of u ; this is a matter of proof. Recalling from Theorems 6 and 6a that the functions of the solution of System A, x, y, u, p, q, r, t are of class C^1 and that $f \in C'''$ under hypothesis 3) of Theorem 9, or $f \in C''$ under hypothesis 3)' of Theorem 9a, we obtain by differentiation and consideration of (6.45) that

$$(6.49) \quad \begin{aligned} \psi_{3,\lambda} - \psi_{3,\mu} &= p_\mu x_\lambda + q_\mu y_\lambda - p_\lambda x_\mu - q_\lambda y_\mu \\ &= \psi_{4x_\lambda} + \psi_{5y_\lambda} - \psi_{4x_\mu} - \psi_{5y_\mu} . \end{aligned}$$

Moreover, since $\psi_3 = \psi_4 = \psi_5 = 0$,

$$(6.50) \quad \begin{aligned} f_\lambda &= f_r r_\lambda + f_t t_\lambda + f_p p_\lambda + f_q q_\lambda + f_u u_\lambda + f_x x_\lambda + f_y y_\lambda \\ &= f_r r_\lambda + f_t t_\lambda + [f]_x x_\lambda + [f]_y y_\lambda , \end{aligned}$$

while

$$\begin{aligned}
 (6.51) \quad f_{\mu} &= f_r r_{\mu} + f_t t_{\mu} + f_p p_{\mu} + f_q q_{\mu} + f_u u_{\mu} + f_x x_{\mu} + f_y y_{\mu} \\
 &= f_r r_{\mu} + f_t t_{\mu} + [f]_x x_{\mu} + [f]_y y_{\mu} \\
 &\quad + f_p \psi_4 + f_q \psi_5 + f_u \psi_3.
 \end{aligned}$$

Thus by (6.45), (6.50) and (6.51),

$$\begin{aligned}
 (6.52) \quad \psi_{4,\lambda} - \varphi_{4,\mu} &= r_{\mu} x_{\lambda} + f_{\mu} y_{\lambda} - r_{\lambda} x_{\mu} - f_{\lambda} y_{\mu} \\
 &= y_{\lambda} \{ f_p \psi_4 + f_q \psi_5 + f_u \psi_3 \} \\
 &\quad + \left(\frac{1+\delta}{2}\right) x_{\lambda} \psi_2 - \left(\frac{1+\delta}{2}\right) r_{y\mu} \varphi_2,
 \end{aligned}$$

and

$$\begin{aligned}
 (6.53) \quad \psi_{5,\lambda} - \varphi_{5,\mu} &= f_{\mu} x_{\lambda} + t_{\mu} y_{\lambda} - f_{\lambda} x_{\mu} - t_{\lambda} y_{\mu} \\
 &= x_{\lambda} \{ f_p \psi_4 + f_q \psi_5 + f_u \psi_3 \} \\
 &\quad - \left(\frac{1+\delta}{2}\right) \sigma_{x\lambda} \psi_2 + \left(\frac{1+\delta}{2}\right) y_{\mu} \varphi_2.
 \end{aligned}$$

Taking into account the fact that System A is satisfied, we reduce (6.49), (6.52) and (6.53) to the system

$$\begin{aligned}
 (6.54) \quad \psi_{3,\lambda} &= \psi_4 x_{\lambda} + \psi_5 y_{\lambda} \\
 \psi_{4,\lambda} &= y_{\lambda} \{ f_u \psi_3 + f_p \psi_4 + f_q \psi_5 \} \\
 \psi_{5,\lambda} &= x_{\lambda} \{ f_u \psi_3 + f_p \psi_4 + f_q \psi_5 \}
 \end{aligned}$$

For fixed μ , (6.54) represents a system of linear, homogeneous, first order ordinary differential equations for the functions ψ_3 , ψ_4 and ψ_5 of the variable λ . Moreover, by (6.48),

the homogeneous one point boundary conditions

$$\psi_3(0, \mu) = \psi_4(0, \mu) = \psi_5(0, \mu) = 0$$

must be satisfied. Hence, the unique solution for the system (6.54) is

$$\psi_3 = \psi_4 = \psi_5 = 0$$

wherever the solution of system A is defined.

Consider the linear algebraic system,

$$(6.55) \quad \begin{cases} \psi_3 = u_\lambda - px_\lambda - qy_\lambda = 0 \\ \psi_3 = u_\mu - px_\mu - qy_\mu = 0. \end{cases}$$

The determinant of this system, by (6.20), does not vanish in a neighborhood of the origin, hence in this neighborhood there exists a unique solution for p and q . Since $p = u_x$ and $q = u_y$ satisfy (6.55) they are the solution of (6.55)

Similarly, from

$$(6.56) \quad \begin{cases} \psi_4 = p_\lambda - rx_\lambda - fy_\lambda \\ \psi_4 = p_\mu - rx_\mu - fy_\mu, \end{cases}$$

we obtain $r = u_{xx}$ and $f = u_{xy}$,

while from

$$(6.57) \quad \begin{cases} \psi_5 = q_\lambda - fx_\lambda - ty_\lambda \\ \psi_5 = q_\mu - fx_\mu - ty_\mu, \end{cases}$$

we obtain the additional information that $t = u_{yy}$. Consequently, any solution of System A under the given characteristic initial conditions satisfies the equation

$$u_{xy} = f(x,y; u; u_x, u_y; u_{xx}, u_{yy})$$

in a neighborhood of the point $(0,0; 0; 0,0; 0,0)$ and the proof of Theorems 9 and 9a is now complete.

Let us designate the problem considered in Theorems 9 and 9a as Problem I. By virtue of the exposition of Chapter IV and this present chapter, we may associate to this problem a particular Problem II, of the type considered in Theorems 3 and 3a of Chapter II. As we have shown, any solution of I is a solution of II, and, conversely, any solution of II is a solution of I. Where for I, $P \in C^{(1)}$, Theorem 3 tells us that the solution of the related Problem II is unique. Hence, as is stated in Theorem 9, the solution for I is likewise unique. If, however, for Problem I, $P \in C^{(1)}$ only, then Theorem 3a tells us merely that the related Problem II has at least one solution. Moreover, Example 1, Chapter II, tells us that this solution cannot be shown to be unique.

We must not conclude merely from the above that for $P \in C^{(1)}$ the solution to Problem I cannot be shown to be unique. We can say, though, that any proof for uniqueness, if such can be made at all, will apparently have to be based upon arguments independent of those of this paper.

Chapter VII

The Mixed Boundary Value Problem

$$\text{for } u_{xy} = f(x, y; u; u_x, u_y).$$

In the terminology of J. HADAMARD [11], appendix II, p. 456, the mixed hyperbolic boundary value problem is one in which we prescribe the values of the integral surface along two lines issuing from a point, one of which is characteristic to the surface in question, while the other is nowhere characteristic.

J. HADAMARD, in the reference above, and E. PICARD [7], p.135, prove the existence of a unique solution to the linear equation

$$(7.1) \quad u_{xy} = a u_x + b u_y + c u,$$

a , b and c continuous functions of x and y alone, satisfying the initial conditions

$$(7.2) \quad u(x, 0) = u(x, x) = 0.$$

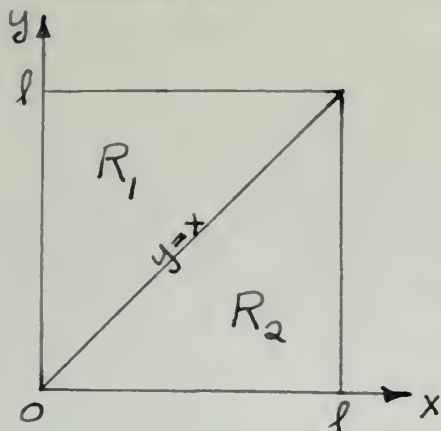
In Theorem 10, below, we extend their conclusions to the equation

$$(7.3) \quad u_{xy} = f(x, y; u; u_x, u_y)$$

maintaining initial conditions (7.2). The result is well known, but does not appear in the literature in the precise form stated. We require this precise statement because we wish to proceed from Theorem 10 by the methods of Chapters II and III in which we relax the Lipschitz condition on the function f to require merely

that f be partially Lipschitzian. Thus we obtain the improved statement of Theorem 10a.

Theorem 10



$$1) f(x,y; u; p,q) \in C(B), B: \begin{cases} 0 \leq x \leq l \\ 0 \leq y \leq l \\ -a \leq u \leq a \\ -b \leq p \leq b \\ -b \leq q \leq b \end{cases}$$

2) f is Lipschitzian on B (as defined in Theorem 1.)

3) $M l^2 \leq a, M l \leq b$, where

$M = \max |f|$ on B

4) There exists one and only one function $u(x,y) \in C^1(R)$, $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq l \\ 0 \leq y \leq l \end{cases}$, such that for each

$(x,y) \in R$, the point $(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in B$, and

$$u_{xy}(x,y) = f(x,y; u(x,y); u_x(x,y), u_y(x,y)),$$

$$u(x,0) = u(x,l) = 0 \quad \text{for each } (x,y) \in R.$$

Proof

This proof is based upon FIGARD's variation of the method of successive approximations, [1] p. 359 or [7] p. 117. Here the uniform convergence of the approximating functions to the solution is verified by means of a majorant series. The majorant series used is that obtained from the approximating functions converging uniformly to the solution for the particular linear equation

$$(7.4) \quad W_{xy} = K (W + W_x + W_y)$$

with the same initial conditions. K is the Lipschitz constant for the function f of (7.3). PICARD applied this technique to the characteristic initial value problem, obtaining Theorem 1 of Chapter II. He thus obtained the theorem for the characteristic initial value problem for the non-linear equation (7.3) from the theorem for the characteristic initial value problem for the linear equation (7.1).

For the mixed boundary value problem under consideration, a curious situation arises. We do not obtain a majorant series from equation (7.4) under mixed initial conditions. However, we do find that PICARD's majorant series for the characteristic initial value problem serves as well for this problem. Thus Theorem 10 follows not from the theorem for the mixed boundary value problem for the linear equation (7.1) but from the theorem for the characteristic initial value problem for equation (7.1).

It is sufficient, as we shall demonstrate later, to show existence of a unique solution in region $R_2: \begin{cases} 0 \leq x \leq l \\ 0 \leq y \leq x \end{cases}$. Assuming $(x, y) \in R_2$, we may express the problem as the integral equation

$$(7.5) \quad u(x, y) = \int_y^x d\xi \int_0^y f(\xi, \eta; u; u_x, u_y) d\eta.$$

By differentiation,

$$(7.6) \quad u_x(x, y) = \int_0^y f(x, \eta; u; u_x, u_y) d\eta,$$

and

$$(7.7) \quad u_y(x,y) = \int_y^x f(\xi, y; u; u_x, u_y) d\xi - \int_0^y f(y, \eta; u; u_x, u_y) d\eta.$$

We form the successive approximations

$$(7.8) \quad \begin{cases} u_1(x,y) = \int_y^x d\xi \int_0^y f(\xi, \eta; 0; 0,0) d\eta \\ u_2(x,y) = \int_y^x d\xi \int_0^y f(\xi, \eta; u_1; u_{1,x}, u_{1,y}) d\eta \\ \vdots \\ u_n(x,y) = \int_y^x d\xi \int_0^y f(\xi, \eta; u_{n-1}; u_{n-1,x}, u_{n-1,y}) d\eta \\ \vdots \end{cases}$$

where, by differentiation,

$$(7.9) \quad u_{n,x}(x,y) = \int_0^y f(x, \eta; u_{n-1}; u_{n-1,x}, u_{n-1,y}) d\eta, \\ (n = 1, 2, \dots),$$

$$(7.10) \quad u_{n,y}(x,y) = \int_y^x f(\xi, y; u_{n-1}; u_{n-1,x}, u_{n-1,y}) d\xi \\ - \int_0^y f(y, \eta; u_{n-1}; u_{n-1,x}, u_{n-1,y}) d\eta, \\ (n = 1, 2, \dots).$$

Since the point $(x,y; 0; 0,0) \in B$ for $(x,y) \in R_2$, by hypothesis 3),

$$\begin{aligned} |u_1(x,y)| &\leq M |x-y| \cdot |y| \leq M l^2 \leq a, \\ |u_{1,x}(x,y)| &\leq M |y| \leq M l \leq b, \\ |u_{1,y}(x,y)| &\leq M \{|x-y| + |y|\} \\ &= M|x| \leq M l \leq b \end{aligned}$$

Thus, by induction, for all n and for any $(x,y) \in R_2$

$$(7.11) \quad \begin{cases} |u_n(x,y)| \leq M l^2 \leq a, \\ |u_{n,x}(x,y)| \leq M l \leq b, \\ |u_{n,y}(x,y)| \leq M l \leq b. \end{cases}$$

Our purpose is to show that on R_2

$$(7.12) \quad \{u_n\} \xrightarrow{\text{unif}} u, \quad \{u_{n,x}\} \xrightarrow{\text{unif}} u_x \quad \text{and} \quad \{u_{n,y}\} \xrightarrow{\text{unif}} u_y$$

such that the function u and its derivatives satisfy conclusion 4) for $(x,y) \in R_2$. To accomplish this we consider the successive approximations

$$(7.13) \quad \begin{aligned} w_1(x,y) &= \int_0^x d\xi \int_0^y M d\eta \\ w_2(x,y) &= \int_0^x d\xi \int_0^y K(w_1 + w_{1,x} + w_{1,y}) d\eta \\ &\vdots \\ w_n(x,y) &= \int_0^x d\xi \int_0^y K(w_{n-1} + w_{n-1,x} + w_{n-1,y}) d\eta \\ &\vdots \end{aligned}$$

where, by differentiation,

$$(7.14) \quad w_{n,x}(x,y) = \int_0^y K [w_{n-1} + w_{n-1,x} + w_{n-1,y}] (x, \eta) d\eta, \quad (n = 1, 2, \dots),$$

$$(7.15) \quad w_{n,y}(x,y) = \int_0^x K [w_{n-1} + w_{n-1,x} + w_{n-1,y}] (\xi, y) d\xi, \quad (n = 1, 2, \dots).$$

Here $M = \max |f|$ on B while K is the Lipschitz constant of hypothesis 2).

Now $w_1(x,y) = Mxy$, hence $w_1(x,y) = w_1(y,x)$. Moreover, $w_{1,x}(x,y) = My$, $w_{1,y}(x,y) = Mx$, hence $w_{1,x}(x,y) = w_{1,y}(y,x)$.

Let us make the inductive hypothesis that for some fixed positive integer n ,

$$(7.16) \quad w_n(x,y) = w_n(y,x), \quad w_{n,x}(x,y) = w_{n,y}(y,x).$$

But this implies that

$$(7.17) \quad [w_n + w_{n,x} + w_{n,y}] (x,y) = [w_n + w_{n,x} + w_{n,y}] (y,x)$$

and thus, by (7.13),

$$w_{n+1}(x,y) = w_{n+1}(y,x).$$

Also, by (7.14) and (7.15), (7.17) implies that

$$\begin{aligned} w_{n+1,x}(x,y) &= \int_0^y K [w_n + w_{n,x} + w_{n,y}] (x,\eta) d\eta \\ &= \int_0^y K [w_n + w_{n,x} + w_{n,y}] (\xi,x) d\xi \\ &= w_{n+1,y}(y,x). \end{aligned}$$

Hence, by induction, (7.16) holds for $n = 1, 2, \dots$.

PICARD, in the references quoted above, shows that

$$(7.18) \quad \sum_{n=1}^{\infty} w_n = w, \quad \sum_{n=1}^{\infty} w_{n,x} = w_x, \quad \sum_{n=1}^{\infty} w_{n,y} = w_y,$$

each uniformly convergent on R , where the function w and its derivatives satisfy

$$(7.19) \quad \begin{aligned} w_{xy} &= K(w + w_x + w_y), \\ w(x,0) &= w(0,y) = 0. \end{aligned}$$

We now show that these series are majorant to the series

$$(7.20) \quad \sum_{n=1}^{\infty} (u_n - u_{n-1}), \quad \sum_{n=1}^{\infty} (u_{n,x} - u_{n-1,x}), \quad \sum_{n=1}^{\infty} (u_{n,y} - u_{n-1,y}),$$

respectively, for each $(x,y) \in R_2$, (with $u_0 = 0$).

Now, for $(x,y) \in R_2$,

$$|u_1(x,y)| \leq \int_y^x d\xi \int_0^y |f(\xi, \eta; 0; 0,0)| d\eta \leq \int_0^x d\xi \int_0^y M d\eta = w_1(x,y)$$

$$|u_{1,x}(x,y)| \leq \int_0^y |f(x, \eta; 0; 0,0)| d\eta \leq \int_0^y M d\eta = w_{1,x}(x,y)$$

$$\begin{aligned}
 |u_{1,y}(x,y)| &\leq \int_y^x |f(\xi,y; 0; 0,0)| d\xi + \int_0^y |f(y,\eta; 0; 0,0)| d\eta \\
 &\leq \int_y^x M d\xi + \int_0^y M d\eta \\
 &= \int_0^x M d\xi = w_{1,y}(x,y).
 \end{aligned}$$

Also, abbreviating our notation somewhat,

$$\begin{aligned}
 |u_2 - u_1| &\leq \int_y^x d\xi \int_0^y |f(\xi,\eta; u_1; u_{1,x}, u_{1,y}) \\
 &\quad - f(\xi,\eta; 0; 0,0)| d\eta \\
 &\leq \int_y^x d\xi \int_0^y K [|u_1| + |u_{1,x}| + |u_{1,y}|] (\xi,\eta) d\eta \\
 &\leq \int_0^x d\xi \int_0^y K [w_1 + w_{1,x} + w_{1,y}] (\xi,\eta) d\eta
 \end{aligned}$$

$$= w_2,$$

$$|u_{2,x} - u_{1,x}| \leq \int_0^y K [w_1 + w_{1,x} + w_{1,y}] (x,\eta) d\eta = w_{2,x}$$

$$|u_{2,y} - u_{1,y}| \leq \int_y^x K [w_1 + w_{1,x} + w_{1,y}] (\xi,y) d\xi$$

$$+ \int_0^y K [w_1 + w_{1,x} + w_{1,y}] (y,\eta) d\eta$$

$$= \int_y^x K [w_1 + w_{1,x} + w_{1,y}] (\xi,y) d\xi$$

$$+ \int_0^y K [w_1 + w_{1,x} + w_{1,y}] (\xi,y) d\xi$$

$$= \int_0^x K [w_1 + w_{1,x} + w_{1,y}] (\xi,y) d\xi$$

$$= w_{2,y}.$$

Hence, by induction, we obtain for $n = 1, 2, \dots$

$$|u_n - u_{n-1}| \leq w_n, \quad |u_{n,x} - u_{n-1,x}| \leq w_{n,x},$$

$$(7.21) \quad |u_{n,y} - u_{n-1,y}| \leq w_{n,y} \quad \text{for each } (x,y) \in R_2.$$

Thus the series of (7.18) are majorant to the corresponding series of (7.20). Moreover, the requirements for termwise differentiation of an infinite sum are satisfied since each of the series of (7.20) is now known to be uniformly convergent on R_2 . Hence, for $(x, y) \in R_2$,

$$(7.22) \left\{ \begin{array}{l} \sum_{n=1}^{\infty} (u_n - u_{n-1}) = u \\ \sum_{n=1}^{\infty} (u_{n,x} - u_{n-1,x}) = u_x \\ \sum_{n=1}^{\infty} (u_{n,y} - u_{n-1,y}) = u_y ; \end{array} \right.$$

or, in other terms, since each of these series telescopes,

$$(7.22)' \left\{ u_n \right\} \xrightarrow{\text{unif}} u, \quad \left\{ u_{n,x} \right\} \xrightarrow{\text{unif}} u_x, \quad \left\{ u_{n,y} \right\} \xrightarrow{\text{unif}} u_y$$

on R_2 .

We now verify that the function u and its derivatives u_x and u_y satisfy the integral equation statement of the problem (7.5):

$$(7.23) \quad \begin{aligned} & \left| u(x, y) - \int_y^x d\xi \int_0^y f(\xi, \eta; u; u_x, u_y) d\eta \right| \\ & \leq |u(x, y) - u_n(x, y)| + \int_y^x d\xi \int_0^y |f(\xi, \eta; u; u_x, u_y) \\ & \quad - f(\xi, \eta; u_{n-1}; u_{n-1,x}, u_{n-1,y})| d\eta \\ & \leq |u(x, y) - u_n(x, y)| \\ & \quad + \int_y^x d\xi \int_0^y K \left[|u - u_{n-1}| + |u_x - u_{n-1,x}| + |u_y - \right. \\ & \quad \left. u_{n-1,y} \right] (\xi, \eta) d\eta \end{aligned}$$

Thus, by (7.22)', given $\epsilon > 0$, there exists a positive integer N , depending on ϵ alone, such that $n > N \Rightarrow$

$$|u(x,y) - \int_y^x d\xi \int_0^y f(\xi, \eta; u; u_x, u_y) d\eta| < \epsilon (1+3K\sqrt{x^2}),$$

for $(x,y) \in R_2$. But ϵ is arbitrary, hence the integral equation is satisfied.

By (7.11) and (7.22)' we see that for any $(x,y) \in R_2$, the point $(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in B$. Thus existence of a solution on R_2 is now proved.

To prove uniqueness, let us suppose that u_1 and u_2 are two solutions on R_2 , then

$$(7.24) \quad |u_1(x,y) - u_2(x,y)| \leq \int_y^x d\xi \int_0^y |f(\xi, \eta; u_1; u_{1,x}, u_{1,y}) - f(\xi, \eta; u_2; u_{2,x}, u_{2,y})| d\eta \\ \leq \int_y^x d\xi \int_0^y K[|u_1 - u_2| + |u_{1,x} - u_{2,x}| + |u_{1,y} - u_{2,y}|] (\xi, \eta) d\eta.$$

$$(7.25) \quad |u_{1,x}(x,y) - u_{2,x}(x,y)| \leq \int_0^y |f(x, \eta; u_1; u_{1,x}, u_{1,y}) - f(x, \eta; u_2; u_{2,x}, u_{2,y})| d\eta \\ \leq \int_0^y K[|u_1 - u_2| + |u_{1,x} - u_{2,x}| + |u_{1,y} - u_{2,y}|](x, \eta) d\eta.$$

$$(7.26) \quad |u_{1,y}(x,y) - u_{2,y}(x,y)| \leq \int_y^x |f(\xi, y; u_1; u_{1,x}, u_{1,y}) - f(\xi, y; u_2; u_{2,x}, u_{2,y})| d\xi \\ + \int_0^y |f(y, \eta; u_1; u_{1,x}, u_{1,y}) - f(y, \eta; u_2; u_{2,x}, u_{2,y})| d\eta.$$

Let $\Psi(x, y) = [|u_1 - u_2| + |u_{1,x} - u_{2,x}| + |u_{1,y} - u_{2,y}|](x, y)$.

With $R^* = \begin{cases} 0 \leq x \leq l^* \\ 0 \leq y \leq x \end{cases}$, $l^* = \min(1, l, \frac{1}{6K})$, we have

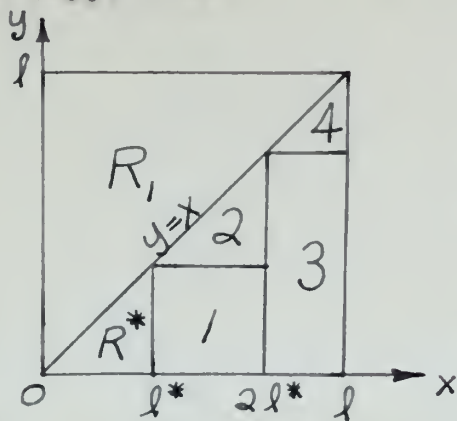
$\Psi(x, y) \in C(R^*)$. Moreover, there exists a point $(x^*, y^*) \in R^*$ such that $\Psi(x^*, y^*) = \mu$ where $\mu = \max \Psi(x, y)$ on R^* . But, adding (7.24), (7.25) and (7.26) we obtain

$$\begin{aligned} \Psi(x, y) &\leq K\mu \{ (x-y)y + y + (x-y) + y \} \\ &\leq K\mu (xy + x + y) \\ &\leq K\mu \cdot \frac{3}{6K} = \frac{\mu}{2}, \end{aligned}$$

hence $\Psi(x^*, y^*) = \mu \leq \frac{\mu}{2}$, which implies $\mu = 0$ and thus

$$(7.27) \quad u_1(x, y) = u_2(x, y)$$

for $(x, y) \in R^*$



To extend this uniqueness proof to the domain R_2 , we subdivide R_2 as shown in the diagram. We know that the solution u is unique on R^* and hence determines $u(l^*, y)$ for $0 \leq y \leq l^*$.

But $u(x, 0) = 0$ by hypothesis, consequently, by Theorem 1, Chapter II, we have a unique solution u_1 to the characteristic initial value problem on sub-region 1. Since $u_x(l^*, 0) = u_{1,x}(l^*, 0)$, we have from the differential equation that $u_x(l^*, y) = u_{1,x}(l^*, y)$ for $0 \leq y \leq l^*$, i.e. u and u_1 have a first order contact across the line $x = l^*$ and hence together represent a unique solution for the region $R^* + 1$. Analogously, by the preceding "in the

small" uniqueness proof for the mixed boundary value problem, the solution u_2 is unique in sub-region 2 and has a first order contact with u_1 across the line $y = x$. We continue obtaining unique solutions for characteristic initial value and mixed initial value problems, alternatively as indicated by the numerical sequence in the diagram. These solutions have first order contacts with each other across the characteristics forming the boundaries of the sub-regions, hence we have extended our uniqueness proof from the region R_1 to the region R_2 .

Having thus determined the existence of a unique solution satisfying conclusion 4) throughout R_2 , we now consider the Cauchy problem for region R_1 with the same equation and hypotheses thereon and with the initial conditions

$$(7.28) \begin{cases} u^0(x,x) = 0, u_x^0(x,x) = u_{x+}(x,x), \text{ and} \\ u_y^0(x,x) = u_{y-}(x,x) \quad \text{for } x \in [0,1]. \end{cases}$$

In (7.28) u_{x+} and u_{y-} are the right-hand x and lower y derivatives, respectively, determined at each point of the line $y = x$ by the known solution u on R_2 . By Theorem 4, Chapter III, there exists a unique solution u^0 to this Cauchy problem for each $(x,y) \in R_1$, hence

$$u_1(x,y) = \begin{cases} u_0(x,y) & \text{for } (x,y) \in R_1 \\ u(x,y) & \text{for } (x,y) \in R_2 \end{cases}$$

is the unique solution valid for each $(x,y) \in R = R_1 + R_2$, since u_0 and u have, by prescription, a first order contact across the line $y = x$. This completes the proof of Theorem 10.

Relaxing only hypothesis 2) of Theorem 10, we obtain the following improvement:

Theorem 10a

- 1)
- 2)' f is partially Lipschitzian on B (as defined in Theorem 1a.)
- 3)
- \Rightarrow 4)' There exists at least one function, etc. (as in Theorem 10.)

Outline of the proof:

As in the proof of Theorem 10, we may, without loss, prove existence on R_2 only. For, prescribing Cauchy conditions on $y = x$ as before, we may extend the solution from R_2 to R_1 , by use of Theorem 4a, Chapter III.

In this proof we follow very closely the derivation of Theorem 1a, Chapter II; hence only the differences between the two proofs will be noted.

WEIERSTRASS' theorem tells us that there exists a sequence of polynomials, $\{g_\lambda\}$, converging uniformly to f on B . We extend

the g_λ , ($\lambda = 1, 2, \dots$), and f from B to

$$B': \begin{cases} 0 \leq x \leq \infty \\ 0 \leq y \leq \infty \\ -\infty < u < \infty \\ -\infty < p < \infty \\ -\infty < q < \infty \end{cases}$$

by definitions analogous to (2.1). There

exists a constant $L > 0$ such that $|g_\lambda| \leq L$ in B' and for all λ . More-

over, the g_λ are "fully" Lipschitzian in B' . Hence by Theorem 10, (with $a \rightarrow \infty$, $b \rightarrow \infty$), for each g_λ there exists a unique function u_λ such that for $(x, y) \in R_2$

$$(7.29) \quad u_\lambda = \int_y^x d\xi \int_0^y g_\lambda(\xi, \eta; u_\lambda; u_{\lambda, x}, u_{\lambda, y}) d\eta,$$

and thus

$$(7.30) \quad u_{\lambda, x} = \int_0^y g_\lambda(x, \eta; u_\lambda; u_{\lambda, x}, u_{\lambda, y}) d\eta,$$

$$(7.31) \quad u_{\lambda, y} = \int_y^x g_\lambda(\xi, y; u_\lambda; u_{\lambda, x}, u_{\lambda, y}) d\xi \\ - \int_0^y g_\lambda(y, \eta; u_\lambda; u_{\lambda, x}, u_{\lambda, y}) d\eta.$$

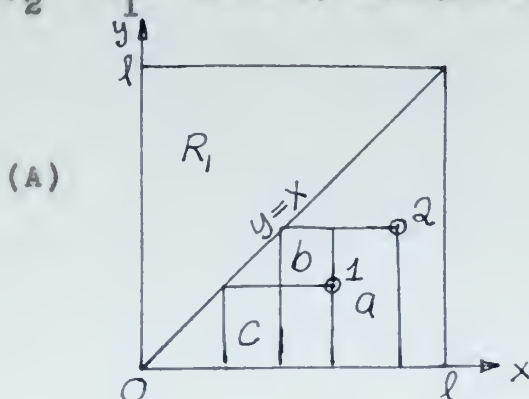
For $(x, y) \in R_2$, by (7.29), (7.30) and (7.31),

$$(7.32) \quad \left. \begin{aligned} |u_\lambda(x, y)| &\leq L \ell^2 \\ |u_{\lambda, x}(x, y)| &\leq L \ell \\ |u_{\lambda, y}(x, y)| &\leq L \{ (x-y) + y \} \\ &\leq L \ell \end{aligned} \right\} (\lambda = 1, 2, \dots)$$

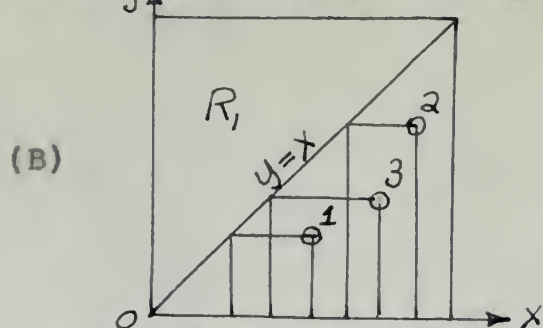
i.e. the sequences $\{u_\lambda\}$, $\{u_{\lambda, x}\}$ and $\{u_{\lambda, y}\}$ are uniformly bounded on R_2 .

Given two points, $(x_1, y_1) \in R_2$, $(x_2, y_2) \in R_2$, we may assume, without loss, that $x_1 \leq x_2$. Then, if $y_1 \leq y_2$, let us assume that $y_2 < x_1$. Then by integrating over the regions a, b and c in

diagram (A) we obtain



$$(7.33) \quad |u_{\lambda}(x_2, y_2) - u_{\lambda}(x_1, y_1)| \leq L \{ \ell(x_2 - x_1) + 2\ell(y_2 - y_1) \}.$$

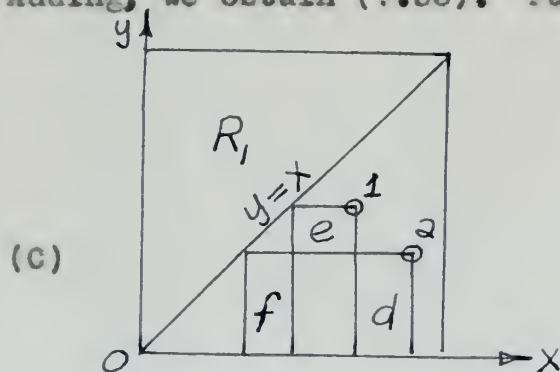


If $y_2 \geq x_1$ we may always choose a point (x_3, y_3) with $y_2 < x_3 < x_2$ and $y_1 < y_3 < x_1$ (as in diagram (B)). Then, as above,

$$|u_{\lambda}(x_2, y_2) - u_{\lambda}(x_3, y_3)| \leq L \{ \ell(x_2 - x_3) + 2\ell(y_2 - y_3) \}$$

$$|u_{\lambda}(x_3, y_3) - u_{\lambda}(x_1, y_1)| \leq L \{ \ell(x_3 - x_1) + 2\ell(y_3 - y_1) \}.$$

Adding, we obtain (7.33). Further if $y_1 \geq y_2$, we have the case



shown in diagram (C). Here by integrating over the regions d, e and f we again obtain (7.33). Hence the sequence

$\{u_{\lambda}\}$ is equicontinuous on R_2 .

Now, for $(x, y_2) \in R_2$, $(x, y_1) \in R_2$, by (7.30)

$$(7.34) \quad |u_{\lambda, x}(x, y_2) - u_{\lambda, x}(x, y_1)| \leq L|y_2 - y_1|.$$

Likewise, for $(x_2, y) \in R_2$, $(x_1, y) \in R_2$, by (7.31)

$$(7.35) \quad |u_{\lambda, y}(x_2, y) - u_{\lambda, y}(x_1, y)| \leq L|x_2 - x_1|.$$

Moreover, by precisely the same argument as that used to prove Lemma 2 of Chapter II, given $\mu > 0$, $\zeta > 0$, there exist $\delta > 0$, $N > 0$, depending only on μ and ζ , respectively, such that for $(x_2, y) \in R_2$, $(x_1, y) \in R_2$,

$$\lambda > N \text{ and } |x_2 - x_1| < \delta$$

$$\begin{aligned} \Rightarrow \\ (7.36) \quad & |u_{\lambda,x}(x_2,y) - u_{\lambda,x}(x_1,y)| \\ & \leq K \int_0^y |u_{\lambda,x}(x_2,\eta) - u_{\lambda,x}(x_1,\eta)| d\eta + \mu + \zeta. \end{aligned}$$

Thus by (7.34), (7.36) and Lemma 1, Chapter II, the sequence

$\{u_{\lambda,x}\}$ is equicontinuous on R_2 .

We need the following refinement of the argument in order to show that the sequence $\{u_{\lambda,y}\}$ is equicontinuous on R_2 :

Let us suppose $(x,y_2) \in R_2$, $(x,y_1) \in R_2$. Without loss, we may assume that $x \geq y_2 \geq y_1$. Then

$$\begin{aligned} & u_{\lambda,y}(x,y_2) - u_{\lambda,y}(x,y_1) \\ (7.37) \quad & = \int_{y_2}^x [g_\lambda(\xi,y_2; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) - g_\lambda(\xi,y_1; u_\lambda; u_{\lambda,x}, u_{\lambda,y})] d\xi \\ & \quad - \int_{y_1}^{y_2} g_\lambda(\xi,y_1; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\xi \\ & \quad - \int_0^{y_1} [g_\lambda(y_2,\eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) - g_\lambda(y_1,\eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y})] d\eta \\ & \quad - \int_{y_1}^{y_2} g_\lambda(y_2,\eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\eta \end{aligned}$$

We have just proved that the sequences $\{u_\lambda\}$ and $\{u_{\lambda,x}\}$ are equicontinuous on R_2 . The sequence $\{g_\lambda\}$ is certainly equicontinuous on B' . Hence, considering (7.35), given $\mu > 0$, there exists $\delta > 0$, depending upon μ alone, such that $|y_2 - y_1| < \delta$

$$\Rightarrow (7.38) \quad \left| \int_0^{y_1} [g_\lambda(y_2,\eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) - g_\lambda(y_1,\eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y})] d\eta \right| < \mu,$$

$$(7.39) \quad \left| \int_{y_2}^x [g_\lambda(\xi,y_2; u_\lambda(\xi,y_2); u_{\lambda,x}(\xi,y_2), \underline{u_{\lambda,y}(\xi,y_2)}) - g_\lambda(\xi,y_1; u_\lambda(\xi,y_1); u_{\lambda,x}(\xi,y_1), \underline{u_{\lambda,y}(\xi,y_2)})] d\xi \right| < \mu,$$

for $\lambda = 1, 2, \dots$.

Also, since $\{\varepsilon_\lambda\} \xrightarrow{\text{unif}} f$ on B' , given $\zeta > 0$, there exists $N > 0$, depending upon ζ alone, such that $\lambda > N$

\Rightarrow

$$(7.40) \left| \int_{y_2}^x [\varepsilon_\lambda - f](\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda, x}(\xi, y_1), \underline{u_{\lambda, y}(\xi, y_2)}) d\xi \right| < \zeta,$$

$$\left| \int_{y_2}^x [f - \varepsilon_\lambda](\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda, x}(\xi, y_1), \underline{u_{\lambda, y}(\xi, y_1)}) d\xi \right| < \zeta.$$

By hypothesis 2)',

$$(7.41) \left| \int_{y_2}^x [f(\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda, x}(\xi, y_1), \underline{u_{\lambda, y}(\xi, y_2)}) - f(\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda, x}(\xi, y_1), \underline{u_{\lambda, y}(\xi, y_1)})] d\xi \right|$$

$$\leq \int_{y_2}^x K |u_{\lambda, y}(\xi, y_2) - u_{\lambda, y}(\xi, y_1)| d\xi.$$

Moreover, since $|\varepsilon_\lambda| \leq L$, ($\lambda = 1, 2, \dots$),

$$(7.42) \left| \int_{y_1}^{y_2} \varepsilon_\lambda(\xi, y_1; u_\lambda; u_{\lambda, x}, u_{\lambda, y}) d\xi \right| \leq L |y_2 - y_1|$$

$$\left| \int_{y_1}^{y_2} \varepsilon_\lambda(y_2, \eta; u_\lambda; u_{\lambda, x}, u_{\lambda, y}) d\eta \right| \leq L |y_2 - y_1|.$$

Thus by equations (7.37) through (7.41), given $\mu > 0$, $\zeta > 0$, there exists $\delta > 0$, $N > 0$, depending only upon μ and ζ , respectively, such that $|y_2 - y_1| < \delta$ and $\lambda > N$



$$\begin{aligned}
 \Rightarrow \\
 (7.43) \quad & |u_{\lambda,y}(x,y_2) - u_{\lambda,y}(x,y_1)| \\
 & \leq K \int_{y_2}^x |u_{\lambda,y}(\xi,y_2) - u_{\lambda,y}(\xi,y_1)| d\xi \\
 & \quad + 4\mu + 2\zeta.
 \end{aligned}$$

By Lemma 1, Chapter II, inequalities (7.35) and (7.43) imply that the sequence $\{u_{\lambda,y}\}$ is equicontinuous on R_2 .

From this point on the proof is practically identical with that for Theorem 1a. Since the sequences $\{u_\lambda\}$, $\{u_{\lambda,x}\}$ and $\{u_{\lambda,y}\}$ are uniformly bounded and equicontinuous on R_2 , we may apply ARZELA's theorem to obtain a subsequence of each, uniformly convergent on R_2 . Hence, as for Theorem 1a, by successive extractions of subsequences we obtain a subsequence $\{u_\lambda^*\}$ of $\{u_\lambda\}$ converging uniformly on R_2 to a solution u of the integral equation

$$u(x,y) = \int_y^x d\xi \int_0^y f(\xi,\eta; u; u_x, u_y) d\eta,$$

and such that for $(x,y) \in R_2$

$(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in B$. The proof for Theorem 10a is now complete.

Following E. PICARD [7] p. 135 and p. 139, we show that the general statement of the mixed boundary conditions, (i.e. where u is prescribed along two intersecting curves, one characteristic and the other nowhere characteristic), can be reduced to the statement found in Theorems 10 and 10a, (i.e. where $u(x,0) = u(x,x) = 0$ for $x \in [0,1]$).

First, let us suppose that we prescribe



$$(7.44) \quad \begin{cases} u(x,0) = \varphi(x) \\ u(x,x) = \psi(x) \end{cases}$$

for $x \in [0, l]$, $\varphi(x)$ and $\psi(x) \in C^1[0, l]$ and $\varphi(0) = \psi(0)$.

Consider

$$(7.45) \quad w(x,y) = \varphi(x) + \psi(y) - \varphi(y).$$

We have $w_{xy} = 0$ on R while

$$(7.46) \quad \begin{cases} w(x,0) = \varphi(x) \\ w(x,x) = \psi(x) \end{cases}$$

for $x \in [0, l]$. Hence, instead of the problem with non-homogeneous boundary conditions (7.44), by setting

$$(7.47) \quad v = u - w$$

we may consider the problem

$$(7.48) \quad \begin{cases} v_{xy} = f(x,y; v+w; v_x + w_x, v_y + w_y) \\ v(x,0) = 0 \\ v(x,x) = 0, \end{cases}$$

a problem of the type covered by Theorems 10 and 10a.

Second, suppose we prescribe u along the characteristic $y = 0$ and the nowhere characteristic curve $y = F(x)$, where $F(x) \in C^1([0, l_1])$, $F'(x) \neq 0$ for $x \in [0, l_1]$ and $F(0) = 0$.

The coordinate transformation

$$(7.49) \quad \begin{cases} \bar{x} = F(x) \\ \bar{y} = y \end{cases}$$

reduces the curve $y = F(x)$ to the diagonal $\bar{y} = \bar{x}$ since the inverse F^{-1} exists and is of class C^1 on $[0, F(l_1)]$. Moreover,

$$(7.50) \quad u_{xy} = F'(x) u_{\bar{x}\bar{y}}.$$

Since $F'(x) \neq 0$, the form of the differential equation remains unchanged and we reduce the problem to one with initial conditions in the form (7.44).

Thus the general statement of the mixed boundary value problem for

$$(7.3) \quad u_{xy} = f(x, y; u; u_x, u_y)$$

can be reduced to the form treated in Theorems 10 and 10a. We note that whatever continuity and Lipschitz conditions are satisfied by (7.3) before transformation (7.49) and substitution (7.47) are satisfied as well after these operations are performed.

CHAPTER VIII

EXISTENCE THEOREMS BASED ON THE
CONCEPT OF UPPER AND LOWER BOUNDING FUNCTIONS

For the ordinary differential equation $y' = f(x, y)$ with $y(x_0) = y_0$, O. PERRON [18], assuming f merely continuous, gives an existence proof that is entirely independent of the classical proofs and contains them as special cases. He bases his proof on the concept of under and over functions, defining $\varphi(x)$ to be an under function if $\varphi(x_0) = y_0$ and

$$(8.1) \quad D_{\pm} \varphi(x) < f(x, \varphi(x))$$

and defining $\psi(x)$ to be an over function if $\psi(x_0) = y_0$ and

$$(8.2) \quad D_{\pm} \psi(x) > f(x, \psi(x)).$$

The solutions are found to lie between the upper limit function g of the set of underfunctions and the lower limit function G of the set of overfunctions, g and G themselves being solutions.

M. MÜLLER [4] shows that PERRON's proof will not carry over directly to apply to a system.

$$(8.3) \quad y_i' = f_i(x, y_1, \dots, y_n) \quad , \quad (i = 1, \dots, n).$$

However, he is able to extend the classical theorem, obtaining a statement which is similar to that of PERRON and which reduces to the direct analogue of PERRON's theorem in the particular case where the functions f_i are monotonically increasing in the arguments y_1, \dots, y_n .

In this chapter we return to the characteristic initial value problem for

$$(8.4) \quad u_{xy} = f(x, y; u; u_x, u_y).$$

We obtain results similar to those of MÜLLER above. In the following Theorems 11 and 11a we improve the statements of Theorems 1 and 1a, Chapter II, by the introduction of upper and lower bounding functions Ω and ω .

Theorem 11 (11a)

$$1) \quad f(x, y; u; p, q) \in C(T), \quad T: \begin{cases} 0 \leq x \leq l \\ 0 \leq y \leq l \\ \omega(x, y) \leq u \leq \Omega(x, y) \\ \omega_x(x, y) \leq p \leq \Omega_x(x, y) \\ \omega_y(x, y) \leq q \leq \Omega_y(x, y) \end{cases}$$

2) (2)' f is Lipschitzian (partially Lipschitzian) on T (as defined in Theorems 1 and 1a).

3) The functions $\omega(x, y)$ and $\Omega(x, y) \in C^1(R)$, $R: \begin{cases} 0 \leq x \leq l \\ 0 \leq y \leq l \end{cases}$
with $\omega_{xy}(x, y)$ and $\Omega_{xy}(x, y) \in C(R)$. Moreover,

$$\omega(x, 0) = \Omega(x, 0) = 0 \quad \text{for } x \in [0, l],$$

$$\omega(0, y) = \Omega(0, y) = 0 \quad \text{for } y \in [0, l],$$

and, for each $(x, y) \in R$,

$$(8.5) \quad \omega_{xy}(x, y) \leq \min_{S(x, y)} [f(x, y; u; p, q)],$$

$$(8.6) \quad \Omega_{xy}(x, y) \geq \max_{S(x, y)} [f(x, y; u; p, q)]$$

where

$$(8.7) \quad \mathfrak{B}(x,y): \begin{cases} x = x \\ y = y \\ \omega(x,y) \leq u \leq \Omega(x,y) \\ \omega_x(x,y) \leq p \leq \Omega_x(x,y) \\ \omega_y(x,y) \leq q \leq \Omega_y(x,y) \end{cases}$$

\Rightarrow 4) (4)' There exists one and only one (at least one) function $u(x,y) \in C^1(R)$, $u_{xy} \in C(R)$ such that for each $(x,y) \in R$ the point $(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in T$, and

$$u_{xy}(x,y) = f(x,y; u(x,y); u_x(x,y), u_y(x,y)),$$

$$u(x,0) = u(0,y) = 0 \quad \text{for each } (x,y) \in R.$$

Proof

We extend the domain of definition of the function f over T to B' :

$$\left\{ \begin{array}{l} 0 \leq x \leq l \\ 0 \leq y \leq l \\ -\infty < u < \infty \\ -\infty < p < \infty \\ -\infty < q < \infty \end{array} \right. \quad \text{by defining } f(x,y; u; p, q)$$

$= f(x,y; \bar{u}; \bar{p}, \bar{q})$, where

$$\begin{aligned} \bar{u} &= u \text{ if } \omega(x,y) \leq u \leq \Omega(x,y), \quad \bar{p}=p \text{ if } \omega_x(x,y) \leq p \leq \Omega_x(x,y), \\ (8.8) \quad \bar{u} &= \omega(x,y) \text{ if } u < \omega(x,y) \quad \bar{p} = \omega_x(x,y) \text{ if } p < \omega_x(x,y) \\ \bar{u} &= \Omega(x,y) \text{ if } \Omega(x,y) < u \quad \bar{p} = \Omega_x(x,y) \text{ if } \Omega_x(x,y) < p \\ \text{and} \quad \bar{q} &= q \text{ if } \omega_y(x,y) \leq q \leq \Omega_y(x,y) \\ \bar{q} &= \omega_y(x,y) \text{ if } q < \omega_y(x,y) \\ \bar{q} &= \Omega_y(x,y) \text{ if } \Omega_y(x,y) < q. \end{aligned}$$

By definition (8.8), f is uniformly continuous and uniformly bounded in B' . Moreover, by hypothesis 2)(2)' and (8.8) f satisfies a Lipschitz (partial Lipschitz) condition in B' .

Hence, by Theorem 1 (1a) Chapter II, there exists one and only one (at least one) function satisfying conclusion 4)(4)' except that for $(x,y) \in R$ we are assured only that the point $(x,y;u(x,y);u_x(x,y),u_y(x,y)) \in B'$. To complete the proof we must show that this point actually lies in T ; i.e. we must show that for each $(x,y) \in R$,

$$(8.9) \quad \begin{cases} \omega(x,y) \leq u(x,y) \leq \Omega(x,y) \\ \omega_x(x,y) \leq u_x(x,y) \leq \Omega_x(x,y) \\ \omega_y(x,y) \leq u_y(x,y) \leq \Omega_y(x,y) . \end{cases}$$

To accomplish this, we first prove the following lemma:

Lemma 3 i) $\omega_{xy}(x,y) \leq u_{xy}(x,y)$ for all $(x,y) \in R$
 \Rightarrow $\omega(x,y) \leq u(x,y)$ "
 $\omega_x(x,y) \leq u_x(x,y)$ "
 $\omega_y(x,y) \leq u_y(x,y)$ " ,

ii) $\Omega_{xy}(x,y) \geq u_{xy}(x,y)$ for all $(x,y) \in R$
 \Rightarrow $\Omega(x,y) \geq u(x,y)$ "
 $\Omega_x(x,y) \geq u_x(x,y)$ "
 $\Omega_y(x,y) \geq u_y(x,y)$ " .

Proof: For i),

$$\begin{aligned} \omega(x,y) &= \int_0^x dx \int_0^y \omega_{xy} dy \leq \int_0^x dx \int_0^y u_{xy} dy = u(x,y) \\ \omega_x(x,y) &= \int_0^y \omega_{xy} dy \leq \int_0^y u_{xy} dy = u_x(x,y) \\ \omega_y(x,y) &= \int_0^x \omega_{xy} dx \leq \int_0^x u_{xy} dx = u_y(x,y) . \end{aligned}$$

The proof for ii) is analogous.

To prove (2.9) it only remains to verify that hypothesis 1) and ii) of Lemma 3 are satisfied by u . By hypothesis 3) and definition (2.3), for each $(x,y) \in M$,

$$\begin{aligned}\omega_{xy}(x,y) &\leq \min_{S(x,y)} [f(x,y; u; p,q)] \\ &\leq f(x,y; u(x,y); u_x(x,y), u_y(x,y)) \\ &= u_{xy}(x,y)\end{aligned}$$

and

$$\begin{aligned}\Omega_{xy}(x,y) &\geq \max_{S(x,y)} [f(x,y; u; p,q)] \\ &\geq f(x,y; u(x,y); u_x(x,y), u_y(x,y)) \\ &= u_{xy}(x,y).\end{aligned}$$

Thus, by Lemma 3, requirement (2.9) is satisfied for each $(x,y) \in M$ and the proof of Theorems 11 and 11a is complete.

It is evident upon inspection of Theorems 11 and 11a that if, instead of homogeneous initial conditions, we prescribe

$$u(x,0) = U(x) \quad \text{with } U(x) \in C^1([0, \ell]),$$

$$u(0,y) = V(y) \quad \text{with } V(y) \in C^1([0, \ell]),$$

where $U(0) = V(0)$, then we must require

$$\omega(x,0) = \Omega(x,0) = U(x),$$

$$\omega(0,y) = \Omega(0,y) = V(y).$$

The proof then goes through as before.

The following example is an illustration of Theorem 11:

Example 4

For the problem

$$(8.10) \quad u_{xy} = (2^{1/m} - u_x)^{1/m+1}, \quad u(x,0) = u(0,y) = 0,$$

we may readily verify that

$$(8.11) \quad \omega(x,y) = \left(\frac{1}{m+1}\right)^{1/m+1} \cdot 2^{1/m(m+1)} xy$$

and

$$(8.12) \quad \Omega(x,y) = 2^{1/m(m+1)} xy$$

satisfy the hypotheses of Theorem 11 for all $x \geq 0$ and

$$0 \leq y \leq C_m^* = \frac{m}{m+1} 2^{1/m+1}$$

In Chapter II we obtained the exact solution

$$(2.42) \quad u(x,y) = x \left\{ 2^{1/m} - \left[\frac{m}{m+1} (C_m - y) \right]^{m+1/m} \right\}$$

where

$$(2.43) \quad C_m = \frac{m+1}{m} 2^{1/m+1}$$

is a branch point of the solution. We observe that as m increases indefinitely ω and Ω approach u from below and above, respectively, while C_m^* approaches C_m from below.

We see from this example that it is possible to obtain approximate solutions, with known limits of error, and to locate singularities in the actual solution by use of Theorem 11, provided that suitable functions ω and Ω can be obtained. For problems where explicit solutions cannot be obtained in "closed form", the procedure is to alter the right-hand side of the equation

$$u_{xy} = f(x,y; u; u_x, u_y)$$

so that an explicit solution of the altered equation can be ob-

tained satisfying the boundary conditions. This may lead to functions ω and Ω satisfying the hypotheses of Theorem 11. (See W. H. WHYBURN [19] and [20].) The motivation for equations (8.11) and (8.12) of Example 4 is now evident.

When we consider the possibility of applying, as explained below, the PERRON method using under and over functions to the characteristic initial value problem under consideration, we find the situation much the same as that in the case of a system of first order ordinary differential equations. We arrive at the unsatisfactory state of affairs wherein there is no assurance that the under functions remain below the over functions throughout the entire region on which a solution is known to exist. In fact, we shall presently give an example where an under function exceeds an over function within the domain of existence of a solution.

Recalling inequalities (8.1) and (8.2), we may express the application of the PERRON method as follows: We require both the under and over functions to satisfy the given characteristic initial conditions and to be continuously differentiable and to possess a mixed second derivative at each point of the domain $R: \begin{cases} 0 \leq x \leq \lambda \\ 0 \leq y \leq \lambda \end{cases}$. We further stipulate that each under function, φ , shall satisfy

$$(8.13) \quad \varphi_{xy}(x,y) < f(x,y; \varphi(x,y); \varphi_x(x,y), \varphi_y(x,y)),$$

and that each over function, ψ , shall satisfy

$$(8.14) \quad \psi_{xy}(x,y) > f(x,y; \psi(x,y); \psi_x(x,y), \psi_y(x,y))$$

for each $(x,y) \in R$.

Analogous arguments to those used by FERROK for the ordinary differential equation $y' = f(x,y)$ lead to the inequalities

$$\begin{aligned} \varphi_x(0,y) < \psi_x(0,y) & \quad \text{for} \quad 0 < y \leq l, \\ \varphi_y(x,0) < \psi_y(x,0) & \quad \text{for} \quad 0 < x \leq l, \end{aligned}$$

for any under function φ and any over function ψ . These inequalities, together with the requirement that φ and ψ satisfy the characteristic initial data on the positive x and y axes, insure that $\psi > \varphi$ in a sufficiently small "L" shaped strip in the first quadrant adjacent to the initial characteristics. Unfortunately, this is inadequate as the following example demonstrates.

Example 5

Consider the problem

$$(8.15) \quad u_{xy} = C, \quad u(x,0) = u(0,y) = 0.$$

This problem has the unique solution $u \equiv 0$ throughout the finite plane. Let

$$(8.16) \quad \begin{cases} \psi_{xy} = Ax - By^2 + C \\ \varphi_{xy} = -D, \end{cases}$$

where A, B, C and D are positive constants. By integration in (8.16) we may obtain functions ψ and φ satisfying the initial conditions of (8.15). Obviously, φ is an under function for all (x,y) . Moreover, $\psi_{xy} > 0$ for all (x,y) lying in the portion of the first quadrant below the parabolic arc

$$y = +\sqrt{\frac{A}{B}x + \frac{C}{B}};$$

and hence ψ meets the requirements for an over function on a domain R_ℓ : $\begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \sqrt{\frac{C}{B}} \end{cases}$ where ℓ is arbitrarily large but finite.

Defining $h = \psi - \varphi$ we have

$$h_{xy}(x,y) = Ax - By^2 + C + D.$$

Since $h(x,0) = h(0,y) = 0$, we obtain by integration

$$h(x,y) = \frac{A}{2} x^2 y - \frac{B}{2} x^2 y^2 + (C+D) xy.$$

We note that $h > 0$ in that portion of the first quadrant below the hyperbola branch

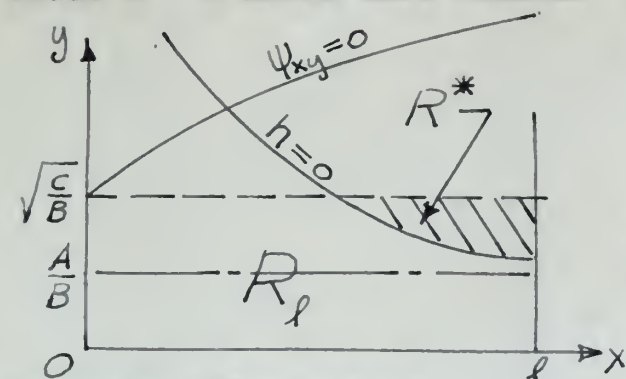
$$y = \frac{A}{B} + \frac{2(C+D)}{Bx}$$

while $h < 0$ above this branch. From the diagram it is evident

that if we require

$$\frac{A}{B} < \sqrt{\frac{C}{B}}$$

then there exists a positive constant ℓ such that within the corresponding domain R_ℓ we have a



subregion R^* on which $\varphi > \psi$. Hence the FERRON method is not directly applicable to this class of problems.

Returning to Theorems 11 and 11a, we observe that if, for fixed (x,y) , f is a monotonically increasing function for the arguments u , p and q , then

$$\begin{aligned} f(x,y; \omega(x,y); \omega_x(x,y), \omega_y(x,y)) \\ = \min_{S(x,y)} [f(x,y; u; p,q)] . \end{aligned}$$

and

$$f(x,y; \Omega(x,y); \Omega_x(x,y), \Omega_y(x,y)) = \max_{S(x,y)} [f(x,y; u; p,q)] .$$

In this case we may alter hypothesis 3) to require merely that

$$\begin{aligned}\omega_{xy}(x,y) &\leq f(x,y; \omega(x,y); \omega_x(x,y), \omega_y(x,y)) \\ \Omega_{xy}(x,y) &\geq f(x,y; \Omega(x,y); \Omega_x(x,y), \Omega_y(x,y))\end{aligned}$$

for each $(x,y) \in R$. This is the direct analogue to PERMANN's theorem (see [18]) and corresponds to the previously mentioned result of MÜLLER for a system (8.3).

We close this chapter with the remark that Theorems 11 and 11a can be extended immediately in two ways. First, the method is directly applicable to the Cauchy problem. We require the functions ω and Ω to satisfy the Cauchy initial data and observe that the proof of Lemma 3 is essentially unchanged. Second, the method extends to apply to a system

$$u_{i,xy} = f_i(x,y; u_j; u_{j,x}, u_{j,y}), \quad (i = 1, \dots, n)$$

for both characteristic and Cauchy initial value prescriptions. The modifications in the hypotheses and proof for Theorems 11 and 11a are obvious.

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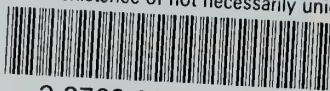
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