MULTI-PHASE-MISSION RELIABILITY OF MAINTAINED SYSTEMS

Merlin Gene Bell

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THESIS

MULTI-PHASE-MISSION RELIABILITY
OF MAINTAINED SYSTEMS

by

Merlin Gene Bell

December 1975

Thesis Advisor J. D. Esary

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Multi-Phase-Mission Reliability
of Maintained Systems

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TABLE OF CONTENTS

1. INTRODUCTION

1.1 BACKGROUND

The area of reliability theory known as structural reliability, with which this paper deals, is the study of relationships between the performance characteristics of systems and those of their components. This field has grown in importance along with the sophistication of the devices upon which modern society depends. The cost of complex devices makes the analytical approach afforded by structural reliability an essential element of system design. Even if it were always possible to build a prototype and test its performance, structural reliability would be an important tool for designers and engineers. There are many instances, moreover, when it is impossible to observe system performance directly. It is not unusual for the customer, who frequently must bear a substantial portion of the cost of system development, to require an evaluation of system performance before any construction is authorized. Even when systems are available for testing, it may be impractical to create the environment necessary to obtain the desired observations or the consequences of testing may be unacceptable. Most importantly, the knowledge of the qualitative and quantitative relationships in system design provided by structural reliability allows the reliability engineer to study the sensitivity of system performance to changes in structure and component performance characteristics. Such parametric studies permit meaningful examination of the trade-offs made to satisfy a constraint on system design such as a weight limitation or a budgetary restriction.

There are several essentially different approaches to the reliability analysis of a complex system. The approach taken here is an analytical one, but as Ziehms [1974] noted, computational and simulation methods are also employed in applications. Each technique has its merits and its shortcomings. Advances on the analytical front can be profitably incorporated into other methods when appropriate, making them both more efficient and more accurate. Monte Carlo techniques can cope with complexities for which analytical methods are unavailable. The best approach for any application will likely be a synthesis of these methods which is more robust than any single technique.

The scope of this paper is confined to analysis of systems which are designed to accomplish a series of tasks during each period of operation. This situation is what Esary and Ziehms [1975] called the *phased-mission* problem.

1.2 THE PHASED MISSION PROBLEM

The concept of a phased mission was introduced in the early papers of Rubin [1964] and Weisburg and Schmidt [1966] which were motivated by the need for reliability and crew-safety predictions in the manned space programs. Ziehms [1974] summarizes the approaches taken by these and other authors and proceeds to show that the performance of a system of non-maintained components with a multi-phase mission can be analyzed by considering an equivalent system with a single-phase mission. The transformation from a multi-phase mission to a single-phase mission makes possible the application of standard structural reliability techniques.

The elements of the phased-mission problem to be considered here are described in the following situation.

A *system* consists of several *components* which perform independently of each other. Each component is said to be *functioning* (up) if it is performing satisfactorily; otherwise it is said to be *failed* (down). The system performs a *mission* which can be divided into an active portion and a standby or readiness portion. The standby portion of the mission is called the *operational readiness (O R) phase*. The active portion is further divided into consecutive time periods or *phases* of known duration during which the functional organization of the system does not change. Implicit in the specification of the mission are the levels of performance which are satisfactory and the environment in which it will be undertaken. The functional organization for each phase can be represented as a *block diagram* built of a subset of the system's components (or equivalently as a *fault tree*). The system is designed to accomplish a specific *task* during each phase and the mission has one or more goals or *objectives* about which it is desired to make probability statements.

The phased-mission problem as described above introduces two concepts not previously considered in this context--the O R phase and the possibility of multiple mission objectives. The duration of the O R phase will generally be unknown since the timing of the event which causes its termination will be difficult to forecast. Some system components will usually be required to function during the O R phase in order to maintain the system's readiness. Because of the prolonged nature of this phase it is highly likely that failures will occur among these active components and possibly among those that are dormant. Thus it is reasonable to expect that some means of monitoring components and correcting failures will be provided. Previous work on the phased-

mission problem has been limited to non-maintained systems. When systems are not maintained, the usual measure of the performance of a system or component is its reliability--that is, the probability that it will perform satisfactorily in the prescribed environment for the period intended. If a system's components are subject to maintenance, the mission reliability cannot be expressed as a function of component reliabilities alone. Consequently, the incorporation of maintenance actions in a model of system performance adds significantly to the complexity of the problem. Fortunately, the increased complexity does not cause severe problems when maintenance is limited to the O R phase.

The notion of multiple objectives serves primarily as a means of recognizing more than two levels of system performance. It is a general concept which the reliability engineer can tailor to the individual application. It provides means with which he can formulate more incisive and meaningful reliability statements than those afforded by the binary success/failure measure of system performance. This concept is used in a highly-structured context in Chapter 3.

Examples of situations which fit the description of the phased-mission problem are numerous. Safety systems, in particular, provide prime examples. Even as simple a system as the local fire station contains all of the ingredients described above. Many military systems, especially strategic weapon systems, are designed to remain in readiness until activated in response to a threat to the national security. Indeed, many of these systems are never activated throughout their entire lifespans.

Much of the work in this paper was motivated by the author's conception of the Navy's Fleet Ballistic Missile (FBM) weapon system.

This system consists of a nuclear-powered submarine and its associated subsystems and sixteen ballistic missiles, each containing many subsystems of its own. The O R phase for any one of the systems commences when it relieves its predecessor in a patrol area and terminates when it is relieved in turn or when a command to launch a missile strike is issued. System components are monitored and maintained throughout the O R phase. Upon activation the system proceeds through phases during which missiles are prepared for launch, the submarine is positioned, and the missiles ejected. As each missile is ejected, it becomes independent of the submarine-borne subsystems and itself progresses through several phases of flight. Each missile is assigned to a target (or more than one target if it has multiple warheads) and destruction of that target is one of the mission objectives. Thus this system is one with sixteen or more objectives and several active phases following an extended O R phase. While maintenance is not performed on the missile components during flight, it is conceivable that limited maintenance can take place during submarine-borne phases. This situation is one in which it is natural to seek the expected value or even the probability distribution of the number of objectives attained (targets destroyed) during the mission. These notions are among those which are discussed in Chapter 3.

1.3 CONTENTS AND SUMMARY

A mathematical model of a single-objective mission with an operational readiness phase and multiple active phases is introduced in Chapter 2. In this model, system maintenance is permitted only during the O R phase. The concepts of component and system availability are used to provide a mathematical statement of mission reliability, which

can be transformed into an equivalent statement for a single-phase mission. The lower bounds discussed in Ziehms [1975] can be applied directly to provide lower bounds on mission reliability. A reduction in the number of components in the equivalent system for any phased mission is shown to result from the use of component availabilities.

The model presented in Chapter 3 is an extension of that developed in Chapter 2 which incorporates the concept of multiple-objective missions. General measures of system performance are discussed which permit great flexibility in applications. Derivation of the distribution of the number of successful objectives is illustrated for the case in which all objectives are of the same type. This also provides a straightforward method for calculating the expected number of objectives successfully completed during the mission.

Chapter 4 develops the theoretical machinery needed to deal with systems which can be maintained throughout a multi-phase mission. The approach extends and combines results for maintained systems performing single-phase missions contained in Esary and Proschan [1970] and results for non-maintained systems performing multi-phase missions obtained by Ziehms [1975]. Two lower bounds on system reliability are developed for the case when the joint component performance process has a property known as association in time. Finally, it is shown that the joint component performance process which results when independent components have exponential times to failure and exponential repair times within each phase of the mission has this property.

The reliability of a parallel system of independent components with the exponential-exponential performance process discussed above is considered in Chapter 5. The distribution of component states at the

beginning of the period during which the parallel system performs is allowed to be arbitrary to permit application of the results to the phased-mission problem. Lower bounds on the reliability of a parallel structure are developed which, when used in conjunction with the results of Chapter 4, provide lower bounds on the reliability of a system with multiple mission phases.

Suggestions for possible extensions of these results and areas for further research are discussed in Chapter 6.

2. THE SINGLE-OBJECTIVE, PHASED-MISSION WITH AN O R PHASE

The situation considered in this chapter is that of a system which performs a single-objective mission with an O R phase and several active phases. A mathematical model is constructed in several stages which relates system performance to that of its components. The development uses the standard tools of structural reliability and extends and modifies the model of Esary and Ziehms [1975]. The various aspects of the extended phased-mission problem are illustrated in the following section by an example motivated by the Fleet Ballistic Missile system.

2.1 THE SLBM SYSTEM EXAMPLE

The following example introduces a hypothetical system which will serve both as motivation and illustration of the model development throughout this chapter and again in Chapter 3.

Example 2.1. A hypothetical submarine-launched ballistic missile system (SLBM) consists of the following components:

--the submarine (S) which provides propulsion, stability, power, and household services.

--the inertial navigation subsystem (N) which provides information on platform position and orientation.

--the communication subsystem (C) which provides the link between the submarine and its command center.

--the fire control subsystem (FC) which provides trajectory information to the missile guidance computer.

--the missile ejection subsystem (E) which launches the missile from the submarine while the latter is submerged.

--the missile guidance component (G) which computes and transmits to the rocket engines the control commands required to maintain the trajectory stored within its memory and triggers stage separation.

--two missile internal power sources (VP and VS).

--the first and second stage rocket engines (RF and RS).

--the first-stage igniters (IP and IS).

--the second-stage igniter (J).

--the missile warhead (W).

The operational characteristics of the system can be summarized as follows:

(a) During the operational readiness phase the submarine patrols its assigned area, maintaining current position information with the inertial navigation subsystem. Should the inertial component fail, then position information can be obtained periodically from a navigation satellite, providing the data necessary for calibration after repairs are completed. The communication subsystem is used continually during the phase for routine ship-shore message traffic. The fire control subsystem is exercised periodically during the O R phase to monitor its status. Similarly, the performance of the missile power sources and guidance component is checked through routine tests. All components which are monitored can be repaired or replaced if found to be failed during the O R phase. Other failures go undetected. In order for the system to be ready to commence active operations, it must have submarine services and current navigation information available, and it must be able to receive the launch command via the communication subsystem.

(b) When a launch command is received, all maintenance actions

cease, and launch preparations commence. The fire control subsystem transmits trajectory data to the missile guidance component, and the submarine is positioned for launch.

(c) During the launch phase the submarine is held stable while the missile is ejected, severing its link with the platform and causing it to switch to internal power. The power sources, although activated, are not required to supply power during this phase.

(d) The first-stage engine ignites as the missile breaks through the surface of the water and boosts the missile along its trajectory. The port igniter can be powered by only the port power source and the starboard igniter by only the starboard power source, but one igniter is sufficient to fire the engine. The guidance component, which can take power from either source, must function throughout the phase.

(e) The second-stage igniter, second-stage engine, guidance component, and at least one power source must function during the flight phase.

(f) Shutdown of the second-stage engine marks the beginning of the terminal phase during which the warhead follows a ballistic trajectory to the target. □

2.2 THE EXTENDED PHASED-MISSION MODEL

The mission consists of an O R phase followed by m active phases. The O R phase commences at time t=0 and continues until time t_0 when active operations begin. For j=1,...,m, the duration of active phase j is assumed to be d_j. Recognizing that t_0 is unknown, let

$$t_j = \sum_{i=1}^{j} d_i + t_0,$$

j=1,...,m. Thus t_j is the time at which phase j ends and (except when

$j = m$) the next phase begins.

The system has n components (or subsystems) $C_1, \ldots, C_n$, which function independently of each other. Assigned to each component C_k, $k=1,\ldots,n$, is a Bernoulli *performance state indicator variable* $X_k(t)$ defined for all $t \geq 0$ by

$$X_k(t) = \begin{cases} 1 \text{ if } C_k \text{ is functioning at time } t, \\ 0 \text{ otherwise.} \end{cases}$$

The stochastic process $\{X_k(t), t \geq 0\}$ is called the *performance process* of component C_k, and the multivariate stochastic process

$$\{\underline{X}(t), t \geq 0\} = \{[X_1(t), \ldots, X_n(t)], t \geq 0\}$$

is the *joint component performance process* of the system.

As in Example 2.1, it will generally be the case that some portion of the system's components can be repaired or replaced upon failure during the O R phase; however it is assumed that no system maintenance is performed after time t_0. Then the performance process of each repairable component C_k has the properties:

$$\text{(2.2.1)} \quad \begin{aligned} X_k(t) = 0 &\Leftrightarrow X_k(s) = 0, \text{ for all } s > t \geq t_0 \\ X_k(t) = 1 &\Leftrightarrow X_k(s) = 1, \text{ for all } s \geq t_0 \text{ such that } s \leq t. \end{aligned}$$

The performance processes of the remaining components satisfy these relations when t_0 is replaced by 0. Thus a sample path of the performance process for a non-repairable component is non-increasing and continuous from the right, and that for a repairable component is also continuous from the right and is non-increasing after time t_0 as shown in Figure 2.1.

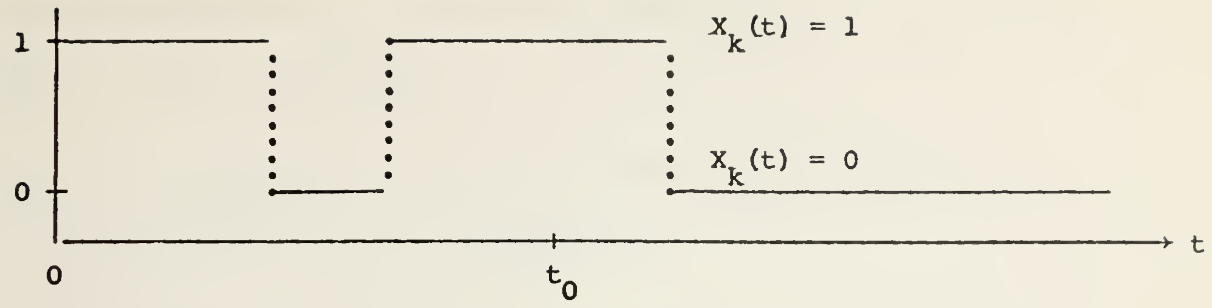


Figure 2.1. Performance process sample path of repairable component C_k.

The joint component performance process is a complete mathematical description of the performance of the system's components. It is useful to summarize the characteristics of a component's performance process in the form of probability statements. The reliability of a component is a statement about its performance *over a period of time*. Thus the reliability of component C_k during the period [t, t + d] is defined as $P[X_k(s) = 1, t < s \leq t+d]$. There are also instances when it is desired to make statements about the performance of a component *at a point in time*. Hence the *availability* of component C_k *at time t* is defined by

$$\alpha_k(t) = P[X_k(t) = 1]$$

If the performance process is non-increasing over the interval [t, t+d] then the reliability of component C_k over the period is equal to its availability at time t+d by Relations 2.2.1.

The state of the system at any time is assumed to be completely determined by the states of its components. The system structure is the connecting link. In a phased mission this structure does not remain fixed throughout the mission, but is allowed to change from phase to phase. Thus, letting the O R phase be phase zero, there is a binary

structure function ϕ_j of the binary variables $x_1, \ldots, x_n$ for each phase j, j=0,...,m, defined by

$$\phi_j(x_1, \ldots, x_n) = \begin{cases} 1 \text{ if the system functions, and} \\ 0 \text{ otherwise.} \end{cases}$$

The composition $\phi[\underline{X}(t)]$ where ϕ is defined by

$$\phi[\underline{X}(t)] = \begin{cases} \phi_0[\underline{X}(t)], & 0 < t \leq t_0 \\ \phi_1[\underline{X}(t)], & t_0 < t \leq t_1 \\ \vdots \\ \phi_m[\underline{X}(t)], & t_{m-1} < t \leq t_m \end{cases}$$

is itself a Bernoulli random variable called the *system performance indicator variable*. The corresponding stochastic process $\{\phi[\underline{X}(t)], t>0\}$ is called the *system performance process*. Although the sample paths of $\{\phi[\underline{X}(t)], t > 0\}$ are not necessarily right continuous, the right continuity of the sample paths of the component performance processes leads to right continuity of the system performance process sample paths within each phase.

In order for the system to satisfactorily complete its mission, it must function throughout each active phase. The O R phase, however, is different, for it is merely a readiness period. It is not necessary that the system function throughout this phase. The single requirement is that the system be available when the O R phase ends. Thus the mission reliability is given by

(2.2.2)
$$p = P\{\phi_0[\underline{X}(t_0)] = 1, \phi_1[\underline{X}(s_1)] = 1, t_0 < s_1 \leq t_1, \ldots, \phi_m[\underline{X}(s_m)] = 1, t_{m-1} < s_m \leq t_m\}$$

The structure of a system is typically represented as a *block diagram* of its components (or equivalently as a *fault tree*). Structures which can be so depicted belong to a special class whose structure functions are said to be *coherent*. Birnbaum-Esary-Saunders [1961] defined a coherent structure function to be one for which

(2.2.3)

(a) $\phi(\underline{x}) \geq \phi(\underline{y})$ whenever $x_k \geq y_k$, $k=1,\ldots,n$,

(b) $\phi(\underline{0}) = \phi(0,0,\ldots,0) = 0$, and

(c) $\phi(\underline{1}) = \phi(1,1,\ldots,1) = 1$.

Since nearly all physical systems have a block-diagram representation, it is assumed that the structure function for each active phase is coherent.

The general model makes provision for the inclusion of a structure function for the O R phase. The system of Example 2.1 provides an illustration of circumstances in which a phase 0 structure function is appropriate, since active operations cannot commence unless certain components are available. The following example shows a case in which use of the function $\phi_0 = 1$ is appropriate.

Example 2.2. A system with a three-phase mission has two components, C_1 and C_2, both of which are dormant during the O R phase. During the first active phase the system functions only if both components function, but during the second phase the functioning of either component will allow the system to function. The structure functions for this mission are:

for phase 0, $\phi(x_1, x_2) = 1$

for phase 1, $\phi(x_1, x_2) = x_1 x_2$

for phase 2, $\phi(x_1, x_2) = x_1 \vee x_2$

where the symbol $\vee$ is the arithmetic "or" operator defined by

$$x_1 \vee x_2 = \begin{cases} 1 \text{ if } x_1 = 1 \text{ or } x_2 = 1, \\ 0 \text{ if } x_1 = 0 \text{ and } x_2 = 0 \end{cases}$$

Computationally, $x_1 \vee x_2 = x_1 + x_2 - x_1 x_2$. The corresponding block diagrams for the phases of this mission are shown in Figure 2.2. □

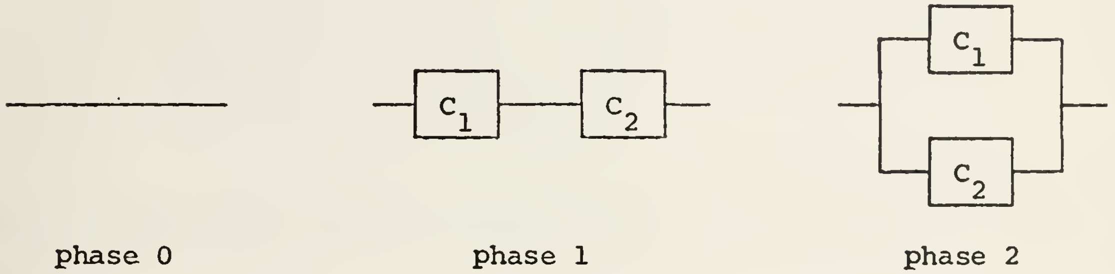


Figure 2.2. Block diagrams for the mission of Example 2.2.

In general, the O R phase structure function will be $\phi_0 = 1$ unless one or more of the components is actively required at time t_0, in which case ϕ_0 will be a coherent structure function. Thus it is assumed that ϕ_0 is at least *semi-coherent* (satisfies Relation 2.2.3a) in all cases.

Since the joint component performance process is non-increasing after time t_0, it follows from Relation 2.2.3a that the system performance process is also non-increasing within each phase. Thus the mission reliability as given by Equation 2.2.2 reduces to the less complex expression

$$p = P\{\phi_0[\underline{X}(t_0)] = 1,\ \phi_1[\underline{X}(t_1)] = 1, \ldots, \phi_m[\underline{X}(t_m)] = 1\}$$

or more simply

(2.2.4) $$p = P\{\textstyle\prod_{j=0}^{m} \phi_j[\underline{X}(t_j)] = 1\} = E\{\textstyle\prod_{j=0}^{m} \phi_j[\underline{X}(t_j)]\}$$

The SLBM system of Example 2.1 can now be used to illustrate the extended phased-mission model. The verbal description of that system's operation translates into the following mathematical structure:

for phase 0, $\phi_0(\underline{x}) = x_S x_C x_N$

for phase 1, $\phi_1(\underline{x}) = x_S x_{FC} x_G$

for phase 2, $\phi_2(\underline{x}) = x_S x_E$

for phase 3, $\phi_3(\underline{x}) = [(x_{VP} x_{IP}) \vee (x_{VS} x_{IS})] x_G x_{RF}$

for phase 4, $\phi_4(\underline{x}) = (x_{VP} \vee x_{VS}) x_G x_J x_{RS}$

for phase 5, $\phi_5(\underline{x}) = x_W$

The equivalent block-diagram representation of the system is shown in Figure 2.3.

A mathematical statement of mission reliability for this system results from substitution of the phase structure functions into Equation 2.2.4. Thus, mission reliability is

$$
\begin{aligned}
p = E\Big\{ & [X_S(t_0)X_C(t_0)X_N(t_0)][X_S(t_1)X_{FC}(t_1)X_G(t_1)] \\
& [X_S(t_2)X_E(t_2)][X_{VP}(t_3)X_{IP}(t_3) \vee X_{VS}(t_3)X_{IS}(t_3)] \\
& [X_G(t_3)X_{RF}(t_3)][X_{VP}(t_4) \vee X_{VS}(t_4)] \\
& [X_G(t_4)X_J(t_4)X_{RS}(t_4)][X_W(t_5)]\Big\}
\end{aligned} \tag{2.2.5}
$$

Evaluation of Equation 2.2.5 is not straightforward. Even though the components perform independently, the performance indicator variables for the same component at different times are obviously not independent. Hence it is not clear at this stage how to proceed. In the next section the transformation due to Esary and Ziehms [1975] will be used to convert expressions such as Equation 2.2.5 into the expectation of sums and

products of independent random variables. The cost associated with this procedure--that of a significant increase in the number of variables--will be made readily apparent when the system of Example 2.1 is transformed.

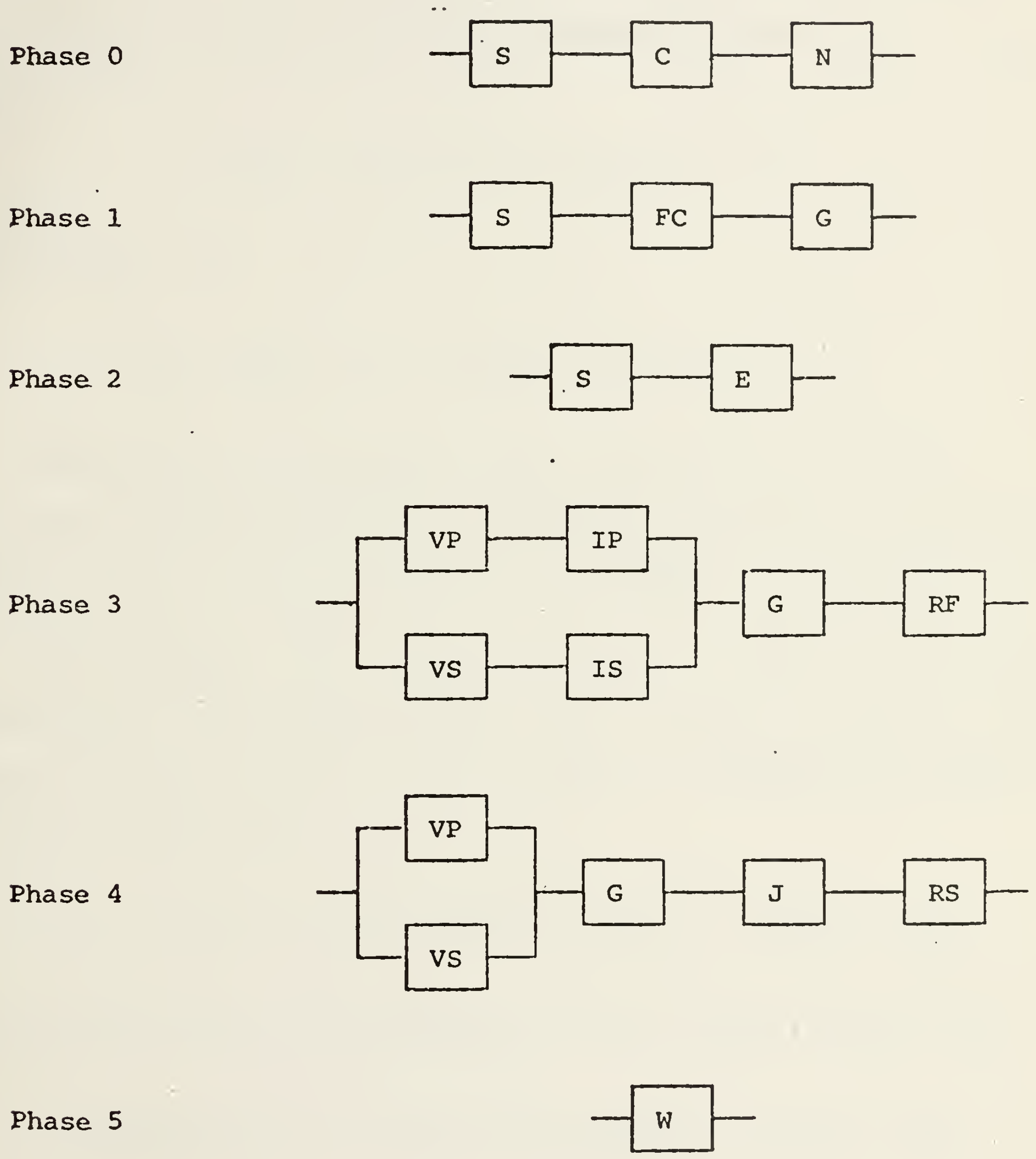


Figure 2.3. Phase configurations for the system of Example 2.1.

2.3 TRANSFORMATION OF THE MULTI-PHASE MISSION PROBLEM

In the context of the phased-mission problem under investigation, the transformation suggested by Esary and Ziehms [1975] consists of the following steps:

(a) Replace each component C_k in the configuration for phase j, $j=0,1,\ldots,m$, by a series arrangement of independent pseudo components $C_{k0},\ldots, C_{kj}$ with performance state indicator variables $U_{k0},\ldots, U_{kj}$, where

$$\alpha_k(t_0) = P[U_{k0} = 1] = P[X_k(t_0) = 1]$$

and, for $i=1,\ldots,j$,

$$\pi_{ki} = P[U_{ki} = 1] = P[X_k(t_i) = 1 | X_k(t_{i-1}) = 1]$$

(b) Connect the transformed phase configurations in series.

The result of this procedure is an *equivalent system* of at most $n(m+1)$ pseudo components which is coherent and performs a single-phase mission.

Since $P[X_k(t_j) = 1] = P[U_{k0}U_{k1}\cdots U_{kj} = 1]$, $X_k(t_j)$ has the same distribution as the product $U_{k0}U_{k1}\cdots U_{kj}$. ($X_k(t_j)$ is said to be stochastically equal to ($\overset{st}{=\!=}$) $U_{k0}U_{k1}\cdots U_{kj}$.) It follows from Theorem 3.1 of Esary-Ziehms [1975] that

$$[X_k(t_0),X_k(t_1),\ldots,X_k(t_m)] \overset{st}{=\!=} [U_{k0},U_{k0}U_{k1},\ldots,U_{k0}U_{k1}\cdots U_{km}]$$

and by independence of the components of the original system that

$$[\underline{X}(t_0),\underline{X}(t_1),\ldots, \underline{X}(t_m)] \overset{st}{=\!=} [\underline{U}_0,\underline{U}_0\underline{U}_1,\ldots, \underline{U}_0\underline{U}_1\cdots \underline{U}_m]$$

where

$$\underline{U}_j = [U_{1j}, U_{2j},\ldots, U_{nj}]$$

and

$$\underline{U}_j\underline{U}_k = [U_{1j}U_{1k}, U_{2j}U_{2k},\ldots, U_{nj}U_{nk}]$$

Thus the reliability of the original system as given by Equation 2.2.4 is the same as the reliability of the equivalent system given by

(2.3.1) $$p = P\{\prod_{j=0}^{m} \phi_j[\underline{U}_0 \cdots \underline{U}_j] = 1\}$$

or more compactly as

(2.3.2) $$p = E\{\prod_{j=0}^{m} \phi_j[\underline{U}_0 \cdots \underline{U}_j]\}$$

The random variables in Equations 2.3.1 and 2.3.2 are mutually independent by construction, and hence computing the expected value is theoretically routine. Application of this transformation is illustrated in Figure 2.4,which shows the transformed phase configurations for the system of Example 2.1.

Figure 2.4 provides a graphic demonstration of the practical difficulty to be encountered when applying the transformation. The number of components in the equivalent system will likely be large for any moderately complex system. Even though computer algorithms for evaluation of block diagrams and fault trees are available, the computation time and memory requirements associated with such a large number of components would be excessive.

Rubin [1964] and Weisburg and Schmidt [1966] pointed out a technique which results in simplified configurations for the early phases of a phased mission. Esary and Ziehms [1975] provide justification for this procedure called *cut cancellation*. It is well known that every coherent system can be represented as a series structure of subsystems, each of which consists of the components belonging to a *minimal cut set* connected in parallel. (See, for example, Barlow and Proschan [1975a].) A cut set is a set of components which by all failing causes the system

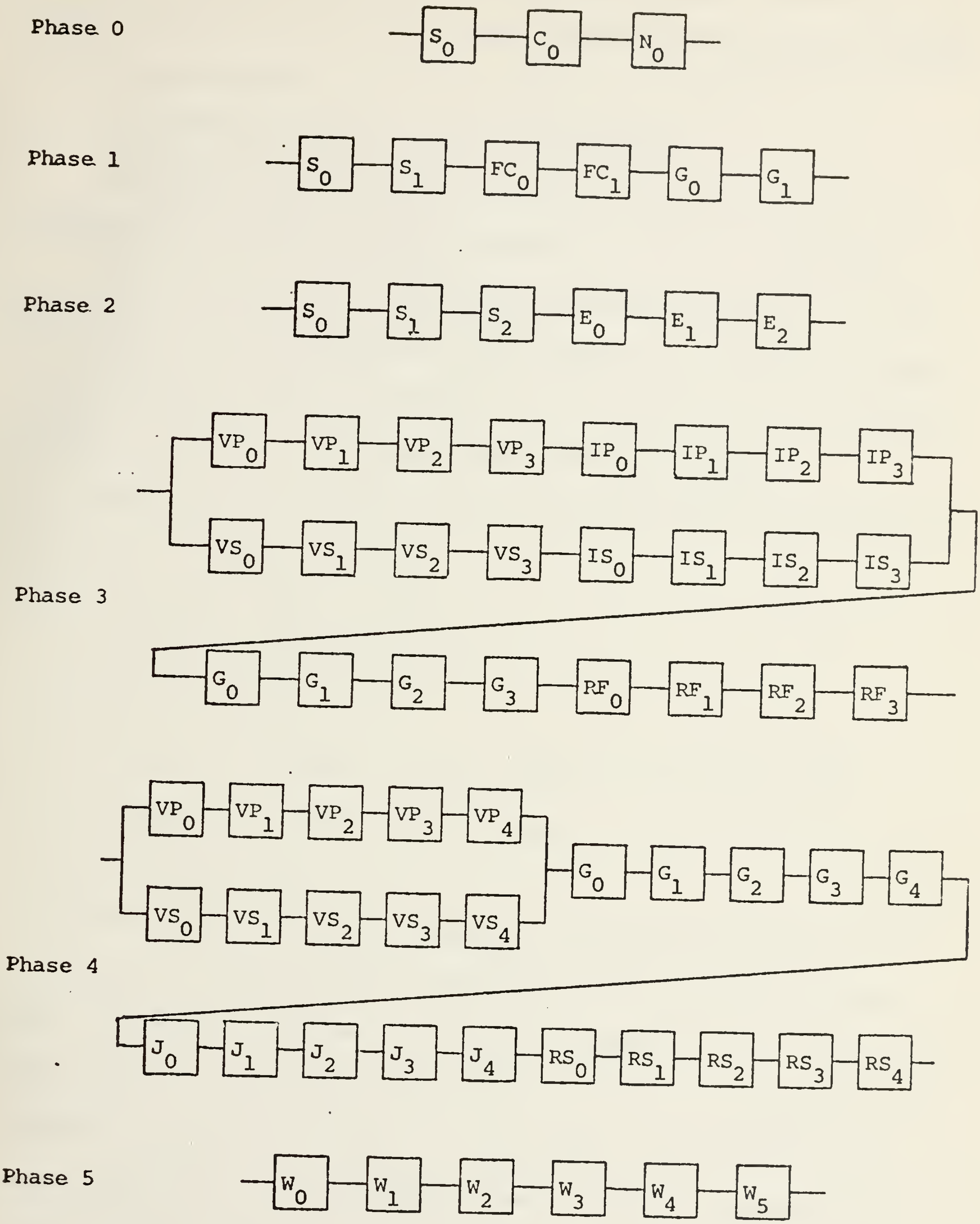


Figure 2.4. Transformed phase configurations for system of Example 2.1.

to fail. A minimal cut set is one which is no longer a cut set if any of the components are removed. The cut-cancellation procedure involves the following steps:

(a) Find the minimal cut sets for each phase.

(b) Remove from the list of minimal cut sets for any phase each minimal cut set which contains a minimal cut set for a later phase.

(c) Reconstitute the system from the remaining minimal cut sets. The reliability of the resulting configuration is unchanged from that of the original system. Performing cut cancellation does not, however, reduce the number of components in the equivalent system.

Example 2.3. The minimal cut sets and indicated cancellations for the system of Example 2.1 are:

for phase 0, ~~{S}~~, {C}, {N}

for phase 1, ~~{S}~~, {FC}, ~~{G}~~

for phase 2, {S}, {E}

for phase 3, ~~{VP,VS}~~, {VP,IS}, {VS,IP}, {IP,IS}, ~~{G}~~, {RF}

for phase 4, {VP,VS}, {G}, {J}, {RS}

for phase 5, {W}

□

Examination of Figure 2.4 suggests a procedure which does result in a reduction in the number of components in the equivalent system. Look, for example, at phase 5. The pseudo components $W_0, \ldots, W_5$ appear nowhere else in the system configuration. Thus it is not necessary to transform component W into $W_0, \ldots, W_5$ in order to gain independence among components of the equivalent system. Leaving W untransformed, however, would nullify one of the other appealing features of the transformation technique. It is completely natural to use conditional phase reliabilities to describe component performance in each active phase

and to keep separate the question of initial availability. A compromise which retains this aspect and yet reduces the number of components results from lumping pseudo components $W_0, \ldots, W_4$ together into a new pseudo component $W_{\vec{4}}$ where

$$P[U_{W\vec{4}} = 1] = P[U_{W4} = 1]P[U_{W3} = 1]P[U_{W2} = 1]P[U_{W1} = 1]P[U_{W0} = 1]$$

$$= P[X_W(t_4) = 1]$$

In general, a component is said to be *relevant* to system operation during any phase only if it appears in the configuration for that phase. If component C_k becomes relevant for the first time in phase j of a mission, a general rule for *component reduction* is to replace pseudo components $C_{k0}, \ldots, C_{k,j-1}$ with a new pseudo component $C_{k,\vec{j}-1}$ wherever they appear in the equivalent system. The new pseudo component has reliability given by

$$P[U_{k,\vec{j}-1} = 1] = \alpha_k(t_{j-1}) = \alpha_k(t_0)\pi_{k1} \cdots \pi_{k,j-1}$$

and the reliability of the reduced equivalent system is the same as that of the original system.

The equivalent system configuration for the SLBM system of Example 2.1 which results from the use of cut cancellation and component reduction is shown in Figure 2.5. In this case the number of pseudo components is reduced from 53 to 26.

In spite of efforts to reduce the number of components, many systems are so large and complex that direct calculation of mission reliability is infeasible. In these cases the upper and lower bounds on mission reliability contained in Ziehms [1975] can be applied directly to the model presented here. Ziehms provides a detailed development

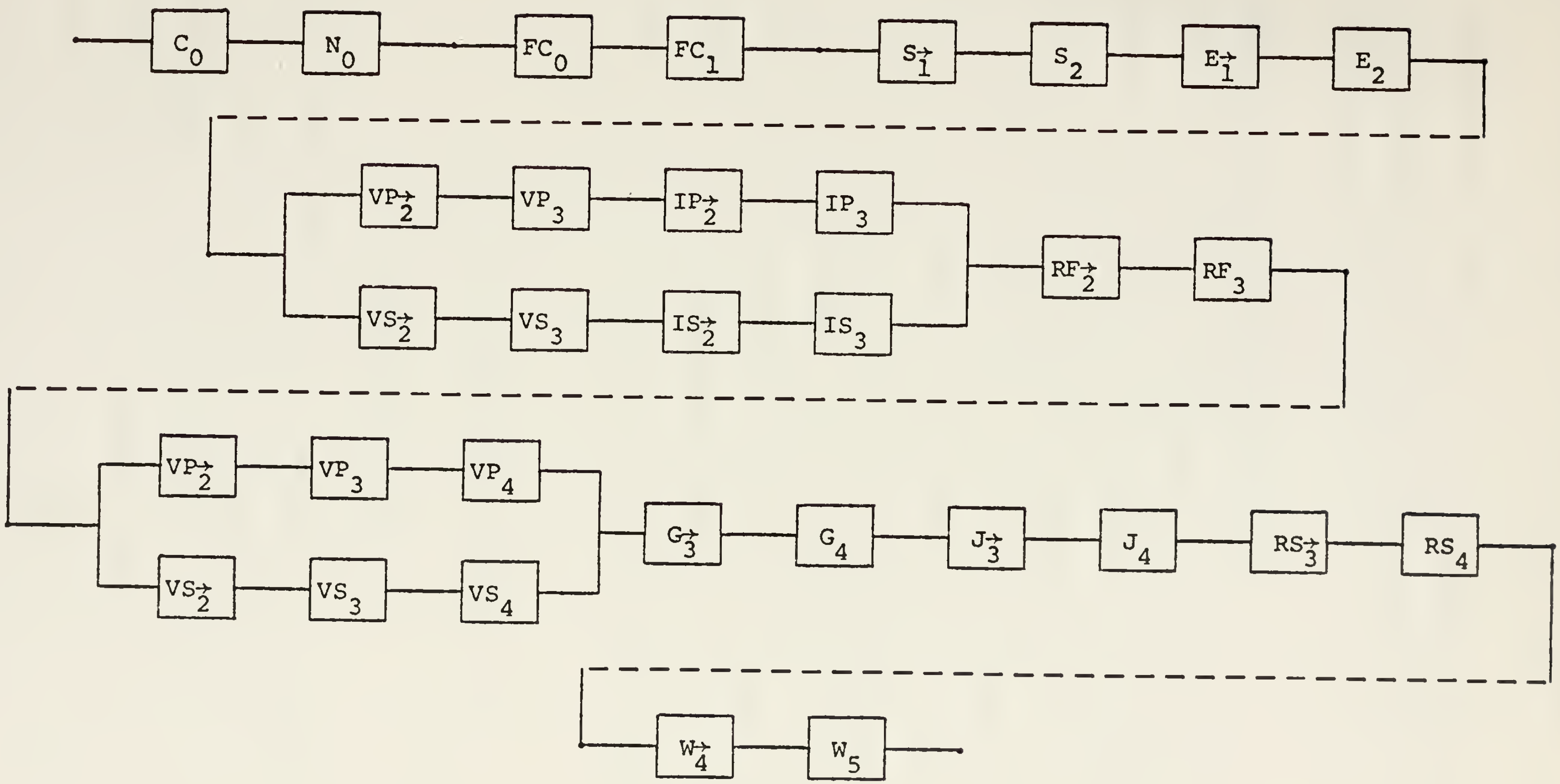


Figure 2.5. Equivalent system for Example 2.1 after component reduction and cut cancellation.

of these bounds and offers some criteria for choosing the best among them.

Before mission reliability or bounds thereon can be computed, the component availabilities and phase reliabilities must be calculated. This is the subject of the next section.

2.4 AVAILABILITIES AND PHASE RELIABILITIES

No attempt is made in this section to catalog existing models for component availabilities, nor are any new models developed. Standard models are used to illustrate a typical approach to the development of component availabilities. Barlow and Proschan [1975a] and Cox [1962] are good sources for additional details on this subject.

There are two different situations to be explored in connection with component availabilities. First is the issue of the availability of components which remain dormant during the O R phase (and thus are not maintained). Let the random variable T_k be the *active* lifelength of component C_k, that is, the time from t_0 until the component fails. Then, for all $t > 0$,

$$P[T_k > t] = P[T_k > t | T_k > 0] P[T_k > 0]$$

$$= \alpha_k(t_0) P[T_k > t | T_k > 0]$$

Thus $\alpha_k(t_0)$ is the probability that component C_k is available when first activated, and $[1 - \alpha_k(t_0)]$ is the probability that it will fail to operate because of a manufacturing defect, mishandling, or some other cause unrelated to service failure. Since failures of this type will generally be independent of the length of the O R phase, the argument t_0 can usually be dropped to yield the constant availability α_k.

The second case is that of repairable components. There are as many models for the performance processes of repairable components as there are different maintenance schemes; however unless it is assumed that components have exponentially-distributed times to failure, there are serious difficulties in accounting for the residual time to failure for those components functioning at time t_0. Thus here, as in most applications of reliability theory, constant component failure rates are assumed. It is convenient (but not necessary) to assume that component repair times are exponentially distributed as well. The resulting performance process for each component is an alternating renewal process, hereafter called the *exponential-exponential* performance process. Specifically, it is assumed that during the O R phase component C_k has a constant failure rate λ_{k0} and a constant repair rate μ_{k0}. A standard renewal theory argument (see Cox [1962] or Barlow and Proschan [1975a]) shows that if component C_k is functioning at time t=0, then its availability at a later time t is given by

$$\alpha_k(t) = [\lambda_{k0} + \mu_{k0}]^{-1}\left(\mu_{k0} + \lambda_{k0} e^{-(\lambda_{k0} + \mu_{k0})t}\right) \tag{2.4.1}$$

and if it is down at time t=0 then

$$\alpha_k(t) = \mu_{k0}[\lambda_{k0} + \mu_{k0}]^{-1}\left(1 - e^{-(\lambda_{k0} + \mu_{k0})t}\right) \tag{2.4.2}$$

If, as generally assumed, time t_0 is unknown, then Equations 2.4.1 and 2.4.2 are of little use in providing the required numerical value for $\alpha_k(t_0)$. Unless component availabilities are known from some other source (similar systems or testing programs) there is no analytic alternative to the use of bounds. The most common approach is to approximate $\alpha_k(t_0)$ by the (long-run) *availability* given by

(2.4.3) $$\alpha_k = \lim_{t\to\infty} \alpha_k(t) = \mu_{k0}[\lambda_{k0} + \mu_{k0}]^{-1}$$

This is equivalent to assuming that the performance process is in equilibrium or steady state at time t=0, i.e. that $P[X_k(0) = 1] = \alpha_k$. It is easy to see that the availability α_k as given by Equation 2.4.3 is a lower bound on $\alpha_k(t)$ as long as $P[X_k(0) = 1] \geq \alpha_k$.

A better lower bound on $\alpha_k(t_0)$ is available if there is an upper limit τ on the duration of the O R phase. Since $\alpha_k(t)$ as given by Equation 2.4.1 is a decreasing function of time, $\alpha_k(t_0)$ is bounded below by

(2.4.4) $$\underset{\sim}{\alpha}_k = [\lambda_{k0} + \mu_{k0}]^{-1}(\mu_{k0} + \lambda_{k0}e^{-(\lambda_{k0} + \mu_{k0})\tau}$$

provided component C_k is functioning at time t=0.

<u>Example</u> <u>2.4</u>. Consider the navigation component of the SLBM system of Example 2.1, and suppose it is subject to failure at rate λ_{NO} and repair at rate μ_{NO} during the O R phase. Assume that the submarine patrols for a maximum of τ days. Upon completion of its patrol, the submarine is relieved on station by another of the same type and returns to its home base for an upkeep period, during which all repairable components are restored to working condition. Then a conservative estimate of the availability of the navigation component at time t_0 is given by Equation 2.4.4. □

Other bounds of this type can be found when there is random selection of the component initial state. As long as the availability of component C_k at time t=0 is at least α_k, $\alpha_k(t)$ is decreasing with time, and $\alpha_k(\tau)$ provides a lower bound. Otherwise, the initial availability is a lower bound on $\alpha_k(t_0)$.

It is not unrealistic to assume that every component has a constant failure rate within each phase. In this case the phase reliabilities, which are the final ingredients needed to perform calculations, take on a particularly simple form. Let the failure rate of component C_k in active phase j be λ_{kj}, k=1,...,n; j=1,...,m. Then the conditional phase reliability is given by

$$\pi_{kj} = P[X_k(t_j) = 1 | X_k(t_{j-1}) = 1] = e^{-\lambda_{kj} d_j}$$

$$k=1,\ldots,n; \; j=1,\ldots,m.$$

After the component availabilities and conditional phase reliabilities have been determined, the final step is calculation of mission reliability. This chapter is concluded with a brief sketch of this step for the SLBM system of Example 2.1. Similar calculations are shown in much greater detail for a less complex system in Example 3.2.

Assume that the initial availability and conditional phase reliabilities have been determined for each component of the SLBM system. The statement of mission reliability can be written down in reduced form (after cut cancellation and component reduction) directly from Figure 2.5. Thus mission reliability is given by

$$\begin{aligned} p = \; & E\{U_{CO}U_{NO}U_{FCO}U_{FC1}U_{S\vec{1}}\,U_{S2}U_{E\vec{1}}\,U_{E2} \\ & \times \, [(U_{VP\vec{2}}\,U_{VP3}U_{IP\vec{2}}\,U_{IP3}) \vee (U_{VS\vec{2}}\,U_{VS3}U_{IS\vec{2}}\,U_{IS3})] \\ & \times \, U_{RF\vec{2}}\,U_{RF3}[(U_{VP\vec{2}}\,U_{VP3}U_{VP4}) \vee (U_{VS\vec{2}}\,U_{VS3}U_{VS4})] \\ & \times \, U_{G\vec{3}}\,U_{G4}U_{J\vec{3}}\,U_{J4}U_{RS\vec{3}}\,U_{RS4}U_{W\vec{4}}\,U_{W5}\} \end{aligned} \tag{2.4.5}$$

Let Π_{kj} be the *unconditional* reliability of component C_k at the end of

phase j where

$$\Pi_{kj} = \alpha_k(t_0)\pi_{k1} \cdots \pi_{kj}$$

Then, after expanding Equation 2.4.5 and performing idempotent cancellations ($U_{kj}U_{kj} = U_{kj}$), the mission reliability as a function of component availabilities and reliabilities is

$$\begin{aligned} p = {} & \alpha_C(t_0)\alpha_N(t_0)\Pi_{FC1}\Pi_{S2}\Pi_{E2}\Pi_{RF3}\Pi_{G4}\Pi_{J4}\Pi_{RS4}\Pi_{W5} \times \\ & \{\Pi_{VP3}\Pi_{VS3}(\Pi_{IP3}\pi_{VS4}(1-\pi_{VP4}) + \Pi_{IS3}\pi_{VP4}(1-\pi_{VS4}) \\ & - \Pi_{IP3}\Pi_{IS3}(\pi_{VP4} + \pi_{VS4} - \pi_{VP4}\pi_{VS4}) \\ & + \Pi_{VP4}\Pi_{IP3} + \Pi_{VS4}\Pi_{IS3}\} \end{aligned}$$

Extension of the model presented in this chapter to the case of a multiple-objective mission is discussed in Chapter 3. The methods for bounding component availabilities and calculating phase reliabilities discussed in this section are equally relevant there.

3. MULTIPLE OBJECTIVE MISSIONS

The performance of systems having more than one objective is considered in this chapter. There are obviously many ways in which a mission statement could be written to recognize multiple objectives. Here, the investigation is limited to those cases in which the objectives are all of approximately the same importance or rank. Further, it is assumed that all objectives are of the same type--that is, are in some sense repetitive in nature. Even with these restrictions, there remains too much latitude in system organization and performance characteristics to permit the development of a universally applicable model. The mathematical model developed in the following sections, which is motivated by the Fleet Ballistic Missile system, is one particularization. Nevertheless, the approach is sufficiently general to allow its adaptation to other situations.

3.1 TEMPORAL STRUCTURE OF THE MULTI-OBJECTIVE MISSION

The system to be considered is assumed to have a multi-phase mission as before. The mission consists of $r \geq 1$ objectives, each containing several tasks--one per phase. The performance of the tasks associated with any one of the objectives involves the use of some components which must also be used (either simultaneously or at another time) in the performance of tasks associated with other objectives. It is assumed that associated with each objective is a subset of components which are used only in the performance of tasks related to that objective. Components associated with more than one objective will be said to comprise the *master system* and components unique to objective i, $i=1,\ldots,r$, will

make up the i^{th} *subsystem*. All objectives are assumed to have the same structure. Thus the block diagram (or structure function) for the j^{th} phase of objective i is the same as that for the j^{th} phase of any other objective (but may involve physically different components.

The time sequence of phases is shown in Figure 3.1. Those phases whose configurations involve master system components are depicted on the horizontal time line and are called *trunk phases*. Those phases in which only subsystem i components are relevant are shown on the i^{th} vertical time line and are called *branch i phases*. The sequence of trunk phases consists of

--the O R phase, shared in common by all objectives and involving only master system components;

--a active phases shared in common by all objectives, involving only master system components;

--b phases associated with objective 1, in which master system and subsystem 1 components are relevant;

--b phases associated with objective 2, in which master system and subsystem 2 components are relevant;

⋮

--b phases associated with objective r, in which master system and subsystem r components are relevant.

Each branch consists of c-b phases so that for each objective there are a total of a+c+1 phases.

The trunk phases shared in common by all objectives are denoted F_{oj}, j=0,1,...,a, and those phases unique to objective i, i=1,...,r, are denoted successively by F_{ij}, j=1,...,c.

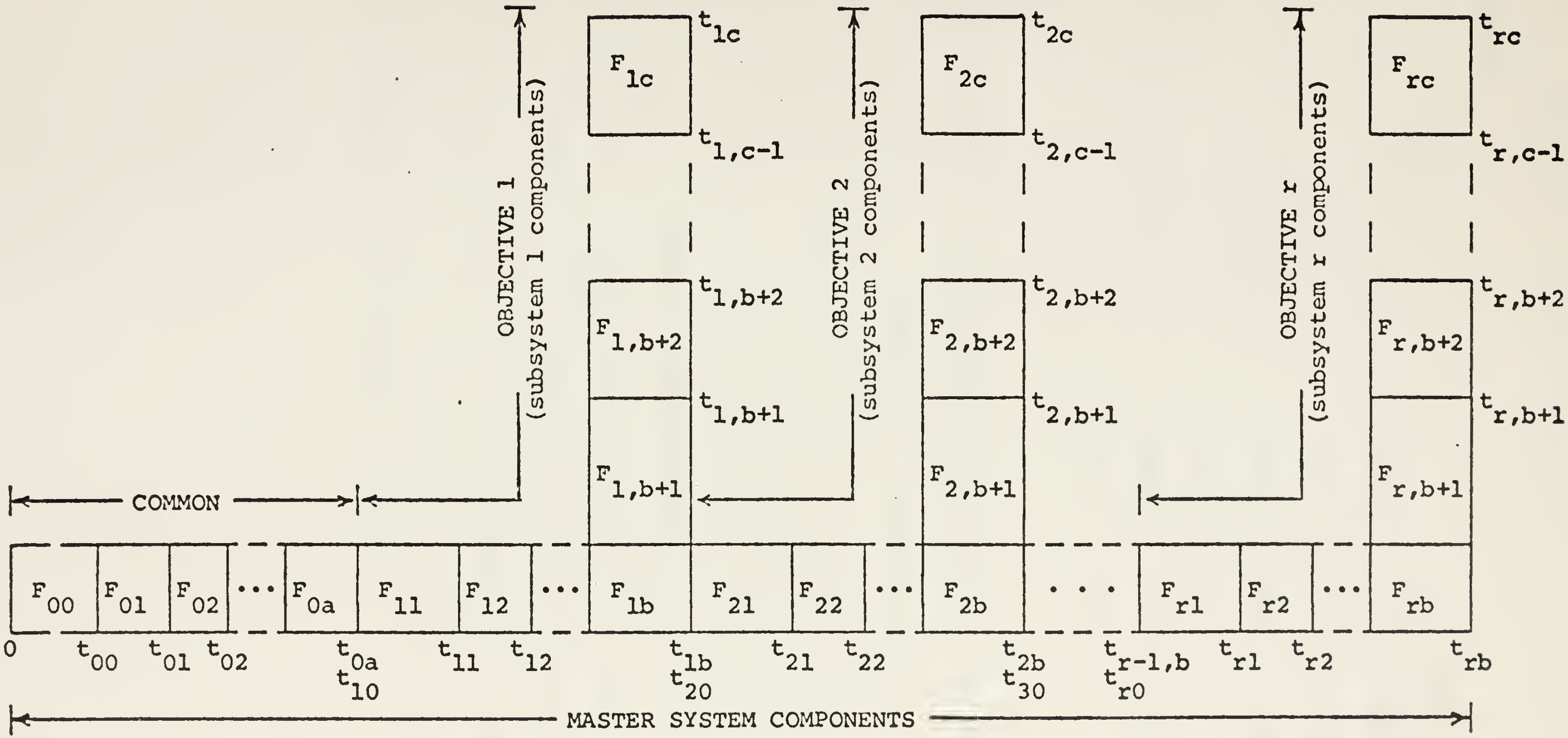


Figure 3.1. Multi-objective mission phase sequence

The time labels displayed in Figure 3.1 are based on the following conventions:

TIME	EVENT
t_{0j}, $j=0,\ldots,a-1$	Phase F_{0j} ends and phase $F_{0,j+1}$ begins.
$t_{0a} = t_{10}$	Phase F_{0a} ends and phase F_{11} begins.
t_{ij}, $i=1,\ldots,r$; $j=1,\ldots,c-1$	Phase F_{ij} ends and phase $F_{i,j+1}$ begins.
t_{ic}, $i=1,\ldots,r$	Phase F_{ic} ends.
$t_{ib} = t_{i+1,0}$, $i=1,\ldots,r$	Phase F_{ib} ends and phase $F_{i+1,1}$ begins.
t_{rb}	Phase F_{rb} ends.

For objective i to be successfully completed, the system must be available at the end of phase F_{00} (O R phase), and it must function satisfactorily throughout phases $F_{01},\ldots, F_{0a}$, and phases $F_{i1},\ldots, F_{ic}$. System performance in other phases is irrelevant to this objective. Aside from the changes in phase arrangement, this is precisely the problem considered in Chapter 2, and the methods presented there could be used to calculate the probability of successfully completing objective i, $i=1,\ldots,r$. Because of the obvious dependencies among the objectives, some additional mathematical structure is required to support joint probability statements about two or more of the objectives. This structure is developed in the following section.

3.2 MATHEMATICAL MODEL OF THE MULTI-OBJECTIVE MISSION

The master system is assumed to have n components, $C_1, \ldots, C_n$, and subsystem i to have m components, $D_1^{(i)}, \ldots, D_m^{(i)}$, $i=1, \ldots, r$. The master system performance state indicator vector at time t is

$$\underline{X}(t) = [X_1(t), \ldots, X_n(t)]$$

and, for $i=1,\ldots,r$, the performance state indicator vector for subsystem i at time t is

$$\underline{Y}^{(i)}(t) = [Y_1^{(i)}(t), \ldots, Y_m^{(i)}(t)].$$

A vector of objective indicator variables $\underline{J} = [J_1, \ldots, J_r]$ is defined by

$$J_i = \begin{cases} 1 & \text{if objective } i \text{ is successful,} \\ 0 & \text{otherwise.} \end{cases}$$

The O R phase is assumed to have a semi-coherent structure function ψ_0, phase F_{0j}, $j=1,\ldots,a$, to have a coherent structure function ψ_j, and phase F_{ij}, $i=1,\ldots,r$; $j=1,\ldots,c$, to have a coherent structure function ϕ_j. For notational convenience, let $X_k(t_{ij}) = X_{kij}$, $Y_k^{(i)}(t_{ij}) = Y_{kij}$, $\underline{X}(t_{ij}) = \underline{X}_{ij}$, and $\underline{Y}^{(i)}(t_{ij}) = \underline{Y}_{ij}$. Then the probability of successfully completing objective i, $i=1,\ldots,r$, is given by

$$\text{(3.2.1)} \qquad p_i = P\{\prod_{j=0}^{a}\psi_j(\underline{X}_{0j}) \prod_{j=1}^{b}\phi_j(\underline{X}_{ij}, \underline{Y}_{ij}) \prod_{j=b+1}^{c}\phi_j(\underline{Y}_{ij}) = 1\}.$$

Thus, for $i=1,\ldots, r$,

$$\text{(3.2.2)} \qquad J_i \overset{st}{=} \prod_{j=0}^{a}\psi_j(\underline{X}_{0j}) \prod_{j=1}^{b}\phi_j(\underline{X}_{ij}, \underline{Y}_{ij}) \prod_{j=b+1}^{c}\phi_j(\underline{Y}_{ij}) .$$

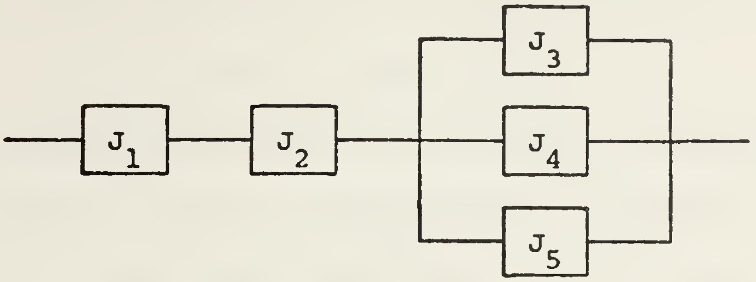
In Chapter 2, the system was required to function satisfactorily in all active phases in order for the mission to be successful. Thus, after each phase structure had been transformed, the resulting structures were connected in series to form the equivalent system. It seems natural, then, to consider a generalization of this procedure which permits the transformed phase structures to be connected in other configurations. Such an arrangement is appropriate in the case of the multi-objective mission.

Clearly, the transformed structures for the phases associated with any single objective should still be connected in series; however only when mission success is defined as successful completion of all objectives should the phase structures associated with one objective be connected in series with those for all other objectives. A link between mission success and success in each of the objectives is required so that the method of connecting transformed phase structures can be prescribed. Accordingly, it is assumed that mission success is completely determined by the outcomes on the r objectives and that there is a binary function η of the binary random variables $J_1, \ldots, J_r$ defined by

$$\eta[J_1,\ldots,J_r] = \begin{cases} 1 & \text{if the mission is successful,} \\ 0 & \text{otherwise.} \end{cases}$$

It is further assumed that this *mission success structure function* is coherent. Then mission success can be represented pictorially as a block diagram in which the "components" are objectives. This is a mild assumption since most measures of mission success will be increasing functions whose ranges can be partitioned into regions defining success and failure which can then be re-scaled to satisfy the requirements for coherence. The following example illustrates this concept.

Example 3.1. A multi-component system performs a mission which has five objectives. The mission is considered a success if objectives 1 and 2 and at least one of the remaining objectives are accomplished. Then the mission success block diagram is



□

In those instances when a mission success structure is not or can not be specified, artificial mission success structures can be used to obtain other quantities of interest. In such cases it may be particularly useful to obtain the probability distribution which governs objective accomplishment. When, as assumed in this chapter, objectives are of the same type and carry the same rank, it is sufficient to determine the distribution of the number of objectives accomplished during the mission. If N is a random variable representing the number of objectives accomplished and η_k is the k out of r structure function, k=1,...,r, then

$$P[N \geq k] = P[\eta_k(\underline{J}) = 1] = E[\eta_k(\underline{J})],\ k=1,\dots,r.$$

Thus the probability distribution of N is given by

(3.2.3)
$$P[N = 0] = 1 - E[\eta_1(\underline{J})],$$
$$P[N = k] = E[\eta_k(\underline{J})] - E[\eta_{k+1}(\underline{J})],\ k=1,\dots,r-1,$$
$$P[N = r] = E[\eta_r(\underline{J})].$$

It may frequently be useful to summarize system performance by specifying the expected number of successes among the r objectives.

This expected value can be found using the k out of r structure functions or directly from the objective success probabilities, since

$$(3.2.4) \qquad E[N] = \sum_1^r P[N \geq k] = \sum_1^r E[\eta_k(\underline{J})] = \sum_1^r E[J_k] \ .$$

Once the mission success structure function to be used is specified, the mathematical description of the mission is complete. If the structure function is based on actual mission requirements (rather than being a device to obtain other quantities) then the probability of mission success is given by

$$(3.2.5) \qquad p = P[\eta(J_1, \ldots, J_r) = 1].$$

Substituting for $J_1, \ldots, J_r$ from Expression 3.2.2 yields the probability of mission success in terms of the component performance indicator variables. Of course the same procedure is appropriate even when the structure function η is just an intermediate device, but the resulting expression is not the probability of mission success.

As was the case in Chapter 2, the expression for the mission success probability is the expected value of a combination of component performance indicator variables which are not mutually independent, since a component's state at one time point is correlated with its state at another. The next section provides the details of the transformation which yields the probability of mission success as the expected value of a function of mutually independent random variables.

3.3 TRANSFORMATION OF THE MULTI-OBJECTIVE MISSION

The procedure for transforming the system with a multi-objective mission into one with an equivalent single-objective, single-phase mission is essentially the same as that presented in Chapter 2. Some of the details are changed because of the modified layout of the mission; however the basic concept remains that of breaking each component of the system into a series of pseudo components. The transformation procedure consists of the following steps:

(a) Replace master system component C_k, $k=1,\ldots,n$, in phase F_{ij} by a series system of pseudo components $C_{k00}, \ldots, C_{kij}$, $i=0$, $j=0,\ldots,a$; $i=1,\ldots,r$, $j=1,\ldots,c$, which perform independently.

(b) Replace component $D_k^{(i)}$ of the i^{th} subsystem in every phase F_{ij} in which it appears by a series configuration of independent pseudo components $D_{ki0}, \ldots, D_{kij}$, $i=1,\ldots,r$, $j=1,\ldots,c$, $k=1,\ldots,m$.

(c) Connect the transformed phase structures for all phases associated with objective i in series to form the equivalent system for objective i.

(d) Connect the equivalent systems for all objectives in the manner prescribed by the mission success structure function.

To see that this transformation procedure yields an equivalent system having a single-objective, single-phase mission and the same mission success probability as the original system, performance state indicator variables for the pseudo components must be introduced. For $k=1,\ldots,n$, let $U_{k00},\ldots, U_{kij}$ be independent performance state indicator variables for pseudo components $C_{k00},\ldots, C_{kij}$ with

$$P[U_{k00} = 1] = P[X_{k00} = 1]$$

and, for $i=0$, $j=1,\ldots,a$ and $i=1,\ldots,r$, $j=1,\ldots,b$,

$$P[U_{kij} = 1] = P[X_{kij} = 1 | X_{ki,j-1} = 1].$$

For $k=1,\ldots,m$ and $i=1,\ldots,r$, let $V_{ki0}, \ldots, V_{kij}$ be independent performance state indicator variables for pseudo components $D_{ki0}, \ldots, D_{kij}$ with

$$P[V_{ki0} = 1] = P[Y_{ki0} = 1] \text{ and}$$

$$P[V_{kij} = 1] = P[Y_{kij} = 1 | Y_{ki,j-1} = 1], \; j=1,\ldots,c.$$

It is immediate that $X_{kij} \overset{st}{=} U_{k00} \cdots U_{kij}$ and that $Y_{kij} \overset{st}{=} V_{ki0} \cdots V_{kij}$. Further, from Theorem 3.1 of Esary-Ziehms [1975], it follows that for $k=1,\ldots,n$,

$$[X_{k00}, X_{k01}, \ldots, X_{krb}] \overset{st}{=} [U_{k00}, U_{k00}U_{k01}, \ldots, U_{k00}U_{k01} \cdots U_{krb}]$$

and, for $i=1,\ldots,r$ and $k=1,\ldots,m$, that

$$[Y_{ki0}, Y_{ki1}, \ldots, Y_{kic}] \overset{st}{=} [V_{ki0}, V_{ki0}V_{ki1}, \ldots, V_{ki0}V_{ki1} \cdots V_{kic}].$$

Then, since the original components perform independently,

$$\text{(3.3.1)} \quad \begin{aligned} &[\underline{X}_{00}, \underline{X}_{01}, \ldots, \underline{X}_{rb}] \overset{st}{=} [\underline{U}_{00}, \underline{U}_{00}\underline{U}_{01}, \ldots, \underline{U}_{00}\underline{U}_{01} \cdots \underline{U}_{rb}] \\ &[\underline{Y}_{i0}, \underline{Y}_{i1}, \ldots, \underline{Y}_{ic}] \overset{st}{=} [\underline{V}_{i0}, \underline{V}_{i0}\underline{V}_{i1}, \ldots, \underline{V}_{i0}\underline{V}_{i1} \cdots \underline{V}_{ic}], \end{aligned}$$

$i=1,\ldots,r$, where

$$\underline{U}_{ij} = [U_{1ij}, U_{2ij}, \ldots, U_{nij}]$$

$$\underline{V}_{ij} = [V_{1ij}, V_{2ij}, \ldots, V_{mij}]$$

$$\underline{U}_{ij}\underline{U}_{k\ell} = [U_{1ij}U_{1k\ell}, U_{2ij}U_{2k\ell}, \ldots, U_{nij}U_{nk\ell}]$$

$$\underline{V}_{ij}\underline{V}_{i\ell} = [V_{1ij}V_{1i\ell}, V_{2ij}V_{2i\ell}, \ldots, V_{mij}V_{mi\ell}] .$$

Further, any vector created by combining the left-hand sides of Equations 3.3.1 is stochastically equal to the same arrangement of the right-hand sides. Substituting into Equation 3.2.2 yields the result:

$$J_i \overset{st}{=} \prod_{j=0}^{a}\psi_j(\underline{U}_{00}\cdots\underline{U}_{0j})\prod_{j=1}^{b}\phi_j(\underline{U}_{00}\cdots\underline{U}_{ij},\underline{V}_{i0}\cdots\underline{V}_{ij}) \times \prod_{j=b+1}^{c}\phi_j(\underline{V}_{i0}\cdots\underline{V}_{ij}),\ i=1,\ldots,r. \tag{3.3.2}$$

For $i=1,\ldots,r$, let $\Phi_{ij} = \phi_j(\underline{U}_{00}\cdots\ \underline{U}_{ij},\ \underline{V}_{i0}\cdots\ \underline{V}_{ij})$, $j=1,\ldots,b$, and $\Phi_{ij} = \phi_j(\underline{V}_{i0}\cdots\ \underline{V}_{ij})$, $j=b+1,\ldots,c$, and let $\Psi_{0j} = \psi_j(\underline{U}_{00}\cdots\ \underline{U}_{0j})$, $j=0,\ldots,a$. Then Equation 3.3.2 can be written more compactly as

$$J_i \overset{st}{=} \prod_{j=0}^{a} \Psi_{0j} \prod_{j=1}^{c} \Phi_{ij}. \tag{3.3.3}$$

Finally, the reliability of the equivalent system for its single-objective, single-phase mission as given by

$$p = P\{\eta(\prod_{j=0}^{a}\Psi_{0j} \prod_{j=1}^{c}\Phi_{1j},\ldots, \prod_{j=0}^{a}\Psi_{0j} \prod_{j=1}^{c}\Phi_{rj}) = 1\} \tag{3.3.4}$$

is equal to the probability of success in the original mission given by Equation 3.2.5.

The major benefit of the transformation which results in the equivalent system whose reliability is given by Equation 3.3.4 is the elimination of dependencies among the performance state indicator variables. Thus the probability of mission success can be obtained as the expected value of sums, products, and differences of independent random variables--a task which is conceptually straightforward. The procedure suffers from the same drawback as the transformation of Chapter 2, however,

since the number of pseudo components in the equivalent system is likely to be quite large. Some techniques for reducing the complexity of the equivalent system are discussed along with approximation methods in the next section.

3.4 SIMPLIFICATIONS AND APPROXIMATIONS

The cut cancellation procedure (appropriately modified) and the component reduction technique, which were discussed in Chapter 2, can also be applied to the multi-objective mission problem. The change in the cut cancellation method amounts to limiting the cancellations to only those minimal cut sets which contain a minimal cut set for a later phase *associated with the same objective*. Thus the step-by-step procedure for the multi-objective mission problem as formulated in this chapter is:

(a) Find the minimal cut sets for each phase associated with objective 1. (The minimal cut sets for the phases of any other objective i are the same with component $D_k^{(1)}$ replaced by $D_k^{(i)}$.)

(b) Remove from the list of minimal cut sets for phase F_{0j}, $j=0,\ldots,a$, each minimal cut set which contains a minimal cut set for phase $F_{0\ell}$, $\ell=j+1,\ldots,a$ or phase F_{1k}, $k=1,\ldots,c$.

(c) Remove from the list of minimal cut sets for phase F_{1j}, $j=1,\ldots,c-1$, each minimal cut set which contains a minimal cut set for phase F_{1k}, $k=j+1,\ldots,c$, and remove the corresponding minimal cut sets from the list for phase $F_{\ell j}$, $\ell=2,\ldots,r$.

(d) Reconstitute the system from the remaining minimal cut sets.

It follows from the proof of Remark 4.2 of Esary-Ziehms [1975] that this cut cancellation procedure does not affect the probability of

mission success.

Some component reductions have been incorporated into the transformation of Section 3.3. A potentially large number of pseudo components has been eliminated by automatically leaving component $C_k^{(i)}$, $k=1,\ldots,m$; $i=1,\ldots,r$, untransformed over the period from time $t=0$ until time t_{i0}, during which the component is irrelevant to system operation. Additional reductions will be possible in most applications. Thus if master system component C_k first becomes relevant to system operation in phase F_{ij} then the series arrangement of pseudo components $C_{k00},\ldots,$ $C_{ki,j-1}$ can be replaced wherever it appears in the equivalent system by a single pseudo component $C_{ki,\vec{j}-1}$ with performance state indicator variable $U_{ki,\vec{j}-1}$, where

$$P[U_{ki,\vec{j}-1} = 1] = P[X_{ki,j-1} = 1] = P[U_{k00}\cdots U_{ki,j-1} = 1].$$

If subsystem i component $D_k^{(i)}$ first becomes relevant in phase F_{ij} then the series configuration of pseudo components $D_{ki0},\ldots,D_{ki,j-1}$ can be replaced wherever it appears in the equivalent system by the pseudo component $D_{ki,\vec{j}-1}$ with performance state indicator variable $V_{ki,\vec{j}-1}$ where

$$P[V_{ki,\vec{j}-1} = 1] = P[Y_{ki,j-1} = 1] = P[V_{ki0}\cdots V_{ki,j-1} = 1].$$

Although component reduction is a worthwhile technique, it cannot always be expected to reduce the number of pseudo components in the equivalent system to a manageable level. The last analytical resort when the number of pseudo components is too large to permit direct calculations is to approximate or bound the mission success probability.

The series organization of transformed phase structures, which was appropriate in Chapter 2 and in the work of Ziehms [1975], led to convenient upper and lower bounds on mission reliability. Unfortunately the general nature of the mission success structure η rules out bounds based on either phase reliabilities or objective success probabilities. It is tempting to try to bound the mission success probability from below by

$$p = E[\eta(J_1,...,J_r)] \geq \eta(EJ_1,...,EJ_r)$$

because this inequality holds when η is a series structure function. This is *not true*, however, since it is well known that the direction of the inequality is reversed when η is a parallel structure function. When η is other than a series or parallel structure function, such an inequality does not usually exist.

Although less convenient, it is still possible to place upper and lower bounds on the mission success probability by finding the minimal cut sets and minimal path sets of the equivalent system as a whole. (A minimal path set is a minimal set of components which by all functioning cause the system to function.) Then the minimal cut lower and minimal path upper bounds due to Esary and Proschan [1963] can be used to bound the reliability of the equivalent system. (See Barlow and Proschan [1975a] for a development of these bounds.)

It should be noted that in order to use Equations 3.2.3 to obtain the distribution of the number of objectives accomplished, Equation 3.3.4 must be computed exactly for each structure function $\eta_1,...,\eta_r$. Any ordering established by the bounds discussed above would be destroyed by the subtraction required in Equations 3.2.3. Such is not the case,

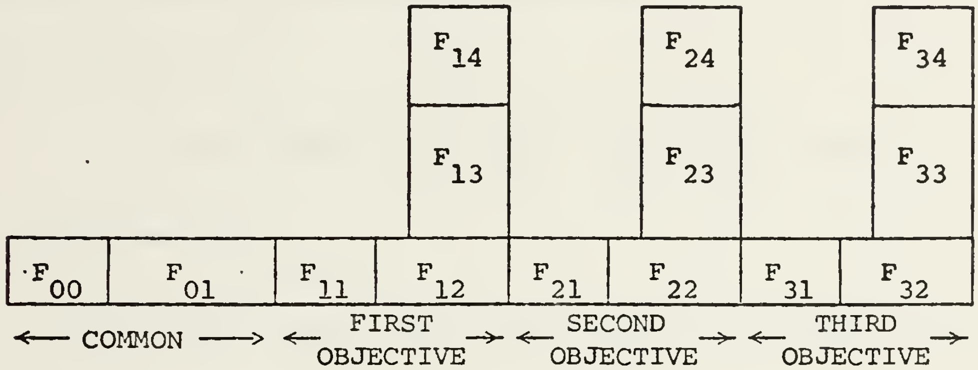
however, when using Equation 3.2.4 to find the expected number of objectives accomplished. In this case the orderings are maintained so that lower bounds on $E[\eta_k(\underline{J})]$ or $E[J_k]$, $k=1,\ldots,r$, yield a lower bound on $E[N]$ and upper bounds on $E[\eta_k(\underline{J})]$ or $E[J_k]$, $k=1,\ldots,r$, yield an upper bound on $E[N]$.

3.5 EXAMPLE

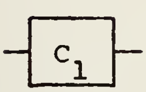
It was stated at the beginning of this chapter that the multi-objective mission problem formulation was motivated by the Navy's Fleet Ballistic Missile system. The SLBM system of Example 2.1, which can be viewed as a hypothetical version of the FBM system, can be extended to provide an illustration of the basic notions of this chapter. Let the submarine of the SLBM system now carry r missiles. Then the system can be viewed as having r objectives--each of which is the destruction of a designated target. The master system would consist of those components which remain aboard the submarine, and the components of each missile would make up one subsystem. Those phases of the mission which take place aboard the submarine would be trunk phases, and branch i phases would commence when the i^{th} missile is launched.

This extended version of the SLBM system, while ideally suited for illustrating the basic elements of the multi-objective mission problem, is somewhat more complex than necessary for the purpose of demonstrating the mechanics of the transformation, cut-cancellation, and component-reduction procedures. The following example introduces a very simple system and then tracks it through the formulation and computation steps discussed in the earlier sections of this chapter.

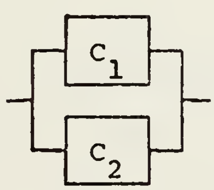
Example 3.2. A system with a three-objective mission has eight components. Components C_1 and C_2 make up the master system, and components $D_1^{(i)}$ and $D_2^{(i)}$ make up subsystem i, i=1,2,3. Each objective entails the successful completion of an O R phase and five active phases as shown in the following diagram.



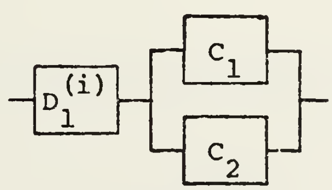
Thus, for this example, n=2, m=2, a=1, b=2, and c=4. The block diagrams for the trunk phases are



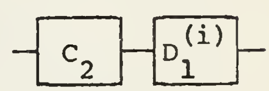
F_{00}



F_{01}

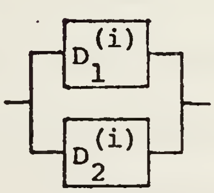


F_{i1}, i=1,2,3

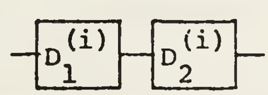


F_{i2}, i=1,2,3

and those for the branch phases are

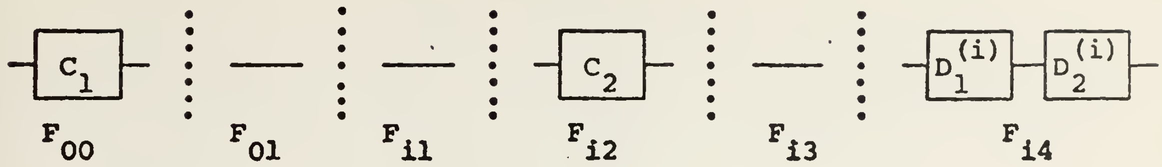


F_{i3}, i=1,2,3



F_{i4}, i=1,2,3

After cut cancellation, the phases of objective i become



After transformation and incorporation of component reduction, the equivalent structures for the objectives are:

for objective 1,

$$-C_{100}-C_{21\vec{1}}-C_{212}-D_{11\vec{3}}-D_{114}-D_{21\vec{3}}-D_{214}-$$

for objective 2,

$$-C_{100}-C_{21\vec{1}}-C_{212}-C_{221}-C_{222}-D_{12\vec{3}}-D_{124}-D_{22\vec{3}}-D_{224}-$$

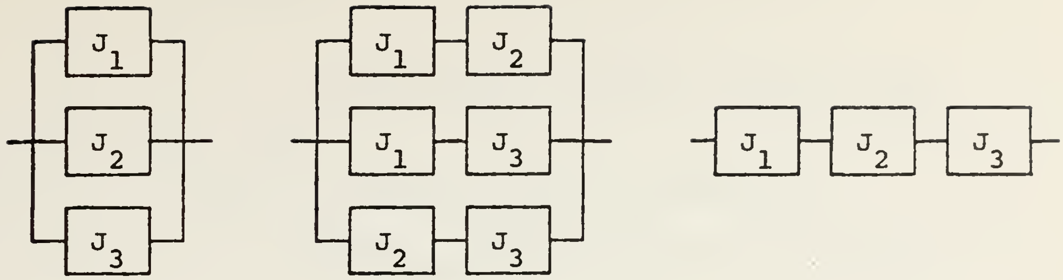
for objective 3,

$$-C_{100}-C_{21\vec{1}}-C_{212}-C_{221}-C_{222}-C_{231}-C_{232}-D_{13\vec{3}}-D_{134}-D_{23\vec{3}}-D_{234}-$$

Thus,

(3.5.1)
$$J_1 \overset{st}{=} U_{100}U_{21\vec{1}}U_{212}V_{11\vec{3}}V_{114}V_{21\vec{3}}V_{214}$$
$$J_2 \overset{st}{=} U_{100}U_{21\vec{1}}U_{212}U_{221}U_{222}V_{12\vec{3}}V_{124}V_{22\vec{3}}V_{224}$$
$$J_3 \overset{st}{=} U_{100}U_{21\vec{1}}U_{212}U_{221}U_{222}U_{231}U_{232}V_{13\vec{3}}V_{134}V_{23\vec{3}}V_{234}$$

Instead of specifying a particular mission success structure for this example, the distribution and expected value of the number of objectives accomplished, N, will be obtained. The structures needed to obtain the distribution are η_1, the 1 out of 3 structure; η_2, the 2 out of 3 structure; and η_3, the 3 out of 3 structure, whose block diagrams are



η_1 η_2 η_3

Computationally,

$$\eta_1(\underline{J}) = J_1 \vee J_2 \vee J_3$$

$$= J_1 + J_2 + J_3 - J_1J_2 - J_1J_3 - J_2J_3 + J_1J_2J_3$$

(3.5.2)
$$\eta_2(\underline{J}) = (J_1J_2) \vee (J_1J_3) \vee (J_2J_3)$$

$$= J_1J_2 + J_1J_3 + J_2J_3 - 2J_1J_2J_3$$

$$\eta_3(\underline{J}) = J_1J_2J_3$$

Then, from Equations 3.2.3, the distribution of N is given by

$$P[N=0] = 1 - E[J_1 + J_2 + J_3 - J_1J_2 - J_1J_3 - J_2J_3 + J_1J_2J_3]$$

$$P[N=1] = E[J_1 + J_2 + J_3 - 2J_1J_2 - 2J_1J_3 - 2J_2J_3 + 3J_1J_2J_3]$$

(3.5.3)
$$P[N=2] = E[J_1J_2 + J_1J_3 + J_2J_3 - 3J_1J_2J_3]$$

$$P[N=3] = E[J_1J_2J_3]$$

Expressions for the product terms in Equations 3.5.2 and 3.5.3 can be obtained by multiplying the right-hand sides of Equations 3.5.1 and using idempotent cancellation. Thus,

$$J_1J_2 \overset{st}{=} [U_{100}U_{21\vec{1}}\ U_{212}U_{221}U_{222}] \times$$

$$[V_{11\vec{3}}\ V_{114}V_{21\vec{3}}\ V_{214}V_{12\vec{3}}\ V_{124}V_{22\vec{3}}\ V_{224}]$$

$$J_1J_3 \overset{st}{=} [U_{100}U_{21\vec{1}}\ U_{212}U_{221}U_{222}U_{231}U_{232}] \times$$
$$[V_{11\vec{3}}\ V_{114}V_{21\vec{3}}\ V_{214}V_{13\vec{3}}\ V_{134}V_{23\vec{3}}\ V_{234}]$$

(3.5.4)
$$J_2J_3 \overset{st}{=} [U_{100}U_{21\vec{1}}\ U_{212}U_{221}U_{222}U_{231}U_{232}] \times$$
$$[V_{12\vec{3}}\ V_{124}V_{22\vec{3}}\ V_{224}V_{13\vec{3}}\ V_{134}V_{23\vec{3}}\ V_{234}]$$

$$J_1J_2J_3 \overset{st}{=} [U_{100}U_{21\vec{1}}\ U_{212}U_{221}U_{222}U_{231}U_{232}V_{11\vec{3}}\ V_{114}] \times$$
$$[V_{21\vec{3}}\ V_{214}V_{12\vec{3}}\ V_{124}V_{22\vec{3}}\ V_{224}V_{13\vec{3}}\ V_{134}V_{23\vec{3}}\ V_{234}]$$

Let the availability of component C_k at time t_{00} be α_{k00}, its conditional reliability for phase F_{ij}, i=0, j=1,...,a; i=1,...,r, j=1,...,b, be π_{kij}, and its unconditional reliability through the end of the same phase be Π_{kij}, where

(3.5.5)
$$\Pi_{kij} = P[X_{kij}=1] = P[X_{kij}=1|X_{ki,j-1}=1] \cdots P[X_{k00}=1]$$
$$= \pi_{kij}\pi_{ki,j-1} \cdots \pi_{k01}\alpha_{k00}$$

Similarly, let the availability of component $D_k^{(i)}$, i=1,...,r, at time t_{i0} be β_{ki0}, its conditional reliability for phase F_{ij}, j=1,...,c, be ω_{kij}, and its unconditional reliability through the end of that phase be Ω_{kij}, where

(3.5.6)
$$\Omega_{kij} = P[Y_{kij}=1] = P[Y_{kij}=1|Y_{ki,j-1}=1] \cdots P[Y_{ki0}=1]$$
$$= \omega_{kij}\omega_{ki,j-1} \cdots \omega_{ki1}\beta_{ki0}$$

Then taking expected values in Equations 3.5.1 and 3.5.4 yields

$$E(J_1) = \alpha_{100}\Pi_{212}\Omega_{114}\Omega_{214}$$

$$E(J_2) = \alpha_{100}\Pi_{222}\Omega_{114}\Omega_{214}\Omega_{124}\Omega_{224}$$

$$E(J_3) = \alpha_{100}\Pi_{232}\Omega_{134}\Omega_{234}$$

(3.5.7) $$E(J_1J_2) = \alpha_{100}\Pi_{222}\Omega_{114}\Omega_{214}\Omega_{124}\Omega_{224}$$

$$E(J_1J_3) = \alpha_{100}\Pi_{232}\Omega_{114}\Omega_{214}\Omega_{134}\Omega_{234}$$

$$E(J_2J_3) = \alpha_{100}\Pi_{232}\Omega_{124}\Omega_{224}\Omega_{134}\Omega_{234}$$

$$E(J_1J_2J_3) = \alpha_{100}\Pi_{232}\Omega_{114}\Omega_{214}\Omega_{124}\Omega_{224}\Omega_{134}\Omega_{234}$$

If the component availabilities and conditional phase reliabilities are as given in Figure 3.2, then the unconditional reliabilities shown in Figure 3.3 can be calculated using Equations 3.5.5 and 3.5.6. Then, from Equations 3.5.7, $E(J_1) = .403$, $E(J_2) = .344$, $E(J_3) = .295$, $E(J_1J_2) = .184$, $E(J_1J_3) = .157$, $E(J_2J_3) = .157$, and $E(J_1J_2J_3) = .084$. Substitution into Equations 3.5.3 yields the distribution

$$P[N = 0] = .372$$

$$P[N = 1] = .298$$

$$P[N = 2] = .246$$

$$P[N = 3] = .084$$

Finally, from Equation 3.2.4, the expected number of objectives accomplished during the mission is $E[N] = 1.042$. □

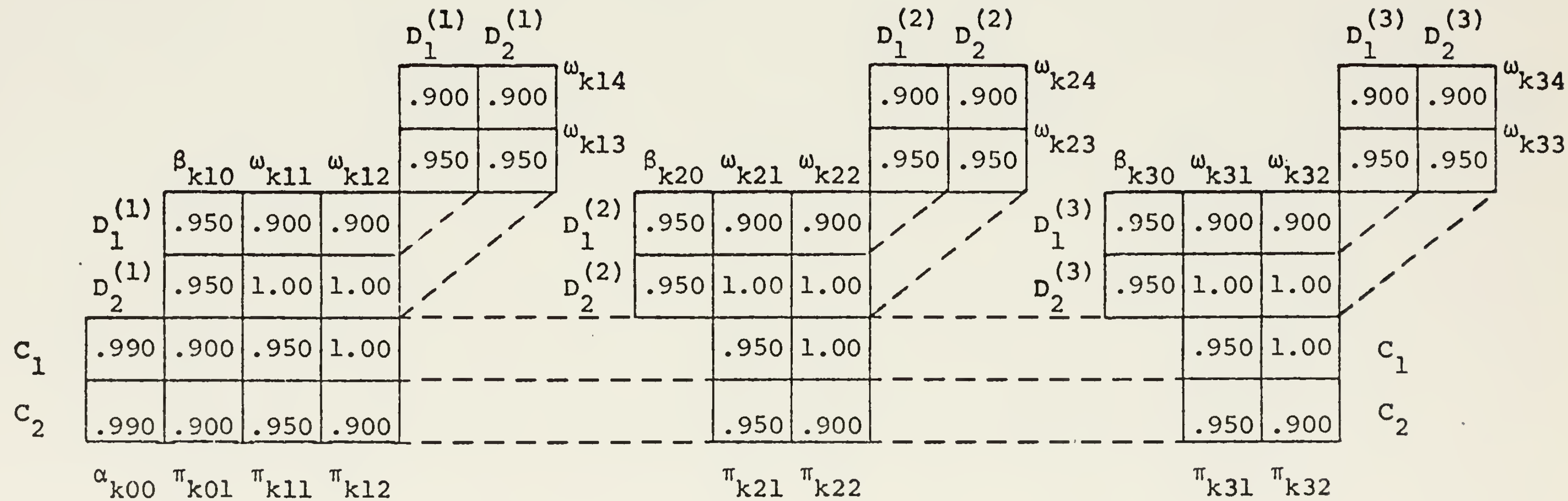


Figure 3.2. Component availabilities and conditional phase reliabilities for Example 3.2.

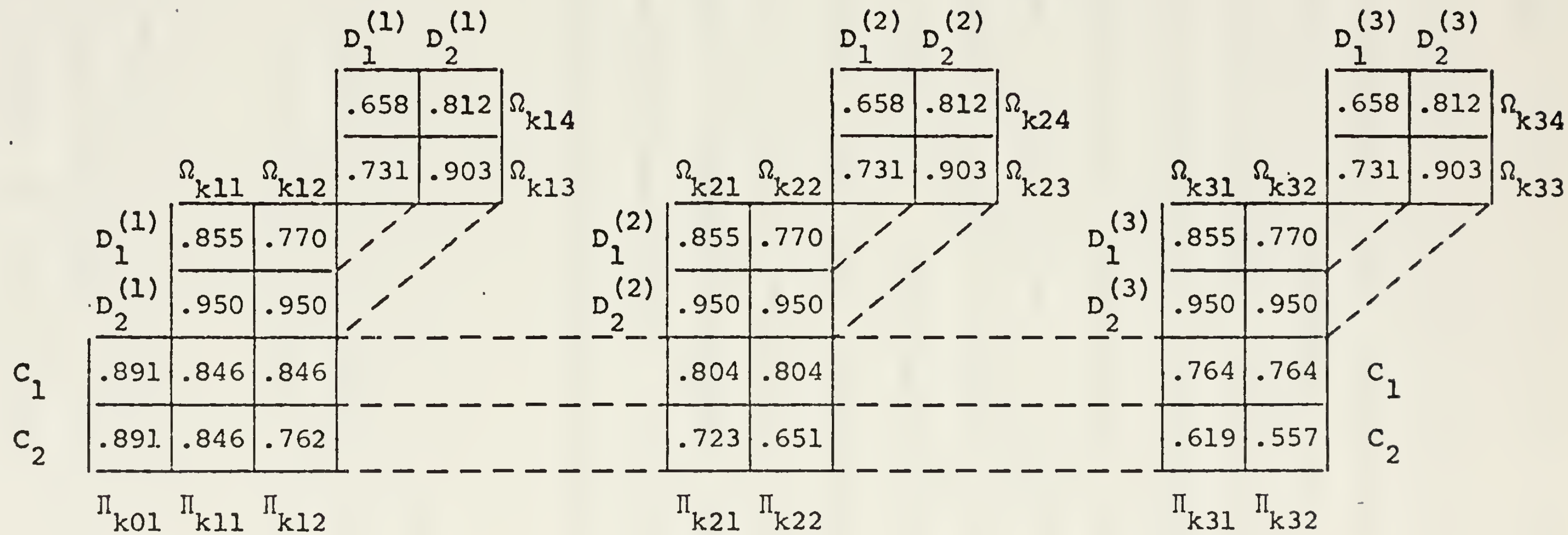


Figure 3.3. Component unconditional phase reliabilities for Example 3.2.

4. PHASED MISSIONS FOR MAINTAINED SYSTEMS

In the preceding chapters it has been assumed that all system maintenance actions ceased at the end of the operational readiness phase and that components were not repairable during any of the active phases. If, in a particular application, some system maintenance is performed during active phases, then use of the models presented in Chapters 2 and 3 leads to results which, though conservative, are adequate for most purposes. Nevertheless, it is conceivable that some circumstances may justify an attempt to account for maintenance efforts in a model of system performance. Esary and Proschan [1970], hereafter referred to as E-P [1970], showed that the minimal cut lower bound on (single phase) mission reliability, derived originally for coherent, non-maintained systems of independent components in Esary-Proschan [1963], also holds under some maintenance policies. The results in E-P [1970] are extended to the phased mission problem in this chapter.

4.1 BACKGROUND

The development in this chapter and the one which follows uses several concepts and results from the literature which were not required in previous chapters. This section provides a brief review and summary of these results.

Basic to the investigation of the reliability of maintained systems for multi-phase missions is the concept of association introduced by Esary-Proschan-Walkup [1967], who present proofs of properties P_1 - P_6 below. Random variables $T_1,\ldots,T_n$ are *associated* if $\mathrm{Cov}[f(\underline{T}),g(\underline{T})] \geq 0$ for every pair of increasing (non-decreasing) functions f,g for which

the covariance exists. Associated random variables have the following basic properties:

(P_1) Any subset of a set of associated random variables is itself a set of associated random variables.

(P_2) If two sets of associated random variables are independent of one another, then the union of the two sets is a set of associated random variables.

(P_3) Any set containing a single random variable is a set of associated random variables.

(P_4) Increasing functions of associated random variables are associated.

(P_5) The limit in distribution of a sequence of sets of associated random variables is a set of associated random variables.

(P_6) If $X_1, \ldots, X_n$ are associated binary random variables, then

$$P\{\prod_{i=1}^{n} X_i = 1\} \geq \prod_{i=1}^{n} P[X_i = 1].$$

Association is a very general kind of positive dependence which includes the boundary cases of independence (by properties P_3 and P_2) and that of positive total dependence (by properties P_3 and P_4). (A set of random variables is *positively totally dependent* if each element is an increasing function of the same random variable. [E-P 1970])

The concept of association can also be applied to stochastic processes, and E-P [1970] say that the joint performance process $\{\underline{X}(t), t \in \tau\}$ of a set of components is *associated in time* if for each set of times $\{t_1, \ldots, t_k\} \subset \tau$ the binary random variables

$$X_1(t_1), \ldots, X_1(t_k), \ldots, X_n(t_1), \ldots, X_n(t_k)$$

are associated.

Several of the results contained in the remarks and examples of E-P [1970] will be needed in the course of investigating the multi-phase-mission reliability of maintained systems. These results are summarized below for easy reference.

(R_1) The performance process of a component with a life is associated in time.

(R_2) The joint performance process of a set of components with lives is associated in time if and only if the lifetimes of the devices are associated.

(R_3) The joint performance process formed from independent performance processes which are associated in time is itself associated in time.

(R_4) The joint performance process of a set of coherent systems is associated in time if the joint performance process of all their components is associated in time.

In an unpublished paper, Esary and Proschan [1968] defined a stochastic process $\{X(t),\ t \in \tau\}$ to be *stochastically increasing in time* if, for all $\{t_1 < \cdots < t_k\} \subset \tau$,

$$P[X(t_2) > x_2 | X(t_1) = x_1] \text{ is increasing in } x_1,$$

$$P[X(t_3) > x_3 | X(t_1) = x_1,\ X(t_2) = x_2] \text{ is increasing in } x_1, x_2,$$

$$\vdots$$

$$P[X(t_k) > x_k | X(t_1)=x_1,\ldots,X(t_{k-1})=x_{k-1}] \text{ is increasing in } x_1,\ldots, x_{k-1}.$$

They showed the following results:

(R_5) A stochastic process which is stochastically increasing in time is associated in time.

(R_6) Let $\{X(t), t \in \tau\}$ be a Markov performance process. If for all $t_1 < t_2 \in \tau$, $P[X(t_2)=1|X(t_1)=1] \geq P[X(t_2)=1|X(t_1)=0]$, then the process is stochastically increasing in time.

A proof of R_5 may be found in Barlow and Proschan [1975a] (Theorem 4.7, page 146). The result R_6 is just a special case of the definition of a process which is stochastically increasing in time.

Armed with this background, it is now possible to proceed with the development of a model for the performance of a maintained system during a phased mission.

4.2 PHASED MISSIONS WITH TIME-ASSOCIATED JOINT COMPONENT PERFORMANCE PROCESSES

The phased mission problem to be considered in this chapter is similar to that of Chapter 2 in structure. A system of n components, $C_1, \ldots, C_n$, performs a single-objective mission which consists of m phases. For simplicity, the mission phases are all assumed to be active--that is, there is no O R phase structure function for the mission. It is assumed that there is a coherent structure function ϕ_j relating system performance to component performance during phase j, which begins at time t_{j-1} and ends at time t_j, j=1,...,m. The mission reliability of the system is given by

(4.2.1)
$$p = P\{\phi[\underline{X}(s)] = 1, 0 < s \leq t_m\} ,$$

where:

(4.2.2)
$$\phi = \begin{cases} \phi_1, & 0 = t_0 < s \leq t_1, \\ \phi_2, & t_1 < s \leq t_2, \\ \vdots & \\ \phi_m, & t_{m-1} < s \leq t_m. \end{cases}$$

In previous chapters, the absence of maintenance during the active phases of the mission led to non-increasing sample paths for the component performance processes, and this in turn resulted in statements, such as Equation 2.3.2, relating mission success to the state of the system at the end of each phase. With the incorporation of maintenance into the model of system performance, the component performance processes are no longer non-increasing, and mission success is a function of the state of the system at every time point during the mission, as indicated in Equation 4.2.1.

Attention is restricted in this and subsequent chapters to maintenance policies which result in time-associated joint component performance processes. The following lemma states that such joint component performance processes lead to time-associated system performance processes.

<u>Lemma 4.1</u>. If the joint component performance process $\{\underline{X}(s),\ 0 < s \leq t_m\}$ is associated in time, then the system performance process $\{\phi[\underline{X}(s)],\ 0 < s \leq t_m\}$, with ϕ defined by Equation 4.2.2, is associated in time.

Proof. Since the process $\{\underline{X}(s),\ 0 < s \leq t_m\}$ is associated in time and the structure functions $\phi_1, \ldots, \phi_m$ are coherent, the stochastic process

$$\{\phi_1[\underline{X}(s)],\ \phi_2[\underline{X}(s)], \ldots, \phi_m[\underline{X}(s)],\ 0 < s \leq t_m\}$$

is associated in time by R_4. Thus for every set of times $\{s_1, \ldots, s_k\}$ such that $0 < s_j \leq t_m$, $j=1,\ldots,k$, the random variables in the array

$$\begin{array}{ccc} \phi_1[\underline{X}(s_1)], & \ldots, & \phi_m[\underline{X}(s_1)] \\ \vdots & & \vdots \\ \phi_1[\underline{X}(s_k)], & \ldots, & \phi_m[\underline{X}(s_k)] \end{array}$$

are associated, and by P_1 so are the random variables in the set

$$\{\phi[\underline{X}(s_1)], \ldots, \phi[\underline{X}(s_k)]\} .$$

Thus the stochastic process $\{\phi[\underline{X}(s), 0 < s \le t_m\}$ is associated in time. □

Theorem 4.2, below, contains one of the main results of this chapter. It states that the procedure of calculating the reliability of the system for each phase separately and then multiplying the results together gives a conservative bound on mission reliability when the joint component performance process is associated in time. It is a generalization of the lower bound given in Theorem 5.1 of Ziehms [1974] for non-maintained systems of independent components.

Theorem 4.2. If the joint component performance process $\{\underline{X}(s), 0 < s \le t_m\}$ is associated in time, then the mission reliability as given by Equation 4.2.1 is bounded below by the product of the phase reliabilities, i.e.,

$$(4.2.3) \qquad p \ge \prod_{j=1}^{m} P\{\phi_j[\underline{X}(s_j)] = 1, s_j \in (t_{j-1}, t_j]\} .$$

Proof. Let $S^1, \ldots, S^m$ be countable subsets of $(0,t_1], \ldots, (t_{m-1}, t_m]$ which are dense in their respective intervals. Since the sample functions of the stochastic process $\{\phi_j[\underline{X}(s_j)], s_j \in (t_{j-1}, t_j]\}$ are continuous from the right, it follows that, for $j=1,\ldots,m$,

$$P\{\phi_j[\underline{X}(s_j)] = 1, s_j \in (t_{j-1}, t_j]\} = P\{\phi_j[\underline{X}(s_j)] = 1, s_j \in S^j\}$$

and

$$p = P\{\phi_1[\underline{X}(s_1)] = 1, s_1 \in S^1, \ldots, \phi_m[\underline{X}(s_m)] = 1, s_m \in S^m\} .$$

Let $S_k^j = \{s_{j1}, \ldots, s_{jk}\}$, $k=1,2,\ldots$; $j=1,\ldots,m$, be subsets of $S^1, \ldots, S^m$ such that $S_k^j \subset S_{k+1}^j$ and $S_1^j \cup S_2^j \cup \cdots = S^j$. By monotone convergence,

$$P\{\phi_j[\underline{X}(s_j)] = 1, s_j \in S_k^j\} \downarrow P\{\phi_j[\underline{X}(s_j)] = 1, s_j \in S^j\},$$

$j=1,\ldots, m$, and

$$P\{\phi_1[\underline{X}(s_1)] = 1,\ s_1 \in S_k^1, \ldots, \phi_m[\underline{X}(s_m)] = 1,\ s_m \in S_k^m\} \downarrow$$

$$P\{\phi_1[\underline{X}(s_1)] = 1,\ s_1 \in S^1, \ldots, \phi_m[\underline{X}(s_m)] = 1,\ s_m \in S^m\}\ .$$

It will be convenient to write

$$P\{\phi_1[\underline{X}(s_1)] = 1,\ s_1 \in S_k^1, \ldots, \phi_m[\underline{X}(s_m)] = 1,\ s_m \in S_k^m\}$$

as

$$P\Big\{\prod_{\ell=1}^{k} \phi_1[\underline{X}(s_{1\ell})] = 1, \ldots, \prod_{\ell=1}^{k} \phi_m[\underline{X}(s_{m\ell})] = 1\Big\}$$

or more compactly as

$$P\Big\{\prod_{j=1}^{m} \prod_{\ell=1}^{k} \phi_j[\underline{X}(s_{j\ell})] = 1\Big\}\ .$$

Since the joint component performance process is associated in time, the system performance process is also associated in time by Lemma 4.1. Thus $\{\phi_j[\underline{X}(s_{j\ell})],\ \ell=1,\ldots,k;\ j=1,\ldots,m\}$ is a set of associated random variables, and by P_4 the random variables $\prod_{\ell=1}^{k} \phi_j[\underline{X}(s_{j\ell})]$, $j=1,\ldots,m$, are also associated. Then from P_6 it follows that

$$P\Big\{\prod_{j=1}^{m} \prod_{\ell=1}^{k} \phi_j[\underline{X}(s_{j\ell})] = 1\Big\} \geq \prod_{j=1}^{m} P\Big\{\prod_{\ell=1}^{k} \phi_j[\underline{X}(s_{j\ell})] = 1\Big\}$$

$$= \prod_{j=1}^{m} P\Big\{\phi_j[\underline{X}(s_j)] = 1,\ s_j \in S_k^j\Big\}\ .$$

Since this holds for all values of k, it also holds in the limit as k approaches infinity.

Thus,

$$P\{\phi_1[\underline{X}(s_1)] = 1,\ s_1 \varepsilon S^1, \ldots, \phi_m[\underline{X}(s_m)] = 1,\ s_m \varepsilon S^m\} \geq$$

$$\prod_{j=1}^{m} P\{\phi_j[\underline{X}(s_j)] = 1,\ s_j \varepsilon S^j\}.$$

It follows directly that

$$p \geq \prod_{j=1}^{m} P\{\phi_j[\underline{X}(s_j)] = 1,\ s_j \varepsilon (t_{j-1}, t_j]\}. \qquad \Box$$

The following result is a restatement of Theorem 5.2 of E-P [1970]. It relates the reliability of a coherent system with a time-associated joint component performance process to the reliabilities of its minimal cut structures.

Theorem 4.3. If the joint performance process of the components of a coherent system with structure function ϕ and minimal cut (parallel) structure functions $\kappa_1, \ldots, \kappa_c$ is associated in time, then

$$(4.2.4) \qquad P\{\phi[\underline{X}(t)] = 1,\ t \varepsilon \tau\} \geq \prod_{i=1}^{c} P\{\kappa_i[\underline{X}(t)] = 1,\ t \varepsilon \tau\}$$

Theorem 4.3 can be used in conjunction with Theorem 4.2 to yield a lower bound on multi-phase-mission reliability which is based on the reliability of the phase minimal cut structures, as stated in the following theorem.

Theorem 4.4. If in a multi-phase mission the joint component performance process $\{\underline{X}(s),\ 0 < s \leq t_m\}$ is associated in time and $\kappa_{j1}, \ldots, \kappa_{jc_j}$ are the minimal cut (parallel) structure functions for phase j, $j=1,\ldots,m$, then the mission reliability given by Equation 4.2.1 is bounded below by the product of the phase minimal cut reliabilities, i.e.,

$$(4.2.5) \qquad p \geq \prod_{j=1}^{m} \prod_{i=1}^{c_j} P\{\kappa_{ji}[\underline{X}(s_j)] = 1,\ s_j \varepsilon (t_{j-1}, t_j]\}.$$

Proof. From Equation 4.2.3,

$$p \geq \prod_{j=1}^{m} P\{\phi_j[\underline{X}(s_j)] = 1,\ s_j \in (t_{j-1}, t_j]\}\ .$$

It follows from Equation 4.2.4 that, for $j=1,\ldots,m$,

$$P\{\phi_j[\underline{X}(s_j)]=1,\ s_j \in (t_{j-1},t_j]\} \geq \prod_{i=1}^{c_j} P\{\kappa_{ji}[\underline{X}(s_j)]=1,\ s_j \in (t_{j-1},t_j]\}.$$

Thus,

$$p \geq \prod_{j=1}^{m} \prod_{i=1}^{c_j} P\{\kappa_{ji}[\underline{X}(s_j)] = 1,\ s_j \in (t_{j-1}, t_j]\} \qquad \square$$

Theorems 4.2 and 4.4 are significant results in that they permit the use of a "divide and conquer" approach in the analysis of multiphase missions for maintained systems. There is little hope of obtaining the exact mission reliability analytically in other than the most trivial applications. Depending on the size and complexity of the problem, simulation offers a possible alternative means of estimating the mission reliability directly.

Theorem 4.2 allows the original problem to be broken down into m separate problems--calculation of the system reliability within each of the phases; however this gain in simplicity is obtained at the expense of an exact solution. Unfortunately, the remaining problems are, in general, still too complex to be solved analytically. Here again, computer simulation methods are an alternative.

Theorem 4.4 allows an even finer breakdown of the original problem. It provides for bounding mission reliability by the product of the reliabilities of parallel structures which typically will be small to moderate in size. The problem of calculating the reliability of a parallel structure of maintained components, though still complex, is much more

amenable to analytic methods. Bounds on the reliability of such parallel structures are developed in Chapter 5.

The circumstances which lead to time-associated performance processes have yet to be discussed. A system of independent components which are not maintained has a time-associated joint components performance process by R_1 and R_3. The time association of Markov component performance processes is explored in the next section.

4.3 TIME ASSOCIATION OF MARKOV PERFORMANCE PROCESSES

The conditions leading to time-associated Markov performance processes in a single-phase mission context were discussed in E-P [1970]. It remains to be verified that time association is not destroyed at phase boundaries.

The following lemma, which is needed in the proof of the main theorem of this section, uses the notion of stochastically increasing random variables. Esary and Proschan [1968] defined the random variable T to be *stochastically increasing* (↑st) in the random variable S if $P[T > t|S = s]$ is non-decreasing in s for each fixed t.

<u>Lemma 4.5</u>. If the sequence of binary random variables $X_1, \ldots, X_n$ is Markov and if X_j ↑st in X_{j-1}, $j=2,\ldots,n$, then X_j ↑st in X_k, $k < j$, $j=2,\ldots,n$.

Proof. The lemma is vacuously true if n=1 or n=2. Consider the case when n=3. By assumption, $P[X_2=1|X_1=x_1]$ is increasing in x_1 and $P[X_3=1|X_2=x_2]$ is increasing in x_2. It must be shown that $P[X_3=1|X_1=x_1]$ is increasing in x_1. By the law of total probability,

$$P[X_3=1|X_1=1] = P[X_3=1|X_2=1]P[X_2=1|X_1=1] + P[X_3=1|X_2=0]P[X_2=0|X_1=1].$$

Similarly,

$$P[X_3=1|X_1=0] = P[X_3=1|X_2=1]P[X_2=1|X_1=0] + P[X_3=1|X_2=0]P[X_2=0|X_1=0].$$

Thus,

$$P[X_3=1|X_1=1] - P[X_3=1|X_1=0]$$

$$= P[X_3=1|X_2=1]\{P[X_2=1|X_1=1] - P[X_2=1|X_1=0]\}$$

$$+ P[X_3=1|X_2=0]\{P[X_2=0|X_1=1] - P[X_2=0|X_1=0]\}$$

$$\geq P[X_3=1|X_2=0]\{P[X_2=1|X_1=1] + P[X_2=0|X_1=1]$$

$$- P[X_2=1|X_1=0] - P[X_2=0|X_1=0]\}$$

$$= 0$$

Now assume the lemma is true for n=m. Then X_m ↑st in X_1. By hypothesis, X_{m+1} ↑st in X_m, so that, from the case when n=3, X_{m+1} ↑st in X_1. Thus the lemma holds for all $n \geq 1$. □

The following theorem establishes that if a component performance process is Markov and stochastically increasing in time within each phase, then the resultant association is not disrupted at phase boundaries, and the component performance process is associated in time throughout the mission.

Theorem 4.6. If the Markov performance processes $\{X(s),\ t_{j-1} \leq s \leq t_j\}$, $j=1,\ldots,m$, are each stochastically increasing in time, then the combined stochastic process $\{X(s),\ t_0 \leq s \leq t_m\}$ is stochastically increasing in time and thus associated in time.

Proof. Let $t_0 \leq s_1 < s_2 \leq t_m$. If $s_1, s_2 \in [t_{j-1}, t_j]$, for some j, then $X(s_2)$ ↑st in $X(s_1)$ since $\{X(s),\ t_{j-1} \leq s \leq t_j\}$ is stochastically increasing in time by assumption. If $s_1 \in [t_{j-1}, t_j]$ and $s_2 \in [t_{\ell-1}, t_\ell]$, $\ell > j$, then by assumption, $X(s_2)$ ↑st in $X(t_{\ell-1})$, $X(t_{\ell-1})$ ↑st in $X(t_{\ell-2})$, ..., $X(t_{j+1})$ ↑st in $X(t_j)$, and $X(t_j)$ ↑st in $X(s_1)$. Thus the sequence of random variables $X(s_1), X(t_j), \ldots, X(t_{\ell-1}), X(s_2)$ satisfies the conditions

of Lemma 4.5, and hence $X(s_2) \uparrow st$ in $X(s_1)$. Then the stochastic process $\{X(s), t_0 \le s \le t_m\}$ is stochastically increasing in time by R_6 and is associated in time by R_5. □

E-P [1970] showed in Example 4.7 that if a component has a performance process that is an alternating failure-repair process in which the time to failure and the repair time are exponentially distributed, then the performance process is associated in time. Example 4.1 below demonstrates a similar result for multi-phase missions.

<u>Example 4.1</u>. Let $\{X_k(s), t_{j-1} \le s \le t_j\}$ be the performance process of component C_k, $k=1,\ldots,n$, in phase j of a multi-phase mission. Let the process be the exponential-exponential process discussed in Section 2.4 in which the time to failure is exponentially distributed with rate parameter λ_{kj} and the time to repair is exponentially distributed with rate parameter μ_{kj}. It follows from Equations 2.4.1 and 2.4.2 that, for $t_{j-1} \le s \le t \le t_j$; $j=1,\ldots,m$; $k=1,\ldots,n$,

$$P[X_k(t)=1|X_k(s)=1] = (\lambda_{kj}+\mu_{kj})^{-1}[\mu_{kj}+\lambda_{kj}e^{-(\lambda_{kj}+\mu_{kj})(t-s)}]$$

(4.3.1)

$$P[X_k(t)=1|X_k(s)=0] = \mu_{kj}(\lambda_{kj}+\mu_{kj})^{-1}[1-e^{-(\lambda_{kj}+\mu_{kj})(t-s)}]$$

Then,

$$P[X_k(t)=1|X_k(s)=1] - P[X_k(t)=1|X_k(s)=0]$$

$$= e^{-(\lambda_{kj}+ \mu_{kj})(t-s)} \ge 0$$

The processes $\{X_k(s), t_{j-1} \le s \le t_j\}$, $j=1,\ldots,m$, are each stochastically increasing in time by R_6. Thus, from Theorem 4.6, the processes $\{X_k(s), t_0 \le s \le t_m\}$, $k=1,\ldots,n$, are each associated in time. If the component performance processes are independent of one another, then the

joint component performance process $\{\underline{X}(s),\ t_0 \leq s \leq t_m\}$ is associated in time by result R_3. $\square$

The reliability of parallel structures built of independent components which have exponential-exponential performance processes is explored in Chapter 5.

5. RELIABILITY BOUNDS FOR A PARALLEL SYSTEM OF INDEPENDENT EXPONENTIAL COMPONENTS

Many authors have studied the reliability of parallel systems in which the component states are governed by the independent alternating failure-repair processes described in Example 4.1. Most recent are the inter-related works of Barlow and Proschan [1975b], Brown [1975], Keilson [1975], and Ross [1974] and that of Vesely [1970]. All of these authors considered systems in which all components were functioning at time t=0. Such an assumption is inappropriate in the context of a multi-phase mission in which the parallel structures to be studied are the phase minimal cut parallel systems. In his Example 1.2, Ziehms [1974] illustrated how this assumption could lead to erroneous results in the case of non-repairable components. Brown [1975] indicated that his approach could also be used in the case of an arbitrary distribution over the initial component states, and much of the following work parallels his development.

5.1 THE EMBEDDED RENEWAL PROCESS

Even though the development of bounds on the reliability of a parallel structure is motivated by the results of Chapter 4, it will be more convenient to proceed in a general framework and then show how to couple the results with those obtained previously. Thus, consider a parallel system of n components which fail and undergo repair independently of one another. Let the time to failure of component C_k be exponentially distributed with mean $(\lambda_k)^{-1}$ and the repair time be similarly distributed with mean $(\mu_k)^{-1}$, and assume that repair effort begins

immediately upon failure. If the initial availability of component C_k is given by

$$\alpha_k = P[X_k(0) = 1],\ k=1,\ldots,n,$$

then the initial system availability is

$$\alpha = 1 - \prod_{k=1}^{n} (1 - \alpha_k).$$

and,, from Equations 2.4.1 and 2.4.2, the availability of component C_k at time t is

$$\alpha_k(t) = \frac{\alpha_k}{\lambda_k + \mu_k}\{\mu_k + \lambda_k e^{-t(\lambda_k + \mu_k)}\}$$

$$+ \frac{(1-\alpha_k)}{\lambda_k + \mu_k}\{\mu_k[1 - e^{-t(\lambda_k + \mu_k)}]\},\ t \geq 0,$$

or,

(5.1.1) $$\alpha_k(t) = \frac{\mu_k}{\lambda_k + \mu_k} + \left(\alpha_k - \frac{\mu_k}{\lambda_k + \mu_k}\right)e^{-t(\lambda_k + \mu_k)},\ t \geq 0.$$

The characteristic of a parallel system which makes it more amenable to analysis than other structures is that it fails only when all of its components are failed. The epochs at which the system fails form a *delayed* or *modified renewal process*. (See, e.g., Cox [1962].) Let the time to the first renewal, T, have distribution function F. Then the reliability of the parallel system for a mission of duration t is given by

(5.1.2) $$p = P[T > t] = 1 - F(t).$$

Since it is possible for the system to fail at time t=0, the distribution F has the form

$$F(t) = (1 - \alpha) + \alpha G(t), \; t \geq 0,$$

where $G(t)$ is a proper distribution function with $G(0) = 0$.

Let $M(t)$ be the renewal function for the delayed renewal process, defined as the expected number of renewals in the interval $[0,t]$, i.e.,

$$M(t) = E[N(t)].$$

Note that any renewal at time $t=0$ is counted in $N(t)$. The renewal function will be exploited to obtain a bound on the unknown distribution function F in Section 5.2. It follows from the definition of the renewal function and the form of $F(t)$ that

$$\begin{aligned} M(t) &= M(0) + \int_0^t M'(u)du \\ &= (1 - \alpha) + \int_0^t M'(u)du \end{aligned}$$

The physical interpretation of the *renewal density*, $M'(t)$, is that $M'(t)\Delta t$ represents the limiting probability of a renewal in the small interval $(t, t+\Delta t)$. (See Cox [1962], p. 26.)

The embedded delayed renewal process experiences a renewal in the interval $(t, t+\Delta t)$ if and only if the parallel system fails in the interval. The probability of a system failure in $(t, t+\Delta t)$ is the probability that all components but one are down at time t, the surviving component fails in $(t, t+\Delta t)$, and no failed components are repaired in the interval plus the probability of a series of other events which is small with respect to the length of the interval, Δt. That is,

$$M'(t)\Delta t = P\{\text{All components but one down at } t\text{, no failed components repaired in } (t, t+\Delta t)\text{, and surviving component fails in } (t, t+\Delta t)\} + o(\Delta t)$$

where $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0.$

Conditioning on the surviving component this becomes

$$M'(t)\Delta t = \sum_{i=1}^{n} P[\text{System failure in } (t, t+\Delta t) | C_i \text{ is lone survivor at } t] \, P[C_i \text{ is lone survivor at } t] + o(\Delta t)$$

Thus,

$$M'(t)\Delta t = \sum_{i=1}^{n} \lambda_i \Delta t \left(\prod_{j \neq i} [1-\mu_j \Delta t]\right) \alpha_i(t) \prod_{j \neq i} [1-\alpha_j(t)] + o(\Delta t)$$

or

$$M'(t) = \lim_{\Delta t \to 0} \sum_{i=1}^{n} \frac{\lambda_i \Delta t \; \alpha_i(t) \prod_{j \neq i} [1 - \alpha_j(t)] + o(\Delta t)}{\Delta t}$$

Then,

$$M'(t) = \sum_{i=1}^{n} \lambda_i \alpha_i(t) \prod_{j \neq i} [1 - \alpha_j(t)]$$

and

$$M(t) = 1 - \alpha + \int_0^t \sum_{i=1}^{n} \lambda_i \alpha_i(u) \left\{\prod_{j \neq i} [1 - \alpha_j(u)]\right\} du.$$

Brown [1975] obtains this result for a parallel system with all components functioning at time t=0 using a slightly different approach.

It is easily shown that

$$\sum_{i=1}^{n} \lambda_i \alpha_i(t) \left\{\prod_{j \neq i} [1 - \alpha_j(t)]\right\}$$

$$= \frac{d}{dt} \prod_{j=1}^{n} [1 - \alpha_j(t)] + \left(\sum_{i=1}^{n} \mu_i\right) \prod_{j=1}^{n} [1 - \alpha_j(t)]$$

and consequently,

$$M(t) = \prod_{j=1}^{n} [1 - \alpha_j(t)] + \left(\sum_{i=1}^{n} \mu_i\right) \int_0^t \prod_{j=1}^{n} [1 - \alpha_j(u)] du.$$

Substituting from Equation 5.1.1 and letting

$$a_j = \alpha_j + \frac{\mu_j}{\lambda_j}(\alpha_j - 1)$$

and

$$\frac{1}{b} = \prod_{j=1}^{n} \frac{\lambda_j}{\lambda_j + \mu_j} \quad ,$$

the integral becomes

$$\int_0^t \prod_{j=1}^{n} [1 - \alpha_j(u)]\, du = \int_0^t \prod_{j=1}^{n} \left(1 - a_j e^{-u(\lambda_j + \mu_j)}\right) du$$

$$= \frac{1}{b} \int_0^t \left\{1 + \sum_{r=1}^{n} (-1)^r \sum_{i_1 < \cdots < i_r} \left(\prod_{j=1}^{r} a_{i_j}\right) e^{-u \sum_1^r (\lambda_{i_j} + \mu_{i_j})} \right\} du$$

where the symbol $\sum_{i_1 < \cdots < i_r}$ represents summation over the $\binom{n}{r}$ subsets of size r from the set {1,2,...,n}. After performing the integration it follows directly that

$$\text{(5.1.3)} \quad M(t) = \frac{1}{b} \left\{ \prod_{j=1}^{n} \left(1 - a_j e^{-t(\lambda_j + \mu_j)}\right) + c\left(t - \sum_{r=1}^{n} (-1)^r \sum_{i_1 < \cdots < i_r} \frac{\prod_{j=1}^{r} a_{i_j}}{\sum_{j=1}^{r} (\lambda_{i_j} + \mu_{i_j})} \left[1 - e^{-t \sum_1^r (\lambda_{i_j} + \mu_{i_j})}\right]\right)\right\}$$

where

$$c = \sum_{j=1}^{n} \mu_j \; .$$

5.2 A RELIABILITY BOUND BASED ON THE RENEWAL FUNCTION

The renewal function as given by Equation 5.1.3 provides the basis for the first of several lower bounds on the reliability of a parallel system. By definition, the renewal function is

$$M(t) = 1\ P[N(t) = 1] + 2\ P[N(t) = 2] + 3\ P[N(t) = 3] + \cdots$$

and thus it is bounded below by

$$\begin{aligned} M(t) &\geq P[N(t) \geq 1] \\ &= 1 - P[N(t) = 0] = F(t) \end{aligned}$$

From Equation 5.1.2,

$$p = 1 - F(t) \geq 1 - M(t)$$

and thus the first lower bound on the system reliability is given by

$$\ell_1 = 1 - M(t) \tag{5.2.1}$$

The bound ℓ_1 suffers from several obvious drawbacks. First, its computation, while entirely feasible by machine, is more complex than is desirable. More seriously, it may well be an extremely poor bound, since for even moderately large values of t it can be negative. Its usefulness in any application will depend on the values of the parameters, but it should not be expected to be of any value unless the component mean times to failure and mean repair times are large relative to the length of the mission. Alternate bounds are explored in the following sections.

5.3 A RELIABILITY BOUND BASED ON THE INITIAL SET OF WORKING COMPONENTS

Brown [1975] showed that the reliability of a parallel system of n components which are all functioning at time t=0 is bounded below by

$$p \geq e^{-t[c/(b-1)]} \tag{5.3.1}$$

This can provide the basis for a second bound on the reliability of a parallel system of components with arbitrary initial availabilities. If the set W is the set of indices of components which are functioning at time t=0, then, using Equation 5.3.1 and ignoring components which are not functioning at time t=0, the system reliability is bounded below by

$$\exp -t\left\{\left(\sum_{i\in W} \mu_i\right)\left(\left[\prod_{i\in W} \frac{\lambda_i+\mu_i}{\lambda_i}\right] -1\right)^{-1}\right\}$$

It follows then that

$$p \geq \sum_{r=1}^{n} \sum_{i_1<\cdots<i_r} \left(\prod_{j=1}^{r} \alpha_{i_j}\right)\left(\prod_{k\neq i_1,\ldots,i_r} (1-\alpha_k)\right) \times$$

$$\exp -t\left\{\left(\sum_{j=1}^{r} \mu_{i_j}\right)\left(\left[\prod_{j=1}^{r} \frac{\lambda_{i_j}+\mu_{i_j}}{\lambda_{i_j}}\right] - 1\right)^{-1}\right\}$$

This bound can be improved by using the true reliability for the case when r=1, giving

$$p \geq \ell_2 = \sum_{i=1}^{n} \alpha_i e^{-\lambda_i t} \prod_{j\neq i} (1-\alpha_j)$$

$$+ \sum_{r=2}^{n} \sum_{i_1<\cdots<i_r} \left(\prod_{j=1}^{r} \alpha_{i_j}\right)\left(\prod_{k\neq i_1,\ldots,i_r} (1-\alpha_k)\right) \tag{5.3.2}$$

$$\times \exp -t\left\{\left(\sum_{j=1}^{r} \mu_{i_j}\right)\left(\left[\prod_{j=1}^{r} \frac{\lambda_{i_j}+\mu_{i_j}}{\lambda_{i_j}}\right] - 1\right)^{-1}\right\}$$

Although the bound ℓ_2 is still computationally complex, it is always positive and thus offers an alternative to ℓ_1 in those cases when the latter is unsuitable. The bound developed in the next section is easily computed but, unfortunately, is not universal in applicability.

5.4 A BOUND BASED ON STEADY-STATE RESULTS

The final bound on the reliability of a parallel system which will be considered is again based on the exponential lower bound contained in Equation 5.3.1, which Brown [1975] showed to be a lower bound on system reliability when all components are in steady state and the system is functioning at time t=0. It seems reasonable that if each component's availability at time t=0 is no smaller than its steady-state or equilibrium availability, then the time to the first system failure should be stochastically larger than the steady-state time to first system failure, and thus the system reliability can be bounded below using Equation 5.3.1. To demonstrate that this is indeed true, some additional results are needed.

A random vector $\underline{X}$ is said to be stochastically larger than the random vector $\underline{Y}$ if and only if $f(\underline{X}) \geq^{st} f(\underline{Y})$, i.e., $P[f(\underline{X})>z] \geq P[f(\underline{Y})>z]$ for all z, for every increasing, real-valued function f. Proschan and Pledger [1973] extended this concept to stochastic processes and defined the stochastic process $\{X(t), t \geq 0\}$ to be stochastically larger than the process $\{Y(t), t \geq 0\}$ if

$$[X(t_1),\ldots, X(t_n)] \geq^{st} [Y(t_1),\ldots, Y(t_n)]$$

for every choice of $0 \leq t_1 \leq \cdots \leq t_n$, n=1,2,... .

Lemmas 5.1 and 5.2 and Theorem 5.3, below, also come from Proschan and Pledger [1973]. Lemma 5.1 is a restatement of a result due to

Veinott [1965], and Lemma 5.2 is attributed to Esary and Proschan [1968].

Lemma 5.1. Let $\underline{X}$ and $\underline{Y}$ be n-dimensional random vectors such that

(a) $X_1 \geq^{st} Y_1$,

(b) $Y_1, \ldots, Y_n$ are stochastically increasing in sequence--that is, $P[Y_j > z | Y_1 = a_1, \ldots, Y_{j-1} = a_{j-1}]$ is increasing in $a_1, \ldots, a_{j-1}$, for $2 \leq j \leq n$ and for all z--and

(c) $\{X_j | [X_1=a_1, \ldots, X_{j-1}=a_{j-1}]\} \geq^{st} \{Y_j | [Y_1=a_1, \ldots, Y_{j-1}=a_{j-1}]\}$ for each $a_1, \ldots, a_{j-1}$ and $2 \leq j \leq n$.

Then $\underline{X} \geq^{st} \underline{Y}$.

Lemma 5.2. Let $\underline{X}$ and $\underline{Y}$ be n-dimensional random vectors and $\underline{X}'$ and $\underline{Y}'$ be m-dimensional random vectors such that $\underline{X} \geq^{st} \underline{Y}$ and $\underline{X}' \geq^{st} \underline{Y}'$ with $\underline{X}$ and $\underline{X}'$ independent and $\underline{Y}$ and $\underline{Y}'$ independent. Then

$$[X_1, \ldots, X_n, X_1', \ldots, X_m'] \geq^{st} [Y_1, \ldots, Y_n, Y_1', \ldots, Y_m'] .$$

Theorem 5.3. Let f be a "well-behaved" continuous, increasing functional, and let

$$\{X(t),\ t \geq 0\} \geq^{st} \{Y(t),\ t \geq 0\} .$$

Then,

$$f\big(\{X(t),\ t \geq 0\}\big) \geq^{st} f\big(\{Y(t),\ t \geq 0\}\big) ,$$

and thus

$$Ef\big(\{X(t),\ t \geq 0\}\big) \geq Ef\big(\{Y(t),\ t \geq 0\}\big) .$$

Now let $\{X_i(t),\ t \geq 0 | \alpha_i\}$ represent the exponential – exponential performance process of component C_i when its initial availability is α_i and $\{X_i(t),\ t \geq 0 | e_i\}$ be its performance process when its initial availability is the equilibrium or steady-state availability given by $e_i = \mu_i(\lambda_i + \mu_i)^{-1}$. The following theorem states the conditions under

which the equilibrium time to the first system failure provides a bound on system reliability.

<u>Theorem 5.4</u>. If the initial component availabilities are each no smaller than the component steady-state availabilities, i.e., if $\alpha_i \geq e_i$, $i=1,\ldots,n$, then the reliability of a coherent system of independent components with performance processes which are stochastically increasing in time is bounded below by the reliability of the system when all components are in steady state at time $t=0$.

Proof. Choose arbitrary time points $0 = t_1 \leq \cdots \leq t_k$, $k=1,2,\ldots$. By assumption, $[X_i(t_1)|\alpha_i] \geq^{st} [X_i(t_1)|e_i]$. Further, the process $\{X_i(t),\ t \geq 0|e_i\}$ is stochastically increasing in time, and thus by definition $\{X_i(t_1),\ldots,X_i(t_k)|e_i\}$ are stochastically increasing in sequence. Finally, since condition (c) of Lemma 5.1 is obviously satisfied,

$$[X_i(t_1),\ldots,\ X_i(t_k)|\alpha_i] \geq^{st} [X_i(t_1),\ldots,\ X_i(t_k)|e_i],\ i=1,\ldots,n.$$

Since the component performance processes are assumed to be independent, it follows from Lemma 5.2 that

$$[X_1(t_1),\ldots,\ X_1(t_k),\ldots,\ X_n(t_1),\ldots,\ X_n(t_k)|\underline{\alpha}]$$
$$\geq^{st} [X_1(t_1),\ldots,\ X_1(t_k),\ldots,\ X_n(t_1),\ldots,\ X_n(t_k)|\underline{e}].$$

Then

$$\left(\phi[\underline{X}(t_1)],\ldots,\ \phi[\underline{X}(t_k)]\,|\underline{\alpha}\right) \geq^{st} \left(\phi[\underline{X}(t_1)],\ldots,\ \phi[\underline{X}(t_k)]\,|\underline{e}\right),$$

since for any increasing f,

$$f\left(\phi[\underline{X}(t_1)],\ldots,\ \phi[\underline{X}(t_k)]\,|\underline{\alpha}\right) \geq^{st} f\left(\phi[\underline{X}(t_1)],\ldots,\ \phi[\underline{X}(t_k)]\,|\underline{e}\right).$$

Consider the functional defined by $f\{\phi[\underline{X}(t)],\ t \geq 0\} = \inf_{0 \leq s \leq t} \phi[\underline{X}(s)]$.

Then by Theorem 5.3,

$$E\left(\inf_{0\le s\le t} \phi[\underline{X}(s)]\,|\,\underline{\alpha}\right) \ge E\left(\inf_{0\le s\le t} \phi[\underline{X}(s)]\,|\,\underline{e}\right) ,$$

or

$$P[T > t|\underline{\alpha}] \ge P[T > t|\underline{e}] ,$$

where T is the time to the first system failure. □

Brown [1975] showed that for a parallel system of independent components with exponential-exponential performance processes,

$$P[T > t|\underline{e}] \ge \frac{b-1}{b} e^{-ct/(b-1)}$$

where $b = \prod_{j=1}^{n} \frac{\lambda_j + \mu_j}{\lambda_j}$ and $c = \sum_{j=1}^{n} \mu_j$. Thus, when the joint component performance process is exponential-exponential and $\alpha_i \ge e_i$, $i=1,\ldots,n$, it follows from Example 4.1 and Theorem 5.4 that

(5.4.1) $$p \ge \ell_3 = \frac{b-1}{b} e^{-ct/(b-1)} .$$

5.5 APPLICATION TO PHASED-MISSION PROBLEMS

To apply the results of the preceding sections of this chapter to the phased-mission problem, it is only necessary to recognize that the mission is one phase of the multi-phase mission and the system is a minimal cut structure. To use any of the bounds it is necessary tc calculate the availability of each component of the system at the beginning of each phase. Starting with the initial component availabilities at time t=0, these phase-initiation availabilities can be computed recursively using Equation 5.1.1. Thus, for j=1,...,m, and k=1,...,n,

$$\alpha_k(t_j) = \frac{\mu_{kj}}{\lambda_{kj}+\mu_{kj}} + \left(\alpha_k(t_{j-1}) - \frac{\mu_{kj}}{\lambda_{kj}+\mu_{kj}}\right) e^{-d_j(\lambda_{kj}+\mu_{kj})}$$

where λ_{kj} (μ_{kj}) is the failure (repair) rate of component C_k during phase j and d_j is the phase duration.

Next it is necessary to obtain the minimal cut sets for each phase of the mission using any of the standard methods. Then a lower bound on the reliability of each of these phase minimal cut structures can be calculated using one of the bounds presented here or the bound which results from assuming that the components are not repairable during the phase. In these calculations, the α_j's are understood to be the component availabilities at the beginning of the appropriate phase, and the mission length t to be the phase duration d_j.

Once this has been done, the mission reliability of the system can be bounded below using Equation 4.2.5, giving

$$(5.5.1) \qquad p \geq \prod_{j=1}^{m} \prod_{i=1}^{c_j} \ell_{ji}$$

where ℓ_{ji} is a lower bound on the reliability of the i^{th} minimal cut structure for phase j and c_j is the number of minimal cut structures for the phase.

If, as may be the case, a minimal cut set contains both repairable and non-repairable components, all is not lost. Examination of the bound ℓ_1 reveals that in the case when no components are repairable (μ_k=0, k=1,...,n), ℓ_1 gives the correct reliability of the structure. Further, the bound continues to be valid for any combination of repairable and non-repairable exponential components.

6. SUGGESTIONS FOR FURTHER RESEARCH

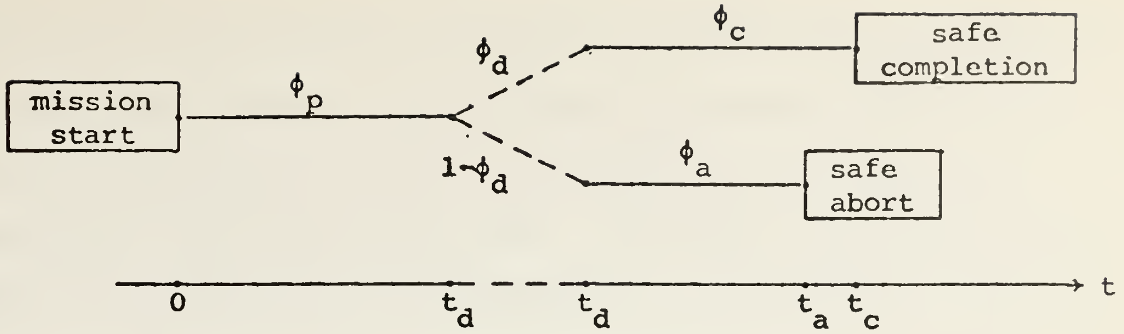
The phased-mission models of Chapters 2 and 3 appear to be well suited for the reliability analysis of systems which have the additional flexibility of real-time configuration selection. In their crudest form such systems may be visualized as traversing a network from mission origin to mission success in which each node represents a point at which a configuration decision is made and the arcs emanating from the node represent the alternatives. A classic example of this situation is provided by what is called the crew safety problem. Reliability engineers connected with U.S. manned spaceflight efforts have long been concerned with assessing the probability of safe return of the crew in addition to mission reliability. Procedures for obtaining crew-safety predictions were discussed in the early papers of Rubin [1964] and Weisburg and Schmidt [1966]. The following example is a much-simplified version of this crew safety problem.

Example 6.1. The designers of a manned space mission are concerned about the safety of the crew and have provided an abort option at a critical point in the mission. The decision to abort or continue the mission is to be made based on the current system status as reflected by the value of a coherent function of the component performance indicator variables, i.e.,

the mission continues if $\phi_d[\underline{X}(t_d)] = 1$, and

the mission is aborted if $\phi_d[\underline{X}(t_d)] = 0$.

The network representation for the purpose of determining crew safety is



where ϕ_p represents the structures of phases prior to the abort decision (and $\{\Phi_p = 1\}$ is understood to mean successful completion of all prior phases), ϕ_c represents the structures of phases subsequent to the decision when the mission is completed, and ϕ_a represents the structures of the abort phases.

The decision is included as a dummy phase of zero duration, and thus the probability of safe crew return is given by

$$p_s = P\{\Phi_p = 1, \Phi_d = 1, \Phi_c = 1\} + P\{\Phi_p = 1, [1 - \Phi_d] = 1, \Phi_a = 1\}$$

or

$$p_s = P\{\Phi_p\Phi_d\Phi_c = 1\} + P\{\Phi_p\Phi_a = 1\} - P\{\Phi_p\Phi_d\Phi_a = 1\}$$

Each term in this expression is of the same form as Equation 2.2.4, and thus could be evaluated using the methods of Chapter 2. □

The mathematical structure of the phased-mission models could be used to aid in determination of the decision functions to be used in a mission with real-time configuration selection using criteria designed to maximize the probability of mission success. If, for example, the objective in the crew safety problem is to maximize mission reliability subject to a lower bound on crew safety, the phased-mission model could aid in selecting among alternative decision functions.

The area of reliability of maintained systems appears to be fertile with opportunities for research. Little is known of the properties

of time-associated performance processes, and the limited results available have mostly been obtained by relying on processes which are stochastically increasing in time. It is conjectured that this is an unnecessarily restrictive requirement. For example, the proofs of Theorems 4.6 and 5.4 require the processes to be stochastically increasing in time when time-association may well be sufficient.

The practicality of the assumption of exponential repair times in much of Chapter 4 and all of Chapter 5 is subject to challenge. It is likely a difficult problem (but one worthy of study) to generalize to other repair distributions. The resulting performance processes must be shown to be time-associated in order to use either the phase lower bound or the minimal cut lower bound. As indicated in the preceding paragraph, a catalog of such processes is not yet available. In a recent paper, however, Barlow and Proschan [1975b] study a system of independent components in which each repairable component has an exponentially distributed failure time and a repair time distribution which has a decreasing repair rate (a DFR distribution). They show that the resulting joint component performance process is stochastically increasing in time and hence time-associated. In addition, they develop several bounds on the reliability of a coherent system with this joint component performance process for the case when all components are new (and thus functioning) at time t=0. Aside from the question of whether the assumption of decreasing repair rates for all components is any more realistic than assuming them to be constant, it is not clear that their results could be modified to allow components to be used (and thus possibly failed) at time t=0 as would be the case at the beginning of phases in a phased mission. It would be necessary to specify not

only the availability of each component at the beginning of each phase but also the distribution of the length of the current repair period for each failed component.

It may be possible to obtain additional information about the time to first failure of a parallel system of exponential components like that studied in Chapter 5 by obtaining the Laplace transform of its distribution. Then, by proceeding as Brown [1975] does, it should be possible to obtain the moments of the system failure time, from which an approximate distribution function could be constructed. Examination of the Laplace transform may also suggest additional lower bounds on the system reliability.

The foregoing discussion has been directed primarily to the study of maintenance policies which prescribe immediate maintenance upon failure of a component. A slightly more complicated policy, which seems to be very realistic for a multi-objective mission model like that of Chapter 3, is possible when the mission has some slack time available. That is, more time is allotted for mission accomplishment than is required in the absence of system failures. This concept is best illustrated by example.

Consider once again the multiple-objective version of the SLBM system and suppose the time allowed to complete a launch sequence exceeds the time required if no system failures occur. Assume that maintenance can be performed during any submarine-borne phase of the mission, but that whenever undertaken, the group of components involved (master system or a subsystem) must be taken out of service. Thus if a master system failure occurs, all active operations cease aboard the submarine. If a subsystem (missile) failure occurs during a submarine-borne phase,

then there are two alternatives--cease operations until repairs are completed or cycle to the next missile and continue the mission while commencing repairs to the defective missile. In any of these cases it is possible to successfully complete all objectives if the system downtime does not exceed the slack time available.

It should not be necessary to continue this discussion of possible extensions and variations to convince the reader that there is a bona-fide need for additional research. The cases for which analytical results are available represent but a small fraction of those worthy of investigation. Hopefully, the methodology and theoretical results presented in this thesis will serve to stimulate additional efforts in this field.

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