SENSITIVITY ANALYSIS OF LINEAR NETWORKS

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THESIS

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ABSTRACT

Network functions and sensitivity expressions for linear networks are reviewed for networks with single unilateral or bilateral immittances and the corresponding extension to the general case of n variable immittances. The sensitivity expression is exact and suitable for the analysis of networks with either small or large variation of parameters.

A method is proposed for minimizing sensitivity using the exact expression. The general linear network is studied in various aspects like changes on frequency, variation of elements or mixed variation of frequency and network components. By proper network transformation and synthesis, it is possible to design a network with minimum sensitivity even if the variation in immittance is large. Generally, for small or incremental variation of immittance, the sensitivity of the network function remains invariant. However, this theorem does not hold in cases of large variation.

Normally, the range of sensitivity for which a network is to operate, is specified. It follows that the elements and parameters composing the network have certain ranges. It was determined that with specified range of sensitivity, the allowable range of values for the parameters of the network can be computed.

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I. INTRODUCTION

Sensitivity analysis is recognized as one of the major considerations in the study of modern networks. The advent of integrated circuits, LSI and MSI, thin film technology, etc., pose a need for an exact sensitivity evaluation on possible variations of parameters. Even with the tremendous advancement of miniaturization, the goal towards component perfection is still far from reality. Instead, the problem has grown complex as the components become smaller and miniaturized. The effect of temperature variations, for example, could cause variation of parameters that is too excessive to be completely ignored. The seriousness of the problem is apparent when circuits fail due to changes in element values. Sensitivity analysis provides information on the acceptable tolerances for the network to operate within the limits of the design specification.

Throughout this thesis, linearity of the network is assumed. This class of circuits involves network functions whose expressions are linear combinations of the variables contained in the network. The second chapter reviews the results in references [2], [6], [9] and [12] which express the exact sensitivity of linear networks with either unilateral or bilateral variable immittances.

The next chapter presents a comparison between the expressions used for incremental sensitivity and exact sensitivity.

Incremental sensitivity uses a truncated Taylor series expansion to find the amount by which the network function deviates from its original value when any or several elements vary.

The exact sensitivity expression is applied in seeking the minimum sensitivity network. The characteristics of a minimum sensitivity network are discussed. The results may be used in synthesis of networks where sensitivity is a major consideration. The simplification of the criteria for minimum sensitivity eases the burden of choosing the best network suitable for a specific purpose.

Sensitivity evaluation by the classical approach, even in simple networks, may become fairly complex. However, using the exact expression, the usual change of procedure from the incremental to large variation analysis is eliminated.

II. GENERAL REVIEW

A. INTRODUCTION

The materials in this chapter are derived from references [2], [6], [9], and [12], which serve as the theoretical basis for this paper.

Two kinds of network functions are of interest, namely, the transfer function and the driving point immittance. A transfer function is defined as the transform ratio of the response to the excitation. This applies to, in general, two-port networks. In network theory, the response and excitation can take the form of either voltage or current. A network can then have a transfer function as the ratio of voltage-to-voltage, voltage-to-current, current-to-current, or current-to-voltage. To define the transfer function of a network, it is necessary to specify the required response and the given excitation.

A driving-point immittance is defined as the ratio of the response to the excitation in a one-port network. The response and excitation must be of different types so that only the ratio of voltage-to-current and current-to-voltage are allowed.

In a linear network, the network function is a multi-linear function of the parameters involved. This means that the network function can be formulated in terms of the network parameters with at most the first power of any

variable. A network parameter may be any passive or active component upon which the network function depends.

The charactersitics of physical devices are subject to change for various reasons. Perfect systems with stable components that can maintain their design value under all conditions are difficult to obtain in practice. Take for example a simple active device, the transistor. It is almost impossible to obtain a transistor with the exact parameters specified by the manufacturer. Even if this "ideal" device is obtainable, factors like temperature variations, aging, radiation etc. may eventually cause the parameters to vary and produce undesirable results on the system of which it is a component.

A measure of the effects of parameter variation upon a network or system has been denoted as sensitivity. Basically, there are two types of parameter variations studied by various investigators. The most popular type is the incremental variation case, where the change in parameter values is assumed to be very small. The other type considers the effect on the network function due to a large variation of parameters. The study of the effects of a large variation is hindered by the need to use series expansion in system analysis. For example, a truncated Taylor series expansion may be used to find the deviation of the network function caused by large parameter variations. However, the technique of truncation entails error and an increase in accuracy requires a corresponding increase in

computation. Further, Taylor series computation requires the computation of derivatives which requires much computation time. These problems in sensitivity analysis and synthesis pose a need for a more sophisticated approach toward a simple and easy solution of network.

B. NETWORK FUNCTIONS

It was derived in [2], [6] and [12] athat a network function, relating the Laplace transform of the response to the Laplace transform of the excitation, of a linear network (Fig. 1) with a single variable immittance or controlled source, free of all independent sources except one and initally at rest, can be expressed as:

$$T(x) = \frac{WT(0) + xT(\infty)}{W + x} \tag{2-1}$$

where

W = the Thevenin immittance seen looking back into the network from the terminals of x.

x = a variable immittance or controlled source.

T(x)= the network function relating a response to an excitation.

T(0)= Lim T(x) as x approaches zero.

T(∞)= Lim T(x) as x approaches infinity.

Eq. (2-1) is valid provided W, T(0), and T(∞) are finite.

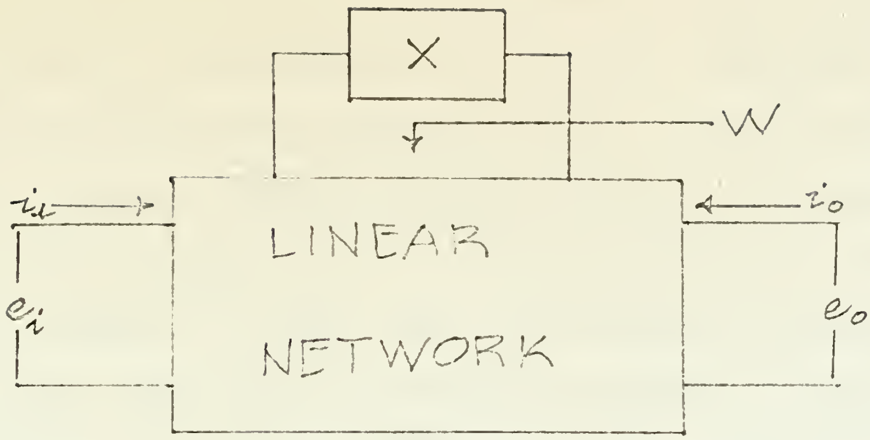

Figure 1

A Linear Network with Variable Immittance or Controlled Source x.

Manipulation of Eq. (2-1) with the introduction of minute or large changes of the variable immittance or controlled source leads to the derivation of a sensitivity equation which will be discussed later.

A basic application of Eq. (2-1) is to consider that the variable x represents a resistor element with Laplace transform r(s) as shown in Fig. 2. It is desired to determine the network function of the linear network N designated as the ratio of the response R(s) over the excitation C(s). Now short the excitation force if it is a current source or open it if its is a voltage source and apply an auxiliary driving force at port 3-3, i.e. at the terminals of the variable element. In this case, the auxiliary driving force is a voltage source and the measured response at the same port terminal is the current flowing to it. This Thevenin immittance measured is designated as W(s), the ratio of voltage over current. Then the network function of the linear network N in Fig. 2 is given as

$$\frac{R(s)}{C(s)} = \frac{W(s)T(0) + r(s)T(\infty)}{W(s) + r(s)} \tag{2-2}$$

In many complex types of networks and with the proper choice of a group of network elements to represent the variable x, Eq. (2-1) offers a much simpler way of determining the network function. Another example worth mentioning is when x represents a dependent source coupling factor

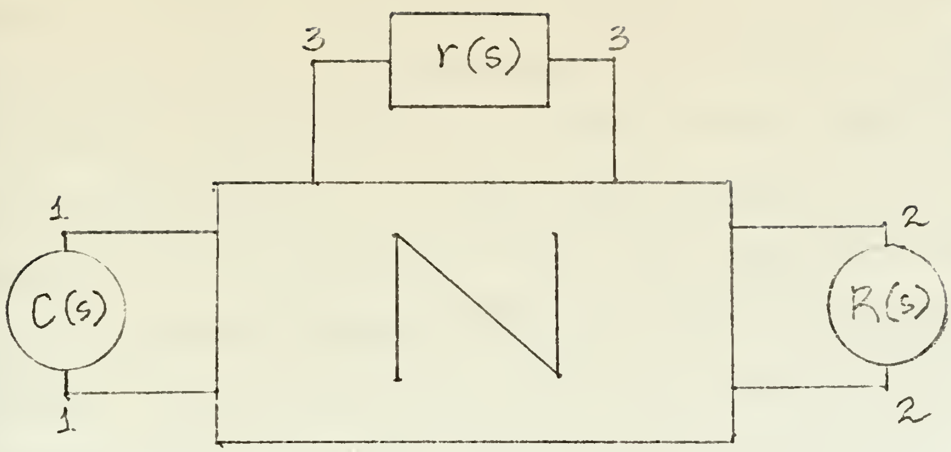

Figure 2

A Three Port Network with Variable Resistance r(s)

designated by β as depicted in Fig. 3. Table II gives the network function for different relationships of the controlled source and controlling quantity.

In general, Eq. (2-1) is a powerful tool in network analysis whether the network has passive or active elements. Further discussions are found in references [6], [7], [8], and [12].

Troop and Peskin [9] extended the expression in Eq. (2-1) to consider n variable immittances. Using the loop or node method of analysis, the network function was derived and found to be a ratio of determinants with elements containing all possible combinations of variables except those involving powers greater than one. Since any immittance can be represented as a controlled source, the resulting expression includes general network containing passive and active elements. The general expression for a transfer function of a network with n variable immittances or controlled sources which are considered as ports, is given by

$$T(x_1, \ldots, x_n) = \frac{\sum\limits_{i_1, \ldots, i_n = 0}^{1} T(c_1, \ldots, c_n) \prod\limits_{j=1}^{n} x_j^{i_j} [W_{\infty, \ldots, \infty, j, c_{j+1}, \ldots, c_n}]^{1-i_j}}{\sum\limits_{i_1, \ldots, i_n = 0}^{1} \prod\limits_{j=1}^{n} x_j^{i_j} [W_{\infty, \ldots, \infty, j, c_{j+1}, \ldots, c_n}]^{1-i_j}} \tag{2-3}$$

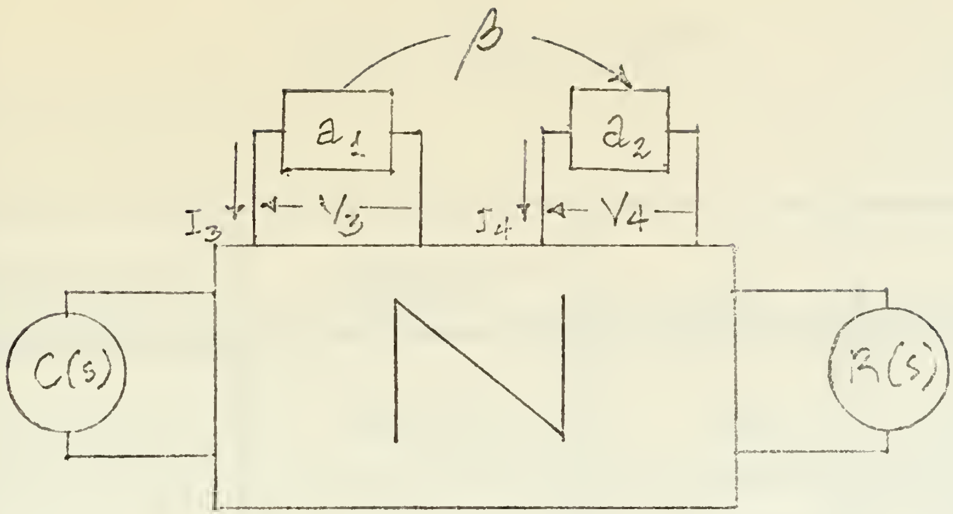

Figure 3

Four-Port Network with Variable Amplification Factor. a_1 and a_2 are the controlling source and controlled quantity, respectively.

Table II

Network Function with Single Variable Amplification Factor

$$T(\beta) = \frac{A\ T(0) + \beta\ T(\infty)}{A + \beta}$$

Controlling Quantity	Controlled Source	β	A*
Voltage V_3	Voltage V_4	V_4/V_3	$-V_3/V_4$
Voltage V_3	Voltage I_4	I_4/V_3	$-V_3/I_4$
Current I_3	Voltage V_4	V_4/I_3	$-I_3/V_4$
Current I_3	Current I_4	I_4/I_3	$-I_3/I_4$

*A is derived from the network with excitation set at zero.

See Fig. 2 for subscript notations.

where

x_j = the variable transmittance or transfer ratio of the controlled source across point j

c_i = 0,∞ for i_j = 0,1 respectively

$T(c_1 \ldots, c_n)$ = Lim $T(x_1, \ldots, x_n)$ as $x_j \to c_j$, j = 1,2,...,n

$W_{c_1 \ldots, k, \ldots, c_n}$ = the Thevenin transmittance or "Thevenin Transfer Ratio" for port K. At all the remaining ports, it is required that $x_j = c_j (j \neq K)$ hence c_1 to c_{k-1} are infinite.

Accordingly, Eq. (2-2) is valid provided the limiting conditions of the network function $T(c_1, \ldots, c_n)$ and the Thevenin transmittance as $W_{\infty, \ldots, \infty, j, c_{j+1}, \ldots, c_n}$ are finite.

Certain conditions have to be satisfied in order to assure the finiteness of the Thevenin immittance W. They are the following:

a. No open circuit should appear in series with the port at which the Thevenin impedance is to be calculated (measured) when other ports are open circuited.

b. No parallel path should be formed across the port at which the Thevenin admittance is to be calculated (measured) when the other ports are short circuited.

The conditions above therefore require that when W is an impedance function, then the branches across the ports (or any subset of these) do not form a cutset. If W is an admittance function, then the branches across the ports (or any subset of these)do not form a closed path.

C. EXACT SENSITIVITY

The definition of sensitivity varies among investigators. Generally, it is construed as a measure of the amount by which the system's network function deviates from its original or nominal value when one or more of its parameters vary. From this definition, sensitivity is mathematically given as:

$$S = \frac{\Delta T(x)}{T(x)} \qquad (2\text{-}4)$$

where T(x) is the nominal network function of the network whose elements are at their designed value without any deviation. In network analysis requiring sensitivity evaluation, the nominal network function is first determined by assuming no deviation of network element. When one or more elements are allowed to vary, a new network function is then computed, after which the ratio of the amount of deviation of the network function to the nominal network function is taken. The result is the sensitivity of the network for the given amount of variation exhibited by the variable elements.

Another definition of sensitivity given by Bode is

$$S_b = \frac{d[\mathrm{Ln}\ T(x)]}{d[\mathrm{Ln}\ x]} \qquad (2\text{-}5)$$

where x is the variable parameter and T(x) is the network function. Unlike Eq. (2-4), the Bode definition offers

simplicity only when a single variable parameter is involved. For multi-variable parameter sensitivity analysis, the normal procedure is to take the summation of the sensitivities of the network for all the variable parameters taken independently. The limitation in this method of analysis is that for practical networks, variable parameters do not change individually but are of combined variation. However, if combined variation is assumed, the normal Bode Method becomes too complex for computation.

A recently developed method of analyzing sensitivity was given by Parker [6]. The method considered linear networks with single variable immittance. The sensitivity expression is given as

$$S = \frac{\Delta T(x)}{T(x)} = \frac{T(\infty) - T(x)}{W + x + \Delta x} \cdot \frac{\Delta x}{T(x)} \tag{2-6}$$

which was derived to be an exact expression whether the variation of the variable Δx is either large or small. The terms in the expression of Eq. (2-6) are as previously defined in this chapter.

To check the validity of Eq. (2-6) in computing the sensitivity consider the case when the variation of x becomes small. Then

$$\lim_{\Delta x \to 0} \frac{\Delta T(x)}{\Delta x} = \frac{dT(x)}{dx} = \frac{T(\infty) - T(x)}{W + x} \tag{2-7}$$

Now, the original network function T(x) stated in Eq. (2-1) is differentiated with respect to the variable x as follows:

$$\frac{d[T(x)]}{dx} = \frac{d}{dx}\left[\frac{W\,T(0) + x\,T(\infty)}{W + x}\right]$$

$$= \frac{[W + x][T(\infty)] - [W\,T(0) + x\,T(\infty)]}{[W + x]^2} \qquad (2\text{-}8)$$

$$= \frac{W}{[W + x]^2}\,[T(\infty) - T(0)]$$

but

$$T(\infty) - T(x) = T(\infty) - \frac{W\,T(0) + x\,T(\infty)}{W + x}$$

$$= \frac{W}{W + x}\,[T(\infty) - T(0)] \qquad (2\text{-}9)$$

therefore

$$\frac{d[T(x)]}{dx} = \frac{T(\infty) - T(x)}{W + x} \qquad (2\text{-}10)$$

It is apparent that comparison of Eqs. (2-10) and (2-6) shows similar expression for sensitivity. Obviously, this is an increment variation analysis when small changes in x was assumed. When large variation is expected in the elements of the network, Eq. (2-6) offers an exact analysis.

Troop and Peskin [9] extended the works of Parker et al. and Sorensen [12] in the analysis of networks containing two ore more variable elements. Following the same techniques used by Parker et al. [7] in deriving Eq. (2-6) the resulting

general expression is given as:

$$S = \frac{\Delta T(x)}{T(x)} = \frac{\Sigma\{[T(p_1,\ldots, p_n)] - T(x_1,\ldots, x_n)\} \prod_{j=1}^{n} x_j^{i_j} [x_j + W_{\infty,\ldots,j,p_{j+1},\ldots,p_n}]^{1-i_j}}{T(x_1,\ldots,x_n) \sum_{i_1,\ldots,i_n=0} \prod_{j=1}^{n} \Delta x_j^{i_j} [x_j + W_{\infty,\ldots,\infty,j,p_{j+1},\ldots,p_n}]^{1-i_j}} \tag{2-11}$$

where

$p_j = x_j + c_j$; $p_j = x_j, \infty$ $c_j = 0, \infty$ for $i_j = 0,1$

All the variables are as defined previously in this chapter.

III. MINIMIZATION OF LINEAR NETWORK SENSITIVITY

A. INTRODUCTION

Sensitivity has been recognized as one of the main criteria in modern network design. Various proposals have been presented to minimize the sensitivity of networks within the framework of assuming small changes in parameters. Schoeffer [10] proposed the method of continuous equivalent theory to transform the original circuit into the best minimum sensitivity network. The method had been found to be a powerful tool in network synthesis except that in the original proposal, a long computation of sensitivity is required at some stages of the transformation. Unfortunately, sensitivity evaluation using classical approach requires too much computer time for practical application in many instances. When there are several equivalent networks to evaluate, there is a need for easier comparison of their sensitivities. There is also the problem of large changes in parameters to consider which can complicate the solution.

Hakimi and Cruz [3] presented measures of sensitivity for linear systems with large multiple parameter variations. They studied a method of obtaining an upper bound and lower bound of the maximum and minimum network function respectively. A problem brought forward is to synthesize a network such that the effect of the variation of the element values on the network is minimized in some sense.

This chapter establishes a general criteria of seeking the minimum sensitivity of linear networks using the exact expression previously stated in Chapter II. Critical regions in the sensitivity values are discussed and eventually lead to the derivation of the minimum sensitivity theorem. Critical region in sensitivity as herein stated is defined as that condition wherein the varying parameters have values such that the sensitivity of the network is either maximum or minimum. A knowledge of these critical regions helps in the solution of the following problems normally encountered in network design.

1. Comparing the sensitivities of equivalent networks. Classically, it would normally require long computation to perform sensitivity comparison but a simpler way is possible as will be shown.

2. Minimizing sensitivity of networks under the assumption of constant network function.

3. Analyzing network under worst case conditions where large or small variations of parameters might be expected.

B. EXACT SENSITIVITY EXPRESSION

As discussed in Chapter II and presented by Parker et al. [7] Eq. (2-6) gives the sensitivity expression of a general linear network with single variable immittance. Another form of expression used which is basically suited for computer application for incremental variation Δx is given by

$$S = \frac{\Delta T(x,s)}{\Delta x(s)} = \frac{T(\infty,s) - T(x,s)}{W(s) + x(s) + \Delta x(s)} \quad (3\text{-}1)$$

Expression (3-1) will be studied under three different sets of conditions:

Case I - frequency is fixed, Δx varies.
Case II - Δx is fixed, frequency varies.
Case III - both Δx and frequency vary.

Sensitivity is normally a complex quantity at a given frequency. However, the usual area of interest is on the magnitude of sensitivity. The magnitude-squared of the sensitivity is given by:

$$F = |S|^2 = \frac{|T(\infty,s) - T(x,s)|^2}{|W(s) + x(s) + \Delta x(s)|^2} \quad (3\text{-}2)$$

C. VARIATION OF VARIABLE IMMITTANCE AT CONSTANT FREQUENCY

Equation (3-1) gives the sensitivity expression of a general linear network with single variable immittance. The equation can also be expressed in terms of $T(0,s)$ and $T(\infty,s)$ by substituting in Eq. (3-1) the expression $T(x,s)$ in Eq. (2-1), resulting in

$$S = \frac{\Delta T(x,s)}{\Delta x(s)} = \frac{W(s)[T(\infty,s) - T(0,s)]}{[W(s) + x(s)][W(s) + x(s) + \Delta x(s)]} \quad (3\text{-}3)$$

At constant frequency, all the terms in Eq. (3-3) are constant except $\Delta x(s)$ which is a complex variable. It is to

be emphasized here that all the terms in the expression above are complex resulting to a complex sensitivity value.

Two problems in practical application may be encountered in dealing with variable immittance at constant frequency. These are:

1. If a certain immittance in a given network is made to vary, at what change in the variable immittance results to a minimum sensitivity of the network?

2. If the network is required to operate at a specific frequency where the sensitivity of the system is greatly affected by the variable immittance, what equivalent network will give the minimum sensitivity due to the variable immittance? This problem is by no means easy as it will involve some techniques in equivalent network synthesis starting from the basic equation in Chapter II and which is not yet available to the writer. However, some discussions are presented and possible solutions are foreseeable provided proper equivalent network synthesis is available.

<u>Definition III-1</u>

Minimum sensitivity is the state of the network that for a given excitation, the response is least affected by any changes of the parameters within the network. The parameters in a network can take the form of an element, a controlled source or any combination thereof.

In trying to search for the answer to the first question posed, it is necessary to refer back to Eqs. (3-1) or (3-3). Since both equations are equivalent to each other, it will

be adequate to analyze only the expression in Eq. (3-1) in the succeeding paragraphs and refer to Eq. (3-3) in cases where necessary. At constant frequency, the expression is investigated in two parts, the numerator and denominator expressions. The result of the investigation can be stated as:

1. The numerator expression $[T(\infty,s) - T(x,s)]$ must be a complex constant term whose value depends only on the frequency and the values of the original parameters or elements of the network. It is noted that the expression is independent of the varying parameter denoted by $\Delta x(s)$.

2. The denominator expression $[W(s) + x(s) + \Delta x(s)]$ is a complex variable expression whose value depends upon the variable parameter $\Delta x(s)$. It was mentioned that $W(s)$ and $x(s)$ are complex constants, hence the complex value of $\Delta x(s)$ mainly determines the value of the expression.

Therefore, in order to minimize the sensitiviy of the network, the parameter deviation of $x(s)$, represented by $x(s)$, must be such as to increase the magnitude value of the denominator.

Further analysis can be made when the criterion used is the magnitude-squared of the sensitivity. Let $L(s) = W(s) + x(s)$, from Eq. (3-2).

$$F = |S|^2 = \frac{|T(\infty,s) - T(x,s)|^2}{|L(s) + \Delta x(s)|^2} \qquad (3\text{-}4)$$

or

$$F = |S|^2 = \frac{|T(\infty,s) - T(x,s)|^2}{|L(s)|^2 \left| \frac{\Delta x(s)}{L(s)} + 1 \right|^2} \qquad (3\text{-}5)$$

where $[T(\infty,x) - T(x,s)]$ must be finite as previously discussed. Also the magnitude-squared of $L(s) = W(s) + x(s)$ must be finite.

Let

$$\frac{|T(\infty,s) - T(x,s)|^2}{|L(s)|^2} = K \qquad (3\text{-}6)$$

Then the expression in Eq. (3-5) can be rewritten as:

$$F = \frac{K}{\left| \frac{\Delta x(s)}{L(s)} + 1 \right|^2} = \frac{K}{\left\{ \mathrm{Im}\left[\frac{\Delta x(s)}{L(s)} \right] \right\}^2 + \left\{ \mathrm{Re}\left[\frac{\Delta x(s)}{L(s)} + 1 \right] \right\}^2} \qquad (3\text{-}7)$$

Where Im and Re denote the Imaginary and Real Parts respectively of the complex quantity $x(s)/L(s)$. F is minimum <u>if</u> and only <u>if</u> the real and imaginary parts of the complex ratio $x(s)/L(s)$ is maximum.

<u>Example III-1</u>

Consider the network as shown in Fig. 4 with the capacitor C as the variable element.

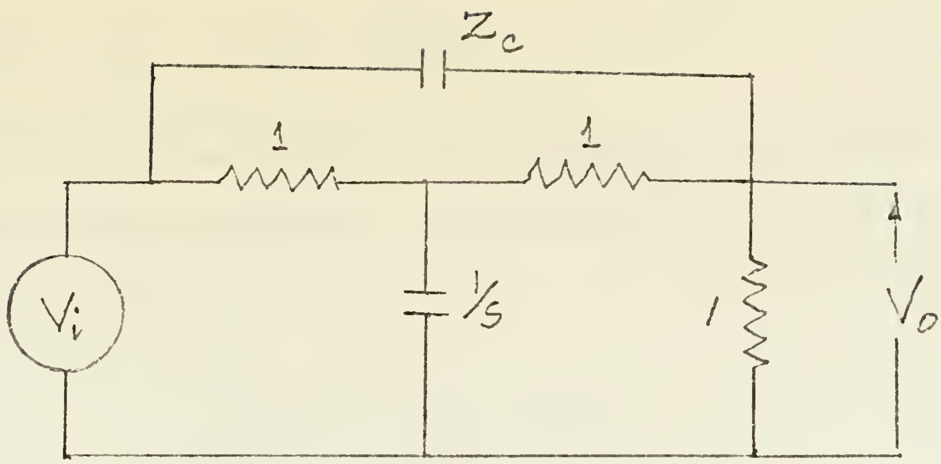

Figure 4

Network for Example III - 1

A nodal or loop analysis of the network yields the following expressions:

$$T(x,s) = \frac{V_o(Z_c,s)}{V_i(s)} = \frac{\frac{s+2}{2s+3} + \frac{1}{2s+3} Z_c}{\frac{s+2}{2s+3} + Z_c}$$

$$T(0,s) = 1 \quad , \qquad T(\infty,s) = \frac{1}{2s+3} \quad ,$$

$$W(s) = \frac{s+2}{2s+3} \quad .$$

At various frequencies, the following results were computed by substituting $s = j\omega$ and assuming the values of C and Δx as indicated.

$\omega = 1$ rad/sec

$Z_c = 1/s$ (one farad capacitance)

a. $\Delta x = 0.0$

$|T(\infty,j1) - T(Z_c,j1)|^2 = 4.5$

$W(j1) + x(j1) = 0.615 - j1.767$

$|W(j1) + x(j1)|^2 = 3.5$

$|S|^2 = 1.28$

b. $\Delta x = -j\ 0.5$

$|S|^2 = 1.99$

c. $\Delta x = j\ 0.5$

$|S|^2 = 0.816$

One of the problems which was brought forward at the beginning of this section involves cases wherein equivalent network synthesis is to be used. The question posed is when equivalent networks having the same immittance x(s) are compared in sensitivity due to changes in the same variable, which network will present the least sensitivity.

Kuh and Lou [13] and de Buda [14] in a recent article proved the sensitivity invariants of equivalent networks. Their theorems were only true on sensitivities of networks involving incremental variations (the limit as $\Delta x(s)$ approaches zero). However, on cases where the variations of the variable x(s) are large, then sensitivity invariant no longer hold. This results from the Eq. (3-1) where it is assumed that T(x,s), x(s) and $\Delta x(s)$ are complex constants. Transformation of the network using equivalent network theory varies $T(\infty,s)$ and W(s) which can result to a circuit having the best sensitivity for a constant frequency and specific $\Delta x(s)$.

D. VARIATION OF FREQUENCY AT SPECIFIC VARIATION OF IMMITTANCE*

When the frequency is varied for a given variation of the variable immittance x(s), then all the terms in Eq. (3-1) change since all of them are functions of frequency. In order to simplify the formula, let $Z(s) = x(s) + \Delta x$ so that Eq. (3-1) can be written as:

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*In this section, Δx is assumed fixed.

$$S = \frac{\Delta T(x,s)}{\Delta x(s)} = \frac{T(\infty,s) - T(x,s)}{W(s) + Z(s)} \qquad (3\text{-}8)$$

Taking the derivative of S in Eq. (3-8) with respect to frequency s,

$$\frac{dS}{ds} = \frac{\partial S}{\partial T(\infty,s)} \cdot \frac{d[T(\infty,s)]}{ds} + \frac{\partial S}{\partial T(x,s)} \cdot \frac{d[T(x,s)]}{ds}$$

$$+ \frac{\partial S}{\partial W(s)} \cdot \frac{dW(s)}{ds} + \frac{\partial S}{\partial Z(s)} \cdot \frac{dZ(s)}{ds} \qquad (3\text{-}9)$$

and

$$\frac{\partial S}{\partial T(\infty,s)} = \frac{1}{W(s) + Z(s)} \qquad (3\text{-}10a)$$

$$\frac{\partial S}{\partial T(x,s)} = - \frac{1}{W(s) + Z(s)} \qquad (3\text{-}10b)$$

$$\frac{\partial S}{\partial W(s)} = - \frac{T(\infty,s) - T(x,s)}{[W(s) + Z(s)]^2} \qquad (3\text{-}10c)$$

$$\frac{\partial S}{\partial Z(s)} = \frac{T(\infty,s) - T(\infty,s)}{[W(s) + Z(s)]^2} \qquad (3\text{-}10d)$$

Equating Eq. (3-9) to zero, the frequencies where the exact sensitivity is either maximum or minimum can be located. Thus we obtain

$$\frac{d[T(\infty,s)]}{ds} - \frac{d[T(x,s)]}{ds} = -\left\{\frac{T(x,s) - T(\infty,s)}{W(s) + Z(s)}\right\}\left\{\frac{d[W(s)]}{ds} + \frac{d[Z(s)]}{ds}\right\},$$

or

$$\frac{\frac{d[T(\infty,s)]}{ds} - \frac{d[T(x,s)]}{ds}}{T(\infty,s) - T(x,s)} = \frac{\frac{d[W(s)]}{ds} + \frac{d[Z(s)]}{ds}}{W(s) + Z(s)},$$

$$\frac{\frac{d}{ds}[T(\infty,s) - T(x,s)]}{T(\infty,s) - T(x,s)} = \frac{\frac{d}{ds}[W(s) + Z(s)]}{W(s) + Z(s)},$$

Thus,

$$\frac{d}{ds} \text{Ln}\,[T(\infty,s) - T(x,s)] = \frac{d}{ds} \text{Ln}\,[W(s) + Z(s)]$$

which requires that

$$T(\infty,s) - T(x,s) = W(s) + Z(s) \tag{3-11}$$

The solution of Eq. (3-11) will result in the frequency at which the sensitivity S is maximum or minimum.

E. MIXED VARIATION OF A SINGLE VARIABLE ELEMENT AND FREQUENCY

Mixed variation of the single variable immittance x(s) and frequency s in a general network provide an insight into the actual behavior of the system under realistic situations. These situations are what really exist in a practical circuit

since all factors whether internal or external, causing the variation in the characteristics of the network come into play as the system is put into operation.

Consider again Eq. (3-1),

$$S(x,s) = \frac{T(\infty,s) - T(x,s)}{W(s) + x(s) + \Delta x(s)}$$

or Eq. (3-8),

$$S(x,s) = \frac{T(\infty,s) - T(x,s)}{W(s) + Z(s)}$$

where

$$Z(s) = x(s) + \Delta x(s)$$

The corresponding partial derivatives of the equation above with respect to either x(s) or s are, respectively,

$$\left.\frac{d[S(x,s)]}{dx}\right|_{s=\text{constant}} = -\frac{2\ T(\infty,s) - T(x,s)]}{[W(s) + Z(s)]^2} \quad (3\text{-}12)$$

$$\left.\frac{d[S(x,s)]}{ds}\right|_{x=\text{constant}} = \frac{W(s) + Z(s)}{T(x,s) - T(\infty,s)} \cdot \frac{d}{ds}\left[\text{Ln}\ \frac{W(s) + Z(s)}{T(x,s) - T(\infty,s)}\right]$$

(3-13)

The general theorem on partial differentiation requires two conditions for a function like S(x,s) to have a realtive minimum as follows:

1. $\frac{\partial S(x,s)}{\partial x} = 0 \quad , \qquad \frac{\partial S(x,s)}{\partial s} = 0$

2. Let

$$F_{11} = \frac{\partial}{\partial x}\left\{\frac{\partial[S(x,s)]}{\partial x}\right\} \quad , \quad F_{12} = \frac{\partial}{\partial x}\left\{\frac{\partial[S(x,s)]}{\partial s}\right\}$$

$$F_{21} = \frac{\partial}{\partial s}\left\{\frac{\partial[S(x,s)]}{\partial x}\right\} \quad , \quad F_{22} = \frac{\partial}{\partial s}\left\{\frac{\partial[S(x,s)]}{\partial s}\right\}$$

then, S(x,s) has a relative minimum at a point when

a. $F_{11} > 0$

b. $[F_{12}]^2 - F_{11}F_{12} < 0$

However, the requirement for condition b is very impractical especially when the aid of the computer is available. The best approach then is to try points taken from condition "a" and subsequently select the minimum sensitivity from the result. In fact, this procedure can easily be determined by letting the computer make the decision where the minimum is and come out with the selected frequency where the sensitivity of the network is

minimum. This is conditioned on the premise taht the variable parameter x and frequency s are both allowed to vary simultaneously.

IV. SENSITIVITY WITH PARAMETER TOLERANCE

A. INTRODUCTION

The efficientcy of any network design depends crucially on the tolerances allowed on its parameters before the network ceases to perform its intended purpose. In any practical circuit, it is normal to find component elements whose values differ from that of the original design. Since this imperfect situation exists, it is imperative that more information be gathered as to the behavior of such elements in network design.

This chapter attempts to discuss the expected changes on sensitivity of the network for a given change of the variable parameter. This will help designers in predicting the possible limits in the magnitude of sensitivity of the network with known variable parameters. Discussion is also made on cases wherein the sensitivity of the network is previously specified and it is now required to find the tolerance or maximum change allowed on the variable parameter within the network.

B. PERCENTAGE CHANGE OF SENSITIVITY MAGNITUDE

It was discussed in Chapter III that the magnitude of sensitivity is one of the main criteria in studying the behavior of networks. It was observed that as the parameters within the network vary, the sensitivity also varies as compared to the original sensitivity of the network when there was no variation of parameters.

Aside from the variation of parameters, it is also obvious that sensitivity values is dependent upon frequency. With this information on hand, a theorem on the magnitude of sensitivity is derived.

<u>THEOREM IV-1</u>: In a linear network with a single variable parameter, the relative change in magnitude of sensitivity caused by the changes in the variable parameter is given by:

$$|C| = \left|\frac{S_f - S_o}{S_o}\right| \geqq \frac{|\Delta x(s)|}{|W(s) + x(s)| + |\Delta x(s)|} \qquad (4\text{-}1)$$

where

C = is the relative change in sensitivity caused by the change in the variable parameter.

S_f = is the exact sensitivity of the linear network when the variable immittance x(s) exhibits a change equivalent to Δx(s).

S_o = is the original sensitivity of the linear network when there is no change in the variable immittance.

all the other terms were as previously defined.

<u>Proof:</u>

The expression of sensitivity S as given in Eq. (3-1) results to the corresponding expressions of S_f and S_o. From the definitions of both terms, the two expressions are:

$$S_o = \frac{T(\infty,s) - T(x,s)}{W(s) + x(s)} \qquad (4\text{-}2)$$

and

$$S_f = \frac{T(\infty,s) - T(x,s)}{W(s) + x(s) + \Delta x(s)} \qquad (4\text{-}3)$$

The difference of the two expressions is:

$$S_f - S_o = \frac{T(\infty,s) - T(x,s)}{W(s) + x(s) + x(s)} - \frac{T(\infty,s) - T(x,s)}{W(s) + x(s)}$$

$$= \frac{-\Delta x(s)[T(\infty,s) - T(x,s)]}{[W(s) + x(s) + \Delta x(s)][W(s) + x(s)]}$$

Substituting the equation in Eq. (4-2) to the expression above, the difference $S_f - S_o$ may be written as:

$$S_f - S_o = -S_o \left[\frac{\Delta x(s)}{W(s) + x(s) + \Delta x(s)}\right] \qquad (4\text{-}4)$$

The magnitude of the whole expression is then taken.

$$|S_f - S_o| = |S_o| \frac{|\Delta x(s)|}{|W(s) + x(s) + \Delta x(s)|} \qquad (4\text{-}5)$$

In complex variable theory, the total sum of the magnitude of the individual terms is greater than or equal to the magnitude of the sum of the individual terms. Applying this theory to the denominator expression of Eq. (4-5) gives

$$|W(s) + x(s) + \Delta x(s)| \leq |W(s) + x(s)| + |\Delta x(s)| \qquad (4\text{-}6)$$

Dividing both sides of Eq. (4-5) by S_o results to the relative change in sensitivity.

$$\left|\frac{S_f - S_o}{S_o}\right| \geqq \frac{|\Delta x(s)|}{|W(s) + x(s)| + |\Delta x(s)|}$$

and this proves the theorem.

Theorem IV-1 shows that for a linear network with a single variable immittance, the change in the magnitude of sensitivity depends upon the frequency and the change in variable immittance equivalent to $\Delta x(s)$.

C. IMMITTANCE TOLERANCE FOR SPECIFIED SENSITIVITY

In circuit designs, the designer may wish to specify the limits of sensitivity wherein the network should operate. Within this framework, it becomes the job of the designer to place the allowed tolerances on the components of the network in order to meet the desired specification.

The previous discussion dealt with the relative change of sensitivity involving single parameter variation. From Eq. (4-1), another theorem can be derived giving the allowed tolerance of a single variable immittance connected to a network when the maximum allowable sensitivity of the circuit is specified.

<u>THEOREM IV-2</u>: In a linear network with a specified maximum sensitivity, the maximum allowable tolerance that the variable immittance should have is given as:

$$|\Delta x(s)| \leqq \frac{|S_f - S_o||W(s) + x(s)|}{|S_o| - |S_f - S_o|} \tag{4-7}$$

where the terms in the expression are as previously defined in Eq. (4-1).

Proof:

Starting from Eq. (4-1),

$$\frac{|S_f - S_o|}{|S_o|} \geqq \frac{|\Delta x(s)|}{|W(s) + x(s)| + |\Delta x(s)|}$$

Multiplying both sides of the equation by $|S_o|\left\{|W(s) + x(s)||\Delta x(s)|\right\}$, we obtain

$$|S_f - S_o|\left\{|W(s) + x(s)| + |\Delta x(s)|\right\} \geqq |\Delta x(s)||S_o| ,$$

or

$$|S_f - S_o||W(s) + x(s)| + |S_f - S_o||\Delta x(s)| \geqq |\Delta x(s)||S_o|$$

Subtracting from both sides the term $|S_f - S_o||\Delta x(s)|$, results in

$$|S_f - S_o||W(s) + x(s)| \geqq |\Delta x(s)||S_o||S_f - S_o||\Delta x(s)| .$$

therefore,

$$|\Delta x(s)| \leqq \frac{|S_f - S_o||W(s) + x(s)|}{|S_o| - |S_f - S_o|}$$

which proves the theorem. All the terms on the right hand side of Eq. (4-7) were previously known so that the maximum tolerance allowed on $x(s)$ can then be computed.

Example IV-1

Consider the network shown in Fig. 4 of Example III-1, from which the following quantities can be calculated:

$$T(Z_c,s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{2s+3}{3s+4} + \frac{1}{3s+4} Z_c}{\frac{2s+3}{3s+4} + Z_c}$$

$$= \frac{2s + 3 + Z_c}{2s + 3 + Z_c(3s+4)} ,$$

$$T(\infty,s) = \frac{1}{3s+4} , \qquad W(s) = \frac{2s+3}{3s+4}$$

Suppose that the network has the specification given as follows:

$Z_c = 1/s$ (one farad capacitance)

frequency $\omega = 1$ rad/sec

The magnitude of sensitivity should not exceed a certain value as will be discussed later.

It is required to find the maximum allowable tolerance on the capacitor to meet the design specification.

The solution is obtained for three specified values of sensitivity:

$$|S_o| = 1.6$$

a. $|S_f| \leqq 2.0$

$|\Delta x| \leqq 0.42$

b. $|S_f| \leqq 1.8$

$|\Delta x| \leqq 0.18$

c. $|S_f| \leqq 5.0$

$|\Delta x| \leqq 2.38$

The result of the example shows that it is now possible to find the maximum tolerance of the element x(s) for a given frequency and specified maximum allowable sensitivity.

<u>Example IV-2</u>

The next circuit that will be studied is the Bootstrapped Darlington circuit shown in Fig. 5a. Assume that the values of the parameters are as follows:

A_x = 1,000 ohms

B_x = 1,000 ohms

C_x = 40,000 ohms

D_x = 4,000 ohms

β_1 = 50 (initial)

β_2 = 50

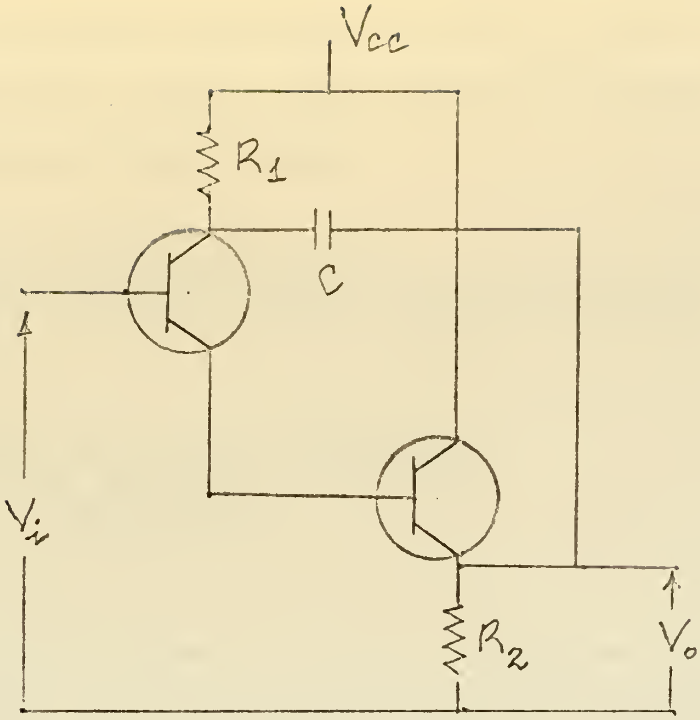

Figure 5a

Bootstrapped Darlington Circuit

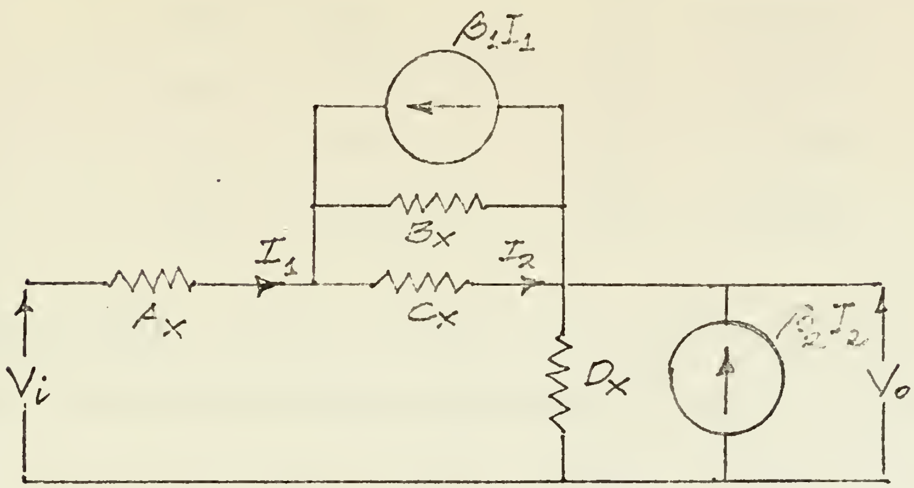

Figure 5b

Low Frequency Equivalent Circuit

The low frequency equivalent circuit is shown in Fig. 5b. From the equivalent circuit and using loop analysis, the following equations are derived;

$$T(\beta_1) = \frac{B_xD_x+C_xD_x+B_xD_x\beta_1+B_xD_x\beta_1\beta_2}{A_xB_x+A_xC_x+B_xC_x+C_xD_x+C_xB_x\beta_1+B_xD_x\beta_2+B_xD_x\beta_1\beta_2}$$

$$W\beta_1 = \frac{A_xB_x+A_xC_x+B_xC_x+B_xD_x+C_xD_x+B_xD_x\beta_2}{C_xB_x+B_xD_x\beta_2}$$

With the given parameter values, using the above expressions, the following quantities can be determined for various values of $\Delta\beta_1$:

$$W\beta_1 = 1.854$$

for	a. $\Delta\beta_1 = -50.0$	$S_f = -3.17309$
	b. $\Delta\beta_1 = -25.0$	$S_f = -0.00541$
	c. $\Delta\beta_1 = 0.0$	$S_o = 0.08484$
	d. $\Delta\beta_1 = 25.0$	$S_f = 0.11608$
	e. $\Delta\beta_1 = 50.0$	$S_f = 0.13192$

From the data above, it can be observed that the original sensitivity of the network S_o is equal to 0.08484.

It is now desired to find the maximum allowed tolerance of the parmeter β_1 for which the maximum percentage change of sensitivity does not exceed 50% of its original value. Using Eq. (4-7) we obtain

$$|S_f - S_o| = 0.04242$$

$$|S_o| = 0.08484$$

$$|W(s) + x(s)| = |W_{\beta_1} + \beta_1| = |1.854 - 50| = 48.146$$

hence

$$|\Delta x(s)| = |\Delta\beta_1| \leqq \frac{(0.04242)(48.146)}{0.04242} = 48.146$$

The sensitivity values taken from Table IV-1 confirm this result.

It might also be necessary to determine the percentage change in the magnitude of sensitivity. Assuming the maximum value of $\Delta\beta_1$ is 25, how much will the sensitivity deviate from its original network sensitivity value? From Eq. (4-1) we obtain

$$|C| = \frac{S_f - S_o}{S_o} \geqq \frac{25}{48.146 + 25}$$

$$= \frac{25}{73.146} = 0.342$$

Example IV-3

Consider the operational amplifier circuit shown in Fig. 6. The voltage transfer function is given as

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{T_A(s)}{T_B(A) + 1/A}$$

TABLE IV-1

$\Delta\beta_1$	$\Delta\beta_2$	Sensitivity
-50.0	-50.0	-7.71812
	-25.0	-5.06164
	0.0	-4.14932
	25.0	-3.57167
	50.0	-3.17309
-25.0	-50.0	-1.43290
	-25.0	-0.35211
	0.0	-0.14331
	25.0	-0.05454
	50.0	-0.00541
0.0	-50.0	-0.85570
	-25.0	-0.13078
	25.0	0.05475
	50.0	0.08484
25.0	-50.0	-0.63423
	-25.0	-0.05231
	0.0	0.05007
	25.0	0.09271
	50.0	0.11608
50.0	-50.0	-0.51709
	-25.0	-0.01213
	0.0	0.07557
	25.0	0.11198
	50.0	0.13192

and

$$W(s) = 1/A$$

$$x(s) = T_B(s)$$

In this example, assume the following values:

$$T_B(s) = 0.9 \text{ at a frequency } \omega = 1 \text{ rad/sec}$$

$$A = -50.0$$

From the information given above, it is required to find the percentage change in the magnitude of sensitivity when $\Delta T_B(s)$ has a maximum value of 0.5 at the operating frequency of $\omega = 1$ radians per second. The computation proceeds as follows:

$$|C| = \frac{S_f - S_o}{S_o} \geq \frac{|0.5|}{|-\frac{1}{50} + .9| + |0.5|}$$

$$= \frac{0.5}{0.5 + .88} = \frac{0.5}{1.38}$$

$$= 0.3623 \quad \text{or} \quad 36.23\%$$

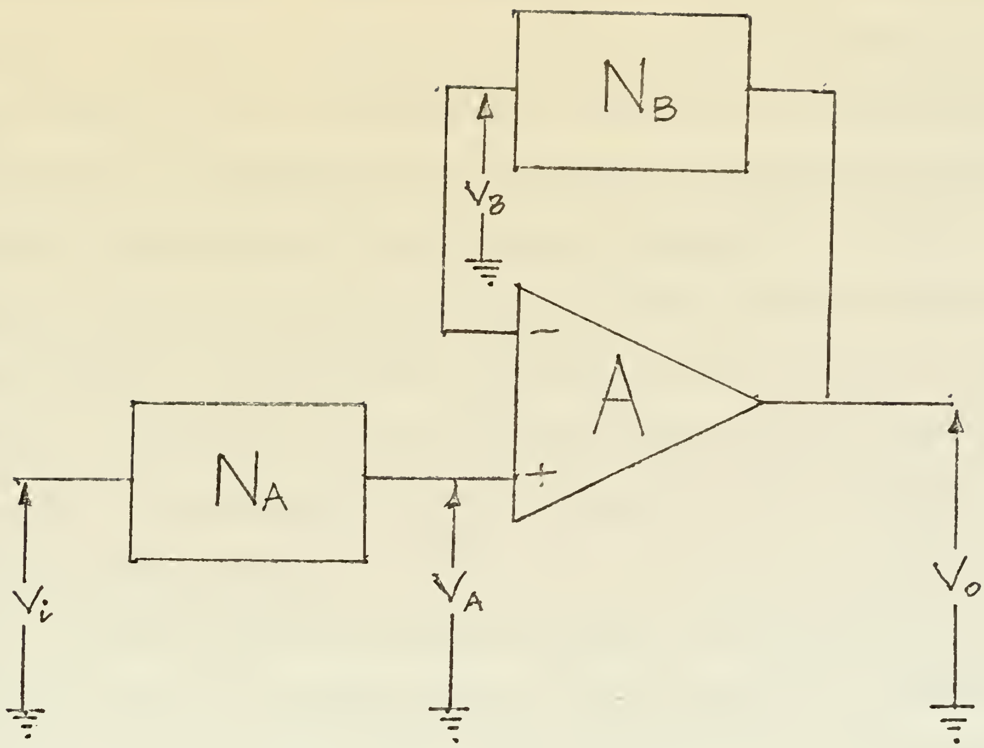

Figure 6

Operational Amplifier Circuit

V. CONCLUSIONS

It is observed that the network function of general linear, passive or active network, could be expressed as a linear combination of the unilateral or bilateral immittances composing the network. This theory was proved by Troop and Peskin [9]. A review was made on the results derived in [2], [6], [9] and [10] as a theoretical foundation of this thesis.

Sensitivity minimization in networks becomes more important as the complexity of the circuit increases. It was determined that it is possible to seek the minimum sensitivity of the network. A possibility exists that with proper transformation, a network can be generated from a design specification such that its sensitivity is minimum at the desired frequency of operation.

It is often normal in network design to encounter problems wherein the range of sensitivity of the circuit is already specified. It was shown that this case can be solved with specific tolerances on the elements and other variable parameters of the network. The limit allowed to the tolerance of the elements can then be specified to meet the design specification of sensitivity.

APPENDIX

SUGGESTED TOPICS FOR FURTHER STUDY

As a result of this investigation several possibilities for continuing research are suggested as listed below;

1. Investigate the conditions of minimum sensitivity of network with multiple variations using the general expression of exact sensitivity.

2. Adopt a procedure to synthesize the network when the given functions are the Thevenin immittances W, the variable immittance x and the limits of the network function T(0) and T(∞) as the variable x approaches zero and infinity respectively. This procedure can easily provide a method to transform a network directly into a minimum, maximum or even zero sensitivity.

3. Investigate the amount of error from the general exact sensitivity expression when mutual coupling between variation are truncated, i.e. products of x_i and x_j and higher product terms. This information can greatly simplify the expression for computational calculation.

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Linear Networks

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Network functions and sensitivity expressions for linear networks are reviewed for networks with single unilateral or bilateral imittances and the corresponding extension to the general case of n variable immittances. The sensitivity expression is exact and suitable for the analysis of networks with either small or large variation of parameters.

A method is proposed for minimizing sensitivity using the

20.

exact expression. The general linear network is studied in various aspects like changes on frequency, variation of elements or mixed variation of frequency and network components. By proper network transformation and synthesis, it is possible to design a network with minimum sensitivity even if the variation in immittance is large. Generally, for small or incremental variation of immittance, the sensitivity of the network function remains invariant. However, this theorem does not hold in cases of large variation.

Normally, the range of sensitivity for which a network is to operate, is specified. It follows that the elements and parameters composing the network have certain ranges. It was determined that with specified range of sensitivity, the allowable range of values for the parameters of the network can be computed.

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