THE RELATIONSHIP BETWEEN A (Q,r) INVENTROY POLICY AND A AGE REPLACEMENT MAINTENANCE POLICY

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THESIS

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and an
Age Replacement Maintenance Policy

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ABSTRACT

Cost models are developed to show the relationship between inventory and maintenance policies when one component of one item of equipment is replaced in accordance with the maintenance policy and the components are stocked in accordance with the inventory policy. A (Q,r) inventory policy and failure replacement and age replacement maintenance policies are used. The necessary conditions for determining optimum values of the reorder quantity, the reorder point, and time to replacement are derived.

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I. INTRODUCTION

The situation to be considered is as follows: There is one piece of equipment which will not operate only if a certain component has failed. Otherwise it will operate satisfactorily. The component fails according to a known probability distribution. There is a maintenance policy which dictates when the component will be replaced and a stock of these components is kept available for use as replacements. There is an inventory policy which determines the procedures for replenishing the stock of components.

When the maintenance policy dictates that the component is to be replaced, it is replaced if there is a component in inventory. If the stock on hand is zero, the equipment becomes inoperative and the time during which the equipment is inoperative is called downtime.

The subject of inventory and inventory policies has been examined in considerable depth in many books and papers. In this paper only one type of inventory policy will be considered, namely a (Q,r) inventory policy. (See Hadley and Whitin [1]). This policy is characterized by specifying a reorder quantity (Q) and a reorder point (r). The policy requires that an order for a quantity Q of the item be placed whenever the stock on hand reaches level r. The length of time from the placing of an order to its receipt is called the lead time and may be fixed or variable. The demand on the inventory system is stochastic and is dependent on the probability distribution of failures and the maintenance policy.

A maintenance policy determines when a piece of equipment will be inspected, repaired or replaced. McCall [2] has published a survey of maintenance policies for stochasticly failing equipment. In what follows, only two types of maintenance policies will be considered, failure replacement and age replacement policies. In a failure replacement policy, a component is replaced upon failure; in age replacement the component is replaced upon failure or after it has been in use a prescribed length of time, whichever comes first. If an item has a continuous, strictly increasing failure rate, Barlow and Proschan [3] have shown that there exists a unique replacement age which may be infinite, (i.e., use failure replacement).

It is easily seen that if the time to replacement of an item is shortened, the quantity used in a given length of time will increase. This means that the demand on the inventory system will increase. It also demonstrates that there is at least one easily seen connection between maintenance and inventory policies. McCall [2] states that the interaction between maintenance and inventory models has never been analyzed and suggests that the question merits additional research. Falkner [4] considered this interaction in the case in which there is a fixed planning horizon and the time to replacement is recalculated after every replacement; i.e., a sequential maintenance policy is used.

The problem is formulated as a dynamic programming problem and methods of determining the optimal values for initial inventory and times to replacement are given.

Although all equipment certainly has a finite life span, and optimal sequential replacement policies have been shown to be superior to optimal periodic policies for a finite life span (see Barlow and Proschan [5]), sequential replacement policies often prove to be impractical. Equipment life span in the military is usually of indefinite length and is sufficiently long that it may be considered infinite. The application of a sequential maintenance policy requires an excessive amount of administrative overhead and requires techniques that are usually not known to individuals performing maintenance on anything but the most sophisticated equipment. For these reasons the infinite horizon models considered in this paper are more realistic in many military situations. The decision variables are expressed in such a way that they can be more easily understood and applied.

The first case considered is the use of a (Q,r) inventory policy and a failure replacement policy. A cost model is developed which is used to determine a minimum cost inventory policy. For those cases in which cost is not considered to be the proper criterion (or sole criterion) for optimization, constraints on the length of downtime and the percentage of downtime are considered.

The second case to be considered is the use of a (Q,r) inventory policy and an age replacement maintenance policy.

A cost model is developed which is similar to the cost model for the first case. The model does contain an additional decision variable, the length of time to scheduled replacement of a component. The first case is actually a special case of the second case in which the age to replacement is infinite. However for clarity we present the two cases separately.

II. COST MODEL WITH FAILURE REPLACEMENT

A. DEFINITION OF TERMS

Reorder Quantity (Q) - The quantity of parts that is ordered each time stock is replenished.

Reorder Point (r) - The inventory level at which the order for the reorder quantity is placed.

Lead Time (T) - The time lag from the time an order is placed until it is received.

Cycle Length (L) - The time from placing of one order to the placing of the next.

Fixed Reorder Cost (A) - The fixed costs of placing an order for more inventory.

Holding Cost (h) - The cost per item per unit time charged on all items used during a cycle over the length of a cycle.

Downtime Cost (π) - The cost of downtime for one unit of time.

Purchase Cost (C) - The cost to buy one item of inventory.

Failure Replacement Cost (C_f) - The cost of replacing the failed component on the equipment. This cost is in addition to the purchase cost C.

Length of Downtime (D) - The length of time in one cycle during which the equipment cannot be operated.

B. SITUATION AND ASSUMPTIONS

Consider the situation as described in the introduction, in which a piece of equipment is maintained under a failure replacement policy and a (Q,r) inventory policy is used to supply the replacement parts. The lifetimes of the components are independent, identically distributed random variables with a known distribution function, F(x). This function is assumed to have an increasing failure rate, that is

$$\frac{f(t)}{\overline{F}(t)} \text{ is increasing in } t,$$

where f(t) is the density function and $\overline{F}(t)=1-F(t)$. The probability of failure of an item with an increasing failure rate increases with time in service, i.e., it exhibits wear-out.

In the development of the model further assumptions are made. It is assumed that the time required for component replacement is zero, thus downtime occurs only when the stock of replacement parts is exhausted. The lead time is assumed to be fixed. The cost of downtime is assumed to be constant throughout the length of the downtime. Figure 1 illustrates the inventory level in a system using a (Q,r) inventory policy and a failure replacement maintenance policy. It is assumed that $Q \geq r+1$, this insures that when an order is received, the quantity on hand is at least r.

Inventory Levels with Failure Replacement Policy

Q = 4, r = 2

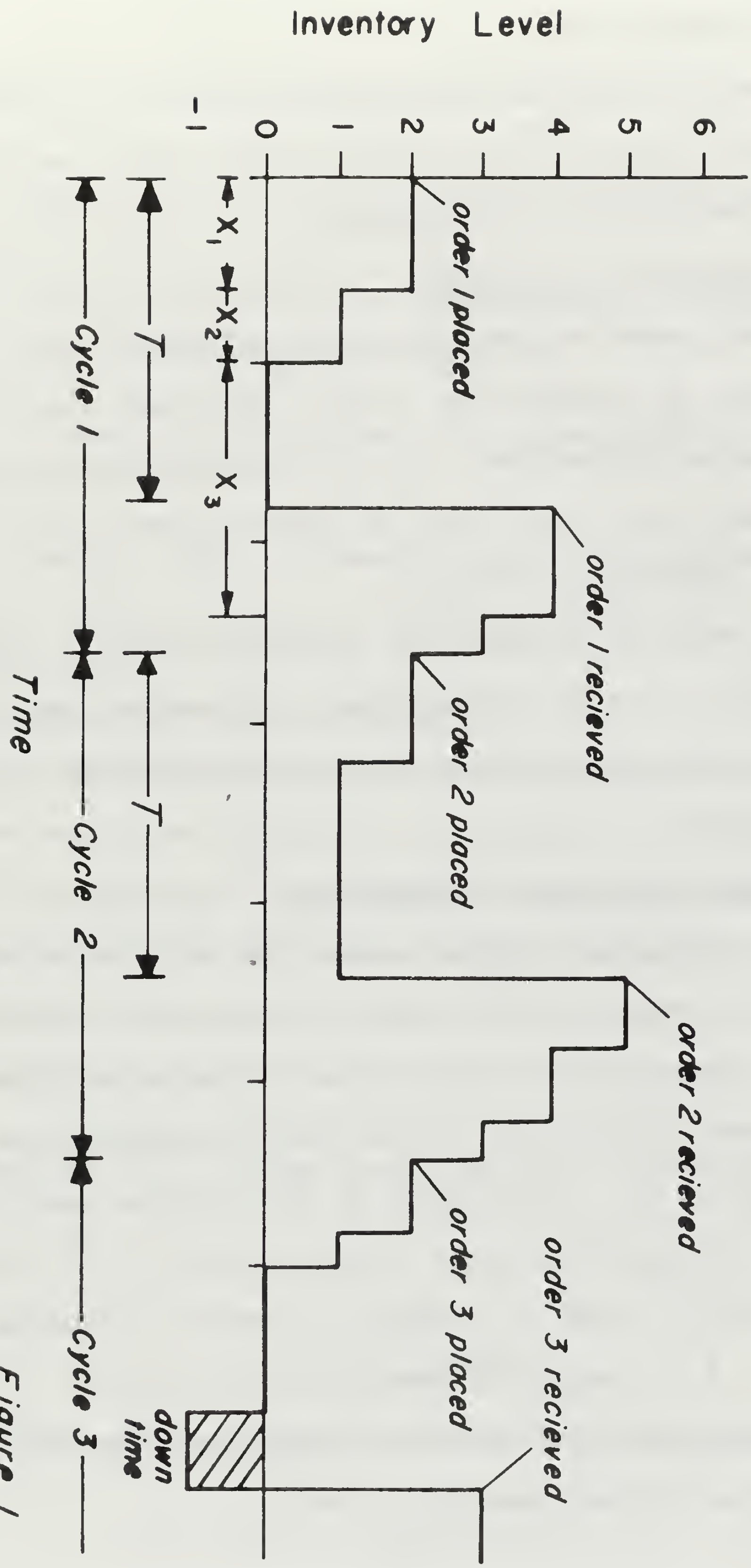


Figure 1

C. MODEL FORMULATION

In formulating the model, an expression is first developed for the costs for one cycle; these costs are then averaged over an infinite horizon.

1. Costs for One Cycle

The costs for one cycle are expressed as:

cost of placing one order + purchase cost + holding costs + costs of downtime + cost of replacing the component.

Thus, the total cost, K, for one cycle is:

$$K=A+CQ+hQL+\pi D+C_f Q \qquad (1)$$

In the above equation, the quantities L, cycle length, and D, length of downtime, are random variables. Expressions for the expectations of these random variables are now found.

2. Expected Length of Downtime

As previously stated, downtime will occur if the component in service fails when the inventory level is zero. Since the inventory level is r when an order is placed, r+1 failures must occur prior to the end of the leadtime T for downtime to occur. An example of this can be seen in figure 1.

Let X_i = time to failure of the ith component,

Y = length of downtime in a cycle.

Since downtime cannot be less than zero and the leadtime T is a fixed quantity, then

$$Y = \text{Max}[0, T - \sum_{i=1}^{r+1} X_i].$$

Defining $S_n = \sum_{i=1}^{n} X_i$, then $Y = \text{Max}[0, T - S_{r+1}]$.

Let the distribution of Y be $G(y) = P[Y \leq y]$. Then

$$G(y) = P[S_{r+1} > T-y] = \overline{F}_{r+1}(T-y) \quad y \leq T, \tag{2}$$

where $F_n(x)$ is the n-th convolution of F(x) with itself.

Let $\delta = E[Y]$ = expected length of downtime per cycle,

$$\delta = E[Y] = \int_0^T [1-G(y)]dy = \int_0^T [1-\overline{F}_{r+1}(T-y)]dy = \int_0^T F_{r+1}(x)dx. \tag{3}$$

3. Expected Length of a Cycle.

A cycle has been defined as the length of time from the placing of one order to the placing of the next order. During a cycle there are exactly Q failures, each preceded by a length of time X_i, during which the equipment is operating.

During a cycle the length of time the equipment is operating is $\sum_{i=1}^{Q} X_i$, the length of time the equipment is not operating is D, the downtime. Thus the length of a cycle is

$$L = \sum_{i=1}^{Q} X_i + D.$$

Taking expectations and letting $\mathcal{L} = E[L]$

$$\mathcal{L} = E[L] = \sum_{i=1}^{Q} E[X_i] + E[D].$$

Let $E[X]=\mu$ the mean time between failures (MTBF).

Thus, $\mathcal{L} = Q\mu+\int_0^T F_{r+1}(x)\,dx$.

4. <u>Costs for an Infinite Horizon</u>

From equation (1), the average costs per cycle may be determined by taking expected values. That is, the costs are averaged over an infinite horizon. Letting K = the cost for one cycle,

$$E[K] = A + CQ + hQE[L] + \pi E[D] + C_f Q$$
$$= A + CQ + hQ\mathcal{L} + \pi\delta + C_f Q \ .$$

The sequence of cycles is a renewal process since the cycle lengths are independent, identically distributed random variables. Let the expected number of renewals in [0,t] be H(t). The expected cost for the interval [0,t] will then be

$E[Cost] = H(t)E[K] + \varepsilon(t)$, where $\varepsilon(t)/t \to 0$ as $t \to \infty$, since $\varepsilon(t)$ represents the cost in the part of a cycle in [0,t] and hence is bounded.

The average cost per unit time is

$$\lim_{t\to 0} \frac{E[Cost]}{t} = \lim_{t\to 0} \left[\frac{H(t)}{t} E[K] + \frac{\varepsilon(t)}{t} \right]$$

$$= E[K] \lim_{t\to 0} \frac{H(t)}{t} + 0 \ .$$

Since $\lim_{t\to 0} \frac{H(t)}{t} = \frac{1}{\mathcal{L}}$ the average cost per unit time is:

$$\mathcal{K} = \frac{E[K]}{\mathcal{L}} = \frac{A + CQ + C_f Q + \pi\delta}{\mathcal{L}} + hQ \ . \qquad (4)$$

Since $\mathcal{K}$ is a function of Q and r equation (4) may be written

$$\mathcal{K}(Q,r) = \frac{A + (C+C_f)Q + \pi \int_0^T F_{r+1}(x)\,dx}{Q\mu + \int_0^T F_{r+1}(x)\,dx} + hQ \; . \tag{5}$$

5. Properties of Cost Function

$\delta(r) = \int_0^T F_{r+1}(x)\,dx$ is a decreasing function of r. It follows from this that $\mathcal{K}(Q,r)$ is either monotone increasing or monotone decreasing in r, depending on the relative magnitudes of certain costs in equation (5). Let $C' = C_f + C$.

If $\frac{A+C'Q}{\pi Q\mu} \geq 1$, $\mathcal{K}$ is monotone increasing in r.

If $\frac{A+C'Q}{\pi Q\pi} \leq 1$, $\mathcal{K}$ is monotone decreasing in r.

Since $-1 \leq r \leq Q-1$ the optimum value of r is

$$\begin{aligned} r &= -1 \text{ if } \frac{A+C'Q}{\pi Q\mu} > 1 \; , \\ r &= Q-1 \text{ if } \frac{A+C'Q}{\pi Q\mu} < 1 \; . \end{aligned} \tag{6}$$

If $\frac{A+C'Q}{\pi Q\mu} = 1$ the function is constant in r.

At $r = -1$, $\delta = T$, thus

$$\mathcal{K}(Q,-1) = \frac{A+C'Q+\pi T}{Q\mu+T} + hQ \; .$$

Assuming $\mathcal{K}$ is continuous and differentiable in Q the optimum value of Q is the solution to $d\mathcal{K}/dQ=0$. From the above,

$$\frac{d\mathcal{K}}{dQ} = \frac{C'}{Q\mu+T} - \frac{A+C'Q+\pi T}{(Q\mu+T)^2} + h = 0 \; ;$$

after solving for Q ,

$$Q = \sqrt{\frac{A\mu+(\pi\mu-C')T}{h\mu^2}} - \frac{T}{\mu} .$$

For large values of r, $\delta \simeq 0$, thus

$$\mathcal{K}(Q,r) = \frac{A+C'Q}{Q\mu} + hQ .$$

In the same manner as above, the optimum Q for large values of r is:

$$Q = \sqrt{\frac{A}{h\mu}} . \qquad (7)$$

This is nearly identical to the well known lot-formula [1] and would be identical if the assumption had been made that holding costs were proportional to Q/2.

For large values of r, the second derivative of $\mathcal{K}$ with respect to Q is

$$\frac{d^2\mathcal{K}}{dQ^2} = \frac{2\mu(\mu A+\mu C'Q)}{(Q\mu)^3} - \frac{2\mu C'}{(Q\mu)^2}$$

$$= \frac{2A}{Q^3\mu} \geq 0 .$$

Thus $\mathcal{K}(Q,r)$ is convex in Q for large r. For r =-1, the second derivative of   is

$$\frac{d^2\mathcal{K}}{dQ^2} = \frac{2\mu(\mu A+\mu C'Q+\mu\pi T)}{(Q\mu+T)^3} - \frac{2\mu C'}{(Q\mu+T)^2}$$

$$= \frac{2\mu^2 A+2\pi\mu^2 T-2\mu C'T}{(Q\mu+T)^2} .$$

This will be positive if $\mu(A+\pi T) > C'T$ which is then the necessary condition to have convexity in Q.

D. DOWNTIME RESTRICTIONS

It is frequently desirable, particularly in the military, to maintain the length of downtime or the percentage of downtime below a certain level. The necessary conditions for applying these restrictions have been developed and the problem of applying them subject to cost constraints has been considered.

1. Restrictions on the Length of Downtime

If D = downtime then, from equation (2),

$$P[D \leq y] = \overline{F}_{r+1}[T-y] \qquad 0 \leq y < T$$

$$= 1 \qquad\qquad y \geq T$$

Note that the length of downtime is a function of r alone, given T as fixed. If it is desired to control the length of downtime this can be done by varying r. For example, since downtime will always be a fraction of T, to keep this fraction less than or equal to α, with probability at least β, we must choose α and β such that $P[Y \leq \alpha T] \geq \beta$.

That is, $1-\beta \geq F_{r+1}[(1-\alpha)T]$.

Since $F_{r+1}(x)$ is a decreasing function of r and $\lim_{r\to\infty} F_{r+1}(x) = 0$, there is a minimum value of r for which the above inequality holds.

2. Fraction of Downtime

The expected fraction of time during which the item of equipment will be down is:

$$\text{fraction downtime} = \frac{\text{expected length of downtime}}{\text{expected length of cycle}} =$$

$$\frac{\delta}{Q\mu+\delta} = \frac{\int_0^T F_{r+1}(x)\,dx}{Q\mu+\int_0^T F_{r+1}(x)\,dx}.$$ We wish to investigate

how this function varies with r and Q.

The following lemma will be needed: If $A>0$ and $B>0$,

then $$\frac{A}{B+A} \geq \frac{A-\varepsilon}{B+A-\varepsilon} \quad \text{for } \varepsilon>0.$$

Proof: assume

$\frac{A}{B+A} < \frac{A-\varepsilon}{B+A-\varepsilon}$, then $A(A+B) - A\varepsilon < A(A+B) - A\varepsilon - B\varepsilon$ and $0 < -B\varepsilon$, but this contradicts the fact that $\varepsilon>0$ and $B>0$, therefore, $\frac{A}{B+A} \geq \frac{A-\varepsilon}{B+A-\varepsilon}$.

The maximum amount of downtime that can occur is T. This will occur when $r= -1$. For any $r>-1$, $\delta<T$.

Thus, from the above lemma

$$\text{fraction downtime} \leq \frac{T}{Q\mu+T}.$$

It can be seen from the above that by increasing Q the fraction of downtime may be made as small as desired. Note also that δ is a decreasing function of r, so, subject to the constraint $Q>r$, by increasing r, the fraction downtime may be made as small as desired.

3. Cost Constraints on Downtime

By increasing Q and r and keeping $Q>r$ both the expected length of downtime and the fraction of downtime may be made as small as desired. Normally there will be a cost constraint of some type; the problem can then be expressed as

$$\text{Minimize (fraction downtime)} = \frac{\delta}{Q\mu+\delta}$$

$$\text{subject to } \mathcal{K}(Q,r) \leq b$$

where b is a cost constraint;

or,

$$\text{Minimize } \mathcal{K}(Q,r)$$

$$\text{subject to } \frac{\delta}{Q\mu+\delta} < p. \qquad 0 \leq p \leq 1$$

in this case if $\frac{T}{Q\mu+T} \leq p$ the constraint is not active and need not be considered in the minimization.

If the length of downtime is of concern the problem is,

$$\text{Minimize } \mathcal{K}(Q,r)$$

$$\text{subject to } 1-\beta \geq F_{r+1}[(1-\alpha)T]$$

or

$$\text{Minimize } F_{r+1}[(1-\alpha)T]$$

$$\text{subject to } \mathcal{K}(Q,r) \leq b \ .$$

III. COST MODEL WITH AGE REPLACEMENT

The cost model with age replacement is a generalization of the model with failure replacement presented in II.

A. DEFINITION OF TERMS

The following definitions are required in addition to those given in II.A.

Time to replacement (t_o) - The time between the placing of a component in service and its scheduled removal assuming the component does not fail before t_o.

Scheduled Replacement Cost (C_s) - The cost of replacing a component at the scheduled time to replacement (t_o).

B. SITUATION AND ASSUMPTIONS

The situation is the same as previously described. The same assumptions still hold. In addition, it is assumed that the cost of scheduled replacement of a component is less than the cost of failure replacement, $C_s < C_f$.

C. MODEL FORMULATION

The development of a cost model for an age replacement policy follows very closely the development in II. There are some additional costs to consider and in addition to the distribution of failures F(x), the distribution of time between replacements must be known.

1. Distribution of Time Between Replacements

The use of an age replacement policy means that an item in service will be replaced upon failure or if the time in service reaches t_o, whichever occurs first. The distribution of time between failures is still F(x), a continuous function with an increasing failure rate. Defining $\hat{F}(x)$ = distribution of time between replacements with an age replacement policy, $\hat{F}(x)$ has the following distribution:

$$\begin{aligned} \hat{F}(x) &= F(x) \qquad 0 \leq x < t_o \\ \hat{F}(x) &= 1 . \qquad x \geq t_o \end{aligned} \tag{8}$$

Let $\hat{\mu}$ be mean time between replacements, then

$$\hat{\mu} = \int_0^{t_o} [1-F(x)]dx .$$

2. Expected Costs for One Cycle

The expected costs for one cycle are:

ordering cost + purchase cost + expected holding costs + expected costs of downtime + expected costs of replacing items upon failure + expected costs of replacing items upon schedule at time t_o.

If X is the time to failure of a component, then the probability of the item failing before the scheduled replacement is $P[X \leq t_o] = F(t_o)$. For one cycle, the expected cost of replacing items upon failure is the cost of a failure replacement times the expected number of items that will fail, which is the number of items used in a cycle, Q, times the probability of failure, $F(t_o)$. Thus, the expected cost of replacing items upon failure = $C_f Q F(t_o)$. Similarly, the expected cost of replacing items upon schedule is $C_s Q[1-F(t_o)]$.

The expected costs for one cycle =

$$A + CQ + hQ\hat{\ell} + \pi\hat{\delta} + C_f QF(t_o) + C_s Q(1-F(t_o)) .$$

3. Costs for an Infinite Horizon

Using the same argument as in II.C.4 the costs per unit time averaged over an infinite horizon are obtained.

$$\mathcal{K} = \frac{A + CQ + C_f QF(t_o) + C_s\ Q(1-F(t_o)) + \pi\hat{\delta}}{\hat{\ell}} + hQ .$$

Recalling that $\hat{\ell}$ and $\hat{F}_{r+1}$ are functions of t_o, this may be written

$$\mathcal{K}(Q,r,t_o) = \frac{A+Q(C+C_f)F(t_o)+Q(C+C_s)\overline{F}(t_o)+\pi\int_0^T \hat{F}_{r+1}(x)\,dx}{Q\hat{\mu} + \int_0^T \hat{F}_{r+1}(x)\,dx} + hQ .$$

4. Properties of the Cost Function

Since $\hat{\delta}(r) = \int_0^T \hat{F}_{r+1}(x)\,dx$ is a decreasing function of r, the same type of argument as in II.C.5 gives the following for optimum values of r;

letting $C'=C+C_f$ as before and letting $C''=C+C_s$

$$r=-1 \quad \text{if} \quad \frac{A+C'QF(t_o)+C''Q\overline{F}(t_o)}{\pi Q\hat{\mu}} > 1 ,$$

$$r=Q-1 \quad \text{if} \quad \frac{A+C'QF(t_o)+C''Q\overline{F}(t_o)}{\pi Q\hat{\mu}} < 1 .$$

Note that the optimum value of r is dependent on both Q and t_o.

At r=-1 $\hat{\delta} = T$, using this fact and assuming $\mathcal{K}$ is continuous in Q, by taking partial derivatives of $\mathcal{K}(Q,-1,t_o)$ with respect to R and t_o the required conditions for optimum Q and t_o can be found.

$$\frac{\partial K}{\partial Q} = h + \frac{C'F(t_o)+C''\bar{F}(t_o)}{Q\hat{\mu}+T} - \frac{A+QC'F(t_o)+QC''\bar{F}(t_o)+\pi T}{(Q\hat{\mu}+T)^2}$$

setting this equal to zero, we find

$$Q = \sqrt{\frac{\hat{\mu}(A+\pi T)-(C'F(t_o)+C''\bar{F}(t_o))T}{h\hat{\mu}^2}} - \frac{T}{\hat{\mu}} . \tag{9}$$

Taking partial derivatives with respect to t_o

$$\frac{\partial K}{\partial t_o} = \frac{QC'f(t_o)-QC''f(t_o)}{Q\hat{\mu}+T} - Q\hat{\mu}'\frac{A+QC'F(t_o)+QC''\bar{F}(t_o)+ T}{(Q\hat{\mu}+T)^2} .$$

note that $\hat{\mu}=\int_0^T F(x)\,dx$

and $\hat{\mu}' = \partial\hat{\mu}/\partial t_o = \bar{F}(t_o)$ by Leibnitz rule

$$0 = \frac{Q(C'-C'')f(t_o)}{Q\hat{\mu}+T} - Q\bar{F}(t_o)\frac{A+QC'F(t_o)+QC''\bar{F}(t_o)+ T}{(Q\hat{\mu}+T)^2} .$$

The optimal t_o is the solution to the above equation.

Providing the necessary conditions for r=-1 hold throughout the region under consideration, an iterative procedure may be used to determine the optimum Q and t_o.

Assuming that r =Q-1 and Q is sufficiently large, the necessary conditions for a minimum $K(Q,r,t_o)$ are:

$$Q = \sqrt{\frac{A}{h\hat{\mu}}} \quad \text{and}$$

$$\frac{Q(C'-C'')f(t_o)}{Q\hat{\mu}} - \frac{\bar{F}(t_o)}{Q\hat{\mu}^2}(A+QC'F(t_o)+QC''\bar{F}(t_o)) = 0 .$$

Rearranging this equation we get

$$\hat{\mu}\frac{f(t_o)}{\overline{F}(t_o)} - \overline{F}(t_o) = \frac{C''}{C'-C''} + \frac{A}{Q(C'-C'')} .$$

This is very similar to the equation given in reference [2] for determining the optimum t_o when an age replacement policy is used. The above equation differs only by the inclusion of the term $\frac{A}{Q(C'-C'')}$ on the right hand side. If A is zero, it is identical to the equation in [2] which does not consider costs due to purchasing and holding inventory. Note also that for large Q or large C the effect of this term is small.

Following the same procedures as in II.C.5 it can be shown that $\mathcal{K}(Q,r,t_o)$ is convex for large values of r and that for r=-1 the following condition must hold for convexity in Q,

$$\hat{\mu}(A+\pi) > (C'F(t_o)+C''\overline{F}(t_o))T .$$

D. DOWNTIME RESTRICTIONS

1. Length of Downtime

Using the same reasoning as in II.D.1 but with the distribution of time to replacement given by (8), to keep the downtime less than or equal to a fraction α of T, the lead time, with probability β; the following must hold.

$$1-\beta \geq F_{r+1}[(1-\alpha)T] .$$

For any r, and letting $k= 1-\alpha$,

$$\hat{F}_{r+1}(kT) = 1 . \qquad kT > (r+1)t_o$$

$$= F_{r+1}(kT) . \quad kT < (r+1)t_o$$

Thus, if $(1-\alpha)T > (r+1)t_o$, β must equal zero, that is if $\alpha T < T-(r+1)t_o$ then $\beta=0$. This means that the lower limit on downtime is $T-(r+1)t_o$.

setting $T-(r+1)t_o$ less than or equal to zero

we get $\quad t_o \geq \frac{T}{r+1}$

This implies that if $t_o \leq \frac{T}{r+1}$ downtime will occur every cycle and it will be no less than $T-(r+1)t_o$.

This result indicates that a strict age replacement policy is defective in some manner. This is the case and this defect will be discussed further in V, where the effects of a more realistic policy will be examined.

For $(1-\alpha)T<(r+1)t_o$,

$1 - \beta \geq F_{r+1}[(1-\alpha)T]$ must hold and as in II.D.1 there is an r' such that

for all $r>r'$

$1-\beta>F_{r+1}[(1-\alpha)T]$ for all α and β .

2. <u>Fraction of Downtime</u>

The expected fraction of downtime is

$$\frac{\hat{\delta}}{Q\hat{\mu}+\hat{\delta}} = \frac{\int_0^T \hat{F}_{r+1}(x)\,dx}{Q\hat{\mu}+\int_0^T \hat{F}_{r+1}(x)\,dx}$$

in a manner similar to II.D.2 it can be shown that

$$\text{fraction downtime} \leq \frac{T}{Q\hat{\mu}+T} .$$

Since $\hat{\delta}$ is nonincreasing in t_o and $\hat{\mu}$ is increasing in t_o, the fraction of downtime is decreasing in t_o. This follows from the above lemma. The fraction of downtime will reach a minimum with respect to t_o at $t_o = \infty$, i.e., when a failure replacement policy is used.

This means that the fraction of downtime can be decreased by increasing Q or r as was seen in II.D.2 and in addition a decrease in downtime can be brought about by increasing t_o.

3. Cost Constraints on Downtime

Constrained minimization problems similar to those given in II.D.3 can easily be developed if required.

IV. NUMERICAL EXAMPLES

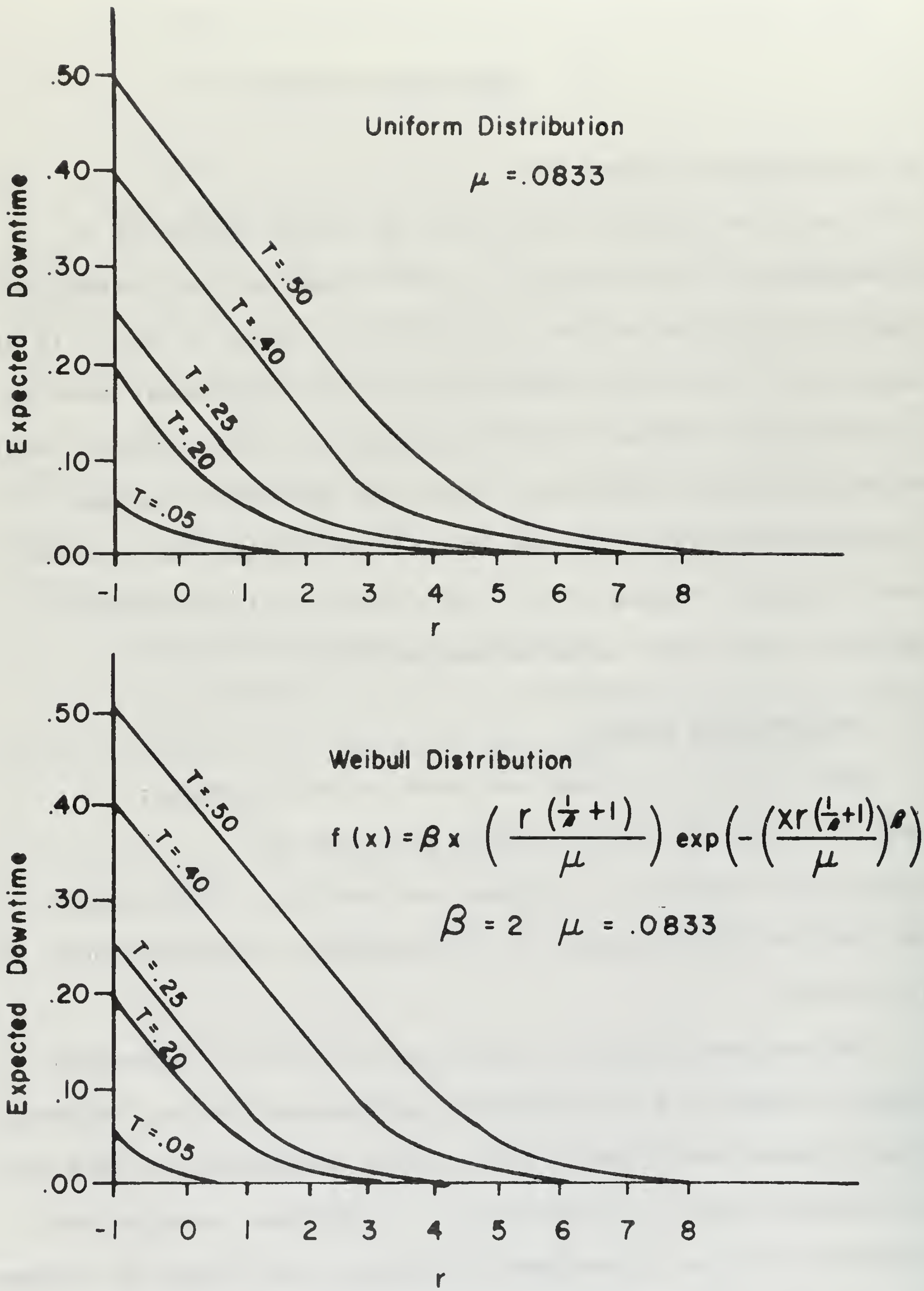
A. INCREASING VALUES OF r

Since the optimal solutions for large values of r obtained in II.C.5 and III.C.4 are based on the assumption that the expected value of downtime is close to zero, it is desirable to see how rapidly this value approaches zero as r increases. Figure 2 contains graphs of the expected value of downtime for increasing r for two different failure distributions with the same mean. The values are graphed for different values of T. Note that in all cases for r greater than 7 the value is negligible.

B. DETERMINING OPTIMA

The following values are used in this example: $A=\$5.00, \pi=\$50.00, C=\$2.00, C_f=\$1.00, C_s=\$0.25$, $h=\$0.10, T=3$ months or .25 year, $\mu=1$ month or .0833 years. The failure distribution of the component being replaced is uniform.

The optimum value of r will be Q-1 for all Q greater than or equal to 5 if a failure replacement policy is used. This follows from equation (6). The optimum value of Q was determined from (7) to be 24.40. A computer routine that searched for the Q that gave a minimum cost found an optimum Q of 25 and a cost of $42.90 for a failure replacement policy.



Uniform Distribution
μ = .0833
Expected Downtime
.50
.40
.30
.20
.10
.00
T = .50
T = .40
T = .25
T = .20
T = .05
-1
0
1
2
3
4
5
6
7
8
r
Weibull Distribution
f(x) = βx (r(1/β+1)/μ) exp(−(xr(1/β+1)/μ)^β)
β = 2 μ = .0833
Expected Downtime
.50
.40
.30
.20
.10
.00
T = .50
T = .40
T = .25
T = .20
T = .05
-1
0
1
2
3
4
5
6
7
8
r

Figure 2

Using the computer search routine the optimum values of Q,r, and t_o were determined to be Q=25, r=24, t_o=0.145 years or about 53 days with a cost of $42.26.

C. SENSITIVITY ANALYSIS

Using the values given above a sensitivity analysis was conducted to determine the effect of changes in the parameters and the variables. Using the optimal Q of 25 the following costs are obtained for the values of t_o shown:

t_o	Cost
.08	50.039
.09	47.736
.10	45.248
.11	44.084
.12	43.238
.13	42.474
.14	42.260
.15	42.343
.16	42.659
∞	42.896

As can be seen from the table it is better to have a failure replacement policy than to replace components too frequently. A failure replacement policy entails a cost increase of 1.2 percent whereas monthly replacement (t_o=.0833) will increase the cost by approximately 18 percent.

The cost increase caused by varying Q plus or minus 5 while t_o was held constant at .145 was less than 0.5 percent, indicating relative insensitivity to changes in Q.

The following table shows costs and downtime percentages at the optimum values of Q and t_o when r is varied through the range of values for which it has an effect on the model.

r	Cost	Percent Downtime
-1	43.43	11.3
0	43.07	7.8
1	42.69	4.2
2	42.43	1.6
3	42.31	0.4
4	42.28	0.1
5	42.27	0.02
6	42.27	0.00
7	42.26	0.00

The effect of varying the parameters in the model was as follows:

Varying h, the holding cost, had a significant effect on the optimum Q and a small effect on the optimum t_o. The changes in cost were significant.

Varying A, the fixed reorder cost, affected the optimum Q and the cost but did not have any sigificant effect on the optimal t_o.

The cost $\mathcal{K}$ varies inversely with changes in μ, the mean time between failures and there are significant changes in optimal t_o. The optimal Q varies by only a small amount as μ changes.

Varying π, the cost of downtime, has no effect until π is low enough that the optimal value of r is -1.

Varying the costs, C, C_f and C_s has a direct effect on $\mathcal{K}$ and causes a significant change in the optimum t_o. The optimum Q changes only slightly.

V. AN IMPROVED AGE REPLACEMENT POLICY

A. DESCRIPTION OF POLICY

The age replacement policy under which the model in III was developed requires that a component be replaced upon failure or when the time in service reaches t_o. This policy is followed regardless of the inventory level, even if the inventory level is zero. This means that the component in service will be removed and the equipment will go down. As was seen in III.D.1 this can lead to a situation in which downtime will occur every cycle.

It is not reasonable to expect that a serviceable component will be removed forcing a piece of equipment to go down when there is no replacement available. For this reason the following improved policy is set forth:

Follow the age replacement policy as set forth in III unless the inventory level is zero. In that case remove the component in service only upon failure or upon receipt of more inventory if the component has more time in service than t_o.

B. MODEL FORMULATION

Figure 3 depicts the inventory levels when such a policy is followed. Comparing this policy with the previous policy, the following can be noted:

(1) Cycle lengths will be identical if r is greater than -1.

Inventory Levels with Improved Age Replacement Policy

$Q = 4,\ r = 2,\ t_o = 3,\ T = 9$

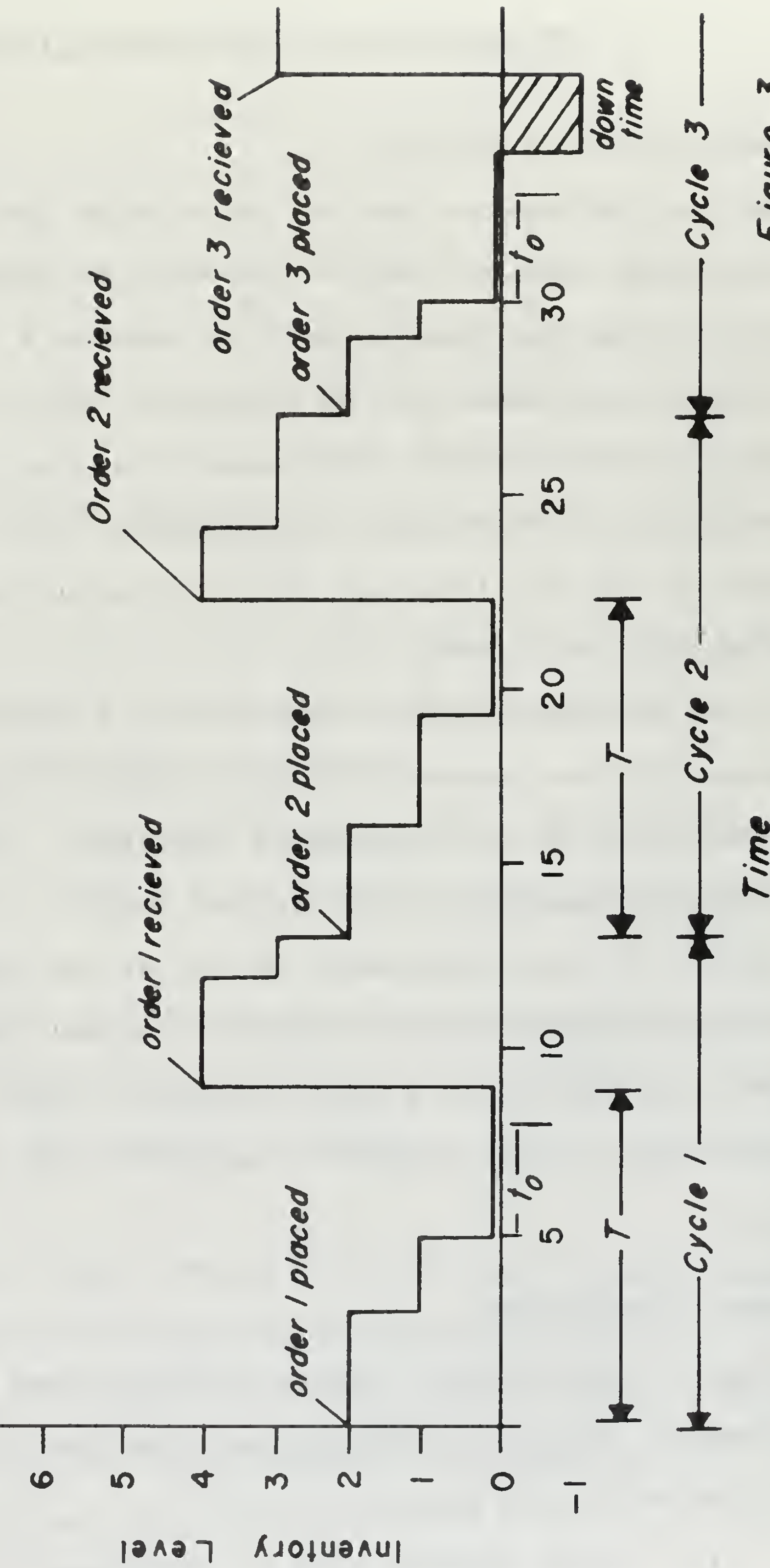


Figure 3

(2) The distribution of downtime will be different.

(3) The distribution of time to replacement of one of the components used in a cycle will be different if the inventory level reaches zero.

1. Identity of Cycle Lengths

If the inventory level does not reach zero or if the level reaches zero, but the age of the item in service does not reach t_o, the length of a cycle will be the same under either policy.

If the inventory level is zero and the age of the item in service reaches t_o before more inventory is received the two policies require different actions: under the previous policy the item in service will be removed, downtime will begin and last until a new component is received; under the improved policy the item will be left in service until failure or receipt of a new component. Note that in both cases the cycle ends when more inventory is received; thus the cycle lengths will be identical although the lengths of downtime will differ.

Since the cycle lengths are identical to the case previously considered, the distributions of cycle lengths are the same and the expected length of a cycle (ℓ) will be the same.

If r=-1 and the item in service when the inventory level is zero is not removed until it fails, the cycle length will be greater than or equal to the cycle length in the previous case. This case will be considered separately.

2. <u>Distribution of Downtime</u>

Let Y = life of item in service when inventory level is zero.

Y is distributed as F(x) if downtime occurs.

Letting D = downtime for a cycle

$$D = \text{Max}[0, T-S_r-Y] \; ,$$

$$P[D \leq t] = P[T-S_r-Y \leq t]$$

$$= P[S_r+Y \geq T-t] \; .$$

Letting $G(t) = \hat{F}_r * F(t)$, where the * denotes convolution

$$P[D \leq t] = \overline{G}(T-t), \qquad 0 \leq t \leq T$$

$$P[D > t] = G(T-t). \qquad 0 \leq t \leq T$$

The expected length of downtime using this policy is then:

$$\hat{\delta}' = E[D] = \int_0^T G(T-t)\,dt = \int_0^T G(x)\,dx \; .$$

Note that $\lim_{r \to \infty} \hat{\delta}' = 0$.

3. <u>Expected Length of a Cycle</u>

As a result of V.B.1 the expected length of a cycle must equal $Q\hat{\mu} + \hat{\delta}$.

Using the improved policy an expected cycle length using the same reasoning as in II.C.2 is

$$(Q-1)\hat{\mu} + E[Y] + \hat{\delta}' \; .$$

Equating the two cycle lengths;

$$Q\hat{\mu} + \hat{\delta} = (Q-1)\hat{\mu} + E[Y] + \hat{\delta}' ,$$

$$E[Y] = \hat{\mu} + \hat{\delta} - \hat{\delta}'$$

$$= \hat{\mu} + \int_0^T \hat{F}_{r+1}(t)\,dt - \int_0^T \hat{F}_r * F(t)\,dt$$

$$= \hat{\mu} + \int_0^T \hat{F}_r * (\hat{F}(t) - F(t))\,dt .$$

Note that: $\lim_{r \to \infty} E[Y] = \hat{\mu}$.

The expected length of a cycle can be expressed as:

$$Q\hat{\mu} + \int_0^T \hat{F}_r * (\hat{F}(t) - F(t))\,dt + \hat{\delta}'$$

or $Q\hat{\mu} + \hat{\delta}$.

4. Cost Model

In the same manner as in III.C.3 we can develop the following cost model:

$$\mathcal{K}(Q,r,t_o) = \frac{A + C'QF(t_o) + C''Q\overline{F}(t_o) + \pi\hat{\delta}'}{Q\hat{\mu} + \hat{\delta}} + hQ .$$

5. Properties of the Cost Model

As in the previous models $\mathcal{K}$ is either monotone increasing or monotone decreasing in r, so the optimum r will be either -1 or Q-1.

For large r the model is identical to the model in III.D.4 and the necessary conditions for optimum Q and t_o are the same.

For r=-1, $\hat{\delta}' = T$ and the expected length of a cycle will be: $(Q-1)\hat{\mu} + \mu + T$.

As mentioned previously, this is not identical to the cycle length under the previous policy, a situation that can arise only if r=-1.

Under this expected length of a cycle and the expected length of downtime from 2, we can determine an optimum value of Q by assuming continuity in Q. It is

$$Q = \sqrt{\frac{\hat{\mu}(A+\pi T)-(C'F(t_o)+C''\bar{F}(t_o))(T+\mu-\hat{\mu})}{h\hat{\mu}^2}} - \frac{\mu-\hat{\mu}+T}{\hat{\mu}} .$$

Since $\mu-\hat{\mu}$ is positive, Q will be less than in equation (9).

The optimal t_o will be the solution to the following equation:

$$0 = \frac{Q(C'-C'')f(t_o)}{Q\hat{\mu}+T+(\mu-\hat{\mu})} - Q\bar{F}(t_o)\frac{A+QC'F(t_o)+QC''\bar{F}(t_o)+\pi T}{(Q\hat{\mu}+T+(\mu-\hat{\mu}))^2} .$$

An iterative solution of these two equations will yield the optimal Q and t_o for the improved policy when the optimum r is -1.

VI. SUMMARY AND CONCLUSIONS

A. SUMMARY

Cost models are developed to show the relationship between inventory and maintenance policies when one component of one item of equipment is replaced in accordance with the maintenance policy and the components are stocked according to the inventory policy. The necessary conditions for determining optimum values of Q, the reorder quantity; r, the reorder point; and t_o, the time to replacement are derived.

The models are monotonic increasing or decreasing in r, thus the optimal r is either -1 or Q-1. If Q is large (approximately 8) further changes in r have little effect, thus r could be considered constant. When r is large the necessary conditions for optimal Q and t_o are very similar to well-known conditions for determining optimal values of Q and t_o.

B. CONCLUSIONS

The cost models can be used to determine optimal values of Q, r, and t_o. The optimal values of Q and t_o are not very sensitive to changes in each other or changes in the parameters. The optimum value of r, having only two values, is very sensitive to changes in Q or t_o or the parameters when near the boundary between increasing and decreasing $\mathcal{H}$. The cost, however, does not appear to vary much even with changes in r and it might be best to use an r large enough to constrain downtime in a specified manner when in this region.

LIST OF REFERENCES

[1] Hadley, G., and Whitin, T. M., Analysis of Inventory Systems, Prentice-Hall, 1963.

[2] McCall, J. J., "Maintenance Policies for Stochastically Failing Equipment: A Survey," Management Science, v. 11, p. 493-524, March 1965.

[3] Barlow, R. E., and Proschan, F., Mathematical Theory of Reliability, Wiley, 1965.

[4] Falkner, C. H., "Jointly Optimal Inventory and Maintenance Policies for Stochastically Failing Equipment," Operations Research, v. 16, p. 587-601, May-June 1968.

[5] Barlow, R. E., and Proschan, F., "Planned Replacement," Chapter in Studies in Applied Probability and Management Science, Arrow, Karlin and Scarf, ed., Stanford University Press, 1962.

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13. ABSTRACT

Cost models are developed to show the relationship between inventory and maintenance policies when one component of one item of equipment is replaced in accordance with the maintenance policy and the components are stocked in accordance with the inventory policy. A (Q,r) inventory policy and failure replacement and age replacement maintenance policies are used. The necessary conditions for determining optimum values of the reorder quantity, the reorder point, and time to replacement are derived.

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