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by

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Submitted in partial fulfillment of
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IN

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# ABSTRACT

One of the many problems in the design of a servomechanism system is the determination of compensation to meet certain specifications. This thesis is concerned with compensation to give "optimum" transient response. The term "optimum" is defined and the required characteristic equations for this response are given by Graham and Lathrop (1). The location of the poles of the "optimum" transfer functions through fifth-order were determined. Types of compensators that can be used in the error channel and types that can be used in the feedback loop have been determined. Finally, equations for determining the parameters of the compensators have been derived.

The topic was suggested by Professor George J. Thaler of the Electrical Engineering Department of the U.S. Naval Postgraduate School.

The writer wishes to acknowledge the invaluable aid given by Professor Orval H. Polk and Professor George J. Thaler during the preparation of this thesis.

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## TABLE OF SYMBOLS AND ABBREVIATIONS

| | |
|---|---|
| AF | Transfer function of the feedback path |
| C | Capacitance |
| $\lvert e \rvert$ | Absolute value of error |
| ITAE | Integral of time-multiplied absolute-value of error |
| KG | Uncompensated transfer function of the feedforward path |
| KG' | Compensated transfer function of the feedforward path |
| K | Gain constant for a system |
| k | Gain constant for a compensator |
| R | Resistance |
| s | Laplace variable |
| t | Time |
| $\alpha$ | A transfer function pole or zero which is to be determined |
| $\omega_0$ | Natural angular frequency |

## GENERAL DISCUSSION

Graham and Lathrop[1] have specified the form of the characteristic equation necessary to obtain "optimum" transient response. The term "optimum" is in quotes because it has been used with so many different connotations that one must clearly define the term. In order to use the forms specified by Graham and Lathrop, their definition of "optimum" will be used. The function used in this definition is known as the integral of time-multiplied absolute-value of error, or ITAE, and is symbolically defined as $\int_0^\infty t\,|e|\,dt$ where $t$ = time and $|e|$ = absolute value of error. The system that gives a minimum value for this integral is then the "optimum" system.

In building a system to meet any given set of specifications (for example, power, space and weight requirements, and cost) certain components will be required. In order to build an "optimum" system, it is necessary that its characteristic equation have the exact form as specified for the minimum ITAE criterion.

There is small probability that a system designed to meet specifications will also meet the "optimum" requirement. Therefore, the system must be altered, or compensated, in some manner to bring it into agreement with the specific "optimum" characteristic equation applicable to the compensated system.

The next problem is to determine the type of compensator to be used and its location in the system. There are three places in any basic system where a compensator may be used, (1) cascade or series before the error detector, (2) cascade in the error channel and (3) in the feedback loop. The location of the compensator will determine some of its required characteristics.

Usually, the transfer function of a system to be compensated is expressed in terms of poles and zeros in the complex(s) plane. The complex variable $s$ is defined by $s = \sigma + j\omega$ where $j = \sqrt{-1}$ and both $\sigma$ and $\omega$ may have any value. A pole is defined as a root of the equation formed when the denominator of the transfer function is set equal to zero, and a zero is a root of the equation formed by equating the numerator of the transfer function to zero.

The first important step in this study was to investigate the location on the complex plane of poles and zeros for compensators to be used in the error channel and in the feedback loop. The locations of the compensator poles and zeros were determined in terms of the poles and zeros of the system to be compensated. It will be shown that when the poles and zeros of a compensator are determined, it will then be possible to determine the type of network to be used for "optimum" transient response, and the values of the parameters required for this network. It was assumed that a passive, resistance-capacitance compensating network will normally be used, and only this type has been considered in this study.

## "OPTIMUM" TRANSFER FUNCTIONS

Before considering the required compensation, a study was made of the location of the poles of the "optimum" transfer functions determined by Graham and Lathrop [1]. This was done so that the general nature of the transfer functions that will meet the "optimum" criterion would be known. With this information and the transfer function for the uncompensated system, it has been possible to specify, or at least limit, the types of compensators that can be used.

In order to locate these "optimum" poles it is necessary to take the characteristic equation for the system, form the system function, and from this determine the transfer function. The system function is defined, for unity feedback systems, as $\frac{KG}{1 + KG}$, where KG is the transfer function and 1 + KG is the characteristic equation. The method for determining transfer functions is outlined in Appendix I, and the results for zero-displacement error, unit-numerator systems up to and including fifth-order are shown in Figure 1.

It should be noted that all of these systems have transfer functions with a single pole at the origin. This is characteristic of the transfer functions for zero-displacement error systems. A study of the nature and location of the poles of the transfer function reveals the type of transfer function required to meet the "optimum" criterion. For example, a third-order system must have, besides the pole at the origin, two complex conjugate poles with negative real parts. Two negative real poles will not satisfy the required characteristic equation. This means that if the transfer function for an uncompensated third-order system has two real negative poles the compensator must cancel these two poles and introduce two complex conjugate poles in the proper locations.

## LOCATION OF THE "OPTIMUM" POLES

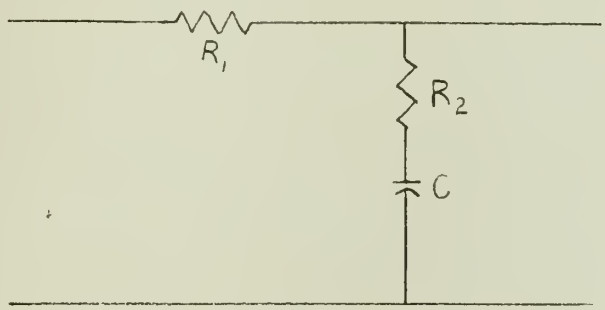
| $\frac{KG}{1+KG}$ | $KG$ | POLES OF KG |
|---|---|---|
| $\frac{1}{S+1}$ | $\frac{1}{S}$ | $S=0$ |
| $\frac{1}{S^2+1.4S+1}$ | $\frac{1}{S(S+1.4)}$ | $S_1=0$<br>$S_2=-1.4$ |
| $\frac{1}{S^3+1.75S^2+2.15S+1}$ | $\frac{1}{S(S^2+1.75S+2.15)}$ | $S_1=0$<br>$S_2=-0.875+j1.177$<br>$S_3=-0.875-j1.177$ |
| $\frac{1}{S^4+2.15S^3+3.4S^2+2.7S+1}$ | $\frac{1}{S(S^3+2.1S^2+3.4S+2.7)}$ | $S_1=0$<br>$S_2=-1.17$<br>$S_3=-0.465+j1.442$<br>$S_4=-0.465-j1.442$ |
| $\frac{1}{S^5+2.8S^4+5.0S^3+5.5S^2+3.4S+1}$ | $\frac{1}{S(S^4+2.8S^3+5.0S^2+5.5S+3.4)}$ | $S_1=0$<br>$S_2=-0.3+j1.32$<br>$S_3=-0.3-j1.32$<br>$S_4=-1.1+j0.79$<br>$S_5=-1.1-j0.79$ |

Fig. 1

## COMPENSATION

For a compensator located in the error channel, the overall transfer function is obtained by multiplying the transfer function of the uncompensated system by the transfer function of the compensator. Therefore, in order to maintain the same order system as the original uncompensated system, it is necessary to use a compensator with an equal number of poles and zeros. It will be shown later that the location of the zeros of this compensator must be at the poles of the uncompensated system. The simplest type of compensator with an equal number of poles and zeros is one whose transfer function is $k\frac{s + k\alpha}{s + \alpha}$. The resistance-capacitance network shown in Figure 2 has this transfer function when

$$\alpha = \frac{1}{C(R_1 + R_2)}, \quad k\alpha = \frac{1}{CR_2} \quad \text{and} \quad k = \frac{R_1 + R_2}{R_2}.$$




Resistance-Capacitance Network

Fig. 2.

In working with this compensator, the coefficient k is ignored since it is a gain factor. It is assumed that the gain for the systems under consideration can be adjusted to any required value. Therefore, when possible, overall gain constants of unity have been used.

In using this resistance-capacitance type of compensator, it is obvious that certain values of k and $\alpha$ cannot be used. These values are $\alpha = 0$, $k = 0$ and $k = 1$. It is seen that; (a) if $\alpha = 0$ the transfer

function of the compensator reduces to $\frac{s}{s}$ with the result of no change in the overall transfer function; (b) if $k = 0$ the compensator transfer function reduces to $\frac{s}{s + \alpha}$ which has the effect of cancelling the $s$ in the denominator of the uncompensated transfer function and giving an overall transfer function which does not represent a zero-displacement-error system; (c) if $k = 1$ the compensator transfer function reduces to $\frac{s + \alpha}{s + \alpha}$. This results in no change in the overall transfer function.

The simplest of all zero-displacement-error transfer functions is a pure integrator which has the transfer function $KG = \frac{K}{s}$ or, assuming $K = 1$, $KG = \frac{1}{s}$. When the compensator of the form $\frac{s + k\alpha}{s + \alpha}$ is used with the pure integrator, the overall transfer function becomes $KG' = \frac{s + k\alpha}{s(s + \alpha)}$ and the system function becomes $\frac{KG'}{1 + KG'} = \frac{s + k\alpha}{s^2 + (\alpha + 1)s + k\alpha}$. In order to maintain the unit-numerator type of function, it is necessary to divide the denominator of the system function by its numerator and set the remainder equal to zero. This procedure is illustrated, using a second-order system, in Appendix II. However, for this basic transfer function, the procedure as outlined leads only to values of $k$ and $\alpha$ which have been ruled out as unacceptable. Therefore, this type of compensator cannot be used with a pure integrator.

A more direct approach to the compensation of this basic integrator is to consider the system where $K \neq 1$, then the transfer function is $KG = \frac{K}{s}$ and the system function is $\frac{KG}{1 + KG} = \frac{K}{s + K}$. Comparing the form of the "optimum" characteristic equation with that for systems having other than unit-numerator transfer functions shows that the system will have the "optimum" transient response if $K = \omega_0$, where $\omega_0$ is the natural frequency of the closed system.

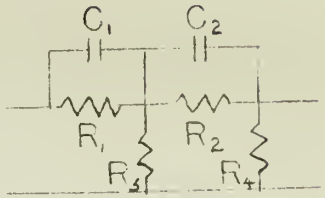
For the next higher order system, represented by the transfer function $KG = \frac{1}{s(s+a)}$, a compensator with one pole and one zero can be used in the error channel. Using the procedure shown in Appendix II and the specified form of the "optimum" characteristic equation, a compensator of the form $\frac{s+k\alpha}{s+\alpha}$, requires a value of $\alpha = 1.4$ and $k = \frac{a}{1.4}$. These results can be obtained with the compensating network shown in Figure 2 by setting $\alpha = 1.4 = \frac{1}{C(R_1 + R_2)}$, $k\alpha = a = \frac{1}{CR_2}$, and $k = \frac{a}{1.4} = \frac{R_1 + R_2}{R_2}$.

Using the procedure of Appendix II on the second and higher order systems it was found that for "optimum" transient response the zeros of the compensator must cancel the undesirable poles of the uncompensated system. In addition, the poles of the compensator must be such as to meet the "optimum" specifications.

Typical transfer functions for two uncompensated third-order systems and the "optimum" characteristic equations are shown in Figure 3. Also listed is one possible compensating network for each system along with the network transfer function. These networks and their transfer functions were chosen from a published list of passive networks and transfer functions [8]. Included with the transfer functions are equations for determining the parameters of the network. These equations were derived from the requirements (1) that the zeros of the compensator cancel the undesired poles of the uncompensated system and, (2) the poles of the compensator must be such as to meet the "optimum" specifications.

It is theoretically possible to design similar compensating networks for higher order systems. These networks, however, will be relatively complex and probably would not be used. Some other type of compensation in another location would be preferable.

# SUMMARY FOR TWO THIRD-ORDER SYSTEMS

| KG | "OPTIMUM" CHARACTERISTIC EQUATION | POSSIBLE COMPENSATOR | COMPENSATOR TRANSFER FUNCTION |
|---|---|---|---|
| (1) $\frac{1}{S(S+a)(S+b)}$ | $S^3 + 1.75 S^2 + 2.15 S + 1$ |  | $\frac{(S+\frac{1}{T_1})(S+\frac{1}{T_2})}{S^2 + \left\{\frac{1}{T_2}(1+\frac{R_2}{R_4}) + \frac{1}{T_1}\left[1+\frac{(R_4+R_3)R_1}{R_3R_4}\right]\right\}S + \frac{1}{T_1T_2G_o}}$ |

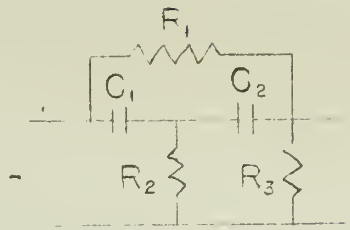


WHERE

$T_1 = R_1C_1 \qquad T_2 = R_2C_2 \qquad G_o = \frac{1}{1 + \frac{R_2}{R_4} + \frac{(R_2+R_3+R_4)R_1}{R_3R_4}}$

$\frac{1}{T_1} = a \qquad \frac{1}{T_2} = b \qquad \frac{1}{T_1T_2G_o} = 2.15 \qquad \frac{1}{T_2}(1+\frac{R_2}{R_4}) + \frac{1}{T_1}\left[1+\frac{(R_4+R_3)R_1}{R_3R_4}\right] = 1.75$

FIG 3a

| KG | "OPTIMUM" CHARACTERISTIC EQUATION | POSSIBLE COMPENSATOR | COMPENSATOR TRANSFER FUNCTION |
|---|---|---|---|
| (2) $\frac{1}{S(S^2+aS+b)}$ | $S^3 + 1.75\,S^2 + 2.15\,S + 1$ |  | $\frac{S^2 + \left[\frac{R_2}{T_2 R_1} + \frac{1}{T_1}\left(\frac{R_3}{R_1+R_3}\right)\right]S + \frac{G_0}{T_1 T_2}}{S^2 + \left[\frac{1}{T_2} + \frac{1}{T_1}\left(1 + \frac{C_2}{C_1}\right)\right]S + \frac{1}{T_1 T_2}}$ |

WHERE

$$T_1 = C_2 \frac{R_1 R_3}{R_1 + R_3} \qquad T_2 = C_1 C_2 \frac{R_1 R_2 R_3}{R_1 + R_3} \qquad G_0 = \frac{1}{1 + \frac{R_1}{R_3}} \qquad \frac{R_2}{T_2 R_1} + \frac{1}{T_1}\left(\frac{R_3}{R_1 + R_3}\right) = a$$

$$\frac{G_0}{T_1 T_2} = b \qquad \frac{1}{T_2} + \frac{1}{T_1}\left(1 + \frac{C_2}{C_1}\right) = 1.75 \qquad \frac{1}{T_1 T_2} = 2.15$$

FIG 3b

The next most frequently used type of compensator, after the compensator in the error channel, is one to be inserted in the feedback loop. This leads to systems whose system functions are of the form $\frac{KG}{1+KGAF}$, where KG is the transfer function of the forward loop and AF is the transfer function of the feedback loop.

Consider a transfer function of the form $KG = \frac{1}{s(s+a)}$ and a feedback compensator of the form $AF = k\frac{s+k\alpha}{s+\alpha}$. The system function was formed from the defining relation, $\frac{KG}{1+KGAF}$. Using the procedure of Appendix I, a unit-numerator system function was determined. The values obtained for k and $\alpha$ are $\alpha = 0$, $k = 0$ and $k = 1$. These values are unacceptable. Therefore, this type of compensator cannot be used in the feedback loop.

Another type of feedback compensator which is often used is a tachometer. For this $AF = s$. With $KG = \frac{1}{s(s+a)}$ the compensated system function is $\frac{KG}{1+KGAF} = \frac{1}{s^2+(a+1)s}$. The denominator of the "optimum" system function for this case must be $s^2 + 1.4s + 1$. The denominator of the compensated system function does not have a constant term corresponding to the one of the "optimum" denominator. The same results were obtained with higher order systems; therefore, a tachometer alone cannot be used as a feedback compensator to obtain "optimum" transient response.

A damping device is another form of compensation that is sometimes used. This device can be either electrical or mechanical or a combination of both. For a damper inserted in the feedback loop $AF = s + \alpha$. However, using this type of compensator with a unit-numerator system function results in a meaningless value for $\alpha$. This value is $\alpha = 1$ regardless of the original transfer function. Therefore, transfer

functions and system functions with numerators other than unity and the "optimum" characteristic equations for systems with other than unit numerators must be used.

The procedure for determining the value of $\alpha$ can be shown by using an uncompensated transfer function of the form $KG = \frac{K}{s(s+a)}$ where K is some specified constant. The system function is

$$\frac{KG}{1 + KGAF} = \frac{K}{s^2 + (a + K)s + K\alpha} .$$

For this system the "optimum" characteristic equation is

$$s^2 + 1.4\,\omega_o\, s + \omega_o^2 = 0$$

where $\omega_o$ is the natural frequency of the system. Setting the denominator of the system function equal to the "optimum" characteristic equation and equating coefficients of equal powers of s gives the equations $K\alpha = \omega_o^2$ and $a + K = 1.4\,\omega_o$. Since a and K are known from the original transfer function, it is possible to solve these equations for $\alpha$ . The solution is

$$\alpha = \frac{a^2 + 2aK + K^2}{1.96K} .$$

Using the same procedure for an original transfer function

$$KG = \frac{K}{s(s+a)(s+b)}$$

and the corresponding "optimum" characteristic equation,

$$s^3 + 1.75\,\omega_o\, s^2 + 2.15\,\omega_o^2\, s + \omega_o^3 = 0,$$

a value of $\alpha$ is found. This value is

$$\alpha = \frac{a^3 + 3a^2b + 3ab^2 + b^3}{(1.75)^3\, K} .$$

This appears to be the only type of commonly used feedback compensator that can be used to obtain the "optimum" transient response.

## CONCLUSIONS

Given the characteristic equation of a system having the "optimum" transient response, it is possible to determine the corresponding transfer function. From this transfer function it is possible to find the location of the "optimum" poles on the complex (s) plane. Knowing these locations and the locations of the poles and zeros of an uncompensated transfer function, it is possible to specify the form of a compensator for this system to give "optimum" transient response.

For error channel compensators of the form $k \frac{s + k\alpha}{s + \alpha}$ the values $\alpha = 0$, $k = 0$, and $k = 1$ are unacceptable. Use of this type of compensator with a unit-numerator ($K = 1$) pure integrator, $KG = \frac{1}{s}$, is not possible since the only values of k and $\alpha$ which will satisfy the "optimum" specification are those values which are unacceptable for this compensator. For a pure integrator with $K \neq 1$, $KG = \frac{K}{s}$, the "optimum" specification can be met by adjusting the gain, K, so that $K = \omega_o$, the natural angular frequency of the system.

For a second-order system with unit-numerator transfer function $KG = \frac{1}{s(s + a)}$, a compensator in the error channel of the form $k \frac{s + k\alpha}{s + \alpha}$ can be used. When this compensator is used the values of k and $\alpha$ for "optimum" transient response are $\alpha = 1.4$ and $k = \frac{a}{1.4}$. This compensator can be obtained using the network shown in Figure 2 by setting

$$\alpha = 1.4 = \frac{1}{C(R_1 + R_2)}, \quad k\alpha = a = \frac{1}{CR_2}, \text{ and } k = \frac{a}{1.4} = \frac{R_1 + R_2}{R_2} .$$

For second and higher order systems using error channel compensators, the zeros of the compensator must cancel the undesirable poles of the uncompensated system. In addition the poles of the compensator must be such as to meet the "optimum" specifications.

For systems of higher than second-order the simple network shown in Fig. 2 is not adequate for compensation. Fig. 3 shows two typical third-order systems and one possible compensating network for each system. Also shown are the equations for determining the parameters of the networks.

For feedback compensation, compensators of the form $k\,\frac{s + k\alpha}{s + \alpha}$ cannot be used since only the unacceptable values of k and $\alpha$ will satisfy the required equations.

A tachometer feedback compensator, AF = s, cannot be used to obtain "optimum" transient response since the characteristic equation of the compensated system cannot have the proper form.

Feedback compensation using a damping device AF = s + $\alpha$ used with unit-numerator transfer functions leads to a value of unity for $\alpha$ regardless of the original system. Since this is meaningless, transfer functions having other than unit-numerators were used. Useful values of $\alpha$ were obtained for some of these systems. For second-order systems, $KG = \frac{K}{s(s + a)}$, $\alpha = \frac{a^2 + 2aK + K^2}{1.96K}$. For third-order systems,

$$KG = \frac{K}{s(s + a)(s + b)}, \quad \alpha = \frac{a^3 + 3a^2b + 3ab^2 + b^3}{(1.75)^3\,K} .$$

Using the method outlined above it is not difficult to determine the theoretical compensator to use with a given system. The only difficulty is the labor involved in the solution of the equations for high order systems. This difficulty can be overcome by the use of a digital computer if one is available.

## RECOMMENDATIONS FOR FURTHER STUDY

Due to time limitations it was not possible to verify, with actual components or analog computer, the theoretical results presented. It is felt that this verification along with the possible extension to higher order systems would be a very worthwhile project.

Another possible extension which seems to offer possibilities is the study of compensation to be used with zero-velocity error and zero-acceleration error systems.

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## APPENDIX I

## DETERMINATION OF TRANSFER FUNCTIONS FROM THE SYSTEM FUNCTIONS

As an example to show the method of finding the transfer function from the system function, consider a third-order system. The characteristic equation for such a system to meet the "optimum" transient criterion [1] would be $s^3+1.75\omega_o s^2+2.15\omega_o^2 s+\omega_o^3$. Then

$$\frac{KG}{1+KG} = \frac{\omega_o^3}{s^3+1.75\omega_o s^2+2.15\omega_o^2 s+\omega_o^3}$$

and, to satisfy the unit-numerator specification, $\omega_o$ would be unity.

This means that $\frac{KG}{1+KG} = \frac{1}{s^3+1.75s^2+2.15s+1}$ . Solving this for KG gives $KG = \frac{1}{s^3+1.75s^2+2.15s}$ . The roots of the denominator polynomial, then, are the poles of the transfer function. These are determined by setting the denominator equal to zero and solving.

$$s^3 + 1.75s^2 + 2.15s = 0$$

$$s(s^2 + 1.75s + 2.15) = 0$$

From this equation $s_1 = 0$

$$s_{2,3} = -0.875 \pm j\sqrt{1.3844} = -0.875 \pm j\,1.177.$$

The same method can be extended to the higher order systems but each higher order involves the solution of an equation of one less order than the system order. For example, a fourth order system involves the solution of a cubic equation. Methods for the solution of the higher order equations by numerical methods are available and fairly straightforward, [6].

For systems where $\omega_o \neq 1$, the roots of the denominator polynomial are multiplied by $\omega_o$ to give the pole locations.

# APPENDIX II

## DETERMINATION OF UNIT NUMERATOR SYSTEM FUNCTIONS AND COMPENSATOR PARAMETERS

Given a unit-numerator zero-displacement error transfer function of the form $KG = \frac{1}{s(s+a)}$ which is to be compensated using series compensation of the form $\frac{s+k\alpha}{s+\alpha}$ in the error channel, it is desired to determine the values of k and $\alpha$ necessary to give "optimum" transient response.

The transfer function for the compensated system will be

$$KG' = \frac{s+k\alpha}{s(s+a)(s+\alpha)}$$

and the system function will be

$$\frac{KG'}{1+KG'} = \frac{s+k\alpha}{s(s+a)(s+\alpha) + (s+k\alpha)} .$$

Expanding the denominator and collecting terms, this becomes

$$\frac{s+k\alpha}{s^3+(a+\alpha)s^2 + (a\alpha+1)s+k\alpha}$$

In order to maintain the unit numerator it is necessary to divide the denominator by the numerator and set any remainder equal to zero. This division yields $\frac{1}{s^2 + (a+\alpha-k\alpha)s + (a\alpha+1 - ak\alpha - k\alpha^2 + k^2\alpha^2)}$

with remainder $\frac{ak\alpha^2 - ak^2\alpha^2 - k^2\alpha^3 + k^3\alpha^3}{s + k\alpha}$ .

In order for the remainder to be zero, the numerator of the remainder must be zero. $ak\alpha^2 - ak^2\alpha^2 - k^2\alpha^3 + k^3\alpha^3 = 0$. Factoring the left side of this equation gives $(\alpha^2)(k)(1-k)(a-k\alpha) = 0$.

From this the alternate solutions are

$\alpha^2 = 0$ from which $\alpha = 0$

$k = 0$

$k = 1$ and

$k\alpha = a$.

The first three are discarded for the reasons given on page 6 of the main text. Using the characteristic equation of the unit-numerator compensated system and equating the coefficients to the required coefficients for "optimum" transient response [1] gives the equations

$$a + \alpha - k\alpha = 1.4$$

$$a\alpha + 1 - ak\alpha - k\alpha^2 + k^2\alpha^2 = 1 .$$

Using the first of these equations along with the value of $k\alpha$ determined previously gives $\alpha = 1.4$. Also, using $k\alpha = a$ the second of these equations reduces to an identity $1 = 1$.

Using the values $\alpha = 1.4$ and $k\alpha = a$ it is possible to determine the parameters for the required compensator.