ELASTIC INSTABILITY OF RELATIVELY THICK CIRCULAR CYLINDRICAL SHELLS SUBJECTED TO HYDROSTATIC PRESSURE

L. D. BALLOW
and
T. M. BARRY

ELASTIC INSTABILITY OF RELATIVELY THICK CIRCULAR CYLINDRICAL SHELLS SUBJECTED TO HYDROSTATIC PRESSURE

A Thesis

Submitted to the Faculty of

Webb Institute of Naval Architecture

In Partial Fulfillment

Of the Requirements for the Degree of

Master of Science

In Naval Architecture

By

LT L. D. Ballow, USN

LT T. M. Barry, USN

[1961]

ABSTRACT

Experimental investigations were undertaken to determime the collapse pressures of relatively thick circular cylindrical shells subjected to hydrostatic pressure. A total of twenty-three fiber glass reinforced plastic cylinders with thickness to diameter ratios ranging from 0.05 to 0.09 was tested to determine collapse pressures and the effective modulus of elasticity of the material. The length to diameter ratio was kept constant. Results were plotted non-dimensionally with P/E as the ordinate and h/D as the abscissa. The results were compared with available theoretical instability formulas and were in very good agreement with theory for ratios of h/D up to about 0.07. Above this value of h/D, experimental collapse pressures were lower than those predicted by theory.

It was concluded that the von Mises equation [equation (6), Ref. 22] is the best instability equation for cylinders with ratios of h/D greater than 0.05. It was also concluded that for almost all practical cases, the thin shell theory is sufficiently accurate for predicting instability failures.

Derivations for the following are included: (1)Expressions for calculation of the moduli of elasticity in the circumferential and longitudinal directions; (2)An expression for the shear modulus and for the effective modulus of elasticity of an orthotropic material; (3)A determinant expressing the critical pressure of simply supported, orthotropic, circular cylinders subjected to hydrostatic pressure; and (4)An expression for calculating the bending stress in a simply supported circular cylinder subjected to hydrostatic pressure.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the assistance of:

Captain Robert A. Hinners, USN (Ret), Head of the Luckenbach Graduate School, Webb Institute of Naval Architecture, whose advice and encouragement contributed in large measure to the fulfillment of the thesis goals;

Professor Lawrence W. Ward, Webb Institute of Naval Architecture, who checked the theoretical portion of the thesis and gave advice for its formulation;

The ZENITH PLASTICS COMPANY, Gardena, California, a Division of the MINNESOTA MINING AND MANUFACTURING COMPANY, which contributed the models and provided useful technical data and information;

The NEW YORK NAVAL SHIPYARD, which assisted in the construction of the test apparatus;

Mr. Warren Berkson, U. S. Navy Material Laboratory, who aided in proof testing the apparatus;

Mr. John G. Pulos, David Taylor Model Basin, who helped in the selection of the thesis topic and who indicated the method of derivation for the formula for bending stress;

LCDR Frank F. Manganaro, Bureau of Ships, who suggested the general area of research;

Mr. Kenneth E. Hom, David Taylor Model Basin, who provided the basic design for the pressure vessel;

Mr. John E. Buhl, David Taylor Model Basin, who gave much useful advice and criticism;

Mrs. Lois Cherwinski, David Taylor Model Basin, who programmed the von Mises equation for high speed computer solution;

Mr. Duncan Robb, Shop Assistant, Webb Institute of Naval Architecture, who aided in constructing and assembling the test apparatus.

NOTATION

D Flexural rigidity, $D = \frac{Eh^3}{12(1-\nu^2)}$

D Mean diameter

D_1 Inside diameter

D_2 Outside diameter **

E Modulus of elasticity

E_e Effective modulus of elasticity for anisotropic material

E_x Modulus of elasticity in the longitudinal direction

E_θ, E_ϕ Modulus of elasticity in the circumferential direction

G Shear modulus, $G = \frac{E}{2(1+\nu)}$

h Thickness of shell

K_1 Non-dimensional term from thick cylinder stress formula,

$$K_1 = \frac{2R_2^2}{R_2^2 - R_1^2}$$

K_2 Non-dimensional term from thick cylinder stress formula,

$$K_2 = \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} = K_1 - 1$$

L Effective length

L_{oa} Actual length

m Number of waves or lobes along effective length of cylinder at time of collapse

m_ϕ Strain-pressure ratio, circumferential strain gage

m_x Strain-pressure ratio, longitudinal strain gage.

M_x Axial bending moment per unit of circumference

M_ϕ Circumferential bending moment per unit of length

$M_{x\phi}$ Twisting moment per unit of length on element cut out by generators

$M_{\phi x}$ Twisting moment per unit of circumference on element cut out by plane perpendicular to axis

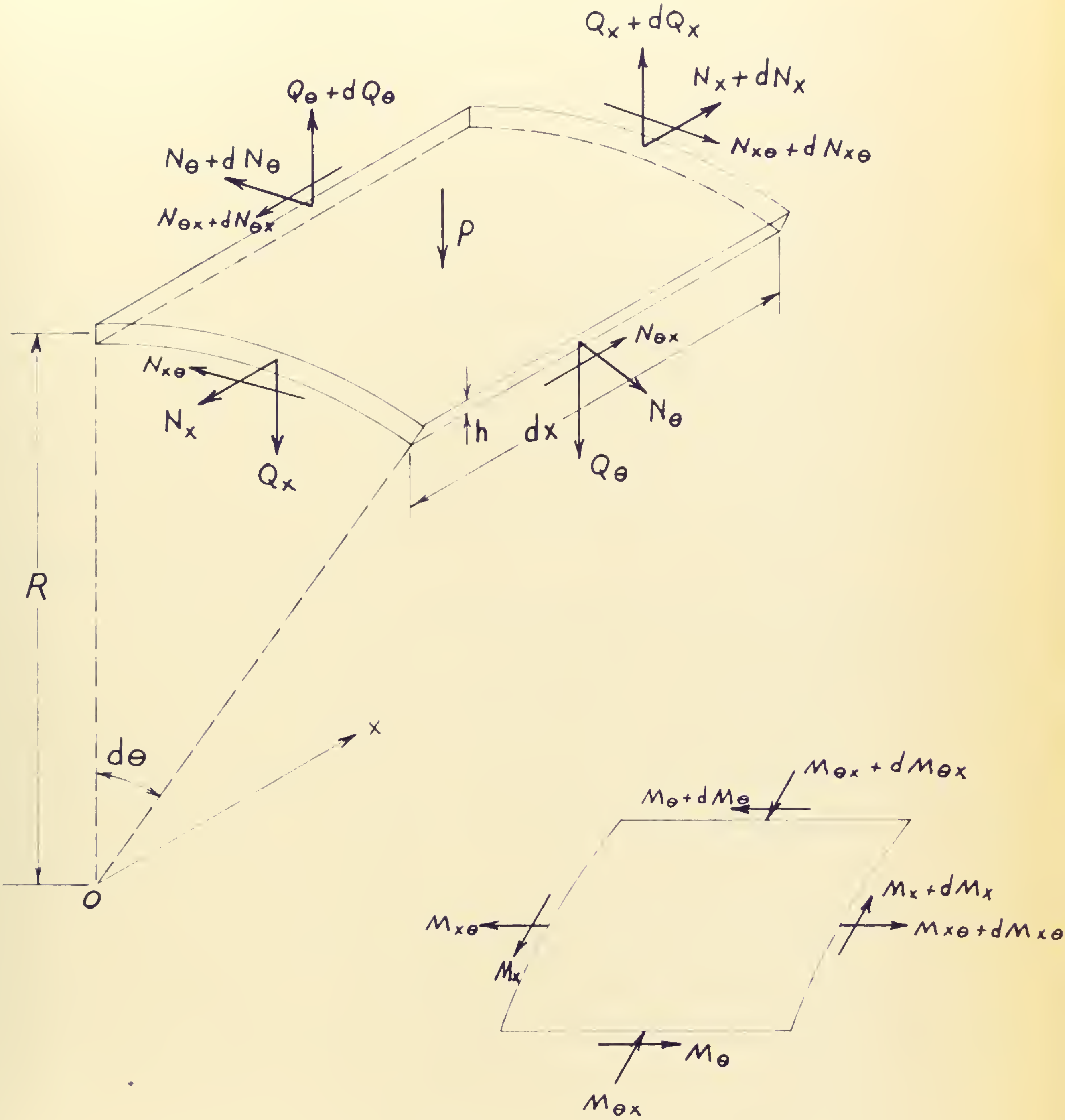
---

** In the notation used in this paper, the subscript "1" refers to the inside of the cylinder, and the subscript "2" refers to the outside of the cylinder.

n	Number of lobes or waves in a complete circumferential belt around cylinder at time of collapse
N_x	Normal force per unit of circumference in axial direction
N_ϕ	Normal force per unit of length in circumferential direction
$N_{x\phi}$	Shear force per unit length on elements cut by generators
$N_{\phi x}$	Shear force per unit of circumference on elements cut by planes perpendicular to axis
p	Pressure
Q_ϕ	Radial shear in the xz plane
Q_x	Radial shear in the yz plane
r	Radial coordinate
R	Mean radius
R_1	Inside radius
R_2	Outside radius
u	Axial displacement
v	Circumferential displacement
w	Radial displacement
z	Perpendicular distance from mid-surface of shell, positive inward
α	Non-dimensional term in buckling equations, $\alpha = \frac{1}{3} \cdot \frac{h^2}{D^2}$
γ	Shearing strain
γ_{ij}	Shearing strain along i-plane in j-direction
ϵ_x	Normal strain in axial direction
ϵ_ϕ	Normal strain in circumferential direction
ϵ_r	Normal strain in radial direction
θ	Cylindrical (angular) coordinate
λ	Non-dimensional term in buckling equations, $\lambda = m\frac{\pi D}{2L}$
ν	Poisson's ratio

σ_x	Normal stress in axial direction
σ_ϕ	Normal stress in circumferential direction
τ_{ij}	Shear stress along i-plane in j-direction
ϕ	Cylindrical (angular) coordinate
χ_x	Change of curvature of generator
χ_ϕ	Change of curvature of circumference
$\chi_{x\phi}$	Twist of element due to $M_{x\phi}$

FIGURE A



SHELL ELEMENT

INTRODUCTION

The absence of experimental results and elastic instability analyses for thick cylinders has been pointed out by Wenk[1]. On the basis of this information and consultation with Mr. John Pulos of the Structural Mechanics Laboratory, David Taylor Model Basin, the authors decided to undertake an experimental investigation of the elastic instability of relatively thick cylinders subjected to external hydrostatic pressure. Thus, the primary purpose of this thesis evolved; which is the experimental evaluation of available theoretical instability formulas as applied to cylinders with thickness to diameter ratios greater than those heretofore tested. From the results of the proposed experiments it was anticipated that the following question posed by Dr. Wenk (see "Discussion" of Ref. 1) could be answered:

"At what ratio of thickness to diameter are errors in the thin-shell approximation unacceptable?"

A prime requisite prior to undertaking the thesis was the making of a careful survey of the literature to determine the extent of theoretical and experimental work. The authors conducted such a survey both prior to beginning the experiments and on a continuing basis throughout the period of the thesis work. The results of this search are presented in the section entitled "Literature Survey."

It was also necessary to design and construct the experimental apparatus and the models. The materials and methods of design, together with the procedures employed, are indicated in the body of this paper. Since the models were fabricated from glass-reinforced plastic, the elastic formulas required in analyzing the data were modified to account for the difference in the values of the principal moduli of elasticity. Derivations for the expressions used are presented in the appendix.

---

[1] References are listed beginning on page 67.

The thesis begins with a section on concepts and assumptions employed in the theory of thin shells. This study was made in order to gain an insight into the differences between a thick and a thin shell theory.

CONCEPTS AND ASSUMPTIONS IN THE THEORY OF THIN SHELLS

A thin shell may be defined as one for which the ratio of the thickness of the shell to the principal radii of curvature is small compared to unity. The problem of classifying a shell as thick or thin therefore reduces to the determination of a value of h/R below which solution errors will be within a prescribed limit. Novozhilov[2] has stated that for ordinary technical calculations it is admissable to have a relative error of 5%, and that for thin shells this corresponds to a maximum $h/R = 0.05$. Timoshenko[3] (p.354) has suggested a value of $h/R = 0.10$ as the dividing point for thin curved bars.

There are three principal assumptions employed in the theory of thin shells in addition to those made for most problems in elasticity. They are

(1) The normal stresses acting on planes parallel to the middle surface may be neglected in comparison with the other stresses.

(2) The normals to the undeformed middle surface are deformed without change of length into the normals to the deformed middle surface.

(3) The thickness h is very small compared to the least radii of curvature (i.e. $h/R \ll 1$).

Assumptions (1) and (2) essentially reduce the problem to the case of plane stress. Consider the generalized matrix form of Hooke's law (see Borg[4], p. 59).

$$T = \lambda I_1 E_3 + 2\mu\eta$$

where

T = the stress tensor
η = the strain tensor
I_1 = the trace of the strain tensor
E_3 = the unit, 3x3, matrix
λ, μ = constants

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \mu = G = \frac{E}{2(1+\nu)}$$

Now at the inside surface of the cylinder (i.e. at $z = h/2$) the stress component σ_z is zero. Then, if the shell is thin, this component can be assumed to be zero throughout the thickness of the shell since it is small compared to the stresses σ_θ and σ_x.

Assumption (2) is equivalent to assuming that the shearing strains γ_{xz} and $\gamma_{\theta z}$ are zero, for the definition of shearing strain is the change in value of an originally right angle from the unstrained state. The assumption is that this right angle is preserved during deformation.

Therefore, letting $\sigma_z = \gamma_{xz} = \gamma_{\theta z}$ be zero, the tensor equation becomes

$$\begin{pmatrix} \sigma_x & \sigma_{x\theta} & \sigma_{xz} \\ \sigma_{x\theta} & \sigma_\theta & \sigma_{\theta z} \\ \sigma_{xz} & \sigma_{\theta z} & 0 \end{pmatrix} = \lambda(\epsilon_x + \epsilon_\theta + \epsilon_z)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2\mu \begin{pmatrix} \epsilon_x & \frac{1}{2}\gamma_{x\theta} & 0 \\ \frac{1}{2}\gamma_{x\theta} & \epsilon_\theta & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix}$$

Expanding the equation gives

$$\sigma_x = \lambda(\epsilon_x + \epsilon_\theta + \epsilon_z) + 2\mu\epsilon_x$$
$$\sigma_\theta = \lambda(\epsilon_x + \epsilon_\theta + \epsilon_z) + 2\mu\epsilon_\theta$$
$$\sigma_z = 0 = \lambda(\epsilon_x + \epsilon_\theta + \epsilon_z) + 2\mu\epsilon_z$$
$$\sigma_{x\theta} = \mu\gamma_{x\theta}$$
$$\sigma_{xz} = \sigma_{\theta z} = 0$$

Eliminating ϵ_z from the first two equations by use of the third gives the desired stress-strain relations

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_\theta)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_x)$$

$$\sigma_{x\theta} = \frac{E}{2(1-\nu)}\gamma_{x\theta}$$

Furthermore, assumptions (1) and (3) are utilized in reducing the strain components to a usable form (see Wang[5], pp. 335-340). When this

is done, the equation for σ_x becomes

$$\sigma_x = \frac{E}{1-\nu^2}\left[\epsilon_{xo} + \nu\epsilon_{\theta o} - z(\chi_x + \nu\chi_\theta)\right]$$

where

$$\epsilon_{xo} = \frac{\partial u}{\partial x} \qquad \chi_x = \frac{\partial^2 w}{\partial x^2}$$

$$\epsilon_{\theta o} = \frac{1}{a}\frac{\partial v}{\partial \theta} - \frac{w}{a} \qquad \chi_\theta = \frac{1}{a}\,\partial/\partial\theta\left(\frac{v}{a} + \frac{1}{a}\frac{\partial w}{\partial \theta}\right)$$

The expressions for the other stress components are similar in degree of complexity.

Finally, to obtain the relations between the stress resultants and couples, the following expressions must be integrated.

$$\begin{Bmatrix} N_\theta \\ M_\theta \\ Q_\theta \\ N_{\theta x} \\ M_{\theta x} \end{Bmatrix} = \int_{-h/2}^{+h/2} \begin{Bmatrix} \sigma_\theta \\ z\,\sigma_\theta \\ \tau_{\theta z} \\ \tau_{\theta x} \\ z\,\tau_{\theta x} \end{Bmatrix} dz \qquad \begin{Bmatrix} N_x \\ M_x \\ Q_x \\ N_{x\theta} \\ M_x \end{Bmatrix} = \int_{-h/2}^{+h/2} \begin{Bmatrix} \sigma_x \\ z\,\sigma_x \\ \tau_{xz} \\ \tau_{x\theta} \\ z\,\tau_{x\theta} \end{Bmatrix} \left(1 + z/R_\theta\right) dz$$

Again, assumption (3) is used and z/R_θ is neglected. Then, $N_{x\theta} = N_{\theta x}$, $M_{x\theta} = M_{\theta x}$, and the expressions may be integrated without difficulty. For example, the expression for N_x becomes

$$N_x = \frac{Eh}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \nu\left(\frac{1}{a}\frac{\partial v}{\partial \theta} - \frac{w}{a}\right)\right]$$

This equation and the remaining ones obtained in a similar manner are known as Love's first approximation.

It is possible to derive expressions for the stress resultants and couples without resorting to the use of assumptions (1) - (3). Naghdi[6], using a variational method due to Reissner[7], has made such a derivation. In order to gain an appreciation for the simplicity of the above expression and of the complexities involved in a thick shell theory, the more exact expression for N_x is given below.

$$N_x = \left\{ \frac{E h}{1-\nu^2} \left[\epsilon_{x0} + \nu \epsilon_{\theta 0} \right] \right\} + \left\{ D \frac{\chi_x}{R_\theta} \right\} + \left\{ - \frac{\nu}{1-\nu} \frac{h^2}{12} \left[\frac{\nu}{2 R_\theta} \left(\chi_x + \chi_\theta \right) \right. \right.$$

$$\left. \left. + \frac{\chi_\theta + \nu \chi_x}{R_\theta} \right] + \frac{\nu}{1-\nu} \frac{h}{2} \left[\left(\sigma_z \right)_{z=+\frac{h}{2}} + \left(\sigma_z \right)_{z=-\frac{h}{2}} \right] \right\}$$

Presumably, these more exact relations could be introduced into the equilibrium equations, and, provided a solution could be effected, a thick shell analysis for the instability problem would become available. An analytical approach to a thick shell instability analysis has not been attempted in this thesis, however.

LITERATURE SURVEY

The theoretical and experimental analysis of the buckling of cylindrical shells subjected to hydrostatic pressure has attracted the attention of numerous investigators dating from experiments conducted by Fairbairn[8] in 1858. All investigators employed the theory of thin shells in arriving at theoretical formulas and most experiments were carried out on models with relatively thin walls. Tables and descriptive information are presented in this section to indicate the extent of previous theoretical and experimental investigations. Particularly helpful in the making of the literature survey were two reports by Nash[9] which contain a listing of 2339 papers and books relevant to experimental and theoretical work on shells and shell-like structures from 1828 through 1956.

Results of Previous Experimental Investigations

Table 1 is a chronological listing of various investigators together with the range of geometries involved and other pertinent data. Since the first analytical solution to the instability problem was not available until 1913, all investigations prior to this time were made principally to determine empirical design formulas. Those investigations made in later years were carried out to test the validity of analytical solutions either in general or as applied to a particular problem.

With the exception of Southwell, all experimental investigators prior to 1929 restricted the loading on the cylinder to a radial pressure only. The axial thrust was eliminated by the use of a relatively stiff rod which connected the end plugs. The use of radial pressure only resulted from the

absence of analytical formulas based on combined loading, and from the fact that one of the principal applications of the results was to the study of boiler flues. Southwell did use a combined loading to simplify the apparatus. He neglected the effect of axial loading, however, in comparing test results with his analytical approach.

Most investigators after 1929 employed models subjected to hydrostatic (combined radial and axial) loads. An analytical solution to the problem of the cylinder under hydrostatic pressure was available by this time, and the submarine presented an immediate application.

From columns 4,5, and 6 of Table 1 the following is noted: Most of the models were relatively small in diameter, as was the h/D ratio. In those cases where the h/D ratio was greater than 0.05 the model was either so long that the values of the critical pressure approached that of an infinite tube, or they were in the range where yielding of the material occurred rather than an instability failure.

The type of support used by the various investigators is indicated in column 7. In general, this information was procured from sketches of the models as prepared for testing. A clamped type of support is indicated only where the author concerned specifically labelled the test condition as such. It is again noteworthy that no attempt was made to clamp the ends until 1941[10] when an analytical solution became available. Also, as indicated by Cook,[11] simply supported ends can be approached experimentally much more closely than fixed ends.

Column 8 indicates the maximum collapse pressures achieved by the various investigators; column 9 the number of models tested; and column 10 the variety of materials which have been utilized in the construction of models.

Several general observations can be made based on the survey of experimental results found in the literature:

(1) Scale effect is not important provided reasonable care is taken in fabricating the models. After testing 100 models representing the strength hull of a submarine, Windenburg[12] concluded that scale effect was not evident. Some the earlier experiments have not given good results when compared to theory because of the inadequacy of materials and unsound fabrication techniques. Most of the results obtained in the last two decades have given very good correlation.

(2) Although the assumed end condition is taken as either simply supported or as clamped, neither condition can be precisely obtained in practice. Based on tests of 258 brass models, Cornell[13] concluded that the end plugs inserted into the ends of the tubes offered a restraint that tended to reduce the effective length. He found that to get close agreement of his test results with theory, it was necessary to use an effective free length which differed from the actual free length by ½% to 1%. (Because of the small value of such a correction, the authors assume the measured free length to be the same as the effective free length for the tests reported in this thesis.)

(3) Several investigators have conducted carefully controlled experiments to determine the influence of other factors, such as eccentricity, on the collapse pressure. Sturm[10] was probably the first to investigate the problem of out-of-roundness from an experimental and analytical point of view. In recent years, Cleaver[14] has conducted very thorough, extensive, and carefully controlled tests on 530 models subjected to radial pressure. He lists the

following factors as affecting the collapse pressure:

a. Eccentricity of bore relative to the external surface.

b. Variations from true circular shape (out-of-roundness).

c. Ovality of the specimen (the difference in the maximum and minimum diameters divided by the nominal diameter).

d. Variation of material properties.

From the results of his tests Sturm concludes that eccentricity of bore is the predominant manufacturing variable affecting collapse strength, but for eccentricities within the limits imposed by current specifications, its effects are small and can be neglected for practical purposes. Variations in circularity exert no measurable systematic influence, and provided they are within current specifications (i.e. for commercial tubes), they too may be neglected.

The authors point out in passing that for a well-made, simply supported model the above effects tend to cancel the effect of any end restraint that might be present. Therefore, neglecting these effects may be justified when attempting to correlate experimental and theoretical results.

Results of Analytical Investigations

Table 2 presents a summary of results of principal theoretical investigations. The method of approach to the problem is indicated.

The differences in the various formulas can be accounted for by the inclusion or omission of certain higher order terms in the differential equations of equilibrium, by the method of approach (i.e. equil-

ibrium, energy, kinetic), and by the degree of simplification of the final result.

All of the more exact formulas are found to have the form

$$p/E = F_1 \ (h/D)^3 + F_2 \ (h/D)$$

where F_1 and F_2 are functions of

(a) The length to diameter ratio.

(b) Poisson's ratio.

(c) The number of lobes in a complete cicumferential belt at the time of collapse.

(d) The number of lobes along a generator of the cylinder at the time of collapse. Von Mises[15] has shown this quantity to be unity for minimum buckling pressure.

In order to facilitate comparison of the buckling equations with test results, values of F_1 and F_2 for the model geometry and material under consideration in this paper have been calculated for the various buckling formulas. These values are given in Table 3. Curves for p/E versus h/D are also included for these cases. (See Fig. 13).

In addition to the formulas for the buckling pressure of cylinders under hydrostatic load, it is of interest to examine formulas for buckling pressure under uniform radial load only. Table 4 gives some of the principal radial load buckling equations. A comparison of the radial load formulas with the hydrostatic load counterparts indicates that the following relationship is approximately true

$$\frac{\text{buckling pressure under hydrostatic load}}{\text{buckling pressure under radial load}} = \frac{n^2 - 1}{n^2 - 1 + \frac{\lambda^2}{2}}$$

Thus as λ approaches zero ($L/D \rightarrow \infty$), the formulas for hydrostatic load should yield the same results as the formulas for radial load. For the case where $\lambda = 0$, n becomes two, and all of the more exact hydrostatic and radial load buckling pressure equations reduce to the Bresse equation (Table 4) for a tube of infinite length.

TABLE I

SUMMARY OF EXPERIMENTAL INVESTIGATIONS IN THE LITERATURE

(1) INVESTIGATOR	(2) DATE	(3) LOAD	(4) D (IN.)	(5) h/D	(6) L/D	(7) SUPPORT	(8) $P_{MAX.}$	(9) NO. TESTED	(10) MATERIAL	(11) REF.
FAIRBAIRN	1858	R	4-12	.0036-.0108	?-15	SS	-	32	WROUGHT IRON	8
CARMAN	1905	R	-	-	>0.90	SS	-	-	BRASS	16
STEWART	1906	-	3-13	.0159-.0813	14-80	SS	5560	256	STEEL	17
CARMAN & CARR	1906	R	-	-	>2.50	SS	-	-	STEEL BRASS	18
SOUTHWELL	1913	H	1	.003-.028	4.18-8.69	SS	4800	23	STEEL	19
COOK	1914	R	3	.0097-.0206	0.69-4.24	SS	1930	39	STEEL	11
REPORTED BY VON MISES	1914	-	38-54	.0046-.0132	0.67-2.30	SS	465	-	STEEL	15
CARMAN	1917	R	.75-3	.010-.064	1.60-14.5	SS	2810	104	STEEL * BRASS	20
COOK	1925	R	1.05	.0084	0.24-6.70	SS	543	23	BRASS	21
VON MISES	1929	H	31-95	.00125	0.15-0.30	SS	139	4	STEEL	22
TOKUGAWA	1929	H	3.94	.005	-	SS	-	-	BRASS	23
A O SMITH CORP.	1931	H	8.6-16	.0163-.0555	8-16	SS	6225	44	STEEL	24
TOKUGAWA	1940	H	2.36	.0033-.0083	-	SS	-	52	STEEL	25
STURM	1941	H, R	6.0-11.9	.0024-.0101	0.835-12	SS, CL	330	33	ALUMINUM	10
KIRKBY	1947	-	-	-	-	-	-	-	-	NOT AVAIL.
CORNELL	1948	H	0.250	.020-.050	5.5-14	SS	4084	258	BRASS	13
CHRISTENSEN	1951	H	3-3.44	.0195-.0216	2-6	SS, CL	1260	24	STEEL ALUMINUM	26
CLEAVER	1955	R	1-2.25	.0098-.056	.51-14	SS	14,000	530	STEEL BRASS	14
BARNET & CARROLL	1956	H	3.06	.025-.035	.25-1.5	SS	-	100	GRP	27
DTMB	1961	H	6	~.052	~1.75	SS	7900		GRP	

R - RADIAL   SS - SIMPLY SUPPORTED   * ALSO ALUMINUM, GLASS, AND HARD RUBBER

H - HYDROSTATIC   CL - CLAMPED

GRP - GLASS REINFORCED PLASTIC

NOTE: THERE ARE A NUMBER OF EXPERIMENTAL RESULTS AVAILABLE FOR RING STIFFENED CYLINDERS WHICH HAVE NOT BEEN INCLUDED IN THIS TABLE FOR EXAMPLE SEE REF. 28.

TABLE 2

THEORETICAL BUCKLING FORMULAS FOR UNSTIFFENED CYLINDERS UNDER HYDROSTATIC LOAD

No.	DATE	INVESTIGATOR	METHOD OF APPROACH	BUCKLING FORMULA	REMARKS	REF
1	1924	PRESCOTT	EQUILIBRIUM	$\frac{P}{E} = \frac{2}{3}\left[\frac{(m^2-1)^2}{m^2-1+\frac{\lambda^2}{2}(1+\frac{1}{m^2})}\right]\left(\frac{h}{D}\right)^3 + \left[\frac{2\lambda^4/m^4}{m^2-1+\frac{\lambda^2}{2}(1+\frac{1}{m^2})}\right]\left(\frac{h}{D}\right)$		35
2	1929	VON MISES	EQUILIBRIUM	$\frac{P}{E} = \frac{2}{3}\left[\frac{(m^2+\lambda^2)^2 - 2\mu_1 \lambda^2 + \mu_2}{(m^2-1+\lambda^2/2)(1-\nu^2)}\right]\left(\frac{h}{D}\right)^3 + \left[\frac{2\lambda^4}{(m^2-1+\lambda^2/2)(m^2+\lambda^2)^2}\right]\left(\frac{h}{D}\right)$ $2\mu_1 = 2 + \rho\left[3+\nu+(1-\nu^2)\rho\right]$ $\mu_2 = (1-\rho\nu)\left[1+(1+2\nu)\rho-(1-\nu^2)\left(1+\frac{1+\nu}{1-\nu}\rho\right)\rho^2\right]$ $\rho = \frac{\lambda^2}{\lambda^2+m^2}$	VON MISES EQUATION (6)	22
3	1929	TOKUGAWA	EQUILIBRIUM	$\frac{P}{E} = \frac{2}{3}\left[\frac{(m^2+\lambda^2)^2 - \frac{m^4(2m^2-1)}{(m^2+\lambda^2)^2}}{(m^2-1+\lambda^2/2)(1-\nu^2)}\right]\left(\frac{h}{D}\right)^3 + \left[\frac{2\lambda^4}{(m^2-1+\lambda^2/2)(m^2+\lambda^2)^2}\right]\left(\frac{h}{D}\right)$	DERIVED FOR RING-STIFFENED CYLINDERS, WHERE $\lambda = \frac{K\pi D}{2L}$, AND $K = 1$ FOR NO STIFFENING	23

TABLE 2 (CONTINUED)

THEORETICAL BUCKLING FORMULAS FOR UNSTIFFENED CYLINDERS UNDER HYDROSTATIC LOAD

NO.	DATE	INVESTIGATOR	METHOD OF APPROACH	BUCKLING FORMULA	REMARKS	REF
4	1934 1936	FLÜGGE TIMOSHENKO	EQUILIBRIUM "	$C_1 + C_2\alpha = C_3\phi_1 + C_4\phi_2$, WHERE: $\alpha = \frac{1}{3}\left(\frac{h}{D}\right)^2 \qquad \phi_1 = \frac{PD}{2Eh}(1-\nu^2) \qquad \phi_2 = \frac{-N_x(1-\nu^2)}{Eh}$ $C_1 = (1-\nu^2)\lambda^4$ $C_2 = (\lambda^2+m^2)^4 - 2\left[\nu\lambda^6 + 3\lambda^4 m^2 + (4-\nu)\lambda^2 m^4 + m^6\right] + 2(2-\nu)\lambda^2 m^2 + m^4$ $C_3 = m^2(\lambda^2+m^2)^2 - (3\lambda^2 m^2 + m^4)$ $C_4 = \lambda^2(\lambda^2+m^2)^2 + \lambda^2 m^2$ $\frac{P}{E} = \frac{2}{3}\left[\frac{C_2}{(C_3+\frac{C_4}{2})(1-\nu^2)}\right]\left(\frac{h}{D}\right)^3 + \left[\frac{2C_1}{(C_3+\frac{C_4}{2})(1-\nu^2)}\right]\left(\frac{h}{D}\right)$	THE FORM GIVEN BY FLÜGGE & TIMOSHENKO. N_x POSITIVE IF TENSILE. REARRANGED, WITH $\phi_1/2$ SUBSTITUTED FOR ϕ_2.	29 30
5	1934	WINDENBURG & TRILLING		$\frac{P}{E} = \frac{2.42}{(1-\nu^2)^{3/4}} \cdot \frac{(h/D)^{5/2}}{\left[\frac{L}{D} - 0.45\left(\frac{h}{D}\right)^{1/2}\right]}$	THIS FORMULA WAS OBTAINED USING VON MISES' EQ (6) BY DERIVING AN APPROXIMATE EXPRESSION FOR THE "MINIMIZING m", AND SUBSTITUTING, WITH FURTHER SIMPLIFICATIONS. THIS REDUCES TO ZERO AS $L \rightarrow \infty$	33

TABLE 2 (CONTINUED)

THEORETICAL BUCKLING PRESSURE FORMULAS FOR UNSTIFFENED CYLINDERS UNDER HYDROSTATIC LOAD

NO.	DATE	INVESTIGATOR	METHOD OF APPROACH	BUCKLING FORMULA	REMARKS	REF.
6	1941	STURM	EQUILIBRIUM	$\frac{P}{E} = \frac{2}{3}\left[\frac{(m^2+\lambda^2)^2 - \left\{\nu\lambda^2 + m^2 + \frac{[m^2+\lambda^2(2-\nu)]m^2}{(m^2+\lambda^2)^2}\left[m^2+(2-\nu)\lambda^2-1\right]\right\}}{(m^2-1+\lambda^2/2)(1-\nu^2)}\right]\left(\frac{h}{D}\right)^3 + \left[\frac{2\lambda^4}{(m^2-1+\lambda^2/2)(\lambda^2+m^2)^2}\right]\left(\frac{h}{D}\right)$		10
7	1944	KIRKWOOD, ET AL.	KINETIC	$\frac{P}{E} = \frac{2h}{D(1-\nu^2)}\cdot\Phi_c$ WHERE $\Phi_c = \frac{b+d\alpha^2}{c}$ $\alpha = \frac{1}{3}\left(\frac{h}{D}\right)^2$ $b = (1-\nu^2)\lambda^4$ $c = (m^2+\lambda^2)^2(m^2-1+\lambda^2/2)$ $d = (m^2+\lambda^2)^4 - 2m^6 + m^4\left[1-2(3+\nu)\lambda^2\right] + m^2\left[4\lambda^2 - \frac{\lambda^4}{(1-\nu)^2}(7-10\nu-3\nu^2+6\nu^3-2\nu^4)\right] + (1+\nu)(1+3\nu)\lambda^4 - \frac{\lambda^6}{(1-\nu)^2}(1-\nu^2+2\nu^3)$	AS PRESENTED IN THE REFERENCE	31
				$\frac{P}{E} = \frac{2}{3}\left[\frac{(m^2+\lambda^2)^2 - \frac{(m^2+\lambda^2)^4 - d}{(m^2+\lambda^2)^2}}{(m^2-1+\lambda^2/2)(1-\nu^2)}\right]\left(\frac{h}{D}\right)^3 + \left[\frac{2\lambda^4}{(m^2-1+\lambda^2/2)(\lambda^2+m^2)^2}\right]\left(\frac{h}{D}\right)$	REARRANGED	

TABLE 2 (CONTINUED)

THEORETICAL BUCKLING FORMULAS FOR UNSTIFFENED CYLINDERS UNDER HYDROSTATIC LOAD

NO.	DATE	INVESTIGATOR	METHOD OF APPROACH	BUCKLING FORMULA	REMARKS	REF.
8	1947	BATDORF		$\frac{P}{E} = \frac{2.42}{(1-\nu^2)} \cdot \frac{(-h/D)^{5/2}}{L/D}$	AN APPROXIMATION OF THE MORE EXACT FORMULA	34
9	1950	SALERNO & LEVINE	ENERGY	$\frac{P}{E} = \frac{2}{3}\left[\frac{(n^2+\lambda^2)^2}{(n^2-1+\lambda^2/2)(1-\nu^2)}\right]\left(\frac{h}{D}\right)^3 + \left[\frac{2\lambda^4}{(n^2-1+\lambda^2/2)(\lambda^2+n^2)^2}\right]\left(\frac{h}{D}\right)$	THIS FORMULA IS ALSO GIVEN BY VON MISES[22] AS A SIMPLIFIED EXPRESSION FOR HIS MORE EXACT EQUATION, EXCEPT THAT $(n^2-1+\lambda^2/2)$ IS REPLACED BY $(n^2+\lambda^2/2)$ IN VON MISES'	32
	1954	NASH	ENERGY			

TABLE 3

CONSTANTS FOR BUCKLING EQUATIONS

$$P/E = F_1 (h/D)^3 + F_2 (h/D)$$

FOR $L/D = 2.53$, $n = 2$, $\nu = 0.15$

EQUATION NO. FROM TABLE 2	INVESTIGATOR	F_1	$F_2 \times 10^4$
1	PRESCOTT	1.851324	57.31102
2	VON MISES	2.389462	48.39826
3	TOKUGAWA	2.864323	48.39826
4	FLÜGGE, TIMOSHENKO	2.384580	48.89380
6	STURM	2.444910	48.39826
7	KIRKWOOD, et. al.	2.511483	48.39826
9	NASH, SALERNO-LEVINE	4.108301	48.39826

TABLE 4

THEORETICAL BUCKLING FORMULAS FOR UNSTIFFENED CYLINDERS UNDER UNIFORM RADIAL LOAD

NO.	DATE	INVESTIGATOR	METHOD OF APPROACH	BUCKLING FORMULA	REMARKS	REF.
1	1859 1888	BRESSE BRYAN	ENERGY	$\frac{P}{E} = \frac{2}{(1-\nu^2)}\left(\frac{h}{D}\right)^3$	FOR CYLINDER OF INFINITE LENGTH, I.E., $\lambda = 0$	36 37
2	1913 1929	SOUTHWELL TOKUGAWA	ENERGY EQUILIBRIUM	$\frac{P}{E} = \frac{2}{3}\left[\frac{(m^2-1)}{(1-\nu^2)}\right]\left(\frac{h}{D}\right)^3 + \left[\frac{2\lambda^4}{(m^2-1)m^4}\right]\left(\frac{h}{D}\right)$		33 23
3	1914	VON MISES	EQUILIBRIUM	$\frac{P}{E} = \frac{2}{3}\left[\frac{(m^2-1)^2 + \lambda_1 m^4 - \lambda_2 m^2 + \lambda_3}{(m^2-1)(1-\nu^2)}\right]\left(\frac{h}{D}\right)^3 + \left[\frac{2\lambda^4}{(m^2-1)(\lambda^2+m^2)^2}\right]\left(\frac{h}{D}\right)$ $\lambda_1 = \frac{\rho(2-\rho)}{(1-\rho)^2}$ $\lambda_2 = \rho\left[3+\nu+(1-\nu^2)\rho\right]$ $\lambda_3 = \rho(1+\nu) - \rho^2\left[\nu(1+2\nu) + (1-\nu^2)(1-\rho\nu)\left(1+\frac{1+\nu}{1-\nu}\rho\right)\right]$ $\rho = \frac{\lambda^2}{\lambda^2+m^2}$		15
4	1914 1936	VON MISES TIMOSHENKO	EQUILIBRIUM EQUILIBRIUM	$\frac{P}{E} = \frac{2}{3}\left[\frac{(m^2-1)(m^2-\lambda^2) + \lambda^2(2m^2-1-\nu)}{(m^2-\lambda^2)(1-\nu^2)}\right]\left(\frac{h}{D}\right)^3 + \left[\frac{2\lambda^4}{(m^2-1)(\lambda^2+m^2)^2}\right]\left(\frac{h}{D}\right)$	AN APPROXIMATION OF FORMULA (3) IN THIS TABLE.	15 30
5	1941	STURM	EQUILIBRIUM	$\frac{P}{E} = \frac{2}{3}\left[\frac{(m^2+\lambda^2)^2 - \left\{\nu\lambda^2+m^2+\frac{[m^2+\lambda^2(2-\nu)]m^2}{(m^2+\lambda^2)^2}\left[m^2+(2-\nu)\lambda^2-1\right]\right\}}{(m^2-1)(1-\nu^2)}\right]\left(\frac{h}{D}\right)^3$ $+ \left[\frac{2\lambda^4}{(m^2-1)(\lambda^2+m^2)^2}\right]\left(\frac{h}{D}\right)$		10

TEST APPARATUS, MODELS, AND TEST PROCEDURE

Test Apparatus

The test apparatus consists of a pressure vessel, pump, tubing, pressure gages, and equipment required for the taking of strain gage data. The arrangement and details of the apparatus are shown in Figures 1 through 5. Specifications which were drawn up for the test apparatus are given below.

Size of chamber: 5" I.D., 13" depth

Working pressure: 10,000 psi

Penetrations: One fitting for pressure tubing, one fitting for four wires from strain gages on the external surface of the model, one fitting for 1/4" tube to vent model to the atmosphere and to lead out internal strain gage wires.

Pump: Hand operated pump with self-contained check valve and reservoir--the type commonly used with hydraulic jacks.

Pressure gages: One 5000 psi 8" Bourdon gage graduated in increments of 50 psi, and one 15,000 psi 6" Bourdon gage, graduated in increments of 100 psi.

Strain Indicator: Baldwin, Type N

The pressure vessel was constructed in accordance with a basic design drawn up by Mr. Kenneth Hom of the David Taylor Model Basin. The design was modified slightly by the authors, to suit the purpose required. The following calculations show the factors of safety, for a working pressure of 10,000 psi. The material is low carbon steel, the yield strength of which was assumed to be 30,000 psi. Formulas used are given in reference (39).

t = wall thickness, inches, = $\begin{cases} 2.25 \text{ for walls} \\ 2.125 \text{ for bottom} \end{cases}$

d = inside diameter of chamber, inches, = 5.00

P = chamber pressure, psi

S = maximum stress, psi, = 30,000

R = inside radius of chamber, inches, = 2.50

F.S. = factor of safety based on yield strength of 30,000 psi

Based on circumferential stress in wall:

$$P = \frac{St}{R + 0.6t}^{*} = \frac{(30,000)(2.25)}{2.50 + (0.6)(2.25)} = 17,532 \text{ psi}$$

$$F.S. = \frac{17,532}{10,000} \cong 1.75$$

Based on longitudinal stress in wall:

$$P = \frac{2St}{R - 0.4t} = \frac{(2)(30,000)(2.25)}{2.50 - (0.4)(2.25)} = 84,375 \text{ psi}$$

$$F.S. = \frac{84,375}{10,000} \cong 8.44$$

Based on stress in bottom:

$$P = \frac{St^2}{d^2(0.162)}^{**} = \frac{(30,000)(2.125)^2}{(5.00)^2(0.162)} = 33,449 \text{ psi}$$

$$F.S. = \frac{33,449}{10,000} \cong 3.34$$

---

* Ref. (39), para. UG-27(c)(1)

** Ref. (39), Section VIII, para. UG-34(1).
Para. UG-36(3) states:

"Single openings in vessels not subject to rapid fluctuations in pressure do not require reinforcement other than that inherent in the construction under the following conditions:
....(b) Threaded, studded, or expanded connections in which the hole cut in the shell or head is not greater than two-inch pipe size."

The pressure vessel was proof tested to 11,000 psi on 30 August, 1960. It was raised to this pressure three times, and strain data were taken on the outside surface using a circumferential strain gage. This test was made prior to the making of the penetrations for fittings "A" and "B" (Fig. 4). The strain data taken plotted linearly from zero to 11,000 psi.

This apparatus should not be used with a working pressure in excess of 10,000 psi, unless the pressure vessel is again proof tested at the higher pressure with strain gage measurements taken in regions of high stress. In any case, it is recommended that a pressure of 15,000 psi not be exceeded, and that for working pressures in excess of 10,000 psi the pressure vessel be placed in a pit or behind a suitable barrier. All tubing and fittings used should be rated at the maximum working pressure employed.

Persons making further use of this apparatus are warned that valve "G", Fig. 4, is rated at 3000 psi. It is recommended that this valve be replaced by one rated at 10,000 psi or higher. As a safety precaution in performing the tests for this thesis, a 1/8" steel plate was attached to the table between the operators and fittings "D" through "H" (Fig. 4). The plate can be seen in the photograph, Fig. 5.

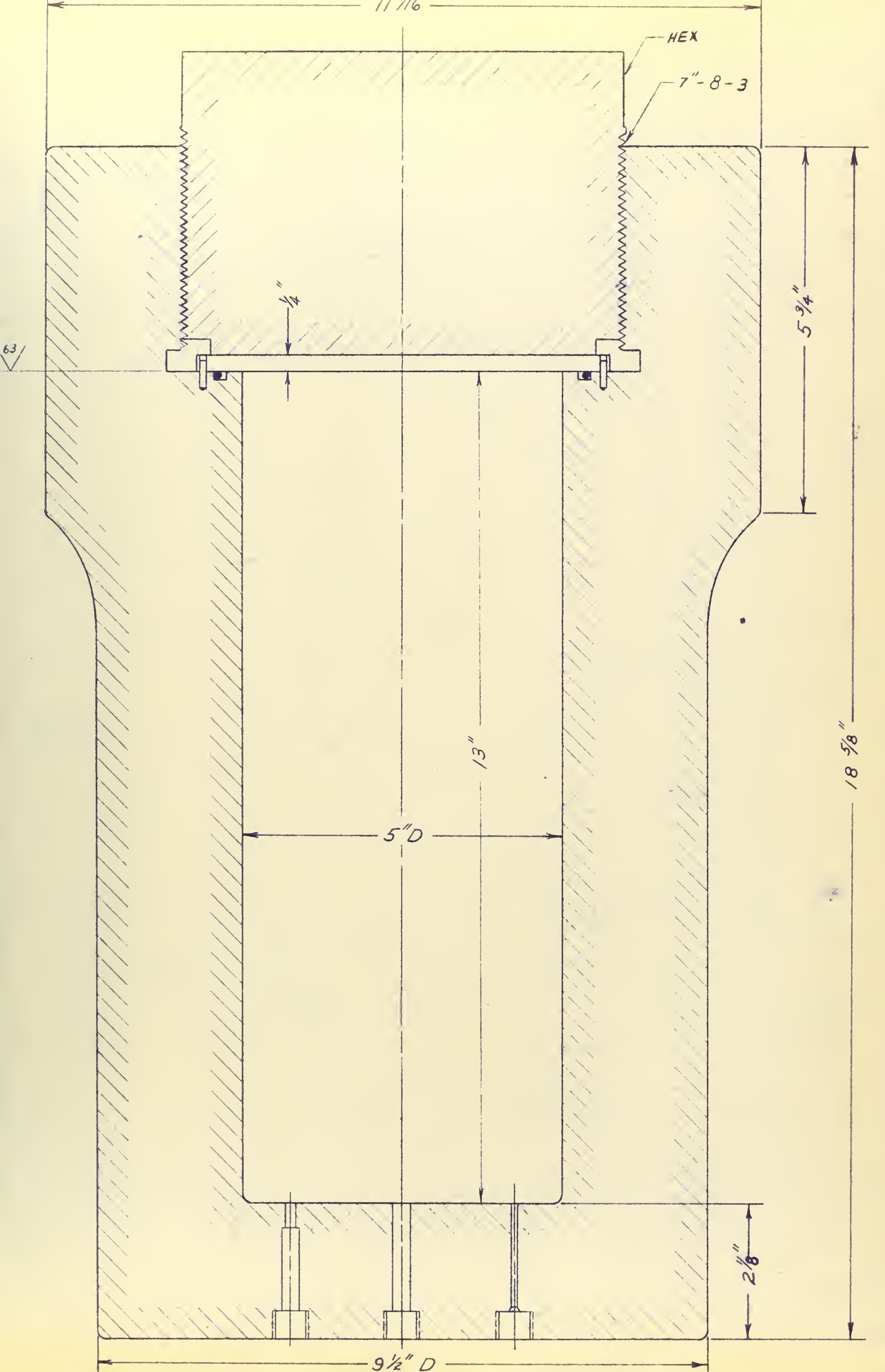
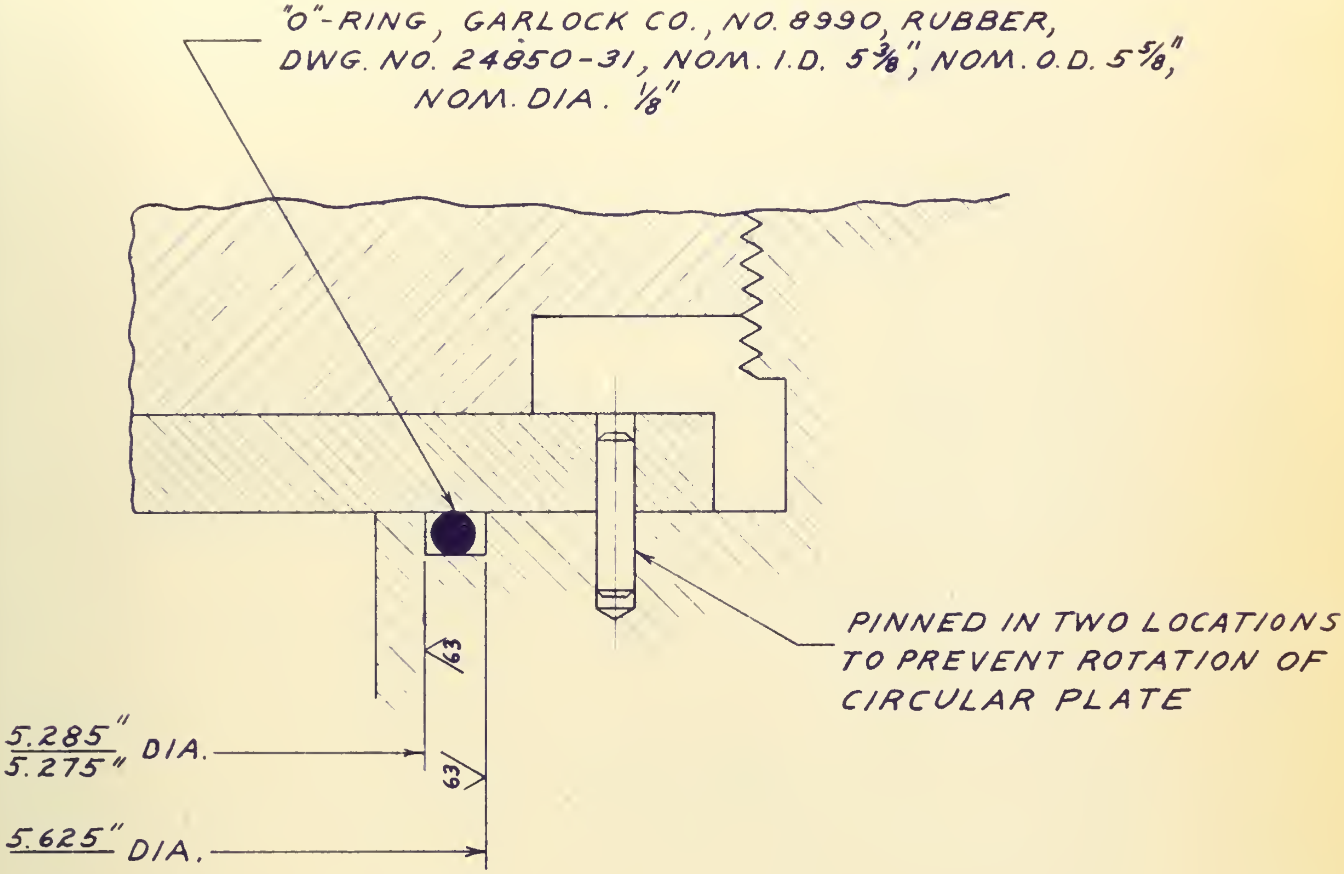
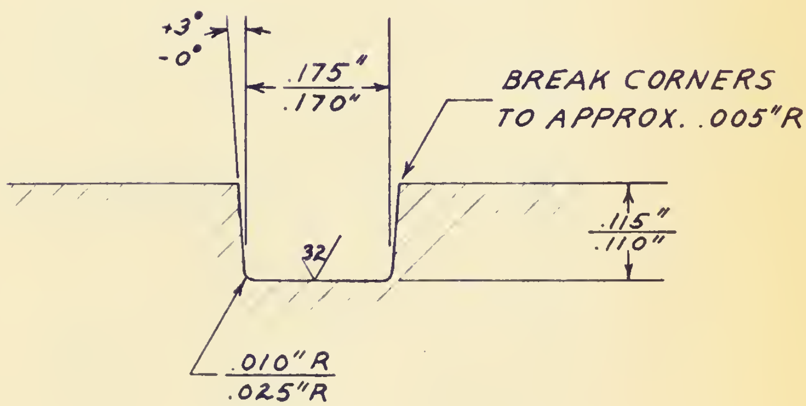


FIGURE I
PRESSURE VESSEL
SCALE: ½"=1"

FIGURE 2



NOTE:
ALL SURFACES AND CORNERS OF "O"-RING GROOVE MUST BE WITHOUT TOOLMARKS, NICKS, OR SCRATCHES.

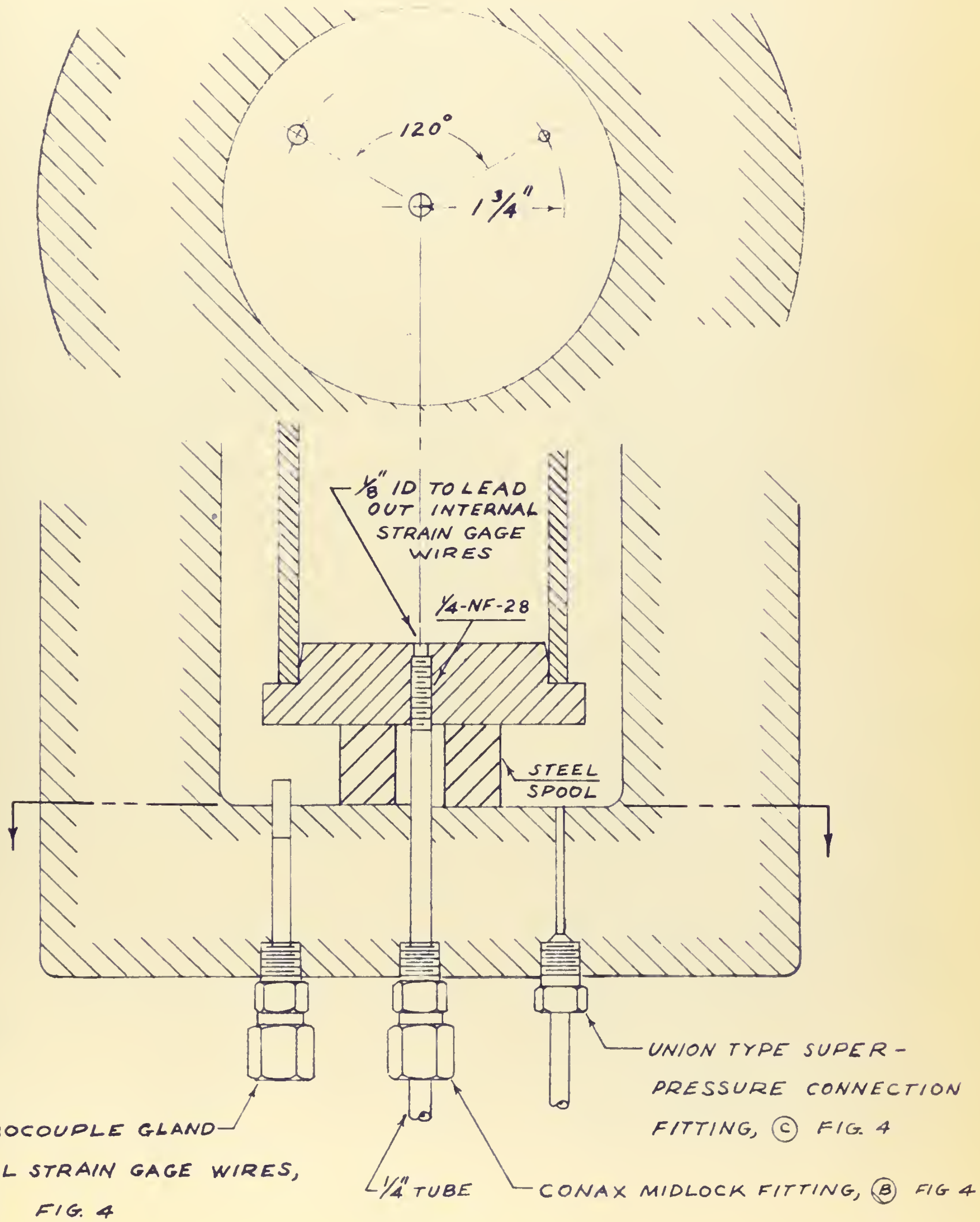


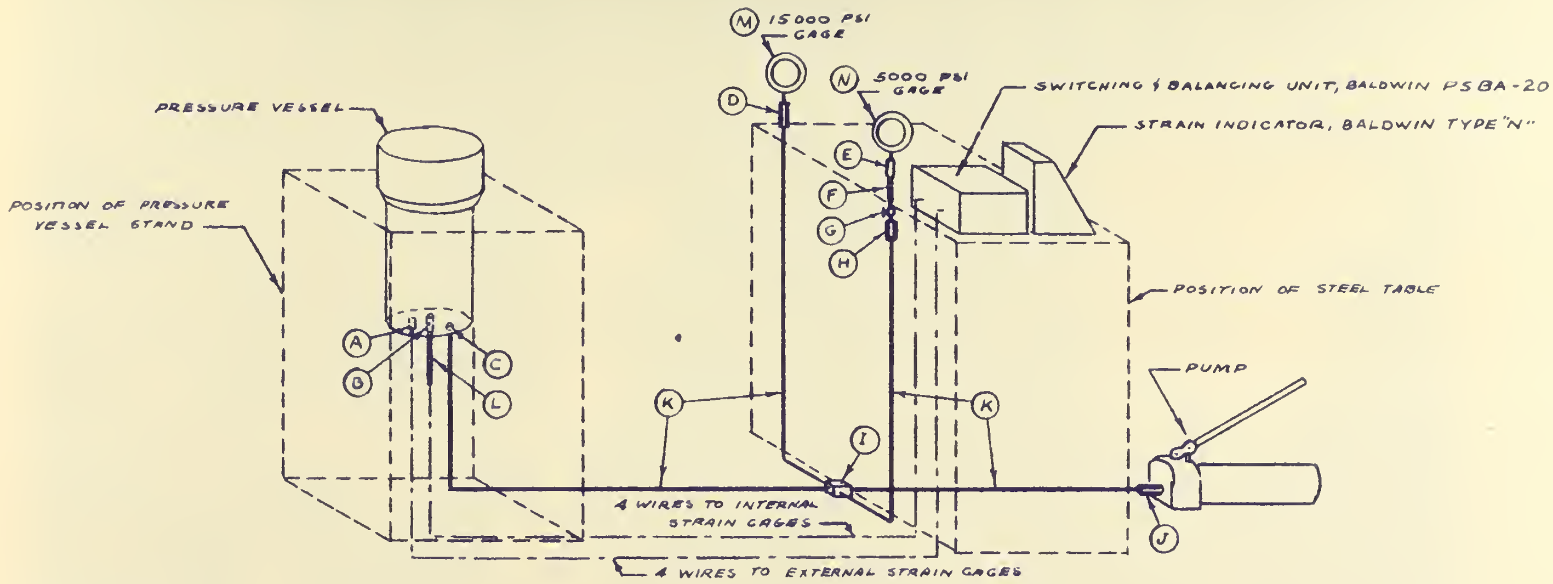
DETAILS OF "O"-RING, GROOVE, & CIRCULAR PLATE

FIGURE 3

ARRANGEMENT OF FITTINGS IN PRESSURE VESSEL

1/2" = 1"





ITEM	QUAN'TY	DESCRIPTION	MANUFACTURER	CAT. OR DWG. NO.
A	1	FOUR-WIRE THERMOCOUPLE GLAND	CONAX CORP.	TG-20-A4
B	1	MIDLOCK GLAND FOR 1/4" O.D. TUBING	CONAX CORP.	MK-4-4
C	1	CONNECTOR	AMINCO	45-1310
D	1	GAGE CONNECTOR	AMINCO	45-7212
E	1	1/8" NPT TO 1/4" NPT ADAPTER		
F	1	1/8" NPT NIPPLE		
G	1	1/8" VALVE	HOKE	331
H	1	FEMALE 1/8" NPT ADAPTER	AMINCO	45-6007
I	1	FOUR-WAY CROSS	AMINCO	45-5311
J	1	MALE 3/8" NPT ADAPTER	AMINCO	45-6433
K		1/4" O.D. 1/16" I.D. CHROME-MOLY. STEEL TUBING	AMINCO	45-1010
L		1/4" O.D. 1/8" I.D. STEEL TUBING		
M	1	15,000 PSI GAGE	AMINCO	47-8310
N	1	5000 PSI GAGE		

ARRANGEMENT OF TEST APPARATUS

FIGURE 4

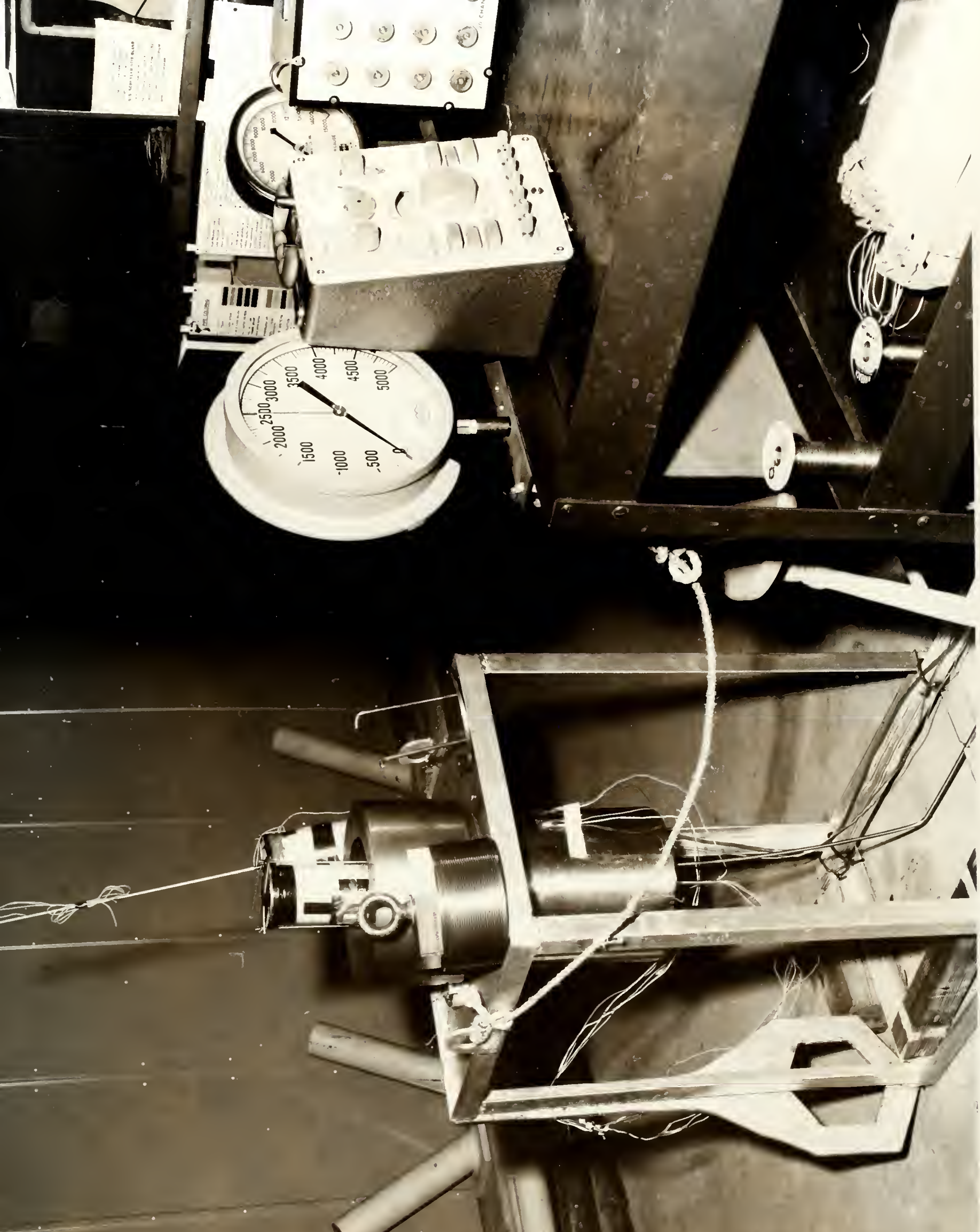


FIGURE 5
Photograph of Assembled Apparatus

Model Design

Design of the models was based on the following considerations:

(1) Maximum working pressure of test apparatus: 10,000 psi

(2) Dimensions of test chamber: 5" I.D. and 13" depth

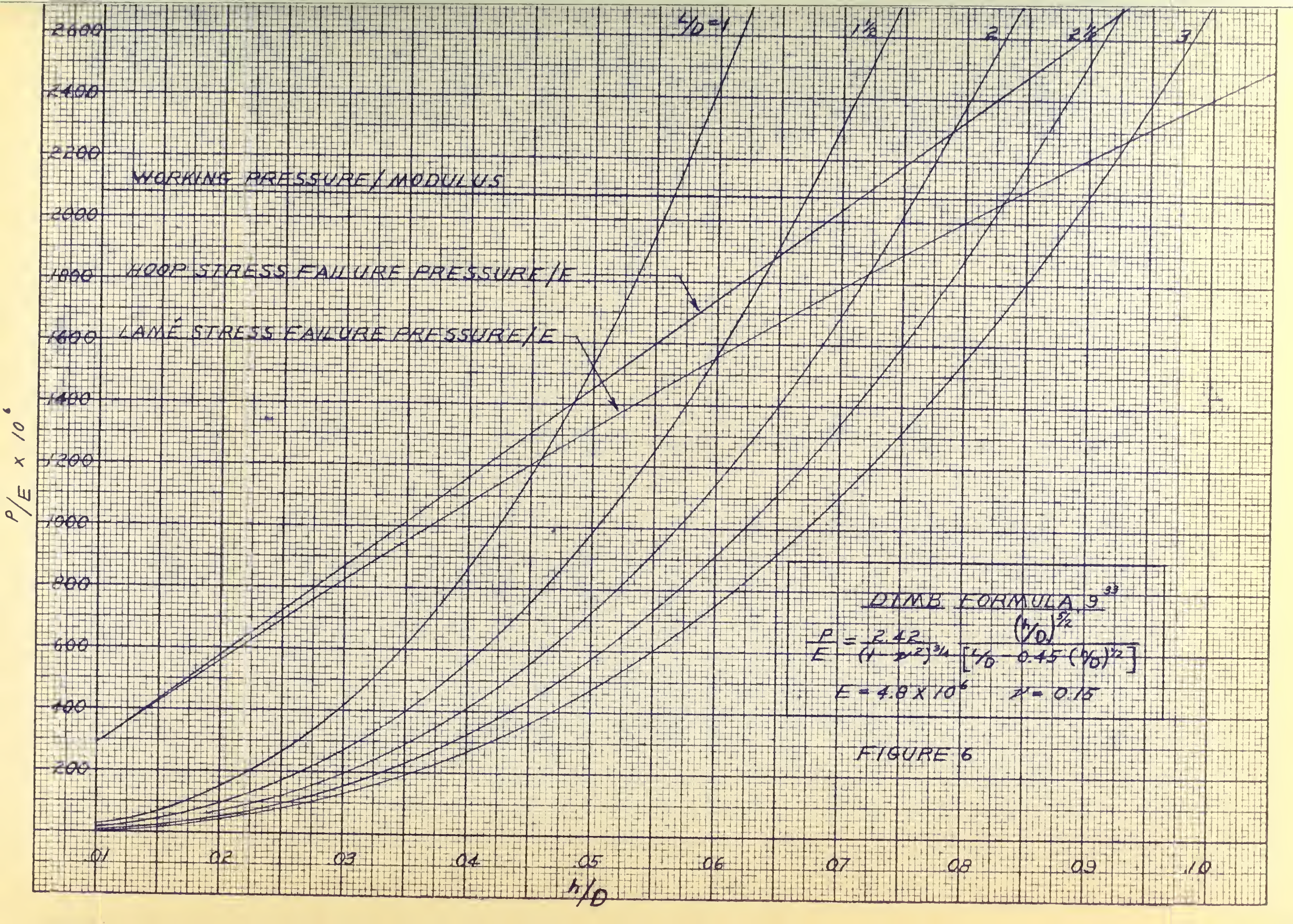
Since the purpose of the thesis is to evaluate the instability formulas for relatively thick-walled cylinders, it was necessary to design the models to fail by instability, and to have a relatively large thickness-diameter ratio. Mr. John Pulos of the David Taylor Model Basin recommended a range of h/D from 0.05 to 0.10. Also, Timoshenko[3] (p.354) has given a value of h/D = 0.05 as an estimated division point between thick and thin curved bars. Accordingly, the range of values for h/D was selected as 0.05 to 0.10.

A calculation revealed that a material with a relatively low modulus of elasticity was required in order to cause instability failures in the range of h/D greater than 0.05, and, at the same time, remain within the upper limit of test pressure of 10,000 psi. Captain Hinners suggested the possible use of glass-reinforced plastic, especially in view of the current interest in this material because of its favorable strength-weight ratio.

The David Taylor Model Basin had published a report[38] (No. 1413) on the hydrostatic pressure tests of a cylindrical shell of a glass fiber reinforced epoxy resin fabricated by the Zenith Plastics Co. of Gardena, California. The circumferential modulus of this material was reported to be 4.8×10^6 psi, and the plot of pressure vs. strain was linear with but a slight departure from linearity near the collapse pressure. This material seemed to meet the requirements well. It had disadvantages, however, in that it was neither homogeneous nor isotropic, thus violating two basic assumptions in the usual approach to the theory of elasticity.

It was felt, nevertheless, that these disadvantages would be overcome by computing an "effective modulus of elasticity" from strain data, and by using models fabricated in such a way as to give maximum dispersion of the glass in the resin. Correspondence with the Zenith Plastics Co. was initiated to obtain information regarding the procurement of the necessary models. Zenith very generously offered to supply the models at no cost.

At this point specifications were drawn up for the model geometries. Using DTMB formula (9) [Formula No. 5, Table 2], a set of curves was drawn for various values of L/D, with buckling pressure as the ordinate and h/D as the abscissa. These curves are shown in Fig. 6. The hoop stress failure pressure was also plotted on these curves using a nominal value of compressive strength of 70,000 psi as obtained from DTMB Report 1413.[38] These curves were then used to select appropriate values for L/D and h/D. It was decided to keep the value of L/D constant at about 2.5 and vary the value of h/D from about 0.05 to 0.09. It was also decided to test a minimum of three models of each geometry in order to determine that test results could be reproduced. Table 5 lists the models received from Zenith and gives their measured average dimensions. Fig. 7 is a sketch of the model showing the end closure plugs. The plugs, made from medium steel stock, were designed to give simple support of the model ends, the condition for the von Mises buckling equation. The effective length, L, is as shown. The inside diameter was chosen as 3.195" because of the availability at the Zenith plant of a mandrel of that diameter.


P/E x 10^6
2600
2400
2200
2000
1800
1600
1400
1200
1000
800
600
400
200
.01
.02
.03
.04
.05
.06
.07
.08
.09
.10
h/D
L/D = 1
1½
2
2½
3
WORKING PRESSURE / MODULUS
HOOP STRESS FAILURE PRESSURE / E
LAMÉ STRESS FAILURE PRESSURE / E
DIMB FORMULA 9 [33]
$\frac{P}{E} = \frac{2.42}{(1-\nu^2)^{3/4}} \frac{(h/D)^{5/2}}{\left[L/D - 0.45 (h/D)^{1/2}\right]}$
$E = 4.8 \times 10^6$ $\nu = 0.15$
FIGURE 6

Model Fabrication

The models were built up using layers of glass fiber reinforced epoxy resin tape. The glass fibers ran in one direction, along the length of the tape strip, and the tape was applied to the mandrel both by circumferential winding under tension (circumferential plies) and by laying on strips parallel to the axis of the mandrel (longitudinal plies). The process was started and ended with two circumferential plies and was continued using alternate layers in a ratio of two circumferential plies to one longitudinal ply. This 2:1 ratio of circumferential to longitudinal plies results in the allowable stresses in the longitudinal and circumferential directions being in close agreement with the induced stresses in these directions. The resulting structure is of course anisotropic. The thickness was varied by varying the number of layers of tape. Five tubes were fabricated, each about 60" in length and each of a different thickness (see Table 5). Upon completion of the curing process, most of the models were cut from the tube to proper length while still on the mandrel at the plant. Some of the models were cut from the remaining portions of tubes after delivery, using a band-saw, and grinding or machining the ends until smooth and normal to the cylinder axis. By weighing and measuring some of the models, the density of the material was determined to be about 0.07 pounds/cubic inch. Values of resin content for the model material were provided by Zenith, and are given in Table 11.

FIGURE 7

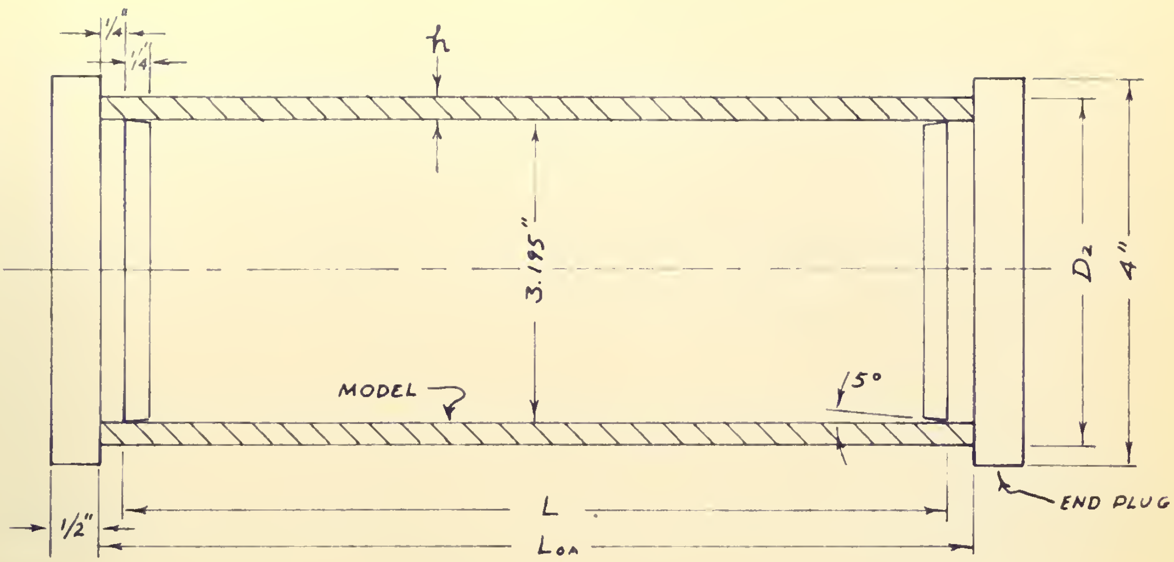


TABLE 5

MODEL	D_2	D^*	h	L_{OA}	L	h/D	L/D
A-1	3.533	3.364	0.169	8.99	8.49	0.05024	2.52
A-2	↓	↓	↓	↓	↓	↓	↓
A-3	↓	↓	↓	↓	↓	↓	↓
A-4	↓	↓	↓	↓	↓	↓	↓
A-5	↓	↓	↓	5.62	5.12	↓	1.52
A-6	↓	↓	↓	3.50	3.00		0.892
B-1	3.605	3.400	0.205	9.04	8.54	0.06029	2.51
B-2	↓	↓	↓	↓	↓	↓	↓
B-3	↓	↓	↓	↓	↓	↓	↓
B-4	↓	↓	↓	7.00	6.50	↓	1.91
C-1	3.675	3.435	0.240	9.17	8.67	0.06987	2.52
C-2	↓	↓	↓	↓	↓	↓	↓
C-3	↓	↓	↓	↓	↓	↓	↓
C-4	↓	↓	↓	5.50	5.00	↓	1.46
D-1	3.757	3.476	0.281	9.32	8.82	0.08084	2.54
D-2	↓	↓	↓	↓	↓	↓	↓
D-3	↓	↓	↓	↓	↓	↓	↓
D-4	↓	↓	↓	↓	↓	↓	↓
D-5	↓	↓	↓	4.62	4.12	↓	1.19
E-1	3.813	3.504	0.309	9.44	8.94	0.08818	2.55
E-2	↓	↓	↓	↓	↓	↓	↓
E-3	↓	↓	↓	↓	↓	↓	↓
E-4	↓	↓	↓	↓	↓	↓	↓

* $D = \frac{1}{2}(3.195 + D_2)$

ALL DIMENSIONS ARE IN INCHES

Instrumentation of Models

Strain data were taken to provide a means for calculating an effective modulus of elasticity for the models. As shown in Appendix C, values of the strain-pressure ratios, taken in the directions of principal stresses, can be used in conjunction with Hooke's Law and thick shell theory to solve for E_ϕ and E_x. A value for Poisson's ratio must be known or assumed.

In order to determine average values for ϵ_ϕ and ϵ_x, strain gages were applied to a number of the models, oriented both circumferentially and longitudinally. The number of gages used was limited both by financial considerations and by the number of leads that could be fed through the pressure fittings in the tank. In general, four gages were applied to each instrumented model. These gages were located at the mid-length of the model to avoid end effects, and were oriented two in a circumferential direction and two in a longitudinal direction. The two circumferential gages were placed "back to back", inside and outside, as were the two longitudinal gages.

The choice of gage types was governed by:

(1) the need to have non-pressure-sensitive gages;

(2) the necessity for measuring strains up to 15,000 microinches/inch;

(3) the desire to use a gage length long enough to reduce the effect of local variations in strain; and

(4) the cost of gages.

On the basis of these considerations, two types of gages were selected. For use on the inside of the model, where the gage would not be subject to the pressure of the fluid, the SR-4, A-5-1 wire gage was used. This gage has a 1/2" gage length, is relatively inexpensive, and is suitable for the

measurement of strains up to $1\frac{1}{2}$ to 2%. For use on the outside of the model, where the gage would be subject to external pressure of the fluid, the SR-4, paper-backed constantan foil gage FAP-50-12 was selected. This gage, with a 1/2" gage length, is not as pressure-sensitive as the A-5-1 gage. It also is suitable for measurements of strain up to $1\frac{1}{2}$ to 2%.

Compensating gages of the same type and lot number were used in conjunction with the active gages.

The gages were applied using Duco cement, and the gage manufacturer's instructions were followed. After the cement had dried, all gages were connected in the strain indicator circuit and pressed with a pencil eraser tip while watching the strain indicator to check for excessive deflections of the needle.

Test Procedure

Strain gages were applied to the inside of the model by using a "harness"--a sheet of graph paper to which the gages were attached. This sheet was trimmed to exact dimensions, rolled into a cylinder, and positioned inside the model with glue applied to the gages. Leads of #28 stranded wire were attached to the strain gages.

The ends of the model were prepared by thorough cleaning with acetone. PR-1321 Class A sealant * was then applied to the ends, and the end plugs inserted, leading the wires from the inside of the model out through the 1/4" tube ** screwed into the lower end plug. After the sealant set up, the model was ready for installation in the tank. Fig. 8 shows a model ready for installation and Fig. 5 shows a model in the tank and partially lowered into testing position. Fig. 3 shows a model inside the chamber.

The model was positioned in the open tank by first leading the wires from the 1/4" end plug tube through the "A" fitting in the center of the tank bottom, then connecting the external strain gage leads to the leads inside the tank from the "B" fitting. The model was then lowered, allowing the 1/4" tube to slide through the "A" fitting until the lower end plug rested on a steel "spool" support about 2" high located in the bottom of the chamber. The lock nut of the "A" fitting was then tightened, taking care not to over-tighten.***

---

* Manufactured by the Products Research Co., 410 Jersey Ave., Gloucester City, New Jersey

** This tube is a semi-permanent attachment to the end plug, its threaded end having been coated with sealant prior to screwing into the tapped end plug.

*** It was observed that excessive tightening of the lock nut permanently deformed the ferrule and tube, making it impossible to slide the tube back out of the fitting after the test.

With the model in place, the strain gage leads coming from the 1/4" tube and from the "B" fitting underneath the pressure vessel were connected to the strain indicator or switching unit. The chamber was filled with transformer oil * to a level slightly above the sealing surface. The circular plate was inserted, causing the excess oil to be squeezed into the plug recess. This precaution was taken to insure that the oil was relatively air-free. The screw plug was then positioned and tightened. Tapping the screw plug wrench handle with a hammer insured firm seating of the circular plate.

Pressure was raised by operating the hand pump. The valve "G" in the line to the 5000 psi gage was shut at 4500 psi. This valve, although rated at 3000 psi, performed satisfactorily.

In some cases, only one strain gage was measured per run, using the strain indicator directly. In most of these tests, at least two runs were made per gage, and the readings for the two runs were found to vary very little. For other tests, the switching unit was employed and a number of gages were read per run.

Upon completion of a test, the screw plug was removed, and the "A" fitting lock nut was loosened to permit removal of the model. As the model was removed, the 1/4" tube slipped out of the "A" fitting, allowing the oil in the tank to drain into a receptacle below.

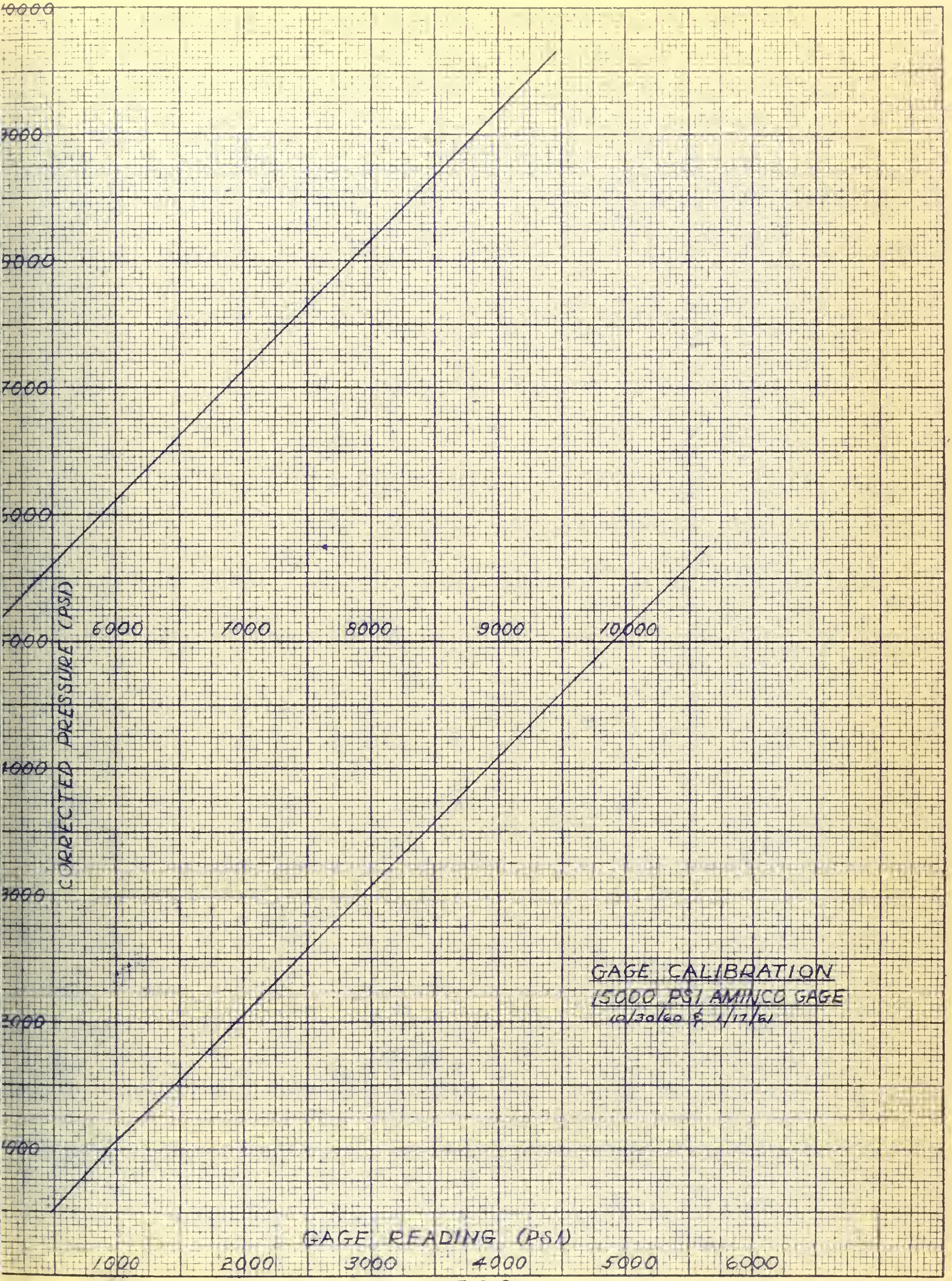
The two pressure gages used during the tests were calibrated at the Material Laboratory before beginning the tests and at DTMB toward the end of testing. Both calibrations were made using dead-weight gage

---

* WEMCO Class C transformer oil was used as the pressurizing fluid, making it unnecessary to waterproof strain gages exposed to the fluid.

testers. No differences were noted in the gage calibrations. The 5000 psi gage was adjusted after the second calibration to give gage readings closer to the actual pressure. The gage calibration curves are shown in Figures 9 and 10.

FIG. 8



CORRECTED PRESSURE (PSI)
GAGE READING (PSI)
1000
2000
3000
4000
5000
6000
6000
7000
8000
9000
10000
GAGE CALIBRATION
15000 PSI AMINCO GAGE
10/30/60 & 1/17/61

FIGURE 9

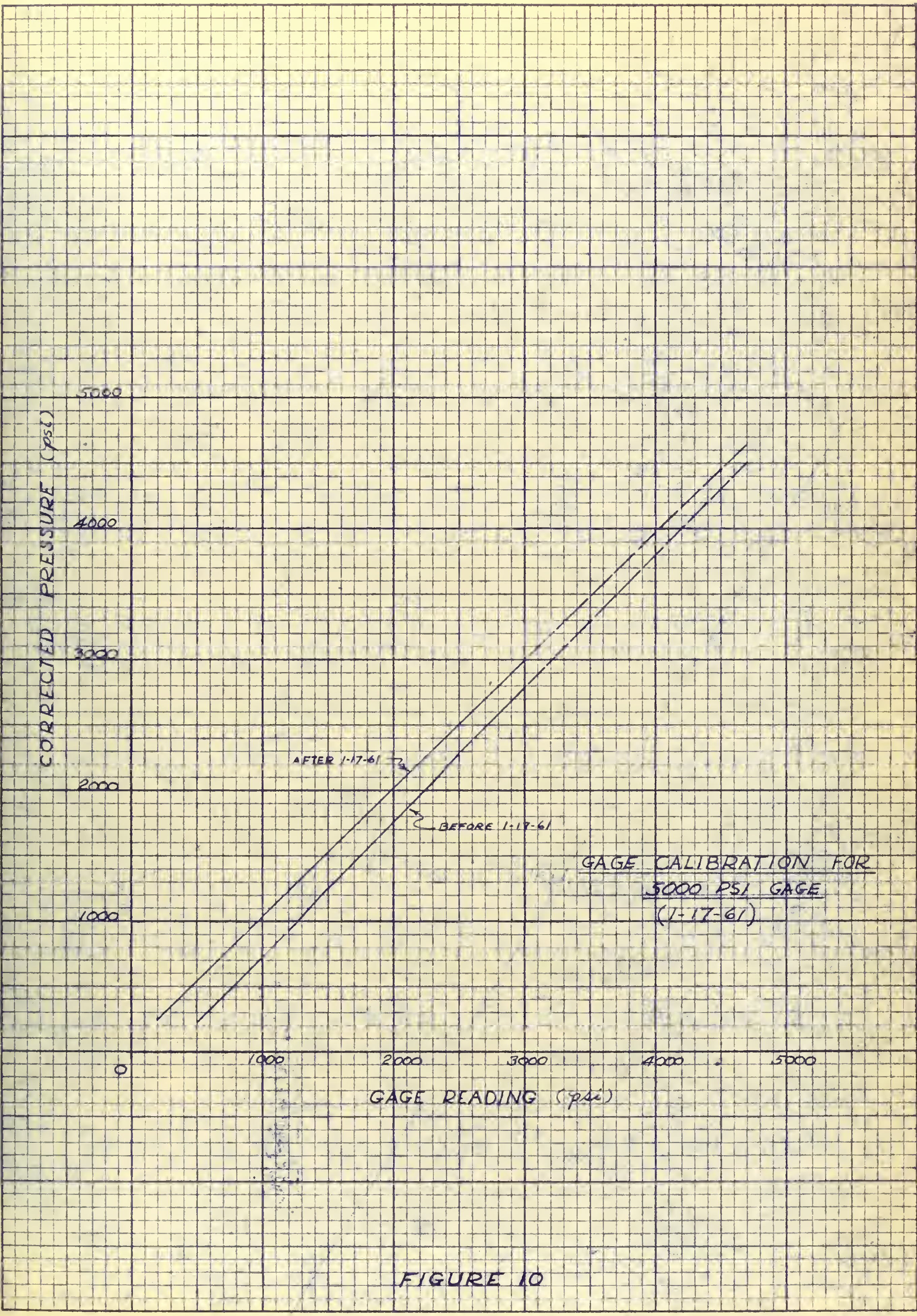

CORRECTED PRESSURE (psi)
5000
4000
3000
2000
1000
0
1000
2000
3000
4000
5000
GAGE READING (psi)
AFTER 1-17-61
BEFORE 1-17-61
GAGE CALIBRATION FOR
5000 PSI GAGE
(1-17-61)

FIGURE 10

EXPERIMENTAL DATA

A total of twenty-three models was tested, twenty-one of which were tested to failure. Seven of the models were instrumented to measure strains in order to determine the moduli of elasticity. Three models were designed to fail by yielding of the material. This was done in order to calculate the maximum compressive stress which the material would sustain and to determine the appearance of models after yield failure. Since the calculated circumferential stress for the "D" and "E" models was close to the nominal strength of the material, this information was used to ascertain whether these models did, in fact, fail by instability alone, or whether some yielding might have taken place.

A summary of the models tested, with the respective collapse pressures, is presented in Table 6. The recorded strain data are in Appendix A, and the pressure-strain plots are in Appendix B. Table 7 summarizes the measured strain-pressure ratios as obtained from the plots. The first models instrumented were tested with the compensating gage located inside the model for inside active gages, and with the compensating gage located in the pressurized fluid for external active gages. As shown in Appendix C, it is theoretically possible to cancel the effect of radial strain on the outside active gages by having the compensating gage mounted on a block which is subject to the hydrostatic pressure. Furthermore, any effect of pressure on the active gage in the fluid would be compensated for , as well as more accurate temperature compensation provided. The later models tested, however, had all gages--inside and outside--balanced against a single dummy gage inside the model. This latter procedure was followed because it had been determined that the error caused by neglecting the Poisson effect of the radial strain on the outside gages was negligible, and therefore the additional complication to the testing procedure was unjustified.

In order to be able to compare and average the strain data taken under the two different methods of compensation, the strain-pressure ratios of active gages read with the dummy under pressure were corrected to values equivalent to those taken with the dummy gage inside the model. This was accomplished by simply adding the value of strain-pressure ratio of a gage mounted on a dummy block under pressure, balanced against a dummy gage inside a model. All the strain-pressure ratios used in computations, therefore, have values which would result from active gages being read against a dummy located on the inside of the model, not under pressure.

TABLE 6

LAMÉ STRESS IN MODELS AT COLLAPSE PRESSURE

(PSI)

MODEL	K_1	COLLAPSE PRESSURE P (PSI)	$\sigma_{\phi_1} = PK_1$	$\sigma_x = \frac{1}{2} PK_1$	$\sigma_{\phi_2} = P(K_1-1)$	REMARKS
A-1	10.978	3090	33,922	16,961	30,832	
A-2		3180	34,910	17,455	31,730	
A-3		3130	34,361	17,180	31,231	
A-4		—	—	—	—	STRAIN DATA ONLY — SEE TABLE 7
A-5		4720	51,816	25,908	47,096	
A-6		5760	63,233	31,616	57,473	YIELD MODEL
B-1	9.323	4360	40,648	20,324	36,288	
B-2		2300	21,443	10,722	19,143	PREMATURE FAILURE
B-3		4330	40,369	20,184	36,038	
B-4		3155	29,414	14,707	26,259	PREMATURE FAILURE
C-1	8.191	5520	45,214	22,607	39,694	
C-2		5480	44,887	22,444	39,407	
C-3		5430	44,477	22,238	39,047	
C-4		7760	63,562	31,781	55,802	YIELD MODEL
D-1	7.225	6870	49,636	24,818	42,766	
D-2		7230	52,237	26,118	45,007	
D-3		7710	55,705	27,852	47,995	
D-4		6920	49,997	24,998	43,077	
D-5		9650	69,721	34,860	60,071	YIELD MODEL
E-1	6.714	8460	56,800	28,400	48,340	
E-2		8830	59,285	29,642	50,455	
E-3		8770	58,882	29,441	50,112	
E-4		—	—	—	—	STRAIN DATA ONLY — SEE TABLE 7

TABLE 7

MEASURED STRAIN-PRESSURE RATIOS

MODEL	m_{ϕ_2}' x 10^6	m_{x_2}' x 10^6	m_D' x 10^6	m_{ϕ_2} x 10^6	m_{x_2} x 10^6	m_{ϕ_1} x 10^6	m_{x_1} x 10^6	AVG. m_{ϕ_1} x 10^6	AVG. m_{ϕ_2} x 10^6	AVG. m_x x 10^6
A-4				1.62	0.867	1.76	0.619			
A-5	1.178	0.755	0.068	(1.246)	(0.823)	1.84				
A AVG.				1.43	0.845	1.80	0.619	1.80	1.43	0.732
C-1	1.063	0.760	0.068	1.131	0.828	1.469	0.798	1.469	1.131	0.813
D-3	0.993	0.625	0.068	(1.061)	(0.693)	1.238				
D-4				0.937		1.287	0.654			
D AVG.				0.999	0.693	1.262	0.654	1.262	0.999	0.674
E-2	0.866		0.068	(0.934)		1.20	0.670			
E-4	0.858		0.068	0.926	0.609		0.468			
E AVG.				0.930	0.609	1.20	0.569	1.200	0.930	0.589

m_D' IS THE MEASURED STRAIN-PRESSURE RATIO OF DUMMY GAGE IN PRESSURIZED FLUID.

$m_D' = 0.068 \times 10^{-6}$ INCHES/INCH.

m_{ϕ_2}' AND m_{x_2}' WERE MEASURED USING DUMMY STRAIN GAGE BLOCK IN THE PRESSURIZED FLUID.

m_{ϕ_2} AND m_{x_2} VALUES IN PARENTHESES WERE COMPUTED USING MEASURED VALUES OF m_{ϕ_2}', m_{x_2}', AND m_D', AND THE FOLLOWING RELATIONSHIPS:

$$m_{x_2} = m_{x_2}' + m_D'$$

$$m_{\phi_2} = m_{\phi_2}' + m_D'$$

m_{ϕ_1} AND m_{x_1} WERE MEASURED USING DUMMY BLOCK NOT SUBJECT TO PRESSURE.

ANALYSIS OF EXPERIMENTAL DATA

Strain Data Analysis

The moduli of elasticity were calculated from the strain-pressure ratios using the formulas presented in Table 8. The theoretical basis for these formulas is shown in Appendix C. A summary for the calculations for E_ϕ and E_x is given in Tables 9 and 10. No values were calculated for the B-model because premature failure of the instrumented model in this series prevented the taking of sufficient strain data. The values of E_ϕ and E_x used for the B-models are averages of the values for the other models.

Since available instability formulas apply to isotropic materials, it was necessary to develop a method for converting the principal moduli E_ϕ and E_x into an effective modulus E_e. This was accomplished by first deriving an expression for the shear modulus of an orthotropic material. The derivation followed a method used by Timoshenko[40] (pp. 54-57) in deriving the expression for the isotropic case and is presented in Appendix D. The shear modulus thus found is

$$G = \frac{E_x E_\phi}{(E_x + E_\phi)(1 + \nu)}$$

Note that for $E_x = E_\phi$ this expression reduces to the isotropic case

$$G = \frac{E}{2(1 + \nu)}$$

In the derivation, Poisson's ratio was assumed to be constant. As noted later, this assumption does not lead to an appreciable error. Then, to obtain an effective modulus, the orthotropic expression for G was equated to that for the isotropic case as follows:

Let $E_x = kE_\phi$, then $G = \frac{E_\phi}{\frac{1+k}{k}(1+\nu)}$. Equating this to

$G = \frac{E_\theta}{2(1+\nu)}$ gives

$$E_\theta = 2E_\phi \frac{k}{1+k}$$

The calculated effective values for the modulus are summarized in Table 11.

In addition, the authors undertook the derivation of an expression for the buckling pressure of an orthotropic cylinder. This derivation is shown in Appendix E, and the buckling pressure is given in the form of a determinant. The calculation of buckling pressure using this determinant is very cumbersome. Since the expression was derived shortly before the completion of the thesis, no evaluation of the determinant was made with respect to the experimental results contained herein. It is believed, however, that it may have merit in computing buckling pressures for orthotropic cylinders. Also, it might be used to verify the assumption of an effective modulus for use with the isotropic equations. It is noted that the determinant derived for the orthotropic case reduces to the expression for the infinite tube, with E_ϕ the only modulus, when L is made to approach infinity. This suggests that, in reality, an effective modulus for use in the buckling equation should depend on the length of the cylinder. It is also noted that the determinant reduces to that given by Timoshenko[30] (p. 449) when E_x is made equal to E_ϕ and G is put equal to $\frac{E}{2(1+\nu)}$, except that the term $-\frac{pa}{h}\frac{\lambda^2}{2}$ is added to the last term in row 3, column C to take care of the end thrust.

Details of the assumptions employed in the calculation of an effective modulus follow.

(1) As previously stated, Poisson's ratio was assumed to be constant.

On the basis of data provided by the David Taylor Model Basin and the Zenith Plastics Company, this value was assumed to be 0.15. The effect of the variation in Poisson's ratio on the value of the calculated modulus is small. For example, using $\nu = 0.10$ instead of 0.15 in the calculation for E_ϕ raises the calculated value about 3% for both the A models and the E models.

(2) The stress field in the model was assumed to be in accordance with the Lamé thick shell theory, and the strain-pressure curves were assumed to be linear up to the point of collapse. As a consequence of this assumption the bending stress due to the simply supported tube of finite length is neglected. As will be shown later, the bending stress prior to buckling is a very small quantity.

(3) The effect of radial strain was neglected in calculating E_{ϕ_2} and E_{x_2}. This effect is small as is shown below:

The radial strain on the outer surface is given by: (see Appendix C)

$$\epsilon_{r_2} = p\left[\frac{1}{E_r} - \nu\left(\frac{K_1/2}{E_x} + \frac{K_2}{E_\phi}\right)\right]$$

Thus,

$$m_{r_2} = \frac{1}{E_r} - \nu\left(\frac{K_1/2}{E_x} + \frac{K_2}{E_\phi}\right)$$

It is seen that the greatest error would occur for the largest value of m_{r_2}, which value would correspond to the thickest cylinder, i.e., the cylinder with the least value of K_1. For the E-model, $K_1 \cong 6.5$; $K_2 = K_1 - 1 \cong 5.5$; $E_\phi \cong 5.5 \times 10^6$; $E_x \cong 4 \times 10^6$; $\nu \cong 0.15$; then assuming a nominal value for $E_r = 3 \times 10^6$, $m_{r_2} \cong 1/3 - (0.15)\left(\frac{3.25}{4} + \frac{5.5}{5}\right) \cong 0.046$

Now the exact formula for computing E using outside strain gage data is:

$$E_\phi = \frac{K_2(1+\nu)(1-2\nu)}{m_{\phi_2}(1-\nu) + \nu m_{x_2} + \nu m_{r_2}}$$

It was found that the difference between the value of E_ϕ calculated by using this equation with $m_{r_2} = 0.046$ and that calculated by neglecting m_{r_2} entirely was less than 1%.

It was further noted that the longitudinal compressive strain was greater on the outside than on the inside of the models. This difference is attributed to the fact that some bending of the shell undoubtedly occurs even for the lower pressures. That this difference is not inherent in the strain relations can be shown by the following development:

From Appendix C,

$$\epsilon_{x_2} = \frac{pK_1}{2E_x} - \nu\left(\frac{pK_2}{E_\phi} + \frac{p}{E_r}\right) \qquad \text{(radial stress} = p\text{)}$$

$$\epsilon_{x_1} = \frac{pK_1}{2E_x} - \nu\left(\frac{pK_1}{E_\phi}\right) \qquad \text{(radial stress} = 0\text{)}$$

Since $K_2 = K_1 - 1$,

$$\epsilon_{x_2} - \epsilon_{x_1} = \frac{pK_1}{2E_x} - \frac{\nu pK_1}{E_\phi} + \frac{\nu p}{E_\phi} - \frac{\nu p}{E_r} - \frac{pK_1}{2E_x} + \frac{\nu pK_1}{E_\phi}$$

$$\epsilon_{x_2} - \epsilon_{x_1} = \frac{\nu p}{E_\phi} - \frac{\nu p}{E_r}$$

from which,

$$m_{x_2} - m_{x_1} = \nu\left(\frac{1}{E_\phi} - \frac{1}{E_r}\right)$$

Since $E_\phi \cong 5 \times 10^6$ $E_r \cong 3 \times 10^6$ and $\nu = 0.15$,

$$m_{x_2} - m_{x_1} \cong (0.15)(1/5 - 1/3) \cong -0.02$$

This difference is small in comparison with the differences actually measured and is of opposite sign. It is concluded, therefore, that the measured difference results from bending of the generators. In order to reduce the errors arising from this effect, an average of m_{x_1} and m_{x_2} was used in all calculations which called for either value.

Table 7 shows that even for identical models there were considerable differences in measured values of strain-pressure ratios. These differences may be attributed to two causes:

(1) The properties of the individual models may vary slightly, even though cut from the same tube.

(2) Even on the same model, the strain-pressure ratios will vary from point to point around the circumference, as shown in reference (38).

The most accurate data for use in computing moduli would be that taken from a number of strain gages spaced around the circumference of each model, using the average values of strain-pressure ratios. Due to limitations discussed previously, this procedure could not be followed. Instead, the averaging process was accomplished by using inside and outside strain data, and averaging the moduli thus calculated. Where circumferential variations in strain-pressure ratios are caused by the tendency to lobe formation, the use of inside and outside data from "back-to-back" gages tends to cancel the variation. In the case of the A, D, and E geometries, two instrumented models of each were tested in order to provide more data for averaging.

Since the tubes were fabricated using the same techniques and material, one would expect the moduli to be about the same for all the models. Such is the case, except that the A models were found to have a slightly higher modulus.(See Tables 10 and 11). As previously noted, the effective modulus for the B models was taken as the average of that of the other models, due to insufficient strain data. The small differences in resin content for the models (Table 11) are not sufficient to account for the differences in moduli.

TABLE 8

SUMMARY OF THICK SHELL FORMULAS FOR CALCULATING MODULI OF ELASTICITY

Inside gages, dummy block not subject to pressure:

$$E_\phi = \frac{K_1(1-\nu^2)}{m_{\phi_1} + \nu m_{x_1}} \qquad E_x = \frac{K_1(1-\nu^2)}{2(m_{x_1} + \nu m_{\phi_1})}$$

Outside gages, dummy block not subject to pressure:

$$E_\phi = \frac{K_2(1+\nu)(1-2\nu)}{m_{\phi_2}(1-\nu) + \nu(m_{x_2} + m_r)} \qquad E_x = \frac{K_1(1+\nu)(1-\nu)}{2\left[m_{x_2}(1-\nu) + \nu(m_{\phi_2} + m_r)\right]}$$

Outside gages, dummy block subject to hydrostatic pressure**:

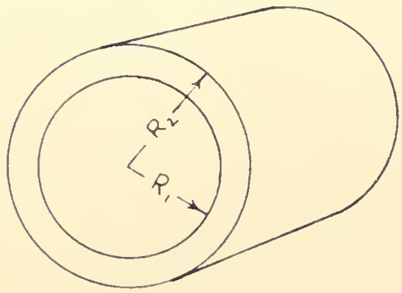
$$E_\phi = \frac{(K_2 - 1)(1-\nu^2)}{m'_{\phi_2} + \nu m'_{x_2}} \qquad E_x = \frac{\left(\frac{K_1}{2} - 1\right)(1-\nu^2)}{m'_{x_2} + \nu m'_{\phi_2}}$$

** Dummy block assumed to be of same material as model; circumferential active gages balanced against circumferential dummy and longitudinal active gages balanced against longitudinal dummy.

TABLE 8 (Continued)

SUMMARY OF THICK SHELL FORMULAS FOR CALCULATING

MODULI OF ELASTICITY



E_ϕ = Circumferential modulus of elasticity

E_x = Longitudinal modulus of elasticity

R = Mean radius, $1/2(R_1 + R_2)$

h = Wall thickness, $(R_2 - R_1)$

$K_1 = \dfrac{2R_2^2}{R_2^2 - R_1^2}$

$K_2 = \dfrac{R_2^2 + R_1^2}{R_2^2 - R_1^2}$

$m_{\phi 1}$ = Avg. strain-pressure ratio, inside circumferential gages, dummy not under pressure.

m_{x1} = Avg. strain-pressure ratio, inside longitudinal gages, dummy not under pressure.

$m_{\phi 2}$ = Avg. strain-pressure ratio, outside circumferential gages, dummy not under pressure.

m_{x2} = Avg. strain-pressure ratio, outside longitudinal gages, dummy not under pressure.

m_r = Avg. strain-pressure ratio, radial, on outer surface.

$m'_{\phi 2}$ = Avg. strain-pressure ratio, outside circumferential gages, dummy under hydrostatic pressure.

m'_{x2} = Avg. strain-pressure ratio, outside longitudinal gages, dummy under hydrostatic pressure.

ν = Poisson's ratio.

TABLE 9

DIMENSIONS OF TUBES FROM WHICH MODELS WERE TAKEN AND CALCULATION FOR K_1

TUBE	A	B	C	D	E
AVG INSIDE DIA. (IN)	3.195	3.195	3.195	3.195	3.195
h AVG. WALL THICKNESS (IN)	0.169	0.205	0.240	0.281	0.309
D AVG. MEAN DIA. (IN)	3.364	3.400	3.435	3.476	3.504
h/D	0.05024	0.06029	0.06987	0.08084	0.08818
R_1 AVG. INSIDE RADIUS (IN)	1.5975	1.5975	1.5975	1.5975	1.5975
R_2 AVG. OUTSIDE RADIUS (IN)	1.7665	1.8025	1.8375	1.8785	1.9065
R_2^2	3.1205	3.2490	3.3764	3.5288	3.6347
R_1^2	2.5520	2.5520	2.5520	2.5520	2.5520
$R_2^2 - R_1^2$	0.5685	0.6970	0.8244	0.9768	1.0827
$K_1 = \frac{2R_2^2}{R_2^2 - R_1^2}$	10.978	9.323	8.191	7.225	6.714

TABLE 10

CALCULATION FOR E_ϕ AND E_x

INSIDE GAGE DATA

$$E_\phi = \frac{K_1(1-\nu^2)}{m_{\phi_1} + \nu m_x} \qquad E_x = \frac{(K_1/2)(1-\nu^2)}{m_x + \nu m_{\phi_1}}$$

$\nu = 0.15$

$1-\nu^2 = 0.9775$

$m_x = (1/2)(m_{x_1} + m_{x_2})$

	①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩
MODEL	K_1	AVE m_{ϕ_1} $\times 10^6$	AVE m_x $\times 10^6$	① $\times(1-\nu^2)$	νm_x $\times 10^6$	νm_{ϕ_1} $\times 10^6$	③ + ⑥	② + ⑤	$E_\phi \times 10^{-6}$ ④ ÷ ⑧	$E_x \times 10^{-6}$ $\frac{1}{2}$[④ ÷ ⑦]
A	10.978	1.800	0.732	10.731	0.1098	0.2700	1.002	1.910	5.62	5.35
B	9.323									
C	8.191	1.469	0.813	8.007	0.1219	0.2204	1.0334	1.5909	5.03	3.87
D	7.225	1.262	0.674	7.062	0.1011	0.1893	0.8633	1.3631	5.18	4.09
E	6.714	1.200	0.589	6.563	0.0884	0.1800	0.769	1.2884	5.09	4.27

OUTSIDE GAGE DATA

$$E_\phi = \frac{K_2(1+\nu)(1-2\nu)}{m_{\phi_2}(1-\nu) + \nu m_x} \qquad E_x = \frac{(K_1/2)(1+\nu)(1-2\nu)}{m_x(1-\nu) + \nu m_{\phi_2}}$$

$\nu = 0.15$

$1-\nu = 0.85$

$(1+\nu)(1-2\nu) = (1.15)(0.70) = 0.805$

$K_2 = K_1 - 1$

	⑪	⑫	⑬	⑭	⑮	⑯	⑰	⑱	⑲	⑳
MODEL	① $\times$ $(1+\nu)(1-2\nu)$	AVE $m_{\phi_2} \times 10^6$	$(1-\nu)\times$⑫	[① − 1] $\times$ $(1+\nu)(1-2\nu)$	$(1-\nu)\times$③	$\nu\times$ ⑫	⑬ + ⑤	⑮ + ⑯	$E_\phi \times 10^{-6}$ ⑭ ÷ ⑰	$E_x \times 10^{-6}$ $\frac{1}{2}$[⑪ ÷ ⑱]
A	8.837	1.430	1.2155	8.0323	0.6222	0.2145	1.3253	0.8367	6.06	5.28
B										
C	6.594	1.131	0.9613	5.7888	0.6910	0.1696	1.0832	0.8606	5.34	3.83
D	5.816	0.999	0.8492	5.0111	0.5729	0.1498	0.9503	0.7227	5.27	4.02
E	5.405	0.930	0.7905	4.5998	0.5007	0.1395	0.8789	0.6402	5.23	4.22

TABLE II

CALCULATION OF EFFECTIVE YOUNG'S MODULUS, E_e

AND RESIN CONTENT OF MATERIAL

MODEL	E_ϕ AVE. $\times 10^{-6}$	E_x AVE. $\times 10^{-6}$	k	$\frac{k}{1+k}$	$E_e \times 10^{-6}$	RESIN CONTENT *
A	5.840	5.315	0.91010	0.47647	5.57	20.4 %
B	–	–	–	–	4.80 **	19.9 %
C	5.185	3.850	0.74253	0.42612	4.42	20.7 %
D	5.225	4.055	0.77608	0.43696	4.57	20.8 %
E	5.160	4.245	0.82267	0.45135	4.66	21.7 %

$$E_e = 2E_\phi\left(\frac{k}{1+k}\right) \quad \text{WHERE} \quad k = E_x/E_\phi$$

(SEE PAGES 44 & 45 FOR DERIVATION)

* VALUES OF RESIN CONTENT WERE PROVIDED BY ZENITH PLASTICS CO.

** B-MODEL E_e TAKEN AS AVERAGE OF E_e FOR A, C, D, & E MODELS.

Stress Analysis

In order to properly evaluate experimental results, it was necessary to verify the mode of failure for each model. Although all models were designed to fail by instability, the predicted instability failure pressure for the D and E models was fairly close to the corresponding compressive yield failure pressure predicted using the thick-shell theory and a nominal value for the yield strength of the material. Also, the thick-shell formulas used to calculate the stress do not take into account the bending effects. Because of these considerations, the following steps were taken in order to make an evaluation for the mode of failure.

(1) In order to evaluate the effects of bending, a formula was derived to calculate the longitudinal bending stress for a simply supported cylinder. The method of derivation was suggested by Mr. John Pulos of DTMB and is presented in Appendix F. The expression for the bending stress at midbay is:

$$\left[\sigma_b\right]_{x=0} = \left[\frac{PR^2(1-\nu/2)}{(1-\nu^2)}\right]\left(\frac{\theta}{L}\right)^2\left[\frac{(\eta_1^2-\eta_2^2)^2}{\eta_1\eta_2}\right]\left[\frac{\sinh\theta\eta_1 \sin\theta\eta_2}{\cosh^2\theta\eta_1\cos^2\theta\eta_2+\sinh^2\theta\eta_1\sin^2\theta\eta_2}\right]$$

Substituting values for the E model, the theoretical longitudinal bending stress at midbay was found to be only 75 psi.

(2) The Lamé (thick-shell) stress corresponding to the failure pressure was calculated for each of the models and compared both with a nominal compressive strength* of 70,000 psi as obtained from DTMB Report 1413[33], and with the lowest calculated value obtained from a yield model (63,233 psi). For the E model, the highest calculated stress was 59,285 psi (see Table 6). This value is 10,715 psi below the value for nominal strength but only 3,948 psi below that obtained from the yield model. It

---

* Yield strength in circumferential direction, σ_ϕ.

should be noted, however, that application of the Hencky-von Mises failure criterion would cause the differences to be greater.

(3) Finally, the physical appearance of the collapsed models was compared to that of the thinnest model, for which a pure instability failure was certain, and with that of a short model which definitely failed by yielding of the material. Figures 11 and 12 show collapsed C, D, and E models together with a short model which failed by yielding. It is noted that the longer models pictured failed along a longitudinal line parallel to the model axis by buckling inward, whereas the short model tended toward failure throughout most of its wall area.

On the basis of these considerations it was determined that the A, B, and C models failed purely in the instability mode. Since the compressive stress for the E models was within about 4000 psi of the value obtained from a yield model, there still may be reason to question whether or not some yielding took place. Also, there was scatter in the points for the D and E models (See Fig. 13) which may be a result of local yielding caused by stress concentrations due to the heterogeneous nature of the material.

FIG. 11

FIG. 12

PRESENTATION AND DISCUSSION OF RESULTS

Results of the experiments are shown graphically in Figure 13. The critical pressures determined from tests were converted to the non-dimensional form p/E_e . These values were plotted against the thickness-diameter ratio. Calculations for p/E_e are shown in Table 12.

Also plotted on Figure 13 are curves representing various theoretical buckling formulas, using an average value of L/D for the models tested. (Average L/D = 2.53).

In spite of the anisotropic nature of the model material and the inaccuracies inherent in the computation for an effective modulus, very good agreement with theory was obtained except for the thicker models. The fact that the points for the D and E models fall below the curves may indicate that the thin shell buckling theory is not adequate for ratios of h/D in this range. As previously noted, however, some yielding may have occurred in these models.

It appears that the von Mises equation is the best one to use in the range of $h/D > 0.05$ since, in general, it yields conservative values for the collapse pressure. Design curves for the buckling of simply supported cylinders were therefore drawn up for the von Mises equation. (See Fig. 14). Data for these curves were calculated using the IBM 7090 computer at DTMB. These data are reproduced in Appendix G.

Since curves were not available for determining the number of circumferential lobes into which a relatively thick cylinder would fail, curves were drawn. (See Fig. 15). These curves were calculated using data in Appendix G by putting the buckling equation in the form

$$p/E = F_1(h/D)^3 + F_2(h/D)$$

for each n and L/D and then solving for the desired intersections. These curves are for $\nu = 0.15$.

For values of n with $\nu = 0.3$, in the range $h/D \leq 0.02$, see reference (33).

As far as is known by the authors, these curves for the von Mises equation and for the number of lobes are the only ones available in the range of h/D greater than 0.05.

MODEL	P_{CR} (PSI)	$E_e \times 10^{-6}$ (PSI)	$(P_{CR}/E) \times 10^{6}$	h/D	REMARKS
A-1	3090	5.57	554.7	0.05024	
A-2	3180		570.9		
A-3	3130		561.9		
A-4	—		—		STRAIN DATA RECORDED, MODEL NOT DESTROYED.
A-5	4720		847.4		$L/D \cong 1.5$
A-6	5760				YIELD MODEL
B-1	4360	4.80	908.3	0.06029	
B-2	2300				PREMATURE FAILURE
B-3	4330		902.1		
B-4	3155				PREMATURE FAILURE
C-1	5520	4.42	1248.9	0.06987	
C-2	5480		1239.8		
C-3	5430		1228.5		
C-4	7760				YIELD MODEL
D-1	6870	4.57	1503.3	0.08084	
D-2	7230		1582.1		
D-3	7710		1687.1		
D-4	6920		1514.2		
D-5	9650				YIELD MODEL
E-1	8460	4.66	1815.4	0.08818	
E-2	8830		1894.8		
E-3	8770		1882.0		
E-4	—		—		STRAIN DATA RECORDED, MODEL NOT DESTROYED.

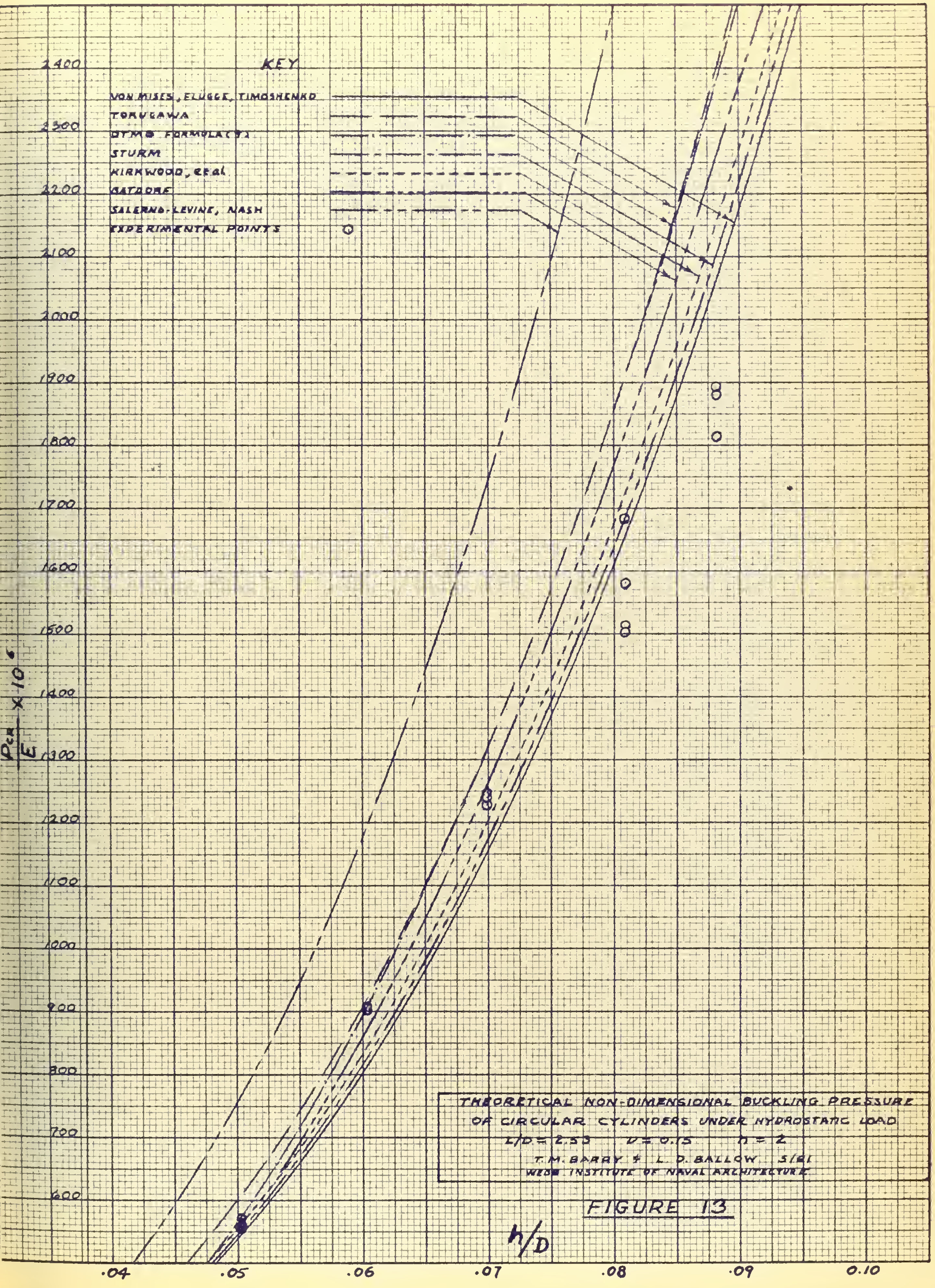

KEY
VON MISES, FLUGGE, TIMOSHENKO
TOKUGAWA
DTMB FORMULA (3)
STURM
KIRKWOOD, et al
BATDORF
SALERNO-LEVINE, NASH
EXPERIMENTAL POINTS
2400
2300
2200
2100
2000
1900
1800
1700
1600
1500
1400
1300
1200
1100
1000
900
800
700
600
$\frac{P_{CR}}{E} \times 10^6$
THEORETICAL NON-DIMENSIONAL BUCKLING PRESSURE
OF CIRCULAR CYLINDERS UNDER HYDROSTATIC LOAD
L/D = 2.53 D = 0.15 n = 2
T.M. BARRY & L. D. BALLOW 5/61
WEBB INSTITUTE OF NAVAL ARCHITECTURE
h/D
.04
.05
.06
.07
.08
.09
0.10

FIGURE 13

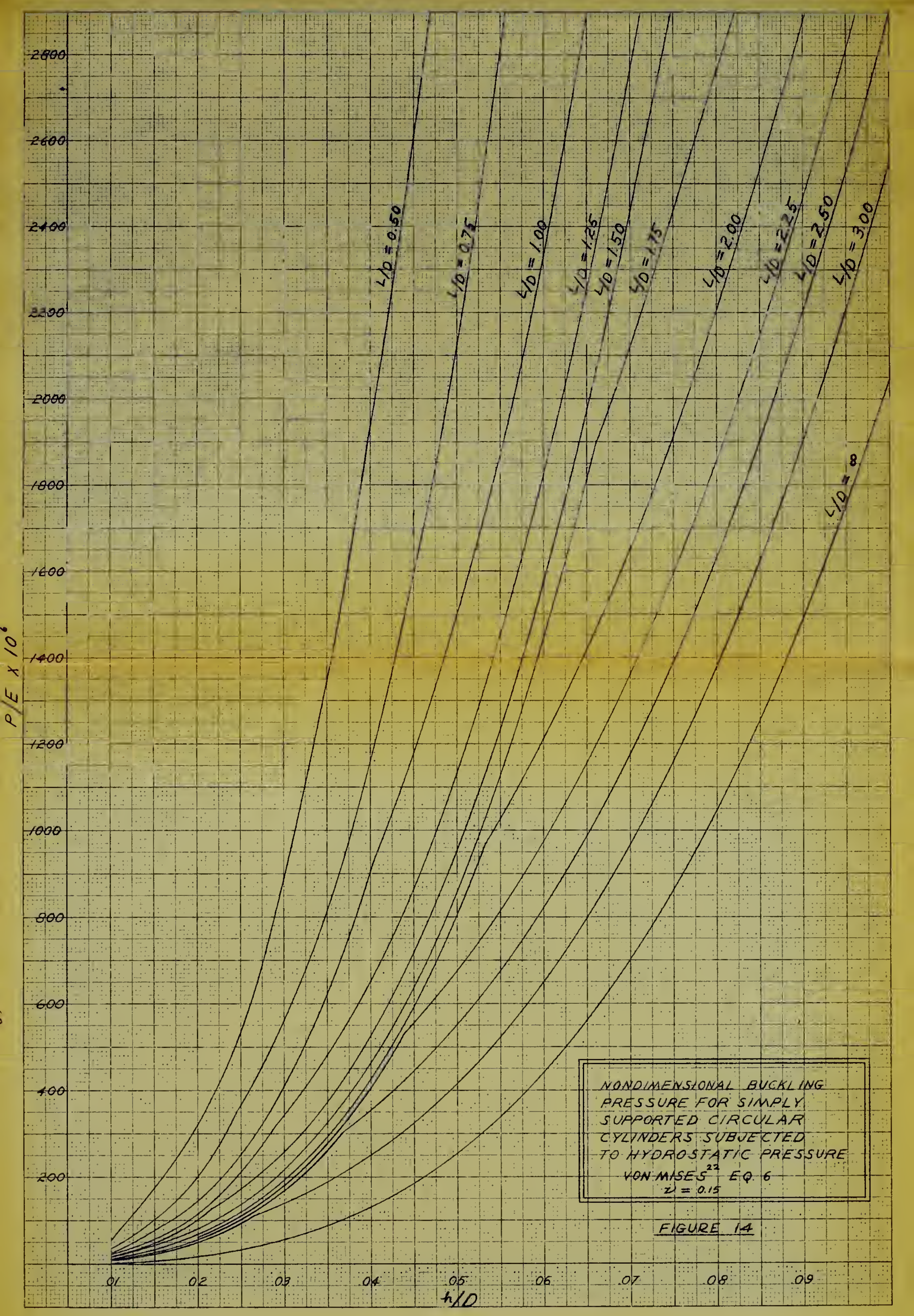

NONDIMENSIONAL BUCKLING PRESSURE FOR SIMPLY SUPPORTED CIRCULAR CYLINDERS SUBJECTED TO HYDROSTATIC PRESSURE
VON MISES 22 EQ. 6
ν = 0.15
P/E x 10⁶
h/D
L/D = 0.50
L/D = 0.75
L/D = 1.00
L/D = 1.25
L/D = 1.50
L/D = 1.75
L/D = 2.00
L/D = 2.25
L/D = 2.50
L/D = 3.00
L/D = ∞
2800
2600
2400
2200
2000
1800
1600
1400
1200
1000
800
600
400
200
.01
.02
.03
.04
.05
.06
.07
.08
.09

FIGURE 14

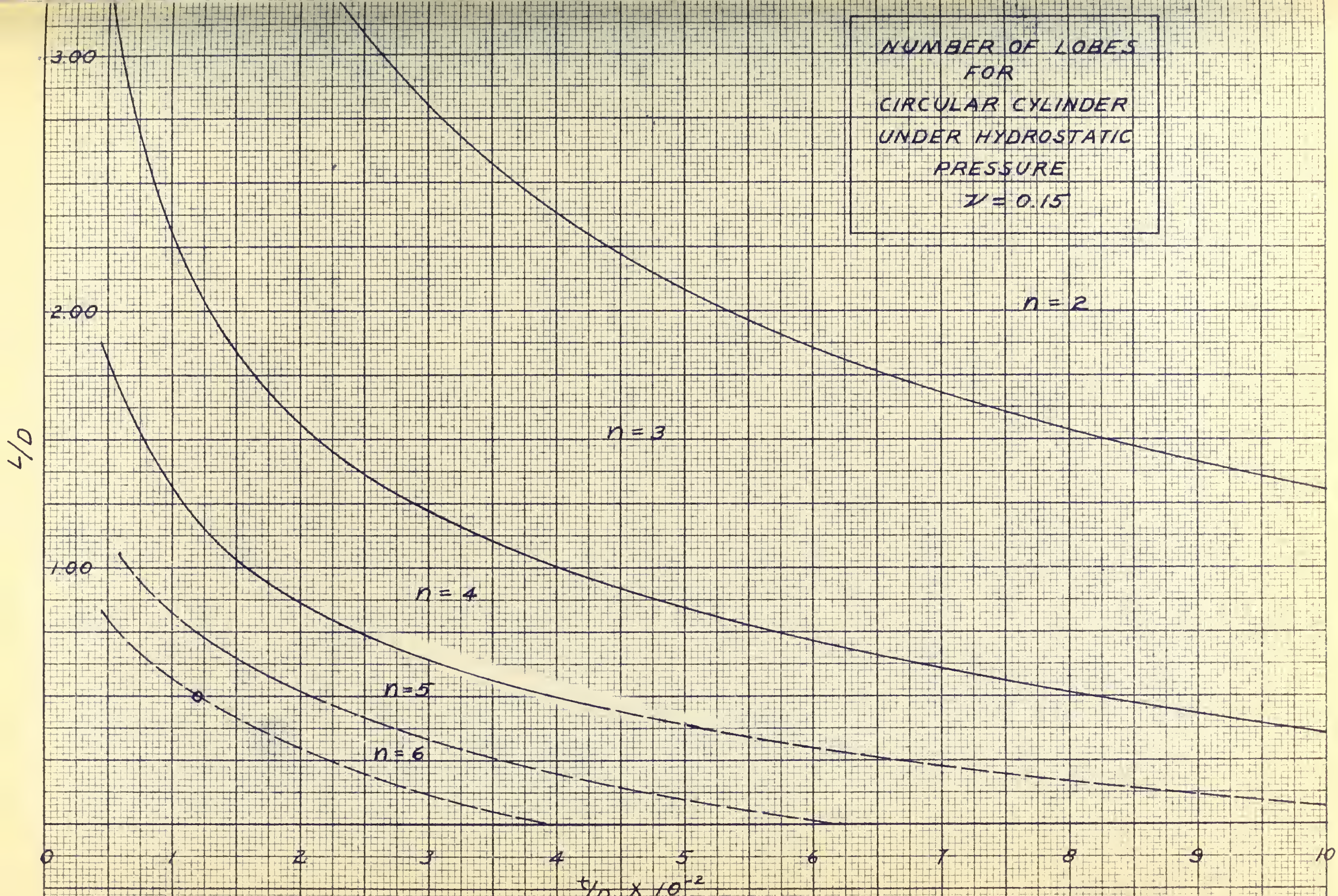

NUMBER OF LOBES
FOR
CIRCULAR CYLINDER
UNDER HYDROSTATIC
PRESSURE
ν = 0.15
n = 2
n = 3
n = 4
n = 5
n = 6
3.00
2.00
1.00
L/D
0
1
2
3
4
5
6
7
8
9
10
t/D x 10^-2

FIGURE 15

CONCLUSIONS

Based on the results of these experiments the following conclusions are drawn:

(1) The von Mises equation, equation (6), Ref. 22 is the best instability equation for cylinders with ratios of h/D greater than 0.05. It gives conservative values for the buckling pressure up to an h/D of about 0.07. Use of presently available simplified forms of buckling equations cannot be expected to give conservative values for the buckling pressure in this range of h/D. It is noted that Windenburg and Trilling[33] had previously reached the conclusion that the von Mises formula was probably the best one; however, their conclusion was based on an analysis of experimental and theoretical results for cylinders with ratios of h/D less than 0.007 in contrast with the much thicker range of 0.05 to 0.088 investigated in this thesis.

(2) For practical cases, the use of stiffening rings, heavy webs, and bulkheads would result in smaller ratios of h/D and L/D for a given weight of structural material. It is concluded, then, that for most practical cases, the thin shell theory is sufficiently accurate for predicting instability failures.

(3) For cylinders built up of fiber glass reinforced plastic layers, the methods employed in this thesis are sufficiently accurate to predict instability failure pressures.

(4) For an orthotropic cylinder, the effective modulus, as used in the buckling equation, is probably a function of the length between supports.

RECOMMENDATIONS

Further Investigations.

The experimental apparatus designed and assembled for this thesis provides a means for the undertaking of a variety of experimental investigations in the field of hydrostatically loaded shells. Some investigations which might be undertaken are:

(1) Investigate the stresses in short, relatively thick, fiber-glass reinforced cylindrical shells. Measure the strain distribution along a generator and determine the effects of bending. Verify the formula for bending stress which is derived in Appendix F.

(2) Using fiber-glass reinforced plastic models, verify the orthotropic buckling determinant as derived in Appendix E. In conjunction with this, evaluate the determinant to arrive at a general expression for buckling pressure similar to the von Mises equation for the isotropic case. Using the result, develop a method of computing an "effective modulus" for use in isotropic buckling formulas.

(3) Perform experiments on models of unusual form, such as:

a. Stiffened spheres

b. Ellipsoids

c. Sphere-cylinder intersections

RECOMMENDATIONS (Continued)

Improvement of Apparatus.

Replace valve (G), (Fig. 4) with one rated at 10,000 psi or higher, such as American Instrument Company valve, Cat. No. 44-1505.

Install a fitting in the line to each gage to damp out the sudden drop in pressure when a model breaks, thus protecting the gage pointer. There are commercial fittings available for this purpose.

If a greater number of strain gage leads are required to be run for the gages mounted on the outside of the model, it is recommended that a CONAX Corporation thermocouple fitting, Cat. No. TG-20-B8, be installed in place of the fitting "A" now installed. The TG-20-B8 will take eight #20 wires.

BIBLIOGRAPHY

1. Wenk, E., Jr., et. al., "An Oceanographic Research Submarine of Aluminum for Operation to 15,000 Feet", Quarterly Transactions of the Royal Institution of Naval Architects, vol. 102, no. 4, pp. 555-578 (1960).

2. Novozhilov, V. V., The Theory of Thin Shells, translated by Lowe, P. G. and Radok, J. R. M., P. Noordhoff Ltd., Groningen, The Netherlands (1959).

3. Timoshenko, S., Theory of Plates and Shells, McGraw-Hill, New York and London (1940).

4. Borg, S., Introduction to Matrix Tensor Methods in Theoretical and Applied Mechanics, Ann Arbor (1956).

5. Wang, C. T., Applied Elasticity, McGraw-Hill, New York, Toronto, London (1953).

6. Naghdi, P. M., "On the Theory of Thin Elastic Shells", Quarterly of Applied Mathematics, vol. XIV, pp. 369-380 (1956).

7. Reissner, E., "Stress Strain Relations in the Theory of Thin Elastic Shells", Journal of Mathematics and Physics, vol. XXXI, no. 2, pp. 109-119 (1952).

8. Fairbairn, W., "The Resistance of Tubes to Collapse", Philosophical Transactions of the Royal Society, vol. 148, pp. 389-413 (1858).

9. Nash, W. A., "Bibliography on Shells and Shell-like Structures", DTMB Report 863 (1954); and "Bibliography on Shells and Shell-like Structures (1954-1956)", Dept. of Eng. Mechanics, U. of Fla., Contract DA-01009-ORD-404, Office of Ordnance Research, U. S. Army.

10. Sturm, R. G., "A Study of the Collapsing Pressure of Thin-Walled Cylinders", U. of Ill. Engineering Experiment Station Bulletin, Series No. 329 (1941).

11. Cook, G., "The Collapse of Short Thin Tubes by External Pressure", Philosophical Magazine (1925).

12. Windenburg, D. F., "The Strength of Thin Reinforced Tubes Under External Pressure", U. S. Experimental Model Basin Report 262 (1930).

13. Cornell, S., "The Collapse of Long Cylindrical Shells Under External Pressure", Proceedings of the Society for Experimental Stress Analysis, vol. VI, no. 1 (1948).

14. Cleaver, P. C., "The Strength of Tubes Under Uniform External Pressure", Royal Aircraft Establishment, Report-Structures 193 (1955).

15. Von Mises, R., "The Critical External Pressure of Cylindrical Tubes", U. S. Experimental Model Basin Trans. 5, Report 309 (1931).

16. Carman, A. P., "Resistance of Tubes to Collapse", Physical Review, vol. 21, pp. 381-387 (1905).

17. Stewart, R. T., "Collapsing Pressures of Bessemer Steel Lap-welded Tubes", Transactions of the American Society of Mechanical Engineers, vol. 27, pp. 730-822 (1906).

18. Carman, A. P. and Carr, M. L., "Resistance of Tubes to Collapse", U. of Ill. Eng. Exp. Sta. Bulletin No. 5 (1906).

19. Southwell, R. V., "On the Collapse of Tubes by External Pressure," Philosophical Magazine, (May, 1913).

20. Carman, A. P., "The Collapse of Short Thin Tubes", U. of Ill. Eng. Exp. Sta. Bulletin No. 99 (1917).

21. Cook, G., "The Collapse of Short Thin Tubes by External Pressure", Philosophical Magazine (Oct. 1925).

22. von Mises, R., "The Critical External Pressure of Cylindrical Tubes Under Uniform Radial and Axial Load", DTMB Report (Translation) No. 366 (1933).

23. Tokugawa, T., "Model Experiments on the Elastic Stability of Closed and Cross-Stiffened Circular Cylinders under Uniform External Pressure", World Engineering Congress Proceedings, vol. xxxix, Part 1, Tokyo (1929).

24. Jasper, T. M., and Sullivan, J. W. W., "The Collapsing Strength of Steel Tubes", A.SM.E. Transactions, vol. 53, pp. 219-245 (1931).

25. Tokugawa, T., "An Approximate Method for Calculating the Collapsing Pressure of Thin Cylindrical, Conical, and Spherical Shells Under Uniform External Pressure", Zōsen Kyōkai, (Nov. 1940).

26. Christensen, H. D., "Strength of Cylindrical Shells Under Hydrostatic Loading", U. S. Naval Ordnance Test Station Technical Memorandum No. 504 (1951).

27. Barnet, F. R., and Carroll, L. L., "Reinforced Plastics in the Construction of External Pressure Vessels", Society of Plastics Engineers Journal (1956).

28. Saunders, H. E. and Windenburg, D. F., "Strength of Thin Cylindrical Shells Under External Pressure", ASME Journal, vol. 53 (1931).

29. Flügge, W., _Stresses in Shells_, Springer-Verlag, Berlin, Göttingen, Heidelberg (1960).

30. Timoshenko, S., _Theory of Elastic Stability_, McGraw-Hill, New York and London (1936).

31. Kirkwood, J. G., et. al., "Buckling Instability of Thin Cylindrical Shells Under External Static Loading", Cornell University, Office of Scientific Research and Development, OSRD No. 3780 (June 1944).

32. Salerno, V. L. and Levine, B., "Buckling of Circular Cylindrical Shells with Evenly Spaced, Equal Strength Circular Ring Frames, Part I, Polytechnic Institute of Brooklyn Report No. 167 (1950); and Part II, PIBAL Report No. 169 (1950).

33. Windenburg, D. F., and Trilling, C., "Collapse by Instability of Thin Cylindrical Shells Under External Pressure", DTMB Report No. 385 (1934).

34. Batdorf, S. B., "A Simplified Method of Elastic-Stability Analysis for Thin Cylindrical Shells," N.A.C.A. Report No. 874 (1947).

35. Prescott, J., Applied Elasticity, Longmans, Green, and Co., London (1924).

36. Bresse, M., "Cours de Mècanique Appliqué", Paris (1859).

37. Bryan, G. H., "Application of the Energy Test to the Collapse of a Long Thin Pipe Under External Pressure", Cambridge Philosophical Society Proceedings, vol. 6 (1888).

38. Pulos, J. G. and Buhl, J. E., "Hydrostatic Pressure Tests of an Unstiffened Cylindrical Shell of a Glass-Fiber Reinforced Epoxy Resin", DTMB Report 1413 (1960).

39. ASME Boiler and Pressure Vessel Code, 1959, Section VIII, "Unfired Pressure Vessels".

40. Timoshenko, S., Strength of Materials, Part I, D. Van Nostrand Co., New York (1941).

41. Conway, H. D., "Some Problems of Orthotropic Plane Stress", ASME Transactions, vol. 75 (1953).

42. Timoshenko, S., Strength of Materials, Part II, D. Van Nostrand Co., New York (1940).

43. Salerno, V. L., and Pulos, J. G., "Stress Distribution in a Circular Cylindrical Shell Under Hydrostatic Pressure Supported by Equally Spaced Circular Ring Frames", PIBAL Report No. 171-A (June 1951).

44. Krenzke, M. A., and Short, R, D., "Graphical Method for Determining Maximum Stresses in Ring-Stiffened Cylinders under External Hydrostatic Pressure", DTMB Report 1348 (1959).

APPENDIX A

RECORD OF STRAIN GAGE DATA

The following tables consist of the recorded strains corresponding to the gage pressures and corrected pressures. The model number, gage type, gage factor, date of test, gage location , cycle number, run number, and pertinent remarks are also indicated.

Three types of Baldwin-Lima-Hamilton gages were used for the strain measurements:

Gage Type	Nominal Gage Length
A-5-1	1/2"
FAP-50-12	1/2"
FA-100-12	1"

The type A-5-1 gage was used on the inside of the model only, since the wire gage is particularly sensitive to hydrostatic pressure. The FAP-50-12, a foil gage, was used both inside and outside the models. The FA-100-12 was used on the outside of model A-4 only. All three types appeared to give reliable data. No particular difficulty was experienced in handling and applying the gages.

In the row entitled Gage Location, "L" stands for a longitudinally mounted gage, "C" for a circumferentially mounted gage, "1" for an inside gage, and "2" for an outside gage.

In some cases the gages were read one at a time requiring a different pressure cycle for each gage. A record was kept of the total number of cycles for each model and the number of runs for a particular gage. These data were taken in order to ascertain the effects of creep, if any. It appeared from the results, however, that creep was not an important factor for these short term tests.

STRAIN GAGE DATA

MODEL NO. A-4 | GAGE TYPE INSIDE FAP-50-12, OUTSIDE FA-100-12 | GAGE FACTOR INSIDE 2.11, OUTSIDE 2.12 | DATE 3-22-61

GAGE LOCATION		L-2			C-2			L-1			C-1								
CYCLE NO.			2			2			2			2							
RUN NO.			2			2			2			2							
P-GAGE	P-CORR.	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ
0	0	9000	—	—	9000	—	—	9000	—	—	9000	—	—						
500	550	8520	480	480	8130	870	870	8650	350	350	8080	920	920						
1000	1030	8085	435	915	7315	815	1685	8360	290	640	7240	840	1760						
1500	1520	7675	410	1325	6520	795	2480	8070	290	930	6360	880	2640						
2000	2010	7270	405	1730	5640	880	3260	7790	280	1210	5330	1030	3670						
2100	2105	7200	70	1800	5450	190	3550	7740	50	1260	5120	210	3880						
2200	2200	7120	80	1880	5250	200	3750	7685	55	1315	4900	220	4100						
2300	2300	7085	35	1915	5070	180	3930	7610	75	1390	4660	240	4340						
2400	2400	7030	55	1970	4800	270	4200	7535	75	1465	4370	290	4630						
2500	2500	7030	0	1970	4575	225	4425	7480	55	1520	4105	265	4895						
2600	2600	7010	20	1990	4170	405	4830	7425	55	1575	3630	475	5370						
2700	2700	7150	-140	1850	3740	430	5260	7390	35	1610	3280	350	5720						
2750	2750	7205	-55	1795	3245	495	5755	7380	10	1620	2650	630	6350						
2800	2800	7420	-215	1580	2500	745	6500	7395	-15	1605	1850	800	7150						

REMARKS: INSIDE DUMMY USED FOR ALL GAGES.

STRAIN GAGE DATA

MODEL NO A-4 | GAGE TYPE: INSIDE FAP-50-12, OUTSIDE FA-100-12 | GAGE FACTOR: INSIDE 2.11, OUTSIDE 2.12 | DATE 3-22-61

GAGE LOCATION		L-2			C-2			L-1			C-1								
CYCLE NO			1			1			1			1							
RUN NO			1			1			1			1							
P-GAGE	P-CORR	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ
0	0	9000	–	–	9000	–	–	9000	–	–	9000	–	–						
250	300	8715	285	285	8530	470	470	8810	190	190	8490	510	510						
500	550	8490	225	510	8125	405	875	8650	160	350	8070	420	930						
750	790	8270	220	730	7725	400	1275	8500	150	500	7650	420	1350						
1000	1030	8060	210	940	7320	405	1680	8345	155	655	7220	430	1780						
1250	1280	7850	210	1150	6940	380	2060	8200	145	800	6795	425	2205						
1500	1520	7650	200	1350	6540	400	2460	8060	140	940	6355	440	2645						
1750	1770	7450	200	1550	6110	430	2890	7905	155	1095	5850	505	3150						
2000	2010	7265	185	1735	5655	455	3345	7770	135	1230	5355	495	3645						

REMARKS: INSIDE DUMMY USED FOR ALL GAGES.

STRAIN GAGE DATA

MODEL NO. A-4 | GAGE TYPE | GAGE FACTOR 2.11 | DATE 1-7-61

GAGE LOCATION		C-1			C-1			C-2			C-2			L-2			L-2		
CYCLE NO.			1			5			2			6			3			4	
RUN NO.			1			2			1			2			1			2	
P-GAGE	P-CORR	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ
0	275	6970	—	—	6965	—	—	9435	—	—	9460	—	—	9280	—	—	9290	—	—
250	525	6070	900	900	5970	990	990	8780	655	655	8840	620	620	8870	410	410	8885	405	405
500	775	5615	455	1355	5495	475	1465	8485	295	950	8550	290	910	8670	200	610	8680	205	[illegible]
750	1025	5160	455	1810	5050	445	1910	8190	295	1245	8255	295	1205	8470	200	810	8480	200	8[illegible]
1000	1280	4615	545	2355	4610	440	2350	7895	295	1540	7970	285	1490	8275	195	1005	8290	200	1010
1250	1500	4250	365	2720	4195	415	2765	7620	275	1815	7685	285	1775	8100	175	1180	8100	180	1190
1500	1750	3780	470	3190	3735	460	3225	7325	295	2110	7370	315	2090	7890	210	1390	7890	210	1400
1750	2000	3325	455	3645	3280	455	3680	7040	285	2395	7070	300	2390	7690	200	1590	7690	200	1600
2000	2230	2820	505	4150	2825	455	4135	6750	290	2685	6780	290	2680	7490	200	1790	7500	190	17[illegible]0
2250	2475				2385	440	4575				6505	275	2955						
2500	2715				1920	465	5040				6190	315	3270						
2750	2950				1450	470	5510				5880	310	3580						
3000	3200				935	515	6025				5555	325	3905						
3250	3450										5210	345	4250						
3500	3880										4920	290	4540						
3750	3930										4610	310	4850						
4050 *	4150																		
4125	4225																		
4300	4410																		
4400	4510																		
4600	4720	COLLAPSE																	

REMARKS: * SHIFTED TO 15,000 PSI GAGE   OUTSIDE DUMMY USED FOR OUTSIDE GAGES

STRAIN GAGE DATA

MODEL NO C-1 | GAGE TYPE A-5-1 | GAGE FACTOR 1.98 | DATE 1-5-61

GAGE LOCATION		C-1			C-1			C-1			L-1			L-1					
CYCLE NO.			3			6			12			4			5				
RUN NO.			1			2			3			1			2				
P-GAGE	P-CORR.	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ
0		10180	–	–	10195	–	–	10230	–	–	10075	–	–	10070	–	–			
250	525	9460	720	720							9650	425	425						
500	775	9095	365	1085	9115	1080	1080	9155	1075	1075	9450	200	625	9430	640	640			
750	1025	8740	355	1440							9250	200	825						
1000	1280	8370	370	1810	8400	715	1795	8450	705	1780	9050	200	1025	9040	390	1030			
1250	1500	8040	330	2140							8865	185	1210						
1500	1750	7670	370	2510	7690	710	2505	7735	715	2495	8660	205	1415	8650	390	1420			
1750	2000	7305	365	2875							8460	200	1615						
2000	2230	6950	355	3230	6980	710	3215	7015	720	3215	8250	210	1825	8240	410	1830			
2250	2475	6605	345	3575							8050	200	2025						
2500	2715	6230	375	3950	6240	740	3955	6310	705	3920	7830	220	2245	7830	410	2240			
2750	2950	5860	370	4320							7635	195	2440						
3000	3200	5485	375	4695	5500	740	4695	5555	755	4675	7410	225	2665	7405	425	2665			
3500	3680				4750	750	5445	4785	770	5445				6985	420	3085			
4000	4170				4075	675	6120	4075	710	6155									
4600 *	4720							3450	625	6780									
5000	5100							2960	490	7270									
5400	5520	COLLAPSE																	

REMARKS: * SHIFTED TO 15,000 PSI GAGE

STRAIN GAGE DATA

MODEL NO. C-1 | GAGE TYPE FAP-50-12 | GAGE FACTOR 2.11 | DATE 1-5-61

GAGE LOCATION		C-2			C-2			L-2			L-2								
CYCLE NO.			1			10			2			11							
RUN NO.			1			2			1			2							
P-GAGE	P-CORR	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ
0		10080	—	—	10050	—	—	10485	—	—	10440	—	—						
250	525	9470	610	610				10090	395	395									
500	775	9195	275	885	9160	890	890	9880	210	605	9835	605	605						
750	1025	8920	275	1160				9700	180	785									
1000	1280	8650	270	1430	8630	530	1420	9510	190	975	9470	365	970						
1250	1500	8405	245	1675				9340	170	1145									
1500	1750	8145	260	1935	8115	515	1935	9150	190	1335	9110	360	1330						
1750	2000	7880	265	2200				8985	165	1500									
2000	2230	7630	250	2450	7570	545	2480	8805	180	1680	8760	350	1680						
2250	2475	7380	250	2700				8630	175	1855									
2500	2715	7130	250	2950	7050	520	3000	8455	175	2030	8390	370	2050						
2750	2950	6860	270	3220				8290	165	2195									
3000	3200	6560	300	3520	6505	545	3545	8100	190	2385	8010	380	2430						
3500	3680				5935	570	4115				7625	385	2815						
4000	4170				5370	565	4680				7240	385	3200						

REMARKS: OUTSIDE DUMMY USED FOR THESE GAGES

STRAIN GAGE DATA

MODEL NO. D-3 | GAGE TYPE: INSIDE A-5-1, OUTSIDE FAP-50-12 | GAGE FACTOR: INSIDE 1.98, OUTSIDE 2.11 | DATE 1-5-61

GAGE LOCATION		C-1			C-1			C-2			C-2			L-2			L-2		
CYCLE NO.			1			6			2			5			3			4	
RUN NO.			1			2			1			2			1			2	
P-GAGE	P-CORR	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ
0		7950	—	—	7900	—	—	9315	—	—	9290	—	—	9515	—	—	9510	—	—
500	775	7000	950	950	6960	940	940	8515	800	800	8510	780	780	8990	525	525	9000	510	[illegible]
1000	1280	6400	600	1550	6350	610	1550	8030	485	1285	8020	490	1270	8670	320	845	8685	315	[illegible]
1500	1750	5800	600	2150	5745	605	2155	7550	480	1765	7545	475	1745	8375	295	1140	8390	295	[illegible]
2000	2230	5185	615	2765	5130	615	2770	7065	485	2250	7065	480	2225	8070	305	1445	8085	305	[illegible]
2500	2715	4580	605	3370	4530	600	3370	6585	480	2730	6590	475	2700	7780	290	1735	7790	295	[illegible]
3000	3200	3940	640	4010	3900	630	4000	6080	505	3235	6080	510	3210	7465	315	2050	7480	310	[illegible]
3500	3680	3310	630	4640	3250	650	4650	5565	515	3750	5560	520	3730	7165	300	2350	7170	310	[illegible]
4000	4170	2690	620	5260	2690	560	5210	5115	450	4200	5070	490	4220	6900	265	2615	6905	265	[illegible]
4600 *	4720				2090	600	5810												
5100	5210				1440	650	6460												
5600	5720				950	490	6950												
7550	7710	COLLAPSE																	

REMARKS: * SHIFTED TO 15,000 psi GAGE. OUTSIDE DUMMY USED WITH OUTSIDE GAGES.

STRAIN GAGE DATA

MODEL NO D-4 | GAGE TYPE - FAP-50-12 | GAGE FACTOR - 2.11 | DATE 3-15-61

GAGE LOCATION		C-1			C-2			L-1			L-2							
CYCLE NO.			1			1			1			1						
RUN NO.			1			1			1			1						
P-GAGE	P-CORR	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ
0	0	9800	—	—	9800	—	—	9800	—	—	9800	—	—					
250	300	9425	375	375	9510	290	290	9610	190	190	9620	180	180					
500	550	9110	315	690	9260	250	540	9450	160	350	9470	150	330					
750	790	8790	320	1010	9020	240	780	9290	160	510	9320	150	480					
1000	1030	8470	320	1330	8790	230	1010	9140	150	660	9160	160	640					
1250	1280	8170	300	1630	8550	240	1250				9020	140	780					
1500	1520	7860	310	1940	8330	220	1470	8820	320	980	8880	140	920					
1750	1770	7540	320	2260	8100	230	1700	8660	160	1140	8720	160	1080					
2000	2010	7220	320	2580	7860	240	1940	8500	160	1300	8580	140	1220					
2250	2250	6910	310	2890	7660	200	2140	8340	160	1460	8420	160	1380					
2500	2500	6590	320	3210	7450	210	2350	8190	150	1610	8290	130	1510					
2750	2750	6280	310	3520	7220	230	2580	8020	170	1780	8130	160	1670					
3000	3000	5930	350	3870	7020	200	2780	7840	180	1960	7990	140	1810					
3250	3250	5580	350	4220	6790	230	3010	7670	170	2130	7820	170	1980					
3500	3500	5230	350	4570	6560	230	3240	7480	190	2320	7660	160	2140					
3750	3750	5030	200	4770	6290	270	3510	7340	140	2460								
4000	4000	4720	310	5080	6070	220	3730	7170	170	2630	7370	290	2430					

REMARKS: SR-4 SWITCHING & BALANCING UNIT USED FOR MODEL D-4 RUNS. INSIDE DUMMY USED FOR ALL GAGES INSIDE AND OUT.

STRAIN GAGE DATA

MODEL NO. D-4		GAGE TYPE FAP-50-12						GAGE FACTOR 2.11						DATE 3-15-61					
GAGE LOCATION		C-1			C-2			L-2			C-1			C-2			L-2		
CYCLE NO.			2			2			2			3			3			3	
RUN NO.			2			2			2			3			3			3	
P-GAGE	P-CORR.	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ
0	0	9800	–	–	9800	–	–	9800	–	–	9800	–	–	9800	–	–	9800	–	–
500	550	9020	780	780	9280	520	520	9470	330	330	9120	680	680	9280	520	520	9480	320	320
1000	1030	8490	530	1310	8790	490	1010	9170	300	630	8500	620	1300	8800	480	1000	9170	310	630
1500	1520	7890	600	1910	8340	450	1460	8880	290	920	7890	610	1910	8340	460	1460	8890	280	910
2000	2010	7260	630	2540	7860	480	1940	8590	290	1210	7260	630	2540	7860	480	1940	8580	310	1220
2500	2500	6650	610	3150	7410	450	2390	8290	300	1510	6640	620	3160	7400	460	2400	8280	300	1520
3000	3000	6010	640	3790	6930	480	2870	7970	320	1830	6010	630	3790	6920	480	2880	7970	310	1830
3500	3500	5330	680	4470	6420	510	3380	7650	320	2150	5320	690	4480	6410	510	3390	7630	340	2170
4000	4000										4710	610	5090	5950	460	3850	7340	290	2460
4400*	4500										4140	570	5660	5570	380	4230	7060	280	2740
4900	5000										3510	630	6290	5120	450	4680	6780	280	3020
5400	5520										2970	540	6830	4720	400	5080	6480	300	3320
5900	6010										2340	630	7460	4240	480	5560	6210	270	3590
6400	6500										2030	310	7770	3580	660	6220	5950	260	3850
6700	6820																5680	270	4120
6800	6920	COLLAPSE																	

REMARKS: GAGE L-1 FAILED DURING RUN NO. 2. * SHIFTED TO 15,000 PSI GAGE.

STRAIN GAGE DATA

MODEL NO. E-2		GAGE TYPE INSIDE A-5-1, OUTSIDE FAP-50-12						GAGE FACTOR INSIDE - 1.98, OUTSIDE - 2.11						DATE 1-4-61					
GAGE LOCATION		C-1			C-1			L-1			L-1			C-2			C-2		
CYCLE NO.			1			2			3			4			5			6	
RUN NO.			1			2			1			2			1			2	
P-GAGE	P-CORR.	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ
0	—	10850	—	—	10865	—	—	10800	—	—	10825	—	—	10330	—	—	10285	—	—
500	775	9910	940	940	9930	935	935	10295	505	505	10290	535	535	9670	660	660	9615	670	670
750	1025	9610	300	1240	9625	305	1240	10130	165	670	10110	180	715				9405	210	880
1000	1280	9315	295	1535	9330	295	1535	9950	180	850	9950	160	875	9255	415	1075	9190	215	1095
1250	1500	9035	280	1815	9050	280	1815	9790	160	1010	9785	165	1040				8990	200	1295
1500	1750	8725	310	2125	8740	310	2125	9620	170	1180	9620	165	1205	8840	415	1490	8770	220	1515
1750	2000	8420	305	2430	8430	310	2435	9465	155	1335	9455	165	1370				8550	220	1735
2000	2230	8115	305	2735	8125	305	2740	9300	165	1500	9300	155	1525	8435	405	1895	8340	210	1945
2250	2475	7810	305	3040	7830	295	3035	9145	155	1655	9135	165	1690				8120	220	2165
2500	2715	7510	300	3340	7520	310	3345	8960	185	1840	8995	140	1830	7990	445	2340	7890	230	2395
2750	2950	7200	310	3650	7220	300	3645	8780	180	2020	8850	145	1975				7680	210	2605
3000	3200	6885	315	3965	6900	320	3965	8580	200	2220	8700	150	2125	7510	480	2820	7445	235	2840
3250	3450	6540	345	4310	6550	350	4315	8350	230	2450	8520	180	2305				7195	250	3090
3500	3680	6240	300	4610				8190	160	2610	8345	175	2480	7045	465	3285	6980	215	3305
3750	3930	5920	320	4930				8045	145	2755	8160	185	2665				6755	225	3530
4000	4170							7850	195	2950	7975	185	2850	6575	470	3755			
4250	4410							7665	185	3135									
4500	4650							7470	195	3330				6055	520	4275			
8650	8830	COLLAPSE																	

REMARKS: OUTSIDE DUMMY USED WITH OUTSIDE GAGES.

STRAIN GAGE DATA

MODEL NO E-4		GAGE TYPE FAP-50-12						GAGE FACTOR 2.11						DATE 3-22-61					
GAGE LOCATION		C-2			L-1			OUTSIDE CIRCUM-FERENTIAL DUMMY			OUTSIDE CIRCUM-FERENTIAL DUMMY			C-2			C-2		
CYCLE NO.			1			1			1			2			2		READINGS TAKEN		
RUN NO.			1			1			1			2			2		WHILE RETURNIN TO ZERO PRESSUR		
P-GAGE	P-CORR	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ
0	0	11800	–	–	11800	–	–	11800	–	–	11800	–	–	11800	–	–	11790	–	–
250	300	11540	260	260	11665	135	135	11780	20	20				11555	245	245	11570	220	22
500	550	11310	230	490	11540	125	260	11760	20	40	11760	40	40				11350	220	44
750	790	11070	240	730	11430	110	370	11740	20	60				11130	425	670	11125	225	66
1000	1030	10850	220	950	11310	120	490	11720	20	80	11730	30	70				10915	210	87
1250	1280	10630	220	1170	11190	120	610	11710	10	90				10720	410	1080	10710	205	108
1500	1520	10420	210	1380	11080	110	720	11680	30	120	11700	30	100	10520	200	1280	10505	205	128
1750	1770	10180	240	1620	10960	120	840	11680	0	120				10300	220	1500	10290	215	150
2000	2010	9950	230	1850	10850	110	950	11665	15	135	11675	25	125	10090	210	1710	10080	210	171
2250	2250	9730	220	2070	10740	110	1060	11660	5	140				9880	210	1920	9870	210	192
2500	2500	9515	215	2285	10620	120	1180	11650	10	150	11650	25	150	9670	210	2130	9670	200	212
2750	2750	9290	225	2510	10500	120	1300	11640	10	160				9450	220	2350	9440	230	23
3000	3000	9075	215	2725	10380	120	1420	11630	10	170	11630	20	170	9250	200	2550			

AGAINST INSIDE DUMMY

AGAINST OUTSIDE DUMMY

REMARKS: SR-4 S & B UNIT USED.

STRAIN GAGE DATA

MODEL NO E-4 | GAGE TYPE FAP-50-12 | GAGE FACTOR 2.11 | DATE 3-22-61

GAGE LOCATION		L-2			L-2			C-2			L-2			L-1		
CYCLE NO.			1		READINGS TAKEN WHILE DECREASING PRESSURE				3			3			3	
RUN NO.			1						2			1			2	
P-GAGE	P-CORR	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ	R	Δ	Σ
0	0	11800	—	—	11860	—	—	11800	—	—	11800	—	—	11800	—	—
250	300	11660	140	140				11530	270	270	11625	175	175	11680	120	120
500	550	11530	130	270	11595	265	265	11295	235	505	11475	150	325	11560	120	240
750	790	11410	120	390	11470	125	390	11065	230	735	11320	155	480	11450	110	350
1000	1030	11280	130	520	11345	125	515	10840	225	960	11170	150	630	11330	120	470
1250	1280	11160	120	640	11225	120	635	10615	225	1185	11030	140	770	11210	120	590
1500	1520	11050	110	750	11005	220	855	10405	210	1395	10880	150	920	11110	100	690
1750	1770	10925	125	875	10980	25	880	10165	240	1635	10725	155	1075	10990	120	810
2000	2010	10815	110	985	10855	125	1005	9930	235	1870	10580	145	1220	10880	110	920
2250	2250	10690	125	1110	10740	115	1120	9700	230	2100	10430	150	1370	10760	120	1040
2500	2500	10580	110	1220	10620	120	1240	9480	220	2320	10295	135	1505	10650	110	1150
2750	2750	10480	100	1320	10490	130	1370	9260	220	2540	10155	140	1645	10530	120	1270
3000	3000	10370	110	1430				9045	215	2755	10025	130	1775	10420	110	1380
3250	3250							8790	255	3010	9880	145	1920	10290	130	1510
3500*	3580	AGAINST OUTSIDE DUMMY						8555	235	3245	9750	130	2050	10160	130	1640
4000	4100							8090	465	3710	9510	240	2290	9900	260	1900
4500	4600							7650	440	4150	9325	185	2475	9670	230	2130
5000	5100							7205	445	4595	9175	150	2625	9415	255	2385
6000	6130							6320	885	5480				8890	525	2910
6500	6615							5820	500	5980						
7000	7140							5340	480	6460						

AGAINST INSIDE DUMMY

REMARKS: * SHIFTED TO 15000 psi GAGE

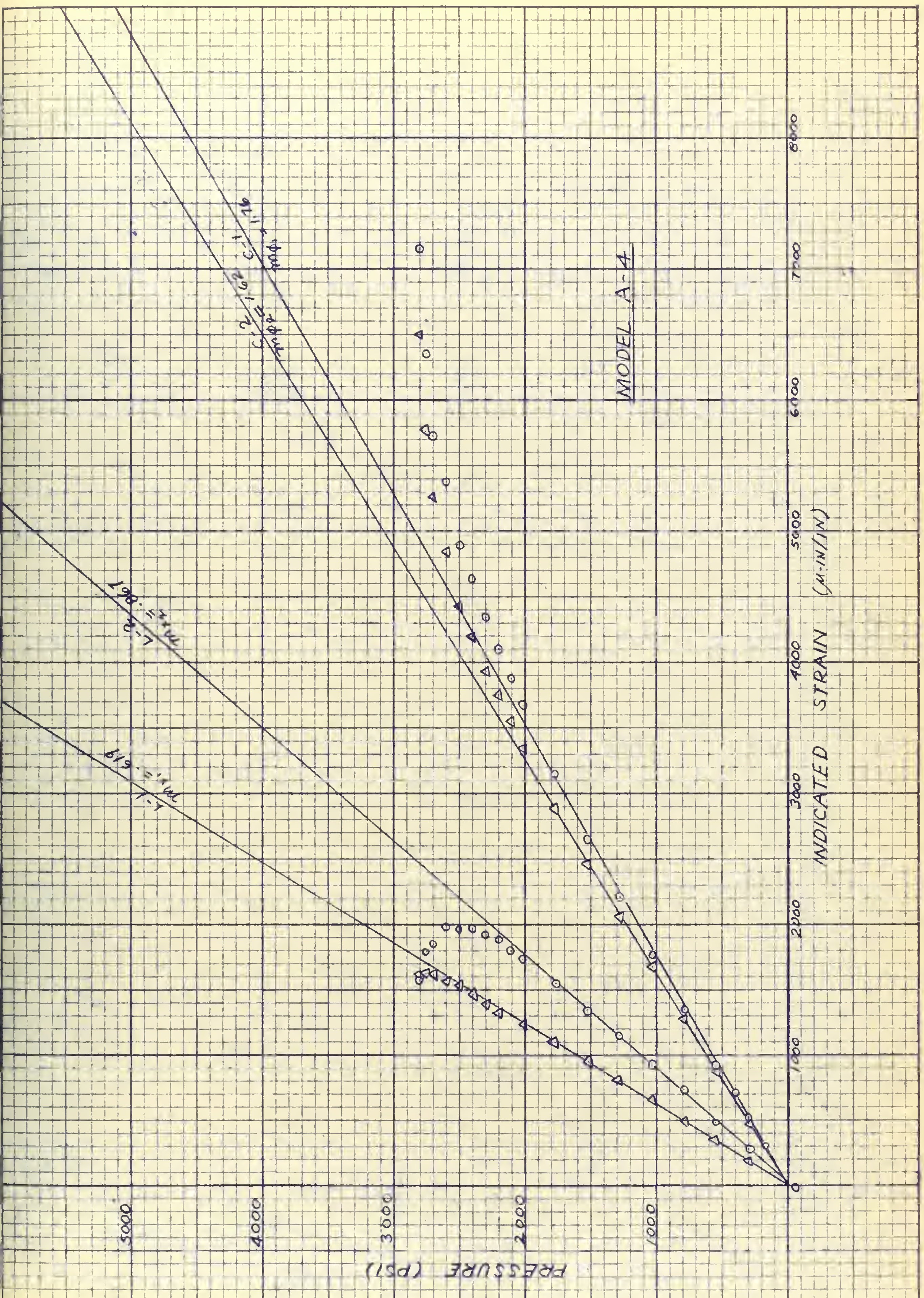
PUMP PLUNGER SEAL FAILED AT 7140 psi (CORRECTED PRESSURE)

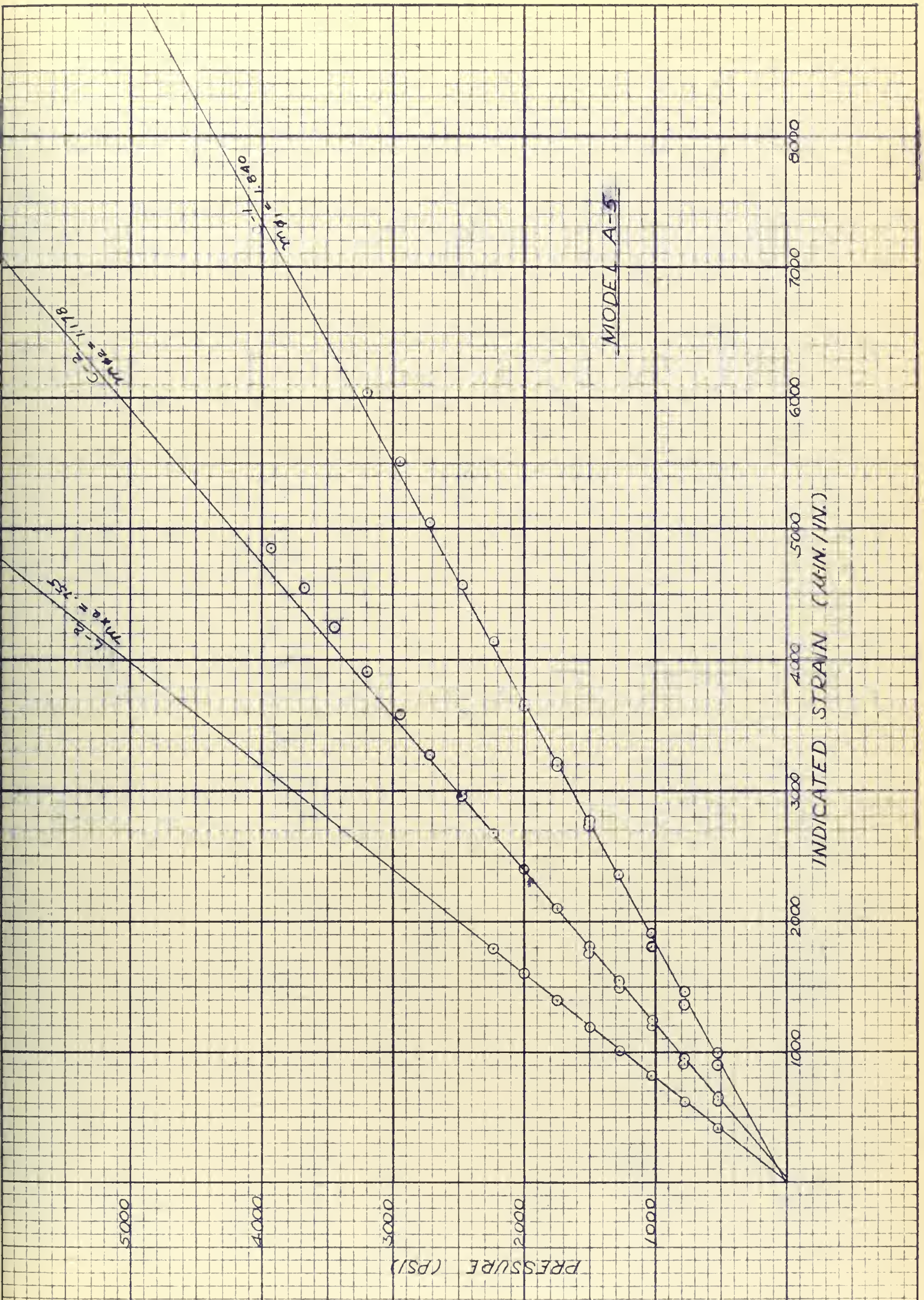
APPENDIX B

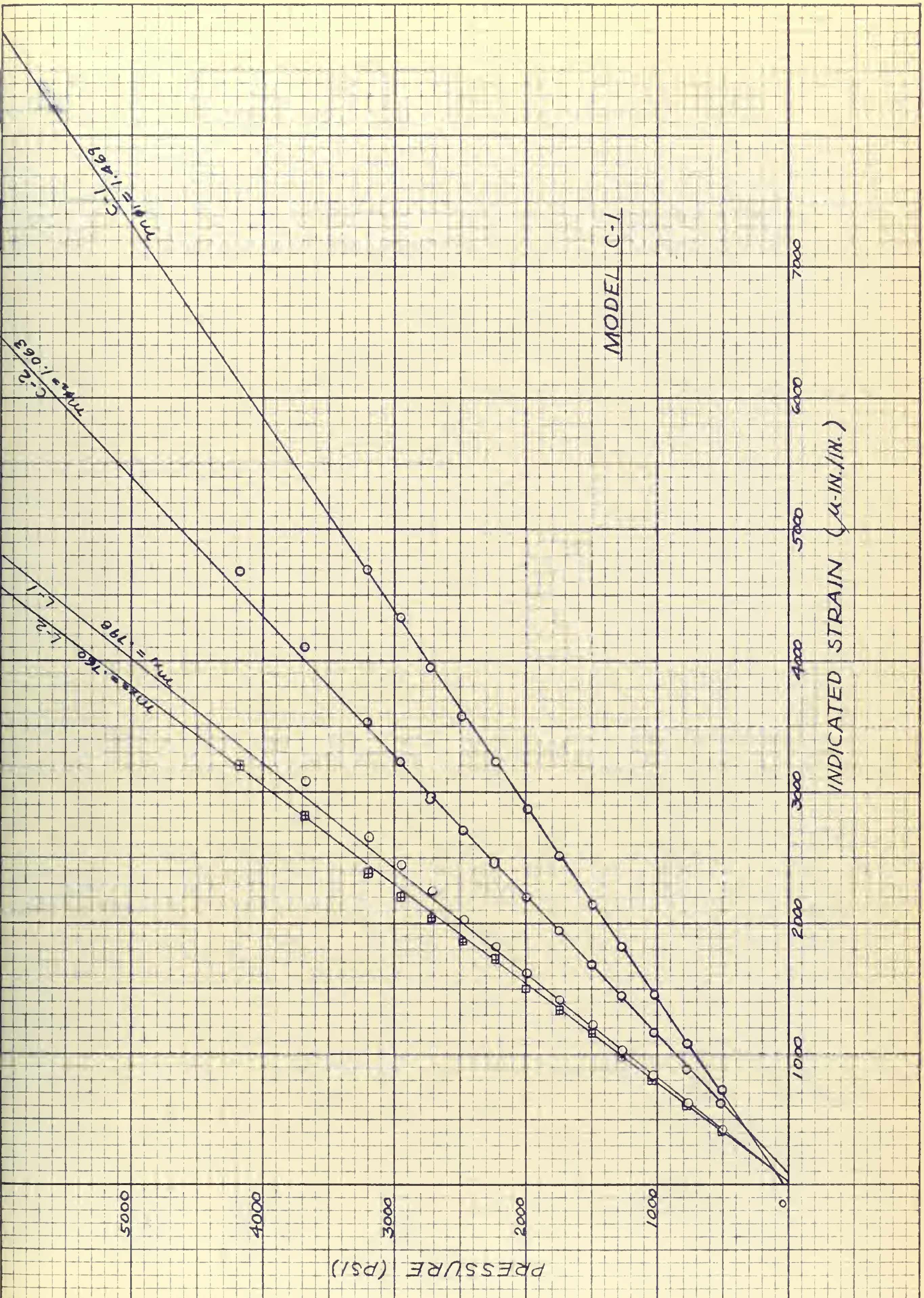
PLOTS OF PRESSURE VERSUS STRAIN

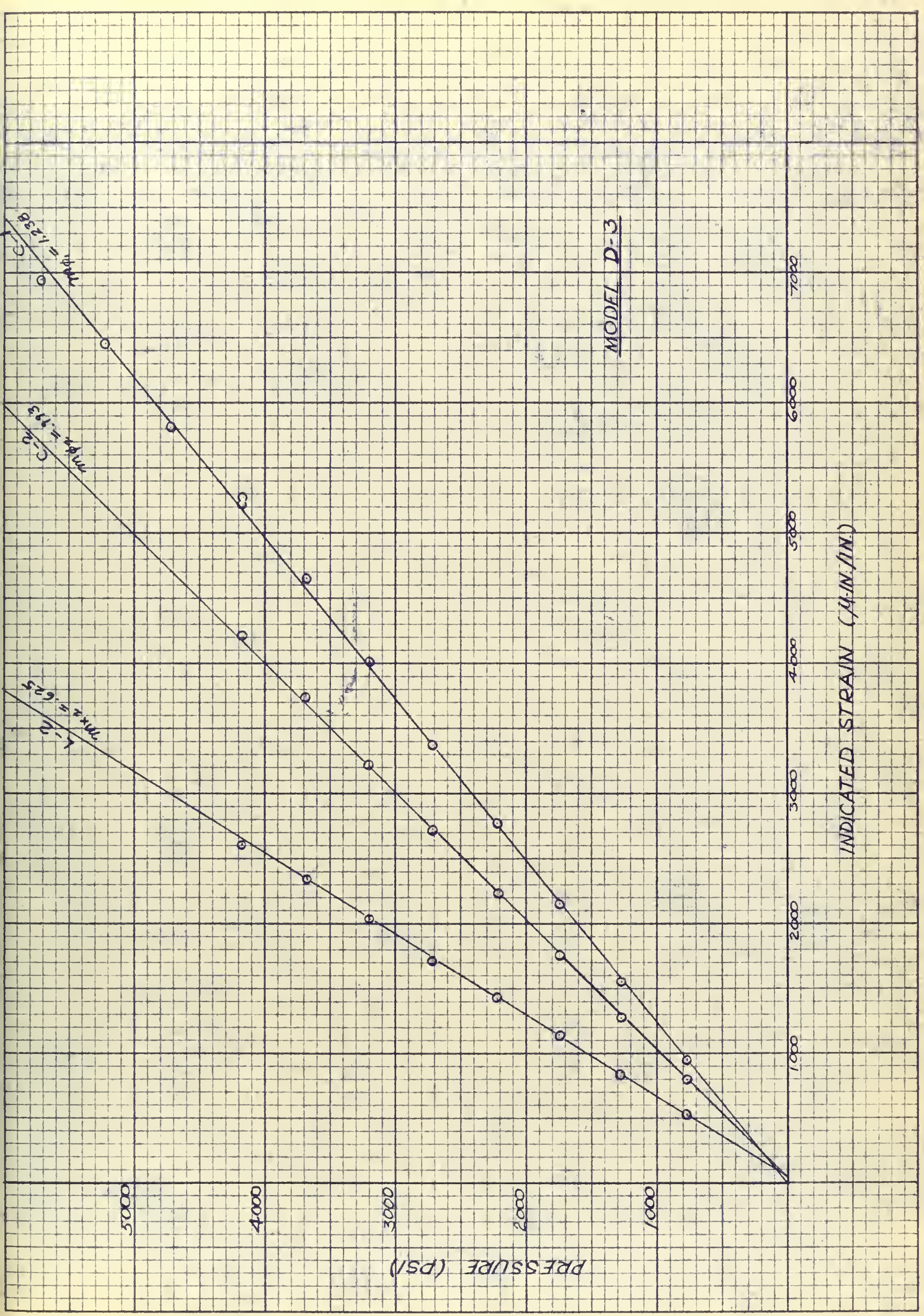
This section contains plots of the measured strains. The resulting slopes, designated by the letter m with appropriate subscripts are indicated on the plots.

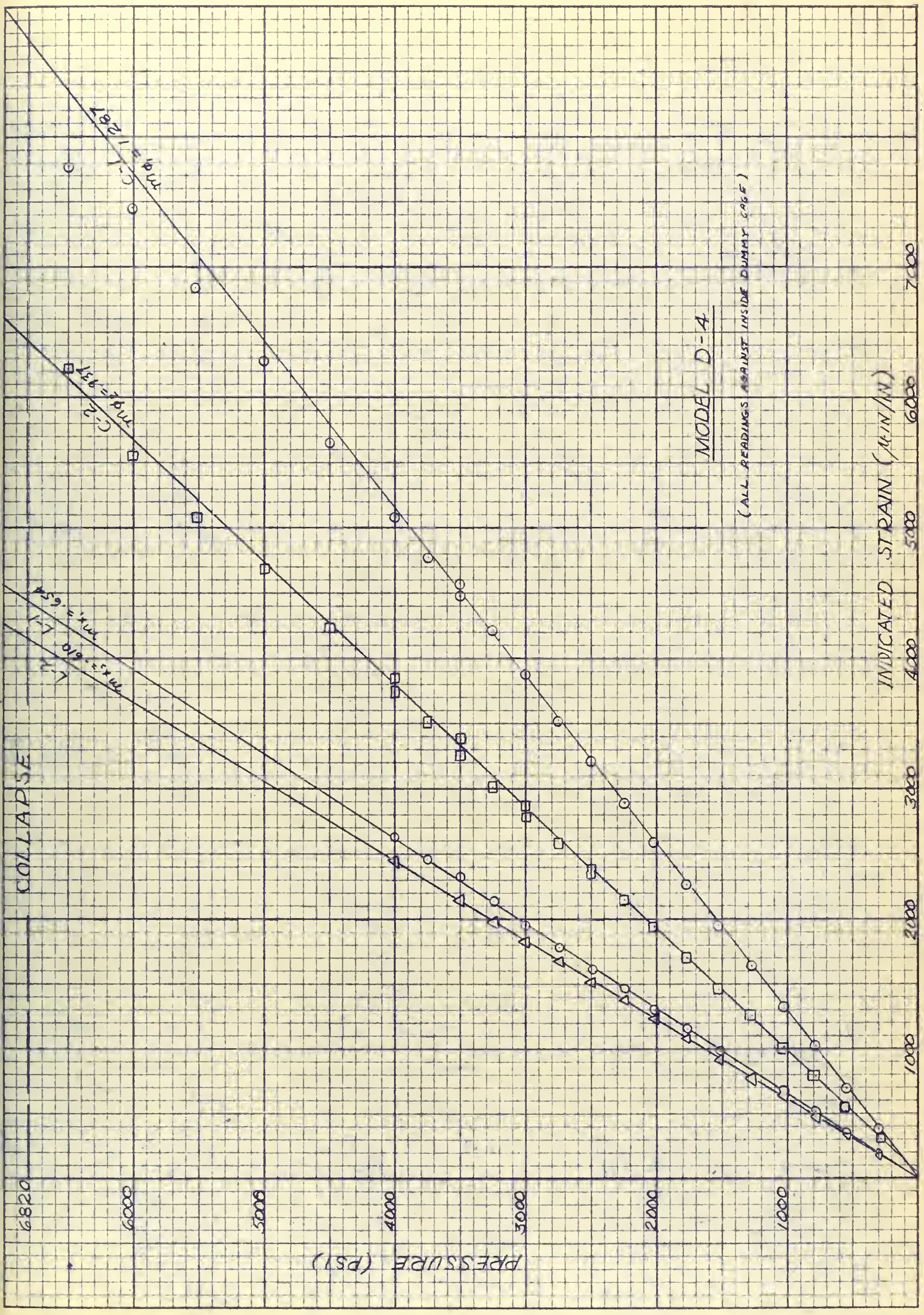
It is noted that all plots are very nearly linear until the collapse pressure is approached. The reversed slope for gage L-2, model A-4 indicates the development of a tensile stress as a result of bending near the collapse pressure.


MODEL A-4
INDICATED STRAIN (μ·IN/IN)
PRESSURE (PSI)
0
1000
2000
3000
4000
5000
6000
7000
8000

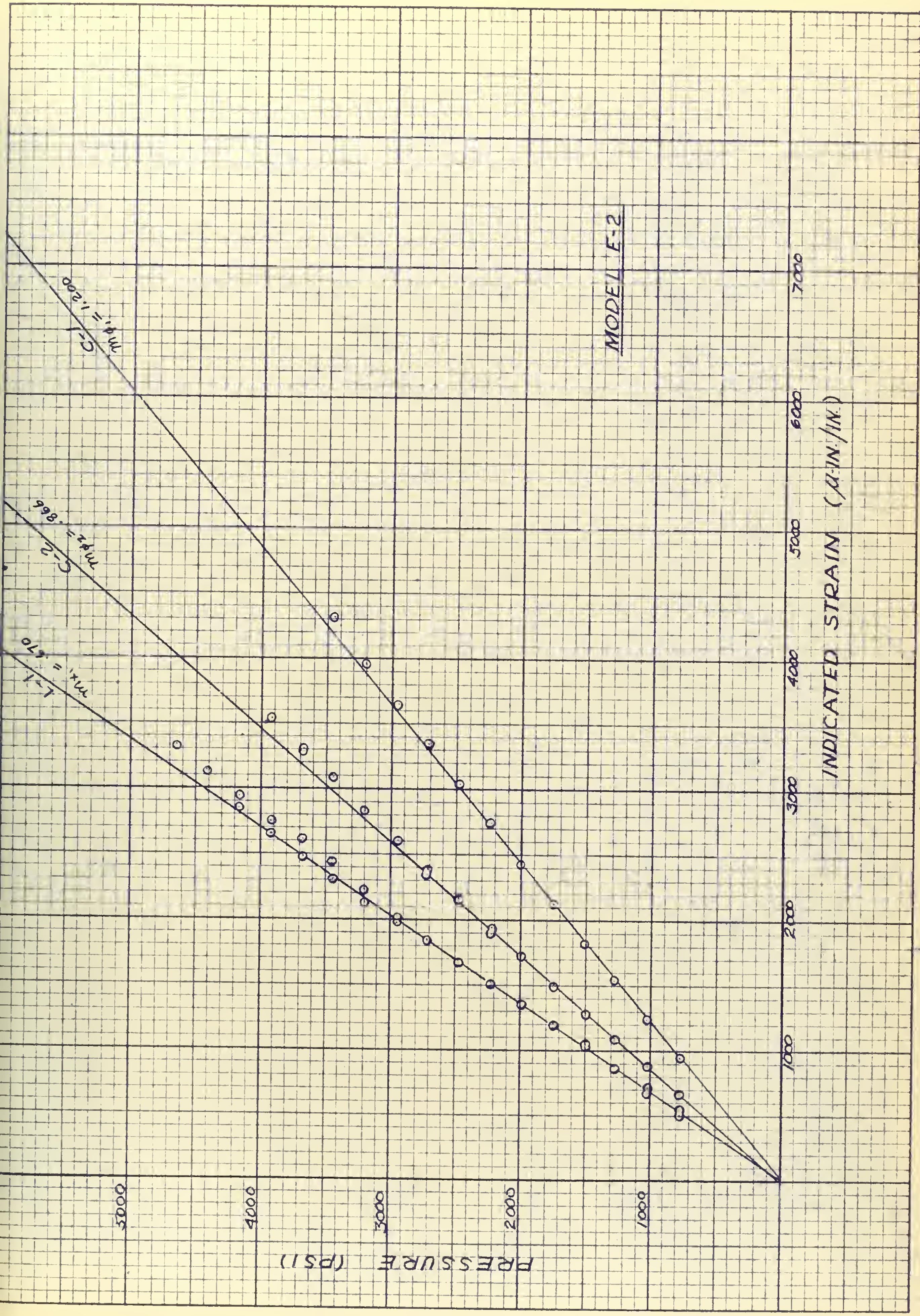

MODEL A-5
INDICATED STRAIN (μIN./IN.)
PRESSURE (PSI)
1000
2000
3000
4000
5000
6000
7000
8000
1000
2000
3000
4000
5000

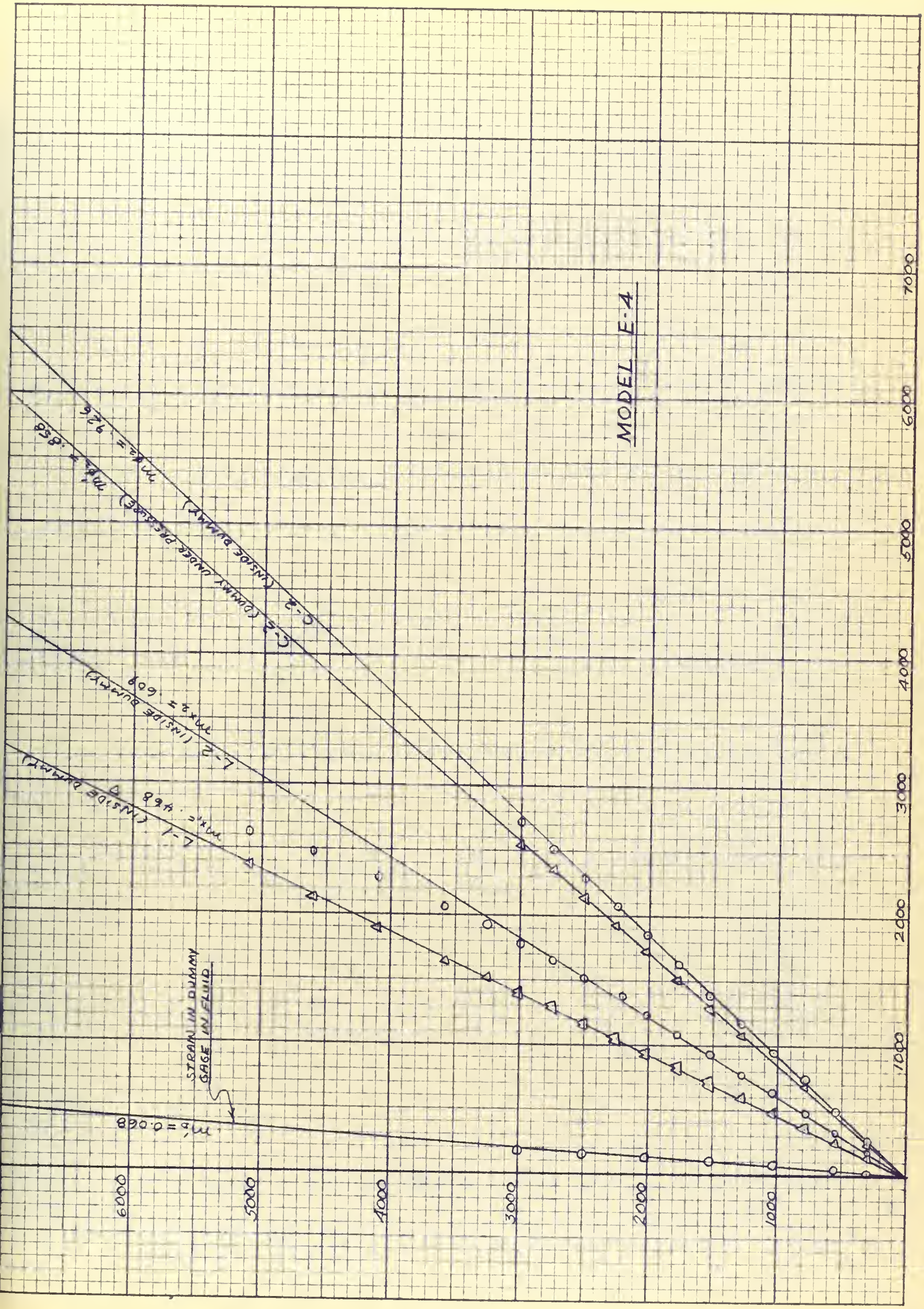

MODEL C-1
$m_{C1} = 1.469$
C-1
$m_{C2} = 1.063$
C-2
$m_{L1} = .798$
L-1
$m_{L2} = .760$
L-2
INDICATED STRAIN (μ-IN./IN.)
1000
2000
3000
4000
5000
6000
7000
PRESSURE (PSI)
0
1000
2000
3000
4000
5000


MODEL D-3
INDICATED STRAIN (μ IN./IN.)
PRESSURE (PSI)
1000
2000
3000
4000
5000
6000
7000
1000
2000
3000
4000
5000
C-1
$m_{\theta\theta}$ = 1.238
C-2
$m_{\theta\theta}$ = .793
L-2
m_{xx} = .625



MODEL D-4
(ALL READINGS AGAINST INSIDE DUMMY GAGE)
COLLAPSE
C-1 m.d=1287
C-2 m.d=937
L-1 m.x1=654
L-2 m.x2=610
INDICATED STRAIN (μ IN/IN)
PRESSURE (PSI)
6820
6000
5000
4000
3000
2000
1000
1000
2000
3000
4000
5000
6000
7000


MODEL E-2
C-1 $m_{\phi}' = 1.200$
C-2 $m_{\phi}'' = .866$
L-1 $m_{x}' = .670$
INDICATED STRAIN (μ IN./IN.)
1000
2000
3000
4000
5000
6000
7000
PRESSURE (PSI)
1000
2000
3000
4000
5000



MODEL E-4
C-2 (INSIDE DUMMY)
m = .926
C-2 (DUMMY UNDER PRESSURE)
m = .858
L-2 (INSIDE DUMMY)
m = .609
L-1 (INSIDE DUMMY)
m = .468
STRAIN IN DUMMY GAGE IN FLUID
m = 0.068
1000
2000
3000
4000
5000
6000
7000
1000
2000
3000
4000
5000
6000

APPENDIX C

THEORY FOR CALCULATION OF E_ϕ AND E_x FROM STRAIN DATA

The circumferential stress in a thick-walled cylinder under external hydrostatic pressure, p, is given by:

$$\sigma_\phi = \frac{pR_2^2 + p\frac{R_1^2 R_2^2}{r^2}}{R_2^2 - R_1^2} \qquad \text{[REF. (42)]}$$

where r is the distance from the center to the point at which the stress is measured. If r is given values of $r = R_1$ and $r = R_2$, the following expressions evolve for the circumferential stress on the inner and outer walls of the cylinder:

$$\sigma_{\phi_1} = p\left(\frac{2R_2^2}{R_2^2 - R_1^2}\right)$$

$$\sigma_{\phi_2} = p\left(\frac{R_2^2 + R_1^2}{R_2^2 - R_1^2}\right) = p\left(\frac{2R_2^2}{R_2^2 - R_1^2} - 1\right)$$

(σ_{ϕ_1} and σ_{ϕ_2} are positive if compressive.)

The longitudinal stress is constant over the cross section, and is given by:

$$\sigma_x = p\frac{R_2^2}{R_2^2 - R_1^2} \qquad \text{(Ref. 42)}$$

Letting $K_1 = \frac{2R_2^2}{R_2^2 - R_1^2}$, we have

$$\left.\begin{aligned} \sigma_{\phi_1} &= pK_1 \\ \sigma_{\phi_2} &= p(K_1 - 1) = pK_2 \\ \sigma_x &= pK_1/2 \end{aligned}\right\} (1)$$

If we assume a material which has three principal values of Young's Modulus and a single value of ν, then from Hooke's Law:

$$\left.\begin{aligned} \epsilon_\phi &= \frac{\sigma_\phi}{E_\phi} - \frac{\nu\sigma_r}{E_r} - \frac{\nu\sigma_x}{E_x} \\ \epsilon_x &= \frac{\sigma_x}{E_x} - \frac{\nu\sigma_r}{E_r} - \frac{\nu\sigma_\phi}{E_\phi} \\ \epsilon_r &= \frac{\sigma_r}{E_r} - \frac{\nu\sigma_x}{E_x} - \frac{\nu\sigma_\phi}{E_\phi} \end{aligned}\right\} \quad (2)$$

Combining Hooke's Law and thick shell theory, we have strains in the inside and outside surfaces of a thick-walled cylinder subject to external hydrostatic pressure:

$$\left.\begin{aligned} \epsilon_{\phi 2} &= p\left(\frac{K_2}{E_\phi} - \frac{\nu}{E_r} - \frac{\nu K_1}{2E_x}\right) \\ \epsilon_{x2} &= p\left(\frac{K_1}{2E_x} - \frac{\nu}{E_r} - \frac{\nu K_2}{E_\phi}\right) \quad \text{(OUTSIDE)} \\ \epsilon_{r2} &= p\left(\frac{1}{E_r} - \frac{\nu K_1}{2E_x} - \frac{\nu K_2}{E_\phi}\right) \end{aligned}\right\} \quad (3)$$

(Note that $\sigma_r = p$ on the outside surface)

$$\left.\begin{aligned} \epsilon_{\phi 1} &= p\left(\frac{K_1}{E_\phi} - \frac{\nu K_1}{2E_x}\right) \\ \epsilon_{x1} &= p\left(\frac{K_1}{2E_x} - \frac{\nu K_1}{E_\phi}\right) \quad \text{(INSIDE)} \\ \epsilon_r &= -\nu p\left(\frac{K_1}{2E_x} + \frac{K_1}{E_\phi}\right) \end{aligned}\right\} \quad (4)$$

(Note that $\sigma_r = 0$ on the inside surface)

The stresses in the surfaces of a block of material subject to external pressure, p, are given by:

$$\sigma_\phi = \sigma_x = \sigma_r = p$$

From Hooke's Law, the strains on the outside surface of the block are given by:

$$\left.\begin{aligned} \epsilon_{\phi D} &= p\left(\frac{1}{E_{\phi D}} - \frac{\nu}{E_{rD}} - \frac{\nu}{E_{xD}}\right) \\ \epsilon_{xD} &= p\left(\frac{1}{E_{xD}} - \frac{\nu}{E_{rD}} - \frac{\nu}{E_{\phi D}}\right) \\ \epsilon_{rD} &= p\left(\frac{1}{E_{rD}} - \frac{\nu}{E_{xD}} - \frac{\nu}{E_{\phi D}}\right) \end{aligned}\right\} \quad (5)$$

The D subscript on a term indicates that it refers to the block rather than to the cylinder.

Consider the case of strain measurements made with strain gages on the inside surface of the cylinder, using a dummy gage not subject to pressure. The strains as read are given by:

$$\left.\begin{aligned} \epsilon_{\phi_1} &= pK_1\left(\frac{1}{E_\phi} - \frac{\nu}{2E_x}\right) \\ \epsilon_{x_1} &= pK_1\left(\frac{1}{2E_x} - \frac{\nu}{E_\phi}\right) \\ \epsilon_{r_1} &= -p\nu K_1\left(\frac{1}{2E_x} + \frac{1}{E_\phi}\right) \end{aligned}\right\} \quad (6)$$

Solving the first two of these equations:

$$\left.\begin{aligned} E_\phi &= \frac{K_1(1-\nu^2)}{m_{\phi_1} + \nu m_{x_1}} \\ E_x &= \frac{K_1(1-\nu^2)}{2(m_{x_1} + \nu m_{\phi_1})} \end{aligned}\right\} \quad (7)$$

Where $m_{x_1} = \epsilon_{x_1}/p$

$m_{\phi_1} = \epsilon_{\phi_1}/p$

Therefore a model instrumented thus can be subjected to hydrostatic pressure and the measured strain-pressure ratios, m_{x_1} and m_{ϕ_1}, with an assumed or known value of ν can be substituted to solve for E_x and E_ϕ. The radial strain does not enter the solution, and the radial stress is zero.

Consider the case of strain measurements made with strain gages on the outside surface of the cylinder, using a dummy gage not subject to pressure. The strains as read are given by:

$$\left.\begin{aligned} \epsilon_{\phi_2} &= p\left(\frac{K_2}{E_\phi} - \frac{\nu}{E_r} - \frac{\nu K_1}{2E_x}\right) \\ \epsilon_{x_2} &= p\left(\frac{K_1}{2E_x} - \frac{\nu}{E_r} - \frac{\nu K_2}{E_\phi}\right) \\ \epsilon_{r_2} &= p\left(\frac{1}{E_r} - \frac{\nu K_1}{2E_x} - \frac{\nu K_2}{E_\phi}\right) \end{aligned}\right\} \quad (8)$$

Solving this system of equations simultaneously gives:

$$\left.\begin{aligned} E_\phi &= \frac{K_2(1+\nu)(1-2\nu)}{m_{\phi 2}(1-\nu)+\nu(m_{x2}+m_{r2})} \\ E_x &= \frac{K_1(1+\nu)(1-2\nu)}{2\left[m_{x2}(1-\nu)+\nu(m_{\phi 2}+m_{r2})\right]} \end{aligned}\right\} \quad (9)$$

Consider the case of strain measurements made with strain gages on the outside surface of the cylinder, using dummy gages mounted on a block subject to the hydrostatic pressure, the longitudinal active gage being balanced against a longitudinally oriented dummy, and the circumferential active gage being balanced against a circumferentially oriented dummy. Assume the dummy block to be of the same material as the cylinder.

The indicated strain readings are:

$$\epsilon'_{\phi 2} = \epsilon_{\phi 2} - \epsilon_{\phi D}$$

$$\epsilon'_{x2} = \epsilon_{x2} - \epsilon_{\phi D}$$

$$\epsilon'_{r2} = \epsilon_{r2} - \epsilon_{rD}$$

where the primes indicate that the strains are the <u>indicated</u> values rather than the actual strains. From (3) and (5):

$$\epsilon'_{\phi 2} = p\left(\frac{K_2}{E_\phi} - \frac{\nu}{E_r} - \frac{\nu K_1}{2E_x}\right) - p\left(\frac{1}{E_\phi} - \frac{\nu}{E_r} - \frac{\nu}{E_x}\right)$$

$$\epsilon'_{x2} = p\left(\frac{K_1}{2E_x} - \frac{\nu}{E_r} - \frac{\nu K_2}{E_\phi}\right) - p\left(\frac{1}{E_x} - \frac{\nu}{E_r} - \frac{\nu}{E_\phi}\right)$$

$$\epsilon'_r = p\left(\frac{1}{E_r} - \frac{\nu K_1}{2E_x} - \frac{\nu K_2}{E_\phi}\right) - p\left(\frac{1}{E_r} - \frac{\nu}{E_x} - \frac{\nu}{E_\phi}\right)$$

$$\left.\begin{aligned} \epsilon'_{\phi 2} &= p\left[\frac{(K_2-1)}{E_\phi} - \frac{\nu(K_1/2-1)}{E_x}\right] \\ \epsilon'_{x2} &= p\left[\frac{(K_1/2-1)}{E_x} - \frac{\nu(K_2-1)}{E_\phi}\right] \\ \epsilon'_r &= 0 \end{aligned}\right\} \quad (10)$$

The terms involving radial strain drop out. A comparison of (10) with (3) shows that they are identical, except that in (10) the radial terms are missing, and $K_2 - 1$ and $K_1/2 - 1$ appear in place of K_2 and $K_1/2$, respectively. Thus by having the compensating gages arranged as in this case, the values of E_ϕ and E_x may be computed from the indicated strains $\epsilon'_{\phi 2}$ and ϵ'_{x2}, and the radial strains do not enter the solution.

Solving (10):

$$\left. \begin{aligned} E_\phi &= \frac{(K_2 - 1)(1 - \nu^2)}{m'_{\phi 2} + \nu m'_{x2}} \\ E_x &= \frac{(K_1/2 - 1)(1 - \nu^2)}{m'_{x2} + \nu m'_{\phi 2}} \end{aligned} \right\} \quad (11)$$

WHERE: $m'_{\phi 2} = \epsilon'_{\phi 2}/\varphi$

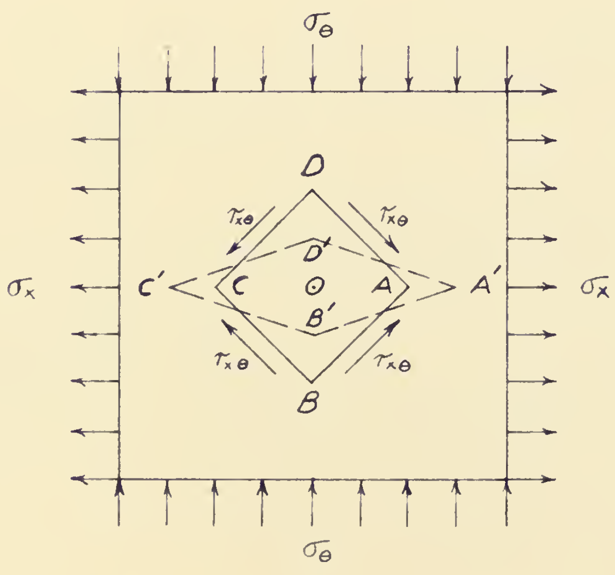
$m'_{x2} = \epsilon'_{x2}/\varphi$

APPENDIX D

DERIVATION OF SHEAR MODULUS FOR AN ORTHOTROPIC MATERIAL

Following a method similar to that used by Timoshenko (Ref. 40, pp. 54-57) for developing the shear modulus for the isotripic case, a formula is derived for the shear modulus of an orthotropic material by assuming two principal moduli of elasticity and a constant value for Poisson's ratio.

Consider an element in pure shear. For this condition $\sigma_x = -\sigma_\theta = \tau_{x\theta}$, as is readily apparent from Mohr's circle. The element can be represented as shown below:



(The deformed element is represented by dotted lines.) The deformed angle at A is $\frac{\pi}{2} - \gamma$; that at B is $\frac{\pi}{2} + \gamma$, where γ is the shearing strain.

Now $\quad \overline{OA}' = \overline{OA}\,(1 + \epsilon_x) \quad$ and $\quad \overline{OB}' = \overline{OB}\,(1 + \epsilon_\theta)$

$$\epsilon_x = \frac{\sigma_x}{E_x} - \frac{\nu\,\sigma_\theta}{E_\theta} \qquad \epsilon_\theta = \frac{\sigma_\theta}{E_\theta} - \frac{\nu\,\sigma_x}{E_x}$$

but $\sigma_x = \tau_{x\theta}$ and $\sigma_\theta = -\tau_{x\theta}$, Then

$$\overline{OA}' = \overline{OA}\left[1 + \frac{\tau_{x\theta}}{E_x} + \frac{\nu\,\tau_{x\theta}}{E_\theta}\right] \qquad \overline{OB}' = \overline{OB}\left[1 - \frac{\tau_{x\theta}}{E_\theta} - \frac{\nu\,\tau_{x\theta}}{E_x}\right]$$

Since $\overline{OA} = \overline{OB}$

$$\tan OA'B' = \tan\left(\pi/4 - \gamma/2\right) = \frac{\overline{OB}'}{\overline{OA}'} = \frac{\left[1 - \frac{\tau_{x\theta}}{E_\theta} - \frac{\nu\,\tau_{x\theta}}{E_x}\right]}{\left[1 + \frac{\tau_{x\theta}}{E_x} + \frac{\nu\,\tau_{x\theta}}{E_\theta}\right]}$$

From trigonometry

$$\tan\left(\pi/4 - \gamma/2\right) = \frac{\tan \pi/4 - \tan \gamma/2}{1 + \tan \pi/4 \tan \gamma/2} \cong \frac{1 - \gamma/2}{1 + \gamma/2}$$

Solving for γ gives

$$\gamma = \frac{2\,(E_x + E_\theta)(1+\nu)}{\frac{2 E_x E_\theta}{\tau_{x\theta}} - (E_x - E_\theta)(1-\nu)}$$

Neglecting the second term in the denominator and solving for $\tau_{x\theta}$

$$\tau_{x\theta} = \frac{E_x E_\theta}{(E_x + E_\theta)(1+\nu)}\,\gamma$$

Then, the orthotropic expression for G becomes

$$G = \frac{E_x E_\theta}{(E_x + E_\theta)(1+\nu)}$$

Conway[41] arrived at a similar expression using an energy analysis.

APPENDIX E

DERIVATION OF EQUATION FOR THE CRITICAL PRESSURE OF A SIMPLY SUPPORTED, ORTHOTROPIC, CIRCULAR CYLINDER SUBJECTED TO HYDROSTATIC PRESSURE

Stress-strain relations.

For isotropic shells, the relation between the stress resultants and couples and the corresponding strains is:

$$N_x = \frac{Eh}{1-\nu^2}\left(\epsilon_{xo} + \nu\epsilon_{\theta o}\right) \qquad M_x = -D\left(\chi_x + \nu\chi_\theta\right)$$

$$N_\theta = \frac{Eh}{1-\nu^2}\left(\epsilon_{\theta o} + \nu\epsilon_{xo}\right) \qquad M_\theta = -D\left(\chi_\theta + \nu\chi_x\right)$$

$$N_{x\theta} = G\gamma_o \qquad M_{x\theta} = D(1-\nu)\,\chi_{x\theta}$$

where the subscript "o" refers to strains in the middle surface.

If is is assumed that the cylinder is orthotropic with principal moduli E_x and E_θ and with a constant value of Poisson's ratio, these equations may be written:

$$N_x = \frac{h}{1-\nu^2}\left(E_x\epsilon_{xo} + \nu E_\theta\epsilon_{\theta o}\right) \qquad M_x = -\frac{h^3}{12(1-\nu^2)}\left(E_x\chi_x + \nu E_\theta\chi_\theta\right)$$

$$N_\theta = \frac{h}{1-\nu^2}\left(E_\theta\epsilon_{\theta o} + \nu E_x\epsilon_{xo}\right) \qquad M_\theta = -\frac{h^3}{12(1-\nu^2)}\left(E_\theta\chi_\theta + \nu E_x\chi_x\right)$$

$$N_{x\theta} = G\gamma_o \qquad M_{x\theta} = G\frac{h^3}{6a}\,\chi_{x\theta}$$

The strains and changes of curvature in terms of the displacements are:

$$\epsilon_{xo} = \frac{\partial u}{\partial x} \qquad \chi_x = \frac{\partial^2 w}{\partial x^2}$$

$$\epsilon_{\theta o} = \frac{1}{a}\frac{\partial v}{\partial \theta} - \frac{w}{a} \qquad \chi_\theta = \frac{1}{a}\,\partial/\partial\theta \left(\frac{v}{a} + \frac{1}{a}\frac{\partial w}{\partial \theta}\right)$$

$$\gamma_o = \frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \theta} \qquad \chi_{x\theta} = \frac{1}{a}\left(\frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x\,\partial \theta}\right)$$

The stress resultants and couples can then be written in terms of the displacements:

$$N_x = \frac{h}{1-\nu^2}\left[E_x \frac{\partial u}{\partial x} + \nu E_\theta \left(\frac{1}{a}\frac{\partial v}{\partial \theta} - \frac{w}{a}\right)\right]$$

$$N_\theta = \frac{h}{1-\nu^2}\left[\frac{E_\theta}{a}\frac{\partial v}{\partial \theta} - \frac{E_\theta}{a} w + \nu E_x \frac{\partial u}{\partial x}\right]$$

$$M_x = -\frac{h^3}{12(1-\nu^2)}\left[E_x \frac{\partial^2 w}{\partial x^2} + \frac{\nu E_\theta}{a^2}\left(\frac{\partial v}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2}\right)\right]$$

$$M_\theta = -\frac{h^3}{12(1-\nu^2)}\left[\frac{E_\theta}{a^2}\left(\frac{\partial v}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2}\right) + \nu E_x \frac{\partial^2 w}{\partial x^2}\right]$$

$$N_{x\theta} = Gh\left[\frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \theta}\right]$$

$$M_{x\theta} = G\frac{h^3}{6a}\left[\frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x\,\partial \theta}\right]$$

The Equilibrium Equations.

The equilibrium equations for a circular cylinder under radial load only are:

$$a^2 \frac{\partial N_x}{\partial x} + a \frac{\partial N_{x\theta}}{\partial \theta} = 0$$

$$\frac{\partial N_\theta}{\partial \theta} + a \frac{\partial N_{x\theta}}{\partial x} - \frac{\partial M_\theta}{a \partial \theta} + \frac{\partial M_{x\theta}}{\partial x} = 0$$

$$a \frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{x\theta}}{\partial x \partial \theta} + \frac{1}{a} \frac{\partial^2 M_\theta}{\partial \theta^2} + N_\theta - p\left(w + \frac{\partial^2 w}{\partial \theta^2}\right) = 0$$

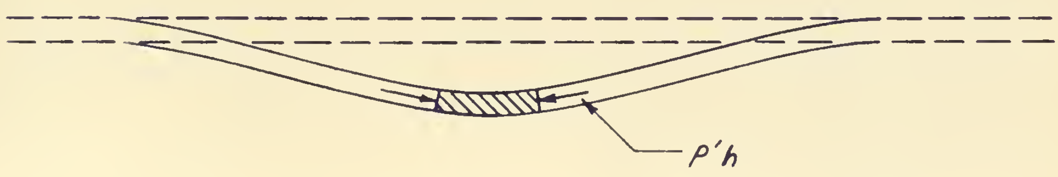
[See Timoshenko, Ref. 30, p.447; note that the term $qa\left(\frac{\partial^2 v}{\partial x \partial \theta} - \frac{\partial w}{\partial x}\right)$ has been neglected, as was done by von Mises[22].]

These equations may be modified to include the effect of axial load as follows. This procedure was used by von Mises.

The total axial force on the ends of the cylinder is equal to $\pi a^2 p$. This force is transmitted to the ends of the cylinder of area $2\pi a h$. Hence the pressure in the cylinder walls is

$$p' = \frac{\pi a^2 p}{2\pi a h} = \frac{a p}{2 h}$$

Now consider an element of the shell in the longitudinal plane.



After deformation, the curvature of the generatrix (neglecting higher order terms) is

$$\frac{\partial^2 w}{\partial x^2}$$

The force per unit length, p'h, thus has a component in the z-direction of

$$p'h \frac{\partial^2 w}{\partial x^2} = \frac{pa}{2} \frac{\partial^2 w}{\partial x^2}$$

Since the radius, a, had been divided out of the original equilibrium equation, this term is written as

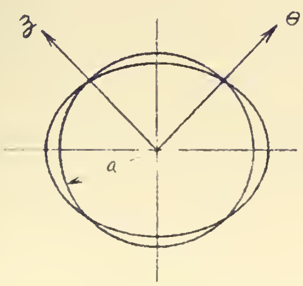
$$\frac{p}{a} \left(\frac{a^2}{2} \frac{\partial^2 w}{\partial x^2} \right)$$

In order to include the effect of the axial load, this term is added to the term in parenthesis in equation (3) to get:

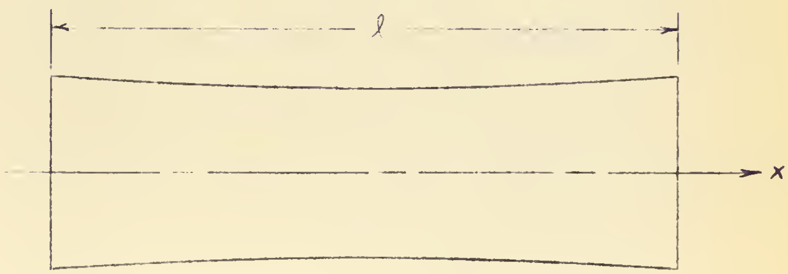
$$a \frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{x\theta}}{\partial x \partial \theta} + \frac{1}{a} \frac{\partial^2 M_\theta}{\partial \theta^2} + N_\theta - p\left(w + \frac{\partial^2 w}{\partial \theta^2} + \frac{a^2}{2} \frac{\partial^2 w}{\partial x^2} \right) = 0$$

Boundary conditions.

The transverse and longitudinal cross sections of the cylinder assume the following shapes after buckling:



$n = 2$



$m = 1$

At x = 0, the displacements may be written:

$$\left.\begin{aligned} u &= A \sin n\theta \\ v &= B \cos n\theta \\ w &= C \sin n\theta \end{aligned}\right\} \quad v \text{ is } 90^\circ \text{ out of phase with } u \text{ and } w.$$

At x = 0, u = 0 and v and w are maximum; hence, the displacements for any value of x become:

$$u = A \sin n\theta \sin \frac{\pi x}{\ell}$$

$$v = B \cos n\theta \cos \frac{\pi x}{\ell}$$

$$w = C \sin n\theta \cos \frac{\pi x}{\ell}$$

Let $\lambda = \frac{\pi a}{\ell}$, then

$$u = A \sin n\theta \sin {}^{\lambda}\!/_{a}\, x$$

$$v = B \cos n\theta \cos {}^{\lambda}\!/_{a}\, x$$

$$w = C \sin n\theta \cos {}^{\lambda}\!/_{a}\, x$$

Determinant for the Critical Pressure.

By substituting the boundary conditions into the relations for the stress resultants and couples, and substituting the resulting expressions into the equilibrium equations, a system of three equations in A, B, and C is determined. In order that an instability condition exist, the constants A, B, and C, which determine the magnitudes of the displacements, must have values other than zero. This condition requires that the determinant of the coefficients of A, B, and C be zero. The determinant for this case is given on the next page. Evaluation of the determinant for a given geometry and material yields the critical buckling pressure.

DETERMINANT FOR BUCKLING PRESSURE OF AN ORTHOTROPIC CIRCULAR CYLINDER UNDER HYDROSTATIC PRESSURE

	A	B	C
1	$-[\lambda^2 E_x + n^2(1-\nu^2) G]$	$[n\lambda(1-\nu^2) G + \nu n \lambda E_\theta]$	$[\nu \lambda E_\theta]$
2	$[n\lambda(1-\nu^2) G + \nu n \lambda E_x]$	$-[n^2(1+\alpha) E_\theta + \lambda^2(1-\nu^2)(1+2\alpha) G]$	$-[n(1+n^2\alpha) E_\theta + \nu n \lambda^2 \alpha E_x + n\lambda^2(1-\nu)\alpha G]$
3	$[\nu \lambda E_x]$	$-[n(1+n^2\alpha + \nu\lambda^2\alpha) E_\theta + 2n\lambda^2(1-\nu)\alpha G]$	$-[(1+n^4\alpha + \nu n^2 \lambda^2 \alpha) E_\theta + (\lambda^4 + \nu n^2 \lambda^2)\alpha E_x + 4n^2\lambda^2(1-\nu^2)\alpha G + \frac{pa(1-\nu^2)}{h}(1-n^2-\frac{\lambda^2}{2})]$

APPENDIX F

DERIVATION OF AN EXPRESSION FOR THE LONGITUDINAL BENDING STRESS IN A SIMPLY SUPPORTED CIRCULAR CYLINDER SUBJECTED TO EXTERNAL HYDROSTATIC PRESSURE

Salerno and Pulos[43] have derived expressions for the stresses in a circular cylindrical shell supported by uniformly spaced circular rings of constant cross section. By using the displacement function obtained in their paper with the appropriate boundary conditions for simple support, an expression for the longitudinal bending stress at midbay is derived.

The longitudinal bending stress is given by:

$$\sigma_b = \frac{Eh}{2(1-\nu^2)} \frac{d^2 w}{dx^2}$$

The displacement function, w, as derived by Salerno and Pulos for deflection in the radial direction, is

$$w = B \cosh(\lambda_1 x) + F \cosh(\lambda_3 x) - \frac{PR^2 [1 - \nu/2]}{Eh}$$

where B and F are constants depending on the boundary conditions, $\lambda_1 \neq \lambda_3$, and

$$\lambda_1 = \left[-\frac{PR}{4D} + \sqrt{\left(-\frac{PR}{4D}\right)^2 - \left(\frac{Eh}{DR^2}\right)} \right]^{1/2}$$

$$\lambda_3 = -\left[\frac{PR}{4D} - \sqrt{\left(-\frac{PR}{4D}\right)^2 - \left(\frac{Eh}{DR^2}\right)} \right]^{1/2}$$

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad \text{(Flexural rigidity of shell)}$$

R = Mean radius of shell.

Taking the x-axis along the axis of the cylinder, with the origin at the mid-length, the boundary conditions for simply-supported ends require that

$$w = 0 \quad \text{at} \quad x = L/2$$

$$\frac{d^2 w}{d x^2} = 0 \quad \text{at} \quad x = L/2$$

from which,

$$0 = B \cosh \lambda_1 L/2 + F \cosh \lambda_3 L/2 - \frac{PR^2}{Eh}\left(1 - \frac{\nu}{2}\right)$$

$$0 = B \lambda_1^2 \cosh \lambda_1 L/2 + F \lambda_3^2 \cosh \lambda_3 L/2$$

Solving simultaneously,

$$B = \left(\frac{\lambda_3^2}{\lambda_3^2 - \lambda_1^2}\right)\left(\frac{1}{\cosh \lambda_1 L/2}\right)\left[\frac{PR^2}{Eh}\left(1 - \nu/2\right)\right]$$

$$F = \left(\frac{\lambda_1^2}{\lambda_1^2 - \lambda_3^2}\right)\left(\frac{1}{\cosh \lambda_3 L/2}\right)\left[\frac{PR^2}{Eh}\left(1 - \nu/2\right)\right]$$

Krenzke and Short [44] have presented a graphical method for calculating the stresses obtained by the analysis of Salerno and Pulos. In this reference, different parameters were employed as a simplification. These parameters, defined below, are used for the remainder of this derivation.

$$\lambda_1 = \frac{2\theta}{L}(\eta_1 + i\,\eta_2) \qquad \lambda_3 = \frac{2\theta}{L}(\eta_1 - i\,\eta_2)$$

$$\eta_1 = \frac{1}{2}\sqrt{1 - \Upsilon} \qquad \eta_2 = \frac{1}{2}\sqrt{1 + \Upsilon}$$

$$\Upsilon = \frac{P}{2E}\left(\frac{R}{h}\right)^2 \sqrt{3(1-\nu^2)}$$

$$\theta = \frac{\sqrt[4]{3(1-\nu^2)}}{\sqrt{Rh}}\, L$$

Using the Krenzke and Short parameters,

$$\frac{\lambda_1^2}{\lambda_1^2 - \lambda_3^2} = \frac{(\eta_1 + i\eta_2)^2}{4i\eta_1\eta_2} \qquad \frac{\lambda_3^2}{\lambda_1^2 - \lambda_3^2} = \frac{(\eta_1 - i\eta_2)^2}{4i\eta_1\eta_2}$$

$$B = -\left[\frac{(\eta_1 - i\eta_2)^2}{4i\eta_1\eta_2}\right] \frac{\frac{PR^2}{Eh}(1 - \nu/2)}{\cosh(\eta_1\theta + i\eta_2\theta)}$$

$$F = \left[\frac{(\eta_1 + i\eta_2)^2}{4i\eta_1\eta_2}\right] \frac{\frac{PR^2}{Eh}(1 - \nu/2)}{\cosh(\eta_1\theta - i\eta_2\theta)}$$

Then,

$$w = \left[\frac{(\eta_1 + i\eta_2)^2}{4i\eta_1\eta_2}\right] \frac{\frac{PR^2}{Eh}(1 - \nu/2)\cosh\frac{2\theta x}{L}(\eta_1 - i\eta_2)}{\cosh(\eta_1\theta - i\eta_2\theta)} - \left[\frac{(\eta_1 - i\eta_2)^2}{4i\eta_1\eta_2}\right] \frac{\frac{PR^2}{Eh}(1 - \nu/2)\cosh\frac{2\theta x}{L}(\eta_1 + i\eta_2)}{\cosh(\eta_1\theta + i\eta_2\theta)} - \frac{PR^2(1 - \nu/2)}{Eh}$$

Taking the second derivative of this expression, simplifying, and substituting into the expression for the longitudinal bending stress gives:

$$\sigma_b = \left[\frac{PR^2(1 - \nu/2)}{(1 - \nu^2)}\right]\left(\frac{\theta}{L}\right)^2 \left[\frac{(\eta_1^2 + \eta_2^2)^2}{\eta_1\eta_2}\right] \times$$

$$\left[\frac{\sinh\theta\eta_1 \sin\theta\eta_2 \cosh\frac{2\theta x}{L}\eta_1 \cosh\frac{2\theta x}{L}\eta_2}{\cosh^2\theta\eta_1 \cos^2\theta\eta_2 + \sinh^2\theta\eta_1 \sin^2\theta\eta_2} - \frac{\cosh\theta\eta_1 \cos\theta\eta_2 \sinh\frac{2\theta x}{L}\eta_1 \sin\frac{2\theta x}{L}\eta_2}{\cosh^2\theta\eta_1 \cos^2\theta\eta_2 + \sinh^2\theta\eta_1 \sin^2\theta\eta_2}\right]$$

At midbay $x = 0$, and

$$\left[\sigma_b\right]_{x=0} = \left[\frac{PR^2(1 - \nu/2)}{(1 - \nu^2)}\right]\left(\frac{\theta}{L}\right)^2 \left[\frac{(\eta_1^2 + \eta_2^2)^2}{\eta_1\eta_2}\right]\left[\frac{\sinh\theta\eta_1 \sin\theta\eta_2}{\cosh^2\theta\eta_1 \cos^2\theta\eta_2 + \sinh^2\theta\eta_1 \sin^2\theta\eta_2}\right]$$

APPENDIX G

SOLUTION OF VON MISES' EQUATION (6)

This appendix contains solutions of the von Mises equation (6)[22] as computed by the IBM 7090 computer at DTMB. Values of p/E were computed for a range of L/D from 0.50 to 3.00 in increments of 0.25, and for a range of h/D from 0.01 to 0.10 in increments of 0.01. The solution for a particular h/D was started with a value of n = 2. The value of n was then incremented by one until a minimum value for p/E was obtained. Columns 3, 4, and 5 give the auxiliary quantities ρ, μ_1, and μ_2 for use in the von Mises equation.

L/D = 0.25

T/C	N	[illegible]	H1	H2	[illegible]
0.01	[illegible]	[illegible]	[illegible]	1.12732431	[illegible]
0.01	[illegible]	[illegible]	[illegible]	1.21135045	[illegible]
0.01	[illegible]	[illegible]	1.36151476	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.03	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.07	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	1.95750083	1.12732431	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	1.21135045	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	1.14172219	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	1.12732431	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.06	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.07	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.07	[illegible]	[illegible]	[illegible]	[illegible]	0.00496710
0.07	[illegible]	[illegible]	1.36151476	[illegible]	[illegible]
0.08	[illegible]	[illegible]	[illegible]	1.12732431	0.01218057
0.08	[illegible]	[illegible]	1.56857589	[illegible]	[illegible]
[illegible]	4.	[illegible]	[illegible]	1.18187042	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	1.12732431	0.01483435
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.09	[illegible]	[illegible]	[illegible]	1.18187042	0.01127736
0.10	[illegible]	[illegible]	[illegible]	1.12732431	[illegible]
0.10	[illegible]	[illegible]	1.56857589	[illegible]	[illegible]
0.10	[illegible]	0.21516655	[illegible]	1.18187042	[illegible]

APL PROBLEM 840-222
SOLUTION OF VON MISES EQUATION (6)

V 0.35

L/D 1.00

T/D	N	RHO	H1	H2	P/E
0.01	2.	0.32151355	1.67202266	1.20498614	[illegible]
0.01	3.	0.21516656	1.36151441	1.18187043	[illegible]
0.01	4.	0.13360347	1.21916810	[illegible]	[illegible]
0.01	5.	0.08983016	1.14542645	[illegible]	0.00002576
0.01	6.	0.06414155	[illegible]	[illegible]	[illegible]
0.02	2.	[illegible]	1.67207256	1.20698614	[illegible]
0.02	3.	0.21516656	1.36151441	1.18187043	[illegible]
0.02	4.	0.13360847	1.21916810	[illegible]	[illegible]
0.02	5.	0.08983016	1.14542645	[illegible]	[illegible]
0.03	2.	[illegible]	1.67207266	[illegible]	[illegible]
0.03	3.	0.21516556	[illegible]	[illegible]	[illegible]
0.03	4.	0.13360847	[illegible]	[illegible]	[illegible]
0.03	5.	[illegible]	[illegible]	[illegible]	[illegible]
0.04	2.	[illegible]	[illegible]	[illegible]	[illegible]
0.04	3.	[illegible]	[illegible]	[illegible]	[illegible]
0.04	4.	[illegible]	[illegible]	[illegible]	[illegible]
0.04	5.	[illegible]	[illegible]	[illegible]	[illegible]
0.05	2.	[illegible]	[illegible]	[illegible]	[illegible]
0.05	3.	[illegible]	[illegible]	[illegible]	[illegible]
0.05	4.	[illegible]	[illegible]	[illegible]	[illegible]
0.06	2.	[illegible]	[illegible]	[illegible]	[illegible]
0.06	3.	[illegible]	[illegible]	[illegible]	[illegible]
0.06	4.	[illegible]	[illegible]	[illegible]	[illegible]
0.07	2.	[illegible]	1.67202266	[illegible]	[illegible]
0.07	3.	[illegible]	[illegible]	[illegible]	[illegible]
0.07	4.	[illegible]	[illegible]	[illegible]	[illegible]
0.08	2.	[illegible]	[illegible]	[illegible]	[illegible]
0.08	3.	[illegible]	[illegible]	1.18187043	[illegible]
0.08	4.	[illegible]	[illegible]	[illegible]	[illegible]
0.09	2.	[illegible]	[illegible]	[illegible]	[illegible]
0.09	3.	[illegible]	[illegible]	[illegible]	[illegible]
0.09	4.	[illegible]	[illegible]	[illegible]	[illegible]
0.10	2.	[illegible]	[illegible]	[illegible]	[illegible]
0.10	3.	[illegible]	[illegible]	[illegible]	0.00899277
0.10	4.	0.13360847	[illegible]	[illegible]	[illegible]

[illegible] PROBLEM 8[illegible]-222

[illegible] OF [illegible] EQUATION (8)

V 0.19

L/D 1.2[illegible]

T/T	N	RHO	U1	U2	P/S
0.01	2.	0.28304320	[illegible]	[illegible]	[illegible]
0.01	3.	0.1492[illegible]	1.[illegible]	1.14172219	0.00005773
0.01	4.	0.08[illegible]	1.14[illegible]	1.0[illegible]	0.000[illegible]
0.01	5.	0.0594[illegible]	1.0[illegible]	1.0[illegible]	0.0000[illegible]
0.01	6.	0.047071[illegible]	1.05[illegible]	1.04[illegible]	0.00002645
0.02	2.	0.2830[illegible]	1.43[illegible]	1.20517619	0.000[illegible]
0.02	3.	0.14[illegible]	1.24[illegible]	1.14[illegible]	0.000[illegible]
0.02	4.	0.08[illegible]	1.14[illegible]	1.09[illegible]0376	0.000[illegible]
0.02	5.	0.0594[illegible]	1.09[illegible]	1.0[illegible]	0.00014936
0.03	2.	0.28[illegible]	1.4[illegible]	1.206[illegible]	0.0013[illegible]
0.03	3.	0.1492[illegible]	1.2[illegible]	1.1417[illegible]	0.00034197
0.03	4.	0.08[illegible]	1.14[illegible]	1.0[illegible]0376	0.00034958
0.04	2.	0.2[illegible]	1.[illegible]	1.2061[illegible]	0.00[illegible]
0.04	3.	0.14[illegible]	1.24[illegible]	1.141[illegible]219	0.0006[illegible]
0.04	4.	0.08[illegible]	1.14[illegible]	1.09[illegible]	0.00079685
0.05	2.	0.28304[illegible]	1.4[illegible]	1.2061[illegible]	0.00[illegible]
0.05	3.	0.14[illegible]	1.[illegible]	1.141[illegible]	0.001[illegible]
0.05	4.	0.0[illegible]	1.1[illegible]	1.0[illegible]0376	0.0012[illegible]
0.06	2.	0.2[illegible]	1.4[illegible]	1.20[illegible]	0.00[illegible]
0.06	3.	0.14926[illegible]	1.24[illegible]	1.141[illegible]	0.00[illegible]
0.06	4.	0.0[illegible]	1.14[illegible]	1.09[illegible]	0.00251272
0.07	2.	0.28304320	1.4[illegible]	1.20[illegible]	0.004[illegible]
0.07	3.	0.149269[illegible]	1.24[illegible]	1.14172219	0.00276792
0.07	4.	0.08983016	1.14542644	1.09200376	0.0043[illegible]
0.08	2.	0.2[illegible]	1.[illegible]	1.20[illegible]	0.0062[illegible]
0.08	3.	0.1[illegible]	1.[illegible]	1.141[illegible]	0.00400[illegible]
0.08	4.	0.0898[illegible]	1.14[illegible]	1.09200376	0.00612952
0.09	2.	0.28[illegible]	1.4[illegible]	1.20[illegible]	0.00[illegible]75
0.09	3.	0.1492[illegible]	1.24[illegible]	1.14172219	0.00558274
0.09	4.	0.08983016	1.14542644	1.09300376	0.00570255
0.10	2.	0.28304320	1.4[illegible]	1.20[illegible]	0.00790211
0.10	3.	0.14926[illegible]	1.24[illegible]	1.141[illegible]	0.00754943
0.10	4.	0.08982016	1.14542644	1.09300376	0.01193422

T/D					
0.01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.02	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.04	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.06	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.07	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

[illegible]

[illegible]

0.1[illegible]

L/D 1.75

T/D	[illegible]	RHO	U1	U2	[illegible]
0.01	2.	0.1676511[illegible]	1.2778921	1.15446[illegible]	0.0001[illegible]
0.01	3.	0.08216481	1.13270913	1.0359306[illegible]	0.0000[illegible]
0.01	4.	0.04794104	1.07663044	1.05230887	0.00[illegible]
0.01	5.	0.03122111	1.04964954	1.0[illegible]72576	0.000[illegible]
0.02	2.	0.167[illegible]	1.27728321	1.15445[illegible]	0.000[illegible]
0.02	3.	0.08216481	1.13270913	1.0359305[illegible]	0.000[illegible]
0.02	4.	0.04794104	1.07663044	1.05230487	0.00[illegible]18
0.03	2.	0.16765196	1.27778931	1.15445[illegible]	0.00015715
0.03	3.	0.08216481	1.13270913	1.03593068	0.00021441
0.03	4.	0.04794104	1.07663044	1.05230887	[illegible]7
0.04	2.	0.16765196	1.27778921	1.15445928	0.00[illegible]
0.04	3.	0.08216481	1.13270913	1.0[illegible]5058	0.00046465
0.04	4.	0.04794104	1.07663044	1.05230887	0.00071[illegible]
0.05	2.	0.16765134	1.27778921	1.15445[illegible]79	0.00011774
0.05	3.	0.08216481	1.13270913	1.03593068	0.00036[illegible]
0.05	4.	0.04794104	1.076630[illegible]4	1.[illegible]30887	0.00139225
0.06	2.	0.1676[illegible]	1.27[illegible]	1.15445[illegible]	0.0015[illegible]
0.06	3.	0.08216481	1.13270913	1.03593068	0.0[illegible]
0.06	4.	0.04794[illegible]	1.07663044	1.05230887	0.00230967
0.07	2.	0.16765196	1.27778921	1.154[illegible]	0.[illegible]012046
0.07	3.	0.08216481	1.13270913	1.035[illegible]068	0.000[illegible]
0.08	2.	0.167[illegible]196	1.27778921	1.15445979	0.00[illegible]
0.08	3.	0.08216481	1.13270913	1.03593068	0.000[illegible]
0.09	[illegible]	0.167[illegible]	1.27778921	1.15445979	0.00[illegible]
0.09	3.	0.08216481	1.13270913	1.03593068	0.00[illegible]169
0.10	2.	0.16765196	1.27778921	1.15445979	0.0041[illegible]
0.10	3.	0.08216481	1.13270913	1.03593068	0.00639[illegible]

[illegible] 0.[illegible]

L/[illegible] 2.00

[illegible]	N	RHO	[illegible]1	[illegible]2	P/L	
0.01	2.	0.1326[illegible]	1.[illegible]	1.[illegible]	0.[illegible]	
0.01	3.	0.06414765	1.10303[illegible]	1.06[illegible]3[illegible]	0.00001505	
0.01	4.	0.0371[illegible]	1.0591[illegible]059	1.041[illegible]470	[illegible]	n = 4
0.01	5.	0.0247[illegible]7	1.03[illegible]17	1.[illegible]2699560	0.[illegible]1[illegible]	
0.02	2.	0.13[illegible]	1.21[illegible]810	1.[illegible]	0.[illegible]	
0.02	3.	0.0641[illegible]	1.10303552	1.0[illegible]309	0.[illegible]	n = 3
0.02	4.	[illegible].03712107	1.05914059	1.[illegible]	[illegible]	
0.03	2.	[illegible].13360847	1.21[illegible]810	1.129[illegible]782	0.[illegible]	
0.03	3.	0.0641[illegible]	1.10303552	1.[illegible]	[illegible]	n = 3
0.03	4.	0.03712107	1.0[illegible]14059	1.04[illegible]	0.000[illegible]	
0.0[illegible]	2.	0.133[illegible]	1.21[illegible]	1.12[illegible]	0.[illegible]	
0.04	3.	[illegible].06414765	1.10[illegible]	1.0[illegible]	0.[illegible]	n = 3
0.04	4.	0.03712107	1.0[illegible]	1.04[illegible]	[illegible]	
0.05	2.	0.13360847	1.21[illegible]810	1.1[illegible]	0.[illegible]	
0.05	3.	0.06414765	1.10303552	1.[illegible]	[illegible]	n = 3
0.05	4.	0.03712107	1.05914059	1.[illegible]	[illegible]	
0.06	2.	0.13360847	1.2[illegible]810	1.1[illegible]	[illegible]	n = [illegible]
0.06	3.	0.06414765	1.10303552	1.0[illegible]	0.[illegible]	
0.07	2.	0.13[illegible]	1.21[illegible]	1.[illegible]	[illegible]	n = 2
0.07	3.	0.06[illegible]14765	1.103[illegible]	1.0[illegible]	0.002[illegible]44[illegible]	
0.08	2.	0.13[illegible]	1.21[illegible]	1.1[illegible]	[illegible]	n = 2
0.08	3.	0.0641[illegible]	1.10[illegible]	1.0[illegible]	0.[illegible]	
0.09	2.	0.13360847	1.21[illegible]	1.[illegible]	0.00[illegible]	n = 2
0.09	3.	0.0[illegible]	1.[illegible]	1.0[illegible]	0.00[illegible]	
0.10	2.	[illegible].13360[illegible]	1.[illegible]	1.[illegible]	[illegible]	n = [illegible]
0.10	3.	0.0641[illegible]	1.10[illegible]	1.0[illegible]	0.00[illegible]	

LRL PROBLEM 840-222
SOLUTION OF VON MISES EQUATION (A)

M 1.15

L/D 1.25

T/D	N	RHO	U1	U2	P/F
0.01	2.	0.10861283	1.17683083	1.10959402	0.000[illegible]
0.01	3.	0.05137219	1.08220105	1.05582562	0.0000[illegible]
0.01	4.	0.02956125	1.04698606	1.03294060	0.[illegible]
0.01	5.	0.01912270	1.03025697	1.02155414	0.00001715
0.02	2.	0.10861283	1.17683083	1.10959402	0.000[illegible]
0.02	3.	0.05137219	1.08220105	1.05582562	0.000[illegible]5018
0.02	4.	0.02956125	1.04698606	1.03294060	0.000[illegible]
0.03	2.	0.10861283	1.17683083	1.10959402	0.00028[illegible]
0.03	3.	0.05137219	1.08220105	1.05582562	0.0001[illegible]
0.03	4.	0.02956125	1.04698606	1.03294060	0.0002[illegible]
0.04	2.	0.10861283	1.17683083	1.10959402	0.[illegible]
0.04	3.	0.05137219	1.08220105	1.05582562	0.00040[illegible]
0.04	4.	0.02956125	1.04698606	1.03294060	0.[illegible]
0.05	2.	0.10861283	1.17683083	1.10959402	0.0006746[illegible]
0.05	3.	0.05137219	1.08220105	1.05582562	0.0007[illegible]
0.06	2.	0.10861283	1.17683083	1.10959402	0.0009[illegible]
0.06	3.	0.05137219	1.08220105	1.05582562	0.001[illegible]766
0.07	2.	0.10861283	1.17683083	1.10959402	0.0013674[illegible]
0.07	3.	0.05137219	1.08220105	1.05582562	0.0020[illegible]
0.08	2.	0.10861283	1.17683083	1.10959402	0.0018555[illegible]
0.08	3.	0.05137219	1.08220105	1.05582562	0.0030[illegible]352
0.09	2.	0.10861283	1.17683083	1.10959402	0.0024[illegible]
0.09	3.	0.05137219	1.08220105	1.05582562	0.0043[illegible]10
0.10	2.	0.10861283	1.17683083	1.10959402	0.0032150[illegible]
0.10	3.	0.05137219	1.08220105	1.05582562	0.00592649

[illegible]

[illegible] OF [illegible] EQUATION ([illegible])

[illegible]	[illegible]
L/[illegible]	2.10

T/[illegible]	N	PWO	U1	U2	S/T
[illegible]	2.	0.09[illegible]2016	[illegible]	[illegible]	[illegible]
[illegible]	1.	[illegible]	1.05704700	1.04616779	0.00001014
[illegible]	4.	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	.	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	3.	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	.	[illegible]	[illegible]02820937	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	3.	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	4.	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	.	[illegible]	[illegible]	[illegible]09300376	[illegible]
[illegible]	1.	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	.	[illegible]	[illegible]	[illegible]	[illegible]00155216
[illegible]	3.	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	.	[illegible]	[illegible]	1.05200376	[illegible]
[illegible]	3.	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	.	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]04202163	[illegible]	1.04616770	[illegible]
[illegible]	.	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	.	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.10	.	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

[illegible]
[illegible] OF [illegible] SOLATION [illegible]

V 0.1[illegible]

L/[illegible] 2.75

T/L	N	RHO	[illegible]	[illegible]	[illegible]
0.01	2.	0.07[illegible]1521	1.121[illegible]	1.0[illegible]	0.[illegible]
0.01	3.	0.03[illegible]	[illegible]	1.[illegible]4[illegible]5	[illegible]976
0.01	4.	0.019[illegible]42[illegible]	1.0316[illegible]0[illegible]	1.0[illegible]	0.[illegible]
0.0[illegible]	[illegible]	0.07[illegible]41[illegible]	1.[illegible]	1.07[illegible]	[illegible]
0.0[illegible]	[illegible]	0.03458376	1.015[illegible]758	1.0[illegible]	[illegible]
0.0[illegible]	4.	0.01[illegible]	1.0316703[illegible]	1.0[illegible]	0.000[illegible]3545
0.0[illegible]	2.	[illegible]07[illegible]41[illegible]	1.1[illegible]5925	1.01[illegible]	[illegible]
0.0[illegible]	3.	0.03[illegible]	[illegible]758	1.0[illegible]	0.0001[illegible]
0.0[illegible]	4.	[illegible]01[illegible]23	1.0316703[illegible]	1.0[illegible]	[illegible]442
0.0[illegible]	[illegible]	0.07541[illegible]	1.121[illegible]	1.[illegible]132	0.000[illegible]32[illegible]
0.04	3.	[illegible]0345[illegible]	1.0[illegible]	1.0[illegible]	0.000[illegible]
0.0[illegible]	[illegible]	[illegible]07541[illegible]	1.121[illegible]	1.[illegible]	0.000[illegible]
0.0[illegible]	[illegible]	[illegible]	1.0[illegible]	1.0[illegible]	0.000[illegible]
0.06	[illegible]	0.07541557	1.121[illegible]	1.07[illegible]	0.00[illegible]
0.06	3.	0.03[illegible]98376	1.0[illegible]62753	1.0[illegible]4625	0.001[illegible]
0.07	2.	0.07[illegible]41[illegible]	[illegible]	1.[illegible]132	[illegible]001[illegible]521[illegible]
0.0[illegible]	3.	0.03[illegible]376	[illegible]	1.0[illegible]4625	0.00129871
0.08	2.	0.0754[illegible]557	1.121[illegible]	1.0[illegible]	[illegible]0014[illegible]
0.08	3.	0.03[illegible]376	1.0[illegible]758	1.03[illegible]625	[illegible]
0.0[illegible]	2.	0.07541557	1.121[illegible]	1.0[illegible]132	0.00[illegible]
0.0[illegible]	3.	[illegible]03[illegible]376	1.0[illegible]	1.0[illegible]625	0.00223166
0.10	[illegible]	0.07541557	[illegible]121[illegible]	1.0[illegible]132	0.00[illegible]
0.10	[illegible]	0.03[illegible]376	1.0[illegible]	1.0[illegible]4625	0.00[illegible]49507

ALL PROBL. 841-222
SOLUTION OF VON [illegible] EQUATION (16)

V 0.15

L/D 1.00

T/D	N	RHO	U1	U2	P/E
0.01	2.	0.06414265	1.10303551	1.06863308	[illegible]
0.01	3.	0.02956125	1.04698606	1.03294058	0.00000756
0.01	4.	0.01584608	1.02867126	1.019[illegible]	[illegible]1047
0.02	2.	0.06414265	1.10303551	1.06863308	0.[illegible]7074
0.02	3.	0.02956125	1.04698606	1.03[illegible]4058	[illegible]
0.02	4.	0.01584608	1.02867126	1.019[illegible]344[illegible]	0.000[illegible]8476
0.03	2.	0.06414265	1.10303551	1.06863308	[illegible]
0.03	3.	0.02956125	1.04698606	1.03294058	0.0001[illegible]
0.04	2.	0.06414265	1.10303551	1.06863308	0.[illegible]381
0.04	3.	0.02956125	1.04698606	1.03294058	0.000[illegible]841
0.05	2.	0.06414265	1.10303551	1.06863308	0.000[illegible]644
0.05	3.	0.02956125	1.04698606	1.03294058	0.00072523
0.06	2.	0.06414265	1.10303551	1.06863308	[illegible]
0.06	3.	0.02956125	1.04698606	1.03294058	0.00[illegible]24753
0.07	2.	0.06414265	1.10303551	1.06863308	0.00[illegible]04767
0.07	3.	0.02956125	1.04698606	1.03294058	0.00197561
0.08	2.	0.06414265	1.10303551	1.06863308	0.0013[illegible]
0.08	3.	0.02956125	1.04698606	1.03294058	0.00[illegible]94375
0.09	2.	0.06414265	1.10303551	1.06863308	[illegible]
0.09	3.	0.02956125	1.04698606	1.03294058	0.00[illegible]8477
0.10	2.	0.06414265	1.10303551	1.06863308	0.0035[illegible]
0.10	3.	0.02956125	1.04698606	[illegible]03294058	0.00573[illegible]