THE DETERMINATION OF THE RANK OF QUADRATIC FORMS USING LINEARLY INDEPENDENT LINEAR RESTRICTIONS ON LINEAR FORMS

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THESIS

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ABSTRACT

A procedure for determining the rank of a quadratic form is outlined by Cramér [1] and Hald [2]. Additional theoretical verification of this procedure is presented and the results are illustrated with applications in the analysis of variance.

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I. INTRODUCTION

A comparison of several sets of observations drawn from normally distributed populations can be performed by means of the statistical procedure known as the analysis of variance. The justification for the statistical procedure in the analysis of variance depends directly upon the application of Cochran's Theorem stated below.

Cochran's Theorem Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be distributed as $N_n(\underline{0}, \underline{I})$ and suppose

$$\sum_{i=1}^{n} \underline{X}_i^2 = \sum_{i=1}^{k} Q_i(\underline{X})$$

where Q_i is a quadratic form of rank n_i, $i=1,\ldots,k$. Then $Q_1(\underline{X}),\ldots,Q_k(\underline{X})$ are mutually independent and $Q_i(\underline{X})$ is distributed with n_i degrees of freedom, $i=1,\ldots,k$, if and only if

$$\sum_{i=1}^{k} n_i = n.$$

Cochran's Theorem formally relates (a) the degrees of freedom of a χ^2 random variable, $Q(\underline{X})$, with the rank of its associated observed variate, $Q(\underline{x})$, when $\underline{X}$ is $N_n(\underline{0}, \underline{I})$ and (b) the sum of the degrees of freedom of a set of χ^2 random variables with the dimension of the random vector $\underline{X}$; the independence among the χ^2 random variables is a consequence of this relationship [2].

Hence, if a statistician is collecting a random sample from a normally distributed population and if the sum of squares of the observations from the random sample, $\underline{x} = (x_1,\ldots,x_n)$, equals the sum of several quadratic forms in $\underline{x}$, say $Q_1,\ldots,Q_K$, he must show that the sum of the ranks of these quadratic forms is equal to the number of total observations <u>before</u> he can conclude independence of the Q_i's, $i=1,\ldots,K$, and assign χ^2 probability distributions to them. In the analysis of variance these quadratic forms, when divided by their ranks, represent independent estimates of an unknown variance σ^2 associated with a random variable vector $\underline{Y}$ which is $N_n(\underline{\mu},\sigma^2\underline{I})$. The $N_n(\underline{0},\underline{I})$ hypothesis of Cochran's Theorem becomes satisfied upon letting

$$\underline{X} = \frac{1}{\sigma}(\underline{Y}-\underline{\mu}).$$

Since the normalized quotient of independent χ^2 variables is an F-random variable, F-statistics, which are used to test hypothesis concerning $\underline{\mu}$, can be formed from the ratio of the quadratic forms in $\underline{x}$.

In turn then, it can be seen that the application of Cochran's Theorem depends upon the determination of the ranks of the quadratic forms in $\underline{x}$ in the relation

$$\sum_{i=1}^{n} x_i^2 = Q_1(\underline{x}) + Q_2(\underline{x}) + \ldots + Q_K(\underline{x}) .$$

Cramér [1] gives a procedure for determining the rank of quadratic forms as follows:

".... If Q may be written in the form $Q = L_1^2 + \ldots + L_k^2$ where the L_i are linear functions of $x_1, \ldots, x_n$ and if there are exactly m independent linear relations[1] between the L_i, then the rank of Q is k-m. ...."

Cramér calls this a proposition.

Likewise, Hald [2] states a similar procedure as a definition.

"The number of degrees of freedom for a set of variables, $L_1, \ldots, L_k$, will be defined as follows: Let the k variables $L_1, \ldots, L_k$ be linear functions of n stochastically independent variables, $x_1, \ldots, x_n$, which are assumed to be normally distributed with parameters (0,1). If m independent linear relations[1] exist between the k variables, $L_1, \ldots, L_k$, the number of degrees of freedom is k-m. The number of degrees of freedom for the sum of squares

$$Q = \sum_{i=1}^{k} L_i^2$$

is defined as the number of degrees of freedom for the k variables $L_1, \ldots, L_k$."

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[1] Both Cramér and Hald point out that the m linear relations between the L's are linearly independent. Formally, a linear relation has the form

$$c_1 L_1 + \ldots + c_k L_k = 0$$

where the c_i are constants and not all zero. When several linear relations of this form exist, they are called independent, if the corresponding vectors $\underline{c} = (c_1, \ldots, c_k)$ are linearly independent.

Cramér's proposition is stated, but not proved. Hald, on the other hand, precludes the necessity of proving the same proposition by calling it a definition. In either exposition no detailed theoretical verification is made of the procedure for determining the rank of quadratic forms. It is the intent of this thesis to (1) state and prove basic theorems which can be used to determine the rank of quadratic forms in the way presented by Cramér and Hald and (2) illustrate the use of these theorems in the analysis of variance.

II. MATHEMATICAL BACKGROUND

The theory which is to be developed concerning the rank of quadratic forms depends upon several mathematical results from matrix algebra [3,4]. This chapter outlines the pertinent mathematical definitions and theorems that are necessary to develop this theory.

A. MATRICES

A matrix $\underline{A}$ has elements denoted by a_{ij} where i refers to the row and j to the column. If $\underline{A}$ denotes the matrix, then $\underline{A}'$ denotes the transpose of $\underline{A}$, and $\underline{A}^{-1}$, the inverse of $\underline{A}$. The symbol $|\underline{A}|$ is used to denote the determinant of $\underline{A}$. The identity matrix is denoted by $\underline{I}$; and $\underline{0}$, the null matrix. The dimension of a matrix is the number of rows by the number of its columns, e.g., nxm. A matrix $\underline{A}$ of dimension nx1 is called a column vector[2]; its transpose $\underline{A}'$, a row vector. The rank of a matrix $\underline{A}$ is denoted by $r(\underline{A})$. Euclidian n-space is symbolized by E_n.

Given matrices $\underline{A} = (a_{ij})$ and $\underline{B} = (b_{ij})$ where the number of columns of $\underline{A}$ equals the number of rows of $\underline{B}$, the product $\underline{AB} = \underline{C} = (c_{ij})$ is defined as the matrix $\underline{C}$ with the pq^{th}

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[2]Column vectors will be indicated by round brackets, as $\underline{x} = (x_1,...,x_n)$; row vectors will be indicated by square brackets, as $\underline{x} = [x_1,...,x_n]$.

element equal to $\sum_{k=1}^{n} a_{pk}b_{kq}$. $\underline{A} + \underline{B} = \underline{C}$ gives $a_{ij} + b_{ij} = c_{ij}$ provided $\underline{A}$ and $\underline{B}$ have the same dimension. If k is a scalar and $\underline{A}$ a matrix, then $k\underline{A}$ means the matrix whose ij^{th} element is ka_{ij}. A <u>diagonal matrix</u> D is a square matrix whose off-diagonal elements are all zero; $D = (d_{ij})$ where $d_{ij} = 0$ if $i \neq j$. A matrix is called symmetric whenever $\underline{A} = \underline{A}'$. If $\underline{C}$ is an nxn matrix such that $\underline{CC}' = \underline{I}$, then $\underline{C}$ is said to be an orthogonal matrix, and $\underline{C}' = \underline{C}^{-1}$.

<u>Theorem 2.1</u> $r(\underline{AA}') = r(\underline{A}) = r(\underline{A}')$.

<u>Theorem 2.2</u>[3] Let $\underline{A}$ be nxn, symmetric and non-negative. Then there exists a non-singular matrix $\underline{C}$ such that $\underline{C}'\underline{AC} = (d_i\delta_{ij})$ where $d_i\{0,1\}$, i=1,...,n and the rank of $\underline{A}$ equals the number of non-zero d_i's.

<u>Theorem 2.3</u> If $\underline{A}$ is an mxn matrix of rank r, and if $\underline{B}$ is an nxq matrix such that $\underline{AB} = \underline{0}$, then the rank of $\underline{B}$ cannot exceed n-r.

<u>Theorem 2.4</u> Consider the sum of k matrices of the same dimension, $A_1 + A_2 + \ldots + A_k$, then

$$r\left(\sum_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} r(A_i) .$$

B. QUADRATIC FORMS

If $\underline{A}$ is an nxn matrix and $\underline{x} = (x_1,\ldots,x_n)$ is an nxl vector with i^{th} element x_i then

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[3]The symbol δ_{ij} is called the <u>Kronecker delta</u> and stands for 1, if i=j; 0, if i≠j.

$$Q(\underline{x}) = \underline{x}'\underline{A}\underline{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_i x_j$$

is called a quadratic form in $\underline{x}$. The quadratic form $\underline{x}'\underline{A}\underline{x}$ and its matrix $\underline{A}$ are called positive definite if whenever $x \neq 0$, $\underline{x}'\underline{A}\underline{x} > 0$; positive semi-definite whenever $\underline{x}'\underline{A}\underline{x} \geq 0$ for all $\underline{x} \neq \underline{0}$ and $\underline{x}'\underline{A}\underline{x} = \underline{0}$ for some $\underline{x} \neq \underline{0}$; and non-negative whenever $\underline{x}'\underline{A}\underline{x}$ (or $\underline{A}$) is either positive definite or positive semi-definite. (Any non-negative matrix $\underline{A}$ of a quadratic form $\underline{x}'\underline{A}\underline{x}$ is assumed symmetric for mathematical convenience and does not alter the value of $\underline{x}'\underline{A}\underline{x}$ since

$$\underline{x}'\underline{A}\underline{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_i x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{(a_{ij}+a_{ji})}{2} x_i x_j$$

$$= \underline{x}' \frac{(\underline{A}+\underline{A}')}{2} \underline{x}$$

where $\frac{A+A'}{2}$ is symmetric.) The rank of a quadratic form $\underline{x}'\underline{A}\underline{x}$ equals the rank of $\underline{A}$.

III. RANK OF QUADRATIC FORMS

This chapter (1) defines linearly independent linear forms and linearly independent linear restrictions on linear forms and (2) develops the theory of rank determination of quadratic forms.

A. LINEAR FORMS

Let $\underline{\lambda} \neq 0$ be an nx1 vector. If $\underline{x} = (x_1,\ldots,x_n)$ is in E_n, then the linear combination

$$L(\underline{x}) = \underline{\lambda}'\underline{x} = \sum_{i=1}^{n} \lambda_i x_i$$

of components of $\underline{x}$ with coefficients from $\underline{\lambda}$ is called a linear form with the associated vector $\underline{\lambda}$. Since $\underline{\lambda}'\underline{x}$ is a scalar, $\underline{\lambda}'\underline{x} = \underline{x}'\underline{\lambda}$. The square of a linear form is a quadratic form since

$$L^2(\underline{x}) = (\underline{\lambda}'\underline{x})(\underline{\lambda}'\underline{x}) = (\underline{x}'\underline{\lambda})(\underline{\lambda}'\underline{x}) = \underline{x}'(\underline{\lambda}\underline{\lambda}')\underline{x}.$$

Here, $\underline{\lambda}\underline{\lambda}'$ is nxn symmetric. The rank of $L^2(\underline{x}) = r(\underline{x}'\underline{\lambda}\underline{\lambda}'\underline{x}) = r(\underline{\lambda}\underline{\lambda}') = r(\underline{\lambda}) = 1$ by Theorem 2.1 since $\underline{\lambda} \neq 0$. So in general, $\underline{\lambda}\underline{\lambda}'$ is not positive definite. However, since $L^2(\underline{x}) = (\underline{\lambda}'\underline{x})^2$, $L^2(\underline{x})$ is always non-negative.

If $L_1, L_2, \ldots, L_k$ are linear forms in E_n, they are said to be <u>linearly independent</u> if their associated vectors $\underline{\lambda}_1, \underline{\lambda}_2, \ldots, \underline{\lambda}_k$ are linearly independent vectors in E_n, i.e., the rank of the matrix $\Lambda = (\underline{\lambda}_1, \underline{\lambda}_2, \ldots, \underline{\lambda}_k)$ equals k.

<u>Theorem 3.1</u> $Q(\underline{x})$ is a non-negative quadratic form in E_n of rank k if and only if there exists linearly independent

linear forms $L_1, L_2, \ldots, L_k$ such that

$$Q(\underline{x}) = \sum_{i=1}^{k} L_i^2(\underline{x}) \text{ for every } \underline{x} \varepsilon E_n.$$

Proof: Suppose $Q(\underline{x}) = \underline{x}'\underline{A}\underline{x}$ is non-negative with rank k and associated matrix $\underline{A}$. By Theorem 2.2, there exists a non-singular matrix $\underline{C}$ such that $\underline{C}'\underline{A}\underline{C} = (d_i\delta_{ij})$, $d_i \varepsilon \{0,1\}$, $i=1,\ldots,n$. Define $L_i: E_n \to E_1$ by $L_i(\underline{x}) = \underline{c}_i'\underline{x}$ where $\underline{c}_i'$ is the i^{th} row of $\underline{C}^{-1}$, $i=1,\ldots,k$. Then, L_i is a linear form and letting $\underline{z} = (L_1(\underline{x}),\ldots,L_n(\underline{x})) = (\underline{c}_1'\underline{x},\ldots,\underline{c}_n'\underline{x})$, it follows that $\underline{z} = \underline{C}^{-1}\underline{x}$ or $\underline{x} = \underline{C}\underline{z}$. In that case,

$$\underline{x}'\underline{A}\underline{x} = \underline{z}'\underline{C}'\underline{A}\underline{C}\underline{z} = \underline{z}(d_i\delta_{ij})\underline{z} = \sum_{i=1}^{n} d_i z_i^2 = \sum_{i=1}^{n} d_i L_i^2(\underline{x}).$$

But, by Theorem 2.2, k is precisely the number of non-zero d_i's. Deleting those d_i such that $d_i = 0$ and renumbering subscripts, if necessary, yields

$$Q(\underline{x}) = \underline{x}'\underline{A}\underline{x} = \sum_{i=1}^{k} L_i^2(\underline{x}).$$

Since $\underline{C}^{-1}$ is non-singular, the vectors $\underline{c}_1',\ldots,\underline{c}_k'$ are linearly independent and so then are the linear forms by definition.

Conversely, suppose $L_1,\ldots,L_k$ are linearly independent linear forms. Then, by definition, there exists k linearly independent non-zero vectors $\underline{\lambda}_1,\ldots,\underline{\lambda}_k$ where $\underline{\lambda}_i$ is associated with L_i, $i=1,\ldots,k$, that is,

$$L_i(\underline{x}) = \sum_{j=1}^{n} \lambda_{ij} x_i$$

for every $\underline{x} = (x_1, \ldots, x_n)$ in E_n. Letting $\underline{\lambda}_i = (\underline{\lambda}_{i1}, \ldots, \underline{\lambda}_{in})$, i-1,...,k the matrix

$$\Lambda = \begin{bmatrix} \underline{\lambda}_{11} & \underline{\lambda}_{21} & \cdots & \underline{\lambda}_{k1} \\ \underline{\lambda}_{12} & \underline{\lambda}_{22} & \cdots & \underline{\lambda}_{k2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \underline{\lambda}_{1n} & \underline{\lambda}_{2n} & & \underline{\lambda}_{kn} \end{bmatrix}_{n\times k}$$

necessarily has rank k because the rows are linearly independent, i.e., $r(\Lambda) = k$.

Consider

$$Q(\underline{x}) = \sum_{i=1}^{k} L_i^2(\underline{x}).$$

Now

$$L_i^2(\underline{x}) = L_i(\underline{x})L_i(\underline{x}) = (\underline{\lambda}_i'x)'(\underline{\lambda}_i'\underline{x}) = \underline{x}'(\underline{\lambda}_i\underline{\lambda}_i')\underline{x}$$

hence

$$Q(\underline{x}) = \sum_{i=1}^{k} \underline{x}'(\underline{\lambda}_i\underline{\lambda}_i')\underline{x} = \underline{x}'(\sum_{i=1}^{k} \underline{\lambda}_i\underline{\lambda}_i')\underline{x}$$

$$= \underline{x}' \sum_{i=1}^{k} \begin{bmatrix} \lambda_{i1}^2 & \lambda_{i1}\lambda_{i2} & \cdots & \lambda_{i1}\lambda_{in} \\ \lambda_{i2}\lambda_{i1} & \lambda_{i2}^2 & \cdots & \lambda_{i2}\lambda_{in} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{in}\lambda_{i1} & \lambda_{in}\lambda_{i2} & & \lambda_{in}^2 \end{bmatrix}_{n\times n} \underline{x}$$

$$Q(\underset{\sim}{x}) = \underset{\sim}{x}' \begin{bmatrix} \sum_{i=1}^{k} \lambda_{i1}^2 & \sum_{i=1}^{k} \lambda_{i1}\lambda_{i2} & \cdots & \sum_{i=1}^{k} \lambda_{i1}\lambda_{in} \\ \sum_{i=1}^{k} \lambda_{i2}\lambda_{i1} & \sum_{i=1}^{k} \lambda_{i2}^2 & \cdots & \sum_{i=1}^{k} \lambda_{i2}\lambda_{in} \\ \vdots & \vdots & \ddots & \\ \sum_{i=1}^{k} \lambda_{in}\lambda_{i1} & \sum_{i=1}^{k} \lambda_{in}\lambda_{i2} & \cdots & \sum_{i=1}^{k} \lambda_{in}^2 \end{bmatrix}_{n\times n} \underset{\sim}{x}$$

$$= \underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \quad \text{where} \quad \underset{\sim}{A} = \sum_{i=1}^{k} \underset{\sim}{\lambda}_i \underset{\sim}{\lambda}_i' .$$

Notice that $\underset{\sim}{A}$ is symmetric. Thus, $Q(\underset{\sim}{x})$ is a quadratic form in $\underset{\sim}{x}$ and it must follow that the rank of $Q(\underset{\sim}{x})$ equals $r(\underset{\sim}{A})$. Observing that $\underset{\sim}{A}$ is, in fact, the product of Λ and Λ',

$$\underset{\sim}{A} = \Lambda\Lambda',$$

and, applying Theorem 2.1, $r(A) = r(\Lambda\Lambda') = r(\Lambda) = k$. Consequently, $r(Q(\underset{\sim}{x})) = r(A) = k$, and since $L_i^2(\underset{\sim}{x})$ is non-negative so also then is

$$\sum_{i=1}^{k} L_i^2(\underset{\sim}{x}) = Q(\underset{\sim}{x}). \quad \text{Q.E.D.}$$

B. LINEAR RESTRICTIONS

A vector $\underset{\sim}{\alpha} = (\underset{\sim}{\alpha}_1,\ldots,\underset{\sim}{\alpha}_k)$ in E_k is called a <u>linear restriction</u> on the set of linear forms, $\{L_1,\ldots,L_k\}$ if $\underset{\sim}{\alpha} \neq 0$ and

$$\underset{\sim}{\alpha}'\underset{\sim}{L}(\underset{\sim}{x}) = \sum_{i=1}^{k} \alpha_i L_i(\underset{\sim}{x}) = 0$$

for every $x \varepsilon E_n$ where $\underset{\sim}{L}(\underset{\sim}{x}) = (L_1,\ldots,L_k)$.

As will be seen, a linear restriction has the effect of reducing the rank of a quadratic form by one. Another linear restriction will further reduce the rank, only if it and the first are linearly independent. In general, it is important to find the total number of <u>linearly independent linear restrictions</u> on a given set of linear forms, $\{L_1, \ldots, L_k\}$, in order to determine the rank of the quadratic form,

$$Q(\underline{x}) = \sum_{i=1}^{k} L_i^2(\underline{x}).$$

<u>Theorem 3.2</u> If there are exactly m linearly independent linear restrictions on the linear forms, $L_1, \ldots, L_k$, $(m<k)$ then the rank of

$$Q(x) = \sum_{i=1}^{k} L_i^2(\underline{x}) \qquad \underline{x} \varepsilon E_n$$

is k-m.

<u>Proof</u>: Suppose $\underline{\alpha}_1, \ldots, \underline{\alpha}_m$ are linearly independent linear restrictions on the linear forms, $L_1, \ldots, L_k$, where $\underline{\alpha}_i = (\alpha_{i1}, \ldots, \alpha_{ik})$ for each i=1,...,m. Then

$$\sum_{j=1}^{k} \alpha_{ij} L_j(\underline{x}) = \sum_{j=1}^{k} \alpha_{ij} \underline{\lambda}_j' \underline{x} = 0$$

for all $\underline{x}$ in E_n, i=1,...,m, where $\underline{\lambda}_j$ is associated with the linear form $L_j(\underline{x})$ for each j=1,...,k. Recall that

$$Q(\underline{x}) = \sum_{i=1}^{k} L_i^2(\underline{x}) = \underline{x}'(\Lambda\Lambda')\underline{x}$$

where

$$\underline{\Lambda}' = \begin{bmatrix} \lambda_{11} & \cdot & \cdot & \cdot & \lambda_{1n} \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \lambda_{k1} & \cdot & \cdot & \cdot & \lambda_{kn} \end{bmatrix} = \begin{bmatrix} \underline{\lambda}_1' \\ \cdot \\ \cdot \\ \cdot \\ \underline{\lambda}_k' \end{bmatrix} .$$

Let $\underline{A} = \begin{bmatrix} \underline{\alpha}_1' \\ \cdot \\ \cdot \\ \cdot \\ \underline{\alpha}_m' \end{bmatrix}$ then $\underline{A}\underline{\Lambda}'\underline{x} = \begin{bmatrix} \sum_{j=1}^{k} \alpha_{1j}\underline{\lambda}_j\underline{x} \\ \cdot \\ \cdot \\ \cdot \\ \sum_{j=1}^{k} \alpha_{mj}\underline{\lambda}_j\underline{x} \end{bmatrix} = \underline{0}$

for all $\underline{x}$ in E_n. Hence $\underline{A}\underline{\Lambda}' = \underline{0}$. $r(\underline{A}) = m$ by hypothesis, and by Theorem 2.3, $r(\Lambda') \leq k-m$.

Suppose m is the maximum number of linearly independent linear restrictions on the linear forms, $L_1,\ldots,L_k$, and $r(\Lambda') < k-m$. Since $r(\underline{A}) = m$, there exists an mxm sub-matrix $\underline{A}^*$ of $\underline{A}$ such that $|\underline{A}^*| \neq 0$. Without loss of generality, let

$$\underline{A}^* = \begin{bmatrix} \alpha_{11} & \cdot & \cdot & \cdot & \alpha_{1m} \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \alpha_{m1} & \cdot & \cdot & \cdot & \alpha_{mm} \end{bmatrix} .$$

Since $r(\Lambda') < k-m$, the last k-m rows of Λ', $\underline{\lambda}_{m+1}',\ldots,\underline{\lambda}_k'$, among others, are linearly dependent. Then there exist scalars $b_1,\ldots,b_k$, not all zero (say $b_k \neq 0$), such that

$$b_1\, \underline{\lambda}_{m+1}' + \cdot\cdot\cdot + b_k\, \underline{\lambda}_k' = \underline{0}.$$

Let $\underline{\beta} = (0,\ldots,0,b_{m+1},\ldots,b_k)$. Then certainly,

$$\sum_{i=1}^{k} \beta_i L_i(\underline{x}) = \sum_{m+1}^{k} b_i \; \underline{\lambda}_i' \underline{x} = (\sum_{m+1}^{k} b_i \; \underline{\lambda}_i')\underline{x} = \underline{0}\underline{x} = 0$$

so that $\underline{\alpha}_1,\ldots,\underline{\alpha}_m,\underline{\beta}$ are linearly independent. For suppose

$$c_1 \; \underline{\alpha}_1 + \ldots c_m \; \underline{\alpha}_m = c_{m+1}\underline{\beta} = \underline{0}.$$

Then

$$\begin{array}{lllll}
c_1\alpha_{11} & + \ldots + c_m\alpha_{m1} & & = 0 \\
\vdots & \qquad\vdots & & \\
c_1\alpha_{1m} & + \ldots + c_m\alpha_{mm} & & = 0 \\
c_1\alpha_{1,m+1} & + \ldots + c_m\alpha_{m,m+1} & + c_{m+1}\beta_{m+1} & = 0 \\
\vdots & \qquad\vdots & \qquad\vdots & \\
c_1\alpha_{1k} & + \ldots + c_m\alpha_{mk} & + c_{m+1}\beta_k & = 0
\end{array}$$

or letting $\underline{c} = (c_1,\ldots,c_m)$, the first m equations can be written $\underline{A}^* \; \underline{c} = \underline{0}$. Since A^* is non-singular, $\underline{c} = \underline{0}$, that is, $c_1 = \ldots = c_m = 0$, and the last k-m equations become

$$\begin{array}{l}
c_{m+1}\beta_{m+1} = 0 \\
\vdots \\
c_{m+1}\beta_k \quad = 0.
\end{array}$$

But $\beta_k \neq 0$, hence $c_{m+1} = 0$. Consequently, $\underline{\alpha}_1,\ldots,\underline{\alpha}_m,\underline{\beta}$ are linearly independent contradicting the maximality of m. Hence $r(\Lambda') = k-m$ and since $r(\Lambda') = r(\Lambda\Lambda')$ by Theorem 2.1 $r(Q(\underline{x})) = k-m$. Q.E.D.

Corollary 3.3 If there is at least m linearly independent linear restrictions on the linear forms, $L_1,\ldots,L_k$, $(m<k)$ then the rank of

$$Q(\underline{x}) = \sum_{i=1}^{k} L_i^2(\underline{x})$$

for all $\underline{x}$ in E_n is less than or equal to $k-m$.

Theorem 3.4 The linear forms $L_1,\ldots,L_k$ are linearly independent if and only if there are no linear restrictions on $L_1,\ldots,L_k$.

Proof: Suppose $L_1,\ldots,L_k$ are linearly independent and there exists at least one linear restriction, $\underline{\alpha} = (\alpha_1,\ldots,\underline{\alpha}_k)$ on $L_1,\ldots,L_k$ such that $\alpha_1 L_1 + \ldots + \alpha_k L_k = 0$. Hence, one of the linear forms, say L_1, can be written as a linear combination of the remaining k-1 linear forms

$$L_1 = -\frac{\alpha_2}{\alpha_1}L_2 - \ldots - \frac{\alpha_k}{\alpha_1}L_k \qquad (\alpha_1 \neq 0)$$

$$= \beta_2 L_2 + \ldots + \beta_k L_k$$

where $\beta_j = -\frac{\alpha_j}{\alpha_1}$, $j=2,\ldots,k$.

Consider the vectors, $\underline{\lambda}_1,\ldots,\underline{\lambda}_k$, associated with the linear forms, $L_1,\ldots,L_k$, and the matrix formed by these vectors,

$$\Lambda' = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{k1} & \cdots & \lambda_{kn} \end{bmatrix} = \begin{bmatrix} \underline{\lambda}_1' \\ \vdots \\ \underline{\lambda}_k' \end{bmatrix}$$

where by hypothesis, $r(\Lambda') = k$. Now

$$L_1 = \beta_2\underline{\lambda}_2\underline{x} + \ldots + \beta_k\underline{\lambda}_k\underline{x}$$

$$= (\sum_{i=2}^{k} \beta_i\lambda_{i1})x_1 + \ldots + (\sum_{i=2}^{k} \beta_i\lambda_{in})x_n$$

which implies

$$\underline{\lambda}_1 = \begin{bmatrix} \sum_{i=2}^{k} \beta_i\lambda_{i1} \\ \cdot \\ \cdot \\ \cdot \\ \sum_{i=2}^{k} \beta_i\lambda_i \end{bmatrix} = [\underline{\lambda}_2,\ldots,\underline{\lambda}_k]\underline{\beta}$$

where $\underline{\beta} = (\beta_2,\ldots,\beta_k)$. Hence, $\underline{\lambda}_1$ is a linear combination of the remaining k-1 rows of Λ', i.e., $r(\Lambda') \leq k-1$ which contradicts the linear independence of the linear forms $L_1,\ldots,L_k$. Therefore, there are no linear restrictions on $L_1,\ldots,L_k$.

Conversely, suppose there are no linear restrictions on the linear forms, $L_1,\ldots,L_k$. Letting $m = 0$ in Theorem 3.2, the rank of

$$Q(\underline{x}) = \sum_{i=1}^{k} L_i^2(\underline{x})$$

for all $\underline{x}$ in E_n equals $k-m = k$. Since $L_i^2(\underline{x})$ is non-negative, so then is $Q(\underline{x})$. Upon applying Theorem 3.1 the linear forms, $L_1,\ldots,L_k$, are seen to be linear independent. Q.E.D.

IV. APPLICATIONS TO ANALYSIS OF VARIANCE

Knowledge of the rank of a quadratic form is essential in testing for the equality of means of k normal populations having the same variance. This statistical method is called the analysis of variance. This chapter demonstrates the use of linearly independent linear restrictions of linear forms to determine the rank of quadratic forms.

A. ONE-WAY CLASSIFICATION

Suppose that there exists k groups of independent observations

$$y_{11},\ldots,y_{1n_1},y_{21},\ldots,y_{2n_2},\ldots,y_{k1},\ldots,y_{kn_k}$$

from normally distributed populations with means $\mu_1,\ldots,\mu_k$ all with the same variance σ^2. Thus the model is

$$y_{ij} = \mu_i + \varepsilon_{ij} \quad i=1,\ldots,k \quad j=1,\ldots,n_i$$

where μ_i are fixed constants and the ε_{ij} are independent random normal deviates with zero mean and variance σ^2.

Let $x_{ij} = \frac{1}{\sigma}(y_{ij}-\mu_i)$ Consider the identity

$$x_{ij} = (x_{ij}-\bar{x}_i) + (\bar{x}_i-\bar{x}) + \bar{x} \tag{1}$$

where

$$\bar{x}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} x_{ij} \quad i=1,\ldots,k$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{k} \bar{x}_i .$$

Squaring both sides of equation (1) and summing over i and j, $i=1,\ldots,k$, $j=1,\ldots,n_i$ yields the identity

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij}^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij}-\bar{x}_i)^2 + \sum_{i=1}^{k} n_k(\bar{x}_i-\bar{x})^2 + n\bar{x}$$

$$Q = Q_1 + Q_2 + Q_3.$$

Since y_{ij} is $N(\mu_i,\sigma^2)$ then x_{ij} is $N(0,1)$ and

$$Q = \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij}^2$$

is $\chi^2(n)$ and has rank

$$n = \sum_{i=1}^{k} n_i.$$

The ranks of Q_1, Q_2, and Q_3, in turn, can be found by finding the number of linearly independent linear restrictions associated with each quadratic form and applying Corollary 3.3 and Theorem 2.4.

Consider first

$$Q_1(\underline{x}) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij}-\bar{x}_i)^2.$$

where $\underline{x} = (x_{11},\ldots,x_{1n_1},\ldots,x_{k1},\ldots,x_{kn_k})$. Let $L_{ij}(\underline{x}) = (x_{ij}-\bar{x}_i) = \underline{\lambda}_{ij}'\underline{x}$ be a linear form in $\underline{x}$ where $\underline{\lambda}_{ij}$ is an nx1 vector

$$\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \underline{0} \\ -\frac{1}{n_i} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad \sum_{u=1}^{i=1} n_u \text{ rows}$$

$$\underline{\lambda}_{ij} = \begin{bmatrix} \vdots \\ -\frac{1}{n_i} \\ 1-\frac{1}{n_i} \\ -\frac{1}{n_i} \\ \vdots \\ -\frac{1}{n_i} \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1} \begin{matrix} \\ \\ \text{----}(\sum_{u=1}^{i=1} n_u+j)^{th} \text{ row} \\ \\ \\ \text{----}(\sum_{u=1}^{i=1} n_u)^{th} \text{ row} \\ \\ \left.\right\} n-(\sum_{u=1}^{i=1} n_u) \text{ rows} \\ \\ \end{matrix}$$

For each i=1.,,,.k there exists an nxl vector $\underline{\alpha}_1$ such that

$$\underline{\alpha}_i'\underline{L}(\underline{x}) = \sum_{j=1}^{n_i} \alpha_{ij}L_{ij}(\underline{x})$$

$$= \sum_{j=1}^{n_i} (x_{ij}-\bar{x}_i) = 0$$

where

$$\underline{\alpha}_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline 1 \\ \vdots \\ 1 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1} \begin{matrix} \left.\right\} \sum_{u=1}^{i=1} n_u \text{ rows} \\ \\ \\ \text{----}(\sum_{u=1}^{i=1} n_u)^{th} \text{ row} \\ \\ \\ \end{matrix}$$

and

$$\underset{\sim}{L}(\underset{\sim}{x}) = \begin{bmatrix} x_{11}-\bar{x}_1 \\ \vdots \\ x_{1n_1}-\bar{x}_1 \\ \hline \vdots \\ \hline x_{k1}-\bar{x}_k \\ \vdots \\ x_{kn_k}-\bar{x}_k \end{bmatrix}_{n\times 1} .$$

Forming the matrix $\underset{\sim}{A} = (\underset{\sim}{\alpha}_1,\ldots,\underset{\sim}{\alpha}_k)$

$$\underset{\sim}{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \\ \hline 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \cdots & 0 \\ \hline & & \vdots & \\ \hline 0 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{matrix} \left.\right\} n_1 \text{ rows} \\ \\ \left.\right\} n_2 \text{ rows} \\ \\ \left.\right\} n_k \text{ rows} \\ n\times k \end{matrix}$$

it can be seen that the columns of $\underset{\sim}{A}$ are linearly independent i.e., $r(\underset{\sim}{A}) = k$; hence by definition, the set of k vectors $\{\underset{\sim}{\alpha}_1,\ldots,\underset{\sim}{\alpha}_k\}$ are linearly independent linear restrictions on the linear forms, $L_{11},\ldots,L_{1n_1},\ldots,L_{k1},\ldots,L_{kn_k}$. Applying Corollary 3.3, $r(Q_1) \leq n-k$.

In a similar manner for

$$Q_2(\bar{\underline{x}}) = \sum_{i=1}^{k} n_i(\bar{x}_i - \bar{x})^2$$

where $\bar{\underline{x}} = (\bar{x}_1, \ldots, \bar{x}_k)$ let

$$L_i(\bar{\underline{x}}) = \sqrt{n_i}\ (\bar{x}_i - \bar{x}) \quad (i=1,\ldots,k)$$

$$= \underline{\lambda}_i' \bar{\underline{x}}$$

be a linear form in $\bar{\underline{x}}$ where $\underline{\lambda}_i$ is a kxl vector

$$\underline{\lambda}_i = \begin{bmatrix} \sqrt{n_i}\ (-\frac{1}{k}) \\ \vdots \\ \sqrt{n_i}\ (-\frac{1}{k}) \\ \sqrt{n_i}\ (1 - \frac{1}{k}) \\ \sqrt{n_i}\ (-\frac{1}{k}) \\ \vdots \\ \sqrt{n_i}\ (-\frac{1}{k}) \end{bmatrix}_{kxl} \begin{matrix} \\ \\ \\ \text{----}i^{th} \text{ row} \\ \\ \\ \\ \end{matrix}$$

Likewise, there exists one kxl vector $\underline{\alpha}$ such that

$$\underline{\alpha}'\underline{L}(\bar{\underline{x}}) = \sum_{i=1}^{k} \alpha_i L_i(\bar{\underline{x}})$$

$$= \sum_{i=1}^{k} \alpha_i \{\sqrt{n_i}(\bar{x}_i - \bar{x})\}$$

$$= \sum_{i=1}^{k} \sqrt{n_i}(\bar{x}_i - \bar{x})$$

$$= 0$$

where

$$\underline{\alpha} = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}_{k\times 1} \quad \text{and } \underline{L}(\bar{\underline{x}}) = \begin{bmatrix} \sqrt{n_1}(\bar{x}_1-\bar{x}) \\ \cdot \\ \cdot \\ \cdot \\ \sqrt{n_k}(\bar{x}_k-\bar{x}) \end{bmatrix}_{k\times 1} .$$

Here $\underline{\alpha}$ represents at least one linear restriction on the linear forms, $L_1(\bar{\underline{x}}),\ldots,L_k(\bar{\underline{x}})$, and by Corollary 3.3 $r(Q_2) \leq k-1$.

Lastly, $Q_3(\bar{\underline{x}}) = n\bar{\bar{x}}^2$ is the square of a single linear form, $L(\bar{x}) = \sqrt{n}\bar{\bar{x}}$, and has rank, $r(Q_3) = 1$.

In summary,

$$\begin{aligned} r(Q) &= n \\ r(Q_1) &\leq n-k \\ r(Q_2) &\leq k-1 \\ r(Q_3) &= 1 \end{aligned}$$

where $Q = Q_1 + Q_2 + Q_3$. Hence

$$r(Q_1) + r(Q_2) + r(Q_3) \leq n - k + k - 1 + 1 = n$$

but by Theorem 2.4

$$r(Q) = n \leq r(Q_1) + r(Q_2) + r(Q_3).$$

Consequently,

$$r(Q_1) + r(Q_2) + r(Q_3) = r(Q)$$

and

$$\begin{aligned} r(Q_1) &= n-k \\ r(Q_2) &= k-1 \\ r(Q_3) &= 1 \end{aligned} .$$

From Cochran's Theorem the quadratic forms Q_1, Q_2, and Q_3 are linearly independent and are χ^2 distributed with n-k, k-1, and 1 degrees of freedom, respectively.

The foregoing procedure for finding at least a lower bound for the rank of a quadratic form

$$Q(\underline{x}) = \sum_{i=1}^{k} L_i^2(\underline{x})$$

endeavors to construct a matrix of known linear restrictions,

$$A = [\underline{\alpha}_1, \ldots, \underline{\alpha}_k]$$

whose rank is necessarily the number of linearly independent linear restrictions on the linear forms $L_1, \ldots, L_k$. It could be shown that if the linear forms $L_1, \ldots, L_k$ are linearly independent then Theorem 3.1 can be applied to give the rank exactly. Such a procedure entails finding the rank of the matrix of coefficient vectors, $\Lambda = [\underline{\lambda}_1, \ldots, \underline{\lambda}_k]$ associated with the linear forms $L_1, \ldots, L_k$. In general, the determination of the rank of Λ is complicated by the odd construction of the $\underline{\lambda}_i$'s. On the other hand the matrix of the known linear restrictions, A, contains only elements equal to zeros or ones. Hence, the easier method of finding information concerning the rank of a quadratic form is to look for the linear restrictions and apply Theorem 2.2 or Corollary 3.3.

In the next section two-way classification of analysis of variance is investigated and it will be seen that although the matrix of known linear restrictions becomes larger determining its rank remains relatively easy.

B. TWO-CLASSIFICATION

In the one-way classification of the analysis of variance k groups of independent normally distributed observations are recorded. The k groups represent k different variations of a particular variable or factor which is needed to yield a desired result, e.g., the observation itself. It is the purpose of the analysis of variance to ascertain if there is any significant differences in the results of any of these variations. The measurement of crop yield using different types of fertilizer is an example of such a grouping or classification.

In a two-way classification of the analysis of variance the results of an experiment are classified according to variations of two influencing variables. In the crop growth example, crop yield can be classified, not only by different types of fertilizer, but also by various soil compositions.

Suppose there exists r variations of one classification and c variations of a second classification. Consider n observations taken from each of rc possible combinations of the two classifications. In tabular form

i \ j	1	. . .	c
1	y_{11}	. . .	y_{1c}
.	.	.	.
.	.	.	.
.	.	.	.
r	y_{r1}	. . .	y_{rc}

where $y = (y_{ij1},\ldots,y_{ijn})$ is an nx1 vector. It is assumed that the rc sets of n observations are random samples from rc separate populations, each normally distributed about mean μ_{ij} but all with the same variance σ^2. The model is

$$y_{ijv} = \mu_{ij} + \varepsilon_{ijv} \qquad \begin{array}{l} i=1,\ldots,r \\ j=2,\ldots,c \\ v=1,\ldots,n \end{array}$$

where ε_{ijv} is $N(0,\sigma^2)$. Let

$$x_{ijv} = \frac{y_{ijv}-\mu_{ij}}{\sigma}$$

and consider the identity

$$x_{ijv} = (x_{ijv}-\bar{x}_{ij}) + (\bar{x}_{ij}-\bar{x}_{i\cdot}-\bar{x}_{\cdot j}+\bar{x}) + (\bar{x}_{i\cdot}-\bar{x}) + (\bar{x}_{\cdot j}-\bar{x}) + \bar{x} \qquad (2)$$

where

$$\bar{x}_{ij} = \frac{1}{n}\sum_{v=1}^{n} x_{ijv}$$

$$\bar{x}_{i\cdot} = \frac{1}{cn}\sum_{j=1}^{c}\sum_{v=1}^{n} x_{ijv} = \frac{1}{c}\sum_{j=1}^{c}\bar{x}_{ij}$$

$$\bar{x}_{\cdot j} = \frac{1}{rn}\sum_{i=1}^{r}\sum_{v=1}^{n} x_{ijv} = \frac{1}{r}\sum_{i=1}^{r}\bar{x}_{ij}$$

$$\bar{x} = \frac{1}{rcn}\sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{v=1}^{n} x_{ijv} = \frac{1}{r}\sum_{i=1}^{r}\bar{x}_{i\cdot} = \frac{1}{c}\sum_{j=1}^{c}\bar{x}_{\cdot j} \quad .$$

Squaring both sides of equation (2) and summing over i, j and v, $i=1,\ldots,r$, $j=1,\ldots,c$, $v=1,\ldots,n$ yields the identity

$$\sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{v=1}^{n} x_{ijv}^2 = \sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{v=1}^{n} (x_{ijv}-\bar{x}_{ij})^2$$

$$+ n\sum_{i=1}^{r}\sum_{j=1}^{c} (\bar{x}_{ij}-\bar{x}_{i.}-\bar{x}_{.j}+\bar{x})^2 + nc\sum_{i=1}^{r} (\bar{x}_{i.}-\bar{x})^2$$

$$+ nr\sum_{j=1}^{c} (\bar{x}_{.j}-\bar{x})^2 + rcn\bar{x}^2$$

$$Q = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 .$$

The rank of Q is rcn since it is χ^2 with rcn degrees of freedom. The ranks of $Q_1,\ldots,Q_5$ can be determined by finding the number of linearly independent linear restrictions associated with each quadratic form and applying Corollary 3.3 and Theorem 2.4.

Following the procedure of the previous section consider first

$$Q_1(\underline{x}) = \sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{v=1}^{n} (x_{ijv}-\bar{x}_{ij})^2$$

where

$$\underline{x} = \begin{bmatrix} \underline{x}_{11} \\ \vdots \\ \underline{x}_{1c} \\ \hline \vdots \\ \hline \underline{x}_{r1} \\ \vdots \\ \underline{x}_{rc} \end{bmatrix}_{rcn\times 1} \quad \text{with } \underline{x}_{ij} = \begin{bmatrix} x_{ij1} \\ \vdots \\ x_{ijn} \\ \\ x_{ijn} \end{bmatrix}_{n\times 1} \quad \begin{matrix} i=1,\ldots,r \\ j=1,\ldots,c \end{matrix}$$

Let $L_{ijv}(\underline{x}) = (x_{ijv}-\bar{x}_{ij})$ $\quad i=1,\ldots,r \quad j=1,\ldots,c \quad v=1,\ldots,n$ be a linear form in $\underline{x}$

$= \underline{\lambda}'_{ijv}\underline{x}$

where $\underline{\lambda}_{ijv}$ is a rcnx1 vector

$$\underline{\lambda}_{ijv} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\frac{1}{n} \\ \vdots \\ -\frac{1}{n} \\ 1-\frac{1}{n} \\ -\frac{1}{n} \\ \vdots \\ -\frac{1}{n} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\right\} \{(i-1)c+j\}n \text{ rows} \\ \\ \text{------}\{(i-1)c+j\}n+v^{\text{th}} \text{ row} \\ \\ \left.\right\} rcn-\{(i-1)c+j+1\}n \text{ rows} \\ rcnx1 \end{matrix}$$

For each i, j, i=1,...,r, j=1,...,c there exists an rcnx1 vector α_{ij} such that

$$\underline{\alpha}'_{ij}L(\underline{x}) = \sum_{v=1}^{n} \alpha_{ijv}L_{ijv}(\underline{x})$$

$$= \sum_{v=1}^{n} (x_{ijv}-\bar{x}_{ij})$$

$$= 0$$

where

$$\underline{\lambda}_{ij} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \hline 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ \hline 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}_{rcn\times 1} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\\0\\0\end{matrix}}\right\} (i-1)c+j \text{ n rows} \\ \\ \\ \left.\vphantom{\begin{matrix}0\\0\\0\\0\\0\end{matrix}}\right\} rcn-(i-1)c+j+1 \text{ n rows} \end{matrix}$$

and $\underline{L}(\underline{x}) = (L_{ijv}(\underline{x}))_{rcn\times 1}$, $i=1,\ldots,r$, $j=1,\ldots,c$, $v=1,\ldots,n$.
Note that there are rc such vectors, $\underline{\alpha}_{ij}$. Forming the matrix

$$\underline{A} = [\underline{\lambda}_{11},\ldots,\underline{\lambda}_{r1},\ldots,\underline{\lambda}_{1c},\ldots,\underline{\lambda}_{rc}]$$

using the nx1 sub-matrices

$$\underline{1} = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}_{n\times 1} \quad \text{and} \quad \underline{0} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}_{n\times 1}$$

$\underline{A}$ becomes

$$\underline{A} = \begin{bmatrix} \underline{1} & \underline{0} & \cdots & \underline{0} & \underline{0} & \underline{0} & \cdots & \underline{0} & \cdots & \underline{0} & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{1} & \cdots & \underline{0} & \underline{0} & \underline{0} & \cdots & \underline{0} & \cdots & \underline{0} & \underline{0} & \cdots & \underline{0} \\ & & \vdots & & & & \vdots & & \vdots & & & \vdots & \\ \underline{0} & \underline{0} & \cdots & \underline{1} & \underline{0} & \underline{0} & \cdots & \underline{0} & \cdots & \underline{0} & \underline{0} & \cdots & \underline{0} \\ & & & & & & \vdots & & & & & & \\ \underline{0} & \underline{0} & \cdots & \underline{0} & \underline{0} & \underline{0} & \cdots & \underline{0} & \cdots & \underline{1} & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{0} & \cdots & \underline{0} & \underline{0} & \underline{0} & \cdots & \underline{0} & \cdots & \underline{0} & \underline{1} & \cdots & \underline{0} \\ & & \vdots & & & & & & & & & & \\ \underline{0} & \underline{0} & \cdots & \underline{0} & \underline{0} & \underline{0} & \cdots & \underline{0} & \cdots & \underline{0} & \underline{0} & \cdots & \underline{1} \end{bmatrix}_{rcn \times rc} .$$

It can be seen that the columns of $\underline{A}$ are linearly independent, i.e., $r(\underline{A}) = rc$; hence by definition, the set of rc vectors, $\{\underline{\alpha}_{ij};\ i=1,\ldots,r,\ j=1,\ldots,c\}$ are linearly independent linear restrictions on the linear forms $\{L_{ijv}(\underline{x});\ i=1,\ldots,r,\ j=1,\ldots,c,\ v=1,\ldots,n\}$. Applying Corollary 3.3, $r(Q_1) \leq rcn-rc = rc(n-1)$.

In a similar manner for

$$Q_2(\bar{\underline{x}}) = n \sum_{i=1}^{r} \sum_{j=1}^{c} (\bar{x}_{ij}-\bar{x}_{i.}-\bar{x}_{.j}+\bar{x})^2$$

where $\bar{\underline{x}} = (\bar{x}_{11},\ldots,\bar{x}_{1c},\ldots,\bar{x}_{r1},\ldots,\bar{x}_{rc})$ let

$$L_{ij}(\bar{\underline{x}}) = (\bar{x}_{ij}-\bar{x}_{i.}-\bar{x}_{.j}+\bar{x}) \qquad \begin{matrix} i=1,\ldots,r \\ j=1,\ldots,c \end{matrix}$$

$$= \underline{\lambda}_{ij}'\bar{\underline{x}}$$

be a linear form in $\bar{\underline{x}}$ where $\underline{\lambda}_{ij}$ is a rcx1 vector

$$
\underline{\lambda}_{ij} = \left[\begin{array}{cl}
\frac{1}{rc} & \\
\vdots & \\
\frac{1}{rc} & \\
-\frac{1}{r} + \frac{1}{rc} & \text{-----} j^{th} \text{ row} \\
\frac{1}{rc} & \\
\vdots & \\
\frac{1}{rc} & \\
\hline
\vdots & \\
\hline
-\frac{1}{c} + \frac{1}{rc} & \text{-----} (i-1)c^{th} \text{ row} \\
\vdots & \\
-\frac{1}{c} + \frac{1}{rc} & \\
1 - \frac{1}{r} - \frac{1}{c} + \frac{1}{rc} & \text{-----} (i-1)c+j^{th} \text{ row} \\
-\frac{1}{c} + \frac{1}{rc} & \\
\vdots & \\
-\frac{1}{c} + \frac{1}{rc} & \text{-----} ic^{th} \text{ row} \\
\hline
\frac{1}{rc} & \\
\vdots & \\
\frac{1}{rc} & \\
-\frac{1}{r} + \frac{1}{rc} & \\
\frac{1}{rc} & \\
\vdots & \\
\frac{1}{rc} & \\
\hline
\vdots &
\end{array}\right]
$$

$$
\begin{bmatrix} \vdots \\ \hline \frac{1}{rc} \\ \vdots \\ \frac{1}{rc} \\ -\frac{1}{r}+\frac{1}{rc} \\ \frac{1}{rc} \\ \vdots \\ \frac{1}{rc} \end{bmatrix}_{rc\times 1} \text{-----}(r-1)c+j^{th} \text{ row} .
$$

Likewise there exist r+c vectors $\{\underline{\alpha}_i ; i=1,\ldots,r\}$ and $\{\underline{\beta}_j ; j=1,\ldots,c\}$ such that

$$
\begin{aligned}
\underline{\alpha}_i' \underline{L}(\bar{\underline{x}}) &= \sum_{j=1}^{c} \alpha_{ij} L_{ij}(\bar{\underline{x}}) \\
&= \sum_{j=1}^{c} (\bar{x}_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}) \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\underline{\beta}_j' \underline{L}(\bar{\underline{x}}) &= \sum_{i=1}^{r} \beta_{ij} L_{ij}(\bar{\underline{x}}) \\
&= \sum_{i=1}^{r} (\bar{x}_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}) \\
&= 0
\end{aligned}
$$

where

$$
\underline{\alpha}_1 = \left[\begin{array}{c}
0 \\ \vdots \\ 0 \\ \hline \vdots \\ \hline 0 \\ \vdots \\ 0 \\ \hline 1 \\ \vdots \\ 1 \\ \hline 0 \\ \vdots \\ 0 \\ \hline \vdots \\ \hline 0 \\ \vdots \\ 0
\end{array}\right]_{rcx1}
\begin{array}{l}
\left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\} \text{c rows} \\ \\ \\
\left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\} \text{c rows} \\
\left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\} i^{th} \text{ block of c rows} \\
\left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\} \text{c rows} \\ \\ \\
\left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\} \text{c rows}
\end{array}
$$

$$
\underline{\beta}_j = \left[\begin{array}{c}
0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ \hline \vdots \\ \hline 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ \hline \vdots \\ \hline 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0
\end{array}\right]_{rcx1}
\begin{array}{l}
\text{c rows: } \text{-----} j^{th} \text{ row} \\
i^{th} \text{ block of c rows: } \text{-----} ic{+}j^{th} \text{ row} \\
\text{c rows: } \text{-----} (r{-}1)c{+}j^{th} \text{ row}
\end{array}
$$

respectively.

Forming the matrix

$$\underset{\sim}{A} = [\underline{\alpha}_1, \ldots, \underline{\alpha}_r, \underline{\beta}_1, \ldots, \underline{\beta}_c]$$

$$= \begin{bmatrix} 1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ & & \vdots & & & & \vdots & \\ 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \\ 0 & 1 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ & & \vdots & & & & \vdots & \\ 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 1 \\ & & \vdots & & & & \vdots & \\ 0 & 0 & \ldots & 1 & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 & 0 & 1 & \ldots & 0 \\ & & \vdots & & & & \vdots & \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 1 \end{bmatrix}_{rc \times (r+c)}$$

it can be seen that

$$\sum_{i=1}^{r} \underline{\alpha}_i = \sum_{j=1}^{c} \underline{\beta}_j ,$$

hence the rank of $\underset{\sim}{A}$ is at most r+c-1. Consider the (r+c-1) x(r+c-1) sub-matrix of $\underset{\sim}{A}$,

$$\underset{\sim}{A}^* = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 & 0 & \ldots & 0 \\ & & & \vdots & & & & & & \vdots & \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 & 1 & 0 & \ldots & 0 \\ & & & \vdots & & & & & & \vdots & \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \begin{matrix} \left.\right\} \text{c rows} \\ \left.\right\} \text{r-1 rows} \end{matrix}$$

Now $|\underset{\sim}{A}^*| = \pm 1$ depending if r+c-1 is odd or even. Consequently, $r(\underset{\sim}{A}) = r+c-1$ and by definition, the set of vector $\{\underset{\sim}{\alpha}_1,\ldots,\underset{\sim}{\alpha}_r,\ \underset{\sim}{\beta}_1,\ldots,\underset{\sim}{\beta}_c\}$ represent r+c-1 linearly independent linear restrictions on the linear forms $\{L_{ij}(\bar{\underset{\sim}{x}});\ i=1,\ldots,r,\ j=1,\ldots,c\}$ and by Corollary 3.3, $r(Q_2) \leq rcn-r-c+1 = (r-1)(c-1)$.

Consider next

$$Q_3(\bar{\underset{\sim}{x}}.) = nc \sum_{i=1}^{r} (\bar{x}_{i.}-\bar{x})^2$$

where $\bar{\underset{\sim}{x}}. = (\bar{x}_{1.},\ldots,\bar{x}_{r.})$. Let

$$\begin{aligned} L_{i.}(\bar{\underset{\sim}{x}}.) &= (\bar{x}_{i.}-\bar{x}) \qquad i=1,\ldots,r \\ &= \underset{\sim}{\lambda}_{i.}'\bar{\underset{\sim}{x}}. \end{aligned}$$

be a linear form in $\bar{\underset{\sim}{x}}.$ where $\underset{\sim}{\lambda}_{i.}$ is a rxl vector

$$\begin{bmatrix} -\frac{1}{r} \\ \cdot \\ \cdot \\ \cdot \\ -\frac{1}{r} \\ 1-\frac{1}{r} \\ -\frac{1}{r} \\ \cdot \\ \cdot \\ \cdot \\ -\frac{1}{r} \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \\ \text{-----}i^{th}\text{ row} \\ \\ \\ \\ \\ \text{rxl} \end{matrix}$$

There exists only one vector $\underset{\sim}{\alpha}$ such that

$$\begin{aligned} \underset{\sim}{\alpha}'\underset{\sim}{L}(\bar{\underset{\sim}{x}}.) &= \sum_{i=1}^{r} \alpha_i L_{i.}(\bar{\underset{\sim}{x}}.) \\ &= \sum_{i=1}^{r} (\bar{x}_{i.}-\bar{x}) \\ &= 0 \end{aligned}$$

where

$$\underset{\sim}{\alpha} = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}_{r\times 1} .$$

Hence $\underset{\sim}{\alpha}$ represents one linear restriction on the linear forms $\{L_{i.}(\bar{\underset{\sim}{x}}.);\ i=1,\ldots,r\}$ and by Corollary 3.3, $r(Q_3) \leq r-1$.

By an analogous argument $r(Q_4) \leq c-1$.

Lastly, $Q_5(\bar{x}) = rcn\bar{x}^2$ is the square of the single linear form, $L(\bar{x}) = \sqrt{rcn}\ \bar{x}$, and has rank, $r(Q_5) = 1$.

In summary,

$$\begin{aligned} r(Q) &= rcn \\ r(Q_1) &\leq rcn-rc \\ r(Q_2) &\leq rc-r-c+1 \\ r(Q_3) &\leq r-1 \\ r(Q_4) &\leq c-1 \\ r(Q_5) &= 1 \end{aligned}$$

where

$$Q = Q_1 + Q_2 + Q_3 + Q_4 + Q_5.$$

Hence,

$$\sum_{k=1}^{5} Q_k \leq rcn-rc+rc-r-c+1+c-1+r-1+1 = rcn$$

but by Theorem 2.4

$$r(Q) = rcn \leq \sum_{k=1}^{5} r(Q_k).$$

Consequently,

$$r(Q_1) + r(Q_2) + r(Q_3) + r(Q_4) + r(Q_5) = r(Q)$$

and

$$r(Q_1) = rc(n-1)$$
$$r(Q_2) = (r-1)(c-1)$$
$$r(Q_3) = r-1$$
$$r(Q_4) = c-1$$
$$r(Q_5) = 1.$$

From Cochran's Theorem the quadratic forms $Q_1,\ldots,Q_5$ are linearly independent and are χ^2 distributed with degrees of freedom equal to their respective ranks.

V. CONCLUSION

In general it may be difficult to verify that the linear forms $L_1(\underline{x}),\ldots,L_k(\underline{x})$ are linearly independent directly, or when they are not, to find the maximum number of linearly independent linear restrictions on them in order to determine the rank of the associated quadratic form

$$Q(\underline{x}) = \sum_{i=1}^{k} L_i(\underline{x}).$$

Fortunately, it is sufficient only to observe that if there are at least m linearly independent linear restrictions, then the rank must be less than or equal to k-m. Enough information is often then available to establish that no more linear restrictions exist. Hence, indirectly, the rank is found to be exactly k-m. The application of rank determination to analysis of variance is an example of this indirect procedure.

It may now be stated that the procedure as outlined by Cramér [1] and Hald [2] for determining the rank of quadratic forms has been explicitly justified in detail and illustrated.

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13. ABSTRACT

A procedure for determining the rank of a quadratic form is outlined by Cramer [1] and Hald [2]. Additional theoretical verification of this procedure is presented and the results are illustrated with applications in the analysis of variance.

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