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Fast generation and covering radius of Reed-Muller Codes

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NAVAL POSTGRADUATE SCHOOL

MONTEREY, CALIFORNIA

THESIS

FAST GENERATION AND COVERING RADIUS OF REED-MULLER CODES

by

Argyrios Alexopoulos

December 2009

Thesis Co-Advisors:	Pantelimon Stanica
	Jon T. Butler

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Reed-Muller codes are known to be some of the oldest, simplest and most elegant error correcting codes. Reed-Muller codes were invented in 1954 by D. E. Muller and I. S. Reed, and were an important extension of the Hamming and Golay codes because they gave more flexibility in the size of the codeword and the number of errors that could be correct.

The covering radius of these codes, as well as the fast construction of covering codes, is the main subject of this thesis. The covering radius problem is important because of the problem of constructing codes having a specified length and dimension. Codes with a reasonably small covering radius are highly desired in digital communication environments.

In addition, a new algorithm is presented that allows the use of a compact way to represent Reed-Muller codes. Using this algorithm, a new method for fast, less complex, and memory efficient generation of 1st and 2nd order Reed - Muller codes and their hardware implementation is possible. It is also allows the fast construction of a new subcode class of 2nd order Reed-Muller codes with good properties. Finally, by reversing this algorithm, we introduce a code compression method, and at the same time a fast, efficient, and promising error-correction process.

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FAST GENERATION AND COVERING RADIUS FOR REED-MULLER CODES

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ABSTRACT

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EXECUTIVE SUMMARY

Error-Correcting codes play a vital role in every real digital communication environment and storage process. Reed-Muller codes are among the oldest, simplest and most elegant error-correcting codes. When information is sent through a network over long distances or through a variety of channels, where errors might occur in the transition, error-correcting codes, like Reed-Muller codes, can correct these errors. This correction process provides our network with an improvement in throughput and efficiency. Therefore, the efficient use of these codes is more than a critical issue.

A contribution of this thesis is a new way of fast generation of 1st and 2nd order Reed-Muller codes and a category of 2nd order subcodes. In addition, this new algorithm allows a compact way to represent 1st and 2nd order Reed-Muller codes.

This expansion algorithm is appropriate where the fast, real-time generation of low order Reed-Muller codes needed. Using this highly compressed form of codewords, we can quickly expand to any full codeword. In this thesis, we also demonstrate the hardware implementation for this algorithm.

It is also shown, that using just eight blocks of 4-bits, all 1st order Reed-Muller codes can be quickly generated. In addition, for 2nd order Reed-Muller codes, a new concatenation method using all sixteen possible 4-bit combinations is presented. Finally, using eight 4-bits words, we can quickly construct, a new category of subcodes of 2nd order Reed-Muller code with minimum distance d=8 and some other good properties.

Additionally, it is proven in this thesis, that the format of the *Algebraic Normal Form* of our fast construction of 2nd order Reed-Muller subcodes is $affine + x_{n-1}x_n$. Combining this property with the low distance of these subcodes, makes them worthy for further investigation concerning their performance.

In addition, by reversing the new algorithm, we demonstrate a new efficient way to correct errors occurring in this word. This is equivalent to compressing the received word. The "reverse" algorithm applies to cases of storage processes and to communication-oriented applications where Automatic Response Request (ARQ) is used.

Furthermore, the state of the art of the covering radius problem for Reed-Muller codes is presented in this thesis. This has been the subject of investigation for many researchers in the area, and a complete resolution of the problem still eludes us. Some recently found results of estimates of covering radius of Reed-Muller codes are summarized and presented. Some of the methods of computations, even without using the help of computers are also presented. In addition, the main properties of Reed-Muller codes are analyzed.

The covering radius problem is very important since it gives insight into the practical problem of constructing codes having a specified length and dimension.

Based on the analysis of this thesis, we conclude that the proposed methods of fast and memory efficient low order Reed-Muller codes, as well as some category of subcodes of 2nd order Reed-Muller codes, is quite challenging and promising.

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I. INTRODUCTION

A.   THESIS OBJECTIVE

A binary code C of length n is a nonempty subset of the set of all binary n-tuples. The (Hamming) distance between two codewords is the number of bits in which they differ. The covering radius of a code C is the smallest integer R such that every binary vector of length n is within distance R from at least one codeword. In other words, the space of all binary n-tuples is completely covered by spheres of radii R having centers at the codewords of our code C.

The covering radius is one of the most important properties of error correcting codes. It will be clarified throughout this thesis. We seek codes having a specified length and dimension with reasonably small covering radius; in this way, no vector of the space is very far from a nearest codeword.

It is worth mentioning that covering radius is a basic geometric parameter of a code. Topics that are currently under research by the coding community are the following:

1.   Given the length and dimension of a linear code, it should be determine what the covering radius is.

2.   Construct efficient codes that have small covering radius.

3.   Develop computational methods to determine the covering radius of well-known error-correction codes.

Specifically, Reed-Muller codes are an extremely interesting class of error-correction codes, and therefore, many researchers have studied Reed-Muller codes. Nevertheless, due to the complexity of computations methods, overall knowledge is still quite limited. We will focus on some of these methods and point out all published results of covering radius of 1st, 2nd and k-th order Reed-Muller codes.

We survey the main properties of the Reed-Muller codes, investigate previous methods for the computation of covering radius, and propose a fast, less complex, and memory efficient algorithm to derive 1st and 2nd order Reed-Muller codes, as well as a

subcode of 2nd order Reed-Muller code with good properties. This new algorithm allows the use of a compact way to represent Reed-Muller codes.

Reversing the new algorithm, in other words, compressing the codewords of low order Reed-Muller codes and of a new subcode, we introduce an efficient way to correct errors occurring in any codeword of these codes. The hardware implementation of expansion algorithm is presented, and analyzed.

B.   THESIS OUTLINE

This thesis is organized as follows: we start with the introduction, background (Chapter II) and four additional chapters. Chapter III contains a detailed analysis of Reed-Muller codes, some applications of these codes, and generation/encoding/decoding methods. In Chapter IV, the covering radius is defined and its importance in error correction is discussed. Some important methods of computations are discussed. Further, in this chapter, some existing covering radius results concerning 1st, 2nd and k-th order Reed-Muller codes are presented. In Chapter V, we develop a new algorithm for fast generation of $1^{st} - 2^{nd}$ order Reed-Muller codes, and a new construction of a subcode is analyzed. The hardware implementation of this algorithm is also presented and analyzed. In addition, a "reverse" of this new algorithm is presented and evaluated. In Chapter VI, the conclusions based on the observations obtained from the analysis in the previous chapters are presented, as well as proposed future work.

II.   BACKGROUND

In this chapter, some background knowledge and concepts for the analysis of Reed-Muller codes and their subcodes are introduced.

A.   DEFINITIONS

1.   Groups

A *group, denoted by* $[G] = (G, *)$ is a set of elements G combined with a binary operation $*$ on G, *satisfying the* following conditions:

a.   Closure

$$\forall a, b \in G; a * b = c \in G$$

b.   Associativity

$$\forall a, b, c \in G; a * (b * c) = (a * b) * c$$

c.   Identity

$$\exists e \in G \mid \forall a \in G; e * a = a * e = a$$

d.   Invertibility

$$\forall a \in G, \exists a^{-1} \in G \mid a * a^{-1} = a^{-1} * a = e$$

Groups that also satisfy the following commutative property are referred to as *commutative* or *Abelian* groups.

e.   Commutativity

$$\forall a, b \in G; a * b = b * a$$

2.   Fields

A *field, denoted by* $[F] = (F, +, *)$, is a set of elements F combined with two binary operations $+$ and $*$ on F, *satisfying* the following conditions:

a. Group Under Addition

$(F,+)$ is an Abelian group with identity 0.

b. Group Under Multiplication

$(F-\{0\},*)$ is an Abelian group with identity 1.

c. Distributive Law

$$\forall a,b,c \in F; a*(b+c)=(a*b)+(a*c)$$

In this thesis, all manipulations will be on the two-element (binary) *field,* $F_2 = \{0,1\}$ in which the usual operations of addition and multiplication modulo 2 hold:

Table 1.　　Addition in F_2

+	0	1
0	0	1
1	1	0

Table 2.　　Multiplication in F_2

*	0	1
0	0	0
1	0	1

3.　Vector Space

A *vector space* over a field F is a non-empty set V together with two binary operations:

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Addition, denoted by +.

Scalar multiplication, denoted by juxtaposition, is a function from F x V to V; that is the scalar product of $a \in F$ and $x \in V$ is written as ax.

Furthermore, these two operations satisfy the following conditions:

Closure under vector addition,

$$\forall u, v \in V; u + v = w \in V$$

Closure under scalar multiplication,

$$\forall u \in V, \forall a \in F; au = v \in V$$

Associative law for vector addition,

$$\forall u, v, w \in V; \ u + (v + w) = (u + v) + w$$

Commutative law for vector addition,

$$\forall u, v \in V; u + v = v + u$$

Identity element in addition,

$$\exists 0 \in V \mid \forall u \in V; u + 0 = u$$

Additive inverse,

$$\forall u \in V, \exists (-u) \in V; u + (-u) = (-u) + u = 0$$

Distributive law for scalar multiplication over vector addition,

$$\forall u, v \in V, \forall a \in F; a(u + v) = au + av$$

Distributive law for vector multiplication over scalar addition,

$$\forall u \in V, \forall a, b \in F; (a + b)u = au + bv$$

Associative law for scalar multiplication with a vector,

$$\forall u \in V, \forall a, b \in F; (ab)u = a(bu)$$

Identity element in vector multiplication,

$$\exists 1 \in V; \forall u \in V, 1u = u$$

The vector spaces $V = F_2^m$ used in this thesis consist of binary strings of length 2^m, where m is a positive integer, with the usual bitwise operations, described below. The codewords of the Reed-Muller codes and other linear subcodes are subspaces of such a vectors space V.

Vectors in such spaces can be manipulated by three main *operations*.

a.   Addition

For two vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, addition is defined by, $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ where each x_i or y_i is either 1 or 0. The *complement* $\bar{x}$ of a vector x is the vector equal to $(1\ 1\ 1 \ldots\ 1) + x$. An example of the complement of a vector is: $\overline{(0\ 0\ 0\ 1\ 1\ 1)} = (0\ 0\ 0\ 1\ 1\ 1) + (1\ 1\ 1\ 1\ 1\ 1) = (1\ 1\ 1\ 0\ 0\ 0)$

b.   Vector Intersection

$x * y = (x_1 * y_1, x_2 * y_2, \ldots, x_n * y_n)$, where each x_i and y_i is either 1 or 0. The multiplication of a vector x by a constant $\alpha \in F_2$ is defined by $\alpha * x = (\alpha * x_1, \alpha * x_2, \ldots, \alpha * x_n)$. An example is $0 * (111001) = (000000)$.

c.   Dot Product

The dot product of x and y is $x \;\square\; y = x_1 * y_1 + x_2 * y_2 + \ldots + x_n * y_n$.

It is clear that addition, vector intersection and dot product require vectors with the same number of coordinates.

B.   REVIEW OF BOOLEAN FUNCTIONS

A Boolean function of m variables $(x_1, x_2, \ldots, x_m)$ is a function $f(x_1, x_2, \ldots, x_m)$ from F_2^m to F_2, where $F_2 = \{0,1\}$. This kind of function can be completely described by its truth table, which is simply the sequence of its outputs, where the input is ordered lexicographically. Precisely, we order F_2^m as $\{v_1 = (0,0,\ldots,0),\ v_2 = (0,0,\ldots,1),\ldots,(1,1,\ldots,1)\}$; the truth table of f is the sequence $f(v_1), f(v_2)\ldots$ Table 3 specifies a Boolean function of four variables.

Table 3.   Truth Table of a Boolean Function

x_1	x_2	x_3	x_4	f
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

From the right most column of this table (beginning from top), we get the binary string (truth table) 1110100001110001 of length 16. For m-variable functions, this string has length 2^m.

The two constant Boolean functions are $(x_1,x_2,...,x_m)=(1,1,...,1)$ and $(x_1,x_2,...,x_m)=(0,0,...,0)$. In this thesis, two logical operations are used on Boolean functions: conjunction (that corresponds to multiplication in F_2) and exclusive OR, or xor (that corresponds to addition in F_2). Consequently, the string versions of these operations are given by:

$$(x_1,x_2,...,x_m) \text{ conjunction } (y_1,y_2,...,y_m)=(x_1,x_2,...,x_m)\cap(y_1,y_2,...,y_m) \text{ and}$$

$$(x_1,x_2,...,x_m) \text{ exclusive OR} (y_1,y_2,...,y_m)=(x_1,x_2,...,x_m)\cup(y_1,y_2,...,y_m).$$

It is obvious that, under the exclusive OR operation, the set of Boolean functions of m variables forms a vector space over F_2, of size 2^{2^m}.

In addition, a Boolean monomial with variables $x_1, x_2, ..., x_m$ is an expression of the form $p = x_{i_1} x_{i_2} ... x_{i_z}$. The reduced form of p is obtained using the rule: $x^2_{i_x} = x_{i_x}$ until the factors become distinct. The degree of p is the number of variables in the reduced version of p.

An example of a Boolean polynomial in reduced form of degree three is $p' = x_1 + x_2 + x_3 + x_1 x_2 x_2$.

On the other hand, a Boolean polynomial is a linear combination of Boolean monomials, with coefficients in F_2. A reduced polynomial is obtained using the rule: $a+a=0$, until all the monomials become distinct.

Since there are $\binom{m}{k}$ distinct Boolean monomials of degree k on m variables, the total number of distinct Boolean monomials is 2^m, and, therefore, the total number of distinct Boolean polynomials in m variables is 2^{2^m}.

At this point, we need to associate a Boolean monomial in m variables to a vector with 2^m elements. The degree-zero monomial is 1, and the degree-one monomials are $x_1, x_2, ...,$ and x_m. First, we define the vectors associated with these monomials. The vector associated with the monomial 1 is simply a vector of length 2^m, whose components are all 1. So, in a space of size 2^4, the vector associated with 1 is (1111111111111111). The vector associated with the monomial x_1 is 2^{m-1} 1's, followed by 2^{m-1} 0's. The vector associated with the monomial x_2 has 2^{m-2} 1's, followed by 2^{m-2} 0's, then another 2^{m-2} 1's, followed by another 2^{m-2} 0's.

In general, the vector associated with a monomial x_i is a pattern of 2^{m-i} ones followed by 2^{m-i} zeros, repeated until 2^m values have been defined. For example, in a space of size 2^4, the vector associated with x_4 is (1010101010101010).

C. AFFINE BOOLEAN FUNCTIONS

An affine Boolean function of m variables $(x_1, x_2, ..., x_m)$ is a function $f(x_1, x_2, ..., x_m) = f_0 + \sum_{1 \leq x \leq m} f_i x_i$ from F_2^m to F_2, where the coefficients f_i belong to $F_2 = \{0,1\}$. The set of all n-variable affine functions is denoted by A_n. As we mentioned previously, in Table 4, the truth tables of every 3-variable affine function is shown.

Table 4.   Truth Table of an Affine Boolean Function

Affine Function	Truth Table
0	00000000
x_1	00001111
x_2	00110011
x_3	01010101
$x_1 + x_2$	00111100
$x_1 + x_3$	01011010
$x_2 + x_3$	01100110
$x_1 + x_2 + x_3$	01101001
1	11111111
$1 + x_1$	11110000
$1 + x_2$	11001100
$1 + x_3$	10101010
$1 + x_1 + x_2$	11000011
$1 + x_1 + x_3$	10100101
$1 + x_2 + x_3$	10011001
$1 + x_1 + x_2 + x_3$	10010110

In Chapter IV the importance of affine Boolean functions in conjunction with Reed-Muller codes is clarified.

D.   NONLINEARITY AND BENT FUNCTIONS

The nonlinearity of a Boolean function f is defined as $N(f) = \min\{d(f, \beta) \mid \beta \in A_n\}$, where d *(Hamming distance)* is the number of different coordinates of vectors in which f differs from β. It is known (see [1]) that the nonlinearity is upper bounded by: $N(f) \le 2^{n-1} - 2^{\frac{n}{2}-1}$. The concept of nonlinearity is a very important cryptographic property.

The Boolean functions on an even n number of variables, whose nonlinearity is maximum, are called *bent functions*. The importance of bent functions is due to their correspondence to the words of length 2^n whose distance to the 1st order Reed-Muller codes is equal to the covering radius of this code. Bent functions play a significant role in cryptographic environments.

E.   HAMMING DISTANCE AND HAMMING WEIGHT

1.   Definition

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two vectors in F^n. The *Hamming distance* $d(x,y)$, between x and y is the number of coordinate places in which they differ. For a fixed length n, the Hamming distance is a metric on the vector space of the words of that length. For words of length 3 and 4, Figures 1 and 2 can be used for calculating this Hamming distance.

$$d(110, 001) = 3$$
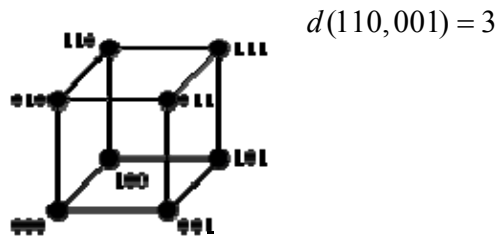
Figure 1.   3-bit binary cube for finding Hamming distance (From [2]).

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$$d(1100,0010) = 3$$
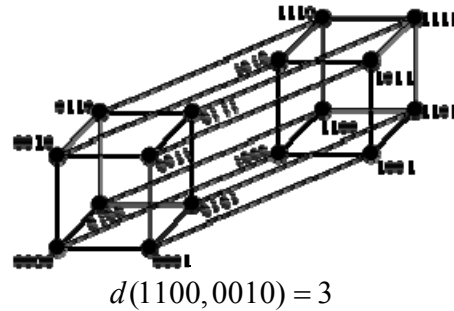
Figure 2.   4-bit binary hypercube for finding Hamming distance (From [2]).

In this thesis, we will refer to the *Hamming distance* as *distance* since it is nonnegative, symmetric, and triangular:

$$d(x, y) \geq 0 \text{ and } d(x, y) = 0 \text{ iff } x=y$$
$$d(x, y) = d(y, x) \text{ for all } x,y \text{ in } F^n$$
$$d(x, y) \leq d(x, z) + d(z, y) \text{ for all } x,y,z \text{ in } F^n$$

2.   Definition

Hamming weight of a binary word w is the number of "1" bits in w. For example wt(11100101110)=7.

F.   ERROR DETECTING AND CORRECTING CAPABILITIES OF CODES

Having defined the Hamming distance of two vectors, we can now clearly describe the distance of a code C as the minimum distance between any two valid codewords of this code.

1.   Definition

Let C be a code. Then, d(C)=$min\{d(x,y) \mid x, y \in C\}$.

2.   Definition

A code C is exactly *t-error-detecting* if and only if $d(C) = t + 1$ and *t-error-correcting* if and only if $d(C) = 2t + 1$ or $d(C) = 2t + 2$.

G. CHAPTER SUMMARY

In this chapter, we discussed the basic principles and properties of the error correction codes, and the background and important concepts necessary to understand their performance. In Chapter III, Reed-Muller codes and their properties, as well as encoding-decoding-generation methods will be introduced and analyzed.

III. REED-MULLER CODES

A. DEFINING REED-MULLER CODES

Let $0 \leq r \leq m$. The *r-th order Reed-Muller code R(r,m)* is the set p of all binary strings of length $n = 2^m$ associated with the Boolean polynomials $p(x_1, x_2, ..., x_m)$ of degree at most *r*.

Consequently, the 0-*th order Reed-Muller code R(0,m)* consists of the binary strings associated with the constant polynomials 0 and 1. This code is the *repetition code* of length 2^m, $R(0,m) = \{0^m, 1^m\} = \{0\ 0...0,\ 1\ 1...1\} = \mathrm{Rep}(2^m)$.

The other extreme situation is the *m-th order Reed-Muller code R(m,m)*, consisting of all binary strings of length 2^m, that is, $R(m,m) = F_2^n$, where $n = 2^m$.

The number of codewords can be found easily from the count of binary monomials in *R(r,m)* of degree at most *r*. There are $k = 1 + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{r}$ such monomials, and so there are 2^k linear combinations of these. It is obvious that the closer *r* is to *m* the more codewords there are. In conclusion, the *r-th order Reed-Muller code R(r,m)* has the following properties:

Length of codewords: 2^m

Number of codewords: $2^{1 + \binom{m}{1} + \binom{m}{2} + ... + \binom{m}{r}}$

Minimum distance between codewords: 2^{m-r} [3]

Reed-Muller codes are among the most useful and interesting binary, linear, block codes. As we will discuss in the next paragraph, first order Reed-Muller codes of length 32 were used in space missions. In order to achieve greater performance than these codes offer, we have to extend their length. The limited bandwidth of communication channels is one thing that we have to take into account. Therefore, the use of very large codes in narrow channels is prohibited. On the other hand, Reed-Muller codes of higher order require significantly less bandwidth than the first order ones.

Many researchers have investigated the weight distribution of Reed-Muller codes, that is, the sequence of codeword weights. The weight spectrum for the first order Reed-Muller codes is found easily, since, as we will see in the next chapter, all codewords in R(1,*m*) codes have the same number of 0's and 1's (are balanced) except for the all 0's and all 1's codewords. For example, in *R*(1,5), there are $2^{1+5} = 2^6 = 64$ codewords of length $2^5 = 32$. Among them, there is a codeword of 32 1's, a codeword of 32 0's and 62 codewords of weight 16 (half 1's, half 0's).

Understanding the weight distribution for higher order Reed-Muller codes is complicated, and very little is known about that. Much work has been done on 2nd and 3rd order Reed-Muller codes [4].

B.   APPLICATIONS OF REED-MULLER CODES

The first order Reed-Muller codes *R*(1,*m*), was used by *Mariner 9* to transmit black and white photographs of Mars in 1972 [5]. A simplified example giving a flavor of code use in digitally transferred data is given below.

The main idea behind applying coding in digital technologies is to break up a picture or a sound into small pieces and to use a binary sequence to represent each of these small pieces, adding at the same time, some redundant bits. This redundancy is used to correct errors that might be caused by noise when the information is sent over a noisy channel.

For example, the pixels (picture elements) shown in Figure 3 could be sent via a channel by coding a white pixel with 111111, a black pixel with 000000 and a gray pixel with 111000. Assuming that the receiver knows the size of the image, in this example 6x6, and that the pixels are being sent row by row, then the picture can be accurately decoded if no more than one error occurs during the transmission process. This happens because the distance between any pair of codewords is at least 3.
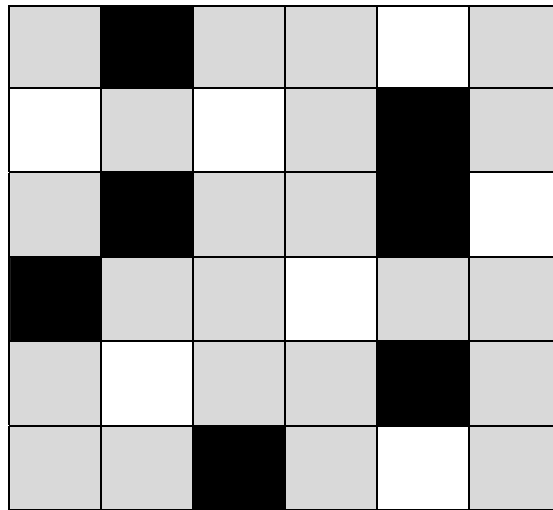
Figure 3.   Picture Elements.

In the case of Mariner 9, the actual scenario is more complicated and finally the error-correcting code used is "heavier." This means that the additional bits used (redundant bits) are repeated information bits. In the case of Mariner 9, the codewords were 32 bits long, consisting of 6 information bits and 26 additional bits.

Another significant application of error-correcting codes is in the *compact disc (CD) technology* [5]. On CDs, the signal is encoded digitally. To protect from errors because of scratches, cracks and similar damage, several kind of codes are used which can correct up to 4,000 consecutive errors (about 2.5 mm of track). Similar error correction techniques are also used on DVDs and Blue-Ray discs.

We cannot ignore the contribution of codes in *compression*. Compression is the process of transforming information from one representation to another smaller representation. In many cases, compression and decompression processes are often referred to as encoding and decoding. It is obvious that data compression has application to data storage and data transmission. Since using a process of reducing the amount of data required to represent a given quantity of information, different amounts of data might be used to communicate the same amount of information. If the exact information can be represented with different amounts of data, it is reasonable to believe that the representation that requires more data contains some kind of data redundancy. Image

compression and coding techniques use three types of redundancies: coding redundancy, spatial redundancy, and psychovisual redundancy.

Another great concern of coding theory is *synchronization*. In many industrial and military activities, such as navigation, mapping, positioning, power distribution, telecommunication, weather station, and digital radio, one of the most important exchanged information is the precise time of action taking place (time tag). Synchronization between these tags is something that can be fixed and controlled by codes. With the use of specific codes any "shift" in phase of a signal can be detected and corrected, enabling the transmission of multiple signals through the same channel.

C.   GENERATION/ENCODING METHODS

1.   Generation Methods

a.   *Using Boolean Polynomials*

An *r*-th order Reed-Muller code $R(r,m)$ is the set of all binary strings of length 2^m associated with Boolean polynomials $x_1, x_2, ..., x_m$ of degree at most *r*. Consequently, the first order Reed-Muller code of length $n = 2^3$ is the set of all binary strings associated with the Boolean polynomials x_1, x_2, and x_3 of degree at most 1. These polynomials have the form $a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3$ where $a_i = 0$ or 1. The binary string corresponding to this polynomial is $a_0(11111111) + a_1(00001111) + a_2(00110011) + a_3(01010101)$.

b.   *Example R(1,3)*

We can list the codewords in $R(1,3)$ as follows:

Table 5.   RM(1,3) codewords

Polynomial	Codeword
0	00000000
x_1	00001111
x_2	00110011
x_3	01010101
$x_1 + x_2$	00111100
$x_1 + x_3$	01011010
$x_2 + x_3$	01100110
$x_1 + x_2 + x_3$	01101001
1	11111111
$1 + x_1$	11110000
$1 + x_2$	11001100
$1 + x_3$	10101010
$1 + x_1 + x_2$	11000011
$1 + x_1 + x_3$	10100101
$1 + x_2 + x_3$	10011001
$1 + x_1 + x_2 + x_3$	10010110

Note that all codewords in $R(1,m)$ except 0 and 1 have weight 2^{m-1}. Thus, in the previous example of R(1,3), the weight of all nontrivial codewords, except 00000000 and 11111111, is $2^{3-1} = 4$.

c.   *Using Direct Sum Construction*

If C_1 is an $R^{(r_1, m_1)}$ code and C_2 is an $R^{(r_2, m_2)}$ code, then the direct sum C_3 is the code $C_3 = \{cd \mid c \in C_1, d \in C_2\}$ with the following parameters:

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Length of codewords: $2^{m_1} + 2^{m_2}$

Number of codewords: $2^{1+\binom{m_1}{1}+\binom{m_1}{2}+...+\binom{m_1}{r_1}} 2^{1+\binom{m_2}{1}+\binom{m_2}{2}+...+\binom{m_2}{r_2}}$

Minimum distance between codewords: $\min\{2^{m_1-r_1}, 2^{m_2-r_2}\}$

d. Using (u,u+v)-Construction

This construction, for many reasons, is more useful than the direct sum construction. If C_1 is an $R(r_1, m_1)$ code and C_2 is an $R(r_2, m_2)$ code, both of which are over the same alphabet (C_1 and C_2 have the same length), then we can define a code $C_1 \oplus C_2$ by: $C_1 \oplus C_2 = \{c(c+d) \mid c \in C_1, d \in C_2\}$ with the following properties [5]:

Length of codewords: $2^{m_1+1} = 2^{m_2+1}$

Number of codewords: $2^{1+\binom{m_1}{1}+\binom{m_1}{2}+...+\binom{m_1}{r_1}} 2^{1+\binom{m_2}{1}+\binom{m_2}{2}+...+\binom{m_2}{r_2}}$

Minimum   distance   between   codewords:

$d(C_1 \oplus C_2) \geq \min\{2^{m_1-r_1+1}, 2^{m_2-r_2}\}$

2. Encoding Methods

To define the encoding matrix of $R(r,m)$, let the first row of the encoding matrix be $11\ldots1$ (the vector with length 2^m with all entries equal to 1). If r is equal to 0, then this row is unique in the encoding matrix. On the other hand, if r is equal to 1, then we add m rows corresponding to the vectors x_1, x_2, ... , and x_m to the $R(0,m)$ encoding matrix.

Thus, in order to form an $R(r,m)$ encoding matrix, where r is greater than 1, we have to add $\binom{m}{r}$ rows to the $R(r-1,m)$ encoding matrix. These added rows consist of all the possible reduced degree r monomials that can be formed using the rows $x_1, x_2, ..., x_m$

a.   *Example R(1,3)*

When *m*=3 we then have:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ x_2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ x_3 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

b.   *Example R(2,3)*

Thus, adding the rows

$x_1 x_2 = 11000000$, $x_1 x_3 = 10100000$ and $x_2 x_3 = 10001000$ we obtain:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ x_2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ x_3 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ x_1 x_2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 x_3 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ x_2 x_3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

c.   *Example R(3,3)*

Note that, the row $x_1 x_2 x_3 = 10000000$ can be added to form: $R(3,3)$

d.   *Example R(2,4)*

Using exactly the same steps, we can obtain:

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$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
x_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
x_3 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
x_4 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
x_1x_2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1x_3 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1x_4 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_2x_3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_2x_4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
x_3x_4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}$$

It is obvious that the number of rows of these encoding matrices is $k = 1 + \binom{m}{1} + \binom{m}{2} + ... + \binom{m}{r}$. So, the sent message must be in blocks of length k. Let $m = (m_1, m_2, ..., m_k)$ be such a block. Then the encoded message M is the sum $\sum_{i=1}^{k} m_i R_i$, where R_i indicates the rows of the encoding matrix of $R(r,m)$.

e. *Example Encoding with R(1,3)*

Using $R(1,3)$ to encode $m=(0011)$ gives:

$0(11111111) + 0(11110000) + 1(11001100) + 1(10101010) = (01100110)$ as the encoded word.

f. *Example Encoding with R(2,4)*

Similarly, using $R(2,4)$ to encode $m=(10101110010)$ gives:

$1*(1111111111111111) + 0*(1111111100000000) + 1*(1111000011110000) + 0*(1100110011001100)$
$+ 1*(1010101010101010) + 1*(1111000000000000) + 1*(1100110000000000) + 0*(1010101000000000)$
$+ 0*(1100000011000000) + 1*(1010000010100000) + 0*(1000100010001000) = (0011100100000101)$

3. Decoding Methods

There are few methods for decoding Reed-Muller codes. In this thesis, the most widely used is analyzed. Decoding is more complex than encoding. The theory behind both encoding and decoding is based on Hamming distance between vectors.

The decoding method checks which row R_i of the encoding matrix was used to form the encoded message. The implementation of this method requires the use of characteristic vectors of the encoding matrix rows. In order to find the characteristic vector, we work on the monomial r associated with the row of the matrix. After that, we take the set of all x_i that are not in r, but only in the encoding matrix. The characteristic vectors are those vectors that correspond to monomials $x_i, \overline{x}_i$, such that exactly one of x_i or $\overline{x}_i$ belongs to each monomial for all elements of the set of all x_i. The dot product of these characteristic vectors with all the rows of the used code matrix yields 0, except the row to which the vector corresponds.

a. Decoding Algorithm

This method is precisely described in the following three steps of an algorithm [3]:

<u>Step 1</u>

Choose a row of the given encoding matrix code and find 2^{m-r} characteristic vectors for that row. Then, form the dot product of these vectors with the encoded message.

<u>Step 2</u>

Compute the majority value (either 1 or 0) of the dot products, and assign it to each row.

<u>Step 3</u>

Executing steps 1, 2 from the bottom of the matrix to the top, multiply the majority value assigned to each row by its corresponding row. Add the results altogether, and then sum this up to the received encoded message. If there is a majority of 1's in the

resulting vector, then assign 1 to the top row. Otherwise, if there is a majority of 0's, then assign 0 to the top row. Adding the top row, multiplied by the assigned value, leads to the original encoded message. Using this algorithm, it is obvious that we can identify the errors occurred during the transmission of encoded message. The vector that is formed using the assigned values of each row, from the top row all the way to the bottom row of the encoding matrix, is the original message.

b.   *Example of Decoding Using R(1,3)*

Assuming an original message $m=(0110)$, using the $R(1,3)$ encoded matrix we get the encoded message $M=(00111100)$. As it is already mentioned, the distance in this code is $2^{3-1}=4$, and therefore, it can correct one error. Assuming that, during message transmission, one error occurred at the first leftmost bit, the encoded message after the error is $M'=(10111100)$. The characteristic vectors of the last row of the encoded matrix are $x_1 x_2, x_1 \overline{x_2}, \overline{x_1} x_2$ and $\overline{x_1}, \overline{x_2}$.

The vector related to x_1 is (11110000), thus $\overline{x_1}$ is (00001111). Similarly, x_2 is (11001100), and thus $\overline{x_2}$ is (00110011). Therefore, $x_1 x_2$ is (11000000), $x_1 \overline{x_2}$ is (00110000), $\overline{x_1} x_2$ is (00001100) and $\overline{x_1}, \overline{x_2}$ is (00000011). Computing the dot product of these vectors with M', we get the values 1,0,0,0 respectively, leading to majority value of 0 for x_3. Repeating the process for the second to last row of the matrix, we get the values 0,1,1,1 respectively, leading to majority value 1 for x_2. Working similarly, we conclude that the coefficient of x_1 is also 1. Adding 0*(10101010) and 1*(11001100) and 1*(11110000) we get $M''=(00111100)$. Then, we notice, that adding M' and M'' we get (10000000), which has more 0's than 1's, leading to 0 for the coefficient of the first row of the used matrix.

Putting together the four coefficients that correspond to four rows 0,1,1,0 we get the original message. Additionally, we can determine the position of the error at the first leftmost bit.

D.   CHAPTER SUMMARY

In this chapter, a detailed discussion of Reed-Muller codes was presented. Some methods of generation, encoding and decoding are also analyzed. This will help us explain later in the thesis the simplicity of a new method of fast construction of these codes. In addition, some examples were examined to help understanding each method. In Chapter IV, the concept of covering radius is presented, and several methods for its computation are examined.

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IV.   COVERING RADIUS

A.   INTRODUCTION

We can trace the origin of error correcting codes in a paper from the 1940s by Claude Shannon [6], who proposed some error detection/correction techniques, to achieve error-free communication through a noisy channel. Data to be sent over a noisy channel is first "encoded," plaintext is turned into a codeword by adding extra data (redundancy). This enlarged codeword is sent via the communication channel and the received data is "decoded" by the receiver. The critical point of this last process is that the decoded data has to be as close as possible to sent data. At this point, covering radius takes its role, since the "quality" of the code, in relation to the channel, depends on how small the code's covering radius is.

In coding theory, the covering radius plays a critical role in every code. In addition, good covering codes have a number of applications in various areas of mathematics and electrical engineering. Though the minimum distance has a more central role for error-correction codes, the covering radius is also related to the error correction capability of the code, since if it is less than the distance, no vector in the space can be added without worsening the code's distance [7].

Since F_{2^n} has a distance metric, it makes sense to use spheres that are centered at a valid codeword x with a given radius ρ. One sample of these spheres is depicted in Figure 4.
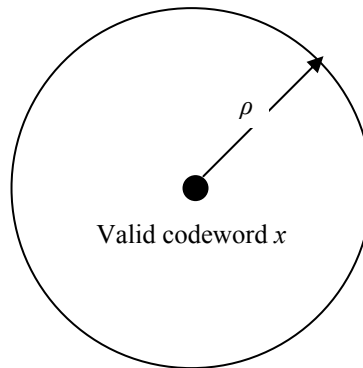
Figure 4.   Sphere of radius ρ.

Let C be a subset of F_{2^n}, in which all the distances are integers. The *covering radius* of a code C is the smallest radius ρ (Figure 5) such that every word of the space is contained in some (at least one) sphere of radius ρ centered at a codeword.

It is obvious that the covering radius problem is important since it helps in investigating the constructing codes having a specified length and dimension such that no vector of the space is very far from the nearest codeword.
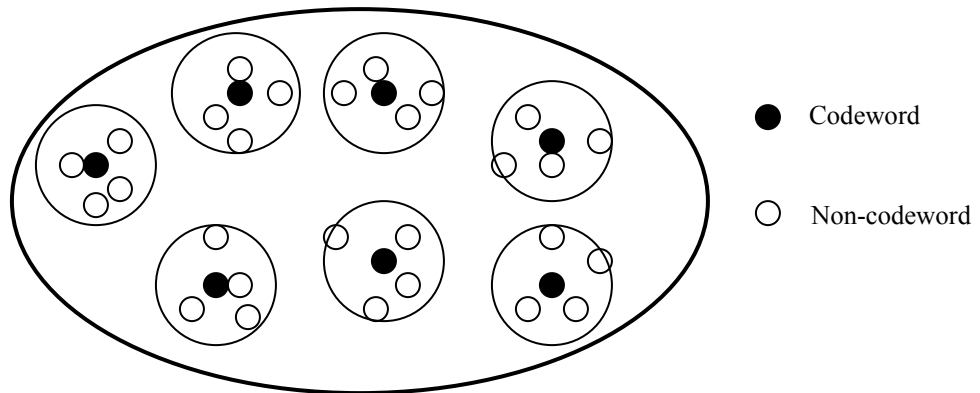
Figure 5.   Covering radius ρ.

Each codeword of a code $C \subset F_{2^n}$ represents a message. When that message is transmitted, errors may occur. However, if the used code C has the property that all the spheres of radius ρ around codewords are completely disjoint, then any received message x that has no more than ρ coordinates in error is within distance ρ from a unique codeword c in C. Therefore, we conclude that the codeword that was originally sent is c. Consequently, we say that C can correct up to ρ errors. It is obvious that the largest value of ρ cannot be greater than d (the distance between any two codewords of C). The critical point here is the ability of constructing error-correcting codes, having specified length and dimension (number of codewords in linear cases) with large minimal d. This is actually one of the central problems in theory of error-correction codes.

In addition, it is worth mentioning that covering radius is a basic geometric parameter of a code. Topics that are currently under research by the coding community are the following:

1. Given the length and dimension of a linear code, it should be determined what the covering radius is.

2. Construct efficient codes that have small covering radius.

3. Develop computational methods to determine the covering radius of well-known error-correction codes.

Specifically, Reed-Muller codes are an extremely interesting class of error-correction codes, and therefore, many researchers have studied Reed-Muller codes [8], [9], [10, [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25]. Nevertheless, due to the complexity of computations methods, overall knowledge is still quite limited. We will focus on some of these methods and point out all published results of covering radius of 1,[st] 2,[nd] and k-th order Reed-Muller codes later in the chapter.

B. METHODS OF COMPUTATIONS OF COVERING RADIUS

1. 1st Method Using Translate

When we construct an error-correction code with large minimum distance d, our focus is in the structure of the code. In addition, the corresponding codewords must be chosen, so that no vector of F_2^n (since in this thesis we only work with binary spaces) has its distance too large from any codeword.

On the other hand, the design of a decoding scheme focuses on the exterior part of the code. If we have a code $C \subset F_n^2$, and we decide to send some data in the form of a codeword, then, on the receiver we may get a vector x that is different from c.

Thus, we can now introduce the concept of a translate $x + C$ of C. This is the set of all codewords of the code C xoring with a specific received word x. The weight of the translate $x + C$ is the minimal weight of any vector in C. Knowing the weights of translates, is very critical in the decoding problem since the covering radius is the largest among all weights of translates.

In the following example, we pick the code $C=\{00000,11000,00111,11111\}$, and we calculate the covering radius using the method we just introduced (see Table 6). Note, that the code we use in this part is an arbitrary code with no specific properties. We use this code for the sake of simplicity of our example.

Having the codewords of the code C and all vectors that can be received (received words), we can calculate the translates of the code, and thus the weights of these translates. Therefore, the maximum of these weights, 2 in our case, is the covering radius of the code. In Table 6, we see this method in detail.

Table 6.   1st method of Covering Radius computation

Codewords of C (transmitted words)	Vectors x (received words)	Minimum $wt(x+C)$ ($x+C$ translates)	{Max[min($wt(x+C)$)]} (covering radius)
00000 11000 00111 11111	00001	1	2
	00010	1	
	00011	1	
	00100	1	
	00101	1	
	00110	1	
	00111	0	
	01000	1	
	01001	2	
	01010	2	
	01011	1	
	01100	2	
	01101	2	
	01110	2	
	01111	1	
	10000	1	
	10001	2	
	10010	2	
	10011	2	
	10100	2	
	10101	2	
	10110	2	
	10111	1	
	11000	0	
	11001	1	
	11010	1	
	11011	1	
	11100	2	
	11101	1	
	11111	0	
	00000	0	

Furthermore, applying the same method to the following code $C'=\{00000,11000,00111\}$ we realize that even though we decrease the dimension of the code from 4 to 3, the covering radius remains the same. This code is actually a trivial sub-code of the given code. In reality, constructing a sub-code with "nice characteristics," and properties is not a triviality.

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In our case, the covering radius of the random code is 2. As we have already mentioned above, the covering radius is also the smallest integer r such that any vector in F_2^n is within distance r from a codeword.

This method works well for codes with small length and dimension. When these parameters become larger, the computation complexity of the method increases exponentially, and the use of computers is necessary.

2.   2nd Method of Using Direct Sum of Codes

Before we describe the 2nd method of covering radius computation [11], we give several definitions.

a.   Definition of Norm of a Code C

Let $C \subset F_n^2$ be a linear code of length r, dimension m and covering radius R. Let j be one of the m coordinates, and C_1 denote the set of codewords in which the j-th coordinate is 0. Similarly, let C_2 denote the set of codewords in which the j-th coordinate is 1. In accordance with [22], if C_2 is not empty, then both C_1, C_2 contain 2^{m-1} codewords. For any vector x in F_2^n, let $d_1 = d(x, C_1)$ and $d_2 = d(x, C_2)$. Also let $D = \max(d_1, d_2)$. Then, D is called the *norm* of C. Norm does not depend on the choice of x or j.

b.   Definition of a Normal Code

A code is normal when its norm satisfies $D \leq 2R + 1$. In other words [22], given a code with norm D, then there is a coordinate i such that, for any vector x, the sum of the d's from x to the nearest codeword having in i-th place 0 and to the nearest codeword having in i-th place 1, cannot be greater than D.

Having defined the critical concepts of the norm of a code, and normal codes, we can now proceed to the second method of computation of covering radius that is combined with a code construction method.

Let C_1 be an $R(r_1, n_1)$ code with covering radius R_1, and C_2 be an $R(r_2, n_2)$ code with covering radius R_2. The direct sum of these codes [11] is another

code of length $2^{n_1} + 2^{n_2}$ vectors $u|v$, where $u \in C_1$ and $v \in C_2$. Then, this direct sum is a

new code with covering radius $R_1 + R_2$. Additionally, if C_1 and C_2 are normal, we can construct their *amalgamated direct sum* [11] that is a code with one less coordinate, one less dimension, and $R_1 + R_2$ covering radius.

c. *Example* $R_1 + R_2$

Consider the code C={00000,11000,00111,11111} that is the direct sum of the following codes: C_1={00,11} and C_2={000,111}. Using the last method of direct sum, we conclude that the covering radius is $R = R_1 + R_2 = 2$.

3. 3rd Method Using Bounds

Let C_1 be any code of length 2^{n_1} and b any vector of the same length. If $b \notin C_1$ and $C = C_1 \cup (b + C_1)$ and if we can find a vector $y \notin C_1$ such that $d(y, C_1) = r$, then from [11] the covering radius of C is at least $[\frac{r}{3}]$.

Pick an arbitrary code, say C_1={0000,1100,0011,1111}, we can calculate the covering radius of code $C = C_1 \cup (b + C_1)$ (Table 7) using the 3rd method we just introduced. The code we use is a random code with no specific properties.

We picked a vector $b \notin C_1$, and we construct the code $C = C_1 \cup (b + C_1)$. Choosing a vector $y \notin C_1$ with $d(y, C_1) = 1$ and using the 3rd method of computation, we conclude that the covering radius of C is at least $\left\lceil \dfrac{1}{3} \right\rceil = 1$. Thus, a lower bound of the covering radius of C is 1.

It is very difficult to find the covering radius of a large code, or even to bound it [18], [22]. Therefore, in the majority of the cases, the 3rd method for the computation of the covering radius is very useful.

Table 7. 3rd method of Covering Radius computation

Codewords of C_1	Vector $b \notin C_1$	Codewords of $C = C_1 \cup (b + C_1)$	Vector $y \notin C_1$ $(d(y, C_1) = 1)$
0000 1100 0011 1111	1010	0000 1100 0011 1111 1010 0110 1001 0101	1000

4. 4th Method Using Norm

This method relies on Theorem 1 [11], which states that every code of norm N has a covering radius $\rho \le \left\lceil \dfrac{N}{2} \right\rceil$. The equality holds for normal codes.

C. COVERING RADIUS FOR 1$^{\text{ST}}$ ORDER REED-MULLER CODES

Recall that, *R(r,m)* is an *r*th order Reed-Muller code of length 2^m, and *ρ(r,m)* is its covering radius. One of the challenging problems in coding theory is to find precisely the covering radius of 1$^{\text{st}}$ order Reed-Muller codes.

The first expression for *ρ(1,m)* was published in 1978 [22] .

$$\rho(1,m) = 2^{m-1} - 2^{\frac{m}{2}-1} \text{ for even } m,$$

$$2^{m-1} - 2^{\frac{m-1}{2}} \le \rho(1,m) \le 2^{m-1} - 2^{\frac{m}{2}-1} \text{ for odd } m.$$

For the first odd values of *m*, we have that *ρ(1,1)*=0, *ρ(1,3)*=2, *ρ(1,5)*=12 (also proved in [8]) and *ρ(1,7)*=56 (also proved in [14]). An easy but unsafe conclusion [14] was that with odd values *ρ(1,2t+1)* is equal to $2^{2t} - 2^t$, thus to the lower bound of last inequality .

In 1983, the last conjecture was finally disproved [24], and it was shown that:

$$\rho(1,m) \ge 2^{m-1} - \frac{27}{32} 2^{\frac{m-1}{2}} \text{ for odd } m \ge 5. \text{ In 1990, a correction of this proof is also provided}$$

[26].

D. COVERING RADIUS FOR 2$^{\text{ND}}$ ORDER REED-MULLER CODES

One of the first detailed studies to find the covering radius for 2$^{\text{nd}}$ order Reed-Muller codes was in [15], where it was proved that *ρ(2,6)*=18. In the same paper, some bounds are also provided:

$$36 \le \rho(2,7) \le 46, and \ \rho(2,8) \ge 72 .$$

Recently, in [27] a new upper bound of 2$^{\text{nd}}$ order Reed-Muller codes was published: $\rho(2,m) \le 2^{m-1} - \sqrt{15} 2^{\frac{m}{2}-1} + O(1) .$

E. COVERING RADIUS FOR *R*TH ORDER REED-MULLER CODES

Some known results for *r*th order Reed-Muller codes are shown below.

In the following trivial cases, we have: $\rho(m,m)=0,\ \rho(m-1,m)=1,$ and $\rho(m-2,m)=2$.

In [13], it is proved that:

$$\rho(m-3,m)=m+2,\ \text{for m even and } m\geq 3\text{, and}$$

$$\rho(m-3,m)=m+1,\ \text{for m odd and } m\geq 3.$$

F. CHAPTER SUMMARY

In this chapter, the concept of covering radius of a code is introduced. In addition, some methods of covering radius computation are presented and finally some known results for Reed-Muller codes are reported. In the next chapter, a new simplified algorithm of fast generation of all 1st order and some of 2nd order Reed-Muller codes is analyzed and a fast construction of a linear subcode with good properties is presented and analyzed. In addition, the "reverse" of this new algorithm is presented.

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V.   FAST ALGORITHM OF GENERATION OF $1^{ST}-2^{ND}$ ORDER REED-MULLER CODES, LINEAR SUBCODES WITH GOOD PROPERTIES, AND THE "REVERSE" ALGORITHM

A.   FAST GENERATION OF 1ST ORDER REED-MULLER CODES

Comparing Tables 4 and 5, and the constructing method of Reed-Muller codes, we notice that all of the 1st order Reed-Muller codes are Affine Boolean Functions. Also, all 1st order Reed-Muller codewords are balanced, except the all 0's and all 1's codewords. The construction of these codewords using a conventional method is time and memory consuming. Therefore, a new algorithm for fast generation is introduced in this chapter. The algorithm is useful for hardware coding applications.

Using a Lemma in [28] which states: "An affine function in more than 2 variables is a linear string made up of the 8 4-bit blocks: $T_1 = \{ A = 0000, \overline{A} = 1111, B = 0011, \overline{B} = 1100, C = 1001, \overline{C} = 0110, D = 0101, \overline{D} = 1010 \}$ in a block sequence $I_1, I_2, \ldots, I_{2^{n-2}}$ given as follows:

The first block I_1 is one of A, $\overline{A}$, B, $\overline{B}$, C, $\overline{C}$, D or $\overline{D}$.

The second block I_2 is either I_1 or $\overline{I_1}$.

The next two blocks I_3, I_4 are I_1, I_2 or $\overline{I_1}, \overline{I_2}$.

The next four blocks I_5, I_6, I_7, I_8 are I_1, I_2, I_3, I_4 or $\overline{I_1}, \overline{I_2}, \overline{I_3}, \overline{I_4}$.

The last 2^{n-3} blocks $I_{2^{n-3}+1}, \ldots, I_{2^{n-2}}$ are $I_1, \ldots, I_{2^{n-3}}$ or $F_2^{n-2} \mid T_1$ ",

We can construct all 1st order Reed-Muller codes using the algorithm described below. In our case when a 1 occurs, we complement and, when 0 occurs, we just copy the block as it is.

1.   New Algorithm for Fast Generating 1st Order RM Codes

Step 1

We begin with the codewords of $R(1,2)$ that are identical to the 4-bit blocks used in previous lemma: $T_1 = \{ A = 0000,\ \overline{A} = 1111,\ B = 0011,\ \overline{B} = 1100,\ C = 1001,\ \overline{C} = 0110,\ D = 0101,\ \overline{D} = 1010 \}$.

Step 2

We construct the following concatenation for each $R(1,2)$ codeword: $F_2^{n-2} \mid T_1$ in order to construct the $R(1,n)$ code. The first part of this structure will play the role of "guide" word.

Step 3

Starting from the leftmost bit of "guide" word, we just complement the bits of right part when we find 1 and just repeating these bits when we find 0, until we take the last rightmost bit of "guide" word.

We repeat step 3 for every block of T_1 using every "guide" word. In Table 8, we generate $R(1,3)$ using the new algorithm for fast generating 1st order RM codes. Therefore, for this case, we repeat step 3 twice since there are two "guide" words for every block of T_1. On the contrary, in Table 9, step 3 is repeated four times, since $F_2^{n-2} = F_2^{4-2} = F_2^2 = \{00, 11, 01, 10\}$.

2.   Example R(1,3)

Using the above algorithm we construct $R(1,3)$.

Table 8.   Fast Generation of $R(1,3)$

Step 1 (R(1,2))	Step 2	Step 3 (R(1,3))
0000	0|0000	00000000
	1|0000	00001111
1111	0|1111	11111111
	1|1111	11110000
1100	0|1100	11001100
	1|1100	11000011
0011	0|0011	00110011
	1|1100	00111100
1001	0|1001	10011001
	1|1001	10010110
0110	0|0110	01100110
	1|0110	01101001
1010	0|1010	10101010
	1|1010	10100101
0101	0|0101	01010101
	1|0101	01011010

3.   Example R(1,4)

Using the same algorithm we construct $R(1,4)$.

Table 9.   Fast Generation of $R(1,4)$

Step 1 (R(1,2))	Step 2	Step 3 (R(1,4))
0000	00|0000	0000000000000000
	11|0000	0000111111110000
	01|0000	0000000011111111
	10|0000	0000111100001111
1111	00|1111	1111111111111111
	11|1111	1111000000001111
…	…	…
1100	00|1100	1100110011001100
	11|1100	1100001100111100
…	…	…
0011	00|0011	0011001100110011
…	…	…
1001	00|1001	1001100110011001
…	…	…
0110	00|0110	0110011001100110
…	…	…
1010	00|1010	1010101010101010
…	…	…
0101	00|0101	0101010101010101
…	…	…

The complexity of constructing the 1st order Reed-Muller codes using this algorithm is significantly lower than the complexity of the method that is introduced in Chapter III, by using Boolean polynomials.

In addition, it is obvious that this compact representation of 1st order Reed-Muller codewords is highly memory efficient because it can actually store a great amount of information in a small word. For example, using a "guide" word of 8 bits, we can compress a codeword of 512 bits to a string of 12 bits. The compression ratio for each codeword in this example is 43:1, and the memory saving is 97.65%. The compressed string includes all the information of the expanded codeword, and moreover, as we analyze below, using the "reverse" algorithm, we can reconstruct a damaged codeword, correcting some errors occurred during the transmission.

B.   HARDWARE IMPLEMENTATION OF ALGORITHM

In general, in our algorithm, in order to store a compact representation of 2^n-bits codeword, $n+1$ bits are needed. This implies that the storage ratio is $\frac{\text{bits of complete codeword}}{\text{bits of compact codeword}} : 1 = \frac{2^n}{n+1} : 1$, and the storage saving is $(1 - \frac{\text{bits of compact codeword}}{\text{bits of complete codeword}})\% = (1 - \frac{n+1}{2^n})\%$. It is obvious that the storage ratio, and the storage saving are very high. The critical point is that the fast generation of complete codewords from compact form cannot be efficiently supported by a program running on a conventional computer. On the other hand, the hardware implementation of this expansion (see Figure 6, for the case of n=3) is faster and more compact.

The logic circuit of Figure 6 generates only one codeword at a time. In order to obtain the whole code, we have to repeat this circuit for each word of either T_1 or T_2 and for each "guide" word.
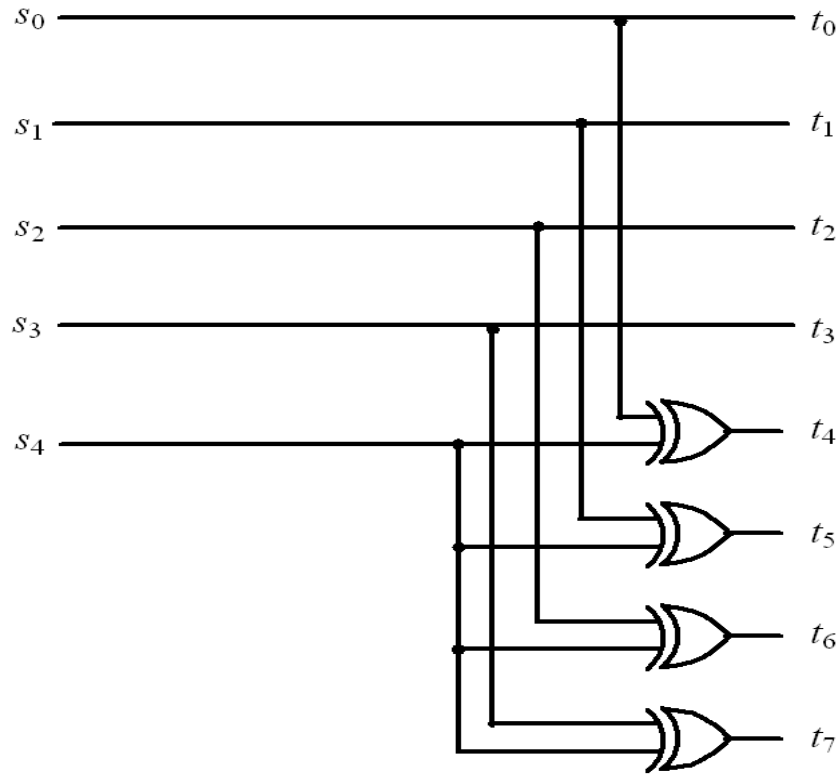
Figure 6.   Hardware implementation of algorithm (*n=3*)

The exclusive OR gates implemented in Figure 6, either complements or leaves uncomplemented the corresponding bits depending on the value of inputs s_4. If $s_4 = 1$, the output of the gate is complemented, otherwise stays unchanged.

The number of two-input exclusive OR gates that are needed for the implementation of our algorithm is $2^n - 4$, where *n* is the number of variables used. Although exponential in *n*, this is close to minimal mostly because 2^n outputs are needed, four of which are driven directly by their inputs and thus, require no gate. The delay associated with this logic circuit is also small.

From the above, we conclude that our conversion algorithm gives to any communication user the ability to produce complete low order Reed-Muller codewords on-the-fly from a compressed representation.

C. FAST GENERATION OF 2ND ORDER REED-MULLER CODES

On the other hand, fast generation of 2nd order Reed-Muller codes is more complicated. This problem is comparable to the construction of all n-variable quadratic functions: $2^{\binom{n+1}{2}+1}$. We just demonstrate the fast construction of $R(2,3)$, since the generation of RM$(2,n)$ for $n>3$ is quite complicated, and we have not been able to achieve it in its generality.

We define $T_2 = \{ E = 1000, \ \overline{E} = 0111, \ F = 0001, \ \overline{F} = 1110, \ G = 0100, \ \overline{G} = 1011,$ $H = 0010, \ \overline{H} = 1101 \}$.

Any codeword of $R(2,3)$ has the structure $T_1 | T_1$ or $T_2 | T_2$, as mentioned in [29]. Thus, in Table 10, we see this fast generation. This way of construction is less complicated and less memory consuming that the normal way.

Table 10.   Fast Generation of $R(2,3)$

	$T_1\|T_1$	$T_2\|T_2$
0000		
1111	00001111	10000111
0011	00000011	10000001
1100	00001100	10001110
1001	00001001	10000100
0110	00000110	10001011
0101	00000101	10000010
1010	00001010	10001101
1000	11111111	01110111
0111	11110011	01110001
0001	11111100	01111110
1110	11111001	01110100
0100	11110110	01111011
1011	11110101	01110010
1011	11111010	01111101
0100	...	...

The left-hand blocks 0000–1010 are grouped as T_1, and the blocks 1000–0100 are grouped as T_2.

D.   FAST GENERATION OF LINEAR SUBCODES WITH GOOD PROPERTIES

Having fast constructed all 1st order Reed-Muller codes using the eight 4-bit blocks: $T_1 = \{A = 0000,\ \overline{A} = 1111,\ B = 0011,\ \overline{B} = 1100,\ C = 1001,\ \overline{C} = 0110,\ D = 0101,\ \overline{D} = 1010\}$ and the given algorithm, we demonstrate a fast generation of $R(2,3)$ using the eight 4-bit blocks: $T_2 = \{E = 1000,\ \overline{E} = 0111,\ F = 0001,\ \overline{F} = 1110,\ G = 0100,\ \overline{G} = 1011,\ H = 0010,\ \overline{H} = 1101\}$.

Again using the algorithm:

1. Algorithm

<u>Step 1</u>

We begin with the 4-bit blocks given above: $T_2 = \{ E=1000, \overline{E}=0111, F=0001,$ $\overline{F}=1110, G=0100, \overline{G}=1011, H=0010, \overline{H}=1101 \}$

<u>Step 2</u>

We construct the following concatenation for each of T_2 blocks: $F_2^{n-2} \mid T_2$ in order to construct a new category of error correction codes with good properties. The first part of this structure plays the role of "guide" word.

<u>Step 3</u>

Starting from the leftmost bit of "guide" word, we complement the bits of right part when we find 1 and repeat these bits when we find 0, until we reach the last rightmost bit of "guide" word.

We repeat step 3 for every block of T_2 using every "guide" word.

We prove that the properties for this construction hold for any n. Therefore, all codewords of any such construction are of the form: $affine + x_{n-1}x_n$. Consequently, the sum of any two codewords is an affine function, and also the sum of any three codewords belongs to the code.

2. Example R(2,3) Subcode

Using the above algorithm, we construct the new subcode as shown in the Table 11.

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Table 11.   Fast Generation of a R(2,3) subcode

Step 1	Step 2	Step 3
1000	0|1000	10001000
	1|1000	10000111
0111	0|0111	01110111
	1|0111	01111000
0001	0|0001	00010001
	1|0001	00011110
1110	0|1110	11101110
	1|1110	11100001
0100	0|0100	01000100
	1|0100	01001011
1011	0|1011	10111011
	1|1011	10110100
0010	0|0010	00100010
	1|0010	00101101
1101	0|1101	11011101
	1|1101	11010010

3.   Example R(2,5) Subcode

Using the same algorithm, we generate another code that has 32 codewords and some important properties, as described below. Table 12 shows the Truth Table and *Algebraic Normal Form* of this construction:

Table 12. Fast Generation of $R(2,5)$ subcode

Step 1	Step 2	Step 3	Algebraic Normal Form
1000	00|1000	1000100010001000	$1 + x_4 + x_3 + x_3 x_4$
	11|1000	1000011101111000	$1 + x_4 + x_3 + x_3 x_4 + x_2 + x_1$
	01|1000	1000100001110111	$1 + x_4 + x_3 + x_3 x_4 + x_1$
	10|1000	1000011110000111	$1 + x_4 + x_3 + x_3 x_4 + x_2$
0111	00|0111	0111011101110111	$x_4 + x_3 + x_3 x_4$
	11|0111	0111100010000111	$x_4 + x_3 + x_3 x_4 + x_2 + x_1$
…	…	…	…
0001	00|0001	0001000100010001	$x_3 x_4$
	11|0001	0001111011100001	$x_3 x_4 + x_2 + x_1$
…	…	…	…
1110	00|1110	1110111011101110	$x_3 x_4 + 1$
…	…	…	…
0100	00|0100	0100010001000100	$x_3 x_4 + x_4$
…	…	…	…
1011	00|1011	1011101110111011	$x_4 + x_3 x_4 + 1$
…	…	…	…
0010	00|0010	0010001000100010	$x_3 + x_3 x_4$
…	…	…	…
1101	00|1101	1101110111011101	$x_3 + x_3 x_4 + 1$
…	…	…	…

All codewords in this construction are affine functions with the term $x_3 x_4$. In addition, there is an even number of 1's in every codeword of this construction. The first property implies that the sum of any two codewords is an affine function. An interesting property of this error-correcting code is that xoring any three codewords gives another codeword. In addition, the minimum distance d of this subcode is 8.

4.   Theorem

All codewords of any subcode generated by this algorithm, are of the form $affine + x_{n-1} x_n$.

Before we prove the theorem we have to present an algorithm for calculating the Algebraic Normal Form from the Truth Table of a function and vice versa [30]. Let $D = [d_0 \ d_1 \ d_2 \ldots d_{2^n-1}]$ be the coefficient vector of the polynomial representing the Boolean function f (the theorem that helps us calculate the coefficient vector is presented below). If $d_i = 1$, where $0 \le i \le 2^n - 1$, then the monomial $x_0^{i_0} x_1^{i_1} x_2^{i_2} \ldots x_{n-1}^{i_{n-1}}$ appears in the *Algebraic Normal Form* of f. On the contrary, when $d_i = 0$, no monomial appears, where $(i_0, i_1, i_2, ..., i_{n-1})$ is the binary representation of pointer i.

a.   *Example of Calculating the ANF of a Function*

$D = [0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1]$,   means

$d_0 = 0$, $d_1 = 0$, $d_2 = 1$, $d_3 = 0$, $d_4 = 0$, $d_5 = 0$, $d_6 = 1$ $d_7 = 1$. We conclude that:

- due to d_2 (i=01000000 in binary representation), one of the monomials that appears in the Algebraic Normal Form is $x_0^0 x_1^1 x_2^0 x_3^0 x_4^0 x_5^0 x_6^0 x_7^0 = x_1$.

- due to d_5 (i=10100000 in binary representation), one of the monomials that appears in the Algebraic Normal Form is $x_0^1 x_1^0 x_2^1 x_3^0 x_4^0 x_5^0 x_6^0 x_7^0 = x_0 x_2$.

- due to d_6 (i=01100000 in binary representation), one of the monomials that appears in the Algebraic Normal Form is $x_0^0 x_1^1 x_2^1 x_3^0 x_4^0 x_5^0 x_6^0 x_7^0 = x_1 x_2$.

- due to d_7 (i=11100000 in binary representation), one of the monomials that appears in the Algebraic Normal Form is $x_0^1 x_1^1 x_2^1 x_3^0 x_4^0 x_5^0 x_6^0 x_7^0 = x_0 x_1 x_2$.

Finally, the Algebraic Normal Form of the given coefficient vector is $x_1 + x_1 x_2 + x_0 x_2 + x_0 x_1 x_2$. Now, we have to connect the coefficient vector with the Truth Table of the function, using the theorem in [30]. This theorem states that if we have an *n*-variable Boolean function *f*, and *D* the coefficient vector of this function, then $D = f * A_n$, where

$$A_1 = \begin{pmatrix} A_0 & A_0 \\ 0 & A_0 \end{pmatrix} \text{ and } A_0 = [1].$$

b. *Example of Calculating the Coefficient Vector*

Given a Truth Table of a 3-variable Boolean function $f = 01100101$ and working in accordance to the above theorem, we can obtain the coefficient vector. Since,

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

we obtain $D = [0\ 1\ 1\ 0\ 0\ 0\ 1\ 0]$.

Our previous theorem claims that all codewords of any such construction are of the form: $affine + x_{n-1} x_n$.

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c. Proof of Theorem

Indeed, every function of our construction corresponds to coefficient vectors of the form: $D = [d_0 \quad d_1 \quad d_2 \quad 1 \quad d_4 \quad 0 \quad 0 \quad 0 \quad d_8...0... d_{x_{2^n}}...0...]$, where:

1. d_0 can be either 1 or 0 since the 1st bit of our codewords is either 1 or 0 and the 1st column of A_n is $[1 \quad 0 \quad 0 \quad 0...]^T$.

2. d_1 can be either 1 or 0 since the first 2 bits of our codewords are 00, 01, 10 or 11 and the 2nd column of A_n is $[1 \quad 1 \quad 0 \quad 0 \quad 0...]^T$.

3. d_2 can be either 1 or 0 since the first three bits of our codewords are 000,001,111,110,100,011,010 or 101 and the 3rd column of A_n is $[1 \quad 0 \quad 1 \quad 0 \quad 0...]^T$.

4. d_3 can only be 1 since T_2 consists of words of odd number of 1's and the 4th column of A_n is $[1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0...]^T$.

5. d_4 can be either 1 or 0 since the first bit and the 2^{n-1} th bit of our codewords are 00, 01, 10 or 11 and 5th column of A_n is $[1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1...]^T$

For the other bits of D, except 2^n th bits, it is obvious that they are all 0's. On the other hand, all 2^n th bits of D can be either 1 or 0. That format of coefficient vector confirms that the *Algebraic Normal Form* of our construction is $affine + x_{n-1}x_n$.

QED

E. THE DECODING "REVERSE" ALGORITHM

Reversing the algorithm introduced at the very beginning of this chapter, we show that not only we can highly compress any codeword of 1st order Reed-Muller codes, and of new construction of subcodes of 2nd order Reed-Muller codes, but we can also reconstruct a damaged codeword, correcting some errors that occurred during transmission or storage.

1. Conjecture

This new correction method can correct *n*-2 errors, or similarly the number of "guide" word bits.

The general compression ratio of this "reverse" algorithm, as it is already mentioned, is $\dfrac{2^n}{n+1}:1$, and the memory saving is $\left(1-\dfrac{n+1}{2^n}\right)\%$. It is obvious that, for high *n*, the compression ratio is extremely high. For example, we can imagine the memory saving in a realistic case for *n*=15, where we can compress $2^{15}=32768$ bits to only 16 bits, without losing any information of the codeword. The memory saving in this particular case is 99.95%. These calculations highlight the importance of this algorithm in environments where the memory efficiency is critical.

2. Algorithm

<u>Step 1</u>

We split the word in two halves, and both of these halves in halves and so on, until we reach 4 bit chunks.

<u>Step 2</u>

We xor bitwise the first two halves, and using the *majority value* of either 0's or 1's, we obtain the rightmost bit of "guide" word. At the same time, the minority of either 1's or 0's indicates the probable positions of errors in both halves of our word. It is obvious that on first xoring we either/both miss some errors due to double errors occurred on two xored bits, or/and over count some of them due to the fact that error candidates are in both halves.

In order to accurately locate and correct all of the errors, we xor the other halves, and we work as described on the second step. The bit positions that take the majority of candidate errors are the errors.

Our decision about the position of errors can be verified by our construction of the Reed-Muller code used. It is known that using our algorithm, all 1st order Reed-Muller codes can be generated by T_1 set of words, and all subcodes of 2nd order Reed-Muller

codes that we quickly generated, by T_2 set of words. Therefore, all 4-bits words, after the very last split, should be either of T_1 set, in our construction of 1st order Reed-Muller codes, or T_2 set, in our construction of subset of 2nd order Reed-Muller codes. Note, all the T_1 set of words has *distance* 1 or 3 from the T_2 set of words.

Having corrected all *n*-2 errors, we finally obtain the complete compressed word.

It is obvious that the compression ratio is as high as the size of "guide" word.

a.   *Example 16 Bits*

Suppose we transmit the codeword 1101001011010010, and we actually receive the word 1111001010010010. This word has two errors on 3rd and 10th bits. Splitting the received word in two halves, we obtain 11110010 | 10010010. Xoring these halves bitwise, we take 01100000. This string gives us the information that the rightmost bit of "guide" word is 0, and errors might occur in the 2nd bits of one of the halves (2nd or 10th bit of the word), and in the 3rd bits of one of the halves (3rd or 11th bit of the word).

In order to detect in which of the first halves the errors located, we xor the subsequent halves bitwise. Thus, for the left halves we obtain 1101, and the information we obtain is that the left most bit of our "guide" word is 1, and the probable errors are in 3rd or 7th bit of our word. Working identically for the right halves, we obtain 1011, and the information we obtain is that the probable errors are in 2nd or 6th bit of right half (10th or 14th bit of our word). Combining the information of three xorings, getting two votes for 3rd and 10th positions, we conclude that the errors are in 3rd, and 10th bits. Thus, the compressed word is 10|1101.

In all combinations of n-2=4-2=2 errors in the received word, the xoring manipulation can inform us for their position. In case there are no errors in the received word, the algorithm proceeds without the correction process.

b.   *Example 32 Bits*

In this example, we use 32-bit codewords, and we correct three errors that occurred in the same set of 4-bits. Suppose we transmit the codeword

1011010010110100010010101101001011, and we actually receive the word 10111010101101000100101101001011. This word has three errors at the 5th, 6th, and 7th bits. Splitting the received word into two halves, we obtain 1011101010110100 | 0100101101001011. Xoring these halves bitwise, we obtain 11110001111111. This string gives us the information that the rightmost bit of the "guide" word is 1, and errors might occur in 5th bits of one of the halves (5th or 21st bit of the word), in 6th bits of one of the halves (6th or 22nd bit of the word), and in the 7th bits of one of the halves (7th or 23rd of the word).

In order to detect in which of the first halves the errors are located, we xor the subsequent halves bitwise. Thus, for the left 16-bits half 1011101010110100 we obtain 00001110, and the information we obtain is that the middle bit of our "guide" word is 0, and the probable errors are in 5th or 13th bit of our word, 6th or 14th bit of our word, and 7th or 15th bit of our word. Working identically on the right 16-bit half 0100101101001011, we obtain 00000000, and there is no useful information.

At this point, the information we have for the position of errors in our word seems sufficient, but for the sake of the completion of our algorithm, we keep on xoring until we get the 4-bits sets. Therefore, xoring the very first 8-bits set we obtain 0001, and the information we obtain is that the leftmost bit of "guide" word is 0, and the probable errors are in the 4th or 8th bit. Xoring the next 8-bits set 10110100 we obtain 1111 and there is no usable information.

Combining the information of the xor processes, we obtain two votes for 5th , 6th and 7th positions, we conclude that the errors are in these bits. Thus, the compressed word is 101|1011.

In all combinations of n-2=5-2=3 errors in the received word, the xor manipulation can inform us of their position. It is obvious that there are many cases where our "reverse" algorithm can correct more than *n*-2 errors, but we cannot generalize based on these cases only. In case there are no errors in the received word, the algorithm proceeds without the correction process.

F.   CHAPTER SUMMARY

In this chapter, a new simplified algorithm of fast generation of all 1st order and some of 2nd order Reed-Muller codes is analyzed and a fast construction of a linear subcode of 2nd order Reed-Muller code with good properties is presented and analyzed. A hardware implementation of this algorithm is also presented for $n=3$. In addition, the "reverse" of the algorithm is introduced, showing at the same time, the process of decoding. In Chapter VI, we summarize the conclusions of this thesis and future work is proposed.

VI.   CONCLUSIONS AND FUTURE WORK

This thesis points out the difficulty of completely estimating a critical property of error-correcting codes, namely the covering radius of a code. This covering radius problem plays a critical role, along with minimum Hamming distance and decoding complexity, to our decision of choosing the most efficient error-correcting code. Nevertheless, even though it is a well-defined property in coding theory, in the majority of the codes, it can only be bounded and not exactly calculated.

Further, in this thesis, a new method of fast generation of 1st order $R(1,n)$ Reed-Muller codes was introduced. This method seems highly memory efficient and fast, since we generate all 1st order Reed-Muller codes using just the T_1 set of 4-bit words and entire F_2^{n-2} field. For example, to generate $2^{1+\binom{7}{1}} = 2^8 = 256$ codewords of $2^7 = 128$ bits length, thus $R(1,7)$ code, we just need the whole set of T_1 (32 bits), as well as the entire $F_2^{n-2} = F_2^{7-2} = F_2^5 = 32$ bits. Furthermore, the fast construction of 2nd order Reed-Muller codes using both T_1, T_2 sets of 4-bits, is another method that can efficiently use memory. In addition, this algorithm allows the use of a compact way to represent low order Reed-Muller codes.

In this thesis, the hardware implementation of the "expansion" algorithm for each codeword is presented (for $n=3$). The complexity of this logic circuit is analyzed and we show that the number of two-input exclusive OR gates that are needed for the implementation of our algorithm is $2^n - 4$, where n is the number of variables used. Although exponential in n, this is close to minimal mostly because 2^n outputs are needed, four of which are driven directly by four inputs and thus, require no gate. Without doubt, this implementation is faster than any common software running on conventional computers.

In addition, it is obvious that this compact representation of 1st order Reed-Muller codewords is highly memory efficient because it can actually stores a large amount of information in a small word. For example, using a "guide" word of 8 bits, we can

expand, and thus compress a codeword of 1024 bits to a string of 12 bits. The compression ratio in this example is 85:1, and the memory saving is 98.83%.

Reversing the algorithm, we show that not only can we highly compress any codeword of 1st order Reed-Muller codes, and of a new construction of subcodes of 2nd order Reed-Muller codes, but we can also reconstruct a damaged codeword, correcting some errors occurring during the transmission or storage. This new correction method can correct n-2 errors, or similarly the number of "guide" word bits. It is obvious that there are many cases where our "reverse" algorithm can correct more than n-2 errors, but we cannot generalize on these cases.

The compressed string includes all the information of expanded codeword. We show that the general compression ratio of this "reverse" algorithm is $\frac{2^n}{n+1}:1$, and the memory saving is $\left(1-\frac{n+1}{2^n}\right)\%$. It is obvious that, for high n, the compression ratio is extremely high. For example, we can imagine the memory saving in a realistic case for n=15, where we can compress $2^{15}=32768$ bits to only 16 bits, without losing any information of the codeword. The memory saving in this particular case is 99.94%. This estimation highlights the importance of this algorithm in environments where memory efficiency is critical.

One of the main contributions of this thesis is the fast generation of a new 2nd order Reed-Muller subcodes of good properties. Even though, this is a non-linear category of subcodes, their low *distance d*, and some other good properties make them worthy of investigation. Their performance in communication-oriented environments can be simulated, and further investigated in a future work. The coding gain of these subcodes must be simulated. It is also recommended to implement these subcodes in devices used for data storage. The usefulness of these 2nd order Reed-Muller subcodes, might be the minimization of memory errors.

In this thesis, it is proven that the format of *Algebraic Normal Form,* for our fast construction of subcodes of 2nd order Reed-Muller codes is *affine* $+ x_{n-1}x_n$. Therefore, the sum of any two codewords is *affine* function, and the sum of any three codewords is another codeword.

The applicability of both algorithms can be tested in a variety of environments. For example in digital repeaters, where store and forward process takes place, and where *Automatic Response Request* (ARQ) processes are needed, instead of Forward Error Correction (FEC), codewords can be stored in compact form, and transmitted in full representation. The storage of the compact form can last until the transmitter receives an acknowledgement. On the other hand, in storage processes, both algorithms' usefulness is indisputable.

In addition, one communication-oriented application that can take advantage of these algorithms, is when, through a process, signal conditions can be measured, and automatically changes the error-correction coding to match current link quality.

In any case, logical extension of this thesis would include a computer simulation on the performance of proposed algorithms, in various operational environments.

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