SINGULAR SYMMETRIC HYPERBOLIC SYSTEMS AND COSMOLOGICAL SOLUTIONS TO THE EINSTEIN EQUATIONS

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DISSERTATION ABSTRACT

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Characterizing the long-time behavior of solutions to the Einstein field equations remains an active area of research today. In certain types of coordinates the Einstein equations form a coupled system of quasilinear wave equations. The investigation of the nature and properties of solutions to these equations lies in the field of geometric analysis. We make several contributions to the study of solution dynamics near singularities. While singularities are known to occur quite generally in solutions to the Einstein equations, the singular behavior of solutions is not well-understood. A valuable tool in this program has been to prove the existence of families of solutions which are so-called asymptotically velocity term dominated (AVTD). It turns out that a method, known as the Fuchsian method, is well-suited to proving the existence of families of such solutions. We formulate and prove a Fuchsian-type theorem for a class of quasilinear hyperbolic partial differential equations and show that the Einstein equations can be formulated as such a Fuchsian system in certain gauges, notably wave gauges. This formulation of Einstein equations provides a convenient general framework with which to study solutions within particular symmetry classes. The theorem mentioned above is applied to the class of solutions with two spatial symmetries – both in the polarized and in the Gowdy cases – in order to prove the existence of families of AVTD solutions. In the polarized case we find families of solutions in the smooth and Sobolev regularity classes in the areal gauge. In the Gowdy case we find a family of wave gauges, which contain the areal gauge, such that there exists a family of smooth AVTD solutions in each gauge.

This dissertation includes previously published and unpublished co-authored material.

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CHAPTER I

INTRODUCTION

1.1. Prelude

Einstein published his theory of gravitation (general relativity) in 1915. Whereas in the Newtonian theory of gravitation massive bodies interact via a gravitational force (instantaneously and with no apparent mechanism) in a global and rigid space and time frame, in the general theory of relativity, spacetime is a dynamical manifold which interacts with energy and matter. The interaction with the spacetime manifold which provides a mechanism for gravitation; the theory says that massive and massless bodies move along time-like and null, respectively, geodesics in the curved spacetime, and it is these motions which we attribute to the gravitational force. Colloquially, the matter informs the spacetime how to curve and the matter moves along paths determined by the curved spacetime.

The interaction between spacetime and matter in general relativity is governed by the Einstein field equations. Solutions to these field equations represent the gravitational field in a physical scenario, such as outside of the earth, or the entire cosmos, and provide some of the most accurate physical models today. Although we understand many explicit solutions to the Einstein equations quite well, in particular those with a high degree of symmetry and which provide the most common physical models, the understanding of the large-scale behavior of general solutions to the Einstein equations is relatively weak. Indeed, much of the present research in general relativity is in exploring the properties of this full space of solutions.

We consider in this dissertation families of solutions which may provide in some sense cosmological models. Within this context one of the particular families of solutions which are well-understood are the homogeneous and isotropic solutions independently worked out by Friedmann, Lemaître, and Robertson and Walker, hence known as the FLRW solutions. These solutions provide a remarkably good model of our observable universe and form the foundation of many studies in cosmology; they possess however a particularly interesting feature. Observers traveling on time-like paths to the past in an FLRW universe will encounter a singular event in which their worldline terminates within a finite amount of proper time and the spacetime curvature and energy density become unbounded. Does such behavior occur in our universe? Observations of the expansion of our visible universe and the cosmic microwave background radiation suggest that it might, and this singular event has come to be called the "big-bang."

A natural question to ask is whether more general (less symmetric), and presumably physically realistic, solutions to the Einstein equations also exhibit this singular behavior, or whether such behavior is a product of the high-degree of symmetry of the FLRW models. This is now a mathematical question about properties of solutions to the Einstein equations which might model some universe, and not a physical question about our particular universe. The evidence, starting with the work of three Russian physicists Belinski, Khalatnikov, and Lifshitz (BKL), suggests that indeed some sort of singular behavior is quite common in solutions to the Einstein equations. To date, the most powerful result is the (mathematical) proof by Hawking and Penrose in the 1970's that singular events in the sense of time-like worldlines which terminate in finite time are general features of solutions to the Einstein equations. However, further details regarding these singularities, such as whether the curvature is unbounded and the general dynamics of the gravitational field, remain unresolved.

The work of BKL, later Misner, and many others more recently employing numerical methods, indicates that the dynamics of the gravitational field in the singular region is quite complicated. The BKL picture, discussed in more detail in Section 1.3.3., is that in the singular region the dynamics are local, vacuum dominated, and oscillatory in particular sense. Verifying this behavior rigorously in general solutions is beyond present mathematical techniques. In order to make progress in understanding the dynamics of solutions near singularities, research has focused on studying restricted classes of spacetimes characterized by symmetries, the presence of certain matter fields, or a particular subclass of the BKL dynamics.

Solutions with a simpler singular dynamics are observed in numerical investigations, particularly in classes of spacetimes which are *polarized*. Like the BKL-type dynamics, these spacetimes are asymptotically local and vacuum dominated, but unlike the BKL case are not oscillatory. Since such solutions can be modeled in the singular region by functions which satisfy a set of ordinary differential equations obtained from the Einstein equations by dropping spatial derivative terms, they are called *asymptotically velocity term dominated* or AVTD. While this behavior is not general, the study of AVTD solutions is accessible by analytical techniques, and thus provides a valuable "window" into the singular nature of solutions.

The projects described in this dissertation contribute to the research program of finding AVTD solutions with various assumed symmetries. There are four different contributions to this program which are made. The first, which is contained in Chapter II, is the formulation and proof existence and uniqueness theorems for a broad class of so-called Fuchsian partial differential equations (PDE). Equations of this type have been the mathematical work-horses for finding families of solutions which are AVTD. The second contribution is the proof of the existence of a families of smooth and Sobolev-regular AVTD solutions in the class of polarized T^2-symmetric spacetimes. These results are presented in Chapter III. The third and fourth contributions, which are smaller in scope, but lay the ground work for future research, are presented in Chapter IV. In the first portion of this chapter we construct a general Fuchsian reduction of the Einstein equations in wave gauge. This reduction, which obtains a symmetric hyperbolic formulation of the equations used in tandem with the existence theory in Chapter II provides a powerful general tool for investigating AVTD behavior. In the second portion of the chapter we use these tools to investigate the gauge-dependence of the AVTD property. It turns out that the notion of AVTD is dependent upon the coordinates one has chosen, and it is unknown whether a solution which is known to be AVTD in one coordinate system is AVTD in other (perhaps families) of coordinate systems. The work in Chapter IV takes a first step in investigating this issue in the class of Gowdy spacetimes.

The Fuchsian theory which is developed in Chapter II is related to that published in [3, 4] in collaboration with Florian Beyer, Jim Isenberg, and Philippe LeFloch. The paper [3] also contains an application of the Fuchsian theorems to the polarized and half-polarized T^2-symmetric spacetimes, and the analysis in Chapter III is based on this work. The results presented in Chapter IV are unpublished; this work is in collaboration with Florian Beyer and Jim Isenberg. Some of the technical results contained or cited in the Appendices are based on results published in [3, 4].

1.2. The Einstein Field Equations and Solutions

1.2.1. The Einstein Field Equations

At the heart of the theory of general relativity are the Einstein equations. The Einstein equations are a geometric relation describing the interplay between the geometry of spacetime and matter, which can be written[1]

$$Ric_{ij}(g) - \frac{1}{2}R(g)g_{ij} = 8\pi T_{ij}. \tag{1.1}$$

Here g is the metric tensor with Lorenztian signature, $Ric(g)$ is the Ricci curvature of the metric, $R(g) = g^{ij}Ric_{ij}(g)$ is the Ricci scalar, and T is the energy momentum tensor. We have used "geometrized units" in which the gravitational constant G and the speed of light c have been set to one. We also make use of the summation convention where identical upper and lower indices are summed over unless explicitly stated otherwise. Throughout the spacetime indices i, j, k etc. run through $0, 1, 2, 3$, while the indices a, b, c etc. correspond to the spatial degrees of freedom and take values $1, 2, 3$. We work in $n = 3$ dimension, although the Einstein equations apply to gravitational phenomena in $n + 1$ dimensions for any $n \geq 2$.

Although many applications of general relativity are concerned with an Einstein-matter system, in which the energy momentum tensor T couples the Einstein equations to relevant matter equations, there are also dynamical solutions to the Einstein equations with no matter terms, corresponding to $T \equiv 0$. The analogue in electromagnetism is the phenomena of electromagnetic radiation. In this dissertation

[1] Taking the trace of these equations we find that we may also write them in the form $Ric_{ij}(g) = 8\pi(T_{ij} - \frac{1}{2}Tg_{ij})$.

we deal only with the **vacuum Einstein equations**

$$Ric_{ij}(g) = 0. \tag{1.2}$$

This is not supposed be physically-motivated assumption, but rather a way to simplify the analysis and ignore complications which arise as a result of the matter equations. A solution to the vacuum Einstein equations consists of a Lorenztian manifold (M, g) such that g satisfies Eq. (1.2), and can be thought of as gravitational radiation.

Written in a system of coordinates the Einstein equations consist of ten (in three dimensions) second-order nonlinear coupled partial differential equations. In an arbitrary system of coordinates, these equations are of indeterminent character. However, in certain types of coordinates, such as the wave-coordinates which we discuss in Section 4.2. below, the equations take hyperbolic form. This complexity of the equations, along with the diffeomorphism invariance make finding general solutions to the Einstein equations, and determining the long-time behavior of solutions difficult. One scheme for obtaining solutions to the Einstein equations and studying their properties is to set-up an initial value formulation of the equations.

1.2.2. The Cauchy Problem in General Relativity

The initial value formulation, or Cauchy problem, for the Einstein equations may be motivated by the fact that, as mentioned, the equations are hyperbolic in certain systems of coordinates. In an initial value formulation one wishes to specify some initial data, possibly satisfying some constraints, and then evolve this data via evolution equations in order to obtain a unique solution. We give only a short synopsis of the relevant results and definitions in the vacuum case here. For more complete treatments see for example [8, 21, 42, 75].

What constitutes appropriate initial data for the vacuum Einstein equations? Given a solution, that is a Lorenztian manifold (M, g), one expects the initial data to be a space-like hypersurface with a Riemannian metric γ and its time-derivative, which is represented by a covariant two-tensor κ. Since the Lorentzian metric g is to satisfy the vacuum Einstein equations, the data must satisfy certain constraint equations (the Gauss and Codazzi equations)

$$\begin{aligned} S(\gamma) - \kappa^2 + (tr\kappa)^2 = 0 \\ D^a\kappa_{ab} - D_b(tr\kappa) = 0. \end{aligned} \tag{1.3}$$

Here D is the Levi-Civita connection with respect to γ, $S(\gamma) = \gamma^{ab}Ric_{ab}(\gamma)$ is the scalar curvature of γ and indices are raised and lowered with γ. The appropriate initial data for the vacuum Einstein equations can thus be defined as the following.

Definition 1.1. *The set of initial data for the vacuum Einstein equations is the triplet (Σ, γ, κ), where Σ is a 3-manifold, γ is a Riemannian metric and κ is a covariant symmetric two-tensor which satisfy the constraint equations Eq.* (1.3).

Given initial data (Σ, γ, κ) as above, we can then formulate the Cauchy problem for the Einstein equations. The initial value problem for the Einstein equations is to find a Lorentz manifold (M, g) satisfying the Einstein equations, and an embedding $i : \Sigma \to M$ such that $\kappa = i^*k, \gamma = i^*g$, where k is the second fundamental form of $i(\Sigma)$. The manifold (M, g, i) (where we have included the embedding i explicitely) is called the development of the data. An important case is when the initial data yields a hypersurface $i(\Sigma)$, which is a Cauchy surface.

Definition 1.2. *A Cauchy hypersurface in a Lorentzian manifold is a subset which is met exactly once by every inextendible time-like curve.*

If M has a Cauchy hypersurface Σ, then it is called **globally hyperbolic**. The development $D(\Sigma)$ of Cauchy hypersurface is all of M and is called the **globally hyperbolic development**. An important question, which was not settled until 1952 is whether there exist a globally hyperbolic development for any given appropriate initial data to the Einstein equations.

Theorem 1.3 (Choquet-Bruhat 1952, [35])**.** *Given initial data as in Definition 1.1 for the vacuum Einstein equations, there is a globally hyperbolic development.*

The issue of uniqueness in general relativity is subtle due to the diffeomorphism invariance of the equations. Recall that if (M, g) is a solution, and if $\varphi \in Diff(M)$ is a diffeomorphism of M, then $h = \varphi^* g$ also satisfies the Einstein equations, though this pulled-back metric may appear very different. In fact there is an equivalence class of solutions, generated by the diffeomorphism group of M. To obtain a statement about uniqueness then we need a criterion which is invariant on this equivalence class of solutions. The concept of a maximal globally hyperbolic development is useful.

Definition 1.4. *A maximal globally hyperbolic development (MGHD) of initial data to the vacuum Einstein equations,* (M, g, i) *is such that if* (N, h, j) *is any other GHD of the same data, then there is a map* $\psi : N \to M$ *that is a diffeomorphism onto its image, and* $\psi^* g = h$*,* $i = \psi \circ j$*.*

With this notion of maximality, Choquet-Bruhat and Geroch proved the stronger existence and uniqueness result in 1969.

Theorem 1.5 (Choquet-Bruhat and Geroch 1969, [23]). *Given initial data as in Definition 1.1 to the vacuum Einstein equations, there is a maximal globally hyperbolic development of the data which is unique up to diffeomorphisms.*

This result establishes that the MGHD is unique in the space of all globally hyperbolic developments. However, it does not establish uniqueness in the space of all developments. In fact, as we discuss below, there are infinite families of initial data such that the corresponding MGHD may be extended, thus violating uniqueness, and in some sense determinism, in general relativity. The extent to which this occurs in the space of all solutions to the Einstein equations is one of the major open research questions today, and is called **strong cosmic censorship**. One might think of this strong cosmic censorship as establishing a "strong" uniqueness result; we discuss this issue and related conjectures further in Section 1.3. below.

1.2.3. Spacetimes with a T^2 Isometry Group

We now identify a class of solutions which is particular interest in general relativistic studies of cosmology.

Definition 1.6 (Bartnik, [7]). *A solution (M, g) to the vacuum Einstein equations is called a* ***vacuum cosmological solution*** *if it is globally hyperbolic, has closed (compact without boundary) Cauchy hypersurfaces, and satisfies $Ric_g(V, V) \geq 0$ for any unit time-like vector V.*

A useful approach in studies of the Einstein equations has been to consider problems (such as strong cosmic censorship) in classes of spacetimes restricted by symmetry assumptions (or in the non-vacuum case by certain matter models), and through gradually relaxing these assumptions develop the techniques and intuition with which to tackle those problems in more general classes of spacetimes. For cosmological solutions one usually assumes some symmetry on the spatial Cauchy hypersurfaces (a symmetry in time yields a stationary solution). In this dissertation we treat spacetimes with Cauchy data which is invariant under a T^2 spatially acting isometry group.

1.2.3.1. The Space of T^2-Symmetric Spacetimes

In this document we consider that each spacetime is the maximal globally hyperbolic development of an initial data set on a compact Cauchy surface, with the data invariant under an effective $U(1) \times U(1) = T^2$ action. Thus we have cosmological solutions (c.f. Definition 1.6) with a two-dimensional isometry group, which we refer to as T^2-symmetric spacetimes; in other literature, for example [87] and references contained therein, these are called G_2 spacetimes. For spacetimes with this symmetry and with spatial orbits on a three-dimensional connected and orientable manifold the spatial topology is restricted to be $\mathbb{T}^3, \mathbb{S}^2 \times \mathbb{S}^1, \mathbb{S}^3$ or a lens space $L(p, q)$ [64]. Since the lens space is covered by $\mathbb{S}^3$, these cases are not usually considered separately.

These spacetimes can be further classified by considering various conditions on the Killing vector fields which generate the two isometries. The space of T^2-symmetric spacetimes is represented Figure 1.1 below. Let Y and Z be the two spatial Killing vector fields which generate the isometry group. The two subclasses are characterized by the following conditions: I) The hypersurface orthogonal condition which says that $g(Y, Z) = c$, is constant. In the literature a spacetime satisfying this relation is known as **polarized**, since this condition effectively turns off one degree of freedom in the metric. II) The second condition involves quantities known as "twists" which are nicely constructed in terms of the generating forms corresponding to Y, Z. Let

$\xi := g(Y, \cdot), \zeta := g(Z, \cdot)$ be the generating forms of a distribution D. Frobenius' theorem (c.f. [22, 53]) states that D is integrable if and only if the *twists*

$$K_Y := \star d\xi \wedge \xi \wedge \zeta \quad \text{and } K_Z := \star d\zeta \wedge \xi \wedge \zeta$$

both vanish, where $\star$ denotes the Hodge star and $\wedge$ the wedge product. As shown by Chruściel [26], the vacuum Einstein equations imply that the twist quantities are constant. The orthogonally transitive,[2] or more commonly **Gowdy** subclass (named after their first discoverer [38]), is characterized by the vanishing of both twist constants. If a solution is both polarized and Gowdy, then the metric may be written in diagonal form.

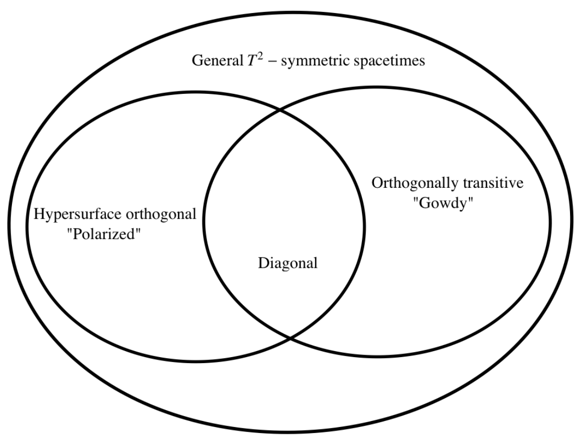

FIGURE 1.1.. The class and subclasses of T^2-symmetric spacetimes. This figure has been adapted from Wainright and Ellis [87].

The Gowdy solutions admit the spatial topologies $\mathbb{T}^3, \mathbb{S}^2 \times \mathbb{S}^1, \mathbb{S}^3$ or a lens space $L(p, q)$. In the case that at least one twist constant is non-vanishing Chruściel has shown that the spatial topology is restricted to be $\mathbb{T}^3$. We call such general solutions simply "T^2-symmetric."

[2]i.e. the two-spaces orthogonal to the group action are surface-forming

1.2.3.2. Areal Coordinates

For spacetimes with a T^2 isometry group, a particularly useful geometrically-defined time coordinate can be specified by setting the time function proportional to the area of the T^2-symmetry group orbits. Coordinates with such a choice of time are called areal coordinates.

To show that these coordinates are well-defined one would like to show that given Cauchy data which is invariant under a spatially acting T^2-isometry group, that the resulting maximal globally hyperbolic development is covered by these coordinates. This is shown for Gowdy intial data with the time t taking values in $(0, \infty)$ in the case of $\mathbb{T}^3$ spatial topology, and $(0, \pi)$ in the remaining cases [26, 62]. A similar result is proved for the T^2-symmetric spacetimes with non-vanishing twist. In [12] the authors show that such spacetimes are covered by areal coordinates with time $t \in (t_0, \infty)$ for some non-negative number t_0. In later work of Weaver and Isenberg [46] this lower bound was clarified to be zero in all cases except that of flat Kasner, in which case $t_0 > 0$.

Beyond the fact that the areal coordinates cover the T^2-symmetric spacetimes, they are useful in studying T^2-symmetric solutions for two additional reasons. The first is that due to the results [12, 26, 46, 62] mentioned above, in these coordinates one approaches the cosmological singularity precisely as the time coordinate approaches $t = 0$. The second is that in these coordinates the Einstein equations can be brought into hyperbolic form. In fact, in the Gowdy spacetimes, the areal coordinates are an example of wave coordinates –c.f. Section 4.2.. Such coordinates arise from a particular gauge choice called a wave gauge, in which the Einstein equations are guaranteed to be hyperbolic.

1.3. Global Properties of Solutions and AVTD Behavior

1.3.1. Singular Solutions to the Einstein Equations

One of the main areas of study in classical general relativity today is in understanding the global properties of general solutions to the Einstein equations. Of particular interest is the study of singularities, which may be roughly thought of as an obstruction to the further development of initial data, or the "boundary" of a globally hyperbolic development.

While the perhaps intuitive notion of a singularity involves relevant quantities (for example metric functions, curvature scalars, etc. in this case) becoming unbounded, the present definition of a singular solution to the Einstein equations is framed in terms of (in)complete geodesics.

Definition 1.7 (Singular solution)**.** *A solution to the Einstein equations is called* singular *if it contains at least one inextendible and incomplete causal geodesic.*

A geodesic is complete if it is defined for all proper time, and incomplete otherwise. The reason for this (perhaps disappointing) definition of a singular solution is simply that one can prove that a solution to the Einstein equations is geodesically incomplete under rather general assumptions – this is the content of the famed singularity theorems of Hawking and Penrose (see [40] for more complete discussion, and Theorem 1.8 below for an example). To make sense the above definition should be restricted to *maximal* solutions. The geodesic must also be inextendible since a geodesic segment which is defined only for a finite range of proper time provides no information on the properties of the spacetime. One might think of the incompleteness as the consequence of "removing" the singularity from the spacetime.

While there is much work devoted to studying such singularity theorems and finding the weakest possible conditions such that one may guarantee a singular spacetime, here we are concerned only with the class of cosmological spacetimes, which in particular are globally hyperbolic. The following is an example of a singularity theorem in this context (this is Theorem 9.5.1 in Wald [88]).

Theorem 1.8 (Cosmological spacetimes are singular)**.** *Let (M, g) be a vacuum cosmological solution (Definition 1.6) with the Cauchy surface Σ such that*

$$\tau \leq C < 0, \quad \textit{everywhere}$$

for $\tau = tr_\gamma \kappa$ is the trace of the extrinsic curvature, and C is a constant.

Then, no past-directed time-like curve from Σ can have a length greater than $3/|C|$, which means in particular that all past-directed time-like geodesics are incomplete.

One may interpret this theorem as follows. If your cosmology is such that at one instant in time it is expanding everywhere at a rate bounded away from zero, then it is singular in the sense of Definition 1.7.

In the case of Einstein-matter systems the condition in Definition 1.6 that $Ric_g(V, V) \geq 0$ for all time-like unit V (i.e. $g(V, V) = -1$) is satisfied if g is a solution to the Einstein equations with a stress energy tensor satisfying the strong energy condition, $T(V, V) \geq -1/2T$. This can be seen from the alternate form of the Einstein equations (footnote below Eq. (1.1)).

The proof of Theorem 1.8 is based on a contradiction argument. Suppose there exists a past-directed time-like curve λ from Σ with a length greater than $3/|C|$, and let p be a point lying beyond $3/|C|$. Then since the spacetime is global hyperbolic there exists a time-like geodesic γ from Σ to p which attains its maximum length (which is in particular greater than $3/|C|$), and further there are no conjugate points for γ between Σ and p. However, this is a contradiction since the expansion condition ensures the existence of a conjugate point within length $3/|C|$ along such a geodesic. For the details of this proof see [88].

While the singularity theorems tell us that a solution to the Einstein equations is singular (in the sense of Definition 1.7) under relatively weak assumptions, they tell us little about the nature of the singularity, and the behavior of the solution near singularities. Indeed, particular examples of solutions show that singular solutions can exhibit very different behavior. The FLRW family of solutions are singular, and the Kretschmann scalar $S_K(g) = Riem^{ijkl}(g)Riem_{ijkl}(g)$ (the square of the Riemann tensor), is unbounded in one or both directions along every time-like geodesic. On the other hand the Taub spacetimes [84], are geodesically incomplete in both the future and past time-like directions, and yet as one approaches the singularity the curvature remains bounded [29]. What's more the spacetime can be extended in inequivalent ways, and the extension need not satisfy the Einstein equations. Because of the original discoverer's of this extension Newman, Unti, and Tamburino, the family of extended spacetimes is called Taub-NUT [65]. The boundary of the globally hyperbolic region in the extended spacetimes is known as a Cauchy horizon.

The known families of singular solutions exhibit one of the above types of behavior, either the curvature (measured by the Kretschmann scalar) is finite in the approach to the singularity and the solution may be extended, leading to a Cauchy horizon, or the curvature is blowing-up. Much of the present work surrounding singularities in general relativity, this dissertation included, is focused on attempting to further resolve the nature of these singularities and the behavior of solutions near them. In the next two sections we discuss several open problems related to these issues.

1.3.2. Strong Cosmic Censorship: "Strong uniqueness"

If a singular solution contains a Cauchy horizon, as in the case of the Taub-NUT solutions, then the solution extends beyond the maximal globally hyperbolic region. Of course if the extension to a given unique MGHD, guaranteed by Theorem 1.5, is unique this might not be so bad, since in that case the entire spacetime could be predicted from initial data. Often however, there exist multiple inequivalent –that is not diffeomorphic– extensions of a given maximal globally hyperbolic development [29]. This type of behavior contradicts our desire that general relativity, a classical theory of physics, should be deterministic. At present all known families of solutions which contain Cauchy horizons also contain symmetries, and therefore do not represent fully general solutions to the Einstein equations. The revised hope then is that fully general (and presumably the most physically relevant) solutions to the Einstein equations are deterministic; this is formulated in the following conjecture.

Conjecture 1.9 (Strong Cosmic Censorship (SCC))**.** *For generic initial data to the vacuum Einstein equations, the maximal globally hyperbolic development is inextendible.*

This conjecture was first proposed by Penrose [66] in 1969. The formulation above is due to Chruściel [27], which in turn is adapted from Moncrief and Eardely [63]. The nomenclature is a bit unfortunate since there is another famous "censorship" conjecture in general relativity pertaining to isolated bodies. This is called the "weak censorship conjecture," (WCC) although SCC does not imply WCC nor vice-versa. A more appropriate name for Conjecture 1.9 might be "strong uniqueness."

There are several approaches to proving a version of Conjecture 1.9 restricted to classes of solutions defined by presence of symmetries, or various matter fields which have been successfully employed. However, proving the conjecture in full generality remains beyond the reach of present techniques.

While the strong cosmic censorship conjecture says that the formation of Cauchy horizons occurs non-generically, we are also interested in the issue of the curvature blow-up at singularities. In a sort of complementary conjecture, this behavior is thought to be generic.

Conjecture 1.10 (Curvature Blow-up)**.** *For generic initial data to the vacuum Einstein equations, curvature blows up in the incomplete directions of causal geodesics in the MGDH.*

The statement of this conjecture comes from [73]. We note that since a C^2-manifold cannot be extended through a curvature singularity, Conjecture 1.10 implies Conjecture 1.9 at least for extensions which are sufficiently smooth. Thus proving a restricted version of Conjecture 1.10 is one pathway to proving restricted strong cosmic censorship. This approach has been successfully employed in the class of Gowdy solutions with $\mathbb{T}^3$ spatial topology (c.f. discussion in Section 1.4.5.).

It should be mentioned that as stated the Conjecture 1.10 and Conjecture 1.9 are not completely clear. First, in a theorem asserting the truth of strong cosmic censorship or curvature blow-up one must specify what is meant by "generic initial data" e.g. a set of non-zero measure in the space of all initial data. One must also specify in such a theorem whether solutions are inextendible as smooth manifolds, or C^2-manifolds etc., and whether the avoided extensions satisfy the Einstein equations (they need not).

The conjectures presented in this section are concerned with whether the singularity is due to a Cauchy horizon or a curvature blow-up. In the next section we discuss related questions concerning the dynamics of solutions in the singular regions.

1.3.3. Generic Singular Behavior and the BKL Proposal

As we have discussed above, the singularity theorems ensure that a solution is singular under rather general assumptions –that is, without very much information. However, there is a sort of conservation of information in that the singularity theorems don't tell us much about the nature of the singularity. Indeed, as we have seen the singularity theorems are unable to distinguish between the formation of a Cauchy horizon and curvature blow-up. One of the main research goals in classical relativity today is to understand the dynamics of the metric field in the region of singularities.

The ideas which drive research on the singular dynamics were put forth by Belinskii, Khalatnikov, and Lifshitz (BKL) in the 1960's and 70's [9, 10, 56]. The BKL proposal, based on heuristic studies, is that generically the spacetime dynamics of an inhomogeneous spacetime is closely approximated by that of a homogeneous model known as a Kasner solution [3] at each spatial point. In this sense the solution is local. According to the BKL proposal the general singular dynamics are also vacuum-dominated in the sense that the matter terms do not significantly contribute, and oscillatory in the sense that at each spatial point the metric is modeled by an infinite sequence of Kasner "epochs" punctuated by transitions in which the particular Kasner-model changes (that is the Kasner exponents p_a introduced below change). Further, the sequence of Kasner "epochs" at each point is uncorrelated.

[3] It should be noted that the Kasner family is just one of several families of homogeneous solutions to the Einstein equations. The homogeneous solutions are organized by the Bianchi-classification, in which the Kasner family is Bianchi I.

The Kasner family is given by a metric of the form

$$g_{Kasner} = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2$$

where the integers $p_a, a = 1, 2, 3$ satisfy $\sum_a p_a = \sum_a (p_a)^2 = 1$. As a result, in a generic Kasner spacetime two of the p_a must be positive while the other is negative. In the resulting approach to the singularity (at $t = 0$), two spatial directions are shrinking, while the third is expanding, leading to a "cigar" type singularity. Hence in the BKL-picture, the spacetime at each spatial point is apparently oscillating: in one Kasner epoch two spatial directions will be shrinking and the third expanding. The spacetime then transitions, changing the local effective values of p_a, and in the subsequent Kasner epoch two generally different spatial directions are contracting –ad infinitum.

A nice illustration of the BKL-type behavior, as well as the simpler asymptotically velocity term domintated (AVTD) behavior which we discuss below is presented in the dynamical systems formulation of the Einstein equations [87]. In this formulation, a solution (within a class of homogeneous solutions) at each time can be represented as a point in a five-dimensional state space with variables $(\Sigma_+, \Sigma_-, N_1, N_2, N_3)$, which roughly correspond the trace-free shear tensor, and the spatial portions of the connection coefficients -see [87] and the references contained therein for a more detailed description of this formulation. The evolution under the Einstein equations then traces out a path in this state-space, and one can bring all the dynamical systems tools to bear on the problem of analyzing the qualitative behavior. The Kasner solutions in this picture are represented by the circle $\Sigma_+^2 + \Sigma_-^2 = 1$, in the $N_1 = N_2 = N_3 = 0$ plane (c.f Figure 1.2 below). There are six "special points" on the Kasner circle represent the case (p_1, p_2, p_3) equal to $(1, 0, 0)$, the "T" points, and $(-1/3, 2/3, 2/3)$, the "Q" points, and permutations. Each of the non-exceptional Kasner points is an unstable equilibrium point.

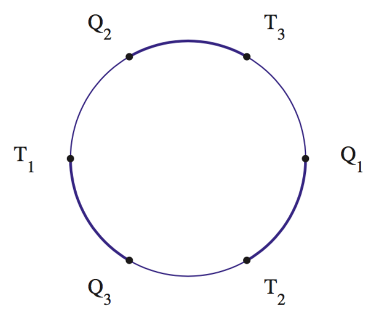

FIGURE 1.2.. The Kasner circle. Figure taken from [6].

Within the dynamical systems formulation, the BKL picture is of the local spacetime at each spatial point repeatedly "bouncing" off the unstable Kasner circle in the approach to the singularity. That is the dynamics of a general spacetime is modeled at each spatial point by a point in this homogeneous state-space. As the singularity is approached, this state-vector approaches a point on the Kasner circle. Yet, since such a point is unstable, the solution transitions, and continues on a trajectory which turns out to steer it towards another point on the Kasner circle, and so on. In this way, an observer traveling towards the singularity in a generic spacetime is expected to experience an infinite sequence of Kasner-like epochs punctuated by "bounces." Further, observers at different spatial points experience generally unrelated sequence of Kasner epochs and bounces. Numerical studies of generic solutions, as well as solutions with symmetries support this picture [11, 13–17, 37].

More recently it has become clear that a phenomena known as "spikes" also play an important role in the dynamics near the singularity. Spikes are when the spatial derivatives of the solution grow very large at isolated points. Explicit spike solutions have been constructed in the Gowdy class [68], and numerical work is underway to further understand their influence [41, 57, 58].

Due to the complicated nature of the BKL-type dynamics, there is very little rigorous work verifying the existence of such solutions. One remarkable exception is in the work of Ringström [69, 70] in the Bianchi type IX homogeneous spacetimes –the simplest class in which this BKL-type oscillatory behavior is observed. Ringström proves the existence of BKL-type oscillatory behavior in this class of spacetimes with a particular matter model, and shows that for generic Bianchi type IX initial data, the Kretschmann scalar is unbounded in the approach to the singularity in the corresponding maximal globally hyperbolic developments –thus establishing restricted curvature blow-up and SCC.

While verifying general BKL-type dynamics for inhomogeneous cosmological solutions has so far proved beyond the reach of analytical techniques, a special case known as "velocity term dominated" or VTD, is more accessible. If a solution to the Einstein equations has VTD dynamics (discussed below) in the singular region, then it is said to be "asymptotically velocity term dominated" or AVTD. In terms of the Kasner circle Figure 1.2. the idea is that under the conditions which lead to VTD behavior, a segment of the Kasner circle becomes stable. Hence the local model solutions at each spatial point in the inhomogeneous cosmology make a few bounces before approaching one Kasner solution asymptotically. The AVTD solutions can be said to be "asymptotically locally Kasner".

In the results of Chapter III and Chapter IV we obtain AVTD solutions with an asymptotic data function k in a particular range. The function k is connected to the Kasner exponents for the model solution, and the indicated range of k corresponds to the stable region of the Kasner circle. More in depth comparison of the dynamical systems formulation and the metric formulation can be found in [6]. We discuss the AVTD dynamics and its relevance to Conjecture 1.10 and Conjecture 1.9 and the BKL proposal in the next section.

We note that while AVTD solutions are not general, verifying the presence of such solutions has provided a useful test-bed with which to develop tools and intuition, and in cases has also been a critical step in proving restricted strong cosmic censorship.

1.3.4. AVTD Solutions

The notion of asymptotically velocity term dominated (AVTD) solutions is introduced and defined in a geometric manner by Isenberg and Moncrief in [44], although the idea might have originated in work of Eardly, Liang and Sachs, [33]. The definition of Isenberg and Moncrief is framed in terms of the ADM field variables (for Arnowitt, Denser, and Misner). We briefly present this formalism here, but for a more complete treatment see for example [61].

1.3.4.1. The ADM Formulation

Although we deal solely with the vacuum Einstein equations in this document, we present the theory in this section for arbitrary matter fields. Let (M, g, ψ) be a globally hyperbolic spacetime with Lorentzian metric g and matter fields ψ. Suppose that $i_t : \Sigma \to M$ is a spatial foliation with the corresponding time vector field $\partial/\partial t$. The 3+1 ADM quantities are: i) A Riemannian 3-metric γ_{ab}. ii) The spatial covariant derivative ∇, and the corresponding Ricci curvature Ric_{ab} and scalar curvature R. iii) The second fundamental form k_{ab} with mean curvature $tr_\gamma k$. iv) The lapse N and the shift M^a. If n is the unit normal vector field to Σ, then we may write $\partial/\partial t = Nn + M$.

The matter quantities are the energy density ρ, the momentum density J_b, and the spatial stress energy tensor S^a_b.

In terms of these fields the Einstein equations take the form of the evolution equations

$$\frac{\partial}{\partial t}\gamma_{ab} = -2Nk_{ab} + \mathcal{L}_M\gamma_{ab} \tag{1.4}$$

$$\begin{aligned}\frac{\partial}{\partial t}k^a_b = {} & N\left(R^a_b + tr_\gamma k k^a_b\right) - \nabla^a\nabla_a N + \mathcal{L}_M k^a_b \\ & + 8\pi N\left(S^a_b + \frac{1}{2}\gamma^a_b(\rho - tr_\gamma S)\right),\end{aligned} \tag{1.5}$$

and the constraint equations

$$R - k^{ab}k_{ab} + (tr_\gamma k)^2 = 16\pi\rho \tag{1.6}$$

$$\nabla_a k^a_b - \nabla_b(tr_\gamma k) = -8\pi J_b. \tag{1.7}$$

1.3.4.2. Definition of AVTD Solutions

The name "asymptotically velocity term dominated" refers to the fact that the solution approaches a model solution that (asymptotic) satisfies a "velocity term dominated" (VTD) system, which is formed from the Einstein equations by dropping spatial derivative terms relative to time-derivative terms. This step encodes the local aspect of the BKL proposal.

To write down the definition of an AVTD solution we form the corresponding VTD system for the Einstein equations. In the ADM fields the VTD system consists of the evolution equations

$$\frac{\partial}{\partial t}\gamma_{ab} = -2Nk_{ab}, \tag{1.8}$$
$$\frac{\partial}{\partial t}k^a_b = N(tr_\gamma k)k^a_b + 8\pi N\left(\mathring{S}^a_b + \frac{1}{2}\gamma^a_b(\mathring{\rho} - tr_\gamma\mathring{S})\right), \tag{1.9}$$

and the constraint equations

$$-k^{ab}k_{ab} + (tr_\gamma k)^2 = 16\pi\mathring{\rho}, \tag{1.10}$$

$$\nabla_a k^a_b - \nabla_b(tr_\gamma k) = -8\pi\mathring{J}_b, \tag{1.11}$$

where the $\mathring{\rho}, \mathring{J}_b$, and $\mathring{S}^a_b$ are corresponding modified versions of the original quantities. This system is obtained from the Einstein system by dropping the terms $\mathcal{L}_M\gamma_{ab}, \mathcal{L}_M k^a_b, R^a_b$, and $\nabla^a\nabla_a N$ from the evolution equations and R from the constraint equations. In a specified system of local coordinates this corresponds to dropping spatial derivative terms in all equations but the momentum constraint, Eq. (1.7). We now give the definition of AVTD solutions adapted from [44].

Definition 1.11 (AVTD Solutions)**.** *A solution to the Einstein equations* (M, g, ψ) *is called* ***asymptotically velocity term dominated*** *(AVTD) if there exists a model spacetime* $(M, \mathring{g}, \mathring{\psi})$ *(same manifold different metric and matter fields), and a foliation* $i_t : \Sigma \to M$ *such that:*

1. *With respect to* i_t*,* $(M, \mathring{g}, \mathring{\psi})$ *satisfies, at least asymptotically, the VTD system Eqs.* (1.8)-(1.11)*.*

2. *The solution* g *approaches* $\mathring{g}$ *in the limit* $t \to t_*$ *(where* t_* *is the singular time) in an appropriate sense: in a suitable norm* $\|\cdot\|$ *on the space of* $3+1$ *quantites* $\{\gamma, k, \rho, J, S\}$ *for any* $\epsilon > 0$*, there exists a* δ *such that for all* t *such that* $|t_* - t| <$

δ

$$\|\{\gamma, k, \rho, J, S\} - \{\mathring{\gamma}, \mathring{k}, \mathring{\rho}, \mathring{J}, \mathring{S}\}\| < \epsilon.$$

It is important to note that the notion of AVTD is coordinate-dependent. This is easily seen from Definition 1.11, since the particular terms which are kept in the VTD equations depend on the choice of coordinates which are used. While it is unknown generally whether a given solution which has been verified to be AVTD in one system of coordinates is AVTD in another system of coordinates, this problem is relatively unstudied in the literature. In Section 4.3. we begin to tackle this question, and related ones in the Gowdy class of spacetimes.

The techniques presented below in Chapter II provide a method for finding *families* of solutions which are AVTD.

1.4. AVTD Behavior in the Literature

The first verification of AVTD solutions to the Einstein equations was in the analytic function class and used a Fuchsian method developed by Kichenassamy and Rendall [51]. The method involves writing a subset of the Einstein equations (roughly the evolution equations) as a first-order system for the first-order fields u, choosing a VTD leading order term u_0, defining the new "remainder" fields w by $u = u_0 + w$, and by inserting this into the system obtain a new system for w. If the system takes the Fuchsian form

$$t\partial_t w(t, x) + N(x)w(t, x) = f(t, x, w, \partial_a w), \tag{1.12}$$

where the singularity is taken to be at at $t = 0$, N is analytic and satisfies a positivity condition, $f[w] = f(t, x, w, \partial_a w)$ is analytic in space, continuous in time, and Lipschitz in the fields w and their spatial derivatives, then the result of [51] shows that a unique solution w exists which vanishes as $t \searrow 0$. This technique has been applied in a wide-range of cosmological classes of solutions in order to verify the existence of families of AVTD solutions. We record some of these results below, along with more recent devopments which establish the existence of smooth (not analytic), and less regular AVTD solutions. Most of the techniques are applied first in the case of the Gowdy spacetimes as these are the simplest of the inhomogeneous classes. We discuss the various methods which have been used, and then in later sections how these techniques have been extended to more general classes of solutions.

1.4.1. AVTD Gowdy Spacetimes

As an application of their Fuchsian theory for analytic functions in [51], Kichenassamy and Rendall find a family of AVTD solutions to the $\mathbb{T}^3$-Gowdy equations. They use the areal foliation described in Section 1.2.3.2., in which the metric has the form Eq. (4.28) below, and treat the equations for $\tilde{P}$ and $\tilde{Q}$. They show that for $\tilde{P}, \tilde{Q}$ of the form Eq. (4.29) with analytic asymptotic data $\{k, P_*, Q_*, Q_{**}\}$, the corresponding six-dimensional first order system forms a Fuchsian system of the form Eq. (1.12) as long as $k \in (0, 1)$ (the "low-velocity" case) or $k > 0$ and $\partial_{\tilde{x}} Q_* = 0$ (the "high-velocity" case).

To go beyond the rather rigid class of analytic functions (see discussion below in Section 1.4.4.) Rendall developed an approach for obtaining smooth solutions to Fuchsian equations [67]. The scheme is based on using a sequence of analytic solutions as approximates to a desired smooth solution, and by reformulating the equations in symmetric hyperbolic form, using associated energy estimates to show that this sequence does in fact converge. While significantly more involved than the analytic Fuchsian "algorithm," Rendall is able to prove the existence of smooth AVTD solutions in the $\mathbb{T}^3$-Gowdy class again of the form Eq. (4.29).

As mentioned above the Gowdy symmetry admits spatial topologies other than $\mathbb{T}^3$, namely $\mathbb{S}^1 \times \mathbb{S}^2$, and $\mathbb{S}^3$, or a lens space which may be covered by $\mathbb{S}^3$. The results of Kichenassamy and Rendall, and Rendall discussed above are actually local in space, and thus apply to these other spatial topologies away from the axis of symmetry. In [77], Ståhl uses the analytic Fuchsian result of [51] and generalizes Rendall's scheme in [67] in order to extend these analytic and smooth AVTD solutions near the symmetry axis in $\mathbb{S}^1 \times \mathbb{S}^2$, and $\mathbb{S}^3$ Gowdy spacetimes. It seems however, that the VTD condition at the axis of symmetry forces the asymptotic velocity k to lie outside of the range $(0, 1)$. Recall that for general AVTD Gowdy solutions away from such an axis, the value k must lie within $(0, 1)$. This is not an issue for the "half-polarized" (or polarized) Gowdy solutions where k may be any real number greater than zero. More work is necessary in order to understand the nature of these solutions near the symmetry axes.

Another approach for obtaining smooth solutions to Fuchsian-type equations has been developed by Beyer and LeFloch [18]. Rather than constructing (less-regular) smooth solutions from (more regular) analytic solutions, this method starts by proving the existence of weak solutions, and by increasing the amount of assumed regularity, constructs solutions which are Sobolev-regular or smooth. The method relies on obtaining a symmetric hyperbolic system and using the associated energy estimates, as well as the existence of solutions to the usual Cauchy problem for these systems. Beyer and LeFloch prove their Fuchsian theorem for semilinear equations which are second-order. Below in Chapter II and the references cited therein, we prove a more general version of this theorem for equations which are quasilinear. Since the Gowdy equations in the areal gauge are semilinear, Beyer and LeFloch are able to apply their theorem in order to obtain families of smooth and Sobolev-regular AVTD $\mathbb{T}^3$-Gowdy solutions. The smooth family which they find coincides with that found by Rendall in [67]. The approach of Beyer and LeFloch is widely applicable, since it is based on a general existence theorem for semilinear Fuchsian PDE's. Similar to the analytic theory, the proof of the existence of AVTD solutions reduces then to verifying certain structural properties of the equations given a VTD leading order term.

1.4.2. AVTD Polarized T^2-Symmetric Spacetimes

All of the Fuchsian techniques presented in the section above have been applied in the polarized T^2-symmetric spacetimes. All AVTD solutions found so far in this class have been in areal coordinates. In [43], Isenberg and Kichenassamy use the analytic Fuchsian theorem to find a family of analytic AVTD polarized T^2-symmetric solutions. Later, Clausen [32] extended this work to prove the existence of analytic AVTD half-polarized T^2-symmetric solutions. The half-polarized condition, as explained below in Section 3.2.3., is a restriction on the asymptotic data. Clausen also generalizes the Fuchsian scheme of Rendall and Ståhl [67, 77] in order to show that there is a corresponding family of AVTD half-polarized T^2-symmetric solutions which are smooth.

In Chapter III we use the Fuchsian theory of Chapter II (an extension of the work of Beyer and LeFloch [18, 19]) to find families of polarized and half-polarized T^2-symmetric solutions which are Sobolev-regular and smooth. The smooth family which we find is the same as that found by Clausen, while the Sobolev-regular family is completely new. The details of these results are discussed in Chapter III.

1.4.3. AVTD Solutions With Fewer Symmetries

As we've mentioned above progress in this field occurs through investigating simple examples and then gradually relaxing the symmetry or matter-field assumptions to obtain results in more general classes of spacetimes. In keeping with this program, the Fuchsian techniques for the analytic functions have been applied to spacetimes with only one Killing vector field –the $U(1)$-symmetric spacetimes.

The $U(1)$-symmetric class is much richer in a variety of ways. The four-dimensional manifold is a $U(1)$ bundle over a $2+1$ Lorentzian manifold, $\Sigma \times R$, where Σ is a 2-dimensional Riemannian manifold. Different cases can be distinguished by the topology of Σ and by the $U(1)$ bundle. The polarization conditions are similar to those in the T^2-symmetric equations. A solution is said to be polarized if one of the metric functions is non-dynamical, and half-polarized if only one of the free functions in the asymptotic data is a fixed constant. While the areal coordinates have proved to be very useful in finding AVTD solutions in the T^2-symmetric class, no such coordinates exist for the $U(1)$-symmetric spacetimes, thus adding another level of complexity.

The simplest case was treated first by Isenberg and Moncrief [45]. The authors assume the spatial topology to be $\mathbb{S}^1 \times \Sigma = \mathbb{T}^3$ for the full solution, and choose a harmonic time coordinate, for which they are able to prove the existence of a families of polarized and half-polarized analytic solutions which are AVTD. Later work with Choquet-Bruhat proved the existence of analytic AVTD solutions with general topology for Σ in the polarized case [25], and in the half-polarized case [24] under an additional assumption that the conformal class of the metric on Σ is independent of t.

While the Fuchsian theory presented in Chapter II is suitable for finding smooth and Sobolev-regular AVTD solutions in the $U(1)$-symmetric class, this work has not been completed. The issue is in finding a gauge and a parametrization of the metric fields in which the Einstein equations written as a first-order system are symmetric hyperbolic. This is the motivation for studying Fuchsian formulation of the Einstein equations in wave gauges as we do in Chapter IV.

There are many more results concerning AVTD solutions and general properties of spacetimes with various matter fields, which we do not mention here in the interest of space and simplicity. Although we concern ourselves only with vacuum spacetimes in this document, we mention one non-vacuum result because of its importance. This is the work of Andersson and Rendall [5] to find a family of analytic AVTD solutions to the general Einstein equations (no assumed symmetry) coupled to a scalar field or stiff fluid. This is an important result since it is one of the only results for general classes of spacetimes in which the singular behavior may be rigorously resolved. Of course the presence of the particular matter fields in this result are necessary, and render the resulting situation certainly not fully general. We believe that the techniques developed in Chapter II and Chapter IV may eventually be applied in order to find corresponding families of general-scalar field AVTD solutions which are smooth and Sobolev-regular.

1.4.4. Smooth Versus Analytic Solutions

As we have seen there are several results concerning the existence of AVTD solutions in the analytic function class. While a great starting point, solutions in this class are not completely satisfactory for a few reasons. One reason has to do with the basic tenant that general relativity be a local theory. That is, given any open, non-empty sets U, V of a connected spacetime (M, g) such that no point of U may be connected by a causal curve to any point of V, we expect that the dynamics in U is independent of that in V. However, for an analytic spacetime, $g|_U$ is essentially determined by events in V. The analytic spacetime is of course still *causal*, it is just *rigid*.

Furthermore, the notion of well-posedness fails to hold for initial value problems in the analytic function class. The Cauchy-Kovalevski theorem says there exists a real analytic solution to the mth order Cauchy problem with real analytic coefficients and initial data, and moreover that the solution is unique in the real analytic class. However, there is no continuous dependence on initial data: Suppose ϕ_k is sequence of real analytic data which converges to the continuous data ϕ. There is no guarantee that the sequence of solutions u_k converges to a solution of the Cauchy problem u with the data ϕ. Moreover, the Cauchy-Kovalevski theorem only claims that the solution is unique in the real analytic class, and does not exclude other solutions in say the smooth class.

To summarize these issues in the present context, we note that this research program is aimed at finding and characterizing general solutions to the Einstein equations. As we have seen the real analytic function class is small and rather rigid, and therefor is not considered very general, and in particular not general enough to study issues such as strong cosmic censorship. It is thus important to extend the results for existence of AVTD solutions in the real analytic class to the smooth or less regular function classes.

1.4.5. Literature Summary and Outlook

We end this literature survey by tabulating the known results for vacuum AVTD solutions and for strong cosmic censorship (strong uniqueness). We also discuss the anticipated progress in the coming years.

1.4.5.1. AVTD Solutions in Increasingly General Classes

The results of the discussions in Section 1.4.1.-Section 1.4.3. can be summarized in Table 1.1 below. As the table indicates, families of AVTD solutions in the analytic function class were found quite rapidly after the theorem of Kichenassamy and Rendall [51] was proved in 1999. Finding families of AVTD solutions in the less regular smooth, or Sobolev function classes is significantly more difficult since stronger structural conditions on the equations are required. The method of Rendall in [67] has proved difficult to generalize to more general classes of spacetimes. We anticipate that with the theory developed in Chapter II and Chapter IV that families of smooth and Sobolev regular AVTD solutions in the polarized U(1)-symmetric class will be forthcoming.

The progress towards the lower right corner of this table is clear. In fact, with the Fuchsian theorems developed in this dissertation, along with the Fuchsian formulation of the Einstein equations in wave gauges (Chapter IV) we expect to complete Table 1.1 That is: in any class of sufficiently regular spacetimes which is polarized and for which each member contains at least one Killing vector field, there exists a family of AVTD solutions. Of course it is another matter to show that AVTD solutions are in some sense generic in such classes of spacetimes –this would constitute a step towards establishing strong cosmic censorship.

TABLE 1.1. Families of AVTD solutions in classes of inhomogeneous vacuum spacetimes.

Vacuum Spacetime	**AVTD** (C^ω)	**AVTD** (C^∞)	**AVTD** (H^q)
Polarized Gowdy	N.R.[11]	1990[1]	1990[1]
Gowdy $\mathbb{T}^3$	1999[2]	2000[3], 2010[4]	2010[4]
Gowdy $\mathbb{S}^2 \times \mathbb{S}^1, \mathbb{S}^3$	2002[5]	2002[5]	N.A.[11]
Polarized T^2**-Symmetric**	1999[6]	2007[7], 2013[8]	2013[8]
Polarized U(1)-Symmetric	2002-2005[9]	In progress[10]	In progress[10]

[1] Isenberg and Moncrief, 1990 [44]. Spatial topologies $\mathbb{S}^1 \times \mathbb{S}^2, \mathbb{S}^3$ and $\mathbb{T}^3$.

[2] Kichenassamy and Rendall, 1999 [51].

[3] Alan Rendall, 2000 [67].

[4] Beyer and LeFloch, 2010 [18]. $\mathbb{T}^3$-spatial topology only.

[5] Frederick Ståhl, 2002 [77]. Ståhl treats the analytic and smooth function classes in the same paper. See discussion in text.

[6] Isenberg and Kichenassamy, 1999 [43].

[7] Adam Clausen, 2007 [32].

[8] Ames, Beyer, Isenberg, and LeFloch, 2013 [3]. These results are also contained in Chapter III of this dissertation.

[9] The results here are contained in three separate papers. Isenberg and Moncrief, 2002 [45] treat the simplest polarized case with $\mathbb{T}^3$-spatial topology. Choquet-Bruhat, Isenberg and Moncrief, 2005 treat the polarized topologically general case in [25]. Finally, later in 2005 Choquet-Bruhat, Isenberg treat the half-polarized case in [24].

[10] Work in progress, see Chapter V.

[11] N.A. stands for "not available". N.R. stands for "not relevant". Because the polarized Gowdy solutions can be computed as an explicit series, the analytic theory is not necessary.

1.4.5.2. AVTD Solutions and Strong Cosmic Censorship

It may seem unintuitive that studying non-generic families of solutions with a particular singular dynamics can provide insight into problems regarding generic solutions. It turns out however that in restricted symmetry-defined classes of solutions, verifying AVTD behavior has been a vital step in proving versions of the curvature blow-up (Conjecture 1.10) and strong cosmic censorship (Conjecture 1.9) conjectures restricted to these symmetry-defined families.

In [44] Isenberg and Moncrief find families of polarized Gowdy solutions (with $\mathbb{T}^3, \mathbb{S}^1 \times \mathbb{S}^2$, and $\mathbb{S}^3$ spatial topologies) which are AVTD, and use the resulting expansions to compute the Kretschmann scalar, and show that it is unbounded. In a later paper with Chruściel [30], they show that such solutions are generic in the space of all polarized Gowdy solutions, thus proving a restricted version of curvature blow-up and strong cosmic censorship in that class of spacetimes. A similar, albeit much more difficult, result has been proved for the fully general Gowdy class with $\mathbb{T}^3$ spatial topology by Ringstöm [71, 72, 74, 76]. In Table 1.2 we summarize the current state of knowledge regarding strong cosmic censorship in classes of inhomogeneous vacuum spacetimes.

TABLE 1.2. For each class of inhomogeneous vacuum spacetimes, we note the largest function class in which families of AVTD solutions have been found, and whether strong cosmic censorship has been verified. More details on the AVTD solutions can be found in Table 1.1.

Vacuum Spacetime	**AVTD**	**SCC**
Polarized Gowdy	Rough	Yes, 1990
Gowdy $\mathbb{T}^3$	Rough	Yes, 2009
Gowdy $\mathbb{S}^2 \times \mathbb{S}^1, \mathbb{S}^3$	Smooth	No
Polarized T^2-Symmetric	Rough	No
Polarized $U(1)$-Symmetric	Analytic	No

While we believe that families of AVTD solutions in the above classes of vacuum spacetimes which are smooth or less regular will be found within the next few years, the verification that such solutions are generic within each class is much more difficult. It can be hoped that the techniques developed by Rinström for the Gowdy class may form a foundation for proving strong cosmic censorship in the class of polarized T^2-symmetric spacetimes, but to the authors knowledge this has not yet been investigated. The issue of proving strong cosmic censorship in the class of polarized $U(1)$-symmetric spacetimes is even further out.

CHAPTER II

FUCHSIAN THEORY FOR SYMMETRIC HYPERBOLIC SYSTEMS

This chapter contains work published in [3, 4]. The calculations were performed by E. Ames and F. Beyer; while writing was done by E. Ames, F. Beyer, and J. Isenberg. P.G. LeFloch contributed editorial changes.

2.1. Prelude

Fuchsian techniques have been used in cosmology since 1999 with the work of Kichenassamy and Rendall [51]. Their use in studying solutions to hyperbolic equations and blow-up phenomena dates back much further; see in particular the work of Kichenassamy [48–50], as well as Tahara [20, 78–83]. To the author's knowledge however, there are only two results in the literature concerning *quasilinear* hyperbolic equations. The first of these, by Claudel and Newman [31], is a well-posedness theorem for the Cauchy problem with initial data on the singularity. In order to be able to prescribe this initial data in a sensible way, severe restrictions on the structure of the equations are needed, and as noted in [67] these conditions are not met in the PDE systems for our application of interest (the T^2-symmetric Einstein equations). The second result concerning quasilinear systems is by Rendall [67]. As discussed in Section 1.4.1. above, Rendall develops an approach in which the steps rely on details of the PDE system under consideration. While this has been generalized to PDE other than the Gowdy system considered by Rendall (e.g. [32, 77]), it has proved difficult to formulate this approach as a general theorem which may then be applied in a large class of PDE.

In this chapter, as well as in [3, 4], we formulate and prove existence and uniqueness theorems for a class of quasilinear Fuchsian PDE. These theorems establish the existence of solutions to the *asymptotic value problem* (in contrast to the initial value problem), in which one obtains a solution that approaches a prescribed model solution within a specified region. This is useful for studying solutions which blow up, since this model solution is allowed to be unbounded. We present a more formal definition of the asymptotic value problem (AVP) below in Section 2.2.4..

There are two main results concerning the existence of solutions to the AVP: Theorem 2.10 establishes the existence and uniqueness of solutions which have a "rough" Sobolev-type regularity. This theorem should be compared to well-posedness results for the initial value problem (IVP) for quasilinear symmetric hyperbolic systems (e.g. [75, 86]). The results for the AVP require one more degree of regularity than those for the IVP. It is unknown whether this is a consequence of our method of proof, or inherent in asymptotic value problems. We also point out that our result does *not* establish well-posedness of the AVP; a result proving continuous dependence of the solution on the asymptotic data is still missing. Our second main result, Theorem 2.28 is designed to "fix an issue" (discussed in more detail here Section 2.6.1.), in which the parameter specifying the control in time on the solution becomes restricted. It turns out that in order to "loosen" this parameter, we must assume greater control over the spatial regularity. We prove the theorem in the smooth (C^∞) case, although similar results could be proved assuming only Sobolev-regularity of sufficiently high order.

The Fuchsian systems, asymptotic value problem and the fundamental Fuchsian theorem Theorem 2.10 are presented in Section 2.2.. The proof, which is outlined in Section 2.3., is contained in Section 2.4. and Section 2.5.. We first prove the existence of solutions to a *linear* Fuchsian system by establishing the existence of weak solutions, and then showing that under stronger assumption on the asymptotic data, that these solutions are in fact strong. This linear existence is then used in a fixed point argument for the quasilinear systems in Section 2.5.. In Section 2.6. we formulate and prove Theorem 2.28 for the smooth systems.

2.2. Quasilinear Symmetric Hyperbolic Fuchsian Systems

2.2.1. Class of Equations

Consider a system for $u : (0, \delta] \times T^n \to \mathbb{R}^d$ of the following form:

$$S^0(t,x,u(t,x))Du(t,x) + \sum_{a=1}^{n} S^a(t,x,u(t,x))t\partial_a u(t,x) + f(t,x,u(t,x)) = 0, \quad (2.1)$$

where each of the $n+1$ maps $S^0, \ldots, S^n$ is a symmetric $d \times d$ matrix-valued function of the spacetime coordinates $(t,x) \in (0,\delta] \times T^n$ and of u (but not of the derivatives of u), while $f = f(t,x,u)$ is a $\mathbb{R}^d$–valued function of (t,x,u). We suppose S^j, f are smooth in t, and H^{q_0} in (x,u) for $q_0 > n/2+1$. We set $D := t\,\partial_t = t\frac{\partial}{\partial t} = x^0\frac{\partial}{\partial x^0}$, while $\partial_a := \frac{\partial}{\partial x^a}$ for[1] $a = 1, \ldots, n$. We list the precise requirements for S^j and f below. This is the class of equations studied in detail in [3] (in the case $n = 1$) and in [4] (for general n). Eq. (2.1) differs from the corresponding equations in [3, 4] in that here we have omitted the $N(t,x,u)u$ term. The distinction between this term and $f(t,x,u)$ is most relevant below when we introduce the notion of quasilinear symmetric hyperbolic Fuchsian systems, Definition 2.7. Therefore we have chosen to simplify the presentation and write the system as in Eq. (2.1).

[1]In all of what follows, indices $i, j, \ldots$ run over $0, 1, \ldots, n$, while indices $a, b, \ldots$ take the values $1, \ldots, n$.

2.2.2. Function Spaces

In order to control and measure the regularity and the decay in time near the singularity $t = 0$ of functions $w(t,x)$ depending on the space and time coordinates, we introduce a family of time-weighted Sobolev spaces. Letting $\mu : T^n \to \mathbb{R}^d$ be a smooth function, we define the matrix

$$\mathcal{R}[\mu](t,x) := \mathrm{Diag}\left(t^{-\mu_1(x)}, \ldots, t^{-\mu_d(x)}\right), \tag{2.2}$$

and use $\mathcal{R}[\mu]$ to define the following norm for functions $w : (0,\delta] \times T^n \to \mathbb{R}^d$:

$$\begin{aligned} ||w||_{\delta,\mu,q} &:= \sup_{t\in(0,\delta]} ||\mathcal{R}[\mu]w||_{H^q(T^n)} \\ &= \sup_{t\in(0,\delta]} \left(\sum_{\alpha,|\alpha|=0}^{q} \int_{T^n} |\partial^\alpha(\mathcal{R}[\mu]w)|^2 dx \right)^{1/2}; \end{aligned} \tag{2.3}$$

whenever this expression is defined. In Eq. (2.3) the spatial derivatives of the $\mathcal{R}[\mu]$-weighted fields are controlled in the usual Sobolev space $H^q(T^n)$ (Definition A.2) of order q on the n-torus T^n; the parameter α denotes a partial derivative multi-index. The behavior in time is controlled by taking the supremum of $t \in (0,\delta]$, and by the explicitly t-dependent weight $\mathcal{R}[\mu]$. Since the spatial derivatives act on this weight as well, logarithms in t to the power $|\alpha|$ are generated; e.g. in the case $d = 1$, and considering one spatial derivative we have the term $(\mu' t^{-\mu} \log t w)^2$. In order for the supremum to be finite then, we require $t^{-\mu}w = O(t^\epsilon)$ for any $\epsilon > 0$ (without the $\log t$, ϵ would be allowed to be zero as well).

Next, we define the function space $X_{\delta,\mu,q}(T^n)$ to be the completion of the set of functions $w \in C^\infty((0,\delta] \times T^n)$ for which the above norm is finite.

Lemma 2.1. *For any $\delta > 0$, exponent vector μ, and integer $q \geq 0$, the space $X_{\delta,\mu,q}(T^n)$ forms a Banach space (Definition A.7).*

This lemma follows from the definition of $X_{\delta,\mu,q}(T^n)$ (that is as the completion) and that fact the Sobolev space $H^q(T^n)$ is a Banach space.

A closed ball of radius r about 0 in $X_{\delta,\mu,q}(T^n)$ is denoted by $B_{\delta,\mu,q}(r)(T^n)$, and for a ball about $f \in X_{\delta,\mu,q}(T^n)$ by $B_{\delta,\mu,q}(r, f)(T^n)$. Note that we often write $X_{\delta,\mu,q}$ in place of $X_{\delta,\mu,q}(T^n)$, with the argument understood to be T^n. To handle functions which are infinitely differentiable and for which we control all spatial derivatives, we also define the space $X_{\delta,\mu,\infty} := \cap_{q=0}^{\infty} X_{\delta,\mu,q}$.

In the following, we refer to parameters μ as **exponent vectors**. We write $\nu > \mu$ for two exponent vectors (of the same dimension) if each component of ν is larger than the corresponding component of μ at each spatial point. If μ is an exponent vector and ϵ a smooth scalar function then $\mu + \epsilon$ refers to the exponent vector with components $\mu_i + \epsilon$.

In working with $d \times d$-matrix-valued functions (such as S^j), we use analogous norms and function spaces. In these cases, we consider d-vector-valued exponents ξ and define the space $X_{\delta,\xi,q}$ of $\mathbb{R}^{d\times d}$-valued functions S in the same way as for $\mathbb{R}^d$-valued functions, but with $\mathcal{R}[\mu]w$ in Eq. (2.3) replaced by $\mathcal{R}[\xi] \cdot S$ (where $\cdot$ denotes the matrix product). According to this definition the i^{th} row of S is controlled by ξ_i, and thus the control is row-wise as opposed to element-wise. This definition is a special case of the definition given in [3, 4], which is sufficient for our needs and simplifies the presentation.

Properties of the spaces $X_{\delta,\mu,q}$ and relations between spaces with different parameters are detailed in Appendix B.

2.2.3. Function Operators

In dealing with nonlinear partial differential equations it is necessary to understand functions of the form

$$F : (0, \delta] \times T^n \times \Omega \to \mathbb{R}^m, \quad (t, x, w) \mapsto F(t, x, w),$$

where Ω is an open set of $\mathbb{R}^d$ containing zero. The functions $w : (0, \delta] \times T^n \to \mathbb{R}^d$ we consider are in a function space $X_{\delta,\mu,q}$, for some $\delta > 0$, $q > n/2$, and an exponent vector μ. We wish to view F as the map

$$F : X_{\delta,\mu,q} \to X_{\delta,\nu,q} \quad w(t, x) \mapsto F(w)(t, x) := F(t, x, w(t, x)),$$

between such function spaces, for some other exponent vector ν. Under what conditions is this map well-defined? Suppose $F(t, x, w)$ is continuous in all its arguments, and suppose $w \in B_{\delta,\mu,q}(s)$ for δ, μ, q as above, and for some $s > 0$. Since $q > n/2$, $w(t, x)$ is continuous in space by the Sobolev inequality. We have

$$\sup_{(t,x)\in(0,\delta]\times T^n} |w(t, x)| = \sup_{t\in(0,\delta]} \|w(t, \cdot)\|_{L^\infty} \leq C(n, q) \sup_{t\in(0,\delta]} \|w(t, \cdot)\|_{H^q}.$$

If $\mu \geq 0$, then $\sup_{t\in(0,\delta]} \|w(t, \cdot)\|_{H^q} \leq \|w\|_{\delta,\mu,q} \leq s$, and there exists an $s_0 \leq s$ (depending in general on n, q) such that all $w \in B_{\delta,\mu,q}(s_0)$ are contained in $\Omega \subset \mathbb{R}^d$. In this case $F(w)(t, x)$ is a well-defined **function operator** from $B_{\delta,\mu,q}(s_0)$ to $X_{\delta,\nu,q}$. If any components of μ are negative, then we must take $\Omega = \mathbb{R}^d$.

Given these observations we make the following definition of function operators. The operators may arise from continuous functions on $(0, \delta] \times T^n \times \Omega$ as discussed above, or they may not.

Definition 2.2 (Well-defined function operator)**.** *Fix positive integers n, d, m, and $q > n/2$, as well as exponent vectors μ, ν. The map $w \mapsto F(w)$ taking functions $w : (0, \delta] \times T^n \to \mathbb{R}^d$ to functions $F(w) : (0, \delta] \times T^n \to \mathbb{R}^m$ is a well-defined function operator provided there exists real numbers $\delta, s_0 > 0$ such that F maps $B_{\delta',\mu,q}(s_0)$ into $X_{\delta',\nu,q}$ for all $\delta' \in (0, \delta]$.*

We note that in the case of the function operator F arising from a continuous function as discussed above it follows that if $F : B_{\delta',\mu,q}(s_0) \to X_{\delta',\nu,q}$ is well-defined for $\delta' = \delta$, then the function operator is also well-defined for all $\delta' \in (0, \delta]$.

The following property is used extensively in the proofs of our main theorems.

Definition 2.3 (Lipschitz property)**.** *A function operator F as in Definition 2.2 is* ***Lipschitz*** *in the $X_{\delta,\nu,q}$ norm provided for all $\delta' \in (0, \delta]$ and for all $s' \in (0, s_0]$ there exists a constant $C > 0$, depending in general on s', q, n such that*

$$\|F(w) - F(\tilde{w})\|_{\delta',\nu,q} \le C\|w - \tilde{w}\|_{\delta',\mu,q}, \tag{2.4}$$

for all $w, \tilde{w} \in B_{\delta,\mu,q}(s')$.

Suppose F satisfies the Lipschitz estimate Eq. (2.4) with regularity q. It does not follow in this case that F satisfies a similar Lipschitz estimate with regulartiy $q - 1$. In applications in which this is a desirable property, it must be shown independently. We say for shorthand that F is Lipschitz in the q-norm, or in the $(q - 1)$-norm as appropriate.

Another useful property of an operator is boundedness.

Definition 2.4 (Bounded operators)**.** *A function operator F as in Definition 2.2 is* ***bounded*** *if for all $w \in B_{\delta,\mu,q}(s_0)$ there exists an $r > 0$ such that $F(w) \in B_{\delta,\nu,q}(r)$.*

In fact, the next lemma shows that if a function operator is Lipschitz, then it is also bounded.

Lemma 2.5. *If F is as in Definition 2.2 and satisfies the Lipschitz property Eq.* (2.4)*, then it is uniformly bounded in the following sense. Let $w \in B_{\delta,\mu,q}(s_0)$ for any $s_0 > 0$ and a $q > n/2$. Then*

$$\|F(w)\|_{\delta,\nu,q} \leq \|F(0)\|_{\delta,\nu,q} + C\|w\|_{\delta,\mu,q} \leq \|F(0)\|_{\delta,\nu,q} + Cs_0.$$

We now make some remarks about the "smooth case" $q = \infty$. By smooth we mean that there is no upper bound for q. A function operator is a **smooth function operator** if it satisfies Definition 2.2 for all $q \geq p$ for some $p > n/2$. In particular, the real numbers δ, s_0 may depend on q. If an operator is Lipschitz as in Definition 2.3, then for each q we have the estimate Eq. (2.4) with a corresponding constant C_q which may depend on q. Although in the smooth case such an estimate must hold at each finite q, the sequence of constants C_q may not be bounded.

More discussion of function operators, and results concerning specific function operators which we encounter in our applications are contained in Appendix C.

2.2.4. The Asymptotic Value Problem and Fuchsian Systems

In this section we introduce the notion of the asymptotic value problem, to be compared with the Cauchy, or initial value problem. The Fuchsian theory which we develop in this chapter provides a scheme for finding solutions to the asymptotic value problem for equations of the type Eq. (2.1).

In the usual Cauchy problem for the partial differential equation $\mathcal{P}[u] = 0$, one seeks a function u, which satisfies the equation, and agrees with data $u(t_*, x) = \phi(x)$

specified at an initial time t_*. In the asymptotic value problem one seeks a function u which satisfies the equation, and which approaches a model solution (called a *leading order term*) $u_0(t,x)$ as $t \to t_*$ in a prescribed way. We give the following more formal definition.

Definition 2.6 (The asymptotic value problem for Eq. (2.1))**.** *For a given choice of a leading order term u_0 and the parameters δ, μ and q, the **asymptotic value problem** consists of finding a unique solution $u = u_0 + w$ to the system Eq.* (2.1) *with **remainder** $w \in X_{\delta,\mu,q}$.*

We note that the terminology used here varies slightly from that in [3, 4], in which the definition above is introduced as the *singular initial value problem*. In fact, the leading order term need not be singular. When convenient we use the shorthand notation AVP, or AVP(u_0) in place of "asymptotic value problem about u_0." At this point no regularity has been specified for the leading order term $u_0 : (0,\delta] \times T^n \to \mathbb{R}^d$. The required regularity of u_0, which contains "asymptotic data functions" of the spatial coordinates, is governed by the required regularity on the coefficients in Definition 2.7 below.

In the proceeding sections within this chapter we prove that solutions to the asymptotic value problem exist for systems Eq. (2.1) with certain structural properties. Some of these properties are encoded in what we call a *quasilinear symmetric hyperbolic Fuchsian system*.

Definition 2.7 (Quasilinear symmetric hyperbolic Fuchsian systems)**.** *Fix positive real numbers δ and s, positive integers $q_0 \geq q > n/2$ (possibly infinite), and an exponent vector $\mu : T^n \to \mathbb{R}^d$, together with an $\mathbb{R}^d$-valued leading-order term $u_0 \in C^\infty((0,\delta]) \cap H^{q_0}(T^n)$. The system Eq.* (2.1) *is said to form a **quasilinear***

symmetric hyperbolic Fuchsian system *around* u_0 *with parameters* $\{\mu, \delta, q, q_0, s\}$ *if for all* $w \in B_{\delta,\mu,q}(s)$*:*

(i) S^0 *is positive definite* [2] *and hence invertible, and both* $S^0(u_0+w)$ *and* $tS^a(u_0+w)$ *for all* $a = 1, \ldots, n$ *are symmetric at every* $(t, x) \in (0, \delta] \times T^n$*.*

(ii) There exists a matrix $S^0_0(u_0)$*, which is positive definite, symmetric, and independent of* t*, contained in the space* $H^{q_0}(T^n)$*, and for*

$$S^0_1(u_0 + w) := S^0(u_0 + w) - S^0_0(u_0),$$

the function operators

$$tS^a, S^0_1 : X_{\delta,\mu,q} \to X_{\delta,\zeta,q}, \quad w \mapsto tS^a(u_0 + w), S^0_1(u_0 + w)$$

satisfy the Lipschitz property (Definition 2.3) in the $(q-1)$*-norm, for some* $\zeta > 0$*.*

(iii) There exists a matrix $N_0(u_0)$*, which is independent of* t *and in* $H^{q_0}(T^n)$*. Further, for*

$$f_1(u_0 + w) := -f(u_0 + w) + N_0(u_0)w$$

the function operator

$$\mathcal{F}(u_0) : X_{\delta,\mu,q} \to X_{\delta,\nu,q}, \quad w \mapsto \mathcal{F}(u_0)[w]$$

[2] Under Condition (ii), and the regularity requirement $q_0 \geq q > n/2$, S^0 is seen to be continuous. Hence, it makes sense to say that S^0 is positive definite pointwise.

defined by

$$\mathcal{F}(u_0)[w] := f_1(u_0 + w) - \sum_{j=0}^{n} tS^j(w)\partial_j u_0, \tag{2.5}$$

satisfies the Lipschitz property (Definition 2.3) in both the q *and* $(q-1)$ *norms, for some* $\nu > \mu$.

If the system Eq. (2.1) is a smooth quasilinear symmetric hyperbolic Fuchsian system as in Definition 2.7 for a choice of leading order term u_0, then it can be written

$$S^0(u_0 + w)Dw + \sum_{a=1}^{n} tS^a(u_0 + w)\partial_a w + N_0(u_0)w = \mathcal{F}(u_0)[w]. \tag{2.6}$$

Note that Condition (i) ensures that the system is symmetric hyperbolic. Conditions (ii) and (iii) are properties expected of a "Fuchsian" type PDE system; namely that near $t \searrow 0$ and for an appropriate leading order term u_0, the system splits into a part which is the same order in t as the fields w, and part which is strictly higher order in t. Since this is a quasilinear system we also expect certain bounded and Lipschitz properties on the nonlinearities, and these are encoded in the definition above. We also note that due to the splitting in Condition (ii), and the fact that $S_1^0(\cdot)$ is not necessarily positive definite, the positivity of S^0 in Condition (i) may require shrinking δ.

If Eq. (2.1) satisfies the properties of Definition 2.7 for $q_0 = \infty$ and if for all $q > n/2$, the operators $\mathcal{F}(u_0)[w], tS^a, S_1^0$ satisfy the Lipschitz estimate then we say Eq. (2.1) is a **smooth quasilinear symmetric hyperbolic Fuchsian system** about u_0 with parameters $\{\delta, \mu, s\}$. Note that due to the regularity assumptions on Eq. (2.1), the functions $f(t, x, u)$ and $S^j(t, x, u)$ are smooth in all arguments for a smooth quasilinear symmetric hyperbolic Fuchsian system.

It is important to clarify the notation used here and below. While the quantities $S_0^0(u_0)$ and $N_0(u_0)$ are explicitly time-independent, they do depend on the $t \to 0$ behavior of the leading order term u_0. For convenience below, if a choice of leading order term u_0 has been fixed, we may omit the explicit dependence on u_0, and write simply S_0^0 and N_0; the dependence on the leading order term is then implicit. We use the same notational shorthand with $S_1^0(u_0 + w)$ and $S^a(u_0 + w)$, omitting the explicit dependence on u_0 so long as the choice of the leading-order term is fixed and unambiguous.

It follows from Definition 2.7 that if Eq. (2.1) is a quasilinear symmetric hyperbolic Fuchsian system, then it is also symmetric hyperbolic for all $t \in (0, \delta]$. Hence for sufficiently regular initial data (i.e. contained in $H^q(T^n)$, with $q > n/2+1$) prescribed at $t_0 \in (0, \delta]$, the Cauchy problem is well-posed in the usual sense (away from $t = 0$), with solutions contained in the space $C(I; H^q(T^n))$ for a sufficiently small interval $I \subset (0, \delta]$; see, for instance, [86]. We note that since solutions to the Cauchy problem are only defined for t bounded away from the singularity at $t = 0$, we know nothing a priori regarding the singular behavior of these solutions, nor whether they are contained in some space $X_{\delta,\mu,q}$.

We note the following differences between Definition 2.7 given above and the corresponding definitions in [3, 4].

(i) In this paper we have omitted splitting S^a into a leading order part, and a higher order part. The reason is that for a Fuchsian system, both as in the definitions of [3, 4] and in Definition 2.7, the important information is that these coefficient matrices decay near the singular time. This decay property, which is indicated by the positivity of the exponent vector ζ, is independent of any splitting. We therefore find the present formulation simpler.

(ii) We have included the condition that S^0 is positive definite (for a sufficiently small δ) in Definition 2.7. Since the positivity of S^0 is part of symmetric hyperbolic systems, it is natural to enforce it at this stage.

(iii) Unlike in the corresponding definition in [3, 4], we have included conditions on the "source" operator $\mathcal{F}(u_0)[w]$ in Definition 2.7. This is natural since one of the key characteristics of a Fuchsian equation is that this operator is higher order in t in a sense described in the definition. Additionally, this operator is constructed from the principle part of the equation and the lower order terms once a leading order term u_0 has been specified.

(iv) The definitions of quasilinear symmetric hyperbolic Fuchsian systems contained in [3, 4] make reference to an operator $N_1(u_0 + w)$. In the present formulation this is lumped in with $\mathcal{F}(u_0)[w]$, or $f_1(w)$ more specifically. This is natural since the two operators play similar roles, and the present formulation simplifies the presentation.

2.2.5. The Fundamental Fuchsian Theorem

Before stating the main theorem for existence and uniqueness of solutions to the asymptotic value problem for quasilinear symmetric hyperbolic systems, we discuss some additional structural properties required of the Fuchsian system Eq. (2.6).

In addition to noting properties of the function operators, it is useful to note the structure of the exponent vectors. In particular we make the following definition.

Definition 2.8. *An $\mathbb{R}^d$-valued exponent vector $\mu(x)$ is has the same **block-diagonal structure** as a $\mathbb{R}^{d\times d}$ matrix $A(t,x)$ if*

$$\mathcal{R}[\mu]A = A\mathcal{R}[\mu],$$

where $\mathcal{R}[\mu]$ is as in Definition 2.2.

For example, if $d = 6$ and A consists of three blocks of size $3, 2, 1$, then μ satisfying having the block-diagonal structure of A is of the form $\mu = (\mu_1, \mu_1, \mu_1, \mu_2, \mu_2, \mu_3)$. We now use this property to characterize the quasilinear symmetric hyperbolic Fuchsian system.

Definition 2.9 (Block diagonality with respect to μ)**.** *Suppose that u_0 is a given leading-order term and μ is an exponent vector. The system Eq.* (2.6) *is **block diagonal with respect to** μ if $S^0(u_0 + w)$ and $S^a(u_0 + w)$ have the same block-diagonal structure as μ for all $w \in X_{\delta,\mu,q}$ for which the expressions are defined, and if*

$$\mathcal{R}[\mu]N_0(u_0)\mathcal{R}[-\mu] \in B_{\delta,0,q}(r_0), \quad \textit{for some } r_0 > 0,$$

where $\mathcal{R}[\mu]$ is defined in Eq. (2.2)*.*

This condition is essential in deriving energy estimates which are fundamental for the proof of Theorem 2.10 below. It ensures that both the matrices $S^j(u)$ and $\mathcal{R}[\mu]S^j(u)\mathcal{R}[-\mu]$ are symmetric. Moreover, it guarantees that the principal part operator only couples those components of the remainder w which decay in t at the same rate. The block diagonal condition here is slightly simpler than in [3, 4] since the terms involving Nw have been redefined. The control we specify here is sufficient for proving our main result.

Another quantity which plays a role in the derivation of energy estimates is the **energy dissipation matrix**

$$M_0 := S_0^0(u_0)\,\mathrm{Diag}(\mu_1, ..., \mu_d) + \mathcal{R}[\mu]N_0(u_0)\mathcal{R}[-\mu]. \tag{2.7}$$

We may now state the main Fuchsian theorem. This theorem provides short-time (as opposed to global) existence and uniqueness of solutions to the asymptotic value problem for equations of the type Eq. (2.1).

Theorem 2.10 (Existence and uniqueness for the asymptotic value problem for quasilinear symmetric hyperbolic Fuchsian systems)**.** *Suppose that Eq.* (2.1) *is a quasilinear symmetric hyperbolic Fuchsian system around* u_0 *with parameters* $\{\delta, \mu, q, q_0, s\}$ *as in Definition 2.7, and is block diagonal with respect to* μ*. Suppose also that* $q > n/2 + 2$ *and* $q_0 > n/2 + 1 + q$*, and that the energy dissipation matrix Eq.* (2.7) *is uniformly positive definite at all* (t, x)*. Then there exists a unique solution* u *to Eq.* (2.1) *whose remainder* $w := u - u_0$ *belongs to* $X_{\widetilde{\delta},\mu,q}$ *with* $Dw \in X_{\widetilde{\delta},\mu,q-1}$ *for some* $\widetilde{\delta} \in (0, \delta]$*.*

If $q = \infty$*, and Eq.* (2.1) *is a smooth quasilinear symmetric hyperbolic Fuchsian system about the leading order term* $u_0 \in C^\infty((0, \delta] \times T^n)$*, then the remainder* w *is contained in* $X_{\widetilde{\delta},\mu,\infty}$*, while* $Dw \in X_{\widetilde{\delta},\mu,\infty}$*.*

Observe that, in the hypothesis of this theorem, the regularity required on S_0^0, and N_0 (specified by q_0) differs slightly from the regularity required of $S_1^0(w)$, $S^a(w)$, and $\mathcal{F}(u_0)$, and of w (specified by q). This gap arises in the course of our proof, and in particular in working with the higher-order energy estimates in Section 2.4.3., and the corresponding Cauchy problems for derivatives of w which are needed to control the regularity of solutions. It is not clear if this gap may be removed by another method of proof. In any case, it vanishes in the C^∞ class of solutions, corresponding to $q = q_0 = +\infty$.

In applications of Theorem 2.10 one often finds an open set of values for the exponent vector μ for which the theorem holds. An upper bound[3] for μ usually originates in the condition that the equation be of symmetric hyperbolic Fuchsian form, and in particular in ensuring that $\mathcal{F}(u_0)[w] \in X_{\delta,\nu,q}$ for some $\nu > \mu$. A lower bound can be generated by enforcing the positivity of the energy dissipation matrix. Both bounds on the set of allowed values for μ provide useful information on the problem. The upper bound for μ specifies the smallest regularity space and, hence, the most precise description of the behavior of w (in the limit $t \searrow 0$), while the lower bound for μ determines the largest space in which the solution u is guaranteed to be unique. This means that while Theorem 2.10 guarantees the existence of a uniques solution w in the space $X_{\delta,\mu,q}$, it does not exclude the possibility that another solution may exist in a larger space, for example, in $X_{\delta,\widetilde{\mu},q}$ with $\widetilde{\mu} < \mu$.

2.3. Outline of Proof

Before presenting a detailed proof of Theorem 2.10 in the following sections we give an overview of the proof here. Supposing that a leading order term u_0 has been specified on $(0, \delta] \times T^n$, the main idea of the proof is to consider a sequence of initial value problems with data prescribed at a sequence of times $\{t_i\}$ which approaches the singular time t_* (we take $t_* = 0$ in this document). The initial data for each problem is chosen in a special way such that $\phi_i(x) = u_0(t_i, x)$, as shown in Figure 2.1 below. We then consider the evolution of this data in the forward t direction, and the theory

[3] A real Λ is defined to be an upper bound for the allowed values of the vector μ if each component μ^a of μ satisfies the condition $\mu^a(t, x) < \Lambda$ for all x in the domain of μ. A similar definition holds for a lower bound for μ.

for symmetric hyperbolic systems provides a sequence of functions $\{v_i\}$ (solutions to the corresponding initial value problems) which we label approximate solutions. We cannot in general control the solution of the Cauchy problem as $t \searrow 0$ –indeed, such control would render our construction of solutions redundant. The work is then to show that this sequence of approximate solutions converges to a solution of the asymptotic value problem Definition 2.6 with leading order term u_0. The proof of

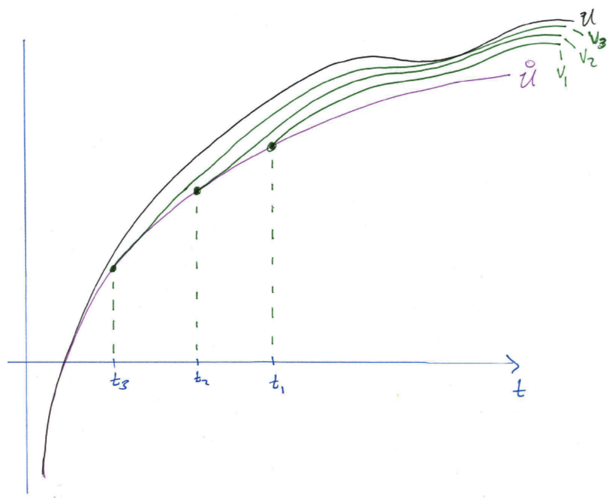

FIGURE 2.1.. Given the leading order term denoted here by $\mathring{u}$, we consider a sequence of approximate solutions $\{v_i\}$, which satisfy in the forward direction an initial value problem with data $\phi_i(x)$ prescribed at t_i according to $\phi_i(x) = \mathring{u}(t_i, x)$. The aim is then to show that this sequence converges to a solution u of the asymptotic value problem.

existence of such solutions in the case of quasilinear equations is done (broadly) in two steps. First we work with a corresponding linear system and establish the existence and uniqueness of solutions to the linear asymptotic value problem; the statements are contained in Proposition 2.20 and Proposition 2.22 below. This linear theory is then used along with a fixed point argument to show existence of solutions to the quasilinear asymptotic value problem.

To show that the sequence of approximate solutions described above converges to a solution in the linear setting we first establish control over the approximate solutions using a family of energy estimates. The energies and the corresponding estimates are contained in Section 2.4.2. and Section 2.4.3.. These estimates allow us to establish existence of first weak (Section 2.4.4.) and then strong (Section 2.4.5.) solutions to the linear asymptotic value problem under the restriction that certain coefficients in the equation are in the smooth subspace of their respective function spaces. We also establish the existence of a map, called the solution operator, which maps a given linear source term to a particular solution of the asymptotic value problem. The smoothness condition is relaxed in Section 2.4.6. using boundedness on the coefficients, and a uniform estimate for the solution operator. At this point we prove in Section 2.4.7. that the solution to the linear asymptotic value problem is unique. This concludes the theory for linear systems. The fixed point argument is contained in Section 2.5..

2.4. Existence and Uniqueness for Linear Systems

2.4.1. Definitions

We start by formally defining the notion of a linear symmetric hyperbolic Fuchsian system. This definition is basically the same as Definition 2.7, but with coefficients independent of the field w. In the linear theory we take the leading order term u_0 to be zero without loss of generality. As a consequence the remainder w agrees with the full field u.

Definition 2.11 (Linear symmetric hyperbolic Fuchsian systems)**.** *Fix positive real numbers δ, r, integers $q \geq 0$, and $q_0 > n/2$ (possibly infinite), and an exponent vector $\mu : T^n \to \mathbb{R}^d$, together with an $\mathbb{R}^d$-valued leading-order term u_0. The system Eq.* (2.1) *is said to form a **linear symmetric hyperbolic Fuchsian system** around u_0 with parameters $\{\delta, \mu, q, q_0, r\}$ if:*

(i) S^0 is positive definite [4] *and hence invertible, and both $S^0(t,x)$ and $tS^a(t,x)$ for all $a = 1, \ldots, n$ are symmetric at every $(t,x) \in (0,\delta] \times T^n$.*

(ii) There exists a matrix $S^0_0(u_0)$, which is positive definite, symmetric, and independent of t, contained in the space $H^{q_0}(T^n)$, and for

$$S^0_1(t,x) := S^0(t,x) - S^0_0(u_0)(x),$$

the matrix-valued maps $tS^a(t,x), S^0_1(t,x)$ are contained in $B_{\delta,\zeta,q}(r)$ for some $\zeta > 0$.

(iii) There exists a matrix $N_0(u_0)$, which is independent of t and in $H^{q_0}(T^n)$. Further, for

$$f_1(u_0 + w) := -f(u_0 + w) + N_0(u_0)w$$

the map $\mathcal{F}(u_0)[w] \in X_{\delta,\nu,q}$ takes the linear form

$$\mathcal{F}(u_0)[w] := f_1(t,x,w) - \sum_{j=0}^{n} tS^j(t,x)\partial_j u_0 = f_0(t,x) + F_1(t,x)w \qquad (2.8)$$

where $f_0(t,x) \in X_{\delta,\nu,q}$ for some $\nu > \mu$ is an $\mathbb{R}^d$-valued function, and $F_1(t,x)$ is an $\mathbb{R}^{d\times d}$-valued map satisfying $\mathcal{R}[\mu]F_1\mathcal{R}[\mu]^{-1} \in B_{\delta,\zeta,q}(r)$, for some $r > 0$.

*The system is defined to be a **smooth linear symmetric hyperbolic Fuchsian system** if these conditions hold for all $q, q_0 > n/2$.*

---

[4] Here q is only required to be non-negative, and thus we cannot guarantee that S^0 is continuous. Hence, we require that S^0 be positive definite in the L^2 sense.

If the system Eq. (2.1) is a linear symmetric hyperbolic Fuchsian system, it can be written as in Eq. (2.6). We define the linear operator $L[w]$

$$L[w](t,x) := S^0(t,x)Dw(t,x) + \sum_{a=1}^{n} tS^a(t,x)\partial_a w(t,x) + N_0(x)w(t,x), \tag{2.9}$$

with respect to which the linear Fuchsian system may be written

$$L[w](t,x) = f_0(t,x) + F_1(t,x)w(t,x).$$

As with quasilinear systems, the matrices S_1^0 and S^a are thought of as perturbations near $t \searrow 0$, which is reflected in the condition $\zeta > 0$. During the course of the proof it becomes necessary to consider bounds which depend on the S_1^0 and S^a. In order that the bounds not depend on the *particular* S_1^0 and S^a, we consider these perturbations to be in bounded subsets of our weighted Sobolev spaces. In order to make precise the dependence of constants on the various parameters and functions we make the following definition:

Definition 2.12. *Suppose that Eq.* (2.1) *is a linear symmetric hyperbolic Fuchsian system for a chosen set of the parameters $\delta, \mu, \zeta, q, q_0$ and r. Suppose that a particular estimate (e.g., the energy estimate Eq.* (2.12)*), involving a collection $\mathcal{C}$ of constants, holds for solutions of Eq.* (2.1) *under a certain collection of hypotheses $\mathcal{H}$. The constants $\mathcal{C}$ are defined to be* ***uniform*** *with respect to the system and the estimate so long as the following conditions hold:*

1. *For any choice of S_1^0, S^a and F_1 contained in the perturbation space $B_{\delta,\zeta,q}(r)$ (see Definition 2.11) which is compatible with the hypothesis $\mathcal{H}$, the estimate holds for the same set of constants $\mathcal{C}$.*

2. *If the estimate holds for a choice of the constants $\mathcal{C}$ for one particular choice of δ, then for every smaller (positive) choice of δ, the estimate remains true for the same choice of $\mathcal{C}$.*

We note that in much of the analysis in this chapter we write a series of estimates involving a constant, generically labeled C, which often changes line to line. Although the change in the constant is not always mentioned during the calculation, it is important that at the end one verifies that the constant is uniform in the sense described above.

As we describe in Section 2.3. the proof proceeds by considering a sequence of Cauchy initial value problems with initial times in the interval $(0, \delta]$ for a linear symmetric hyperbolic Fuchsian system with smooth coefficients in some sense. Suppose S_1^0, S^a, and F_1 are $C^\infty((0,\delta] \times T^n)$ functions contained in the space $B_{\delta,\zeta,q}(r)$. Under this assumption, we will say the system Eq. (2.1) is a linear symmetric hyperbolic Fuchsian system with **smooth perturbations**. Note however, that this does *not* mean that we have control over decay of all spatial derivatives of the perturbations; such control (for q spatial derivatives) is measured by $B_{\delta,\zeta,q}(r)$.

Given a linear symmetric hyperbolic Fuchsian system Eq. (2.1) with smooth perturbations and if in addition $q_0 > n/2 + 1$ and also $f_0 \in X_{\delta,\nu,q} \cap C^\infty((0,\delta] \times T^n)$, then Proposition A.15 for linear symmetric hyperbolic systems shows that the Cauchy problem is well-posed in the sense that for initial data $v_{[t_0]} \in H^{q_0}(T^n)$, there is a unique solution $v : [t_0, \delta] \times T^n \to \mathbb{R}^d$ to this Cauchy problem with $v(t_0) = v_{[t_0]}$ and with $v(t, \cdot) \in H^{q_0}(T^n)$ for all $t \in [t_0, \delta]$. We note in particular that the solution to these linear systems extends forward in time all the way to δ, independent of the initial time t_0.

2.4.2. Energies and Basic Energy Estimate

In order to control the solutions to the Cauchy problem for linear symmetric hyperbolic Fuchsian systems (Definition 2.11) with smooth perturbations in the forward direction, particularly in the limit when the initial time approaches the singular time, we introduce a family of time-dependent energies. Suppose the exponent vector μ is fixed; for any two positive real numbers κ and γ, we define the energy $E_{\mu,\kappa,\gamma}$ for a function $v : [t_0, \delta] \times T^n \to \mathbb{R}^d$ (with $v(t, \cdot) \in L^2(T^n)$ for each $t \in [t_0, \delta]$) by

$$E_{\mu,\kappa,\gamma}[v](t) := \frac{1}{2} e^{-\kappa t^\gamma} \left\langle S^0(t, \cdot)\mathcal{R}[\mu](t, \cdot)v(t, \cdot), \mathcal{R}[\mu](t, \cdot)v(t, \cdot)\right\rangle_{L^2(T^n)}. \tag{2.10}$$

We have used the notation for the L^2-product

$$\langle v, w\rangle_{L^2(T^n)} := \int_{T^n} \langle v, w\rangle \, dx,$$

where $\langle v, w\rangle$ denotes the usual vector inner product. The matrix $S^0(t, \cdot)$ is the same one which appears in Eq. (2.1), and the matrix $\mathcal{R}[\mu](t, \cdot)$ is given by Eq. (2.2). Similar energies, but without the explicit time dependence are common for symmetric hyperbolic systems; see for example [75].

It is useful in our analysis to relate these energies to the L^2-norm of $\mathcal{R}[\mu](t, \cdot)v(t, \cdot)$. We find the following equivalence:

Lemma 2.13. *For any $v : [t_0, \delta] \times T^n \to \mathbb{R}^d$ with $v(t, \cdot) \in L^2(T^n)$, and any $S^0(t, \cdot)$ satisfying the conditions of Definition 2.11 with smooth perturbations, there exist positive constants C_b and C_t, which are uniform in the sense of Definition 2.12 and* *independent of t, such that*

$$C_b\|\mathcal{R}[\mu](t,\cdot)v(t,\cdot)\|_{L^2(T^n)} \leq \sqrt{E_{\mu,\kappa,\gamma}[v](t)} \leq C_t\|\mathcal{R}[\mu](t,\cdot)v(t,\cdot)\|_{L^2(T^n)}. \tag{2.11}$$

Proof. Consider first the upper bound. Since $1/2e^{-\kappa t^\gamma}$ is positive and bounded on $[0,\delta]\times T^n$ by one we have

$$\sqrt{E_{\mu,\kappa,\gamma}[v](t)} \leq \left(\int_{T^n} \langle S^0\mathcal{R}[\mu]v, \mathcal{R}[\mu]v\rangle\right)^{1/2}.$$

Furthermore, we claim that under the hypotheses of the lemma $\sup_{t\in(0,\delta]}\|S^0\|_{L^\infty} < C$ for a constant depending on n, q, ζ, δ, r and u_0. To see this note that by Definition 2.11 and the smooth perturbations hypothesis, $S^0 = S^0_0 + S^0_1$ for $S^0_0 \in H^{q_0}$ and $S^0_1 \in B_{\delta,\zeta,q}(r)\cap C^\infty((0,\delta]\times T^n)$. Since $q_0 > n/2$ the Sobolev inequalities (Theorem A.3) imply that $S^0_0 \in C^0(T^n)$ and $\|S^0_0\|_{L^\infty} \leq C(n,q)\|S^0_0\|_{H^{q_0}}$. To address S^0_1 we note that since ζ is strictly positive it follows from Lemma B.1 that $S^0_1 \in B_{\delta,0,q}(Cr)\cap C^\infty((0,\delta]\times T^n)$ for a constant depending on δ and ζ. We find

$$\sup_{t\in(0,\delta]}\|S^0\|_{L^\infty} \leq C(n,q)\|S^0_0\|_{H^{q_0}} + C(\zeta,\delta)r.$$

It follows that there exists a constant C_t depending in general on $(n, q, \zeta, \delta, r, u_0)$, but in particular independent of the particular S^0_1 such that

$$\sqrt{E_{\mu,\kappa,\gamma}[v](t)} \leq \left(\sup_{t\in(0,\delta]}\|S^0\|_{L^\infty(T^n)}\right)\|\mathcal{R}[\mu]v\|_{L^2} \leq C_t\|\mathcal{R}[\mu]v\|_{L^2}.$$

Next we show the lower bound. We note that by positive definite property of S^0

$$\sqrt{E_{\mu,\kappa,\gamma}[v](t)} \geq \left(\inf_{t\in(0,\delta]} \frac{1}{2}e^{-\kappa t^\gamma}\right)^{1/2} \|\mathcal{R}[\mu]v\|_{L^2(T^n)} \geq C_b\|\mathcal{R}[\mu]v\|_{L^2(T^n)}$$

for some positive constant C_b depending on $\kappa, \gamma, \delta, r, \zeta$ and, u_0. □

These energies have been defined in such a way, including in particular the factor of $e^{-\kappa t^\gamma}$, so that the growth of the energies may be controlled. We obtain the following estimate.

Lemma 2.14 (Fundamental energy estimate)**.** *Suppose that Eq.* (2.1) *is a linear symmetric hyperbolic Fuchsian system for the parameters* $\{\delta, \mu, q, q_0, r\}$ *(as in Definition 2.11), has smooth perturbations, and is block diagonal with respect to* μ*, with* $q \geq 0$ *and* $q_0 > n/2 + 1$*. Suppose also that the energy dissipation matrix Eq.* (2.7) *is positive definite for all* $x \in T^n$ *and, in addition,* $DS_1^0, \partial_b S^a \in B_{\delta,\xi,0}(\tilde{r})$ *for all* $a, b = 1, \dots, n$ *for some* $\tilde{r} > 0$ *and some exponent vector* ξ *with strictly positive entries. Then for any initial data* $v_{[t_0]} \in H^{q_0}(T^n)$ *specified at some* $t_0 \in (0, \delta]$*, there exists a unique solution* v *to the corresponding Cauchy problem in* $C([t_0, \delta]; H^{q_0}(T^n))$*, and there exist positive constants* κ, γ *and* C *such that* v *satisfies the energy estimate*

$$\sqrt{E_{\mu,\kappa,\gamma}[v](t)} \leq \sqrt{E_{\mu,\kappa,\gamma}[v](t)}|_{t=t_0} + C\int_{t_0}^{t} s^{-1}\|\mathcal{R}[\mu](s,\cdot)f_0(s,\cdot)\|_{L^2(T^n)}ds \qquad (2.12)$$

for all $t \in [t_0, \delta]$*. The constants* C*,* κ*, and* γ *may be chosen to be uniform*[5]*. In particular, if one replaces* $v_{[t_0]}$ *specified at* t_0 *by any* $v_{[t_1]}$ *specified at any time* $t_1 \in (0, t_0]$*, then the energy estimate holds for the* same *constants* C*,* κ*,* γ*.*

[5]While the constants C, κ and γ here can be chosen to be uniform in the sense of Definition 2.12, there generally does not exist a choice which holds for all $\delta, S_0^j, N_0, r, \tilde{r}, \zeta, \xi, \mu$ and ν.

In lieu of Eq. (2.11), the estimate Eq. (2.12) can be written

$$\begin{aligned}\|\mathcal{R}[\mu](t,\cdot)v(t,\cdot)\|_{L^2(T^n)} \leq \widetilde{C}\Big(&\|\mathcal{R}[\mu](t_0,\cdot)v_{t_0}\|_{L^2(T^n)} \\ &+ \int_{t_0}^{t} s^{-1}\|\mathcal{R}[\mu](s,\cdot)f_0(s,\cdot)\|_{L^2(T^n)}ds\Big),\end{aligned} \tag{2.13}$$

where the constants C, κ and γ have been absorbed into $\widetilde{C}$.

We also note that the control $DS_1^0, \partial_b S^a \in B_{\delta,\xi,0}(\tilde{r})$ does not follow from the smooth perturbations condition. The latter is the statement that the perturbations are in the smooth subset of the relevant weighted Sobolev spaces, while the former is a statement about the control on the asymptotic behavior of the lowest derivatives of S_1^0 and S^a.

With regards to the proof of Lemma 2.14, we note that the existence of unique solutions to the $n+1$ dimensional Cauchy problem corresponding to Eq. (2.1) (which follows from, e.g., Proposition 1.7 in Chapter 16 of [86]) plays a key role, and the inequality for q_0 stated in the hypothesis is necessary in order to guarantee such existence.

Proof. The basic idea of the proof is to compute $DE_{\mu,\kappa,\gamma}[v](t)$, then bound the terms on the right hand side and finally integrate the equation in time. For simplicity we write $E[v]$ in place of $E_{\mu,\kappa,\gamma}[v]$. Computing[6] $DE[v]$, and using the symmetry of the

[6]In calculating this time derivative, we use the fact that the solution v is C^1 in both time and space.

matrix S^0, we obtain

$$
\begin{aligned}
DE[v](t) = & - \kappa\gamma t^\gamma E[v](t) + \frac{1}{2} e^{-\kappa t^\gamma} \int_{T^n} \left\langle (DS^0)\mathcal{R}[\mu]v, \mathcal{R}[\mu]v \right\rangle dx \\
& + e^{-\kappa t^\gamma} \int_{T^n} \left\langle S^0 (D\mathcal{R}[\mu])v, \mathcal{R}[\mu]v \right\rangle dx \\
& + e^{-\kappa t^\gamma} \int_{T^n} \left\langle S^0 \mathcal{R}[\mu]\, Dv, \mathcal{R}[\mu]v \right\rangle dx.
\end{aligned}
$$

We first analyze the fourth term on the right hand side of this expression, which we label I. Using the assumption that S^0 and $\mathcal{R}[\mu]$ commute (a consequence of the block-diagonal condition), and the fact that v is a (forward) solution of equation Eq. (2.6)[7] with linear source function given by Eq. (2.8) we calculate

$$
\begin{aligned}
I = e^{-\kappa t^\gamma} \int_{T^n} \Big(& \langle \mathcal{R}[\mu] f_0, \mathcal{R}[\mu]v \rangle + \langle \mathcal{R}[\mu] F_1 v, \mathcal{R}[\mu]v \rangle - \langle \mathcal{R}[\mu] N_0 v, \mathcal{R}[\mu]v \rangle \\
& - t \sum_{a=1}^{n} \langle \mathcal{R}[\mu] S^a \partial_a v, \mathcal{R}[\mu]v \rangle \Big) dx.
\end{aligned}
$$

Integration by parts on the last term, along with the assumption that S^a and its spatial derivatives commute with $\mathcal{R}[\mu]$ (by block-diagonality) gives

$$
\begin{aligned}
I = e^{-\kappa t^\gamma} \int_{T^n} \Big(& \langle \mathcal{R}[\mu] f_0, \mathcal{R}[\mu]v \rangle + \langle \mathcal{R}[\mu] F_1 v, \mathcal{R}[\mu]v \rangle - \langle \mathcal{R}[\mu] N_0 v, \mathcal{R}[\mu]v \rangle \\
& + \frac{1}{2} t \sum_{a=1}^{n} \langle (\partial_a S^a)\, \mathcal{R}[\mu]v, \mathcal{R}[\mu]v \rangle \\
& + t \sum_{a=1}^{n} \left\langle (S^a (\partial_a \mathcal{R}[\mu]) \mathcal{R}[\mu]^{-1}) \mathcal{R}[\mu]v, \mathcal{R}[\mu]v \right\rangle \Big) dx.
\end{aligned}
$$

[7] Here we use the existence theorems for symmetric hyperbolic systems, and the regularity condition $q_0 > n/2 + 1$.

Using the Hölder inequality, Lemma A.10, we may then estimate the first term in this expression as follows:

$$e^{-\kappa t^\gamma}\int_{T^1}\langle\mathcal{R}[\mu]f_0,\mathcal{R}[\mu]v\rangle\,dx\leq e^{-\kappa t^\gamma}||\mathcal{R}[\mu]f_0||_{L^2}||\mathcal{R}[\mu]v||_{L^2}.$$

We now argue that for appropriate choices of κ and γ, all the other terms in $DE[v]$ can be neglected in a certain sense. Combining all terms in the expression for $DE[v]$ and using $\mathcal{M}:=\mathrm{Diag}(\mu)=-(D\mathcal{R}[\mu])\mathcal{R}[\mu]^{-1}$ we compute

$$\begin{aligned}DE[v]\leq\,&e^{-\kappa t^\gamma}||\mathcal{R}[\mu]f_0||_{L^2}||\mathcal{R}[\mu]v||_{L^2}\\&-e^{-\kappa t^\gamma}\int_{T^n}\left\langle\left(S^0_0\mathcal{M}+\mathcal{R}[\mu]N_0\mathcal{R}[\mu]^{-1}\right)\mathcal{R}[\mu]v,\mathcal{R}[\mu]v\right\rangle\\&-e^{-\kappa t^\gamma}\int_{T^n}\left\langle\left(\frac{1}{2}\kappa\gamma t^\gamma S^0-\mathcal{K}(t)\right)\mathcal{R}[\mu]v,\mathcal{R}[\mu]v\right\rangle,\end{aligned}$$

where

$$\begin{aligned}\mathcal{K}(t):=&\frac{1}{2}DS^0_1-S^0_1\mathcal{M}+\mathcal{R}[\mu]F_1\mathcal{R}[\mu]^{-1}\\&+t\sum_{a=1}^n\partial_a\mathcal{R}[\mu]S^a\mathcal{R}[\mu]^{-1}+t\frac{1}{2}\sum_{a=1}^n\partial_aS^a.\end{aligned}$$

The first line in the inequality for $DE[v]$ contains the term we keep. The second line contains terms over which we have control only in $X_{\delta,0,q_0}$. This integral is negative definite if the energy dissipation matrix $M_0=S^0_0\mathcal{M}+\mathcal{R}[\mu]N_0\mathcal{R}[\mu]^{-1}$ (Eq. (2.7)) is positive definite, and hence can be neglected. We argue that the last integral can be controlled as well. As a consequence of Definition 2.11, and the assumptions of Lemma 2.14, each term in $\mathcal{K}(t)$ is controlled in $B_{\delta,\zeta,q}(r)$ or $B_{\delta,\xi,0}(\tilde{r})$. Hence we can choose a κ large enough and a γ small enough, and use the positivity of S^0, to ensure that the third line is negative definite. The constants κ, γ may be chosen to be uniform (Definition 2.12) since their value depends only on the norm of quantities in $\mathcal{K}$, each of which may be bounded by r. Thus, κ, γ may be chosen independently of the particular functions within the ball $B_{\delta,\zeta,q}(r)$. In total, we obtain

$$DE[v](t) \leq e^{-\kappa t^\gamma} ||\mathcal{R}[\mu] f_0||_{L^2} ||\mathcal{R}[\mu] v||_{L^2},$$

which implies that

$$\partial_t E[v](t) \leq t^{-1} e^{-\kappa t^\gamma} ||\mathcal{R}[\mu] f_0||_{L^2} ||\mathcal{R}[\mu] v||_{L^2}.$$

Then using the norm equivalence Eq. (2.11), we may rewrite this as

$$\partial_t E[v](t) \leq C t^{-1} e^{-\kappa t^\gamma} ||\mathcal{R}[\mu] f_0||_{L^2} \sqrt{E[v](t)}. \tag{2.14}$$

To integrate this inequality, it would be useful to divide both sides by $\sqrt{E[v](t)}$. However, since the L^2 norm of v may vanish in special cases, we use the following strategy (see, for instance, [75, Page 59]). We set $E_\epsilon := E + \epsilon$ for some constant $\epsilon > 0$, and we check that the last inequality holds if we replace E by E_ϵ. Then dividing, and using $\frac{1}{\sqrt{E_\epsilon}} \partial_t E_\epsilon = 2\partial_t \sqrt{E_\epsilon}$, we obtain

$$\partial_t \sqrt{E_\epsilon[v](t)} \leq C t^{-1} e^{-\kappa t^\gamma} ||\mathcal{R}[\mu] f_0||_{L^2},$$

after rescaling the constant C. We now integrate both sides over $\int_{t_0}^{t} ds$, thereby obtaining

$$\begin{aligned}\sqrt{E_\epsilon[v](t)} &\leq \sqrt{E_\epsilon[v](t_0)} + C\int_{t_0}^{t} s^{-1}e^{-\kappa s^\gamma}||\mathcal{R}[\mu]f_0||_{L^2}(s)ds \\ &\leq \sqrt{E_\epsilon[v](t_0)} + C\Big(\sup_{s\in(t_0,t)} e^{-\kappa s^\gamma}\Big)\int_{t_0}^{t} s^{-1}||\mathcal{R}[\mu]f_0||_{L^2}(s)ds \\ &\leq \sqrt{E_\epsilon[v](t_0)} + C\int_{t_0}^{t} s^{-1}||\mathcal{R}[\mu]f_0||_{L^2}(s)ds,\end{aligned}$$

where we note that the constant C changes from the second to the third line of this calculation. Taking the limit $\epsilon \to 0$ finishes the proof that the inequality (Eq. (2.12)) holds. It also follows directly that the constant C is uniform. □

2.4.3. Higher Order Energy Estimates

We also need to control higher order spatial derivatives of the solutions of the Cauchy problem, for which we establish the following energy estimate.

Lemma 2.15. *Consider a linear symmetric hyperbolic Fuchsian system with parameters $\{\delta, \mu, q, q_0, r\}$, which satisfies all of the conditions in Lemma 2.14, except that here we allow for arbitrary integers $q \geq 1$ and $q_0 > n/2 + 1 + q$. Assume as well that*[8] *$DS_1^0 \in B_{\delta,\xi,0}(\tilde{r})$. Then there exist positive uniform* [9] *constants C, ρ such that*

[8] We note that the condition $\partial_b S^a \in B_{\delta,\xi,0}(r)$ $(a, b = 1, \ldots, n)$ of Lemma 2.14 is now implied by the choice $q \geq 1$.

[9] We note however that C and ρ generally depend on q.

for all $\epsilon > 0$, the solution $v(t,x)$ of Lemma 2.14 satisfies

$$\begin{aligned} \|\mathcal{R}[\mu-\epsilon](t,\cdot)v(t,\cdot)\|_{H^q(T^n)} \le C\Big(&\|\mathcal{R}[\mu](t_0,\cdot)v_{t_0}\|_{H^q(T^n)} \\ &+\int_{t_0}^t s^{-1}\left(\|\mathcal{R}[\mu](s,\cdot)f_0(s,\cdot)\|_{H^q(T^n)} + s^\rho\|\mathcal{R}[\mu]v\|_{H^{q-1}(T^n)}\right)ds\Big). \end{aligned} \tag{2.15}$$

The same choice of constants C and ρ can be used for any initial time $t_0 \in (0,\delta)$.

The inequality for q_0 comes again from the condition for well-posedness for the Cauchy problem in n spatial dimensions, but now also from the fact that in deriving the above estimate we take q spatial derivatives of the coefficients.

Proof. We consider the $q = 1$ case and comment on the case of general q below. The idea is to derive an expression for the first spatial derivatives of $v(t,x)$ and then apply the basic energy estimate Lemma 2.14.

Step 1: Derive equation for the spatial derivatives. Let ∂v be the $n\cdot d$-length vector defined by

$$\partial v := (\partial_1 v,\dots,\partial_n v)^T = \left(\partial_1 v^1,\dots,\partial_1 v^d,\dots,\partial_n v^1,\dots,\partial_n v^d\right)^T.$$

To derive an equation for ∂v, let b be any value in $\{1,\dots,n\}$. Recall, $v(t,x)$ satisfies the equation

$$S^0 Dv + \sum_{a=1}^n tS^a\partial_a v + N_0 v = f_0 + F_1 v$$

within the time interval $[t_0,\delta]$ for some $t_0 \in (0,\delta]$, in accord with the assumption of a linear symmetric hyperbolic Fuchsian system. Letting ∂_b act on this system we obtain

$$
\begin{aligned}
S^0 D(\partial_b v) + \sum_{a=1}^{n} t S^a \partial_a(\partial_b v) + N_0(\partial_b v)& \\
&= \partial_b f_0 + (\partial_b F_1) v + F_1(\partial_b v) \\
&\quad - (\partial_b S^0) D v - \sum_{a=1}^{n} t(\partial_b S^a)\partial_a v - (\partial_b N_0) v.
\end{aligned}
$$

The left-hand side of this expression satisfies the hypotheses of Lemma 2.14; it remains to verify sufficient control over the right-hand side. Since bounds on Dv are not known at this point, we eliminate this term by using the fact that v satisfies the linear system. The resulting expression for the source terms can be written as

$$
\widehat{f}_0^b + (\widehat{F}_1 \partial v)^b
$$

where

$$
\begin{aligned}
\widehat{f}_0^b &= \left(\partial_b - \partial_b S^0 \left(S^0\right)^{-1}\right) f_0 \\
&\quad + \left(\left(\partial_b - \partial_b S^0 \left(S^0\right)^{-1}\right) F_1\right) v - \left(\left(\partial_b - \partial_b S^0 \left(S^0\right)^{-1}\right) N_0\right) v,
\end{aligned}
\tag{2.16}
$$

and $\widehat{F}_1$ is a $(n \cdot d) \times (n \cdot d)$ matrix with components

$$
(\widehat{F}_1)^{ab} = \left(F_1 \delta^{ab} + \partial_b S^0 \left(S^0\right)^{-1} t S^b - t \partial_b S^b\right) + t \sum_{a \neq b} \left(\partial_b S^0 \left(S^0\right)^{-1} S^a - \partial_b S^a\right).
$$

Note that S^0 is invertible according to Definition 2.11, and further that as a consequence of the block-diagonal conditions, the inverse has the block-diagonal structure of μ. The system for the full $n \cdot d$-length vector ∂v can be written in linear symmetric hyperbolic form

$$\widehat{S^0} D\partial v + \sum_{a=1}^{n} t\widehat{S^a}\partial_a \partial v + \widehat{N_0}\partial v = \widehat{f_0} + \widehat{F_1}\partial v, \tag{2.17}$$

where $\widehat{S^0} = \mathrm{Diag}(S^0, \ldots, S^0)$, $\widehat{S^a} = \mathrm{Diag}(S^a, \ldots, S^a)$, and $\widehat{N_0} = \mathrm{Diag}(N_0, \ldots, N_0)$, each contain n-blocks of the corresponding $d \times d$ matrix.

In order to apply Lemma 2.14 we must in particular show that the above system is a Fuchsian system, and more specifically meets the hypotheses of Definition 2.11. As part of this definition, it is required that $\widehat{f_0} \in X_{\delta,\nu,q}$ for some $\nu > \mu$. However, we only have μ-control over $\partial_b N_0$. To deal with this situation, we seek to apply Lemma 2.14 with an exponent vector $\hat{\mu} := \mu - \epsilon$ which is slightly decreased, corresponding to slightly weaker control on the behavior in t. Notice that if Eq. (2.1) is a linear symmetric hyperbolic Fuchsian system with exponent vector μ such that the energy dissipation matrix with respect to μ is positive definite, then there exists an $\epsilon > 0$ such that Eq. (2.1) is also a linear symmetric hyperbolic Fuchsian system with exponent vector $\mu - \epsilon$ and with the energy dissipation matrix with respect to $\mu - \epsilon$ is positive definite. The upper bounds on ϵ in this case come from i) ϵ must be chosen less than $\nu - \mu$, and ii) ϵ must not be so large that the energy dissipation matrix fails to be positive definite with respect to $\mu - \epsilon$. It is easily checked that the remaining hypotheses of the lemma hold. We may apply Lemma 2.14 to the system Eq. (2.17) in order to estimate $E_{\hat{\mu},\hat{\kappa},\hat{\gamma}}[\partial v]$ for in general different uniform constants $\hat{\kappa}$, and $\hat{\gamma}$.

Step 2: q-order energies. We proceed to prove the inequality Eq. (2.15). To this end, consider

$$E^{(1)}_{\hat{\mu}}[v] := \sqrt{E_{\hat{\mu},\kappa,\gamma}[v](t)} + \sqrt{E_{\hat{\mu},\kappa^{(1)},\gamma^{(1)}}[\partial v](t)},$$

where $\kappa^{(1)}$ and $\gamma^{(1)}$ represent the in general different choices of κ and γ for the estimate for ∂v. Notice that the energy is computed with respect to $\hat{\mu}$ in both terms.

By the energy/norm equivalence there exist uniform constants C_1, C_2 such that

$$C_1 \left(\|\mathcal{R}[\hat{\mu}]v\|_{L^2} + \|\widehat{\mathcal{R}}[\hat{\mu}]\partial v\|_{L^2} \right) \leq E^{(1)}_{\hat{\mu}}[v] \leq C_2 \left(\|\mathcal{R}[\hat{\mu}]v\|_{L^2} + \|\widehat{\mathcal{R}}[\hat{\mu}]\partial v\|_{L^2} \right),$$

where $\widehat{\mathcal{R}}[\hat{\mu}] := \mathrm{Diag}\left(\mathcal{R}[\hat{\mu}], \dots, \mathcal{R}[\hat{\mu}]\right)$ consists of n-blocks of $\mathcal{R}[\hat{\mu}]$.

Step 3: Lower bound for $E^{(1)}_{\hat{\mu}}[v]$. We show that $\|\mathcal{R}[\tilde{\mu}]v\|_{H^1(T^n)} \leq C E^{(1)}_{\hat{\mu}}[v]$ for a uniform constant C, and some exponent vector $\tilde{\mu}$. Distributing the spatial derivative in the H^1 norm, we compute

$$\begin{aligned}
\|\mathcal{R}[\tilde{\mu}]v\|_{H^1(T^n)} = &\left(\int_{T^n} \sum_{a=1}^{n} |\partial_a \mathcal{R}[\tilde{\mu}]v|^2 + |\mathcal{R}[\tilde{\mu}]v|^2 \right)^{1/2} \\
\leq &\left(\int_{T^n} \sum_{a=1}^{n} |\mathcal{R}[\tilde{\mu}]\partial_a v|^2 \right)^{1/2} + \left(\int_{T^n} |\mathcal{R}[\tilde{\mu}]v|^2 \right)^{1/2} \\
&+ \left(\int_{T^n} \sum_{a=1}^{n} |(\partial_a \mathcal{R}[\tilde{\mu}])v|^2 \right)^{1/2} \\
&+ \left(2 \int_{T^n} \sum_{a=1}^{n} \langle (\partial_a \mathcal{R}[\tilde{\mu}])v, \mathcal{R}[\tilde{\mu}]\partial_a v \rangle \right)^{1/2}
\end{aligned}$$

Consider the fourth term. The integral can be written

$$\int_{T^n} \sum_{a=1}^{n} \langle (\partial_a \mathcal{R}[\tilde{\mu}])v, \mathcal{R}[\tilde{\mu}]\partial_a v \rangle = \int_{T^n} \left\langle (\partial \mathcal{R}[\tilde{\mu}])\widehat{v}, \widehat{\mathcal{R}}[\tilde{\mu}]\partial v \right\rangle,$$

where $\partial\mathcal{R}[\tilde{\mu}] := \mathrm{Diag}\{\partial_1\mathcal{R}[\tilde{\mu}], \ldots, \partial_n\mathcal{R}[\tilde{\mu}]\}$, and $\widehat{v} := (v, \ldots, v)^T$ (n-copies), and $\widehat{\mathcal{R}}[\tilde{\mu}]$ and ∂v are defined as before. Then, by an application of Hölder's inequality we obtain

$$\left(2\int_{T^n}\sum_{a=1}^{n}\langle(\partial_a\mathcal{R}[\tilde{\mu}])v, \mathcal{R}[\tilde{\mu}]\partial_a v\rangle\right)^{1/2} \leq \sqrt{2\|\partial\mathcal{R}[\tilde{\mu}]\widehat{v}\|_{L^2}\|\widehat{\mathcal{R}}[\tilde{\mu}]\partial v\|_{L^2}}.$$

Now notice that for a, b non-negative real numbers $\sqrt{2}\sqrt{ab} \leq a + b$,[10] which allows us to obtain the bound

$$\|\mathcal{R}[\tilde{\mu}]v\|_{H^1(T^n)} \leq C\left(\|\widehat{\mathcal{R}}[\tilde{\mu}]\partial v\|_{L^2} + \|\partial\mathcal{R}[\tilde{\mu}]\widehat{v}\|_{L^2} + \|\mathcal{R}[\tilde{\mu}]v\|_{L^2}\right). \tag{2.18}$$

The expression $\|\partial\mathcal{R}[\tilde{\mu}]\widehat{v}\|_{L^2}$ occurs in two places in the inequality we have obtained so far. This expression is bounded by $C\|\mathcal{R}[\hat{\mu}]v\|_{L^2}$ for a uniform constant C. Note that

$$\|\partial\mathcal{R}[\tilde{\mu}]\widehat{v}\|_{L^2} = \left(\int_{T^n}\sum_{a=1}^{n}|(\partial_a\mathcal{R}[\tilde{\mu}])v|^2\right)^{1/2} \leq \sum_{a=1}^{n}\|(\partial_a\mathcal{R}[\tilde{\mu}])v\|_{L^2}$$

Fixing an $a \in \{1, \ldots, n\}$, we compute

$$\partial_a\mathcal{R}[\tilde{\mu}] = -\log t\mathrm{Diag}(\partial_a\tilde{\mu}) \cdot \mathcal{R}[\tilde{\mu}] \leq -C\log t\mathcal{R}[\tilde{\mu}]$$

since the $\tilde{\mu}$ are smooth functions of $x \in T^n$. To control the logarithm we extract a positive power of t (say t^ϵ) from $\mathcal{R}[\tilde{\mu}]$, for an ϵ which may be arbitrarily small. As a result,

$$\|\partial_a\mathcal{R}[\tilde{\mu}]v\|_{L^2} \leq C\|\mathcal{R}[\tilde{\mu}+\epsilon]v\|_{L^2} \quad \text{and hence, } \|\partial\mathcal{R}[\tilde{\mu}]\widehat{v}\|_{L^2} \leq C\|\mathcal{R}[\tilde{\mu}+\epsilon]v\|_{L^2}.$$

[10] To show this note that for a, b non-negative real numbers $\sqrt{ab} = \sqrt{(a+b)^2 - a^2 - b^2}/\sqrt{2} \leq \sqrt{(a+b)^2}/\sqrt{2} = (a+b)/\sqrt{2}$.

Setting $\tilde{\mu} = \hat{\mu} - \epsilon$ in Eq. (2.18), using $\mathcal{R}[\hat{\mu} - \epsilon] = t^{\epsilon}\mathcal{R}[\hat{\mu}] \leq \delta^{\epsilon}\mathcal{R}[\hat{\mu}]$, and noting the energy/norm equivalence from Step 2, we obtain the desired lower bound

$$\|\mathcal{R}[\hat{\mu} - \epsilon]v\|_{H^1(T^n)} \leq C\left(\|\mathcal{R}[\hat{\mu}]v\|_{L^2} + \|\widehat{\mathcal{R}}[\hat{\mu}]\partial v\|_{L^2}\right) \leq CE^{(1)}_{\hat{\mu}}[v].$$

Step 4: Upper bound for $E^{(1)}_{\hat{\mu}}[v]$. We now use the fundamental energy estimate Lemma 2.14 to prove an upper bound. By the equivalence of norms and an application of Lemma 2.14 we find

$$\begin{aligned} E^{(1)}[v] \leq{}& C_1\left(\|\widehat{\mathcal{R}}[\hat{\mu}]\partial v\|_{L^2}(t_0) + \|\mathcal{R}[\hat{\mu}]v\|_{L^2}(t_0)\right) \\ &+ C_2\left(\int_{t_0}^{t} s^{-1}(\|\widehat{\mathcal{R}}[\hat{\mu}]\widehat{f_0}\|_{L^2}(s) + \|\mathcal{R}[\hat{\mu}]f_0\|_{L^2}(s))ds\right). \end{aligned} \tag{2.19}$$

Next we bound the terms $\|\widehat{\mathcal{R}}[\hat{\mu}]\widehat{f_0}\|_{L^2}(s)$. Recall the expression for $\widehat{f_0}^{b}$ Eq. (2.16). Since $q_0 > n/2+1$ the operators N_0, S^0, and $(S^0)^{-1}$ are continuous on T^n due to the Sobolev inequalities. Further recall that the perturbations, including F_1, are smooth and bounded in $B_{\delta,\zeta,q}(r)$. Hence there exists a uniform constant C such that

$$\begin{aligned} \|\widehat{\mathcal{R}}[\hat{\mu}]\widehat{f_0}\|_{L^2}(s) &\leq \sum_{a=1}^{n} \|\mathcal{R}[\hat{\mu}]\widehat{f_0}^{a}\|_{L^2} \\ &\leq C\left(\sum_{a=1}^{n} \|\mathcal{R}[\hat{\mu}]\partial_a f_0\|_{L^2} + \|\mathcal{R}[\hat{\mu}]f_0\|_{L^2} + \|\mathcal{R}[\hat{\mu}]v\|_{L^2}\right). \end{aligned}$$

We now claim that for any vector valued function f, (ie f_0 above) there is a uniform constant such that

$$\|\mathcal{R}[\hat{\mu}]\partial_a f\|_{L^2} \leq C\left(\|\partial_a(\mathcal{R}[\hat{\mu}+\epsilon]f)\|_{L^2} + \|\mathcal{R}[\hat{\mu}+\epsilon]f\|_{L^2}\right), \tag{2.20}$$

for any $\epsilon > 0$. To see this, compute

$$\begin{aligned}
\|\mathcal{R}[\hat{\mu}]\partial_a f\|_{L^2} &= \left(\int_{T^n} \langle \mathcal{R}[\hat{\mu}]\partial_a f, \mathcal{R}[\hat{\mu}]\partial_a f\rangle \, dx\right)^{1/2} \\
&= \left(\int_{T^n} |\partial_a(\mathcal{R}[\hat{\mu}]f)|^2 + |(\partial_a\mathcal{R}[\hat{\mu}])f|^2 - 2\,\langle \partial_a(\mathcal{R}[\hat{\mu}]f), (\partial_a\mathcal{R}[\hat{\mu}])f\rangle \, dx\right)^{1/2} \\
&\leq \|\partial_a(\mathcal{R}[\hat{\mu}]f)\|_{L^2} + C\|\mathcal{R}[\hat{\mu}+\epsilon]f\|_{L^2} \\
&\quad + C\sqrt{\|\partial_a(\mathcal{R}[\hat{\mu}]f)\|_{L^2}\|\mathcal{R}[\hat{\mu}+\epsilon]f\|_{L^2}},
\end{aligned}$$

where we have controlled the $\log t$ factors generated in computing $\partial_a\mathcal{R}[\hat{\mu}]$ by extracting a t^ϵ from $\mathcal{R}[\hat{\mu}]$, and used that $\hat{\mu}$ is a smooth function of $x \in T^n$. Again using $\sqrt{2}\sqrt{ab} \leq a + b$ for a, b non-negative real numbers we obtain Eq. (2.20).

Applying this result to $\|\partial_a(\mathcal{R}[\hat{\mu}]f_0)\|_{L^2}$ in the inequality for $\|\widehat{\mathcal{R}}[\hat{\mu}]\widehat{f_0}\|_{L^2}(s)$ above, we find

$$\|\widehat{\mathcal{R}}[\hat{\mu}]\widehat{f_0}\|_{L^2}(s) \leq C\sum_{a=1}^{n}\left(\|\partial_a(\mathcal{R}[\hat{\mu}+\epsilon]f_0)\|_{L^2} + \|\mathcal{R}[\hat{\mu}+\epsilon]f_0\|_{L^2}\right) + \|\mathcal{R}[\hat{\mu}]v\|_{L^2}.$$

In the above expression we have also pulled out a factor of t^ϵ in the first term so that the exponent vector matches that in the second term. Finally we show that the terms in the parenthesis can be bounded by the H^1-norm. Notice that $\sum_{i=1}^n \sqrt{a_i} \leq n\sqrt{\sum_{i=1}^n a_i}$ for all $\{a_i\} \in \mathbb{R}^+ \cup \{0\}$. This is a version of the discrete Hölder inequality; see for example [34] (page 623). When applied to the situation at hand we find that there is a constant depending only on n such that

$$\sum_{a=1}^{n}\|\partial_a(\mathcal{R}[\hat{\mu}+\epsilon]f_0)\|_{L^2(T^n)} + \|\mathcal{R}[\hat{\mu}+\epsilon]f_0\|_{L^2(T^n)} \leq C\|\mathcal{R}[\hat{\mu}+\epsilon]f_0\|_{H^1(T^n)}.$$

We may apply these same arguments to the first term in Eq. (2.19), and as a consequence

$$E_{\hat{\mu}}^{(1)}[v] \leq C\left(\|\mathcal{R}[\hat{\mu}+\epsilon]v\|_{H^1}(t_0) + \int_{t_0}^{t} s^{-1}(\|\mathcal{R}[\hat{\mu}+\epsilon]f_0\|_{H^1}(s) + \|\mathcal{R}[\hat{\mu}]v\|_{L^2}(s)) \right).$$

Combining the upper bound just obtained with the lower bound of Step 3, recalling that $\hat{\mu} = \mu - \epsilon$ and rescaling $\epsilon \to \epsilon/2$ we find

$$\begin{aligned} &\|\mathcal{R}[\mu-\epsilon]v\|_{H^1(T^n)} \\ &\qquad \leq C\left(\|\mathcal{R}[\mu]v\|_{H^1}(t_0) + \int_{t_0}^{t} s^{-1}(\|\mathcal{R}[\mu]f_0\|_{H^1}(s) + s^{\epsilon/2}\|\mathcal{R}[\mu]v\|_{L^2}(s)) \right), \end{aligned}$$

which is the desired inequality for the case $q = 1$. Similar arguments can be made for arbitrary q. □

2.4.4. Weak Solutions to the Asymptotic Value Problem

A useful technique in proving the existence of solutions to partial differential equations is to first establish that solutions exist in an integral or distributional sense; such solutions are called *weak solutions*. In this section we make precise the notion of weak solutions to the asymptotic value problem, and prove the existence of such weak solutions for linear symmetric hyperbolic Fuchsian systems. The proof is based on constructing a sequence of approximate solutions described briefly in Section 2.3. and in detail below. We may then use our control over these approximate solutions, given by the fundamental energy estimate Lemma 2.14, in order to prove that the sequence of approximate solutions converges to a weak solution of the asymptotic value problem.

2.4.4.1. Weak operators

The weak version of the linear symmetric hyperbolic Fuchsian system is its integral or distributional form. Define a **test function** for this system to be any smooth function $\phi : (0, \delta] \times T^n \to \mathbb{R}^d$ for which there is a $T \in (0, \delta)$, such that $\phi(t, x) = 0$ for all $t > T$. For any $w \in X_{\delta,\mu,0}$ and test function ϕ we define the operator $\mathcal{L}[\cdot]$ via

$$\langle \mathcal{L}[w], \phi \rangle := -\int_0^\delta \Bigg(\big\langle \mathcal{R}[\mu] S^0 w, D\phi \big\rangle_{L^2(T^n)} + \sum_{a=1}^{n} \langle \mathcal{R}[\mu] t S^a w, \partial_a \phi \rangle_{L^2(T^n)} + \langle \mathcal{R}[\mu] \mathcal{S} w, \phi \rangle_{L^2(T^n)} \Bigg) dt,$$

where

$$\mathcal{S} := \Bigg(S^0 - N_0 + \mathcal{R}[\mu]^{-1} D\mathcal{R}[\mu] S^0 + DS^0 + \mathcal{R}[\mu]^{-1} \sum_{a=1}^{n} (\partial_a \mathcal{R}[\mu]) t S^a + \sum_{a=1}^{n} t \partial_a S^a \Bigg).$$

This definition is motivated by formally writing

$$\int_0^\delta \langle \mathcal{R}[\mu] L[w], \phi \rangle_{L^2(T^n)} \, dt = \int_0^\delta \int_{T^n} \left\langle \mathcal{R}[\mu] \left(S^0 Dw + \sum_{a=1}^{n} t S^a \partial_a w + N_0 w \right), \phi \right\rangle dx dt$$

and transferring the derivatives to act on ϕ using integration by parts. The terms in $\mathcal{S}$ above are a product of this procedure. The operator $\mathcal{L}[\cdot]$ is called the adjoint of $L[\cdot]$ (recall $L[\cdot]$ is given by Eq. (2.9)). A corresponding weak version of the linear source operator is given by

$$\langle \mathcal{F}[w], \phi \rangle := \int_0^\delta \langle \mathcal{R}[\mu] \left(f_0 + F_1 w \right), \phi \rangle_{L^2(T^n)} \, dt,$$

The next result establishes that these operators are well-defined on the space $X_{\delta,\mu,0}$.

Lemma 2.16. *Suppose that Eq.* (2.1) *is a linear symmetric hyperbolic Fuchsian system with parameters* $\{\delta, \mu, q, q_0, r\}$ *as per Definition 2.11, and is block diagonal with respect to* μ*. Then for every test function* ϕ*, the maps* $\langle \mathcal{L}[\cdot], \phi\rangle$ *and* $\langle \mathcal{F}[\cdot], \phi\rangle$ *are bounded linear functionals on* $X_{\delta,\mu,0}$*.*

Proof. To prove this lemma it is sufficient to show that each term in $\langle \mathcal{L}[w], \phi\rangle$ is bounded by $C||w||_{\delta,\mu,0}$, for some positive constant C and for every $w \in X_{\delta,\mu,0}$. We demonstrate this for the first term, $\int_0^\delta \langle \mathcal{R}[\mu] S^0 w, D\phi\rangle_{L^2}\, dt$. Using Hölder's inequality, the spatial continuity[11] of S^0 and the block-diagonal property, we find that

$$\begin{aligned}\left|\int_0^\delta \left\langle \mathcal{R}[\mu] S^0 w, D\phi\right\rangle_{L^2} dt\right| &\leq \int_0^\delta ||\mathcal{R}[\mu] S^0 w||_{L^2} ||D\phi||_{L^2} dt \\ &\leq \delta \sup_{t\in(0,\delta]} ||\mathcal{R}[\mu] S^0 w||_{L^2}(t) ||D\phi||_{L^2}(t) \leq C||w||_{\delta,\mu,0}.\end{aligned}$$

The constant C, which is used to estimate both the contributions from S^0 and from ϕ, is uniform in the sense defined above. Other terms in $\langle \mathcal{L}[w], \phi\rangle$ follow similarly, and the same arguments hold for the $\langle \mathcal{F}[w], \phi\rangle$ operator. □

We define w to be a **weak solution** of the linear asymptotic value problem corresponding to Eq. (2.1) with vanishing leading term provided it satisfies, for *all* test functions ϕ,

$$\langle \mathcal{P}[w], \phi\rangle := \langle \mathcal{L}[w] - \mathcal{F}[w], \phi\rangle = 0. \tag{2.21}$$

[11] This follows from the definition of a linear symmetric hyperbolic Fuchsian system, and from Sobolev embedding.

2.4.4.2. Existence of weak solutions

Having established a definition of weak solutions to the asymptotic value problem, and that the weak operators are well-defined on the function space of interest, $X_{\delta,\mu,0}$, we now prove that weak solutions exist to the linear symmetric hyperbolic Fuchsian systems.

Proposition 2.17 (Existence of weak solutions of the linear asymptotic value problem with smooth perturbations)**.** *Suppose that Eq.* (2.1) *is a linear symmetric hyperbolic Fuchsian system with parameters* $\{\delta, \mu, q, q_0, r\}$*, for* $q = 0$ *and* $q_0 > n/2+1$ *with smooth perturbations, and is block-diagonal with respect to* μ*. Suppose also that the energy dissipation matrix Eq.* (2.7) *is positive definite for all* $x \in T^n$ *and, in addition,* $DS_1^0, \partial_b S^a \in B_{\delta,\xi,0}(\tilde{r})$ *for all* $a, b = 1, \ldots, n$ *for some* $\tilde{r} > 0$ *and for some exponent vector* ξ *with strictly positive entries (so that we may apply the fundamental energy estimate). Then there exist weak solutions* $w : (0, \delta] \times T^n \to \mathbb{R}^d$ *to the asymptotic value problem (with vanishing leading term) which are elements of* $X_{\delta,\mu,0}$*.*

This is the most general existence result we are able to prove, requiring only weak control over the regularity of the coefficients (measured by q and q_0). However, additional control is necessary to prove uniqueness of solutions; this is done below in Proposition 2.22.

Proof. As stated above, the proof is based on constructing a sequence of approximate solutions. Let $\{t_i\}$ be a monotonically decreasing sequence of times $t_i \in (0, \delta]$ which converges to zero. For each i, we construct a function $v_i : (0, \delta] \times T^n \to \mathbb{R}^d$ which vanishes on $(0, t_i]$, and which is equal on $(t_i, \delta]$ to the solution of the Cauchy problem with zero initial data at t_i[12]. The functions $\{v_i\}$ are called **approximate solutions**. One can verify that $v_i \in C^0((0,\delta]; H^{q_0}(T^n)) \cap X_{\delta,\mu,0}$ for each i. We seek to show that the sequence $\{v_i\}$ forms a Cauchy sequence in $X_{\delta,\mu,0}$. Defining $\xi_{ij} := v_i - v_j$ for $i > j$, we readily see that

$$\xi_{ij}(t,x) = \begin{cases} 0, & t \in (0, t_i], \\ v_i, & t \in (t_i, t_j], \\ v_i - v_j, & t \in (t_j, \delta]. \end{cases} \tag{2.22}$$

From the energy estimate for the Cauchy problem Lemma 2.14 on each subinterval, we then compute

$$||\mathcal{R}[\mu](t,\cdot)\xi_{ij}(t,\cdot)||_{L^2} \begin{cases} = & 0, & t \in (0, t_i], \\ \leq & 0 + C\int_{t_i}^{t} s^{-1}||\mathcal{R}[\mu]f_0||_{L^2}ds, & t \in (t_i, t_j], \\ \leq & ||\mathcal{R}[\mu](t_j,\cdot)v_i(t_j,\cdot)||_{L^2}, & t \in (t_j, \delta], \end{cases} \tag{2.23}$$

where in the last inequality we have used the energy/norm equivalence Eq. (2.11) above, and we have also used the fact that the (linear) PDE system for $v_i - v_j$ has a vanishing source term f_0. Recalling the definition of the norm $||\cdot||_{\delta,\mu,q}$, noting the monotonicity of $\int_{t_i}^{t} s^{-1}||\mathcal{R}[\mu]f_0||_L^2 ds$, and noting the equality $\xi_{ij}(t_j,\cdot) = v_i(t_j,\cdot)$ at $t = t_j$, we now have

$$||\xi_{ij}||_{\delta,\mu,0} = \sup_{t\in(0,\delta]} ||\mathcal{R}[\mu](t,\cdot)\xi_{ij}(t,\cdot)||_{L^2} \leq C\int_{t_i}^{t_j} s^{-1}||R[\mu]f_0||_{L^2}ds.$$

[12]Note that in general we prescribe initial data as $\phi_i(x) = u_0(t_i, x)$ as in Section 2.3.. However for the linear theory presented in this section we have assumed for simplicity, and without loss of generality, that $u_0 \equiv 0$.

To complete the argument that we have a Cauchy sequence, it is useful to introduce

$$G(t) := \int_0^t s^{-1} ||\mathcal{R}[\mu] f_0||_{L^2}(s) ds, \tag{2.24}$$

which is well-defined so long as $f_0 \in X_{\delta,\nu,0}$ for $\nu > \mu$. Choosing $\epsilon > 0$ as a lower bound for the gap between ν and μ among all components, we see that there must exist a constant C such that $G(t) \leq Ct^{\epsilon}$; thence, we have

$$||\xi_{ij}||_{\delta,\mu,0} \leq C|G(t_j) - G(t_i)|, \tag{2.25}$$

from which it easily follows that $\{v_i\}$ is a Cauchy sequence in the Banach space $X_{\delta,\mu,0}$.

Since it has been established (in Lemma 2.16) that $\mathcal{P} = \mathcal{L} - \mathcal{F}$ is a continuous operator on $X_{\delta,\mu,0}$, to show that the limit of the Cauchy sequence $\{v_i\}$ is a weak solution of the system of interest, it is sufficient to show that the limit of the sequence of reals $(\langle \mathcal{P}[v_i], \phi \rangle)$ is zero for all test functions ϕ. Choosing any v_i in our sequence, we know from its definition that v_i vanishes on $(0, t_i]$ and is a solution to the equation $\langle \mathcal{P}[v_i], \phi \rangle = 0$ on $[t_i, \delta]$. Recalling the definition of $\mathcal{P}$, we calculate on the former interval, for any test function ϕ,

$$|\langle \mathcal{P}[v_i], \phi \rangle| = \left| - \int_0^{t_i} \langle \mathcal{R}[\mu] f_0, \phi \rangle_{L^2(T^n)} \, dt \right| .$$

Straightforward calculation then shows that

$$\begin{aligned}\left|-\int_0^{t_i} \langle \mathcal{R}[\mu] f_0, \phi\rangle_{L^2(T^n)}\right| dt &\leq \int_0^{t_i} |\langle \mathcal{R}[\mu] f_0, \phi\rangle_{L^2(T^n)}| dt \\ &\leq \int_0^{t_i} \left(\Big(\int_{T^n} dx |\mathcal{R}[\mu] f_0|^2 \Big)^{1/2} \Big(\int_{T^n} dx |\phi|^2 \Big)^{1/2} \right) dt \\ &= \int_0^{t_i} \left(t^{-1} \Big(\int_{T^n} dx |\mathcal{R}[\mu] f_0|^2 \Big)^{1/2} t \Big(\int_{T^n} dx |\phi|^2 \Big)^{1/2} \right) dt \\ &\leq \sup_{t\in(0,\delta]} ||t\phi||_{L^2} \int_0^{t_i} t^{-1} ||\mathcal{R}[\mu] f_0(t)||_{L^2} dt \leq CG(t_i),\end{aligned}$$

from which it follows (from the properties of $G(t)$), that we have a weak solution. □

2.4.4.3. A solution operator

For use in later parts of the proof of Theorem 2.10, we define an operator which for fixed S^0, S^a, N_0, and F_1 takes any function $f_0 \in X_{\delta,\nu,0} \cap C^\infty((0,\delta] \times T^n)$ to a weak solution $w \in X_{\delta,\mu,0}$ of the linear asymptotic value problem. Then as a next step, we would like to extend this map to all f_0 of $X_{\delta,\nu,0}$, and thereby show that weak solutions exist for all $f_0 \in X_{\delta,\nu,0}$, and not just for those f_0 which are smooth. A potential obstruction to this definition is the lack of a uniqueness result for weak solutions. We avoid this by defining an operator which takes f_0 to the weak solution obtained as the limit of the sequence $\{v_i\}$ and verify that this limit is independent of the sequence of times $\{t_i\}$ which is chosen.

Proposition 2.18. *Presuming the hypotheses listed in Proposition 2.17, there exists an operator $\mathbb{H} : X_{\delta,\nu,0} \to X_{\delta,\mu,0}$ that maps a smooth source function f_0 to the weak solution w of the linear asymptotic value problem (i.e. w satisfies $\langle \mathcal{P}[w], \phi\rangle = 0$) which is obtained as the limit of the sequence of approximate solutions $\{v_i\}$ corresponding to a choice of a monotonic sequence of times $\{t_i\}$ converging to zero. This operator is* *well-defined (independent of the choice of the sequence $\{t_i\}$) and satisfies the estimate*

$$||\mathbb{H}[f_0]||_{\delta,\mu,0} \leq \delta^\rho C ||f_0||_{\delta,\nu,0}, \tag{2.26}$$

for all smooth $f_0 \in X_{\delta,\nu,0}$. The positive constants C and ρ are uniform.

The operator extends to all (not necessarily smooth) $f_0 \in X_{\delta,\nu,0}$, with the estimate (2.26) *holding for all such f_0 with the same constants. Indeed, this extended operator $\mathbb{H}$ maps all $f_0 \in X_{\delta,\nu,0}$ to weak solutions of Eq.* (2.1)*.*

The last paragraph in this proposition generalizes the existence result in Proposition 2.17 to all, not necessarily smooth, source terms $f_0 \in X_{\delta,\nu,0}$. We note, however, that otherwise the system is still assumed to have smooth perturbations in the sense defined above.

Proof. In the first step we show that for $f_0 \in C^\infty((0,\delta] \times T^n) \cap X_{\delta,\nu,0}$, the map $f \mapsto \mathbb{H}[f]$ is a well-defined map to $X_{\delta,\mu,0}$, independent of the choice of time sequence. Let $\{t_i^1\}$ and $\{t_j^2\}$ be two monotonically decreasing sequences of times in $(0,\delta]$ with limit zero, and let $\{v_i^1\}$ and $\{v_j^2\}$ be the corresponding sequences of approximate solutions. We show that the limits of each of these sequences, call them w^1 and w^2 respectively, are identical in $X_{\delta,\mu,0}$. From the union of the two time sequences we construct a third sequence $\{t_k\}$, and obtain the corresponding sequence of approximate solutions $\{v_k\}$. As is the case for $\{v_i^1\}$ and $\{v_j^2\}$, the combined sequence of approximate solutions $\{v_k\}$ must be a Cauchy sequence, so[13] $||v_i^1 - v_j^2||_{\delta,\mu,0}$ must vanish in the limit $i, j \to \infty$. Then it follows from the estimate

$$||w^1 - w^2||_{\delta,\mu,0} \leq ||w^1 - v_i^1||_{\delta,\mu,0} + ||v_j^2 - w^2||_{\delta,\mu,0} + ||v_i^1 - v_j^2||_{\delta,\mu,0},$$

[13]Here, we set δ to be the smallest bound among the two sequences.

that w^1 and w^2 are equal in $X_{\delta,\mu,0}$.

Next we prove the estimate Eq. (2.26) for $\mathbb{H}$, still with smooth f_0. Let $\{v_i\}$ be a sequence of approximate solutions with limit $w = \mathbb{H}(f_0)$. The idea is to use the estimate Eq. (2.25) in order to bound $\|w\|_{\delta,\mu,0}$ by $G(\delta)$, and then argue that this in turn can be bounded by $\|f_0\|_{\delta,\nu,0}$. From Eq. (2.25) and the monotonicity of $G(t)$ we determine that $||w - v_1||_{\delta,\mu,0} \leq CG(t_1) \leq CG(\delta)$, and hence that

$$||w||_{\delta,\mu,0} \leq ||v_1||_{\delta,\mu,0} + CG(\delta).$$

It follows from the energy estimate Eq. (2.12) and the energy/norm equivalence that $||v_1||_{\delta,\mu,0} \leq \widetilde{C}G(\delta)$, and thus for an adapted constant C we find

$$||w||_{\delta,\mu,0} \leq CG(\delta).$$

To relate $G(\delta)$ to the source term, recall $G(\delta) = \int_0^\delta s^{-1}\|\mathcal{R}[\mu]f_0\|_{L^2}(s)ds$. Consider the integrand

$$\begin{aligned}
s^{-1}\|\mathcal{R}[\mu]f_0\|_{L^2}(s) &= s^{-1}\|\mathcal{R}[\mu-\nu]\mathcal{R}[\nu]f_0\|_{L^2}(s) \\
&\leq s^{-1}s^{\rho}\|\mathcal{R}[\nu]f_0\|_{L^2}(s) \\
&\leq s^{\rho-1}\left(\sup_{s\in(0,\delta]}(\|\mathcal{R}[\nu]f_0\|_{L^2}(s))\right) \\
&= s^{\rho-1}\|f_0\|_{\delta,\nu,0}
\end{aligned}$$

where $\rho := \min(\nu - \mu)$, and where the minimum is taken over all $i = 1, \ldots, d$, and $x \in T^n$. Integrating this inequality over $\int_0^\delta ds$ we find

$$G(\delta) \leq \left(\int_0^\delta s^{\rho-1} ds \right) \|f_0\|_{\delta,\nu,0}.$$

The integral on the right hand side is finite since $\rho > 0$; a consequence of $\nu > \mu$. The desired inequality Eq. (2.26) is obtained for some $C \geq 1/\rho$.

Finally we extend the operator $\mathbb{H}$ to fully general (not necessarily smooth) f_0 in $X_{\delta,\nu,0}$. Since the space $C^\infty((0,\delta] \times T^n) \cap X_{\delta,\nu,0}$ is dense in $X_{\delta,\nu,0}$, any element $f_0 \in X_{\delta,\nu,0}$ can be represented as the limit of a convergent sequence $\{f_0^{(j)}\} \in C^\infty((0,\delta] \times T^n) \cap X_{\delta,\nu,0}$. From the continuity of $\mathbb{H}[\cdot]$ on the smooth subspace we conclude that there exists a limit $w = \lim_{j\to\infty} \mathbb{H}[f_0^{(j)}]$, and that w is in $X_{\delta,\mu,0}$ by the completeness of these spaces. Hence we extend $\mathbb{H}[\cdot]$ to the full space by defining

$$\mathbb{H}[f_0] := \lim_{j\to\infty} \mathbb{H}[f_0^{(j)}].$$

Furthermore, the estimate Eq. (2.26) holds for the extended operator. To show this, note that for any j

$$\|\mathbb{H}[f_0^{(j)}]\|_{\delta,\mu,0} \leq \delta^\rho C \|f_0^{(j)}\|_{\delta,\nu,0},$$

and hence

$$\|\mathbb{H}[f_0^{(j)} - f_0 + f_0]\|_{\delta,\mu,0} \leq \delta^\rho C \|f_0^{(j)} - f_0 + f_0\|_{\delta,\nu,0}.$$

By an application of the triangle inequality and the reverse triangle inequality we find

$$\|\mathbb{H}[f_0]\|_{\delta,\mu,0} \leq \delta^\rho C \|f_0\|_{\delta,\nu,0} + \left(C\delta^\rho \|f_0 - f_0^{(j)}\|_{\delta,\nu,0} + \|\mathbb{H}[f_0] - \mathbb{H}[f_0^{(j)}]\|_{\delta,\mu,0} \right).$$

Since the terms in the parenthesis become arbitrarily small for $j \to \infty$ we conclude that the estimate holds for the extended operator (which is denoted using the same symbol). $\square$

2.4.5. Regularity of Solutions to the Asymptotic Value Problem

Having established in Section 2.4.4. the existence of solutions to the asymptotic value problem in the weak sense, we proceed to determine the regularity of these solutions, and prove that they are solutions in a strong sense defined below. See Section A.1. for general comments on finding solutions to partial differential equations. Note that if w is a solution to the weak equation, we have

$$\begin{aligned}
0 &= \langle \mathcal{L}[w] - \mathcal{F}[w], \phi \rangle \\
&= -\int_0^\delta \Big(\langle \mathcal{R}[\mu] S^0 w, D\phi \rangle_{L^2(T^n)} + \sum_{a=1}^n \langle \mathcal{R}[\mu] t S^a w, \partial_a \phi \rangle_{L^2(T^n)} + \langle \mathcal{R}[\mu] \mathcal{S} w, \phi \rangle_{L^2(T^n)} \Big) dt \\
&\quad - \int_0^\delta \langle \mathcal{R}[\mu] \left(f_0 + F_1 w \right), \phi \rangle_{L^2(T^n)} \, dt.
\end{aligned}$$

Now suppose w could be shown to be continuously differentiable in both time and space on $(0, \delta] \times T^n$. Then, by reversing the integration by parts, we find that

$$\begin{aligned}
0 = &\int_0^\delta \Big(\langle \mathcal{R}[\mu] S^0 D w, \phi \rangle_{L^2(T^n)} + \sum_{a=1}^n \langle \mathcal{R}[\mu] t S^a \partial_a w, \phi \rangle_{L^2(T^n)} + \langle \mathcal{R}[\mu] N_0 w, \phi \rangle_{L^2(T^n)} \Big) dt \\
&- \int_0^\delta \langle \mathcal{R}[\mu] \left(f_0 + F_1 w \right), \phi \rangle_{L^2(T^n)} \, dt,
\end{aligned}$$

or equivalently,

$$0 = \int_0^\delta \langle \mathcal{R}[\mu] \left(L[w] - \mathfrak{F}[w] \right), \phi \rangle_{L^2(T^n)} \, dt. \tag{2.27}$$

Since this equation holds for any test function ϕ, we could argue that the differential version of the equation, that is $L[w]-\mathcal{F}[w]=0$, holds pointwise on $(0,\delta]\times T^n$ almost everywhere.

Of course, we cannot always (often) verify that w is continuously differentiable. In such cases, the next best hope would be to show that w is differentiable in a distributional sense (Definition A.1). If we can show that $w \in X_{\delta,\mu,q}$ and that there exists a time derivative $Dw \in X_{\delta,\mu,q-1}$ so that $L[w]$ and, $\mathcal{F}[w]$ are both in $X_{\delta,\mu,q-1}$ and are equivalent as distributions (ie Eq. (2.27) holds), then we say w is a *strong solution.*

For solutions to the asymptotic value problem for linear symmetric hyperbolic Fuchsian systems with smooth perturbations, we obtain the following proposition.

Proposition 2.19 (Regularity of solutions to the Linear AVP)**.** *Suppose that Eq.* (2.1) *is a linear symmetric hyperbolic Fuchsian system with smooth perturbations and with parameters* $\{\delta,\mu,q,q_0,r\}$*, for* $q\geq 1$ *and* $q_0>n/2+1+q$*. Suppose also that the system is block-diagonal with respect to* μ*, and the energy dissipation matrix Eq.* (2.7) *is positive definite for all* $x\in T^n$ *and, in addition,* $DS_1^0\in B_{\delta,\xi,0}(\tilde{r})$ *for some* $\tilde{r}>0$ *and some exponent matrix* ξ *with strictly positive entries.*[14]*. Then, weak solutions* w *of the asymptotic value problem (whose existence has been checked in Proposition 2.17) are differentiable in time and satisfy Eq.* (2.1)*, with* $w\in X_{\delta,\mu,q}$ *and* $Dw\in X_{\delta,\mu,q-1}$*.*

Further, the solution operator $\mathbb{H}$ *defined in Proposition 2.17 maps* $X_{\delta,\nu,q}$ *to* $X_{\delta,\mu,q}$*, and satisfies*

$$\|\mathbb{H}[f_0]\|_{\delta,\mu,q}\leq \delta^{\rho}C\|f_0\|_{\delta,\nu,q}, \tag{2.28}$$

[14]These are the conditions of Proposition 2.17 but with increased values of q, q_0

for all (not necessarily smooth) $f_0 \in X_{\delta,\nu,q}$*. The constants* $C > 0$ *and* $\rho > 0$ *are uniform in the sense of Definition 2.12 (but may depend in particular on* q*).*

Proof. Step 1: Convergence in $X_{\delta,\mu,q}$*.* We begin by establishing that the weak solutions obtained as the limit of the sequence of approximate solutions $\{v_i\}$ introduced in the proof of Proposition 2.17 have q weak spatial derivatives, and thus are in $X_{\delta,\mu,q}$. Due to the higher regularity assumed, each approximate solutions is contained in $C^0((0,\delta]; H^q(T^n)) \cap X_{\delta,\mu,q}$.[15] The same arguments detailed in the proof of Proposition 2.17 can be applied here using the higher order energy estimates Lemma 2.15, in order to show that $\{v_i\}$ is a Cauchy sequence. However, because of the slight loss of control in the higher order energy estimate Lemma 2.15, we obtain only that w, the limit of the sequence $\{v_i\}$, is in $X_{\delta,\mu-\epsilon,q}$ for an arbitrarily small $\epsilon > 0$, and the estimate

$$\|\mathbb{H}[f_0]\|_{\delta,\mu-\epsilon,q} \leq \delta^\rho C \|f_0\|_{\delta,\nu,q}.$$

To regain "μ-control" over the solution we note that since the equation is of linear symmetric hyperbolic Fuchsian form for the choice of μ, then it is also of this form for $\widehat{\mu} := \mu + \epsilon$ if $\epsilon > 0$ is sufficiently small in comparison to $\nu - \mu$. Moreover, the block-diagonality conditions and the energy dissipation matrix positivity hold with respect to $\widehat{\mu}$. Hence the analysis described above can be performed with $\widehat{\mu}$ in place of μ, leading to the conclusion that in fact, the solution w is in $X_{\delta,\widehat{\mu}-\epsilon,q} = X_{\delta,\mu,q}$ (as opposed to $X_{\delta,\mu-\epsilon,q}$ above) and

$$\|\mathbb{H}[f_0]\|_{\delta,\mu,q} \leq \delta^\rho C \|f_0\|_{\delta,\nu,q},$$

[15] In fact, the theory for symmetric hyperbolic systems guarantees that $v_i \in C^0((0,\delta]; H^{q_0}(T^n))$ for each i, and each v_i is also in X_{δ,μ,q_0}. However, since we only control the $t \searrow 0$ behavior of q derivatives of the coefficients, we only hope to control the solution of the asymptotic value problem in $X_{\delta,\mu,q}$.

possibly after a slight change of the constants C and ρ. This verifies that w has the sufficient number of distributional spatial derivatives to be considered a strong solution, and in particular that there exists a $\partial_a w$ such that

$$-\int_0^\delta \left(\sum_{a=1}^n \langle \mathcal{R}[\mu] t S^a w, \partial_a \phi \rangle_{L^2(T^n)} + \sum_{a=1}^n \langle (\partial_a \mathcal{R}[\mu]) t S^a + \mathcal{R}[\mu] t \partial_a S^a, \phi \rangle_{L^2(T^n)} \right) dt$$
$$= \int_0^\delta \sum_{a=1}^n \langle t S^a \mathcal{R}[\mu] \partial_a w, \phi \rangle_{L^2(T^n)}$$

Step 2: Existence of a time derivative. We must also verify that the solution is differentiable in time. This is a consequence of the convergence of the sequence in $X_{\delta,\mu,q}$. Define

$$\widehat{v}_i := \left(S^0\right)^{-1} \left(f_0 + F_1 v_i - \sum_{a=1}^n t S^a \partial_a v_i - N_0 v_i \right).$$

Since $v_i \in X_{\delta,\mu,q}$, we have that $\widehat{v}_i \in X_{\delta,\mu,q-1}$ for all i. Further, $\widehat{v}_i = D v_i$ for all $t \in [t_i, \delta]$. Hence, for any $\delta_I \in (0, \delta)$, there exists a sufficiently large i such that $t_i \leq \delta_I$ and $\widehat{v}_i = D v_i$ for all $t \in [\delta_I, \delta]$. Moreover, due to the convergence of the sequence $\{v_i\}$, we find that

$$||\widehat{v}_i - \widehat{v}_j\|_{\delta,\mu,q-1} \leq C \|v_i - v_j\|_{\delta,\mu,q} \to 0$$

for a uniform constant $C > 0$. Let $\widehat{v}$ denote the limit of $\{\widehat{v}_i\}$, which is in the space $X_{\delta,\mu,q-1}$. At this point we have shown that $D v_i(t) = \widehat{v}_i(t) \to \widehat{v}(t)$ uniformly (that is, independent of t) at every $t \in [\delta_I, \delta]$. An application of Theorem A.9 shows that under the uniform convergences we have established thus far, Dw exists at each $t \in [\delta_I, \delta]$ as a Frechet derivative from $[\delta_I, \delta]$ to $H^{q-1}(T^n)$ and $Dw = \widehat{v}$. Since $\delta_I \in (0, \delta)$ can be made arbitrarily close to zero, the argument just presented applies for all $t \in (0, \delta]$, and thus $Dw \in X_{\delta,\mu,q-1}$.

We now show that Dw is the distributional time derivative of w in the following sense. First note that since $D := t\partial_t$, for any matrix-valued function M and any $\epsilon > 0$

$$\int_\epsilon^\delta D \langle Mw, \varphi \rangle = \int_\epsilon^\delta \partial_t (t \langle Mw, \varphi \rangle_{L^2(T^n)}) - \langle Mw, \varphi \rangle_{L^2(T^n)} \, dt.$$

Evaluating the boundary term we find

$$\int_\epsilon^\delta \partial_t (t \langle Mw, \varphi \rangle_{L^2(T^n)}) dt = -\epsilon \langle Mw, \varphi \rangle_{L^2(T^n)} \Big|_{t=\epsilon},$$

which vanishes in the limit $t \searrow 0$. Thus if it exists Dw, satisfies

$$\int_0^\delta \langle MDw, \varphi \rangle = - \int_0^\delta \big(\langle (DM + M)w, \varphi \rangle + \langle Mw, D\varphi \rangle \big).$$

In the present case $M = \mathcal{R}[\mu]S^0$, and the existence of Dw implies we can reverse the integration-by-parts as in the discussion above Proposition 2.19. Combined with the result from the spatial derivatives we obtain Eq. (2.27).

Step 3: Strong solutions. To complete the argument that $w \in X_{\delta,\mu,q}$ just constructed is a strong solution to the equation we verify that both $L[w]$ and $\mathcal{F}[w]$ are in $X_{\delta,\mu,q-1}$. This follows from the definition of linear symmetric hyperbolic Fuchsian systems Definition 2.11, Lemma C.1, and the block-diagonality condition. □

2.4.6. Extension Argument

So far we have proven the existence of solutions to the asymptotic value problem for linear symmetric hyperbolic Fuchsian systems under the smooth perturbations condition that S^0_1, tS^a and, F_1 are in the smooth subspaces of the relevant weighted Sobolev spaces. In this section we extend this theory to equations where the coefficients are general elements of $B_{\delta,\zeta,q}(r)$.

In this section we consider linear symmetric hyperbolic Fuchsian systems of the form

$$L[w] := S^0 Dw + \sum_{a=1}^{n} tS^a \partial_a w + N_0 w = f_0. \tag{2.29}$$

We have dropped the linear source term $F_1 w$ to simplify the arguments. This is no loss of generality since below we use the result established in this section in a contraction mapping argument, the conclusion of which is the full non-linear theorem. Note that the term $F_1 w$ has been introduced above in order to prove the higher order energy estimates, since such terms arise in the derivation of the equations for ∂v. We obtain the following proposition.

Proposition 2.20. *Suppose Eq.* (2.1) *is a linear symmetric hyperbolic Fuchsian system with parameters* $\{\delta, \mu, q, q_0, r\}$ *(with* $F_1 \equiv 0$*) and thus of form Eq.* (2.29)*, with* $q_0 > q + n/2 + 1$ *and* $q > n/2 + 1$*. Suppose also that with respect to* μ *the energy dissipation matrix is positive definite and the system is block-diagonal. Then for all* $f_0 \in X_{\delta,\nu,q}$ *with* $\nu > \mu$ *there exists a solution* $w : (0, \delta] \times T^n \to \mathbb{R}^d$ *of the linear asymptotic value problem with zero leading order term such that* $w \in X_{\delta,\mu,q}$*, and* $Dw \in X_{\delta,\mu,q-1}$*. Further, the solution operator* $\mathbb{H}[\cdot] : f_0 \mapsto w$ *satisfies*

$$\|\mathbb{H}[f_0]\|_{\delta,\mu,q} \leq \delta^\rho C \|f_0\|_{\delta,\nu,q},$$

for uniform constants C, ρ*.*

The proof of this proposition relies on Proposition 2.19, for the existence of solutions to the asymptotic value problem for linear symmetric hyperbolic Fuchsian systems with smooth perturbations. The basic idea is to approximate the system in the general (non-smooth) case by a sequence of systems which have smooth perturbations. Proposition 2.19 then provides a sequence of solutions to the smooth equations, and we show that this sequence converges to a solution of the non-smooth system.

Proof. Step 1: Construction of the sequence. Having specified a linear symmetric hyperbolic Fuchsian system with zero leading order term, and coefficients $S_0^0, N_0 \in H^{q_0}$ and $S_1^0, S^a \in B_{\delta,\zeta,q}(r)$ (*not* in the smooth subset), let $\{S^0_{1\,[i]}\}$, and $\{tS^a_{[i]}\}$ be sequences in $B_{\delta,\zeta,q}(r) \cap C^\infty((0,\delta] \times T^n)$ which converge to S_1^0 and tS^a respectively. For each i and the corresponding coefficients, define the linear symmetric hyperbolic Fuchsian operator $L_{[i]}$ by

$$L_{[i]}[\tilde{w}] := \left(S_0^0 + S^0_{1\,[i]}\right) D\tilde{w} + \sum_{a=1}^{n} tS^a_{[i]}\partial_a\tilde{w} + N_0\tilde{w},$$

and consider the sequence of linear symmetric hyperbolic Fuchsian equations $\{L_{[i]}[\tilde{w}] = f_0\}$ with the same source f_0 in each iterate. We make the following remarks:

(i) Because for each i, the perturbation coefficients $S^0_{1\,[i]}$ and $tS^a_{[i]}$ are in the space $B_{\delta,\zeta,q}(r) \cap C^\infty((0,\delta] \times T^n)$, each system is a linear symmetric hyperbolic Fuchsian system with the same parameters $\{\delta, \mu, q, q_0, r\}$.

(ii) Since S_0^0 and N_0 are the same for all i, the energy dissipation matrix corresponding to each system is positive definite with respect to the same μ.

(iii) We choose $S^0_{1\,[i]}$ and $tS^a_{[i]}$ such that the system is block-diagonal with respect to μ for each i.

We would like to apply Proposition 2.19 in order to obtain a sequence of solution operators $\{\mathbb{H}_{[i]}[\cdot]\}$, and a sequence of solutions $\{w_{[i]}\}$. To do this we must ensure that the sequence of $DS^0_{1\,[i]}$ is uniformly bounded in $B_{\delta,\xi,0}(\tilde{r})$. It turns out that this property can be proved from the assumptions we have made thus far. The argument is long and quite technical, and we simply cite the result from the proof of Proposition 2.13 in [3]. With this result, an application of Proposition 2.19 provides the sequence of solutions $\{w_{[i]}\}$ given by

$$w_{[i]} = \mathbb{H}_{[i]}[f_0],$$

with the property

$$\|w_{[i]}\|_{\delta,\mu,q} \leq \delta^\rho C \|f_0\|_{\delta,\nu,q}$$

for uniform constants which are also independent of i.

Step 2: Convergence of the sequence in $X_{\delta,\mu,q-1}$. We now show that this sequence converges to a solution w of the asymptotic value problem with general (non-smooth) coefficients in $B_{\delta,\zeta,q}(r)$. For technical reasons we show this convergence first in the space $X_{\delta,\mu,q-1}$. In Step 4 below we extend the convergence to $X_{\delta,\mu,q}$.

Let $\xi_{[ij]} := w_{[i]} - w_{[j]}$, and derive the equation

$$L_{[i]}[\xi_{[ij]}] = -\Delta L_{[ij]}[w_{[j]}],$$

where we have used $L_{[i]}[w_{[i]}] = L_{[j]}[w_{[j]}] = f_0$, and where

$$\begin{aligned} \Delta L_{[ij]}[w] :=& \left(L_{[i]} - L_{[j]}\right)[w] \\ =& \left(S^0_{1\,[i]} - S^0_{1\,[j]}\right) D[w] + \sum_{a=1}^{n} t\left(S^a_{[i]} - S^a_{[j]}\right) \partial_a[w]. \end{aligned}$$

Note that this is a linear symmetric hyperbolic Fuchsian system for $\xi_{[ij]}$ with parameters $\{\delta, \mu, q-1, q_0, r\}$. The change from q to $q-1$ is due to the source term which involves derivatives of $w_{[j]}$. By an application of Proposition 2.19 we obtain the estimate

$$\|\xi_{[ij]}\|_{\delta,\mu,q-1} \leq \delta^\rho C \|f_0\|_{\delta,\nu,q-1} = \delta^\rho C \|\Delta L_{[ij]}[w_{[j]}]\|_{\delta,\nu,q-1}.$$

To show that $\{w_{[i]}\}$ is a Cauchy sequence in $X_{\delta,\mu,q-1}$ we show that the right hand side of this inequality is bounded by a quantity which vanishes as $i, j \to \infty$. Note that

$$\begin{aligned}\|\Delta L_{[ij]}[w_{[j]}]\|_{\delta,\nu,q-1} \leq &\| \left(S^0_{1\,[i]} - S^0_{1\,[j]}\right) D[w_{[j]}]\|_{\delta,\nu,q-1} \\ &+ \sum_{a=1}^{n} \|t \left(S^a_{[i]} - S^a_{[j]}\right) \partial_a [w_{[j]}]\|_{\delta,\nu,q-1}.\end{aligned}$$

Using Lemma C.1, which provides a bound on the product of a matrix with a vector, and the block-diagonal conditions we see that each product is in $X_{\delta,\mu+\zeta,q-1}$. We may bound the first term by

$$\| \left(S^0_{1\,[i]} - S^0_{1\,[j]}\right) D[w_{[j]}]\|_{\delta,\mu+\zeta,q-1} \leq C \|S^0_{1\,[i]} - S^0_{1\,[j]}\|_{\delta,\zeta,q-1} \|Dw_{[j]}\|_{\delta,\mu,q-1}.$$

Since we do not have control over $Dw_{[j]}$ in the limit $j \to \infty$, we must eliminate this term. We do this using the symmetric hyperbolic Fuchsian equation, from which it follows that

$$\begin{aligned}\|Dw_{[j]}\|_{\delta,\mu,q-1} \leq &\| \left(S^0_{[j]}\right)^{-1} f_0\|_{\delta,\mu,q-1} + \| \left(S^0_{[j]}\right)^{-1} N_0 w_{[j]}\|_{\delta,\mu,q-1} \\ &+ \sum_{a=1}^{n} \| \left(S^0_{[j]}\right)^{-1} t S^a_{[j]} \partial_a w_{[j]}\|_{\delta,\mu,q-1}.\end{aligned}$$

We now show that each of these terms can be bounded by $C\|f_0\|_{\delta,\nu,q}$, for a constant C which depends only on r, (the radius of the perturbation space $B_{\delta,\zeta,q}(r)$ associated to the symmetric hyperbolic Fuchsian system) and in particular not on j. By the splitting of S^0, and the form of the inverse matrix Lemma C.20, it follows that $\left(S^0_{[j]}\right)^{-1} \in B_{\delta,0,q}(\tilde{r})$ for some $\tilde{r} > 0$. Further, since S^0 has the block-diagonal structure of μ, it is easily shown that $\left(S^0_{[j]}\right)^{-1}$ commutes with $\mathcal{R}[\mu]$ for each j.[16]

Let us first consider $\| \left(S^0_{[j]}\right)^{-1} f_0\|_{\delta,\mu,q-1}$. Using the properties of $\left(S^0_{[j]}\right)^{-1}$, and Lemma C.1 we compute

$$\begin{aligned}
\| \left(S^0_{[j]}\right)^{-1} f_0\|_{\delta,\mu,q-1} &\leq C\|\mathcal{R}[\mu] \left(S^0_{[j]}\right)^{-1} \mathcal{R}[-\mu]\|_{\delta,0,q-1}\|f_0\|_{\delta,\mu,q-1}, \\
&\leq C(\tilde{r})\|f_0\|_{\delta,\mu,q-1}, \\
&\leq C(\tilde{r})\|f_0\|_{\delta,\nu,q-1}, \\
&\leq C(\tilde{r})\|f_0\|_{\delta,\nu,q},
\end{aligned}$$

where in the second to last line we have used Lemma B.1, and in the last line Lemma B.2.

Next consider the term $\| \left(S^0_{[j]}\right)^{-1} N_0 w_{[j]}\|_{\delta,\mu,q-1}$. By the block-diagonal property and properties of $\left(S^0_{[j]}\right)^{-1}$ there exists an $\hat{r}$ such that $\mathcal{R}[\mu] \left(S^0_{[j]}\right)^{-1} N_0 \mathcal{R}[-\mu] \in B_{\delta,0,q}(\hat{r}) \subset B_{\delta,0,q-1}(\hat{r})$. Hence, it follows that

$$\| \left(S^0_{[j]}\right)^{-1} N_0 w_{[j]}\|_{\delta,\mu,q-1} \leq C\|\mathcal{R}[\mu] \left(S^0_{[j]}\right)^{-1} N_0 \mathcal{R}[-\mu]\|_{\delta,0,q-1}\|w_{[j]}\|_{\delta,\mu,q-1}.$$

We claim $\|\mathcal{R}[\mu] \left(S^0_{[j]}\right)^{-1} N_0 \mathcal{R}[-\mu]\|_{\delta,0,q-1}$ is bounded by a constant independent of the particular perturbations, and depending only on $\hat{r}$. This follows from the

[16] In fact one only needs $\mathcal{R}[\mu] S^0 \mathcal{R}[-\mu] \in B_{\delta,0,q}(s)$ here.

Sobolev inequality and a Moser-type estimate Lemma A.11, and the block-diagonal conditions. Further, $\|w_{[j]}\|_{\delta,\mu,q-1}$ is bounded by $C\|f_0\|_{\delta,\nu,q}$ for a uniform constant C as a consequence of Lemma B.2 and the bound $\|w_{[j]}\|_{\delta,\mu,q} \leq \delta^{\rho} C\|f_0\|_{\delta,\nu,q}$.

Lastly, consider the terms $\| \left(S^0_{[j]}\right)^{-1} tS^a_{[j]}\partial_a w_{[j]}\|_{\delta,\mu,q-1}$. The matrix operator $\left(S^0_{[j]}\right)^{-1} tS^a_{[j]} \in B_{\delta,\hat{\zeta},q}(\hat{r})$ for some positive exponent scalar $\hat{\zeta}$ and some $\hat{r} > 0$ (not necessarily the same as for the previous terms), while $\partial_a w_{[j]} \in B_{\delta,\mu-\epsilon,q-1}(s)$ for an arbitrarily small $\epsilon > 0$ (see Lemma B.6). Hence, similar arguments as for the two terms above allow us to obtain the bound

$$\| \left(S^0_{[j]}\right)^{-1} tS^a_{[j]}\partial_a w_{[j]}\|_{\delta,\mu,q-1} \leq C\|\mathcal{R}[\mu-\epsilon] \left(S^0_{[j]}\right)^{-1} tS^a_{[j]}\mathcal{R}[-\mu+\epsilon]\|_{\delta,\hat{\zeta},q-1}\|\partial_a w_{[j]}\|_{\delta,\mu-\epsilon,q-1},$$

where we note that $\| \left(S^0_{[j]}\right)^{-1} tS^a_{[j]}\partial_a w_{[j]}\|_{\delta,\mu,q-1} \leq C\| \left(S^0_{[j]}\right)^{-1} tS^a_{[j]}\partial_a w_{[j]}\|_{\delta,\mu+\hat{\zeta}-\epsilon,q-1}$. Under the block-diagonality condition the first factor can be bounded by a constant depending on r and $\tilde{r}$. It remains to show that $\|\partial_a w_{[j]}\|_{\delta,\mu-\epsilon,q-1} \leq C\|f_0\|_{\delta,\nu,q}$; this follows from Lemma B.6 and $\|w_{[j]}\|_{\delta,\mu,q} \leq C\|f_0\|_{\delta,\nu,q}$.

At this point we have achieved

$$\begin{aligned}\|\Delta L_{[ij]}[w_{[j]}]\|_{\delta,\mu+\zeta,q-1} \leq C\|S^0_{1\,[i]} - S^0_{1\,[j]}\|_{\delta,\zeta,q-1}\|f_0\|_{\delta,\nu,q} \\ + \sum_{a=1}^{n} \|t\left(S^a_{[i]} - S^a_{[j]}\right)\partial_a[w_{[j]}]\|_{\delta,\mu+\zeta,q-1}.\end{aligned}$$

for a uniform constant (in the sense of Definition 2.12) which is independent of i, j. The second term can be handled in a manner similar as above to obtain

$$\begin{aligned}\|\Delta L_{[ij]}[w_{[j]}]\|_{\delta,\mu+\zeta,q-1} \leq C\|S^0_{1\,[i]} - S^0_{1\,[j]}\|_{\delta,\zeta,q-1}\|f_0\|_{\delta,\nu,q} \\ + C\sum_{a=1}^{n} \|t\left(S^a_{[i]} - S^a_{[j]}\right)\|_{\delta,\zeta,q-1}\|f_0\|_{\delta,\nu,q}.\end{aligned}$$

Since the $\{S^0_{1\,[i]}\}$ and $\{S^a_{[i]}\}$ are Cauchy sequences, the terms on the right hand side of this inequality vanish as $i, j \to \infty$, thus proving that $\{w_{[j]}\}$ is a Cauchy sequence in $X_{\delta,\mu,q-1}$.

Step 4: A solution to the asymptotic value problem. We establish that the limit $w := \lim_{i\to\infty} w_{[i]}$ is a solution in a weak sense to the linear asymptotic value problem with zero leading order term in a similar manner as in Proposition 2.17. Moreover, each $w_{[i]}$ is a strong solution to the linear asymptotic value problem, which possesses a time derivative $Dw_{[i]}$. Arguments like those detailed in the proof Proposition 2.19 show that there exists a time derivative $Dw \in X_{\delta,\mu,q-2}$, and thus that w is a strong solution.

Step 5: Convergence of the sequence in $X_{\delta,\mu,q}$. So far we have shown that the sequence $\{w_{[i]}\}$ converges in $X_{\delta,\mu,q-1}$ to a function w, which has a time derivative Dw in the space $X_{\delta,\mu,q-2}$. Moreover, we also know that each iterate $w_{[i]}$ is contained in the space $X_{\delta,\mu,q}$, with the bound

$$\|w_{[i]}\|_{\delta,\mu,q} \leq C\|f_0\|_{\delta,\nu,q} \tag{2.30}$$

for a constant C independent of i. The constant C depends on the radius r associated with the linear symmetric hyperbolic Fuchsian system, and each system is constructed to have coefficients in a ball of this radius within the appropriate function space. Since this situation comes up in other parts of the proof, we state the following general lemma.

Lemma 2.21. *Let $\{w_i\}$ be a sequence of functions in $X_{\delta,\mu,q}(T^n)$, each of which satisfies a linear symmetric hyperbolic Fuchsian system of the form Eq.* (2.29) *with coefficients in bounded subset of the appropriate function spaces. Suppose that this sequence is known to converge to w in $X_{\delta,\mu,q-1}(T^n)$, which also satisfies a linear* *symmetric hyperbolic Fuchsian system of the same type (in particular the coefficients are in the same bounded subset). Further suppose that for each i, $\|w_i\|_{\delta,\mu,q} \leq M$, for some $M > 0$ independent of i. Then, $w \in X_{\delta,\mu,q}(T^n)$.*

Proof of Lemma 2.21. Fix a $t_0 \in (0, \delta]$ and consider the sequence $\{w_j(t_0)\}$. Since each $w_j \in X_{\delta,\mu,q}$, the map $\mathcal{R}[\mu]w_j$ is a continuous bounded map into H^q. Further, since $\mathcal{R}[\mu]$ is smooth, the sequence $\{w_j(t_0)\}$ is bounded in H^q for any choice of t_0. The bound is uniform in that it is independent of j, but may depend on the choice of t_0. Due to the convergence in $X_{\delta,\mu,q-1}$ we also know that $\{w_j(t_0)\}$ converges to a function $w|_{t_0} \in H^{q-1}$. From these two data and Corollary A.6, it follows that $w|_{t_0} \in H^q$. Of course this argument can be made for any $t_0 \in (0, \delta]$, and so we have shown that $w : (0, \delta] \to H^q$ is bounded. However, we have no information on continuity; this must be gained using the equation.

The limit, w, satisfies an equation of the form Eq. (2.29) where the perturbation coefficients are contained in $B_{\delta,\zeta,q}(r)$. Hence S_1^0 and tS^a are bounded continuous maps of $(0, \delta]$ into H^q. Further, since $\zeta > 0$, the $t \searrow 0$ behavior of S_1^0 and tS^a is well-behaved and in fact $S_1^0, tS^a \in C^0\left((0, \delta]; H^q\right)$. As a consequence, the theory for linear symmetric hyperbolic systems implies that $w \in C^0\left((0, \delta]; H^q\right)$, and since $\mathcal{R}[\mu] \in C^\infty\left((0, \delta] \times T^n\right)$ that $\mathcal{R}[\mu]w \in C^0\left((0, \delta]; H^q\right)$. In fact $\mathcal{R}[\mu]w$ is bounded as a consequence of the $j \to \infty$ limit of Eq. (2.30), and hence $w \in \widehat{X}_{\delta,\mu,q}$.

The argument just given goes through for $\mu \to \mu + \epsilon$ for any ϵ small compared to $\nu - \mu$, and we find $w \in \widehat{X}_{\delta,\mu+\epsilon,q}$. It follows from the embedding Lemma B.5, that $w \in X_{\delta,\mu,q}$. □

Returning to the present step; since the sequence $\{w_{[i]}\}$ is uniformly (in i) bounded by a constant, an application of Lemma 2.21 shows that $\lim_{i\to\infty} w_{[i]} =: w \in X_{\delta,\mu,q}$. This completes the proof of Proposition 2.20. □

2.4.7. Uniqueness

At this point we have established that there exists a solution to the linear asymptotic value problem for equations with coefficients which have perturbations in the spaces $B_{\delta,0,q}(r)$. We now show that this solution is unique in $X_{\delta,\mu,q}$.

Proposition 2.22. *Suppose Eq.* (2.1) *is a linear symmetric hyperbolic Fuchsian system with parameters* $\{\delta, \mu, q, q_0, r\}$ *with* $q > n/2+1$ *and* $q_0 > q+n/2+1$*. Suppose also that with respect to* μ *the system is block-diagonal and the energy dissipation matrix is positive definite. Then the solution of the asymptotic value problem (with zero leading order term) for this system is unique in* $X_{\delta,\mu,q}$*.*

Remark 2.23. *(i) The solution is guaranteed to be unique only in the space* $X_{\delta,\mu,q}$*, and there could in particular be another solution in the larger space* $X_{\delta,\tilde{\mu},q}$ *with* $\tilde{\mu} < \mu$*.*

(ii) We have, without loss of generality, assumed the leading order term to be zero for the linear systems. However, with a non-zero leading order term, one obtains the same uniqueness result for the corresponding (in general different) asymptotic value problem.

Proof. Let w and $\tilde{w}$ be two generally different solutions to the same asymptotic value problem, and define $\Delta = w - \tilde{w}$. By linearity, Δ satisfies $L[\Delta] = f_0^{(\Delta)} = 0$, where $L[\cdot]$ is given in Eq. (2.29), and $f_0^{(\Delta)} = 0$ because the equations for both w and $\tilde{w}$ have the same source term f_0. From the definition, $\Delta \in X_{\delta,\mu,q}$ and $\mathcal{R}[\mu]\Delta : (0, \delta] \to H^q(T^n)$ is a continuous bounded map.[17] It follows that $\Delta \in H^q(T^n)$ at each $t \in (0, \delta]$.

We wish to use the energy estimates in order to control the L^2-norm of Δ for $t \in (0, \delta]$. Fix a $t_0 \in (0, \delta]$. Then $\Delta(t, x)$ is the unique solution to the Cauchy initial

[17] Of course we cannot guarantee that Δ alone is continuous map of $(0, \delta]$ to H^q.

value problem with data $\Delta|_{t_0}(x) := \Delta(t_0, x)$ at t_0. Because the system is a linear symmetric hyperbolic Fuchsian system which satisfies the hypotheses of the basic energy estimate, we have

$$\|\mathcal{R}[\mu]\Delta\|_{L^2}(t) \leq C\|\mathcal{R}[\mu]\Delta\|_{L^2}(t_0),$$

for all $t \in [t_0, \delta]$ (recall $f_0^{(\Delta)} = 0$). Moreover, this estimate holds for $\mu \to \mu - \epsilon$ for any $\epsilon > 0$ for which the energy dissipation matrix is positive definite. Thus, using the definition of $\mathcal{R}[\mu]$, we have

$$\|\mathcal{R}[\mu-\epsilon]\Delta\|_{L^2}(t) \leq C\|\mathcal{R}[\mu-\epsilon]\Delta\|_{L^2}(t_0) \leq Ct_0^{\epsilon}\|\mathcal{R}[\mu]\Delta\|_{L^2}(t_0).$$

Since $\mathcal{R}[\mu]\Delta : (0, \delta] \to L^2$ is bounded, the limit $t_0 \searrow 0$ of both sides is well-defined, and the right-hand most side has a limit zero. To complete the argument, we note that $\mathcal{R}[\mu-\epsilon](t, x)$ is positive and bounded at each t. It follows that $\Delta = 0$ on $(0, \delta] \times T^n$ almost everywhere. □

2.5. Existence and Uniqueness for Quasilinear Systems

In the previous section (Section 2.4.) we have shown that there exists a unique solution to the asymptotic value problem for linear symmetric hyperbolic Fuchsian systems. In this section we use the existence and uniqueness for the linear theory and a fixed point argument to establish existence and uniqueness to the full quasilinear system.

For a specified leading order term u_0, the idea is to construct a sequence of solutions $\{u_i\}$ with $u_i = u_0 + w_i$ via an operator $\mathbb{G}(u_0)[\cdot] : B \to B$ for an appropriate bounded set B of $X_{\delta,\mu,q}$. If we can show that the operator $\mathbb{G}(u_0)[\cdot]$ is bounded and is a contraction, then by the Banach fixed point theorem (Theorem A.13), there exists a unique fixed point. The operator $\mathbb{G}(u_0)\,[\cdot]$ is constructed in such a way that the fixed point is the desired solution to the quasilinear symmetric hyperbolic Fuchsian system.

2.5.1. Construction of the operator $\mathbb{G}(u_0)\,[\cdot]$.

It is notationally convenient to define the following operator

$$\widehat{L}(u_0+v)[w] := S^0(u_0+v)Dw + \sum_{a=1}^{n} tS^a(u_0+v)\partial_a w + N_0 w. \tag{2.31}$$

In terms of this operator the quasilinear symmetric hyperbolic Fuchsian system Eq. (2.6) can be written

$$\widehat{L}(u_0+w)[w] = \mathcal{F}(u_0)[w].$$

Let $\tilde{w}$ be a fixed function in $B_{\delta,\mu,q}(s)$ for some $s>0$, and consider the *linear* equation for w given by

$$\widehat{L}(u_0+\tilde{w})[w] = \phi,$$

for some specified function $\phi \in X_{\delta,\nu,q}$. This equation is linear symmetric hyperbolic Fuchsian as in Definition 2.11, and therefore the solution is given by the solution operator

$$w = \mathbb{H}(u_0+\tilde{w})\,[\phi]\,.$$

In the case $\phi = \mathcal{F}(u_0)[\tilde{w}]$ the system is also linear symmetric hyperbolic Fuchsian, and $w = \mathbb{H}(u_0+\tilde{w})\,[\mathcal{F}(u_0)[\tilde{w}]]$. Define

$$\mathbb{G}(u_0)\,[\tilde{w}] := \mathbb{H}(u_0+\tilde{w})\,[\mathcal{F}(u_0)[\tilde{w}]]\,,$$

so that the solution to the linear asymptotic value problem with coefficients parametrized by $\tilde{w}$ is given by $w = \mathbb{G}(u_0)\,[\tilde{w}]$. Further, the solution to the *quasilinear* asymptotic value problem is a fixed point of the operator $\mathbb{G}(u_0)\,[w]$; that is $w = \mathbb{G}(u_0)\,[w]$.

For a prescribed leading order term u_0, define the sequence $\{w_i\}_{i\in\mathbb{N}}$ by

$$w_0 := 0, \quad w_{i+1} := \mathbb{G}(u_0)\,[w_i]\,.$$

We show that the sequence can be bounded in $B_{\delta,\mu,q}(s)$ and that $\mathbb{G}(u_0)\,[\cdot]$ is a contraction.

Note that each sequence element w_i satisfies the linear symmetric hyperbolic Fuchsian system

$$\widehat{L}(u_0 + w_{i-1})[w_i] = \mathcal{F}(u_0)[w_{i-1}].$$

Further, due to the condition in the hypothesis of Theorem 2.10, that the system be a quasilinear symmetric hyperbolic Fuchsian system as in Definition 2.7 in which $S^0_1(\cdot), tS^a(\cdot)$, and $\mathcal{F}(u_0)[\cdot]$ are bounded operators, Proposition 2.20 may be applied to obtain a strong solution w_i to this equation $w_i \in B_{\delta,\mu,q}(s)$, with a time derivative $Dw_i \in B_{\delta,\mu,q-1}(s)$.

2.5.2. The sequence is bounded.

Suppose that for $j = 0, \ldots, N$ each $w_j \in B_{\delta,\mu,q}(s)$, for (δ, μ, q, s) as specified in Theorem 2.10, and consider $w_{N+1} = \mathbb{G}(u_0)\,[w_N]$. We wish to show that $\|w_{N+1}\|_{\delta,\mu,q} \leq s$, for a sufficiently small choice of δ, and as a consequence the sequence is bounded.

The estimate Eq. (2.28) (for in general non-smooth perturbations) implies that

$$\|w_{N+1}\|_{\delta,\mu,q} = \|\mathbb{H}(u_0 + w_N)\,[\mathcal{F}(u_0)[w_N]]\,\|_{\delta,\mu,q} \le \delta^\rho C \|\mathcal{F}(u_0)[w_N]\|_{\delta,\nu,q},$$

for some $\nu > \mu$. Now, since it follows from the definition of QSHF systems that $\mathcal{F}(u_0)[\cdot]$ is a bounded operator satisfying the Lipschitz property, $\mathcal{F}(u_0)[\cdot]$ satisfies

$$\|\mathcal{F}(u_0)[w_N] - \mathcal{F}(u_0)[0]\|_{\delta,\nu,q} \le C(s,q)\|w_N\|_{\delta,\mu,q}.$$

Hence, by the reverse triangle inequality

$$\|\mathcal{F}(u_0)[w_N]\|_{\delta,\nu,q} \le C(s,q)\|w_N\|_{\delta,\mu,q} + \|\mathcal{F}(u_0)[0]\|_{\delta,\nu,q},$$

and we obtain the bound

$$\|w_{N+1}\|_{\delta,\mu,q} \le \delta^\rho C(s,q)\|w_N\|_{\delta,\mu,q} + \delta^\rho C \|\mathcal{F}(u_0)[0]\|_{\delta,\nu,q}.$$

Note that $\|\mathcal{F}(u_0)[0]\|_{\delta,\nu,q}$ is a non-decreasing function of δ. Hence, for an appropriately smaller $\bar{\delta} \le \delta$,

$$\bar{\delta}^\rho C(s,q)\left(\|w_N\|_{\bar{\delta},\mu,q} + \|\mathcal{F}(u_0)[0]\|_{\bar{\delta},\nu,q}\right) \le s,$$

where we have used $\|w_N\|_{\delta,\mu,q} \le s$. This shows that the sequence $\{w_i\}$ is bounded in $B_{\bar{\delta},\mu,q}(s)$ for a sufficiently small $\bar{\delta}$.

2.5.3. $\mathbb{G}(u_0)\,[\cdot]$ is a contraction.

We show that for any $w, v \in B_{\delta,\mu,q-1}(s)$

$$\|\mathbb{G}(u_0)\,[w] - \mathbb{G}(u_0)\,[v]\,\|_{\bar{\delta},\mu,q-1} \leq \theta \|w - v\|_{\bar{\delta},\mu,q-1},$$

for $\theta \in [0, 1)$. The reason for proving this contraction property in the space $B_{\delta,\mu,q-1}(s)$ is similar to that in the proof of Proposition 2.20, and becomes clear below. Compute

$$\begin{aligned}
&\|\mathbb{G}(u_0)\,[w] - \mathbb{G}(u_0)\,[v]\|_{\bar{\delta},\mu,q-1} \\
&\qquad = \|\mathbb{H}(u_0 + w)\,[\mathcal{F}(u_0)[w]] - \mathbb{H}(u_0 + v)\,[\mathcal{F}(u_0)[v]]\,\|_{\bar{\delta},\mu,q-1} \\
&\qquad \leq \|\mathbb{H}(u_0 + w)\,[\mathcal{F}(u_0)[w] - \mathcal{F}(u_0)[v]]\,\|_{\bar{\delta},\mu,q-1} \\
&\qquad\quad + \|\mathbb{H}(u_0 + w)\,[\mathcal{F}(u_0)[v]] - \mathbb{H}(u_0 + v)\,[\mathcal{F}(u_0)[v]]\,\|_{\bar{\delta},\mu,q-1}.
\end{aligned}$$

We now estimate both terms on the right hand side of this inequality. For the first term, it follows from the estimate Eq. (2.28) and from the Lipschitz property of $\mathcal{F}(u_0)[\cdot]$ that

$$\|\mathbb{H}(u_0 + w)\,[\mathcal{F}(u_0)[w] - \mathcal{F}(u_0)[v]]\,\|_{\bar{\delta},\mu,q-1} \leq C\bar{\delta}^{\rho}\|w - v\|_{\bar{\delta},\mu,q-1}.$$

For a possibly smaller choice of $\bar{\delta}$, we find that $\theta_1 := C\bar{\delta}^{\rho} < 1$.

To estimate the second term, define

$$w_A := \mathbb{H}(u_0 + w)\,[\mathcal{F}(u_0)[v]]\,, \quad w_B := \mathbb{H}(u_0 + v)\,[\mathcal{F}(u_0)[v]]\,,$$

and note that $\widehat{L}(u_0+w)[w_A] = \mathcal{F}(u_0)[v] = \widehat{L}(u_0+v)[w_B]$. Subtracting $\widehat{L}(u_0+w)[w_B]$ from both sides we obtain a linear symmetric hyperbolic Fuchsian system for the difference $w_A - w_B$

$$\widehat{L}(u_0 + w)[w_A - w_B] = \widehat{L}(u_0 + v)[w_B] - \widehat{L}(u_0 + w)[w_B].$$

Notice that the source term for this equation is guaranteed to have only $q - 1$ derivatives. This is the same situation as in the proof of Proposition 2.20, and is the reason we are working in the space $X_{\delta,\mu,q-1}$ rather than $X_{\delta,\mu,q}$. Assuming $q > n/2 + 2$ as in Theorem 2.10 we may apply Proposition 2.20[18] to show that there is a (unique) solution to this equation with the estimate

$$\|w_A - w_B\|_{\bar{\delta},\mu,q-1} \leq C\bar{\delta}^{\rho}\|\Delta L(v, w)[w_B]\|_{\bar{\delta},\nu,q-1},$$

for an appropriate $\nu > \mu$. We have defined the operator

$$\Delta L(v, w)[w_B] := \left(S^0(v) - S^0(w)\right) Dw_B + \sum_{a=1}^{n} t\,(S^a(v) - S^a(w))\,\partial_a w_B.$$

We know that w_B is a strong solution to the linear asymptotic value problem $\widehat{L}(u_0 + v)[\tilde{w}] = \mathcal{F}(u_0)[v]$, and as a result there exists a time derivative map Dw_B and a spatial derivative $\partial_a w_B$, both of which take values in $B_{\bar{\delta},\mu,q-1}(s) \subset B_{\bar{\delta},\mu,q-2}(s)$. Further, the coefficient matrices $S_1^0, S^a \in B_{\bar{\delta},\zeta,q}(r)$ are bounded, satisfy the Lipschitz property, and have the same block-diagonal structure as μ. Hence, we may apply Lemma C.1 in

[18]This is the source of the regularity requirement $q > n/2 + 2$ in Theorem 2.10.

order to estimate

$$\begin{aligned}\|\Delta L(v,w)[w_B]\|_{\bar{\delta},\nu,q-1} &\le C\Big(\|S^0(v)-S^0(w)\|_{\bar{\delta},\zeta,q-1}\|Dw_B\|_{\bar{\delta},\mu,q-1}\\ &\qquad+\sum_{a=1}^{n}\|t\left(S^a(v)-S^a(w)\right)\|_{\bar{\delta},\zeta,q-1}\|\partial_a w_B\|_{\bar{\delta},\mu,q-1}\Big)\\ &\le C\Big(\|S^0(v)-S^0(w)\|_{\bar{\delta},\zeta,q-1}+\sum_{a=1}^{n}\|t\left(S^a(v)-S^a(w)\right)\|_{\bar{\delta},\zeta,q-1}\Big)\\ &\le C\|v-w\|_{\bar{\delta},\mu,q-1},\end{aligned}$$

for an adapted constant C in each step. Thus for a sufficiently small $\hat{\delta}\in(0,\bar{\delta}]$ we find

$$\|w_A-w_B\|_{\hat{\delta},\mu,q-1}\le\theta_2\|v-w\|_{\hat{\delta},\mu,q-1},$$

for some $\theta_2\in[0,1)$.

Combining the estimates for both terms we obtain

$$\begin{aligned}\|\mathbb{G}(u_0)\,[w]-\mathbb{G}(u_0)\,[v]\,\|_{\hat{\delta},\mu,q-1} &\le\theta_1\|w-v\|_{\hat{\delta},\mu,q-1}+\theta_2\|w-v\|_{\hat{\delta},\mu,q-1},\\ &\le\theta\|w-v\|_{\hat{\delta},\mu,q-1},\end{aligned}$$

for $\theta:=1/2\max\{\theta_1,\theta_2\}$. Since $\theta\in[0,1)$ (controlled by the choice of $\hat{\delta}$), it follows that $\mathbb{G}(u_0)\,[\cdot]$ is a contraction.

2.5.4. The fixed point is a solution.

Having shown that the sequence is bounded and that the operator $\mathbb{G}(u_0)\,[\cdot]$ is a contraction, the Banach fixed point theorem (Theorem A.13) shows that there exists a unique fixed point $w=\lim_{i\to\infty}w_i\in B_{\hat{\delta},\mu,q-1}(s)$ such that $w=\mathbb{G}(u_0)\,[w]$. Due to the definition of $\mathbb{H}(u_0)[\cdot]$ (and hence $\mathbb{G}(u_0)[\cdot]$), w is weak solution to $\widehat{L}(u_0+w)[w] = \mathcal{F}(u_0)[w]$.

Before we argue that w is in fact a strong solution to the equation above, we show that $w \in B_{\hat{\delta},\mu,q}(s)$, and not just $B_{\hat{\delta},\mu,q-1}(s)$. The situation is as follows. We know that for each i, $w_i \in B_{\hat{\delta},\mu,q}(s)$ (and is therefore uniformly bounded by s) and further that $\{w_i\}$ converges to w (the limit point) in $B_{\hat{\delta},\mu,q-1}(s)$. We have encountered this same situation above in Step 5 of the proof of Proposition 2.20. There we proved a general lemma, Lemma 2.21, which we apply here as well in order to show that $w \in B_{\hat{\delta},\mu,q}(s)$.

To show that w is a strong solution of Eq. (2.6) we show that there exists first distributional derivatives in time and space. As a result, we may reverse the integration by parts in the weak version of the equation as in Section 2.4.5., to obtain the strong version Eq. (2.27). Since $w \in B_{\hat{\delta},\mu,q}(s)$, there exists first spatial derivatives $\partial_a w \in B_{\hat{\delta},\mu,q-1}(s)$. To show that there exists a time derivative $Dw \in B_{\hat{\delta},\mu,q-1}(s)$, note that for each i, w_i in the sequence constructed above is a strong solution to a linear symmetric hyperbolic Fuchsian system, and hence there exists a time derivative $Dw_i \in X_{\delta,\mu,q-1}$. Due to the uniform (in time) convergence of $\{w_i\}$ to w, we can show as in the proof of Proposition 2.19 that Dw_i converges uniformly to some $\widehat{w} \in X_{\delta,\mu,q-1}$. Applying Theorem A.9 we see that w is differentiable in time, and $Dw = \widehat{w}$ on any set $[\delta_I, \delta] \subset (0, \delta]$. Since we can take δ_I arbitrarily small, $Dw = \widehat{w}$ on $(0, \delta]$ and hence $Dw \in X_{\delta,\mu,q-1}$. This completes the proof that w is a strong solution.

2.5.5. The case $q \to \infty$.

To complete the proof of Theorem 2.10 we consider the case in which $q \to \infty$. For any finite $q > n/2 + 2$, we have now shown that there exists a solution to the asymptotic value problem with remainder $w \in X_{\delta_q,\mu,q}$ for some $\delta_q > 0$. Note that many of the constants in the estimates we have used in principle depend on q. It is conceivable that as $q \to \infty$ it may be necessary to take $\delta_q \to 0$. The extendibility of solutions to the *initial* value problem for symmetric hyperbolic systems under sufficient regularity conditions shows that we can always take $\delta = \delta_{n+2}$.

Suppose a finite $q > n/2 + 2$ has been fixed, and $w(t, x)$ is the resulting unique solution to the asymptotic value problem about u_0 in $X_{\delta_q,\mu,q}$. Then for any $t_0 \in (0, \delta_q]$, $w(t, x)$ satisfies the Cauchy initial value problem on $[t_0, \delta_q] \times T^n$ with data $w|_0(x) := w(t_0, x) \in H^q(T^n)$ at t_0. The well-posedness of the Cauchy initial value problem (Proposition 1.4, Chapter 16 of [85]) shows that $w \in C\left([t_0, \delta_q]; H^q(T^n)\right)$. Since the coefficients in the equation Eq. (2.6) depend smoothly on all arguments[19] we may apply Proposition 1.5 from Chapter 16 of [85] to show that there exists a $\delta_* > \delta_q$ such that $w \in C\left([t_0, \delta_*]; H^q(T^n)\right)$. Since, $q = n + 2 > n/2 + 2$ is the minimum (integer) regularity required in order apply the Theorem 2.10, we can take $\delta_* = \delta_{n+2}$. This shows that for any finite q, which may be taken arbitrarily large, each corresponding solution w can be extended to exists on the interval $(0, \delta_{n+2}]$. This completes the proof of Theorem 2.10.

2.6. A Fuchsian Theorem for Smooth Systems

In this section we develop the theory for the special case in which u_0 depends smoothly on both the spatial and time variables (i.e. u_0 is $C^\infty((0, \delta) \times T^n)$), and the system is a smooth quasilinear symmetric hyperbolic Fuchsian system as in Definition 2.7 and the comment below that definition. The main result is Theorem 2.28, which establishes the existence and uniqueness of solutions to the smooth asymptotic value problem under suitable hypotheses. A similar result could

[19] See definition of smooth quasilinear symmetric hyperbolic Fuchsian systems in paragraph below Definition 2.7.

be proved in the finite differentiability case. However, our aim here is not to prove a Fuchsian theorem in all generality, but to provide a compact, easily checked theorem for the "smooth" case, which is readily applied to our systems of interest.

The version of the theory presented in this section was developed by Florian Beyer and myself. This theory builds upon and improves an earlier set of results with F. Beyer, J. Isenberg, and P.G. LeFloch which is published in [3].

2.6.1. Motivation and Outline

The motivation for developing the theory in this section is that in applications (such as that in Chapter III), it may be impossible to satisfy all the hypotheses of Theorem 2.10 simultaneously, or the conditions that these hypotheses impose on the parameters in the problem are unsatisfactory. The conflict arises when simultaneously satisfying the block-diagonal conditions, the positivity of the energy dissipation matrix, and the desired properties of $\mathcal{F}(u_0)[\cdot]$ in the definition of quasilinear symmetric hyperbolic Fuchsian systems (Definition 2.7). Each of these conditions imposes inequalities, or equalities in the case of the block-diagonal condition, and it may happen that these inequalities directly conflict, or as in the case of the Chapter III application, constrain the asymptotic data functions in u_0. Such breakdown in the applicability of Theorem 2.10 can be taken (as we show in this section) as evidence that the prescribed leading order term does not contain enough information. By adding a higher order correction to the leading order term, such as an *asymptotic solution* which we discuss below, Theorem 2.10 can be successfully applied.

In Section 2.6.2. we introduce the additional structural conditions which are required on Eq. (2.1), and state the existence and uniqueness theorem Theorem 2.28. In the following sections we introduce the mathematical machinery, which is used in proving this theorem. Section 2.6.3. introduces the notion of an asymptotic solution and proves an existence and uniqueness theorem in the case that one has such an asymptotic solution. In Section 2.6.4. we construct an "ODE-formulation" of the Fuchsian system Eq. (2.6). This formulation can be used to construct and improve asymptotic solutions using the equation and prescribed leading order terms via an iterative process. At each iteration one degree of regularity in the spatial derivatives is lost. To ensure that we obtain asymptotic solutions with sufficient regularity we work in the smooth category, although similar results can be obtained by starting with a leading order term in a Sobolev space with sufficiently high regularity. Having developed these tools we implement them in the proof of Theorem 2.28 in Section 2.6.5. and Section 2.6.6.. In the last subsection, Section 2.6.7., we prove a few results from Section 2.6.4. whose proofs were omitted earlier in the interest of streamlining the presentation.

2.6.2. Structural Properties and Statement of the Theorem

We start by introducing additional structural properties which are needed of Eq. (2.6). In this section we consider the situation where the prescribed leading order term u_0 is modified by a function $w \in B_{\delta,\mu,q}(s)$. In proving the results of Section 2.6.3. and Section 2.6.4. it is necessary to control products of the form $(S^j(w) - S^j(w+h))\,\partial_j w$, where $h \in X_{\delta,\hat{\mu},q}$ for some $\hat{\mu} \geq \mu$. The following property make sense in light of Lemma C.1.

Definition 2.24 (Higher-order difference property)**.** *Suppose $F : X_{\delta,\mu,q} \to X_{\delta,\nu,q}$ is a function operator satisfying the Lipschitz property, and let $\hat{\mu} \geq \mu$ be another exponent* *vector. F satisfies the* ***higher-order difference property*** *with respect to $\hat{\mu}$ if*

$$h \mapsto \Delta F_w(h) := F(w+h) - F(w)$$

maps all $h \in B_{\delta',\hat{\mu},q}(s/2)$ to $X_{\delta',\hat{\mu}+\nu-\mu,q}$ for each $w \in B_{\delta',\mu,q}(s/2)$ and all $\delta' \in (0,\delta]$, and satisfies the Lipschitz property.

The next two definitions record where this property is required in the principle part and source terms in the equation.

Definition 2.25 (Product compatibility conditions)**.** *Suppose that Eq.* (2.1) *is a quasilinear symmetric hyperbolic Fuchsian system around u_0 with parameters $\{\delta, \mu, q, q_0, s\}$, as specified in Definition 2.7. Pick another exponent $\hat{\mu}$ with $\hat{\mu} \geq \mu$. This system satisfies the* ***product compatibility conditions*** *with respect to $\hat{\mu}$ provided for all $w \in B_{\delta,\mu,q}(s)$, there is a positive exponent vector $\tilde{\zeta}$ such that*

(i) The function operator

$$w \mapsto \mathcal{R}[\mu] S_1^0(u_0 + w) \mathcal{R}[-\mu] \in X_{\delta,\tilde{\zeta},q}$$

satisfies the higher-order property with respect to $\hat{\mu}$ in both the q and the $(q-1)$ norms.

(ii) The function operator

$$w \mapsto \mathcal{R}[\mu] t S^a(u_0 + w) \mathcal{R}[-\mu] \in X_{\delta,\tilde{\zeta},q}$$

satisfies the higher-order property with respect to $\hat{\mu}$ in both the q and the $(q-1)$ norms.

Similar properties are required of the source term $\mathcal{F}(u_0)[w]$. We have

Definition 2.26 (Higher order source conditions)**.** *The function operator* $w \mapsto \mathcal{F}(u_0)[w] \in X_{\delta,\nu,q}$ *obeys the* ***higher order source conditions*** *with respect to* $\hat{\mu}$ *if for a fixed* $\hat{\mu} \geq \mu$ *it satisfies the higher-order property with respect to* $\hat{\mu}$ *in both the* q *and* $(q-1)$ *norms.*

Next, we note that one of the strengths of the theorem in this section over Theorem 2.10 is that we no longer require the system Eq. (2.1) to be block-diagonal with respect to μ. Instead we demand the following weaker property:

Definition 2.27 (Smooth commutator conditions)**.** *For all* $w \in B_{\delta,\mu,q}(s)$*, there exists an exponent scalar* $\xi > 0$ *and an* $r > 0$ *such that* $\mathcal{R}[\mu]S_0^0 = S_0^0\mathcal{R}[\mu]$, $\mathcal{R}[\mu]N_0 = N_0\mathcal{R}[\mu]$ *and the function operators*

$$w \mapsto \mathcal{R}[\mu]tS^a(u_0 + w)\mathcal{R}[-\mu] \quad \text{and} \quad w \mapsto \mathcal{R}[\mu]S_1^0(u_0 + w)\mathcal{R}[-\mu]$$

take values in $B_{\delta,\xi,q}(r)$ *for all* $q > n/2$.

We may now state the main result of this section.

Theorem 2.28 (Solutions to the asymptotic value problem for smooth Fuchsian systems)**.** *Suppose that Eq.* (2.1) *is a smooth quasilinear symmetric hyperbolic Fuchsian system around* u_0 *with parameters* $\{\mu, \delta, s\}$ *which satisfies the smooth commutator conditions. Further suppose that for all* $q \geq p > n/2$*, for some integer* p *we have*

(i) the system Eq. (2.6) *satisfies the product compatibility conditions Definition 2.25 with respect to both* $\hat{\mu} = \mu + \gamma_0$*, where* γ_0 *is a exponent scalar and* $\hat{\mu}$ *with respect to which Eq.* (2.6) *is block-diagonal.*

(ii) *the system Eq.* (2.6) *satisfies the higher order source conditions Definition 2.26 with respect to both* $\hat{\mu} = \mu + \gamma_0$*, where* γ_0 *is a exponent scalar and* $\hat{\mu}$ *with respect to which Eq.* (2.6) *is block-diagonal.*

(iii) *let* λ *be the vector of eigenvalues of the matrix* $\mathcal{N} := S_0^{0^{-1}} N_0$*. Then the exponent vector* μ *satisfies the positivity condition*

$$\mu > -\Re\lambda.$$

Then there exists a unique solution of the asymptotic value problem for Eq. (2.1) *about* u_0 *with remainder* $w \in X_{\tilde{\delta},\mu,\infty}$*,* $Dw \in B_{\tilde{\delta},\mu,\infty}$ *and for some* $\widetilde{\delta} \in (0, \delta]$*.*

This theorem makes several refinements and improvements upon the corresponding theorem in [3] (Theorem 2.21). First, we replace the condition that $\mathcal{N}$ be in Jordan normal form by the smooth commutator conditions Definition 2.27. This represents a slight loosening of the hypotheses, since under these conditions one can transform to the Jordan basis without destroying the essential structure of the equation. We note that this hypothesis is satisfied by the polarized T^2-symmetric Einstein equations which we consider in [3] and in Chapter III. Second, in Theorem 2.28 we do not require the separate conditions on $(S^0(u_0 + w))^{-1}$ which are included in the hypotheses of Theorem 2.21 in [3]. These properties are shown in Section C.4. to follow from the assumptions on S^0. Finally, the leading order term u_0 in Theorem 2.28 is not required to be an “ODE-leading-order” term as it is in [3].

We remark that the proof of Theorem 2.28 relies on the application of Theorem 2.10. Since the hypotheses of Theorem 2.28 are assumed to hold for all $q > n/2$, and in the smooth systems $q_0 = \infty$, the regularity requirements of Theorem 2.10 are satisfied. The proof of the existence of solutions to the asymptotic value problem in the smooth case is contained in Section 2.6.5., while the proof of the uniqueness of this solution is detailed in Section 2.6.6..

To apply Theorem 2.28 to particular partial differential equations, such as the Einstein equations in Chapter III and Chapter IV, it is convenient to verify conditions Condition (i) and Condition (ii) for large classes of function operators. In Appendix C we show these properties for the types of function operators which appear in our applications in the cases that $\hat{\mu} = \mu + \gamma_0$, where γ_0 is a scalar, and $\hat{\mu}$ is a scalar exponent itself (that is $\hat{\mu}$ is "completely block-diagonal"), or where certain components of $\hat{\mu}$ are allowed to differ by $\pm\epsilon$ –the "nearly scalar" case.

It has recently come to light that there is a technical difficulty in applying this theorem to our equations in Chapter III and Chapter IV. The issue is in satisfying Condition (ii) with respect to a $\hat{\mu}$ which is block-diagonal. It turns out that it is insufficient in our applications to verify this property for a scalar "completely block-diagonal" $\hat{\mu}$. The next approach is to seek a "nearly scalar" $\hat{\mu}$ and ensure that one can find a consistent ordering of the components. While this is possible in the T^2-symmetric application (c.f. Chapter III), it severely limits the range of the asymptotic data function k, thus rendering the application of this theorem mute. For the application to the Gowdy spacetimes (c.f. Chapter IV) it has become clear that one cannot find a consistent ordering of the components without choosing a different leading order term altogether. While this is a serious issue, we believe that it is ultimately technical in nature, and that the results Theorem 3.10 and Theorem 4.4 which we obtain are essentially correct.

The reason we believe our results are essentially correct is that it preliminary calculations seem to indicate that the main obstacles can be overcome by another method. Recall that the essential problem in applying the fundamental Fuchsian theorem to the second order equations in Chapter III and Chapter IV is in simultaneously meeting the block-diagonal conditions and the positivity of the energy dissipation matrix. The first-order reduction which we use forces (via the energy dissipation matrix condition) a unnaturally large lower bound on the block-diagonal exponent vector μ. Our thought up to recently was that this problem was solved in the present section by allowing the exponent vectors which satisfy these conditions to be different (μ vs $\hat{\mu}$). However, this issue can be overcome by another method. It seems that by using a slightly more general first-order reduction we can control the unnatural lower bound from the energy dissipation matrix. The original Fuchsian theorem can then be used with a block-diagonal exponent vector. A more complete treatment of these ideas is forthcoming in [2].

2.6.3. Asymptotic Solutions

Although we state and prove Theorem 2.28 for smooth systems, the concepts and results in this sub-section (Section 2.6.3.) apply in the finite differentiability case. We begin by introducing the following useful concept of an asymptotic solution.

Definition 2.29. *Let μ, σ be exponent vectors with $\sigma > 0$ and let $\delta, s > 0$. The function $\hat{w} \in B_{\delta,\mu,q+1}(s)$ is called an **asymptotic solution of order** σ (or a σ-asymptotic solution) of Eq.* (2.1) *with respect to u_0 if*

$$R(u_0)[\hat{w}] := \sum_{j=0}^{n} S^j(u_0 + \hat{w}) t\partial_j \hat{w} + N_0(u_0)\hat{w} - \mathcal{F}(u_0)[\hat{w}] \tag{2.32}$$

is contained in $X_{\delta,\sigma,q}$.

We call $\hat{w}$ a **smooth asymptotic solution of order** σ if it is in $B_{\delta,\mu,q}(s)$ for all $q > n/2$.

Lemma 2.30 (Boundedness of $R(u_0)[\hat{w}]$)**.** *Suppose Eq.* (2.1) *is a quasilinear symmetric hyperbolic Fuchsian system about* u_0 *with parameters* $\{\delta, \mu, q, q_0, s\}$ *which satisfies the smooth commutator conditions Definition 2.27. Let* $\hat{w} \in B_{\delta,\mu,q+1}(s)$ *be a* σ*-asymptotic solution. Then, there exists an* $r > 0$ *such that* $R(u_0)[\hat{w}] \in B_{\delta,\sigma,q}(r)$*. That is,* $R(u_0)[\hat{w}]$ *is bounded.*

This result follows from the definition of quasilinear symmetric hyperbolic Fuchsian systems, the smooth commutator conditions, and Lemma C.1.

The proof of Theorem 2.28 relies on an application of Theorem 2.10 to the asymptotic value problem for Eq. (2.1) about $\hat{u}_0 = u_0 + \hat{w}$, where $\hat{w}$ is an asymptotic solution of sufficiently high order. We state this result for the existence and uniqueness of solutions to the asymptotic value problem based on asymptotic solutions in Proposition 2.32. We first show that if one has a quasilinear symmetric hyperbolic Fuchsian system about u_0 with parameters $\{\delta, \mu, q, q_0, s\}$, an asymptotic solution $\hat{w}$, and if certain conditions are met, then one may obtain a quasilinear symmetric hyperbolic Fuchsian system in a more tightly controlled space with parameter $\hat{\mu} \geq \mu$.

Lemma 2.31. *Suppose Eq.* (2.1) *is a QSHF system about* u_0 *with parameters* $\{\delta, \mu, q, q_0, s\}$*, and suppose that* $\hat{w} \in B_{\delta,\mu,q+1}(s/2)$ *for some* $q > n/2$ *is a* σ*-asymptotic solution with* $D\hat{w} \in X_{\delta,\mu,q}$*. Then, Eq.* (2.1) *is a QSHF system about* $\hat{u}_0 = u_0 + \hat{w}$ *with parameters* $\{\delta, \hat{\mu}, q, q_0, s/2\}$ *for any* $\mu \leq \hat{\mu} < \sigma$*, provided the higher order source conditions (Definition 2.26) and the product compatibility conditions (Definition 2.25) with respect to* $\hat{\mu}$ *hold.*

Note that we tacitly assume (always in this document) that μ is greater than the exponent vector for the leading order term u_0. We present a proof of Lemma 2.31 below. The following proposition is a consequence of Lemma 2.31 and Theorem 2.10.

Proposition 2.32. *Suppose the conditions of Lemma 2.31 are met. Furthermore suppose the smooth commutator conditions (Definition 2.27) are satisfied with respect to μ, that $q_0 > n/2 + q + 1$, that $q > n/2 + 2$, and that σ is sufficiently large enough so that there exists an exponent vector $\hat{\mu}$ with $\mu \leq \hat{\mu} < \sigma$ for which the system is block-diagonal and with respect to which the energy dissipation matrix Eq.* (2.7) *is uniformly positive definite at all* (t, x)*. Then there exists a unique solution u to the asymptotic value problem for Eq.* (2.1) *about $\hat{u}_0$ with remainder $h := u - u_0 - \hat{w}$ belonging to $X_{\widetilde{\delta},\hat{\mu},q}$ and $Dh \in X_{\widetilde{\delta},\hat{\mu},q}$ for some $\widetilde{\delta} \in (0, \delta]$.*

The regularity conditions on q and q_0 along with the assumption on σ allow us to apply Theorem 2.10 to the AVP of Eq. (2.1) about $\hat{u}_0$. Note that although Proposition 2.32 provides a unique solution with the leading order term $\hat{u}_0 = u_0 + \hat{w}$, and is a solution to the AVP(u_0) under the conditions we have imposed, there could still be other solutions with leading order term u_0. Below we show that only if further conditions are met, is it true that the solution identified in Proposition 2.32 is the unique solution to the AVP about u_0.

Proof of Lemma 2.31. To prove Lemma 2.31 we verify that the conditions of Definition 2.7 are satisfied with the leading order term $\hat{u}_0 = u_0 + \hat{w}$. Condition (i) is clearly satisfied, since the structure of the matrices has not been altered.

Regarding Condition (ii) we note that S_0^0 is unchanged since this depends on the limiting $t \searrow 0$ behavior of the leading order term, which is unchanged. Next we verify that for the fixed $\hat{w} \in B_{\delta,\mu,q+1}(s/2)$, the operators

$$h \mapsto S_1^0(\hat{w} + h), \quad h \mapsto tS^a(\hat{w} + h)$$

map all $h \in B_{\delta,\hat{\mu},q}(s/2)$ to $B_{\delta,\zeta,q}(r)$ (i.e. are bounded operators) for some $\zeta > 0$ and satisfy the $(q-1)$ Lipschitz property. The boundedness property follows from Condition (ii) and from the fact that $\hat{w} + h \in B_{\delta,\mu,q}(s)$. To verify the Lipschitz property note that for all $\hat{w} \in X_{\delta,\mu,q}$, we have

$$\begin{aligned}
\|S_1^0(\hat{w}+h) - S_1^0(\hat{w}+\tilde{h})\|_{\delta,\zeta,q-1} &\leq C\|\hat{w}+h-\hat{w}+\tilde{h}\|_{\delta,\mu,q-1} \\
&= C\|h-\tilde{h}\|_{\delta,\mu,q-1} \\
&\leq C\|h-\tilde{h}\|_{\delta,\hat{\mu},q-1},
\end{aligned}$$

where in the last line we have used Lemma B.1. This shows the desired Lipschitz property for $h, \tilde{h} \in X_{\delta,\hat{\mu},q}$.

Lastly we verify Condition (iii). Again, the limiting function N_0 remains unchanged, and we proceed to verify that

$$\mathcal{F}(\hat{u}_0)[h] = f_1(\hat{u}_0 + h) - \sum_{j=0}^{n} tS^j(\hat{u}_0 + h)\partial_j \hat{u}_0$$

maps all $h \in B_{\delta,\hat{\mu},q}(s/2)$ to a ball in $X_{\delta,\hat{\nu},q}$ for some $\hat{\nu} > \hat{\mu}$, and satisfies the Lipschitz properties in both the q and $(q-1)$ norms. To derive a more useful expression for $\mathcal{F}(\hat{u}_0)[h]$, note that

$$\widehat{L}(u_0 + \hat{w})[h] := \sum_{j=0}^{n} tS^j(\hat{u}_0 + h)\partial_j h + N_0 h = \mathcal{F}(\hat{u}_0)[h],$$

and

$$\widehat{L}(u_0 + \hat{w})[\hat{w} + h] = \mathcal{F}(u_0)[\hat{w} + h],$$

while by linearity in the second argument we have

$$\widehat{L}(u_0+\hat{w})[\hat{w}+h]=\widehat{L}(u_0+\hat{w})[\hat{w}]+\widehat{L}(u_0+\hat{w})[h].$$

Combining these observations we find

$$\widehat{L}(u_0+\hat{w})[\hat{w}]+\mathcal{F}(\hat{u}_0)[h]=\mathcal{F}(u_0)[\hat{w}+h],$$

and thus

$$\mathcal{F}(\hat{u}_0)[h]=\mathcal{F}(u_0)[\hat{w}+h]-\sum_{j=0}^{n}tS^j(\hat{u}_0+h)\partial_j\hat{w}-N_0\hat{w}. \tag{2.33}$$

We now use the fact that $\hat{w}$ is an asymptotic solution to Eq. (2.1). Towards this end we add and subtract $\sum_{j=0}^{n}tS^j(u_0+\hat{w})\partial_j\hat{w}$ to the above equation, and use Eq. (2.32) to obtain

$$\mathcal{F}(\hat{u}_0)[h]=-\Delta\mathcal{F}(\hat{u}_0)[h]-\sum_{j=0}^{n}t\Delta S^j(\hat{u}_0)[h]\partial_j\hat{w}-R(u_0)[\hat{w}], \tag{2.34}$$

where

$$\Delta\mathcal{F}(\hat{u}_0)[h]:=\mathcal{F}(u_0)[\hat{w}]-\mathcal{F}(u_0)[\hat{w}+h] \tag{2.35}$$

$$\Delta S^j(\hat{u}_0)[h]:=S^j(u_0+\hat{w})-S^j(u_0+\hat{w}+h). \tag{2.36}$$

In order to verify the desired properties of $\mathcal{F}(\hat{u}_0)[h]$ needed to satisfy Condition (iii), it is sufficient to verify that these properties are satisfied for each term in Eq. (2.34). We start with the $R(u_0)[\hat{w}]$ term. This can be treated as a function operator $h\mapsto R(u_0)[\hat{w}]$ (independent of h), which is contained in $X_{\delta,\sigma,q}$. In fact, due to Lemma 2.30 there exists $\tilde{r}>0$ such that $R(u_0)[\hat{w}]\in B_{\delta,\sigma,q}(\tilde{r})$. This is a function operator of the appropriate form under the hypothesis that $\hat{\mu} < \sigma$, and satisfies the Lipshitz property trivially. Further, the requisite conditions on $\Delta\mathcal{F}(\hat{u}_0)[h]$ follow from the hypotheses and Definition 2.26.

We now focus our attention on the terms in $\Delta S^j(\hat{u}_0)[h]$. Let

$$\begin{aligned} I[h] &:= \Delta S^0(\hat{u}_0)[h] D\hat{w} = \left(S_1^0(\hat{u}_0) - S_1^0(\hat{u}_0 + h)\right) D\hat{w}, \\ II[h] &:= \sum_{a=1}^{n} t \Delta S^a(\hat{u}_0)[h] \partial_{x^a} \hat{w} = \sum_{a=1}^{n} t \left(S^a(\hat{u}_0) - S^a(\hat{u}_0 + h)\right) \partial_{x^a} \hat{w}. \end{aligned}$$

Then under Condition (i) of Definition 2.25, $I[h]$ maps h to $B_{\delta,\hat{\mu}+\zeta,q}(r)$ for some $r > 0$. Moreover, for all $h, \tilde{h} \in B_{\delta,\hat{\mu},q}(s/2)$,

$$\begin{aligned} &\|I[h] - I[\tilde{h}]\|_{\delta,\hat{\mu}+\zeta,q} \\ &\qquad \leq C\|\mathcal{R}[\mu]\Delta S^0(\hat{u}_0)[h]\mathcal{R}[-\mu] - \mathcal{R}[\mu]\Delta S^0(\hat{u}_0)[\tilde{h}]\mathcal{R}[-\mu]\|_{\delta,\hat{\mu}+\zeta-\mu,q}\|D\hat{w}\|_{\delta,\mu,q} \\ &\qquad \leq C\|h - \tilde{h}\|_{\delta,\hat{\mu},q}, \end{aligned}$$

where in the first inequality we have used Lemma C.1, and in the second we have used that $\hat{w}$ is bounded and Condition (i) of Definition 2.25. A similar computation holds in the case of $(q-1)$ regularity.

Similarly, for each term in $II[h]$, the appropriate map type and boundedness requirements follow from Condition (ii) of Definition 2.25, from the boundedness of $\hat{w}$, and from Lemma C.1. Under the same hypotheses similar calculations as above may be performed to verify the Lipschitz conditions (in q and $q-1$ norms) for $II[h]$.

This completes the proof that $\mathcal{F}(\hat{u}_0)[h]$ in Eq. (2.34) satisfies the properties of Condition (iii), and hence the proof of Lemma 2.31. □

2.6.4. ODE-Formulation and a Sequence of Asymptotic Solutions

In the previous section we have shown that if one has an asymptotic solution of sufficiently high order, then it can be used to modify the prescribed leading order term in order to apply Theorem 2.10 and obtain a unique solution to the asymptotic value problem. However, the solution thus obtained is guaranteed only to be the unique solution of the asymptotic value problem about the modified leading order term, and not the original prescribed one. In this section we develop a formulation, which we call the "ODE-formulation" that is useful for constructing and improving such asymptotic solutions. Using this formulation we are able to prove under certain conditions that the solution to the asymptotic value problem identified above is the unique solution to the asymptotic value problem about the original prescribed leading order term. The results of this section are critical for the proof of Theorem 2.28. While in the section above we have stated results for finite regularity q, here we restrict to the smooth case, $q \to \infty$.

From the definition of quasilinear symmetric hyperbolic Fuchsian systems Definition 2.7, it is clear that the coefficients S_0^0 and N_0 dominate in the limit $t \searrow 0$. As a result, the partial differential equation Eq. (2.6) may under some circumstances be well-approximated near $t \searrow 0$ by the space-parameterized set of ODEs of the form $S_0^0 Dw + N_0 w = O(t^{\mu+\epsilon})$ for some $\epsilon > 0$, and where we take $w \in X_{\delta,\mu,q}$ as usual. We use this observation to introduce "ODE"-operators corresponding to the quasilinear symmetric hyperbolic Fuchsian system, and use these operators to understand the leading order behavior of certain solutions. Given a leading order term u_0, let

$$L_{\text{ODE}}(u_0)[v] := Dv + \mathcal{N}(x)v, \tag{2.37}$$

where $\mathcal{N} := \left(S_0^0(u_0)\right)^{-1} N_0(u_0)$, be the ODE-operator associated to the quasilinear symmetric hyperbolic Fuchsian system Eq. (2.6). Since u_0, and thus $\left(S_0^0(u_0)\right)^{-1}$ and $N_0(u_0)$ are $C^\infty(T^n)$, and since S_0^0 is invertible, $\mathcal{N}$ is well-defined and $C^\infty(T^n)$. For a prescribed leading order term, the system Eq. (2.6) can be written

$$L_{\mathrm{ODE}}(u_0)[w] = \mathcal{F}_{\mathrm{ODE}}(u_0)[w], \tag{2.38}$$

where $L_{\mathrm{ODE}}(u_0)[w]$ is given by Eq. (2.37), and where

$$\begin{aligned} \mathcal{F}_{\mathrm{ODE}}(u_0)[w] := &\left(S^0(u_0+w)\right)^{-1} \mathcal{F}(u_0)[w] \\ &- \sum_{a=1}^{n} \left(S^0(u_0+w)\right)^{-1} S^a(u_0+w) t \partial_{x^a} w \\ &- \left(\left(S^0(u_0+w)\right)^{-1} - \left(S_0^0(u_0)\right)^{-1}\right) N_0(u_0) w. \end{aligned} \tag{2.39}$$

The operator $\mathcal{F}_{\mathrm{ODE}}(u_0)[\cdot]$ has the following properties.

Lemma 2.33 (Properties of $\mathcal{F}_{\mathrm{ODE}}(u_0)[\cdot]$)**.** *Suppose that Eq. (2.1) is a smooth quasilinear symmetric hyperbolic Fuchsian system around u_0 with parameters $\{\delta, \mu, s/2\}$. Suppose further that the smooth commutator conditions Definition 2.27 are satisfied, and that the product compatibility conditions (Definition 2.25) and the higher order source conditions (Definition 2.26) are satisfied with respect to $\hat{\mu} = \mu + \gamma_0$, with some exponent scalar $\gamma_0 \geq 0$, and for all $q > n/2 + 1$.*

Then there exists an exponent scalar γ such that $0 < \gamma < \min\{\nu - \mu, \zeta\}$, and a constant $r > 0$ independent of γ_0, so that for all $w \in B_{\delta', \mu, q}(s/2)$ and $h \in B_{\delta', \hat{\mu}, q}(s/2)$,

$$w \mapsto \mathcal{F}_{ODE}(u_0)[w] \in B_{\delta', \mu+\gamma, q-1}(r) \tag{2.40}$$

and

$$h \mapsto \mathcal{F}_{ODE}(u_0)[w] - \mathcal{F}_{ODE}(u_0)[w+h] \in B_{\delta',\hat{\mu}+\gamma,q-1}(r) \tag{2.41}$$

for all $\delta' \in (0, \delta]$ *and for all* $q > n/2 + 1$.

The inequality $q > n/2 + 1$ in this lemma arises from the condition that the various properties involving $q - 1$ hold for all $q > n/2$. This lemma is proved below in Section 2.6.7..

Under conditions which guarantee that $L_{\mathrm{ODE}}(u_0)[w]$ dominates near $t \searrow 0$, we can use this formulation of the Fuchsian system to construct a sequence of asymptotic solutions as in Definition 2.29. Suppose the right hand side of Eq. (2.38) is just a function of the coordinates $f(t, x)$:

$$L_{\mathrm{ODE}}(u_0)[v] = f. \tag{2.42}$$

If $W(t, x)$ denotes the fundamental solution to the homogeneous equation $L_{\mathrm{ODE}}(u_0)[v] = 0$, then the general solution to Eq. (2.42) may be formally written as

$$v(t, x) = W(t, x)(u_{*,1}(x), \ldots, u_{*,n}(x))^T + W(t, x) \int_0^t s^{-1} W^{-1}(s, x) f(s, x) ds,$$

for a spatially-parameterized $\mathbb{R}^d$-valued "initial data" function $u_*(x)$. We may then formally define the operator

$$\mathbb{H}_{\mathrm{ODE}}(u_0)\,[f]\,(t, x) := W(t, x) \int_0^t s^{-1} W^{-1}(s, x) f(s, x) ds, \tag{2.43}$$

which, if it exists, maps a given source function f to the particular solution $w = \mathbb{H}_{\mathrm{ODE}}(u_0)\,[f]$ of Eq. (2.42) determined by $(u_{*,1}(x), \ldots, u_{*,d}(x)) = 0$.

Lemma 2.34 (Existence and properties of $\mathbb{H}_{\mathrm{ODE}}(u_0)\,[w]$)**.** *Suppose Eq.* (2.1) *is a smooth quasilinear symmetric hyperbolic Fuchsian system with parameters* $\{\delta, \mu, s/2\}$ *such that* μ *has the block-diagonal structure of* $\mathcal{N}$*, and* $\mu > -\Re(\lambda)$*, where* λ *is the vector of eigenvalues of* $\mathcal{N}$*. Then for any* $r > 0$*, for all* $q > n/2$ *and for any scalar exponent* $\gamma > 0$*, there exists* $\tilde{r} > 0$*, and an exponent scalar* $\eta > 0$ *which may be taken arbitrarily small, such that*

$$\mathbb{H}_{ODE}(u_0)\,[\cdot] : B_{\delta,\mu+\gamma,q}(r) \to B_{\delta,\mu+\gamma-\eta,q}(\tilde{r}).$$

In particular, for every $f \in X_{\delta,\mu+\gamma,q}$ *we have the estimate*

$$\|\mathbb{H}_{ODE}(u_0)\,[f]\,\|_{\delta,\mu+\gamma-\eta,q} \leq C\delta^{\kappa}\|f\|_{\delta,\mu+\gamma,q}, \tag{2.44}$$

for constants $C, \kappa > 0$.

The scalar exponent η represents a loss of control due to the presence of $\log t$ terms. It can be chosen to be zero only if all eigenvalues of $\mathcal{N}$ have multiplicity equal to one. In the more general case, η must be chosen to be positive, but it can be chosen arbitrarily small. Details and the proof of Lemma 2.34 are found in Section 2.6.7. below.

The idea for the proof of Theorem 2.28 is to construct a sequence of asymptotic solutions using the composition of $\mathcal{F}_{\mathrm{ODE}}(u_0)[\cdot]$ and $\mathbb{H}_{\mathrm{ODE}}(u_0)\,[\cdot]$; we record some properties of this composition now. The proof of these properties is detailed in Section 2.6.7. below.

Lemma 2.35 (Properties of the composition)**.** *Suppose that Eq.* (2.1) *is a quasilinear symmetric hyperbolic Fuchsian system around* u_0 *with parameters* $\{\delta, \mu, s/2\}$*, as specified in Definition 2.7 such that the hypotheses of Lemma 2.33 and Lemma 2.34* *are satisfied. Let* $\hat{w} \in B_{\delta,\mu,q+1}(s/2)$ *be a smooth* σ*-asymptotic solution of Eq.* (2.1) *around* u_0 *with* $\sigma = \mu + \beta$ *for some strictly positive exponent scalar* β[20] *and for any* $q > n/2$*. Then the function*

$$\check{w} := \mathbb{H}_{ODE}(u_0)\left[\mathcal{F}_{ODE}(u_0)[\hat{w}]\right]$$

is well-defined, and for some $\epsilon > 0$ *is an element of* $B_{\check{\delta},\mu+\epsilon,q}(s/2) \subset B_{\check{\delta},\mu,q}(s/2)$ *for some* $\check{\delta} \in (0, \delta]$*. Furthermore,* $\check{w}$ *is a* $(\sigma + \Delta)$*-asymptotic solution for an exponent scalar* $\Delta > 0$*.*

The exponent scalar Δ is equal to $\gamma - \eta$, where η can be chosen arbitrarily small, and γ is bounded above by $\nu - \mu$, and by ζ. The property specified by Eq. (2.41) is used in showing that the function $\check{w}$ is an asymptotic solution of higher order than $\hat{w}$.

2.6.5. Existence for Theorem 2.28

As mentioned in the outline above, the proof of Theorem 2.28 is based on constructing an improving sequence of asymptotic solutions, verifying that under the hypotheses of the theorem we can generate an asymptotic solution of sufficiently high order, and finally applying Proposition 2.32.

Theorem 2.28, proof of existence. We start by constructing a sequence of asymptotic solutions using the ODE theory from Section 2.6.4., and in particular Lemma 2.35. Define $\{\hat{w}^{(i)}\}_{i\in\mathbb{Z}^+}$ by

$$\hat{w}^{(0)} = 0, \quad \hat{w}^{(i+1)} = \mathbb{H}_{\mathrm{ODE}}(u_0)\left[\mathcal{F}_{\mathrm{ODE}}(u_0)[w^{(i)}]\right].$$

[20]Note that if $\hat{w}$ is a $\tilde{\sigma}$-asymptotic solution for arbitrary $\tilde{\sigma} > \mu$, then there exists a $0 < \beta < \tilde{\sigma} - \mu$ such that $\hat{w}$ is a $(\mu + \beta)$-asymptotic solution.

We claim that $\{\hat{w}^{(i)}\}_{i\in\mathbb{Z}^+}$ is a well-defined sequence of asymptotic solutions in $B_{\tilde{\delta},\mu,q}(s/2)$ for some $\tilde{\delta} \in (0, \delta]$, and for all $q > n/2$. Moreover, there exists an iterate i_{final}, such that $\hat{w}^{(i_{final})}$ is an asymptotic solution of sufficiently high order $\sigma^{(i_{final})}$ so that there exists a $\hat{\mu} < \sigma^{(i_{final})}$, with respect to which the system is block-diagonal, and the energy dissipation matrix is positive definite.

Consider $\hat{w}^{(0)} = 0$. The left-hand side of Eq. (2.6) vanishes, while the right-hand side is hypothesized to be in the space $X_{\delta,\nu,q}$ for some $\nu > \mu$ and for all $q > n/2$. In particular, $\hat{w}^{(0)}$ is a ν-asymptotic solution. In order to apply the lemmas of Section 2.6.4., we track a particular sequence of asymptotic solutions, those of the form $\sigma^{(i)} = \mu + \beta$ for some scalar exponent β. With this in mind, fix β such that $\mu + \beta < \nu$, and note that $\hat{w}^{(0)} = 0$ is a $(\mu + \beta)$-asymptotic solution. We then check if there exists a $\hat{\mu}$ satisfying $\mu < \hat{\mu} < \mu + \beta$ such that with respect to $\hat{\mu}$ the Fuchsian system Eq. (2.6) is block-diagonal and the energy dissipation matrix is positive definite (such a $\hat{\mu}$ satisfies the hypotheses of Proposition 2.32). Of course if ν is large enough (and so β may be chosen large enough) so that we can find a $\mu < \hat{\mu} < \mu + \beta < \nu$ satisfying these criteria, then we may simply apply Theorem 2.10. We presume that this is not the case throughout this section.

From Lemma 2.35 we observe that $\hat{w}^{(1)}$ is a $(\mu + \beta + \Delta)$-asymptotic solution, for some scalar exponent vector Δ which is bounded above by $\nu - \mu$ and by ζ. Again we check if there exists a $\hat{\mu} < \mu + \beta + \Delta$ which satisfies the hypothesis of Proposition 2.32. Since the increase in order of the asymptotic solution at each iterate can be fixed, we find that $\hat{w}^{(i)}$ is a $(\mu + \beta + i\Delta)$-asymptotic solution. Since the order of the asymptotic solution increases (by Δ) at each step of the iteration, it follows that there exists an iteration step i_{final} for which $\hat{w}^{(i_{final})}$ satisfies the conditions of Proposition 2.32. We note that due to the form of $\mathcal{F}_{\mathrm{ODE}}(u_0)[]$ a derivative is lost at each step of the iteration. However, since we work in the smooth setting this is not an issue, and we can perform the iteration as many times as needed. In fact this is the reason for restricting to the smooth setting in this theorem. We apply this proposition to obtain a unique solution to the asymptotic value problem about $u_0 + \hat{w}^{(i_{final})}$, with remainder $h \in X_{\tilde{\delta},\hat{\mu},q}$.

□

2.6.6. Uniqueness for Theorem 2.28

In Section 2.6.5. we construct a unique solution to the AVP($\hat{u}_0$). This is also a solution to the AVP(u_0), although it is not known to be the unique solution to this problem. In this section we perform a "bootstrap" type argument to show that under the hypotheses of Theorem 2.28, the solution $u = u_0 + \hat{w} + h$ constructed above is indeed the unique solution to the AVP(u_0). A bootstrap-type argument is a scheme for iteratively increasing the amount of information one has about a system or property at each step.

Suppose $\tilde{u} = u_0 + \tilde{w}$ is any other solution to the AVP(u_0) with $\tilde{w} \in X_{\delta,\mu,q}$. We show that $\tilde{w} - \hat{w} \in X_{\delta,\hat{\mu},q}$, and hence by the uniqueness in Proposition 2.32, we obtain $\tilde{w} - \hat{w} = h$, and consequently $\tilde{u} = u$.

We know that both $\tilde{w}$ and $\hat{w}$ are contained in a bounded subset of $X_{\delta,\mu,q}$ for all $q > n/2$. Thus, $\tilde{w} - \hat{w} \in B_{\delta,\mu,q}(s/2)$ for some $s > 0$ and for sufficiently small δ. The bootstrap argument which we now employ increases our knowledge of the exponent vector for $\tilde{w} - \hat{w}$, which we expect to be greater than μ. Now $\tilde{w}$ satisfies

$$L_{\mathrm{ODE}}(u_0)[\tilde{w}] = F_{\mathrm{ODE}}(u_0)[\tilde{w}],$$

while $\hat{w}$ satisfies

$$L_{\mathrm{ODE}}(u_0)[\hat{w}] = F_{\mathrm{ODE}}(u_0)[\hat{w}] + \left(S^0(u_0+\hat{w})\right)^{-1} R(u_0)[\hat{w}].$$

Due to the linearity properties of $L_{\mathrm{ODE}}(u_0)[\cdot]$, we find that

$$\begin{aligned} L_{\mathrm{ODE}}(u_0)[\tilde{w}-\hat{w}] &= K_{ODE}[\tilde{w},\hat{w}] \\ &:= F_{\mathrm{ODE}}(u_0)[\tilde{w}] - F_{\mathrm{ODE}}(u_0)[\hat{w}] - \left(S^0(u_0+\hat{w})\right)^{-1} R(u_0)[\hat{w}]. \end{aligned}$$

At this point we would like to apply the $\mathbb{H}_{\mathrm{ODE}}(u_0)\,[\cdot]$ operator in order to increase our knowledge of the exponent vector for $\tilde{w}-\hat{w}$. To do this we must first understand the properties of the right hand side. According to the properties of $F_{\mathrm{ODE}}(u_0)[\cdot]$ (Lemma 2.33) we have

$$F_{\mathrm{ODE}}(u_0)[\tilde{w}] - F_{\mathrm{ODE}}(u_0)[\hat{w}] \in B_{\delta,\mu+\gamma,q-1}(r)$$

for some $r>0$ and for an exponent scalar $0<\gamma<\min\{\nu,\zeta\}$. Furthermore, due to the smooth commutator conditions (Lemma 2.27) and the properties of $(S^0(u_0+\hat{w}))^{-1}$ (Lemma C.21) we have

$$\left(S^0(u_0+\hat{w})\right)^{-1} R(u_0)[\hat{w}] \in B_{\delta,\mu+\beta,q}(\tilde{r})$$

for some $\tilde{r}>0$ and for an exponent scalar $\beta<\sigma-\mu$. It follows that

$$K_{ODE}[\tilde{w},\hat{w}] \in B_{\delta,\mu+\alpha,q-1}(r)$$

for some $r > 0$ and for an exponent scalar $\alpha < \min\{\beta, \gamma\}$. Lemma 2.34 then tell us that

$$\tilde{w} - \hat{w} = \mathbb{H}_{\mathrm{ODE}}(u_0)\left[K_{ODE}[\tilde{w}, \hat{w}]\right] \in B_{\delta,\mu+\alpha-\eta,q-1}(\hat{r})$$

and that

$$\|\tilde{w} - \hat{w}\|_{\delta,\mu+\alpha-\eta,q-1} \leq C\delta^{\kappa}\hat{r}.$$

The exponent scalar $\alpha - \eta$ is positive and can be fixed. Hence for a possibly smaller choice of δ, ($\delta' \in (0, \delta]$) we find that $\tilde{w} - \hat{w} \in B_{\delta',\mu+\alpha-\eta,q-1}(s/2)$. Iterating this argument k times provides us with the information that $\tilde{w} - \hat{w} \in B_{\delta',\mu+k(\alpha-\eta),q-k}(s/2)$, and since we work in the smooth case, that further $\tilde{w} - \hat{w} \in B_{\delta',\mu+k(\alpha-\eta),q}(s/2)$.

There exists an iterate k such that $\mu + k(\alpha - \eta) > \hat{\mu}$, showing that $\tilde{h} := \tilde{w} - \hat{w} \in X_{\delta,\hat{\mu},q}$. That is, any "other" solution to the AVP(u_0) can be written as $\tilde{u} = u_0 + \hat{w} + \tilde{h}$ for $\tilde{h} \in X_{\delta,\hat{\mu},q}$. But by uniqueness to the AVP($\hat{u}_0$), $\tilde{h} = h$ and thus $\tilde{u} = u$.

2.6.7. Proofs of Lemma 2.33, Lemma 2.34, and Lemma 2.35

2.6.7.1. Proof of Lemma 2.33

Proof of Lemma 2.33. It suffices to prove the desired properties for each term in the expression Eq. (2.39).

Step 1: Boundedness of $F_{ODE}(u_0)[w]$

We note that from the smooth commutator conditions Definition 2.27, and from Lemma C.21, that

$$\mathcal{R}[\mu]\left(S^0(u_0 + w)\right)^{-1}\mathcal{R}[-\mu] \in X_{\delta,0,q} \quad \text{and} \quad \mathcal{R}[\mu]\Sigma_1^0(u_0 + w)\mathcal{R}[-\mu] \in X_{\delta,\underline{\xi},q},$$

for the positive exponent scalar ξ. Since the system is quasilinear symmetric hyperbolic Fuchsian, it then follows from Lemma C.1 that the three terms

$$
\begin{aligned}
T_1[w] &:= \left(S^0(u_0+w)\right)^{-1}\mathcal{F}(u_0)[w],\\
T_2[w] &:= \left(S^0(u_0+w)\right)^{-1} tS^a(u_0+w)\partial_a w,\\
T_3[w] &:= \left(\left(S^0(u_0+w)\right)^{-1} - \left(S^0_0\right)^{-1}\right) N_0 w, = \Sigma^0_1(u_0+w)N_0 w
\end{aligned}
$$

are bounded in $X_{\delta,\mu+\gamma,q-1}$ for some exponent scalar $\gamma < \min\{\nu-\mu,\xi\}$.

Step 2: Boundedness of $\Delta F_{ODE}(u_0,w)[h]$

We verify this property by computing each term. Consider first

$$
\begin{aligned}
&\|T_1[w] - T_1[w+h]\|_{\delta,\hat{\mu}+\gamma,q-1}\\
&\qquad\le \|\left(S^0(w)\right)^{-1}\left(\mathcal{F}(u_0)[w] - \mathcal{F}(u_0)[w+h]\right)\|_{\delta,\hat{\mu}+\gamma,q-1}\\
&\qquad\quad + \|\left(\left(S^0(w)\right)^{-1} - \left(S^0(w+h)\right)^{-1}\right)\mathcal{F}(u_0)[w+h]\|_{\delta,\hat{\mu}+\gamma,q-1}.
\end{aligned}
$$

Now by Definition 2.26, Lemma C.21, and the assumption that $\hat{\mu} = \mu + \gamma_0$ for an exponent scalar γ_0, the first term can be bounded by

$$
C\|\mathcal{R}[\mu]\left(S^0(w)\right)^{-1}\mathcal{R}[-\mu]\|_{\delta,0,q-1}\|\Delta\mathcal{F}(u_0,w)[h]\|_{\delta,\hat{\mu}+\gamma,q-1}.
$$

For the second term, we have that $\mathcal{F}(u_0)[w+h] \in X_{\delta,\mu+\gamma,q}$ for some $\gamma < \nu - \mu$. Lemma C.22 and the smooth commutator conditions then imply that the second term can be bounded

$$
\begin{aligned}
&\| \left(\left(S^0(w)\right)^{-1} - \left(S^0(w+h)\right)^{-1} \right) \mathcal{F}(u_0)[w+h] \|_{\delta,\hat{\mu}+\gamma,q-1} \\
&\quad \leq \| \left(\left(S^0(w)\right)^{-1} - \left(S^0(w+h)\right)^{-1} \right) \mathcal{F}(u_0)[w+h] \|_{\delta,\hat{\mu}+\gamma+\tilde{\zeta},q-1} \\
&\quad \leq C \| \left(\left(S^0(w)\right)^{-1} - \left(S^0(w+h)\right)^{-1} \right) \|_{\delta,\hat{\mu}+\tilde{\zeta}-\mu,q-1} \| \mathcal{F}(u_0)[w+h] \|_{\delta,\mu+\gamma,q-1}.
\end{aligned}
$$

Each norm is bounded under the hypotheses of Lemma 2.33. Using similar arguments we compute

$$
\begin{aligned}
&\|T_2[w] - T_2[w+h]\|_{\delta,\hat{\mu}+\gamma,q-1} \\
&\qquad \leq \| \left(S^0(w)\right)^{-1} \left(tS^a(w)\partial_a w - tS^a(w+h)\partial_a(w+h) \right) \|_{\delta,\hat{\mu}+\gamma,q-1} \\
&\qquad\quad + \| \left(\left(S^0(w)\right)^{-1} - \left(S^0(w+h)\right)^{-1} \right) tS^a(w+h)\partial_a(w+h) \|_{\delta,\hat{\mu}+\gamma,q-1} \\
&\qquad \leq \|\mathcal{R}[\mu] t \left(S^a(u_0,w) - S^a(u_0,w+h) \right) \mathcal{R}[-\mu]\|_{\delta,\hat{\mu}-\mu+\zeta,q-1} \\
&\qquad\quad \times C \|\mathcal{R}[\mu] \left(S^0(w)\right)^{-1} \mathcal{R}[-\mu]\|_{\delta,0,q-1} \|\partial_a w\|_{\delta,\mu,q-1} \\
&\qquad\quad + C \|\mathcal{R}[\mu] \left(S^0(w)\right)^{-1} \mathcal{R}[-\mu]\|_{\delta,0,q-1} \\
&\qquad\quad \times \|\mathcal{R}[\mu] tS^a(w+h) \mathcal{R}[-\mu]\|_{\delta,\zeta,q-1} \|\partial_a h\|_{\delta,\hat{\mu},q-1} \\
&\qquad\quad + C \|\mathcal{R}[\mu] \left(\left(S^0(w)\right)^{-1} - \left(S^0(w+h)\right)^{-1} \right) \mathcal{R}[-\mu]\|_{\delta,\hat{\mu}+\tilde{\zeta}-\mu,q-1} \\
&\qquad\quad \times \|\mathcal{R}[\mu] tS^a(w+h) \mathcal{R}[-\mu]\|_{\delta,\zeta,q-1} \|\partial_a w\|_{\delta,\mu,q-1} \\
&\qquad\quad + C \|\mathcal{R}[\mu] \left(\left(S^0(w)\right)^{-1} - \left(S^0(w+h)\right)^{-1} \right) \mathcal{R}[-\mu]\|_{\delta,\hat{\mu}+\tilde{\zeta}-\mu,q-1} \\
&\qquad\quad \times \|\mathcal{R}[\mu] tS^a(w+h) \mathcal{R}[-\mu]\|_{\delta,\zeta,q-1} \|\partial_a h\|_{\delta,\hat{\mu},q-1}.
\end{aligned}
$$

and

$$
\begin{aligned}
\|T_3[w] - T_3[w+h]\|_{\delta,\hat{\mu}+\gamma,q-1} \\
&\le \|\Sigma_1^0(w) N_0 h\|_{\delta,\hat{\mu}+\gamma,q-1} + \| \left(\Sigma_1^0(w) - \Sigma_1^0(w+h)\right) N_0(w+h)\|_{\delta,\hat{\mu}+\gamma,q-1} \\
&\le C\|\mathcal{R}[\mu]\Sigma_1^0(w)\mathcal{R}[-\mu]\|_{\delta,\tilde{\zeta},q-1}\|\mathcal{R}[\mu]N_0\mathcal{R}[-\mu]\|_{\delta,0,q-1}\|h\|_{\delta,\hat{\mu},q-1} \\
&\quad + C\|\mathcal{R}[\mu]\left(\Sigma_1^0(w) - \Sigma_1^0(w+h)\right)\mathcal{R}[-\mu]\|_{\delta,\hat{\mu}-\mu+\tilde{\zeta},q-1} \\
&\quad \times \|\mathcal{R}[\mu]N_0\mathcal{R}[-\mu]\|_{\delta,0,q-1}\|w\|_{\delta,\mu,q-1} \\
&\quad + C\|\mathcal{R}[\mu]\left(\Sigma_1^0(w) - \Sigma_1^0(w+h)\right)\mathcal{R}[-\mu]\|_{\delta,\hat{\mu}-\mu+\tilde{\zeta},q-1} \\
&\quad \times \|\mathcal{R}[\mu]N_0\mathcal{R}[-\mu]\|_{\delta,0,q-1}\|h\|_{\delta,\hat{\mu},q-1}.
\end{aligned}
$$

Each of these terms is bounded under the hypotheses of Lemma 2.33, thus completing the proof. □

2.6.7.2. Proof of Lemma 2.34

Proof of Lemma 2.34. It is useful to work with the quantities corresponding to the Jordan normal form of Eq. (2.37); we denote quantities for the ODE equation in Jordan normal form using an underbar. Let T be the matrix which takes $\mathcal{N}$ to its Jordan normal form, $\underline{\mathcal{N}} := T \cdot \mathcal{N} \cdot T^{-1}$. Hence the inhomogeneous equation Eq. (2.42) can be written

$$D(Tv) + \underline{\mathcal{N}}(Tv) = Tf,$$

and the corresponding solution operator can be written

$$\mathbb{H}_{\text{ODE}}^{\text{Jordan}}(u_0)\,[Tf] := \underline{W} \int_0^t s^{-1} \underline{W}^{-1}(s,x) Tf(s,x) ds.$$

We find that since $\underline{W} = T \cdot W$, we have

$$\mathbb{H}_{\mathrm{ODE}}(u_0)\,[f] = T^{-1}\mathbb{H}_{\mathrm{ODE}}^{\mathrm{Jordan}}(u_0)\,[Tf]\,.$$

Working with the Jordan normal form is useful because in this basis the $\underline{W}$ matrices have a well-understood structure. Let λ denote the $\mathbb{R}^d$-vector of eigenvalues of $\mathcal{N}$; due to the smoothness assumption, $\lambda \in C^\infty(T^n)$. Then $\underline{\mathcal{N}}$ and $\underline{W}$ take a block-diagonal form with each block corresponding to a particular eigenvalue of $\mathcal{N}$. Let λ_i be an eigenvalue of multiplicity m.

Then, the blocks corresponding to λ_i are $m \times m$ matrices with the form

$$\underline{\mathcal{N}}|_{block} = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ & \vdots & & & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda_i \end{pmatrix},$$

$$\underline{W}|_{block} = t^{-\lambda_i} \begin{pmatrix} 1 & -\log t & \frac{1}{2}\log^2 t & \dots & \frac{(-1)^{m-1}}{(m-1)!}\log^{m-1} t \\ 0 & 1 & -\log t & \dots & \frac{(-1)^{m-2}}{(m-2)!}\log^{m-2} t \\ \dots & & & & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.45}$$

The inverse of this block, i.e., the corresponding block of $\underline{W}^{-1}$, is

$$\underline{W}^{-1}|_{block} = t^{\lambda_i} \begin{pmatrix} 1 & \log t & \frac{1}{2}\log^2 t & \dots & \frac{1}{(m-1)!}\log^{m-1} t \\ 0 & 1 & \log t & \dots & \frac{1}{(m-2)!}\log^{m-2} t \\ \dots & & & & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.46}$$

Using this formulation of the quantities, we now prove that $f \mapsto \mathbb{H}_{\mathrm{ODE}}(u_0)[f]$ is a well-defined bounded function operator with the estimate Eq. (2.44). The majority of the work is in showing that the function-operator

$$Z[f](t,x) = Z(t,x,f(t,x)) := \int_0^t s^{-1}\underline{W}^{-1}(s,x)T(x)f(s,x)ds$$

is a well-defined function operator from $X_{\delta,\nu,q}$ to $X_{\delta,\rho,q}$ for some exponent vector ρ.

Consider Tf with $f \in X_{\delta,\nu,q}$, where ν is a general exponent vector $\nu > \mu$. Since T is independent of t, we expect (naively) that Tf should also be in $X_{\delta,\nu,q}$. However, the components of Tf are formed by linear combinations of the components of f, and even with the assumption that T has the same block-diagonal structure as μ, the best we can hope for is that there exists a $\underline{\nu}$ (consisting of permuted elements of ν), such that $Tf \in X_{\delta,\underline{\nu},q}$. To avoid this issue we accept the slight loss of generality[21] and assume that $\nu = \mu + \gamma$ for a scalar exponent γ as in the statement of the lemma. With this choice, ν shares the same block-diagonal structure as μ, and thus $\mathcal{R}[\nu]T\mathcal{R}[-\nu] = T \in X_{\delta,0,q}$. We apply Lemma C.1 (for the product of a matrix and a vector) in order to show that $Tf \in X_{\delta,\mu+\gamma,q}$ for all $q > n/2$. This form of ν is sufficient to prove the existence of solutions for Theorem 2.28 in Section 2.6.5..

[21] We also accept a slight loss of control, since in general $\nu \geq \mu + \gamma$. However, since we construct a sequence of solutions over which we have increasing control, this slight loss is of no consequence.

Due to the particular structure of $\underline{W}^{-1}$ we see that these are $\mathbb{R}^{d\times d}$-valued functions of (t,x) in $X_{\delta,\lambda-\epsilon,\infty}$ for some $\epsilon \geq 0$. The case $\epsilon > 0$ is required to control the $\log t$ terms that occur if any of the eigenvalues $\lambda_1,\ldots,\lambda_d$ have multiplicity greater than 1 (see Eq. (2.45)). In the special case that all λ_i have multiplicity one, we can take a $\epsilon = 0$. Since $\mathcal{R}[\mu+\gamma]\underline{W}^{-1}\mathcal{R}[-\mu-\gamma] = \underline{W}^{-1} \in X_{\delta,\lambda-\epsilon,\infty}$, due to assumptions that μ has block-diagonal structure of $\mathcal{N}$, we can again apply Lemma C.1, to show that $\underline{W}^{-1}Tf \in X_{\delta,\mu+\lambda+\gamma-\epsilon,q}$ for all $q > n/2$.

Now we compute

$$\|\mathcal{R}[\rho](t,\cdot)Z[f](t,\cdot)\|_{H^q}^2 = \sum_{|k|\leq q}\int_0^t\int_0^t s^{-1}s'^{-1}\left\langle \partial_x^k\mathcal{R}[\rho](t,\cdot)z(s,\cdot), \partial_x^k\mathcal{R}[\rho](t,\cdot)z(s',\cdot)\right\rangle_{L^2} ds'ds$$

where we have defined for convenience $z(s,x) := \underline{W}^{-1}(s,x)T(x)f(s,x)$. Due to the Hölder inequality (Lemma A.10) we have

$$\begin{aligned}
&\|\mathcal{R}[\rho](t,\cdot)Z[f](t,\cdot)\|_{H^q}^2 \\
&\leq \sum_{|k|\leq q}\int_0^t\int_0^t s^{-1}s'^{-1}\left\|\partial_x^k\mathcal{R}[\rho](t,\cdot)z(s,\cdot)\right\|_{L^2}\left\|\partial_x^k\mathcal{R}[\rho](t,\cdot)z(s',\cdot)\right\|_{L^2} ds'ds \\
&= \sum_{|k|\leq q}\left(\int_0^t s^{-1}\left\|\partial_x^k\mathcal{R}[\rho](t,\cdot)z(s,\cdot)\right\|_{L^2} ds\right)^2 \\
&= \left(\int_0^t s^{-1}\left\|\mathcal{R}[\rho](t,\cdot)z(s,\cdot)\right\|_{H^q} ds\right)^2.
\end{aligned}$$

Using the Moser estimate (Lemma A.11), we find that for any exponent vector σ for which the norm $\|z\|_{\delta,\sigma,q}$ is defined ,

$$\begin{aligned}\sum_{|k|\le q}\left(\int_0^t s^{-1}\|\partial_x^k\mathcal{R}[\rho](t,\cdot)z(s,\cdot)\|_{L^2}ds\right)^2 \\ \le C\left(\int_0^t s^{-1}\left\|\mathcal{R}[\rho](t,\cdot)\mathcal{R}[-\sigma](s,\cdot)\right\|_{H^q}\left\|\mathcal{R}[\sigma](s,\cdot)z(s,\cdot)\right\|_{H^q}ds\right)^2 \\ \le C\left(\int_0^t s^{-1}\left\|\mathcal{R}[\rho](t,\cdot)\mathcal{R}[-\sigma](s,\cdot)\right\|_{H^q}ds\right)^2\|z\|^2_{\delta,\sigma,q}.\end{aligned}$$

The constant C, which comes from the Moser estimate, depends only on q and n, and is thus independent of ρ, σ and z. To estimate the integral over s, let $r = s/t$, so that we may write

$$\begin{aligned}\int_0^t s^{-1}\left\|\mathcal{R}[\rho](t,\cdot)\mathcal{R}[-\sigma](s,\cdot)\right\|_{H^q}ds &= \int_0^1\left\|\mathcal{R}[\rho-\sigma](t,\cdot)\mathcal{R}[-\sigma+1](r,\cdot)\right\|_{H^q}dr \\ &\le C\left\|\mathcal{R}[\rho-\sigma](t,\cdot)\right\|_{H^q}\int_0^1\left\|\mathcal{R}[-\sigma+1](r,\cdot)\right\|_{H^q}dr,\end{aligned}$$

where the constant C again originates in the Moser estimate. The remaining integral is finite for $\sigma = \mu+\gamma+\lambda-\epsilon > 0$, and hence can be estimated by a constant which only depends on σ and q. Since the constant ϵ can always be chosen so that $\epsilon < \gamma$, it follows that $Z[f]$ is well-defined under the hypothesis $\mu > -\Re\lambda$. For any $\rho \le \sigma$ the factor $\|\mathcal{R}[\rho-\sigma](t,\cdot)\|_{H^q} \le C\delta^\kappa$ for C,κ positive and depending on q and $\rho-\sigma$. Thus we have $Z[f] \in X_{\delta,\mu+\gamma+\lambda-\epsilon,q}$ and

$$\|Z[f](t,\cdot)\|_{\delta,\mu+\gamma+\lambda-\epsilon,q} \le C\delta^\kappa\|f\|_{\delta,\mu+\gamma,q},$$

for all $q > n/2$.

To finish, recall that $\mathbb{H}^{\text{Jordan}}_{\text{ODE}}(u_0)\,[Tf] = \underline{W}Z[f]$, and $\mathbb{H}_{\text{ODE}}(u_0)\,[f] = T^{-1}\mathbb{H}^{\text{Jordan}}_{\text{ODE}}(u_0)\,[Tf]$. Since ϵ and γ are scalar exponents, μ has the block-diagonal structure of $\mathcal{N}$, and λ has the same block-diagonal structure of $T^{-1}\underline{W}$, we have

$$
\begin{aligned}
\mathcal{R}[\lambda+\mu+\gamma-\epsilon]T^{-1}\underline{W}\mathcal{R}[-\lambda-\mu-\gamma+\epsilon] &= \mathcal{R}[\lambda+\mu]T^{-1}\underline{W}\mathcal{R}[-\lambda-\mu] \\
&= \mathcal{R}[\lambda]T^{-1}\underline{W}\mathcal{R}[-\lambda] \\
&= T^{-1}\underline{W}.
\end{aligned}
$$

Hence, we use Lemma C.1 and the fact that $T^{-1}.\underline{W} \in X_{\delta,-\lambda-\epsilon,\infty}$ to show that $\mathbb{H}_{\text{ODE}}(u_0)\,[f] \in X_{\delta,\mu+\gamma-\eta,\infty}$, for some $\eta > 0$ (η is basically 2ϵ). Since both T^{-1} and $\underline{W}$ are smooth in T^n, the estimate Eq. (2.44) holds. □

2.6.7.3. Proof of Lemma 2.35

Proof of Lemma 2.35. 1. $\breve{w} \in B_{\check{\delta},\mu+\epsilon,q}(s/2)$. We apply Lemma 2.33 to show that $\mathcal{F}_{\text{ODE}}(u_0)[\hat{w}] \in B_{\delta',\mu+\gamma,q}(r)$ for some $\gamma > 0$, which is bounded above by $\nu - \mu$ and by ζ, and for some $\delta' \in (0,\delta]$, and some $r > 0$. Next, we apply Lemma 2.34, which shows that there exists an $\eta > 0$, such that $\breve{w} := \mathbb{H}_{\text{ODE}}(u_0)\,[\mathcal{F}_{\text{ODE}}(u_0)[\hat{w}]] \in X_{\delta',\mu+\gamma-\eta,q}$, and such that

$$
\|\breve{w}\|_{\delta',\mu+\gamma-\eta,q} \le C\delta'^{\kappa}\|\mathcal{F}_{\text{ODE}}(u_0)[\hat{w}]\|_{\delta',\mu+\gamma,q} \le C\delta'^{\kappa} r,
$$

for positive constants C, κ. Hence, we can shrink δ' if necessary to $\check{\delta} \in (0,\delta]$ such that $\|\breve{w}\|_{\check{\delta},\mu+\gamma-\eta,q} \le s/2$ for any $s > 0$, as specified in the smooth quasilinear symmetric hyperbolic Fuchsian system. Since η can be chosen arbitrarily small, it follows that there exists $\epsilon > 0$ such that $\breve{w} \in B_{\check{\delta},\mu+\epsilon,q}(s/2)$.

2. Control $\breve{w} - \hat{w}$. In order to prove that $\breve{w}$ is an asymptotic solution of higher order than $\hat{w}$, we need to determine in which space the difference $h := \hat{w} - \breve{w}$ lives. Note that

$$\breve{w} = \mathbb{H}_{\text{ODE}}(u_0)\left[\mathcal{F}_{\text{ODE}}(u_0)[\hat{w}]\right]$$

while $\hat{w}$ satisfies

$$\begin{aligned} &\widehat{L}(u_0 + \hat{w})[\hat{w}] - \mathcal{F}(u_0)[\hat{w}] = R(u_0)[\hat{w}] \\ \Longrightarrow\ &L_{\text{ODE}}(u_0)[\hat{w}] - \mathcal{F}_{\text{ODE}}(u_0)[\hat{w}] = \left(S^0(\hat{w})\right)^{-1} R(u_0)[\hat{w}] \end{aligned}$$

and thus

$$\hat{w} = \mathbb{H}_{\text{ODE}}(u_0)\left[\left(S^0(\hat{w})\right)^{-1} R(u_0)[\hat{w}] + \mathcal{F}_{\text{ODE}}(u_0)[\hat{w}]\right].$$

It follows from the linearity of $\mathbb{H}_{\text{ODE}}(u_0)[\cdot]$ that

$$h = \mathbb{H}_{\text{ODE}}(u_0)\left[\left(S^0(\hat{w})\right)^{-1} R(u_0)[\hat{w}]\right].$$

Now by definition $R(u_0)[\hat{w}] \in X_{\delta,\sigma,q}$, and we apply Lemma C.1 in order to control $\left(S^0(\hat{w})\right)^{-1} R(u_0)[\hat{w}]$. Due to Lemma C.21 we know that $\mathcal{R}[\mu + \beta]\left(S^0(\hat{w})\right)^{-1} \mathcal{R}[-\mu - \beta] \in X_{\delta,0,q}$. Hence, $\left(S^0(\hat{w})\right)^{-1} R(u_0)[\hat{w}] \in X_{\delta,\mu+\beta,q}$, and thus by Lemma 2.34, $h \in X_{\delta,\mu+\beta-\eta,q}$.

3. $\check{w}$ is an asymptotic solution. To verify that $\check{w}$ is an asymptotic solution, we compute

$$\begin{aligned} R(u_0)[\check{w}] &= \widehat{L}(u_0+\check{w})[\check{w}] - \mathcal{F}(u_0)[\check{w}] \\ &= S^0(u_0+\check{w})\left(L_{\mathrm{ODE}}(u_0)[\check{w}] - \mathcal{F}_{\mathrm{ODE}}(u_0)[\check{w}]\right) \\ &= S^0(u_0+\check{w})\left(\mathcal{F}_{\mathrm{ODE}}(u_0)[\hat{w}] - \mathcal{F}_{\mathrm{ODE}}(u_0)[\check{w}]\right). \end{aligned}$$

From Step 2 and the properties of $\mathcal{F}_{\mathrm{ODE}}(u_0)[\cdot]$ (Lemma 2.33), we find that $\mathcal{F}_{\mathrm{ODE}}(u_0)[\hat{w}] - \mathcal{F}_{\mathrm{ODE}}(u_0)[\check{w}] = \mathcal{F}_{\mathrm{ODE}}(u_0)[\hat{w}] - \mathcal{F}_{\mathrm{ODE}}(u_0)[\hat{w}+h] \in B_{\delta,\mu+\beta-\eta+\gamma,q}(s/2)$. Since $\eta > 0$ can be chosen arbitrarily small, there exists a scalar exponent $\Delta = \gamma - \eta$ which is bounded above by $\nu - \mu, \zeta$ such that $\check{w}$ is a smooth asymptotic solution of order Δ greater than that of $\hat{w}$. □

CHAPTER III

AVTD BEHAVIOR IN POLARIZED T^2-SYMMETRIC SPACETIMES

This chapter contains work published in [3]. The calculations were performed by E. Ames and F. Beyer; while writing was done by E. Ames, F. Beyer, and J. Isenberg. P.G. LeFloch contributed editorial changes.

3.1. Prelude

In this chapter we prove two theorems which establish the existence and uniqueness of AVTD T^2–symmetric solutions to the Einstein equations. The first one, Theorem 3.3, obtains solutions in a weighted Sobolev space of Section 2.2.2. with finite regularity. This result represents the minimal regularity assumptions needed to obtain existence and uniqueness via our method of proof. In our second result, Theorem 3.10, we find a family of smooth AVTD solutions. The family of smooth solutions turns out to be slightly larger in that a constraint on the asymptotic data which parametrizes the rough family of solutions is lifted. Both this constraint for the rough solutions, and its removal in the smooth case is due to the difference in the Fuchsian theorems which are used to prove the respective theorems.

Our results extend existing results in the literature for AVTD solutions to the Einstein equations in the presence of T^2-symmetry. As discussed above Isenberg and Kichenassamy use the analytic Fuchsian theory to find a family of analytic AVTD T^2-symmetric solutions in the polarized class. Later, Clausen extended the work of Rendall in [67] to obtain a families of smooth AVTD T^2-symmetric solutions in both the polarized and half-polarized classes [32]. The theory we develop in this dissertation allows us to confirm the smooth results of Clausen, and also show that similar families of solutions can be found with finite regularity.

3.2. T^2-Symmetric Spacetimes

We write the metric for the T^2-symmetric spacetimes Section 1.2.3., in areal coordinates Section 1.2.3.2., and write down the Einstein equations.

3.2.1. Polarized T^2-Symmetric Metric and Einstein Equations

Let y, z be coordinates on T^2, and let x be the remaining spatial coordinate, which takes values in S^1. Further, let t denote the areal time coordinate of Section 1.2.3.2.. We write the metric as in [46], and make the same gauge choice so that the two shift quantities M_y and M_z vanish. This metric is obtained from a general form of the metric on spacetimes with $U(1) \times U(1)$ symmetry and T^3 spatial topology derived by Chrúsciel [26]. We have

$$g = e^{2(\eta - U)}\Big(-\alpha dt^2 + dx^2 \Big) + e^{2U}\Bigg(dy + Adz + \Big(G + AH \Big) dx \Bigg)^2 + e^{-2U} t^2 \Bigg(dz + H dx \Bigg)^2, \tag{3.1}$$

where the metric functions $\{\eta, U, \alpha, A, G, H\}$ depend only on t and x.

From the form of the metric it is clear that the polarized case introduced in Section 1.2.3., in which the generators Y, Z can be chosen to be g-orthogonal corresponds to $A = const$. While the polarized spacetimes are characterized by a geometric condition, another subclass we consider, called the *half-polarized* T^2–symmetric spacetimes, are defined by a restriction on the asymptotic behavior of the metric fields. We introduce this subclass below in Section 3.2.3..

When working with the Einstein equations it is convenient to make a particular choice of the spatial coordinates y, z on T^2. First note that we may always choose a linear combination of the generators $Y = a\partial_y + b\partial_z$ and $Z = c\partial_y + d\partial_z$ with constants a, b, c, d such that $det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$ so that $K_Y = 0, K_Z \equiv K$ (recall the definitions of the twist constants from Section 1.2.3.). Since we are considering here the case of T^2-symmetric solutions and not the Gowdy solutions, we restrict to solutions with $K \neq 0$. The transformation $K \to -K$ preserves all conditions imposed, and so we restrict further to consider just the case of $K > 0$.

Next we choose coordinates y, z on T^2 so that the coordinate derivatives align with the generators specified above $Y = \partial_y$ and $Z = \partial_z$. Since the form of the metric Eq. (3.1) holds for any smooth coordinates on T^2, it is preserved for this choice of coordinates. With these choices we write the Einstein equations Eq. (1.1) as the following system of partial differential equations, computed in [12]. We have a set of second-order equations

$$U_{tt} + \frac{U_t}{t} - \alpha U_{xx} = \frac{\alpha_x U_x}{2} + \frac{\alpha_t U_t}{2\alpha} + \frac{e^{4U}}{2t^2}\left(A_t^2 - \alpha A_x^2\right), \tag{3.2}$$

$$A_{tt} - \frac{A_t}{t} - \alpha A_{xx} = \frac{\alpha_x A_x}{2} + \frac{\alpha_t A_t}{2\alpha} - 4A_t U_t + 4\alpha A_x U_x, \tag{3.3}$$

$$\begin{aligned} \eta_{tt} - \alpha\eta_{xx} = {} & \frac{\alpha_x \eta_x}{2} + \frac{\alpha_t \eta_t}{2\alpha} - \frac{\alpha_x^2}{4\alpha} + \frac{\alpha_{xx}}{2} - U_t^2 + \alpha U_x^2, \\ & + \frac{e^{4U}}{4t^2}\left(A_t^2 - \alpha A_x^2\right) - \frac{3e^{2\eta}\alpha}{4t^4}K^2, \end{aligned} \tag{3.4}$$

a set of first-order equations

$$\eta_t = tU_t^2 + t\alpha U_x^2 + \frac{e^{4U}}{4t}(A_t^2 + \alpha A_x^2) + \frac{e^{2\eta}}{4t^3}\alpha K^2, \tag{3.5}$$

$$\eta_x = 2tU_tU_x + \frac{e^{4U}}{2t}A_tA_x - \frac{\alpha_x}{2\alpha}, \tag{3.6}$$

$$\alpha_t = -\frac{e^{2\eta}}{t^3}\alpha^2 K^2, \tag{3.7}$$

plus a set of auxiliary equations

$$G_t = -e^{2\eta}\sqrt{\alpha}\,Kt^{-3}, \quad H_t = e^{2\eta}\sqrt{\alpha}\,AKt^{-3}. \tag{3.8}$$

We have used the short-hand notation $U_t := \partial_t U$, etc. for the partial derivatives. The auxiliary equations come from the definition of the twist constants K_Y and K_Z and from setting $K_Y = 0$.

3.2.2. VTD System

In order to prove AVTD behavior of solutions we must first identify the VTD model metric functions, which the solution is to approach in the singular region. In the language of the Fuchsian theory of Chapter II, we are determining the appropriate/desireable leading order term for the solution. To this end we compute the VTD system as in Section 1.3.4.. With a specified system of coordinates, computing the VTD system from the associated Einstein system reduces to dropping the spatial derivative terms relative to time derivative terms. This means that while in an equation with both space and time derivative terms we drop the spatial ones, these terms are not dropped in an equation in which each term contain spatial derivatives, such as Eq. (3.6).

From Eq. (3.2)-Eq. (3.8), we find the VTD system to be composed of the following equations. We multiply the evolution equations by a power of t equal to the order of the highest time derivative in order to write the equations in Fuchsian form, that is with the derivative operators $D := t\partial_t$, and $D^2 = t^2\partial_t^2 + t\partial_t$. We have the second-order evolution equations

$$
\begin{aligned}
D^2U &= \frac{D\alpha DU}{2\alpha} + \frac{e^{4U}}{2t^2}(DA)^2, && (3.9)\\
D^2A - 2DA &= \frac{D\alpha DA}{2\alpha} - 4DADU, && (3.10)\\
D^2\eta - D\eta &= \frac{D\alpha D\eta}{2\alpha} - (DU)^2 + \frac{e^{4U}}{4t^2}(DA)^2 - \frac{3e^{2\eta}\alpha}{4t^2}K^2, && (3.11)
\end{aligned}
$$

and the first-order evolution equations

$$
\begin{aligned}
D\eta &= (DU)^2 + \frac{e^{4U}}{4}(DA)^2 + \frac{e^{2\eta}}{4t^2}\alpha K^2, && (3.12)\\
D\alpha &= -\frac{e^{2\eta}}{t^2}\alpha^2 K^2, && (3.13)\\
DG &= -e^{2\eta}\sqrt{\alpha}\,Kt^{-2}, && (3.14)\\
DH &= e^{2\eta}\sqrt{\alpha}\,AKt^{-2}. && (3.15)
\end{aligned}
$$

Note that the equations for α, G, H are unchanged except for multiplication by t. We also have the constraint equation, Eq. (3.6) which is unchanged; the asymptotic analysis of this equation is addressed in Section 3.3.5. below.

It can be shown [32, 43] that Eq. (3.9)-Eq. (3.15) are asymptotically satisfied in the limit $t \searrow 0$ by the following expansions

$$\mathring{U}(t,x) = \frac{1}{2}(1-k(x))\log t + U_{**}(x), \tag{3.16}$$

$$\mathring{A}(t,x) = A_*(x) + A_{**}(x)t^{2k(x)}, \tag{3.17}$$

$$\mathring{\eta}(t,x) = \frac{1}{4}(1-k(x))^2\log t + \eta_*, \tag{3.18}$$

$$\mathring{\alpha}(t,x) = \alpha_*(x), \tag{3.19}$$

$$\mathring{G}(t,x) = G_*(x), \tag{3.20}$$

$$\mathring{H}(t,x) = H_*(x), \tag{3.21}$$

provided $k(x) > 3$. The functions of $x \in T^1$, $\{k, U_{**}, A_*, A_{**}, \alpha_*, G_*, H_*\}$ are called asymptotic data functions. The function α_* is expected to be positive definite in order for the asymptotic metric to maintain the appropriate signature. Further conditions on the asymptotic data are imposed in the main theorems below.

In the polarized T^2–symmetric spacetimes the A-field is non-dynamical, corresponding to $A \equiv const$. In terms of the asymptotic data above, we have in this case A_* a constant and $A_{**} \equiv 0$, so that there is no free asymptotic data to choose. The polarized class also determines the singular character of the solution through the asymptotic data function $k(x)$. From the expressions above we see that if an AVTD solution is not polarized, then there is a power-law type blow-up in the A-field if and only if $k(x) < 0$. For polarized AVTD solutions on the other hand, there is no power-law type blow-up for any sign of $k(x)$.

Note that since we expect locally Kasner-like behavior, the logarithmic terms for U and η are consistent with those fields appearing in exponentials in the form of the metric. The function $k(x)$ determines the Kasner exponents p_1, p_2, p_3 of the VTD metric at each spatial point

$$p_1 = (k^2-1)/(k^2+3), \quad p_2 = 2(1-k)/(k^2+3), \quad p_3 = 2(1+k)/(k^2+3).$$

These are computed as the eigenvalues of $(trk)^{-1}k$, where k is the second fundamental form expressed in an orthonormal frame.

3.2.3. Half-Polarized T^2-Symmetric Solutions

While the polarization condition corresponding to a non-dynamical A-field is a geometric condition, relating to the structure of the two symmetry generators Y and Z, the half-polarized T^2-symmetric solution which we now introduce refers to a restriction on the space of asymptotic data. We note that a fully general T^2-symmetric solution which is AVTD (if such exist) has free asymptotic data functions A_* and A_{**}. On the other hand the polarized T^2-symmetric solution which is AVTD has, as discussed above, no free asymptotic data. This discussion motivates the following definition.

Definition 3.1. *Let g be a T^2-symmetric solution of the Einstein equations which is AVTD and hence has the metric field expansions Eq.* (3.16)*-Eq.* (3.21)*. The solution is called a **half-polarized** T^2**-symmetric solution** if the asymptotic data are such that $\partial_x A_*(x) = 0$, and $A_{**}(x)$ is freely specified.*

This arises as a meaningful class of solutions since in our analysis below we find that the Fuchsian theory of Chapter II cannot be applied unless the condition $\partial_x A_* = 0$ holds. Of course this condition alone does not imply the full polarized class. T^2–symmetric solutions which are AVTD and which have half-polarized asymptotic data have been shown to exist [32]. The term "half-polarized" to label solutions with "half" the number of freely-specifiable asymptotic data functions first appears in a discussion of $U(1)$-symmetric solutions with AVTD behavior, in [45].

While we show in the following sections that there exists families of AVTD half-polarized and polarized T^2-symmetric solutions, fully general solutions are not expected to be AVTD. To the extent that it makes sense to speak of the data functions A_* and A_{**} in this general context, such functions would be non-vanishing and non-constant. Numerical studies and heuristic considerations indicate however that fully general T^2–symmetric solutions exhibit Mixmaster-like BKL behavior at the $t = 0$ singularity, or behavior which is dominated by spikes –see the discussion and references in Section 1.3.3..

3.3. AVTD Solutions to the T^2-Symmetric Einstein Equations: The Finite Regularity Case

The first result is an application of the fundamental Fuchsian theorem, Theorem 2.10 to the polarized and half-polarized T^2-symmetric Einstein equations.

3.3.1. Statement of Theorem

It is useful to specify the appropriate set of asymptotic data.

Definition 3.2. *Let $\mathcal{K}^q$ denote the set of asymptotic data $\{k, U_{**}, A_*, A_{**}, \alpha_*, G_*, H_*\}$ such that A_* is a constant, $k, U_{**}, \alpha_* \in H^{q+2}(T^1)$ (with $\alpha_*(x) > 0$), $A_{**} \in H^{q+1}(T^1)$ and $G_*, H_* \in H^q(T^1)$ for any $q \geq 3$[1], and which satisfy the integrability condition*

$$\int_0^{2\pi} \Big((1 - k(x)) U'_{**}(x) - \frac{1}{2} (\log \alpha_*)'(x) \Big) dx = 0,$$

[1]The inequality $q \geq 3$ comes from the condition $q > n/2 + 2$ in Theorem 2.10.

together with, at each point $x \in T^1$, either

1. $k(x) > 1 + \sqrt{6}$ *for arbitrary* A_{**} *(the* ***half-polarized case****),*

2. $k(x) > 1 + \sqrt{6}$ *or* $k(x) < 1 - \sqrt{6}$ *for* $A_{**} \equiv 0$ *(the* ***polarized case****).*

The integrability condition above arises in demanding that the constraint equation Eq. (3.6) hold in the limit $t \searrow 0$, as well as in considering the closed spatial topology. The derivation of this constraint, and more details are found in Section 3.3.5.. The inequalities on k become clear in the proof of Theorem 3.3 below; they are required in order to make the conditions on the exponent vector μ consistent. The different regularity conditions on the asymptotic data functions arise depending on which parts of the equation each function plays a role, and on the particular non-linear terms in the Einstein equations. In the definition above, as well as below we use the notation $U'_{**}(x), (\log \alpha_*(x))'$, etc., to denote the derivatives of functions which only depend on the spatial variable x.

Our first existence and uniqueness theorem for the T^2–symmetric vacuum solutions is as follows.

Theorem 3.3 (AVTD (half)-polarized T^2–symmetric vacuum solutions: finite differentiability case)**.** *For any twist constant $K \in \mathbb{R}$, constant $\eta_0 \in \mathbb{R}$, and asymptotic data in $\mathcal{K}^q$ there exists a $\delta > 0$, and a T^2–symmetric solution $(U, A, \eta, \alpha, G, H)$ of Einstein's vacuum field equations with twist K of the form*

$$(U, A, \eta, \alpha, G, H) = (\mathring{U}, \mathring{A}, \mathring{\eta}, \mathring{\alpha}, \mathring{G}, H) + W,$$

with leading-order term $(\mathring{U}, \mathring{A}, \mathring{\eta}, \mathring{\alpha}, \mathring{G}, \mathring{H})$ given by Eqs. (3.16)–(3.21)*, with*

$$\eta_*(x) := \eta_0 + \int_0^x \left((1 - k(y))U'_{**}(y) - \frac{1}{2}(\log \alpha_*)'(y) \right) dy, \qquad (3.22)$$

and remainder $W \in X_{\delta,\mu,q}$ *(and* $DW \in X_{\delta,\mu,q-1}$*) for any exponent vector* $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6)$ *satisfying*

$$\begin{aligned}
1 &< \mu_1(x) < \min\{2, \frac{1}{2}(k(x)-3)(k(x)+1)\}, \\
\frac{1}{2}\left(2k(x) + \sqrt{1+4k(x)^2}\right) &< \mu_2(x) < 1 + 2k(x), \\
0 &< \mu_3(x) < \mu_1(x), \\
0 &< \mu_4(x), \mu_5(x) < \frac{1}{2}(k(x)-3)(k(x)+1), \\
0 &< \mu_6(x) < \frac{1}{2}(k(x)+3)(k(x)-1).
\end{aligned} \tag{3.23}$$

This solution is unique among all solutions with the same leading-order term and with remainder $W \in X_{\delta,\mu,q}$*.*

This result does not imply uniqueness of the solutions within the largest function space of interest. For a given choice of asymptotic data in $\mathcal{K}^q$, the ideal result would establish the existence of a unique remainder $W \in X_{\delta,\mu,q}$ with a lower bound for μ given by the exponent of the corresponding leading order term, that is

$$\mu_1, \mu_2 - 2k, \mu_3, \mu_4, \mu_5, \mu_6 > 0.$$

However, Theorem 3.3 requires a stricter lower bound on μ which is specified in Eq. (3.23). It follows then that for given choice of asymptotic data in $\mathcal{K}^q$ there may exist solutions of the asymptotic value problem about $(\mathring{U}, \mathring{A}, \mathring{\eta}, \mathring{\alpha}, \mathring{G}, \mathring{H})$ with remainder in the larger space $X_{\delta,\tilde{\mu},q}$, for $\tilde{\mu}$ such that $0 < \tilde{\mu}_1 < 1$ and $2k < \tilde{\mu}_2 < \frac{1}{2}\left(2k(x) + \sqrt{1+4k(x)^2}\right)$. We discuss this issue further below, and prove a result, Theorem 3.10, which establishes uniqueness in the largest space compatible with the leading order expressions (i.e. a space with exponent vector μ with the ideal lower bounds above).

In proving Theorem 3.3, it is useful to first focus on a subset of the T^2–symmetric Einstein vacuum equations. Inspecting, Eqs. (3.2)-(3.8) we see that the equations Eqs. (3.2), (3.3), (3.5) and (3.7) together form a coupled evolution system for the variables U, A, η, and α, while Eq. (3.6) serves as a constraint equation for this system. We call this system the **main evolution equations**. We treat the second-order evolution equation for η as a constraint on the main evolution equations, although it plays an insignificant role. The remaining evolution equations for G and H, Eqs. (3.8), can be integrated after we have dealt with the main evolution system.

In the next few sections we focus on the main evolution equations. The central task is to formulate a first-order symmetric hyperbolic system, and verify that, for our choice of leading order term, this system is Fuchsian in the sense of Definition 2.7. The leading order term is chosen to be consistent with the VTD expansions Eq. (3.16)-Eq. (3.21) so that the solutions we eventually obtain are AVTD. To obtain solutions we seek to apply Theorem 2.10. The remaining hypotheses of Theorem 2.10 are checked in Section 3.3.4., and we formulate our central result for the main evolution equations in Proposition 3.5. In Section 3.3.5. we return to the remaining Einstein equations and show that given a solution to the first-order main evolution system, we obtain a solution to the full Einstein system as in Theorem 3.3.

3.3.2. Main Evolution Equations.

In order to apply the Fuchsian theory of Chapter II we formulate the main evolution equations as a first-order symmetric hyperbolic system of the form

$$S^0(u)Du + tS^a(u)\partial_a u + f(t,x,u) = 0$$

as in Eq. (2.1) for some first-order fields u. To obtain a symmetric hyperbolic system it is necessary to define a new field from the spatial derivative of α. As is done in [43], we set

$$\beta := \partial_x \alpha. \tag{3.24}$$

The evolution equation for β may be obtained by taking the spatial derivative of Eq. (3.7) and by using the constraint Eq. (3.6) to eliminate η_x; we find

$$\beta_t = -\frac{e^{2\eta}}{t^4}\alpha K^2 \left(t\beta + \alpha\left(e^{4U}A_x A_t + 4t^2 U_x U_t\right)\right).$$

The first derivatives of α which appear in other evolution equations are now replaced using Eq. (3.7) for α_t and β for α_x.

We also introduce at this stage a redefinition of the variables U and η which is performed in [3]. In that paper we define

$$\widehat{U} := U - \frac{1}{2}(1-k(x))\log t \quad \text{and} \quad \widehat{\eta} := \eta - \frac{1}{4}(1-k)^2\log t.$$

The reasons for these two seemingly similar redefinitions is actually different. The variable $\widehat{U}$ is introduced because in the Fuchsian theory of [3] it is desirable to have an "ODE leading order term." This means that the leading order term for the first-order system u_0 should satisfy

$$L_{\mathrm{ODE}}(u_0)[u_0] := Du_0 + \mathcal{N}(u_0)u_0 = 0.$$

One must balance this condition on u_0 with the desire that the first order leading order term, which is derived from the VTD expansions Eqs. (3.16)–(3.21) above, be compatible with the first-order field definitions. For a second-order field $u(t,x)$, the first-order fields in a Fuchsian equation are typically defined by $u_1 = u, u_2 = Du, u_3 = t\partial_x u$. In Section 2.4.4 of [3] we show that for an equation of the type which U satisfies (a non-linear Euler-Poisson-Darboux equation), the ODE-leading order term condition is incompatible with the first-order field definitions. This issue is rectified by working with the new field $\widehat{U}$ obtained by subtracting off the $\log t$-term.

The field $\widehat{\eta}$ is introduced more as convenience. Note that the logarithmic term $1/4(1-k)^2 \log t$ leads to an $O(1)$ term under the action of the $D = t\partial_t$ operator. In Eq. (3.12) this term cancels with the $O(1)$ contribution from $(DU)^2$ term on the right hand side. By defining $\widehat{\eta}$ as we do, and in light of the redefinition of U, this cancelation occurs at this stage.

The Fuchsian theory in [3] has since been improved. Although neither of the field definitions $\widehat{U}$ or $\widehat{\eta}$ is necessary for our current formulation of the Fuchsian theory, we keep them here in order to avoid redoing the analysis completely. The full set of first-order fields are defined as follows

$$u_1 = \widehat{U}, \qquad u_2 = D\widehat{U}, \qquad u_3 = t\partial_x\widehat{U}, \tag{3.25}$$

$$u_4 = A, \qquad u_5 = DA, \qquad u_6 = t\partial_x A, \tag{3.26}$$

$$u_7 = \widehat{\eta}, \qquad u_8 = \alpha, \qquad u_9 = \beta. \tag{3.27}$$

The main evolution system Eqs. (3.2), (3.3), (3.5) and (3.7) can now be written as a first-order symmetric hyperbolic system as in Eq. (2.1). We find

$$Du_1 = u_2, \tag{3.28}$$

$$\begin{aligned} Du_2 - u_8 t\partial_x u_3 = &\frac{1}{2}tu_9(u_3 - \frac{1}{2}t\log tk') + \frac{1}{2}e^{4u_1}t^{-2k}\left(u_5^2 - u_8 u_6^2\right) \\ &- \frac{1}{4}e^{2u_7}t^{1/2(1-k)^2-2}u_8K^2(1-k+2u_2) \\ &- \frac{1}{2}t^2\log tk'' u_8, \end{aligned} \tag{3.29}$$

$$u_8 Du_3 - u_8 t\partial_x u_2 = u_8 u_3, \tag{3.30}$$

$$Du_4 = u_5, \tag{3.31}$$

$$\begin{aligned} Du_5 - u_8 t\partial_x u_6 = &2ku_5 - 4u_5u_2 + \frac{1}{2}tu_9u_6 + 2u_8u_6(2u_3 - t\log tk') \\ &- \frac{1}{2}e^{2u_7}t^{1/2(1-k)^2-2}u_8u_5K^2, \end{aligned} \tag{3.32}$$

$$u_8 Du_6 - u_8 t\partial_x u_5 = u_8 u_6, \tag{3.33}$$

$$\begin{aligned} Du_7 = &(1-k)u_2 + u_2^2 + \frac{1}{4}u_8(2u_3 - t\log tk')^2 \\ &+ \frac{1}{4}t^{-2k}e^{4u_1}\left(u_5^2 + u_8u_6^2\right) + \frac{1}{4}e^{2u_7}t^{1/2(1-k)^2-2}u_8K^2, \end{aligned} \tag{3.34}$$

$$Du_8 = -e^{2u_7}t^{1/2(1-k)^2-2}u_8^2K^2, \tag{3.35}$$

$$\begin{aligned} Du_9 = &-e^{2u_7}t^{1/2(1-k)^2-2}u_8K^2 \\ &\cdot\Big(\frac{(1-k+2u_2)(2u_3 - t\log tk')u_8}{t} + t^{-1-2k}u_5u_6u_8e^{4u1} + u_9\Big), \end{aligned} \tag{3.36}$$

The coefficient matrices are

$$S^0(u) = \mathrm{Diag}(1, 1, u_8, 1, 1, u_8, 1, 1, 1), \tag{3.37}$$

and

$$S^1(u) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -u_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -u_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -u_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -u_8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.38}$$

while the source term $f(t, x, u)$ is given by the negative of the right-hand-side of Eqs. (3.28)–(3.36).

Note that the system has a block-diagonal structure, and the first two blocks have a structure typical of second-order equations reduced to first-order by field definitions of the type Eq. (3.25). To make these blocks symmetric we have multiplied the third and sixth equations by u_8.

The function $k(x)$ that appears in the components of $f(t, x, u)$ as a result of the definitions of the variables $u_1 = \widehat{U}$, and $u_7 = \widehat{\eta}$ is, at this stage, an arbitrary function.

Finally, we note that the corresponding equations in [3] included terms Nu for a matrix $N(u)$. As is discussed in Section 2.2. we no longer partition the equations in this way at this stage in order to simplify the presentation; the system Eqs. (3.28)–(3.36) is however, equivalent to that in [3] .

3.3.3. Evolution Equations as a QSHF System

In this section we prove that for a specified leading order term $\mathring{u}$, the first-order system corresponding to the main evolution equations, which is introduced in the previous section, in fact forms a quasilinear symmetric hyperbolic Fuchsian system as in Definition 2.7. A leading order term for the first-order fields can be derived from the VTD expansions Eqs. (3.16)–(3.19) using the definitions of the first-order fields. We find

$$\begin{aligned}\mathring{u} =&(\mathring{u}_1, \mathring{u}_2, \mathring{u}_3, \mathring{u}_4, \mathring{u}_5, \mathring{u}_6, \mathring{u}_7, \mathring{u}_8, \mathring{u}_9)\\ =&\left(U_{**}, 0, tU'_{**}, A_* + A_{**}t^{2k}, 2kA_{**}t^{2k}, 0, \eta_*, \alpha_*, \xi_*\right).\end{aligned} \tag{3.39}$$

The choice $\mathring{u}_6 = 0$ may seem incorrect since from the definition of u_6 and the VTD expansion for the A-field we compute

$$t\partial_x \mathring{A}(t,x) = t^{1+2k(x)}\left(A'_{**}(x) + 2A_{**}(x)k'(x)\log t\right),$$

assuming A_* is independent of x as in $\mathcal{K}^q$. We note however that since in the half-polarized case[2] $k(x)$ is positive definite, this leading order term vanishes as $t \searrow 0$. Our choice above is therefore consistent with this computation, and in addition simplifies the analysis. Although not needed for the analysis here, we note that this choice of $\mathring{u}$ is an ODE-leading-order term.

To check that we have a quasilinear symmetric hyperbolic Fuchsian system, we also specify an exponent vector μ. Considering the block-diagonal conditions

[2]In the polarized case A is not a dynamical field, rendering this discussion mute.

Definition 2.9, we choose

$$\mu = (\mu_1, \mu_1, \mu_1, \mu_2, \mu_2, \mu_2, \mu_3, \mu_4, \mu_4). \tag{3.40}$$

As per the Fuchsian method we make the ansatz that the solution takes the form $u = \mathring{u} + w$, where w is a remainder controlled in $X_{\delta,\mu,q}$. The precise control which we may obtain on w is determined by the leading order term $\mathring{u}$ and the equation. From the VTD expansions we hope to obtain control in spaces with μ given by

$$\mu_1, \mu_3, \mu_4 > 0, \quad \text{and} \qquad \mu_2 > 2k.$$

However, as the lemma below shows, we can not guarantee at this stage that w is in so large a space. The first step in the Fuchsian analysis is to verify that for the given choice of leading order term one obtains, possibly under certain conditions, a quasilinear symmetric hyperbolic Fuchsian system.

Lemma 3.4. *Choose any asymptotic data functions $\{k, U_{**}, A_*, A_{**}, \alpha_*\}$ such that A_* is constant, α_* is positive definite, α_* and η_* are functions in $H^q(T^1)$, A_{**} is contained in $H^{q+1}(T_1)$, k and U_{**} are elements of $H^{q+2}(T^1)$, and k satisfies*

$$k(x) > 1 + \sqrt{5}, \ \textit{or}, \ k(x) < 1 - \sqrt{5}$$

Then there exists sufficiently small $\delta, s > 0$ such that for any $q \geq 3$ the symmetric hyperbolic system formed by Eqs. (3.28)–(3.36) *forms a quasilinear symmetric hyperbolic Fuchsian system as in Definition 2.7 about $\mathring{u}$ given by Eq.* (3.39) *and with*

μ as in Eq. (3.40), satisfying at each point $x \in T^1$

$$\max\{0, 1-(k(x)-3)(k(x)+1)/2\} < \mu_1(x) < \min\{2, (k(x)-3)(k(x)+1)/2\},$$
$$2k(x) < \mu_2(x) < \min\{1+2k(x), \mu_1(x)+2k(x)\},$$
$$0 < \mu_3(x) < \mu_1(x),$$
$$0 < \mu_4(x) < \min\{(k(x)-3)(k(x)+1)/2, \mu_1(x)-1+(k(x)-3)(k(x)+1)/2\}.$$

Moreover, we find the matrices

$$S_0^0 = Diag(1, 1, \alpha_*, 1, 1, \alpha_*, 1, 1, 1), \tag{3.41}$$

and

$$N_0 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.42}$$

Proof. Step 1: By construction the matrices S_0^0, S^0 and S^1 are all symmetric. Further, provided $\alpha_* > 0$ at all $x \in T^1$, there exists a $\delta >$ and $s > 0$ for which both S_0^0, and S^0 are positive definite.

Step 2: From the form of Eq. (3.37) above, and expansion for $u_8 = \alpha$ it is clear that

$$S_0^0 = \text{Diag}(1, 1, \alpha_*, 1, 1, \alpha_*, 1, 1, 1), \quad \text{and} \quad S_1^0(w) = \text{Diag}(0, 0, w_8, 0, 0, w_8, 0, 0, 0).$$

As $S_1^0(w)$ is linear in w it is clearly bounded and satisfies the desired Lipschitz property. Further, $S_1^0(w) \in X_{\delta,\zeta,q}$ for any $0 < \zeta < \mu_4$.

Similarly, we find from Eq. (3.38) that $tS_1^a(w)$ is linear in w, and $tS_1^a(w) \in X_{\delta,\zeta,q}$ for any $0 < \zeta < 1$.

Step 3: Lastly we check Condition (iii) of Definition 2.7. To decompose $f(t, x, u)$ into $f_1(t, x, w)$ and $N_0 w$ we insert the Fuchsian expansion ansatz $u = \mathring{u} + w$ in the expression for $f(t, x, u)$ above in Eqs. (3.28)-(3.36). Inspecting these expressions it is clear that terms of only a few different types are present. There are terms of the form $\prod_{i=1}^{d} u_i^{p_i}$ for some positive integers p_i, there are terms of the form e^{ru_i} for some positive real number r, and component u_i, and there are products of these two types of terms, as well as such terms being multiplied by functions of space and time. Lemma C.3 shows that any such term $\psi(u)$ can be expanded in the form

$$\psi(u) = \psi_0(t, x) + \psi_1(t, x)w + \psi_2(t, x, w),$$

and that it is a well-defined function operator. We then form $N_0 w$ by considering all terms linear in w with an $O(1)$ coefficient, and which respect the desired block-diagonal structure. Although N_0 need not have the same block-diagonal structure as μ at this stage (ie in order for the system to be a quasilinear symmetric hyperbolic Fuchsian system), we construct it to be consistent with this structure for later convenience. An example is the term $(1-k)u_2$ which appears in the evolution equation for u_7, Eq. (3.34). While this term generates a linear term in w_2 with an order one coefficient, it breaks the block-diagonal structure of N_0. The terms not selected for N_0 (the Fuchsian principle part) go into $f_1(t, x, w)$.

The source operator for the reduced equation, $\mathcal{F}(u_0)[w]$, is then obtained by subtracting off $\sum_{j=0}^{n} tS^j(w)\partial_j u_0$, as in Eq. (2.5). The objective of this step is to remove terms which are unbounded in the limit $t \searrow 0$ from the equation.

Our next goal is to verify that $\mathcal{F}(u_0)[w] \in X_{\delta,\nu,q}$ for some $\nu > \mu$. Similar to in the Fuchsian reduction step above the terms in each component of $\mathcal{F}(u_0)[w]$ are of three types: I) Terms which are independent of the w fields and depend only on asymptotic data functions and (t, x). II) Products of w-fields multiplied by some function of (t, x), such as a combination of asymptotic data functions. III) Terms of type (II) multiplied by a factor e^{rw_i} for some real number r and component w_i. The discussion and lemmas in Appendix C show that for the asymptotic data with the indicated regularity, each such term is a bounded operator on $X_{\delta,\mu,q}$ with target $X_{\delta,\nu,q}$ for some $\nu > \mu$ which satisfies the requisite Lipschitz estimates.

The constraint on the asymptotic data $\partial_x A_* = 0$, which defines the half-polarized class arises in this step. In analyzing $\mathcal{F}(u_0)[w]$ we find terms which blow-up as some function of $k(x)$ in the limit $t \searrow 0$, and which are proportional to $\partial_x A_*$ –e.g. $\partial_x A_* t^{-2k}$. Since these terms violate the condition that $\mathcal{F}(u_0)[w]$ is contained in $X_{\delta,\nu,q}$ for some $\nu > \mu$ (recall $\mu > 0$), we eliminate them by restricting our asymptotic data to the set with $\partial_x A_* = 0$.

Due to the large number of terms in the expression for $\mathcal{F}(u_0)[w]$ (this is especially large after the Fuchsian ansatz $u = u_0 + w$ has been implemented) this analysis is performed with the aid of a computer program. This program is written in Mathematica. □

3.3.4. AVTD Solutions of the Main Evolution System.

We now show as an application of Theorem 2.10, and as a step towards proving Theorem 3.3, that there exists a unique solution the first-order main evolution system with leading order term given by Eq. (3.39). We state this result formally in the following proposition.

Proposition 3.5. *For any twist constant $K \in \mathbb{R}$, for any Sobolev differentiability index $q \geq 3$, and for any choice of the asymptotic data functions such that A_* is an arbitrary constant, $\alpha_*(x) > 0$, $k, U_{**}, \alpha_* \in H^{q+2}(T^1)$, $A_{**} \in H^{q+1}(T^1)$ and $\eta_* \in H^q(T^1)$, and k satisfies (at each $x \in T^1$) either*

1. *$k(x) > 1 + \sqrt{6}$ (for arbitrary A_{**} the half-polarized case),*

2. *$k(x) > 1 + \sqrt{6}$ or $k(x) < 1 - \sqrt{6}$ (for $A_{**} \equiv 0$ the polarized case),*

there exists a $\delta_1 \in (0, \delta]$, and a unique solution of the first-order main evolution system Eqs. (3.28)–(3.36) with leading-order term $\mathring{u}$ and remainder $w \in X_{\delta_1,\mu,q}$ (and $Dw \in X_{\delta_1,\mu,q-1}$) so long as the exponent vector μ given by Eq. (3.40) satisfies the following inequalities at all $x \in T^1$:

$$
\begin{gathered}
1 < \mu_1(x) < \min\{2, (k(x)-3)(k(x)+1)/2\}, \\
\frac{1}{2}\left(2k(x) + \sqrt{1 + 4k(x)^2}\right) < \mu_2(x) < 1 + 2k(x), \\
0 < \mu_3(x) < \mu_1(x), \\
0 < \mu_4(x) < \frac{1}{2}(k(x)-3)(k(x)+1).
\end{gathered}
$$

While in Proposition 3.5 and elsewhere in this section we have written the results for the polarized and half-polarized solutions together, it should be clear that in these statements any mention of μ_2, the A-field, and the corresponding first order fields w_4, w_5 and w_6 only pertains to the half-polarized case.

As noted above, this proposition is an application of Theorem 2.10 to the quasilinear symmetric hyperbolic system established in Lemma 3.4. The following lemma regarding the positivity of the energy dissipation matrix is an essential part of the proof of Proposition 3.5.

Lemma 3.6. *The energy dissipation matrix M_0 defined in Eq.* (2.7) *corresponding to the quasilinear symmetric hyperbolic system of Lemma 3.4 is positive definite at every x, provided that*

$$\begin{aligned}
&\alpha_*(x) > 0, \quad \mu_1(x) > 1,\\
&\mu_2(x) > \max\left\{1, k(x) + \frac{1}{2}\sqrt{1+4k(x)^2}\right\}, \quad \mu_3(x), \mu_4(x) > 0,
\end{aligned}$$

holds for all $x \in T^1$.

Proof. We compute from the definition

$$M_0 = \begin{pmatrix}
\mu_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \alpha_*(\mu_1-1) & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \mu_2 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \mu_2-2k & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \alpha_*(\mu_2-1) & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \mu_3 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4
\end{pmatrix}.$$

This matrix is guaranteed to be positive definite if the eigenvalues of $sym(M_0) = 1/2\left(M_0 + M_0^T\right)$ are positive definite; this property holds if the inequalities above are satisfied. □

Proof of Proposition 3.5. We note that the quasilinear symmetric hyperbolic Fuchsian system formed by Eqs. (3.28)–(3.36) and the leading-order term Eq. (3.39) is block-diagonal with respect to the exponent vector Eq. (3.40). Thus to apply Theorem 2.10 and complete the proof of Proposition 3.5 it remains to verify that the hypotheses of Lemma 3.4 and Lemma 3.6 can be satisfied simultaneously, and that the matrices S_0^0 and N_0 are contained in $H^{q_0}(T^1)$ for $q_0 > 1/2 + 1 + q$ (since $n = 1$). This later condition is satisfied provided α_* and k (the asymptotic data appearing in these matrices) are contained in $H^{q+2}(T^1)$.

The hypotheses of Lemma 3.4 and Lemma 3.6 can be satisfied simultaneously only if $k(x) > 1 + \sqrt{6}$ in the half-polarized case, and either $k(x) > 1 + \sqrt{6}$ or $k(x) < 1 - \sqrt{6}$ in the polarized case. In particular, the constraint $k(x) > 1 + \sqrt{6}$ comes from combining the inequalities on μ_1.

This establishes that the hypotheses of Theorem 2.10 are satisfied. An application of this theorem completes the proof of Proposition 3.5. □

3.3.5. The Full Set of Einstein's Vacuum Field Equations.

Thus far, using Theorem 2.10, we have constructed solutions u of the first-order main evolution system. In this section we show that under additional restrictions on the asymptotic data: I) The first-order fields given by the solutions to the first-order main evolution system in fact correspond to second-order fields, that is the first-order field definitions propagate. II) Given the solutions to the main evolution system, the Einstein equations, Eq. (3.4) and Eq. (3.6), which we treat as constraints, as well as the definition of the β-field Eq. (3.24) are satisfied asymptotically and propagated by the evolution. Provided these first two points can be shown, it follows that the solutions to the first-order main evolution system in fact give solutions to the Einstein equations Eqs. (3.2), (3.3), (3.5), and (3.7) by reversing the first-order field definitions. III) The auxiliary equations for G and H (Eqs. (3.8)) can be integrated. We state these results formally in the following proposition.

Proposition 3.7. *For any solution of Proposition 3.5 with asymptotic data in* $\mathcal{K}^q$*, as in Definition 3.2, the full set of Einstein's vaccum field equations Eqs.* (3.2) *–* *Eq.* (3.7) *are satisfied, and Eqs.* (3.8) *can be integrated for* G *and* H*.*

Proof. We start by showing that the solution to the first-order main evolution system corresponds to a solution of the original second-order Einstein equations. Define the constraint violation quantities

$$\mathcal{C}_1(u) := u_2/t - \partial_t u_1, \qquad \mathcal{C}_2(u) := u_3/t - \partial_x u_1, \tag{3.43}$$

$$\mathcal{C}_3(u) := u_5/t - \partial_t u_4, \qquad \mathcal{C}_4(u) := u_6/t - \partial_x u_4. \tag{3.44}$$

The propagation of $\mathcal{C}_1(u)$ and $\mathcal{C}_3(u)$ follows directly from Eq. (3.28) and Eq. (3.31) respectively, since t times these constraint violation quantities is equal to the indicated equations. For $\mathcal{C}_2(u)$ and $\mathcal{C}_4(u)$ we use Eq. (3.28),Eq. (3.30), Eq. (3.31), and Eq. (3.33) to derive the expressions $D\mathcal{C}_2(u) = 0$, and $D\mathcal{C}_4(u) = 0$. Then, since the form of the leading order term for the main system implies that $C_2(u), C_4(u)$ must asymptotically vanish, it follows that each must vanish for all time. The results of these analyses are that we may make appropriate substitutions of the fields in the analysis below, e.g. we may replace instances of u_2 by $t\partial_t u_1$. This is also a step towards proving that the second-order Einstein equations for U and for A are satisfied.

To complete the argument that the remaining Einstein equation are satisfied, we consider the following constraint violation quantities. From Eq. (3.4), we define

$$\begin{aligned}\mathcal{E} :=& -\eta_{tt} + \alpha\eta_{xx} + \frac{\alpha_x \eta_x}{2} + \frac{\alpha_t \eta_t}{2\alpha} - \frac{\alpha_x^2}{4\alpha} + \frac{\alpha_{xx}}{2} \\ & - U_t^2 + \alpha U_x^2 + \frac{e^{4U}}{4t^2}\left(A_t^2 - \alpha A_x^2\right) - \frac{3e^{2\eta}\alpha}{4t^4}K^2\end{aligned} \tag{3.45}$$

while, from the constraint Eq. (3.6), we have

$$\mathcal{Q}_1 := -\eta_x + 2tU_tU_x + \frac{e^{4U}}{2t}A_tA_x - \frac{\alpha_x}{2\alpha}. \tag{3.46}$$

Additionally, we must show that the constraint based on the definition of β is satisfied; we define

$$\mathcal{Q}_2 := -\alpha_x + \beta. \tag{3.47}$$

We note that if the constraint violation quantities $\mathcal{E}, \mathcal{Q}_1$, and $\mathcal{Q}_2$ can be shown to vanish identically, then, along with the propagation of $\mathcal{C}_i, i = 1, \ldots, 4$ above, we obtain that a solution u of the main evolution system Eqs. (3.28) – (3.36) corresponds to a solution of the Einstein equations Eqs. (3.2)–(3.7). The following lemma is an important step in establishing this result.

Lemma 3.8. *Let u be a solution of Eqs.* (3.28) – (3.36)*. Then,*

$$\begin{aligned}\mathcal{E} =& \frac{1}{2}(\mathcal{Q}_2 - u_9)\mathcal{Q}_1 - u_8\mathcal{Q}_{1,x} \\ & + \frac{1}{4}\Big(e^{4u_1}t^{-1-2k}u_5u_6 + (k-1-2u_2)(\log tk' - 2t^{-1}u_3)\Big)\mathcal{Q}_2.\end{aligned} \tag{3.48}$$

A consequence of this lemma is that if we can show that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ vanish identically under the hypotheses of Theorem 3.3, then it follows that $\mathcal{E}$ vanishes identically as well. Before treating $\mathcal{Q}_1$ and $\mathcal{Q}_2$ we prove the above lemma.

Proof. The proof consists of substituting the results of the first-order main evolution system in Eq. (3.45), and using the field definitions

$$U = u_1 + \frac{1}{2}(1 - k(x)) \log t, \quad \eta = u_7 + \frac{1}{4}(1 - k(x))^2 \log t.$$

In order to apply our knowledge of the solution u to the first-order main evolution system, we put the constraint violation quantities in terms of these quantities. We find

$$\mathcal{Q}_2 = -u_{8,x} + u_9 \tag{3.49}$$

$$\begin{aligned} \mathcal{Q}_1 &= -u_{7,x} + \frac{u_3}{t}(1 - k + 2u_2) + \frac{1}{2}e^{4u_1}t^{-1-2k}u_5u_6 - k' \log t u_2 - \frac{u_{8,x}}{2u_8} \\ &\equiv -u_{7,x} - \frac{u_{8,x}}{2u_8} + S(u) \end{aligned} \tag{3.50}$$

$$\begin{aligned} \mathcal{E} \equiv & - u_{7,tt} + u_8 u_{7,xx} + T(u) \\ & + \frac{1}{2}u_{8,x}u_{7,x} + \frac{1}{2}u_{8,xx} - \frac{u_{8,x}^2}{4u_8} - \frac{1}{4}k'(1 - k) \log t u_{8,x}, \end{aligned} \tag{3.51}$$

where $S(u)$, and $T(u)$ contain the remaining terms, all of which depend on the first-order fields u, but in particular do not depend on spatial derivatives of u_8 –we write such terms explicitly since they involve the constraint violation quantity $\mathcal{Q}_2$ later in the analysis. Notice that in the expressions for $\mathcal{E}$ and $\mathcal{Q}_1$, the terms independent of u which arise due to the definition of U and η cancel.

We now write the second-derivative terms appearing in the expression for $\mathcal{E}$ as

$$
\begin{aligned}
u_{7,tt} &= -\frac{Du_7}{t^2} + \frac{1}{t}\partial_t Du_7 \\
u_8 u_{7,xx} &= -u_8 \mathcal{Q}_{1,x} - \frac{1}{2}u_{8,xx} + \frac{u_{8,x}^2}{2u_8} + u_8 \partial_x S(u),
\end{aligned}
$$

and we also compute

$$
\frac{1}{2}u_{8,x}u_{7,x} = -\frac{1}{2}u_{8,x}\mathcal{Q}_1 - \frac{u_{8,x}^2}{4u_8} + \frac{1}{2}u_{8,x}S(u).
$$

Inserting these expressions into $\mathcal{E}$ we find

$$
\begin{aligned}
\mathcal{E} = & -u_8 \mathcal{Q}_{1,x} - \frac{1}{2}u_{8,x}\mathcal{Q}_1 - \frac{1}{4}k'(1-k)\log t u_{8,x} + \frac{1}{2}u_{8,x}S(u) \\
& + \frac{Du_7}{t^2} - \frac{1}{t}\partial_t Du_7 + u_8 \partial_x S(u) + T(u).
\end{aligned}
$$

Finally, we use the constraint equation for $u_{8,x}$, from which it follows that

$$
\begin{aligned}
\mathcal{E} = & -\frac{1}{2}(-\mathcal{Q}_2 + u_9)\mathcal{Q}_1 - u_8 \mathcal{Q}_{1,x} - \frac{1}{4}\left(k'(1-k)\log t - 2S(u)\right)\mathcal{Q}_2 \\
& + \frac{Du_7}{t^2} - \frac{1}{t}\partial_t Du_7 - \frac{1}{4}k'(1-k)\log t u_9 + u_8 \partial_x S(u) + \frac{1}{2}u_9 S(u) + T(u).
\end{aligned}
$$

Presuming that u satisfies the first-order evolution system Eqs. (3.28) – (3.36) we find that the terms in the second line cancel, giving the form of $\mathcal{E}$ stated in the lemma. □

We now proceed to show that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ vanish under appropriate conditions on the asymptotic data; these conditions are encoded in the Definition 3.2 for $\mathcal{K}^q$. We start by writing evolution equations for $D\mathcal{Q}_1$ and $D\mathcal{Q}_2$. To this end it is useful to compute the following mixed derivatives. From Eq. (3.34) and Eq. (3.35) we obtain

$$
\begin{aligned}
u_{7,xt} =& \frac{1}{t}\partial_x D u_7 \\
=& -\frac{1}{4}K^2\log(t)t^{\frac{1}{2}(1-k)^2-3}k'e^{2u_7}u_8 + \frac{1}{4}K^2 k\log(t)t^{\frac{1}{2}(1-k)^2-3}k'e^{2u_7}u_8 \\
&+ \frac{1}{2}K^2 t^{\frac{1}{2}(1-k)^2-3}e^{2u_7}u_{7,x}u_8 + \frac{1}{4}K^2 t^{\frac{1}{2}(1-k)^2-3}e^{2u_7}u_{8,x} \\
&- \log(t)k''u_3u_8 - \frac{1}{2}\log(t)t^{-2k-1}k'e^{4u_1}u_5^2 - \frac{1}{2}\log(t)t^{-2k-1}k'e^{4u_1}u_6^2u_8 \\
&- \frac{k'u_2}{t} - \log(t)k'u_{3,x}u_8 - \log(t)k'u_3u_{8,x} + \frac{1}{4}t\log^2(t)k'^2u_{8,x} \\
&+ \frac{1}{2}t\log^2(t)k'k''u_8 + t^{-2k-1}e^{4u_1}u_{1,x}u_5^2 + t^{-2k-1}e^{4u_1}u_{1,x}u_6^2u_8 \\
&+ \frac{1}{2}t^{-2k-1}e^{4u_1}u_5u_{5,x} + \frac{1}{2}t^{-2k-1}e^{4u_1}u_6u_{6,x}u_8 + \frac{1}{4}t^{-2k-1}e^{4u_1}u_6^2u_{8,x} \\
&- \frac{ku_{2,x}}{t} + \frac{2u_2u_{2,x}}{t} + \frac{u_{2,x}}{t} + \frac{2u_3u_{3,x}u_8}{t} + \frac{u_3^2u_{8,x}}{t} \\
u_{8,xt} =& \frac{1}{t}\partial_x D u_8 \\
=& K^2 t^{\frac{1}{2}\left(k^2-2k-5\right)}\left(-e^{2u_7}\right)u_8\left(u_8\left(2u_{7,x} + (k-1)\log(t)k'\right) + 2u_{8,x}\right).
\end{aligned}
$$

By using these expressions for the mixed derivatives, the constraints Eq. (3.50) and Eq. (3.49) to eliminate $u_{7,x}$ and $u_{8,x}$, and finally the evolution system Eqs. (3.28) – (3.36) we obtain

$$
\begin{aligned}
D\mathcal{Q}_1 =& -\frac{1}{2}u_8K^2e^{2u_7}t^{(k-3)(k+1)/2}\mathcal{Q}_1 \\
&+ \left(u_3^2 - t\log t u_3 k' + \frac{1}{4}\left(e^{4u_1}t^{-2k}u_6^2 + t^2(\log t)^2(k')^2\right)\right)\mathcal{Q}_2, \\
D\mathcal{Q}_2 =& -2K^2e^{2u_7}t^{(k-3)(k+1)/2}u_8\left(\mathcal{Q}_2 + 2u_8\mathcal{Q}_1\right).
\end{aligned}
$$

The intermediate expressions are long in length, and so we refrain from writing them out here.

These describe, at each spatial point, a system of linear homogeneous ordinary differential equations for $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on the interval $(0, \delta]$. Due to the leading order terms for the first-order fields u,[3] and the condition $k(x) > 3$ or $k(x) < -1$ at each $x \in T^1$, we observe that the coefficients of this system are well-defined and vanish in the limit $t \searrow 0$. Hence, at each spatial point, this system is Fuchsian *ODE* that can be considered a special case of Definition 2.7. The N_0 matrix for this system is identically zero. Hence if the leading order terms, $\mathring{\mathcal{Q}}_1(x)$ and $\mathring{\mathcal{Q}}_2(x)$ can be chosen to vanish, then the quantities $\mathcal{Q}_1$ and $\mathcal{Q}_2$ vanish for all $(t, x) \in (0, \delta] \times T^1$. Moreover, $\mathcal{Q}_1 = \mathcal{Q}_2 = 0$ is the unique solution to this ODE AVP in a space $X_{\delta,\mu}$, (a space similar to $X_{\delta,\mu,q}$ but without the spatial regularity parameter) for a $\mu > 0$.

The question becomes under what conditions on the asymptotic data functions k, U_{**}, A_*, A_{**}, η_*, α_* and ξ_* can we set $\mathring{\mathcal{Q}}_1(x)$ and $\mathring{\mathcal{Q}}_2(x)$ to zero. By inserting the leading order expressions for u into Eq. (3.50) and Eq. (3.49) and taking the $t \searrow 0$ limit we find

$$\mathring{\mathcal{Q}}_1 = -\eta_*' + (1-k)U_{**}' - \frac{\alpha_*'}{2\alpha_*}, \quad \text{and} \quad \mathring{\mathcal{Q}}_2 = \alpha_*' - \beta_*.$$

It follows that $\mathring{\mathcal{Q}}_2 = 0$ if and only if

$$\beta_* = \alpha_*', \tag{3.52}$$

and $\mathring{\mathcal{Q}}_1 = 0$ if and only if, for an arbitrary constant η_0,

$$\eta_*(x) = \eta_0 + \int_0^x \left((1-k(y))U_{**}'(y) - \frac{1}{2}(\log\alpha_*)'(y)\right) dy. \tag{3.53}$$

[3]Recall that in this section we assume u to be a solution obtained in Proposition 3.5

In particular, due to the closed spatial topology we must choose the asymptotic data k, U_{**} and α_*, as in Definition 3.2, to satisfy

$$\int_0^{2\pi} \left((1 - k(x'))U'_{**}(x') - \frac{1}{2}(\log \alpha_*)'(x') \right) dx' = 0.$$

At this point we have shown that under the hypotheses of Theorem 3.3 a solution u of Proposition 3.5 is a solution of Eqs. (3.2)–(3.7) if and only if the asymptotic data functions satisfy the conditions of Definition 3.2, and in particular Eqs. (3.52) and (3.53).

It remains to treat the Eqs. (3.8) for G and H. Using the known solutions of Eqs. (3.2)–(3.7) to evaluate the right hand sides of Eqs. (3.8), we see that both are $O(t^\xi)$ for a power $\xi > -1$ uniformly in space. It follows that the Eqs. (3.8) may be integrated over $t \in [0, \delta]$ at every spatial point, giving

$$G(t,x) = G_*(x) - \int_0^t e^{2\eta(t',x)} \sqrt{\alpha(t',x)}\, K t'^{-3} dt',$$
$$H(t,x) = H_*(x) + \int_0^t e^{2\eta(t',x)} \sqrt{\alpha(t',x)}\, A(t',x) K t'^{-3} dt'.$$

From the control on α, η, and A which we have established –via control on u, and the above constraints– we observe that the remainder functions $G - G_* \in X_{\delta_1,\mu_5,q}$ and $H - H_* \in X_{\delta_1,\mu_6,q}$ for any of exponents satisfying

$$0 < \mu_5(x) < 1/2(k(x) - 3)(k(x) + 1), \quad 0 < \mu_6(x) < 1/2(k(x) + 3)(k(x) - 1).$$

It is sufficient to take $G_{1*}, G_{2*} \in H^q(T^1)$.

This completes the proof of Theorem 3.3. □

3.4. AVTD Solutions to the T^2-Symmetric Einstein Equations: The Smooth Case

In this section we prove that there exists smooth AVTD T^2-symmetric solutions to the Einstein equations Eqs. (3.2) –(3.7) and Eqs. (3.8). This result improves on Theorem 3.3 in that it establishes uniqueness of the remainder part of the solution, W in a larger function space. On the other hand, in order to prove this theorem we must make stronger regularity assumptions; in fact we go to the smooth case. While it is not necessary to assume smoothness, a similar result could be established with sufficiently high Sobolev regularity, we do so here because this assumption simplifies the analysis and the end result.

To clarify how Theorem 3.10 below achieves the improvement in the uniqueness statement, we briefly discuss the obstacles to such a result which arise in Theorem 3.3. In that theorem we must satisfy a number of conditions which constrain the exponent vector μ. In particular, the block-diagonal conditions imply that we must choose μ as in Eq. (3.40), while for an exponent vector with this structure the positivity of the energy dissipation matrix Lemma 3.6 requires the (non-optimal) lower bounds $\mu_1 > 1$ and $\mu_2 > \frac{1}{2}\left(2k(x) + \sqrt{1 + 4k(x)^2}\right)$. This is the origin of the lower bound "gap" for the exponent vector obtained in Theorem 3.3. The theorem in this section is based on an application of Theorem 2.28 to the main evolution system. Because we not require the system to be block-diagonal for the theory applied in this section, we can in particular choose μ^3 to be different from $\mu^1 = \mu^2 = \mu_1$[4]. This flexibility allows us to choose a μ^3 as to optimize the positivity condition of Theorem 2.28 –see the proof of Proposition 3.11 below. The non-optimal lower bound on μ_2, is improved

[4]Here we have used upper indices to enumerate the component of μ for the nine-dimensional first-order main evolution system, while we have kept the convention of the lower indices to denote components of the six-dimensional exponent vector corresponding to the Einstein metric fields.

in the positivity condition of Theorem 2.28 without needing to alter the form of the exponent vector.

A related issue with Theorem 3.3 is the constraint on the asymptotic velocity $k(x)$, which is stricter than the bounds $k(x) > 3$ in the half-polarized case, and $k(x) > 3$ or $k(x) < -1$ in the polarized case, which are expected based on numerical and heuristic studies. The constraints on $k(x)$ in Theorem 3.3 come from ensuring that the upper bound for μ_1 is in fact larger than the corresponding lower bound. Since in this section we improve the lower bounds on the exponent vector, the constraints on $k(x)$ are also improved to the results expected.

We now proceed to present the set of appropriate asymptotic data for the smooth case, and the main result of this section Theorem 3.10.

Definition 3.9. *Let $\mathcal{K}^\infty$ denote the set of asymptotic data $\{k, U_{**}, A_*, A_{**}, \alpha_*, G_*, H_*\}$ such that A_* is a constant, and $k, U_{**}, \alpha_*, A_{**}, G_*, H_* \in C^\infty(T^1)$ with $\alpha_*(x) > 0$ and which satisfy the integrability condition*

$$\int_0^{2\pi} \Big((1 - k(x)) U'_{**}(x) - \frac{1}{2} (\log \alpha_*)'(x) \Big) dx = 0,$$

together with, at each point $x \in T^1$, either

1. *$k(x) > 3$ for arbitrary A_{**} (the **half-polarized case**),*

2. *$k(x) > 3$ or $k(x) < -1$ for $A_{**} \equiv 0$ (the **polarized case**).*

Theorem 3.10 (Existence of smooth AVTD solutions to the (half)-polarized T^2-symmetric vacuum Einstein equations)**.** *For any twist constant $K \in \mathbb{R}$, constant $\eta_0 \in \mathbb{R}$, and asymptotic data in $\mathcal{K}^\infty$ there exists a $\delta > 0$, and a T^2–symmetric solution*

$(U, A, \eta, \alpha, G, H)$ of Einstein's vacuum field equations with twist K of the form

$$(U, A, \eta, \alpha, G, H) = (\mathring{U}, \mathring{A}, \mathring{\eta}, \mathring{\alpha}, \mathring{G}, H) + W,$$

with leading-order term $(\mathring{U}, \mathring{A}, \mathring{\eta}, \mathring{\alpha}, \mathring{G}, H)$ given by Eqs. (3.16)–(3.21)*, with*

$$\eta_*(x) := \eta_0 + \int_0^x \left((1 - k(y)) U'_{**}(y) - \frac{1}{2} (\log \alpha_*)'(y) \right) dy,$$

and remainder $W \in X_{\delta,\mu,\infty}$ (and $DW \in X_{\delta,\mu,\infty}$) for any exponent vector $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6)$ satisfying

$$\begin{aligned} &0 < \mu_1(x) < \min\{2, 1/2(k(x) - 3)(k(x) + 1)\}, \\ &0 < \mu_2(x) - 2k(x) < \min\{1, \mu_1(x)\}, \\ &0 < \mu_3(x) < \mu_1(x), \\ &0 < \mu_4(x), \mu_5(x) < 1/2(k(x) - 3)(k(x) + 1) \\ &0 < \mu_6(x) < 1/2(k(x) + 3)(k(x) - 1) \end{aligned} \tag{3.54}$$

This solution is unique among all solutions with the same leading-order term and with remainder $W \in X_{\delta,\mu,\infty}$.

The proof of this theorem is similar to that of Theorem 3.3; the most notable difference being that in the proof of existence of solutions to the first-order main evolution system we apply Theorem 2.28 rather than Theorem 2.10. We discuss the proof and other minor differences next.

Proof of Theorem 3.10. As mentioned the heart of this proof is the application of Theorem 2.28 to the first order main evolution system. In order to apply this theorem we first verify that Eqs. (3.28) – (3.36) form a smooth quasilinear symmetric hyperbolic Fuchsian system about the leading order term Eq. (3.39). The main evolution system is the same as in Section 3.3.2., except that now the function $k(x)$ introduced in the definition of the fields $\widehat{U}$ and $\widehat{\eta}$ is smooth on T^1. As a result, the coefficients $S^0(t, x, u), S^a(t, x, u)$ and the function $f(t, x, u)$ are smooth in all arguments.

To show that this system forms a smooth quasilinear symmetric hyperbolic Fuchsian system, we specify the leading order term Eq. (3.39) as before, although now the asymptotic data is taken to be $C^\infty(T^1)$. The functions S_0^0 and N_0 associated to the Fuchsian system have the same structure as in Eq. (3.41) and Eq. (3.42), but are in this case smooth in space. Finally, as above the operators $S_1^0(w), tS^a(w)$ and $\mathcal{F}(u_0)[w]$ can be shown, using results of Section C to be Lipschitz operators for all $q > n/2$. Further, $\mathcal{F}(u_0)[w] \in X_{\delta,\nu,q}$ for some $\nu > \mu$ provided μ is bounded above as in Eq. (3.54); these bounds are similar to those found in the case of finite regularity (cf. Lemma 3.4).

For reasons listed in the discussion above we may choose the exponent vector for this smooth system to be of the form

$$\mu = (\mu_1, \mu_1, \mu_1 + 1 - \epsilon, \mu_2, \mu_2, \mu_2, \mu_3, \mu_4, \mu_4) \tag{3.55}$$

for some $\epsilon > 0$ which may be taken arbitrarily small. This seemingly odd choice of the third component becomes clear in the proof of Proposition 3.11 below. The next proposition shows that there exists smooth solutions to this Fuchsian system.

Proposition 3.11. *For any twist constant $K \in \mathbb{R}$, and smooth asymptotic data functions $\{k, U_{**}, A_*, A_{**}, \alpha_*, G_*, H_*\}$ such that A_* is a constant and k satisfies*

1. $k(x) > 3$, *(in the half-polarized case),*

2. $k(x) > 3$ *or* $k(x) < -1$, *(in the polarized case)*

there exists a $\widehat{\delta} \in (0, \delta]$, *and a unique solution* $u = \mathring{u} + w$ *of the asymptotic value problem of Eqs.* (3.28) – (3.36) *about* $\mathring{u}$ *given by Eq.* (3.39) *with remainder* $w \in X_{\widehat{\delta},\mu,\infty}$, *and* $Dw \in X_{\widehat{\delta},\mu,\infty}$. *The exponent vector Eq.* (3.55) *must satisfy*

$$0 < \mu_1 < \min\{2, 1/2(k-3)(k+1)\},$$

$$0 < \mu_2 - 2k < \min\{1, \mu_1\},$$

$$0 < \mu_3 < \mu_1,$$

$$0 < \mu_4 < 1/2(k-3)(k+1)$$

We prove this proposition below, after we complete the proof of Theorem 3.10.

Now that we have obtained smooth solutions to the first-order main evolution system, it remains to show that there exists smooth solutions to the full Einstein system Eqs. (3.2) –Eq. (3.7) and Eqs. (3.8). The argument proceeds exactly as it does in the finite regularity case detailed in Section 3.3.5., as all of the arguments presented in that section extend to the case when the fields are smooth. We find the same constraints on the asymptotic data, which are now taken to be smooth; this leads to the definition of appropriate data $\mathcal{K}^\infty$. □

We now prove the main step in the proof of Theorem 3.10, Proposition 3.11.

Proof of Proposition 3.11. We verify that the smooth quasilinear symmetric hyperbolic Fuchsian system about the leading order term Eq. (3.39) and with the exponent vector Eq. (3.55) satisfies the hypotheses of Theorem 2.28.

1. Smooth commutator condition: The matrix coefficients must satisfy the conditions outlined in Definition 2.27. We verify from the form of N_0 given by Eq. (3.42), and the diagonality of S^0_0 (Eq. (3.41)) that these coefficient-matrices have the block-diagonal structure of μ given in Eq. (3.55). Similarly $S^0_1(w)$ is diagonal and hence commutes with $\mathcal{R}[\mu]$. Since $S^0_1(w) \in B_{\delta,\zeta,q}(r)$ for $\zeta > 0$, as long as $\mu_4 > 0$ the corresponding condition in Definition 2.27 is satisfied. To prove that the condition on $tS^1(w)$ is satisfied we compute $\mathcal{R}[\mu] \cdot tS^1(w) \cdot \mathcal{R}[-\mu]$. Because of the structure of $tS^1(w)$ (cf. Eq. (3.38)), and the structure of μ in Eq. (3.55), we find this product to be in $B_{\delta,\xi,q}(\tilde{r})$ for some exponent scalar $\xi < \epsilon$. This condition is the reason we can only have $(\mu_1 + 1 - \epsilon)$-control (and not $(\mu_1 + 1)$-control) over the field $t\partial_x U$. Without the positive ϵ the product $\mathcal{R}[\mu] \cdot tS^a(w) \cdot \mathcal{R}[-\mu]$ would only be contained in some $B_{\delta,0,q}(\tilde{r})$.

2. The product compatibility, and the higher-order source conditions: We verify that the product compatibility conditions outlined in Definition 2.25 hold for the operators $S^0_1(\cdot)$ and $tS^1(\cdot)$ given in Eq. (3.37) and Eq. (3.38) respectively. The matrix-valued operator $S^0_1(\cdot)$ is diagonal, and hence

$$\mathcal{R}[\mu]\left(S^0_1(w) - S^0_1(w+h)\right)\mathcal{R}[-\mu] = S^0_1(w) - S^0_1(w+h) = \mathrm{Diag}(0,0,h^8,0,0,h^8,0,0,0)$$

for any $h \in X_{\delta,\hat{\mu},q}$. We consider only $\hat{\mu}$ of the form $\hat{\mu} = \mu + \gamma_0$ for a positive scalar exponent γ_0. Thus, $\hat{\mu}^8 = \mu^8 + \gamma_0 = \mu_4 + \gamma_0$ and we have control in $X_{\delta,\tilde{\zeta}+\hat{\mu}-\mu,q}$ for some positive exponent scalar $\tilde{\zeta}$ as desired, since $\mu_4 > 0$.

For the similar condition on $tS^1(\cdot)$, we consider each of this matrices three blocks separately. The third block is identically zero, so the property holds trivially. For the second block, the 3×3 matrix has the same block-diagonal structure as the relevant portion of the exponent vector μ (namely (μ_2, μ_2, μ_2)), and hence commutes with $\mathcal{R}[\mu]^{(2)}$. The same argument that is used above for the $S_1^0(\cdot)$ operator can be applied to this block. For the first block, we find due to the structure of μ (c.f. Eq. (3.55))

$$\mathcal{R}[\mu]^{(1)} \left(tS^1 w - tS^1 w + h\right)^{(1)} \mathcal{R}[-\mu]^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -h^8 t^{2-\epsilon} \\ 0 & -h^8 t^{\epsilon} & 0 \end{pmatrix}.$$

Again, the same argument as before can be used to show that we have appropriate control on this quantity for $\hat{\mu} = \mu + \gamma_0$.

The additional properties needed of the source operator $\mathcal{F}(u_0)[\cdot]$, which are specified in Definition 2.26 are shown in Section C.3.3. to hold for the types of function operators present in this application.

3. Positivity condition: Finally we verify Condition (iii) of Theorem 2.28. From the expressions for N_0 and S_0^0 (Eq. (3.42) and Eq. (3.41) respectively) we compute the block-diagonal matrix $\mathcal{N} = (S_0^0)^{-1} N_0$, with blocks

$$\mathcal{N}^{(1)} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathcal{N}^{(2)} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -2k & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and $\mathcal{N}^{(3)} = 0_{3\times 3}$. Since the matrices are upper triangular we can read the eigenvalues off the diagonal. Let λ denote the $\mathbb{R}^9$-vector of eigenvalues. The condition $\mu > -\Re\lambda$ then gives

$$\mu_1 > \epsilon \quad \mu_2 > \max\{0, 1 - 2k\} = 0 \quad \mu_3, \mu_4 > 0.$$

Since $\epsilon > 0$ can be taken arbitrarily small, we take $\mu_1 > 0$, and note that for any such μ_1 we can find an $\epsilon > 0$ such that the conditions discussed above are satisfied.

This verifies the hypotheses of Theorem 2.28; the proposition follows as a direct application of this theorem. □

Note that in [3] we needed u_0 to be an ODE leading order term in order to apply the smooth Fuchsian theorem (an earlier version of Theorem 2.28) in that paper. This condition is still reflected in the leading order term we choose here, given by Eq. (3.39), although the fact that it is ODE is no longer a necessary hypothesis of Theorem 2.28.

CHAPTER IV

AVTD GOWDY SOLUTIONS IN WAVE GAUGES

The work presented in this chapter is unpublished; all calculations were performed by E. Ames with the guidance of F. Beyer and J. Isenberg.

4.1. Prelude

In this chapter we establish the existence of a family of AVTD $\mathbb{T}^3$ Gowdy solutions in a family of so-called wave gauges. This family of gauge choices has been used in proving many important results in mathematical relativity, most notably the local existence of solutions to Einstein equations by Yvonne Choquet-Bruhat [35] (see also [23] for a global existence result, [36, 75] for general expositions, and [52, 59] for more recent uses). The wave gauges are particularly useful because they guarantee that the Einstein equations take hyperbolic form, and one can employ the methods and techniques which have been developed for this type of partial differential equation. Of particular importance is the first-order symmetric hyperbolic form, on which our Fuchsian theory is based.

The original motivation for this Fuchsian formulation of the Einstein equations in wave gauges is to develop a tool with which we can study the $U(1)$-symmetric class of solutions. As discussed in Section 1.4.3., families of AVTD $U(1)$-symmetric solutions have been found in the analytic function class. However to prove the existence of (only) smooth solutions, the present methods (such as ours presented in Chapter II) require the structural property of hyperbolicity. While the natural, geometric, time coordinate for T^2-symmetric solutions –the areal time– provides a hyperbolic formulation of the Einstein equations in that class, no such time coordinate has been identified for the $U(1)$-symmetric class of solutions. Further, the gauges chosen in the studies of analytic $U(1)$-symmetric solutions [24, 25, 45] do not yield a symmetric hyperbolic formulation of the equations. As a warm-up problem to using this formalism in the $U(1)$-symmetric class, we have used this formalism to investigate the gauge-dependence of the AVTD property in the simpler case of $\mathbb{T}^3$-Gowdy solutions.

The AVTD property is by definition dependent upon the choice of coordinates – recall Definition 1.11 (and the surrounding discussion), which is taken from [44]. The results discussed above in Section 1.4.1. show that in both the smooth and analytic AVTD Gowdy solutions, there exists a family of surface-orthogonal observers (those following worldlines with zero shift relative to the foliation) relative to the areal foliation that experience AVTD behavior. It is not clear if non-stationary observers (corresponding to coordinates with a non-vanishing shift) in these spacetimes would experience AVTD behavior. Similar statements can be made regarding the polarized and half-polarized T^2-symmetric solutions, as well as the polarized $U(1)$-symmetric solutions. We have the following open questions: *I) Suppose a solution in some symmetry class is AVTD in one system of coordinates. Is this solution AVTD in any other system of coordinates? II) What characterizes a family of coordinate systems in which a particular solution (in a particular symmetry class) can be shown to be AVTD?*

The only results in the literature which begin to address these questions are those by Isenberg and Moncrief in [45]. Most of that paper is devoted to showing that there is a family of analytic polarized and half-polarized $U(1)$-symmetric solutions in which surface-orthogonal observers in a harmonic time foliation experience AVTD behavior. In Section 5 the authors investigate the gauge-dependence of their result. They find a two-parameter family of harmonic time foliations in which the surface-orthogonal observers experience AVTD behavior, and further, they show that in each of these foliations, the world-lines of the surface-orthogonal observers become asymptotically parallel. That is the shift vector describing the path of the observers in one harmonic time asymptotically vanishes when expressed in another harmonic time within the family. It is unknown whether this property of asymptotically parallel worldlines is a general property of coordinate systems in which AVTD behavior can be verified.

In this Chapter we take a step towards understanding these questions. We construct in Section 4.3.2. a two-parameter family of wave gauges which contains the areal gauge. These gauges are asymptotically areal in the sense that the gauge source functions approach those of the areal gauge near the singularity. In Section 4.3.3. we state, and prove in Section 4.4., that for each fixed gauge choice in the family, there exists a family of smooth AVTD solutions parametrized by a set of asymptotic data. In each of our solutions the worldlines of the surface-orthogonal observers are asymptotically parallel to those of the observers in the areal gauge. This last result lends support to the idea that asymptotically parallel worldlines is a general feature of coordinate systems in which AVTD behavior can be verified.

Our results so far do not address question (I) posed above. Although for a fixed (non-areal) gauge in our family, we find a family of AVTD solutions, we do not yet know how this family compares to the family of AVTD solutions in areal gauge. This important question is currently under investigation.

We give a brief outline to the chapter. In Section 4.2.1. we review the wave-gauge formalism for the Einstein equations. This is followed in Section 4.2.2. and Section 4.2.3. by writing both the evolution and constraint propagations systems as first-order symmetric hyperbolic systems (in the case of one relevant spatial coordinate) and writing down a general Fuchsian reduction as in Chapter II for these systems. In Section 4.3. we review in more detail the known AVTD solutions in areal gauge, in particular we state a theorem for existence of these solutions with the remainders in weighted Sobolev spaces of Section 2.2.2.. We also set-up and state our main results in this section. The Sections 4.4. and 4.5. are devoted to proving the main results; the proof of existence of solutions is based on the Fuchsian theory developed above.

4.2. Wave Gauge Formalism

4.2.1. Vacuum Einstein Equations

The vacuum Einstein equations Eq. (1.2), can be written in an arbitrary system of coordinates as

$$Ric_{ij} = -\frac{1}{2} g^{kl} \partial_k \partial_l g_{ij} + \nabla_{(i} \Gamma_{j)} + g^{kl} g^{mn} \left(\Gamma_{kmi} \Gamma_{lnj} + \Gamma_{kmi} \Gamma_{ljn} + \Gamma_{kmj} \Gamma_{lin} \right) = 0, \quad (4.1)$$

where

$$\Gamma_{kmi} := \frac{1}{2} \left(\partial_k g_{mi} + \partial_i g_{mk} - \partial_m g_{ki} \right), \quad (4.2)$$

and

$$\Gamma_m := g^{ki} \Gamma_{kmi}.$$

Clearly if one could choose coordinates so that $\Gamma_j \equiv 0$ the equations would take hyperbolic form; such a choice is called *wave gauge*.

One can actually make a more general gauge choice by choosing arbitrary gauge source functions $\mathcal{F}_i$ so that $\mathcal{F}_i = \Gamma_i$, which may depend on the metric functions but not its derivatives. The Einstein equations are then equivalent to the following system of evolution equations:

$$-\frac{1}{2}g^{kl}\partial_k\partial_l g_{ij} + \nabla_{(i}\mathcal{F}_{j)} + g^{kl}g^{mn}\left(\Gamma_{kmi}\Gamma_{lnj} + \Gamma_{kmi}\Gamma_{ljn} + \Gamma_{kmj}\Gamma_{lin}\right) = 0 \tag{4.3}$$

and constraint equations:

$$\mathcal{F}_j - \Gamma_j = 0 \tag{4.4}$$

$$\nabla(\mathcal{F}_j - \Gamma_j) = 0. \tag{4.5}$$

It is necessary that the constraints be propagated by the evolution equations. Let $\mathcal{D}_i := \Gamma_i - \mathcal{F}_i$ be the constraint violation quantity. In terms of this quantity the Einstein evolution equations Eq. (4.3) are written

$$Ric_{ij} + \nabla_{(i}\mathcal{D}_{j)} = 0. \tag{4.6}$$

We can take the trace of this equation to compute the scalar curvature

$$R = -\nabla^j\mathcal{D}_j,$$

and us the divergence free property of the Einstein tensor $\nabla^j G_{ji} = 0$ to derive a linear wave-type constraint propagation equation

$$\nabla^i\nabla_i\mathcal{D}_j + R_j{}^k\mathcal{D}_k = 0. \tag{4.7}$$

Thus, in a Cauchy problem if $\mathcal{D}_i = 0$ and $\nabla_j \mathcal{D}_i = 0$ on a slice Ω_{t_*}, then $\mathcal{D}_i \equiv 0$ in the domain of dependence $D(\Omega_{t_*})$ (see for example [75] Chapter 12).

4.2.2. Fuchsian Formulation of the Vacuum Einstein Evolution Equations

In this section we write the second order Einstein evolution equations in the wave gauge as a first-order symmetric hyperbolic system like that in Chapter II. We also perform a general Fuchsian reduction for a certain class of leading order terms; in specific applications one must check that the reduced system obtained is in fact a Fuchsian system in the sense of Definition 2.7. Because we find it necessary in applications such as to the Gowdy equations in Section 4.3., we add multiples of the constraint violation quantity $\mathcal{D}_k$ to the evolution equations. The coefficients ${C_{ij}}^k$ multiplying $\mathcal{D}_k$ can be chosen in applications to modify the principle part of the quasilinear symmetric hyperbolic Fuchsian equation, since $\mathcal{D}_k$ contains first derivatives of the metric. The extent to which the principle part can be modified depends, of course, on which derivatives of which metric components appear in $\mathcal{D}_k$ in a given application. Write the evolution equations as

$$-\frac{1}{2} g^{kl} \partial_k \partial_l g_{ij} + \nabla_{(i} \mathcal{F}_{j)} + \mathcal{H}_{ij} - {C_{ij}}^k \mathcal{D}_k = 0 \tag{4.8}$$

where we have defined

$$\mathcal{H}_{ij} := g^{km} g^{ln} \left(\Gamma_{kli} \Gamma_{mnj} + \Gamma_{kli} \Gamma_{mjn} + \Gamma_{klj} \Gamma_{min} \right). \tag{4.9}$$

for brevity. Since $n = 1$, the principle part is

$$-\frac{1}{2}g^{kl}\partial_k\partial_l g_{ij} = -\frac{1}{2}g^{00}\partial_0^2 g_{ij} - g^{01}\partial_1\partial_0 g_{ij} - \frac{1}{2}g^{11}\partial_1^2 g_{ij}.$$

To bring the evolution system into Fuchsian form, involving the Fuchsian derivative $D := t\partial_0$, and to avoid a singular coefficient in the principle part of the eventual Fuchsian system we multiply through by $-2t^2(g^{00})^{-1}$. Noting that $t^2\partial_0^2 u = D^2 u - Du$ we obtain

$$D^2 g_{ij} + 2t\beta(g)\partial_1 D g_{ij} - t^2\alpha(g)\partial_1^2 g_{ij} - 2t^2(g^{00})^{-1}H(t,x,g) = 0$$

where we have defined

$$\begin{aligned} \alpha(g) &:= -g^{11}/g^{00}, \quad \beta(g) := g^{01}/g^{00} \\ H(t,x,g) &:= 2t^2(g^{00})^{-1}\left(\nabla_{(i}\mathcal{F}_{j)} + \mathcal{H}_{ij} - C_{ij}{}^k\mathcal{D}_k\right) \end{aligned} \tag{4.10}$$

The system is put into first order form by introducing new fields for the first derivatives of the metric fields. In the following we focus our attention on one metric field by fixing i, j, and derive the first order system corresponding to this field. Let $U = (U_1, U_2, U_3)$ be defined by

$$U_1 := g_{ij}, U_2 := Dg_{ij}, U_3 := t\partial_1 g_{ij} \tag{4.11}$$

The first order equations for g_{ij} (i, j still fixed) are then

$$DU_1 - U_2 = 0, \tag{4.12}$$

$$DU_2 + 2\beta(U)t\partial_1 U_2 - \alpha(U)t\partial_1 U_3 - U_2 + H[U] = 0, \tag{4.13}$$

$$DU_3 - t\partial_1 U_2 - U_3 = 0, \tag{4.14}$$

where $\alpha(U), \beta(U)$, and $H[U]$ are the functions of (t, x) and g introduced in Eq. (4.10) written in terms of the first order fields. By multiplying the third equation through by $\alpha(U)$ the system Eq. (4.12)-Eq. (4.14) can be written as a symmetric hyperbolic system as in Eq. (2.1) with

$$S^0(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha(U) \end{pmatrix}, \quad \text{and} \quad S^a(U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\beta(U) & -\alpha(U) \\ 0 & -\alpha(U) & 0 \end{pmatrix}, \tag{4.15}$$

and

$$f[U] = \begin{pmatrix} -U_2 \\ -U_2 + H[U] \\ -\alpha(U)U_3 \end{pmatrix}.$$

To obtain a the Fuchsian system we must choose a leading order term. Suppose u_0 is the prescribed leading order term for the metric field g_{ij}. Then we let

$$\mathring{U}_1 = u_0, \mathring{U}_2 = Du_0, \mathring{U}_3 = t\partial_1 u_0, \tag{4.16}$$

and write the Fuchsian equation for the remainder field $W := U - \mathring{U}$ contained in some space $X_{\delta,\mu,q}$ for $\delta > 0$, and integer q possibly infinite, and an exponent vector $\mu : T^1 \to \mathbb{R}^3$. Suppose $u_0 \in X_{\delta,\kappa,q}$ for some $\kappa : T^1 \to \mathbb{R}$, then we take μ to have the form

$$\mu = (\mu_1 + \kappa, \mu_1 + \kappa, \mu_1 + \kappa + 1 - \epsilon),$$

where $\epsilon \geq 0$. The value of $1 - \epsilon$ measures the additional control we have over the t-weighted spatial derivatives (the component $W_3 = t\partial_1 W_1$). One may expect that we should have one additional power of control corresponding to $\epsilon = 0$. However, we leave ϵ arbitrary at this stage since it turns out that this quantity is constrained in the Fuchsian analysis below.

The problem of obtaining a Fuchsian equation, given a choice of leading order term, is in separating the various terms in $H[U] = H(\mathring{U})[W]$ into those function operators belonging in the Fuchsian principle part and those belonging in the Fuchsian source. This sorting requires knowledge of the types of function operators which appear in each term. We suppose at this point that $H(\mathring{U})[W]$, and α have expansions of the form

$$\alpha(\mathring{U} + W) = \alpha_0(x) + \alpha_1(W) \tag{4.17}$$

$$H[\mathring{U} + W] = h_0(t,x) + h_1^1(x)W_1 + h_1^2(x)W_2 + h_2(t,x,W), \tag{4.18}$$

where the functions $h_1^1(x)$ and h_1^2 depend on the functions C_{ij}^k.

Since the function operator $\beta(\cdot)$ appears only in $S^a(\cdot)$, we do not bother with expanding it as above, but we still need to know in which weighted Sobolev space (Section 2.2.2.) it takes values. Below we discuss in slightly more detail the nature of the function operators which make up $H(\mathring{U})[W]$, and motivate the above expansions. We now write out the Fuchsian quantities in terms of the expansions Eq. (4.17) and Eq. (4.18).

Recalling Definition 2.7 we write

$$S_0^0 = \mathrm{Diag}(1, 1, \alpha_0(x)), \quad S_1^0(W) = \mathrm{Diag}(0, 0, \alpha_1(W)),$$

from which it is clear that α_0 must be positive definite. It must be checked in a given application that $S_1^0(W)$ and $S_1^a(W)$ are function operators which satisfy the requisite properties listed in Condition (ii) of Definition 2.7 above. Next, we seek to write $f[U]$

as $-f_1(u_0+W)+N_0(u_0)W$ and find

$$-f_1(u_0+W)=\begin{pmatrix} -\mathring{U}_2 \\ -\mathring{U}_2+h_0(t,x)+h_2(t,x,W) \\ -(\alpha_0(x)+\alpha_1(W))\mathring{U}_3 \end{pmatrix},$$

and

$$N_0(u_0)=\begin{pmatrix} 0 & -1 & 0 \\ h_1^1(x) & h_1^2(x)-1 & 0 \\ 0 & & -\alpha_0(x) \end{pmatrix}. \tag{4.19}$$

Finally we compute

$$\mathcal{F}(u_0)[W]=$$
$$\begin{pmatrix} 0 \\ -h_0(t,x)-h_2(t,x,W)-D\mathring{U}_2-2t\beta(\mathring{U}+W)\partial_1\mathring{U}_2+t\alpha(\mathring{U}+W)\partial_1\mathring{U}_3+\mathring{U}_2 \\ 0 \end{pmatrix},$$

which must be verified to satisfy Condition (iii) of Definition 2.7 in particular applications.

The expansion of the function operator $H(\mathring{U})[W]$ in Eq. (4.18) can be understood in more detail. Recall from Eq. (4.10) that there are three types of terms, $\mathcal{H}_{ij}$, $\nabla_{(i}\mathcal{F}_{j)}$, and those proportional to $\mathcal{D}_k$. In the following, we rely heavily on the discussion of function operators in Section C of the Appendix. To understand all of these terms, it is necessary to understand the components of the inverse metric, and in particular their properties as function operators. Lemma C.10 shows that if $\mu>0$, the inverse metric components are smooth function operators on $w\in B_{\delta,\mu,q}(s)$ for a sufficiently small s, each of which can be expanded in the form

$$g^{ij}(W) = y_0(t,x) + y_1^n(t,x)W_n + y_2(t,x,W),$$

where the sum over $n = 1, \ldots d$ is implied and y_2 is a rational function of the components of W. Once the components of the inverse metric are understood as function operators, the various contractions of the Christoffel symbols are easily analyzed. Note from Eq. (4.2) that for any i, j, k, Γ_{ijk} is a linear function operator. Consequently each quadratic term in Γ is a quadratic function operator, and it follows that $\mathcal{H}_{ij}$ is a polynomial function operator which has an expansion as in Eq. (C.1).

Similarly, each Γ^k_{ij} consists of inverse metric components multiplying an expression linear in the first-order fields, and the same for Γ_k. Since each component of the inverse metric has an expansion as above, as do the linear function operators in the remainder of the Christoffel symbols, it follows that $\nabla_{(i}\mathcal{F}_{j)}$ and $\mathcal{D}_k$ are also polynomial function operators, each of which has an expansion of the form Eq. (C.1). We conclude that $H(\mathring{U})[W]$ takes the form

$$H(\mathring{U})[W] = \sum_{j=1}^{l}\prod_{i=1}^{18} c_j(t,x)W_i^{p_i^j} = h_0(t,x) + h_1^1(x)W_1 + h_1^2(x)W_2 + h_2(t,x,W). \quad (4.20)$$

Note that while in general terms linear in components of W other than W_1 and W_2 appear in the expression, these do not contribute to the Fuchsian principle part of the equation for W_2 –as this would break the block-diagonal structure– such terms are instead contained in $h_2(t,x,W)$.

We remark that such a reduction must be done for each metric field, resulting in a $3 \times N$-dimensional first-order symmetric hyperbolic system, which is organized into N blocks. Here N denotes the number of independent metric fields, and has a maximum of ten. A similar reduction can be performed in situations where the metric depends on more than one spatial coordinate. In such scenarios, the first-order system will be $(2+n)\times N$-dimensional, where n denotes the number of dynamically relevant spatial coordinates.

4.2.2.0..1 Propagation of first-order field definitions Suppose that in a particular application one can verify that the function operators satisfy the requisite properties in Definition 2.7 and Theorem 2.10, (or Theorem 2.28 in the smooth case), and hence obtain a unique solution to the first-order symmetric hyperbolic Fuchsian system. To show that this solution corresponds to a solution of the original second-order system we must show that the constraints obtained from the first-order field definitions are propagated by the evolution equations. We have

$$\mathcal{C}_1(U) := U_2/t - \partial_t U_1, \quad \text{and} \quad \mathcal{C}_2(U) := U_3/t - \partial_x U_1.$$

The preservation of the first constraint, $\mathcal{C}_1 = 0$ for all $t \in (0, \delta]$, follows directly from the first evolution equation Eq. (4.12), which implies

$$t\mathcal{C}_1 \equiv 0.$$

Further, from Eq. (4.12) and Eq. (4.14) we derive

$$\begin{aligned} D\mathcal{C}_2 &= \frac{1}{t} DU_3 - U_3/t - \partial_x DU_1 \\ &= \frac{1}{t}\left(t\partial_x U_2 + U_3\right) - U_3/t - \partial_x U_2 \\ &= 0 \end{aligned}$$

It follows that since $\mathcal{C}_2 = 0$ is a solution to this equation, it is the unique solution provided $\mathcal{C}_2$ vanishes asymptotically, that is $\mathcal{C}_2(\mathring{U}) = 0$. This condition is satisfied by the definition of the leading order terms for the first order fields Eq. (4.16).

4.2.3. Fuchsian Formulation of the Vacuum Einstein Constraint Propagation Equations

While in the Cauchy formulation discussed above in Section 4.2. it is possible to show that the constraints vanish if they vanish on an initial data surface, here we must show that the constraints vanish in a neighborhood of the singularity based only on the knowledge that they are satisfied asymptotically. This requires a formulation of the propagation equations as a symmetric hyperbolic Fuchsian system. Since these equations are linear, we seek to form a system of the type in Definition 2.11, and to prove the existence of solutions only the hypotheses of Proposition 2.20 must be met.

With the addition of constraints to the evolution equations, so that Eq. (4.6) has the form

$$Ric_{ij} + \nabla_{(i}\mathcal{D}_{j)} = C_{ij}{}^{k}\mathcal{D}_k, \tag{4.21}$$

where $C_{ij}{}^{k}$ are functions of the spacetime coordinates, we obtain the corresponding linear wave equation for the constraint propagation by the same process as in Section 4.2.. The result is

$$\nabla^i\nabla_i\mathcal{D}_j + R_j{}^l\mathcal{D}_l = \left(2\nabla_i C^i{}_j{}^k - \nabla_j C_l{}^{lk}\right)\mathcal{D}_k + \left(2C^i{}_j{}^k - C_l{}^{lk}\delta^i_j\right)\nabla_i\mathcal{D}_k. \tag{4.22}$$

Comparing to Eq. (4.7), we see that the left-hand side is the same, while additional terms depending on the coefficients $C_{ij}{}^{k}$ have been added to the right-hand side. Expanding the wave operator and the covariant derivatives in terms of the metric and Christoffel symbols Eq. (4.22) can be written

$$g^{jk}\partial_j\partial_k\mathcal{D}_i - A_i^{jk}\partial_j\mathcal{D}_k - B_i^k\mathcal{D}_k = \left(2\nabla_j C^j{}_i{}^k - \nabla_i C_l{}^{lk}\right)\mathcal{D}_k - \left(2C^j{}_i{}^k - C_l{}^{lk}\delta_i^j\right)\Gamma_{jk}^m\mathcal{D}_m + \left(2C^j{}_i{}^k - C_l{}^{lk}\delta_i^j\right)\partial_j\mathcal{D}_k$$

with

$$\begin{aligned} A_i^{jk} =& g^{ml}\Gamma^j_{ml}\delta_i{}^k + 2g^{mj}\Gamma^k_{mi}, \\ B_i^k =& g^{mj}\left(\partial_m\Gamma^k_{ji} - \Gamma^l_{mj}\Gamma^k_{li} - \Gamma^l_{mi}\Gamma^k_{jl}\right) - R_i{}^k. \end{aligned}$$

Multiplying this equation by t^2, using the definition of the Fuchsian derivative operator $D := t\partial_t$ and the identity $D^2u(t) = t^2\partial_t^2 u(t) + t\partial_t u(t)$ we find

$$\begin{aligned} g^{00}D^2\mathcal{D}_i + 2g^{01}t\partial_1 D\mathcal{D}_i + g^{11}t\partial_1 t\partial_1\mathcal{D}_i & \\ = & \, tA_i^{0k}D\mathcal{D}_k + tA_i^{1k}t\partial_1\mathcal{D}_k + t^2B_i^k\mathcal{D}_k + D\mathcal{D}_i \\ & + t\left(2g^{0m}C_{mi}{}^k - g^{lm}C_{lm}{}^k\delta_i^0\right)D\mathcal{D}_k \\ & + t\left(2g^{1m}C_{mi}{}^k - g^{lm}C_{lm}{}^k\delta_i^1\right)t\partial_1\mathcal{D}_k \\ & + t^2\left(2\nabla_j C^j{}_i{}^k - \nabla_i C_l{}^{lk} - 2C^j{}_i{}^m\Gamma^k_{jm} + C_l{}^{lm}\delta_i^j\Gamma^k_{jm}\right)\mathcal{D}_k \end{aligned}$$

We now construct a first-order symmetric hyperbolic system as in Section 4.2.2., and choosing a leading order term, the corresponding Fuchsian system. Since the system in this section is linear (the system in Section 4.2.2. is quasilinear), we are able to give a slightly more detailed presentation. Elements of the reduction however are very similar to that in Section 4.2.2.. We derive a first-order system for the first-order fields

$$(V_1, \dots, V_{12})^T := (\mathcal{D}_0, D\mathcal{D}_0, t\partial_1\mathcal{D}_0, \dots, \mathcal{D}_3, D\mathcal{D}_3, t\partial_1\mathcal{D}_3)^T ,$$

which is block-diagonal. Each block has the form

$$\begin{aligned} DV_{3A-2} - V_{3A-1} &= 0 \\ DV_{3A-1} + 2\beta(g)t\partial_1 V_{3A-1} - \alpha(g)t\partial_1 V_{3A} - V_{3A-1} + H_A[V](t,x) &= 0 \\ DV_{3A} - t\partial_1 V_{3A-1} - V_{3A} &= 0 \end{aligned} \tag{4.23}$$

where $A = 1, \dots, 4$ indexes the blocks in this case. The $H_A[V]$ are given by

$$\begin{aligned} H_{A=i+1}[V] = & -t/g^{00}\left(A_i^{0k} + \widetilde{C^{(0)}}_i^{\,k}\right)V_{3k+2} - t/g^{00}\left(A_i^{1k} + \widetilde{C^{(1)}}_i^{\,k}\right)V_{3k+3} \\ & - t^2/g^{00}\left(B_i^k + \widetilde{C}_i^k\right)V_{3k+1}, \end{aligned} \tag{4.24}$$

where

$$\widetilde{C}_i^k := \left(2\nabla_j C^j{}_i{}^k - \nabla_i C_l{}^{lk} - 2C^j{}_i{}^m\Gamma^k_{jm} + C_l{}^{lm}\delta^j_i\Gamma^k_{jm}\right),$$

$$\widetilde{C^{(0)}}_i^{\,k} := \left(2g^{0m}C_{mi}{}^k - g^{lm}C_{lm}{}^k\delta^0_i\right), \quad \text{and} \quad \widetilde{C^{(1)}}_i^{\,k} := \left(2g^{1m}C_{mi}{}^k - g^{lm}C_{lm}{}^k\delta^1_i\right).$$

The coefficient matrices are known in this case since they depend upon the solutions to the evolution equations.

Next we seek the reduced system formed by inserting $V = \mathring{V} + W$ into the first-order system. We choose a leading order term $\mathring{V} = 0$ since this a linear system and we assume the constraints hold asymptotically. If this is to form a *linear* symmetric hyperbolic Fuchsian system about $\mathring{V} = 0$, we should find that

$$f(\mathring{V} + W) = N_0 W - \mathcal{F}(\mathring{V})[W],$$

(since $\sum_{j=0}^{n} tS^j\partial_j\mathring{V} = 0$), where N_0 is a matrix-valued function of x, and $\mathcal{F}(\mathring{V})[W]$ satisfies the properties in Definition 2.7. As in Section 4.2.2. the main burden of the analysis is in the terms $H_A[V]$. Here, each $H_A[V]$ is a linear combination of the components of V, with coefficients determined by the metric fields (solutions to the evolution equations which are taken to be found). The analysis simplifies in this (linear) case to checking the exponent of t in the coefficient of each term. Those terms with coefficients which are $O(1)$ as $t \searrow 0$ will placed into the N_0 part, while any other term must be higher order and placed in $\mathcal{F}(\mathring{V})[W]$. In the later case, we may obtain inequalities on the exponent vector for W.

The contributions to the N_0 part of the Fuchsian principle part can be identified as the $t \searrow 0$ behavior of the corresponding coefficients. To this end we define

$$a^{(0)}{}^{j}_{i} := \lim_{t\searrow 0} t(g^{00})^{-1}A^{0j}_{i}, \quad a^{(1)}{}^{j}_{i} := \lim_{t\searrow 0} t(g^{00})^{-1}A^{1j}_{i} \quad b^{j}_{i} := \lim_{t\searrow 0} t^2(g^{00})^{-1}B^{j}_{i}, \tag{4.25}$$

and similarly,

$$c^{k}_{i} := \lim_{t\searrow 0}\left\{\frac{t^2}{g^{00}}\widetilde{C}^{k}_{i}\right\}, \quad c^{(0)}{}^{k}_{i} := \lim_{t\searrow 0}\left\{\frac{t}{g^{00}}\widetilde{C^{(0)}}{}^{k}_{i}\right\}, \quad c^{(1)}{}^{k}_{i} := \lim_{t\searrow 0}\left\{\frac{t}{g^{00}}\widetilde{C^{(1)}}{}^{k}_{i}\right\}. \tag{4.26}$$

Thus, $a^{(0)}{}^{j}_{i} + c^{(0)}{}^{j}_{i}$ describes the $O(1)$ coefficient of V_{3j+2} in the function operator $H_{i+1}[V]$, and similar interpretations are made for the other coefficients (see Eq. (4.24)). The $(i+1)^{th}$ block[1] of the N_0 matrix can then be written

$$N_0^{(i+1)} = \begin{pmatrix} 0 & -1 & 0 \\ -b^{i}_{i} - c^{i}_{i} & -a^{(0)}{}^{i}_{i} - c^{(0)}{}^{i}_{i} - 1 & -a^{(1)}{}^{i}_{i} - c^{(1)}{}^{i}_{i} \\ 0 & 0 & -\alpha_0 \end{pmatrix}. \tag{4.27}$$

[1]Recall $i = 0, \ldots 3$.

Note that in most of our applications $a^{(1)}{}_i^i = c^{(1)}{}_i^i = 0$, and $\alpha_0 = 1$, and this matrix has the same form as N_0 found in Eq. (4.19) above.

4.3. AVTD Gowdy Solutions

4.3.1. Review: AVTD Solutions in Areal Gauge

In this section we review in more detail than Section 1.4.1., what is known regarding AVTD behavior in the Gowdy solutions in areal coordinates. The original results may be found in [18, 44, 67]. The purpose of this section is to state an existence theorem for the smooth AVTD Gowdy solutions in terms of our present theory, most importantly we establish control of the remainder fields in terms of the weighted Sobolev spaces Section 2.2.2.. We expect this formulation of the result to be useful in comparing the AVTD solutions in areal gauge with those obtained in Theorem 4.4 below. As stated above, this comparison is ongoing work.

In areal coordinates $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$, the $\mathbb{T}^3$−Gowdy spacetimes (M, g) are given by $M = (0, \infty) \times \mathbb{T}^3$ and

$$g = \frac{1}{\sqrt{\tilde{t}}} e^{\tilde{\lambda}/2}(-d\tilde{t}^2 + d\tilde{x}^2) + \tilde{t}(e^{\tilde{P}} d\tilde{y}^2 + 2e^{\tilde{P}} \tilde{Q} d\tilde{y} d\tilde{z} + (e^{\tilde{P}} \tilde{Q}^2 + e^{-\tilde{P}}) d\tilde{z}^2), \tag{4.28}$$

where $\tilde{t} > 0$ is the areal time coordinate, and $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ are standard coordinates on $\mathbb{T}^3$. The functions $\tilde{\lambda}$, $\tilde{P}$, and $\tilde{Q}$ are functions of $\tilde{t}$ and $\tilde{x}$ only. It is shown in [18, 67] that a family of smooth solutions to the Einstein equations, which we denote $\mathcal{A}$, exist in which the metric functions have the following expansions at every point $\tilde{x} \in \mathbb{T}^1$ in a neighborhood of $\tilde{t} = 0$, (the singular region)

$$\begin{aligned}\tilde{P}(\tilde{t},\tilde{x}) &= -k(\tilde{x})\log\tilde{t} + P_{**}(\tilde{x}) + w_P(\tilde{t},\tilde{x}),\\ \tilde{Q}(\tilde{t},\tilde{x}) &= Q_*(\tilde{x}) + Q_{**}(\tilde{x})t^{2k(\tilde{x})} + w_Q(\tilde{t},\tilde{x}),\\ \tilde{\lambda}(\tilde{t},\tilde{x}) &= k(\tilde{x})^2\log\tilde{t} + \lambda_{**}(\tilde{x}) + w_\lambda(\tilde{t},\tilde{x}).\end{aligned} \tag{4.29}$$

The functions $w_P(\tilde{t},\tilde{x}), w_Q(\tilde{t},\tilde{x}), w_\lambda(\tilde{t},\tilde{x})$ in Eq. (4.29) decay in a controlled way as $\tilde{t} \searrow 0$. The asymptotic data functions $k(\tilde{x})$, $P_{**}(\tilde{x})$, $Q_*(\tilde{x})$, and $Q_{**}(\tilde{x})$ depend only on spatial variable $\tilde{x}$, and must satisfy certain constraints listed in Theorem 4.1 below. The work of Ringstrom [74] proves that in fact the solutions $\mathcal{A}$ are generic in the space of all solutions with T^3 Gowdy symmetry.[2] Solutions with expansions given by Eq. (4.29) are AVTD since the leading order terms in the expansions of the metric fields satisfy (at least asymptotically) the corresponding VTD system [44]. We have the following theorem, which is essentially Theorem 4.4 from [18] formulated in our present notation.

Theorem 4.1 (Existence and uniqueness of smooth AVTD solutions to the $\mathbb{T}^3$-Gowdy-Einstein system)**.** *Let* $\{k, P_{**}, Q_*, Q_{**}, \lambda_{**}\}$ *be any smooth asymptotic data with* $k \in (0,1)$*, and satisfying*

$$\lambda'_{**}(\tilde{x}) = -2kP'_{**}(\tilde{x}), \quad \int_0^{2\pi} k(\tilde{x})\left(-\partial_{\tilde{x}}P_{**}(\tilde{x}) + 2e^{2P_{**}(\tilde{x})}Q_{**}(\tilde{x})\partial_{\tilde{x}}Q_*(\tilde{x})\right)d\tilde{x} = 0.$$

[2]More precisely, Ringstrom shows that there exists an open and dense set of initial data for the $\mathbb{T}^3$-Gowdy equations for which the corresponding maximal globally hyperbolic developments have the expansions Eq. (4.29) with $k \in (0,1)$ in the direction of the singularity about all but a finite collection of points.

There exists a solution of the full $\mathbb{T}^3$*-Einstein-Gowdy system of the form Eq.* (4.29) *with* $w_P \in X_{\delta,\mu_P,\infty}$*,* $w_Q \in X_{\delta,\mu_Q,\infty}$ *and* $w_\lambda \in X_{\delta,\mu_\lambda,\infty}$ *for* $\mu_P, \mu_Q, \mu_\lambda$ *satisfying*

$$\begin{aligned} 0 <&\mu_P < \min\{2-2k, 2k, 1+\mu_Q\} \\ 0 <&\mu_Q < \min\{2, 2k, 1+\mu_P\} \\ 0 <&\mu_\lambda < \min\{3-2k, 2k\}. \end{aligned} \tag{4.30}$$

4.3.2. Asymptotically Areal Wave Gauges

Before presenting our main results in Section 4.3.3. below, we introduce the metric and gauge fields which are used in the analysis. Due to the symmetries in the Gowdy class we consider gauge source functions depending only on (t, x), and further we restrict to the case $\mathcal{F}_2 = \mathcal{F}_3 = 0$. Given this choice of gauge source functions we find that the metric for the vacuum $\mathbb{T}^3$ Gowdy Einstein equations in general takes a block-diagonal form with one block corresponding to the (t, x)-part of the metric (which we call the γ-block), and one block to the (y, z)-part of the metric (the τ-block). In the special case of the areal gauge ($\mathcal{F}_0 = -1/t, \mathcal{F}_1 = 0$), the (t, x) block can be chosen to be diagonal. However, for more general families of gauge source functions the shift component γ_{01} is non-vanishing under Einstein evolution. For the wave gauge formalism a parametrization based on the metric components is most natural. However, for technical reasons we choose a non-metric component parametrization for the τ-block. We parametrize the metric in terms of the fields

$\gamma_{00}, \gamma_{01}, \gamma_{11}$, and $\tau_{11}, \tau_{12}, \tau_{22}$ as

$$g = \begin{pmatrix} \gamma_{00} & \gamma_{01} & 0 & 0 \\ \gamma_{01} & \gamma_{11} & 0 & 0 \\ 0 & 0 & \tau_{11} & \tau_{11}\tau_{01} \\ 0 & 0 & \tau_{11}\tau_{01} & \tau_{11}\tau_{01}^2 + \tau_{22} \end{pmatrix}. \tag{4.31}$$

Now that we have specified the metric fields, we make a formal definition for the class of gauge source functions we consider.

Definition 4.2 (Asymptotically areal wave gauge)**.** *Let $t > 0$, and let x^a be coordinates on $\mathbb{T}^3$. Furthermore, let $\mathring{\tau}$ be the leading order term of the τ-block, and suppose that $Vol(\mathring{\tau}) = t$. The gauge is called* ***asymptotically areal wave gauge*** *if the gauge source functions take the form*

$$\mathcal{F}_0 = -\frac{1}{t} + F_0(t, x), \quad \mathcal{F}_1 = F_1(t, x), \quad \mathcal{F}_2 = \mathcal{F}_3 = 0$$

for $F_0 \in X_{\delta,\xi_0,\infty}$ and $F_1 \in X_{\delta,\xi_1,\infty}$ with $\xi_0 > -1, \xi_1 > 0$.

Notice that the gauge source functions correspond to those for the areal gauge at leading order –hence the name. The areal gauge can be recovered by taking the limit $\xi_0, \xi_1 \to \infty$.

4.3.3. Main Results

We begin by proving the following theorem for the existence and uniqueness of a family of solutions to the $\mathbb{T}^3$-Gowdy Einstein equations in a class of wave gauges. The following set of asymptotic data plays an important role in the results of this section.

Definition 4.3. *Suppose gauge source functions $\mathcal{F}$ have been chosen as in Definition 4.2. Let $\mathcal{K}$ denote the set of asymptotic data $\{k(x), \gamma_*(x), \gamma_{**}(x), \tau_*(x), \tau_{**}(x)\}$ which is $C^\infty(\mathbb{S})$, and such that $k \in (0,1)$ and*

$$\gamma_{**} = -\frac{\gamma_* \varphi_1}{1+\xi_1}, \quad \gamma_*'/\gamma_* = -k(x)\tau_*'/\tau_*.,$$

where $\varphi_1 := \lim_{t \searrow 0} F_1(t,x)/t^{\xi_1}$.

Theorem 4.4 (AVTD Gowdy solutions in asymptotically areal wave gauges)**.** *Choose any gauge source functions as in Definition 4.2 with parameters ξ_0, ξ_1 satisfying $\xi_0 > 3/2(1+k^2), \xi_1 > 1/2(1+3k^2)$, and any smooth asymptotic data in $\mathcal{K}$ for the metric fields in Eq. (4.31). There exists a solution $g = \mathring{g} + \hat{g}$ of the form Eq. (4.31) to the full Einstein-wave system given by the evolution equations Eq. (4.3) and the constraints Eq. (4.4), with leading order term $\mathring{g}$ given by*

$$\mathring{\gamma}_{00} = -\gamma_*(x) t^{1/2(k^2-1)} \tag{4.32}$$

$$\mathring{\gamma}_{01} = \gamma_{**}(x) t^{1/2(k^2-1)+1+\xi_1} \tag{4.33}$$

$$\mathring{\gamma}_{11} = \gamma_*(x) t^{1/2(k^2-1)} \tag{4.34}$$

$$\mathring{\tau}_{11} = \tau_*(x) t^{1-k} \tag{4.35}$$

$$\mathring{\tau}_{12} = \tau_{**}(x) \tag{4.36}$$

$$\mathring{\tau}_{22} = \tau_*(x)^{-1} t^{k+1}. \tag{4.37}$$

and with remainder $\hat{g} \in X_{\delta,\mu,\infty}$ *for* μ *satisfying*

$$\begin{aligned}
(1-k^2)/2 < \mu_1 &< \mu_3, \\
0 < \mu_2 &< \min\{\xi_1, \mu_1, \mu_4, \mu_6\}, \\
(1-k^2)/2 < \mu_3 &< 2-2k, \\
0 < \mu_4 &< 2-2k, \\
2k < \mu_5 &< 2, \\
0 < \mu_6 &< 2-2k.
\end{aligned} \tag{4.38}$$

Let $\mathcal{S}(\xi_0, \xi_1)$ denote the family of solutions obtained in Theorem 4.4. Note that for each appropriate pair (ξ_0, ξ_1) specifying the gauge, there is a family of solutions parameterized by the asymptotic data in $\mathcal{K}$. The asymptotic data is coupled to the choice of gauge through the shift function, which imposes the constraint $\gamma_{**} = -\frac{\gamma_* \varphi_1}{1+\xi_1}$. Theorem 4.4 is proved in Section 4.4. below using Fuchsian methods.

The leading order terms for the metric fields given in Eqs. (4.32)-(4.37) are motivated by the leading order terms for the metric fields in areal gauge Eqs. (4.29).

The later can be shown to asymptotically satisfy the VTD Gowdy equations in areal gauge. Similarly, the leading order terms we have selected in the asymptotically areal gauge can be shown to satisfy the VTD equations corresponding to Eqs. (4.3)-(4.5). The following lemma, which states this result, is proved below in Section 4.5..

Lemma 4.5 (Each family $\mathcal{S}(\xi_0, \xi_1)$ is AVTD). *For any two parameters* ξ_0, ξ_1 *satisfying the inequalities in Theorem 4.4, and asymptotic data in* $\mathcal{K}$*, the family of solutions* $\mathcal{S}(\xi_0, \xi_1)$ *obtained in Theorem 4.4 is AVTD.*

Given that we have established the existence of two families of AVTD solutions in different gauges (one gauge being the limiting case of the other), we would like to relate these two families. However at this time we the relationship between $\mathcal{S}(\xi_0, \xi_1)$ and $\mathcal{A}$ is not known. Based on our preliminary computations so far we make the following conjecture:

Conjecture 4.6. *For any choice of parameters ξ_0, ξ_1 satisfying the inequalities in Theorem 4.4*

$$\mathcal{S}(\xi_0, \xi_1) \subset \mathcal{A}.$$

4.4. Existence of Solutions in Asymptotically Areal Wave Gauge

The proof of Theorem 4.4 is based on an application of Theorem 2.28, to the Gowdy equations in asymptotically areal wave gauges. In Section 4.4.1. we use the Fuchsian reduction in Section 4.2.2. to set up a singular initial value problem for the Gowdy equations, and check the criteria of Theorem 4.4. In Section 4.4.2., we analyze the constraint equations in the Gowdy case taking advantage of the general Fuchsian formulation worked out in Section 4.2.3..

4.4.1. Analysis of the Evolution Equations

4.4.1.1. First order system and leading order terms

To prove Theorem 4.4 we begin with the Einstein equations for the metric Eq. (4.31) written in the wave-gauge formalism for a fixed choice of gauge source functions chosen as in Definition 4.2, and derive a quasilinear symmetric hyperbolic Fuchsian system as in Definition 2.7 using the reduction in Section 4.2.2..

There are six non-vanishing metric fields, leading to an eighteen dimensional system for the first order fields

$$(U_1, \dots, U_{18})^T := (\gamma_{00}, D\gamma_{00}, t\partial_1\gamma_{00}, \dots, \tau_{22}, D\tau_{22}, t\partial_1\tau_{22})^T. \tag{4.39}$$

The system is block-diagonal with each block having the form of Eq. (4.12)-Eq. (4.14). Let $A = 1, \ldots, 6$ index the blocks corresponding to the six metric fields. We have the system

$$DU_{3A-2} - U_{3A-1} = 0$$
$$DU_{3A-1} + 2\beta(U)t\partial_1 U_{3A-1} - \alpha(U)t\partial_1 U_{3A} - U_{3A-1} + H_A[U] = 0$$
$$DU_{3A} - t\partial_1 U_{3A-1} - U_{3A} = 0,$$

where the functionals $H_A[U]$ are given by

$$H_1[U] = 2t^2 \frac{\det\gamma(U)}{U_7}\left(\nabla_{(0}\mathcal{F}_{0)} + \mathcal{H}_{00} - C^k_{00}\mathcal{D}_k\right)[U] \tag{4.40}$$
$$H_2[U] = 2t^2 \frac{\det\gamma(U)}{U_7}\left(\nabla_{(0}\mathcal{F}_{1)} + \mathcal{H}_{01} - C^k_{01}\mathcal{D}_k\right)[U] \tag{4.41}$$
$$H_3[U] = 2t^2 \frac{\det\gamma(U)}{U_7}\left(\nabla_{(1}\mathcal{F}_{1)} + \mathcal{H}_{11} - C^k_{11}\mathcal{D}_k\right)[U] \tag{4.42}$$
$$H_4[U] = 2t^2 \frac{\det\gamma(U)}{U_7}\left(\nabla_{(2}\mathcal{F}_{2)} + \mathcal{H}_{22} - C^k_{22}\mathcal{D}_k\right)[U] \tag{4.43}$$
$$H_5[U] = 2t^2 \frac{\det\gamma(U)}{U_7 U_{10}}\Big(\nabla_{(2}\mathcal{F}_{3)} + \mathcal{H}_{23} - C^k_{23}\mathcal{D}_k - U_{13}\left(\nabla_{(2}\mathcal{F}_{2)} + \mathcal{H}_{22} - C^k_{22}\mathcal{D}_k\right)\Big)[U] + \mathcal{G}_5(U) \tag{4.44}$$
$$H_6[U] = 2t^2 \frac{\det\gamma(U)}{U_7}\Big(\nabla_{(3}\mathcal{F}_{3)} + \mathcal{H}_{33} - C^k_{33}\mathcal{D}_k + U_{13}^2\left(\nabla_{(2}\mathcal{F}_{2)} + \mathcal{H}_{22} - C^k_{22}\mathcal{D}_k\right) - 2U_{13}\left(\nabla_{(2}\mathcal{F}_{3)} + \mathcal{H}_{23} - C^k_{23}\mathcal{D}_k\right)\Big)[U] + \mathcal{G}_6(U). \tag{4.45}$$

The functionals $\mathcal{G}_5(U)$ and $\mathcal{G}_6(U)$ defined by

$$\mathcal{G}_5(g) = 2\frac{D\tau_{11}D\tau_{12}}{\tau_{11}} + 2\frac{\gamma_{00}t\partial_1\tau_{11}t\partial_1\tau_{12}}{\gamma_{11}\tau_{11}} - 2\frac{\gamma_{01}D\tau_{12}t\partial_1\tau_{11}}{\gamma_{11}\tau_{11}} - 2\frac{\gamma_{01}D\tau_{11}t\partial_1\tau_{12}}{\gamma_{11}\tau_{11}} \tag{4.46}$$
$$\mathcal{G}_6(g) = 2\tau_{11}(D\tau_{12})^2 + 2\frac{(t\partial_1\tau_{12})^2\gamma_{00}\tau_{11}}{\gamma_{11}} - 4\frac{D\tau_{12}t\partial_1\tau_{12}\gamma_{01}\tau_{11}}{\gamma_{11}}, \tag{4.47}$$

account for the additional terms generated when the wave operator acts upon the τ-block of the metric Eq. (4.31). As in Section 4.2.2. the system is brought to symmetric form with block-diagonal matrices S^0 and S^1 (with each block of the form Eq. (4.15)) by multiplying each DU_{3A}-equation through by $\alpha(U)$.

We now choose a leading order term $\mathring{U}$, for the first order fields, and in the Section 4.4.1.2. below we verify that the resulting system for W

$$S^0(\mathring{U}+W)DW + tS^a(\mathring{U}+W)\partial_a W = -f(\mathring{U}+W) - \sum_{j=0}^{n} tS^j(\mathring{U}+W)\partial_j\mathring{U}, \quad (4.48)$$

is a quasilinear symmetric hyperbolic Fuchsian system as in Definition 2.7. The remainder W is taken to be in $X_{\delta,\mu,q}$, where $\delta > 0$, q is some integer possibly infinite, and for each of the six blocks in the first order equation the exponent vector μ is chosen to have the form

$$\mu^{(A)} = \mu_A + \kappa_A, \mu_A + \kappa_A, \mu_A + \kappa_A + 1 - \epsilon. \quad (4.49)$$

The origin of the $1-\epsilon$ in the third component is explained above in Section 4.2.2.. Here κ_A denotes the exponent corresponding to the appropriate leading order term.

The leading order term for the first order evolution system is chosen to be

$$\mathring{U}_1 = -\gamma_*(x)t^{1/2(k^2-1)} \tag{4.50}$$

$$\mathring{U}_4 = \sigma_{**}(x)t^{1/2(k^2+1)} \tag{4.51}$$

$$\mathring{U}_7 = \gamma_*(x)t^{1/2(k^2-1)} \tag{4.52}$$

$$\mathring{U}_{10} = \tau_*(x)t^{1-k} \tag{4.53}$$

$$\mathring{U}_{13} = \tau_{**}(x) \tag{4.54}$$

$$\mathring{U}_{16} = \tau_*(x)^{-1}t^{k+1}, \tag{4.55}$$

and with leading order terms for the remaining first-order fields chosen in a manner consistent with the definition of the fields as in Eq. (4.16) above. That is, if $\mathring{\psi}_A$, $A = 1, \ldots, 6$ denotes the six functions on the right-hand side of Eq. (4.50) -Eq. (4.55), then we choose the remaining first-order leading order terms according to

$$\mathring{U}_{3A-2} = \mathring{\psi}_A, \quad \mathring{U}_{3A-1} = D\mathring{\psi}_A, \quad \text{and} \quad \mathring{U}_{3A} = t\partial_1\mathring{\psi}_A. \tag{4.56}$$

This leading order term is consistent with that in Eq. (4.32) -Eq. (4.37), except in the case of the shift, $\mathring{U}_4$. The reason is that in the evolution equation for γ_{01} (or DU_5) the most singular terms, given the other leading order terms, are of order $O(t^{1/2(k^2+1)})$. In order to cancel these singular terms we choose the leading order term as in Eq. (4.51) above, where $\sigma_{**}(x)$ is an appropriate function of the spatial coordinate. The origin of the leading order term Eq. (4.33), in Theorem 4.4 is explained below when we analyze the constraint equations.

4.4.1.2. Obtaining a smooth QSHF system

In this section we show that the symmetric hyperbolic system obtained in the previous section is a quasilinear symmetric hyperbolic Fuchsian system as in Definition 2.7 about $\mathring{U}$ given by Eq. (4.56) and Eq. (4.32)-Eq. (4.37). Before we state this result in the following lemma we make a more specific choice for the functions ${C_{ij}}^k$. In Eq. (4.40)-Eq. (4.45) we write the form for the functions $H_A[U]$ for a fully general set of functions ${C_{ij}}^k$. The reason for adding multiples of the constraint violation quantities to the evolution equations is to modify the N_0 matrix in Eq. (4.48) in order to obtain solutions for W in the largest possible space $X_{\delta,\mu,q}$ (i.e. the smallest possible μ). The lower bound on μ is often controlled in our Fuchsian theorems by a positivity condition on N_0, (e.g. Condition (iii) in the case of Theorem 2.28). It turns out, that due to the structure of the Gowdy equations the optimal lower bound on each component μ_A can be obtained without the addition of terms ${C_{ij}}^k\mathcal{D}_k$ for all A except $A = 1$. In the $A = 1$ case, which corresponds to the block of equations for γ_{00}, we find that we must add constraint violation terms to avoid strong restrictions on the asymptotic data $k(x)$ and the exponent vectors ξ_0, and ξ_1 that arise from the (uncontrolled) lower bound on μ_1. Fortunately, $\mathcal{D}_0$ contains terms which contribute to Fuchsian principle part of the γ_{00} evolution equations when multiplied by appropriate coefficients.

Henceforth, we set

$${C_{00}}^0 = \Lambda(x)/t, \quad {C_{ij}}^k \equiv 0, \quad \text{elsewise.}$$

We choose an explicit $1/t$ time dependence for the coefficient so that when multiplied by t^2, as in done to obtain Eq. (4.12)-Eq. (4.14), $C_{00}{}^0\mathcal{D}_0$ has the appropriate order in t to modify the principle part of Eq. (2.6).

Lemma 4.7. *The block-diagonal system Eq.* (4.12)*-Eq.* (4.14) *and Eq.* (4.40)*-Eq.* (4.45) *forms a smooth quasilinear symmetric hyperbolic Fuchsian system as in Definition 2.7 around* $\mathring{U}$ *given by Eq.* (4.56) *and Eq.* (4.50)*-Eq.* (4.55) *with the asymptotic data* $\{k, \gamma_*, \sigma_{**}, \tau_*, \tau_{**}\}$ *satisfying the relations*

$$\sigma_{**} = \frac{2\left(\tau_*\gamma_*' + k\gamma_*\tau_*'\right)}{\tau_*(k^2-1)}, \quad k \in (0,1)$$

and for $W \in X_{\delta,\mu,q}$ *for all* $q > n/2$*, with* μ *given by Eq.* (4.49) *satisfying the following inequalities:*

$$0 < \mu_1 < \mu_3, \tag{4.57}$$

$$0 < \mu_2 < \min\{\xi_1, \mu_1, \mu_4, \mu_6\}, \tag{4.58}$$

$$0 < \mu_3 < \min\{1 + \xi_0, 2 - 2k\}, \tag{4.59}$$

$$0 < \mu_4 < \min\{1 + \xi_0, 2 - 2k\}, \tag{4.60}$$

$$0 < \mu_5 < 2, \tag{4.61}$$

$$0 < \mu_6 < \min\{1 + \xi_0, 2 - 2k\}. \tag{4.62}$$

Furthermore, $S_0^0 = \mathbb{I}_{18}$ *(the identity matrix in eighteen dimensions) and the matrix* N_0 *is block-diagonal with the blocks given by:*

$$N_0^{(1)} = \begin{pmatrix} 0 & -1 & 0 \\ (k^4-1)/2 + \Lambda/2(1-k^2) & -(1+3k^2)/2 + \Lambda & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$N_0^{(2)} = \begin{pmatrix} 0 & -1 & 0 \\ k^2/2(k^2-1) & (1-3k^2)/2 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$N_0^{(3)} = \begin{pmatrix} 0 & -1 & 0 \\ (k^2-1)^2/4 & 1-k^2 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$N_0^{(4)} = \begin{pmatrix} 0 & -1 & 0 \\ (k-1)^2 & -2+2k & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$N_0^{(5)} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -2k & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$N_0^{(6)} = \begin{pmatrix} 0 & -1 & 0 \\ (k+1)^2 & -2-2k & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

Proof. We first show the coefficient matrices S^0 and S^a satisfy the conditions of Definition 2.7. Due to the leading order terms $\mathring{U}$ we find

$$\begin{aligned}
\alpha(U) = &\ -\frac{U_1}{U_7} \\
= &\ -\mathring{U}_1\mathring{U}_7^{-1} - \mathring{U}_7^{-1}W_1 + \mathring{U}_7^{-2}\mathring{U}_1 W_7 + \mathcal{O}(\min\{\mu_1+\mu_3, 2\mu_3\}) \\
= &\ 1 - \gamma_*^{-1}t^{1/2(1-k^2)}W_1 - \gamma_*^{-1}t^{1/2(1-k^2)}W_7 + \mathcal{O}(\min\{\mu_1+\mu_3, 2\mu_3\}) \\
\equiv &\ \alpha_0 + \alpha_1(W)
\end{aligned}$$

and

$$\begin{aligned}
\beta(U) &= \frac{U_4}{U_7} \\
&= \mathring{U}_4\mathring{U}_7^{-1} + \mathring{U}_7^{-1}W_4 - \mathring{U}_7^{-2}\mathring{U}_4 W_7 + \mathcal{O}(\min\{\mu_2+\mu_3, 2\mu_3\}) \\
&= \sigma_{**}\gamma_*^{-1}t + \gamma_*^{-1}t^{1/2(1-k^2)}W_4 - \sigma_{**}\gamma_*^{-2}t^{1/2(3-k^2)}W_7 \\
&\quad + \mathcal{O}(\min\{\mu_2+\mu_3+1, 2\mu_3+1\})
\end{aligned}$$

where $\alpha_0 = 1$, $\alpha_1(\cdot) : X_{\delta,\mu,q} \to X_{\delta,\zeta_\alpha,q}$ for $\zeta_\alpha = \min_{x\in T^n}\{\mu_1, \mu_3\}$ and $\beta(\cdot) : X_{\delta,\mu,q} \to X_{\delta,1,q}$ are Lipschitz operators in the sense of Definition 2.3. Hence, consulting Eq. (4.15) we find $S_0^0 = \mathbb{I}_{18}$, and for each block $A = 1, \ldots, 6$

$$S_1^0(W)^{(A)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_1(W) \end{pmatrix}, \tag{4.63}$$

and

$$S^a(W)^{(A)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\beta(W) & -1-\alpha_1(W) \\ 0 & -1-\alpha_1(W) & 0 \end{pmatrix}. \tag{4.64}$$

It follows that there exists a constant exponent vector ζ with $0 < \zeta < 1$ such that $tS^a(W), S^0(W) \in X_{\delta,\zeta,q}$ for all $W \in B_{\delta,\mu,q}(s)$ and all $q > n/2$.

We now show that $f(\mathring{U}+W)+\sum_{j=0}^{n} tS^j(W)\partial_j\mathring{U} = N_0W - \mathcal{F}(\mathring{U})[W]$, where N_0 is as given in Lemma 4.7 and $\mathcal{F}(\mathring{U})[W]$ maps all $W \in B_{\delta,\mu,q}(s)$ to $X_{\delta,\nu,q}$ for some $\nu > \mu$, and satisfies the Lipschitz property provided the inequalities Eqs. (4.57)-(4.62) hold. In each block the vector $f(\mathring{U}+W)$ from Eq. (2.1) has the form

$$f_{3A-2}(U) = \, -U_{3A-1}, \tag{4.65}$$

$$f_{3A-1}(U) = H_A[U] - U_{3A-1}, \tag{4.66}$$

$$f_{3A}(U) = (U_1/U_7)U_{3A}. \tag{4.67}$$

Clearly the f_{3A-2} and f_{3A} components are quite simple to analyze while the f_{3A-1} components take more work. We start by analyzing Eq. (4.65). With the decomposition $U = \mathring{U} + W$, we find

$$f_{3A-2}(U) = -\mathring{U}_{3A-1} - W_{3A-1}.$$

As a consequence we set $(N_0^{(A)})_{(1,2)} = -1$, for each A. Furthermore, from Eq. (2.5) and Eq. (4.15) we find

$$(\mathcal{F}(\mathring{U})[W])_{3A-2} = \mathring{U}_{3A-1} - D\mathring{U}_{3A-2} = 0,$$

where the second equality holds due to the definition of $\mathring{U}$ (see Eq. (4.56)).

Next, consider the components Eq. (4.67).

$$f_{3A}(U) = (U_1/U_7)U_{3A} = -W_{3A} - \alpha_1(W)W_{3A} - \alpha(U)\mathring{U}_{3A}$$

Again we find $(N_0^{(A)})_{(3,3)} = -1$, for each A. From the definition $(f_1(W))_{3A} = \alpha_1(W)W_{3A} + \alpha(U)\mathring{U}_{3A}$ and hence

$$\begin{aligned}(\mathcal{F}(\mathring{U})[W])_{3A} &= \alpha_1(W)W_{3A} + \alpha(U)\mathring{U}_{3A} - \alpha(U)D\mathring{U}_{3A} + \alpha(U)t\partial_1\mathring{U}_{3A-1} \\ &= \alpha_1(W)W_{3A} - \alpha(U)\left(D\mathring{U}_{3A} + t\partial_1\mathring{U}_{3A-1} - \mathring{U}_{3A}\right)\end{aligned}$$

However, $D\mathring{U}_{3A} + t\partial_1\mathring{U}_{3A-1} - \mathring{U}_{3A} = 0$, by the definition of $\mathring{U}$, so

$$(\mathcal{F}(\mathring{U})[W])_{3A} = \alpha_1(W)W_{3A}.$$

Since $\alpha_1(W)$ is a Lipschitz operator which takes values in $X_{\delta,\zeta_\alpha,q}$ for $\zeta_\alpha = \min_{x\in T^n}\{\mu_1,\mu_3\}$, and the product of Lipschitz operators is again Lipschitz (Lemma C.16), the function operator $\alpha_1(W)W_{3A}$ is Lipschitz, and we obtain control in $X_{\delta,\nu,q}$ with $\nu > \mu$ provided $\mu_1, \mu_3 > 0$.

Finally we treat the $f_{3A-1}(U)$ components Eq. (4.66). The analysis of the $-U_{3A-1}$ term proceeds as before, and we focus our attention on the analysis of the functionals $H_A[U]$. Each $H_A[U]$ has the form

$$H_A[U] = \sum_{j=1}^{l_1^{(A)}}\prod_{i=1}^{18} c_j(t,x)U_i^{p_i^j} + \frac{1}{det(\gamma)[U]}\sum_{j=1}^{l_2^{(A)}}\prod_{i=1}^{18} d_j(t,x)U_i^{p_i^j}$$

where $p_i^j \in \mathbb{Z}$, and $c_j(t,x)$ and $d_j(t,x)$ are constants or functions of the spacetime coordinates. Since $\det(\tau) = \tau_{11}\tau_{22} = U_{10}U_{16}$, this function operator is contained in the expression above. The inverse determinant of the blocks γ and τ, which show up in the components of the inverse metric, appear independently due to the block-diagonal form of the metric. The inverse can be computed

$$g^{-1} = \begin{pmatrix} \gamma_{11}/\det\gamma & -\gamma_{01}/\det\gamma & 0 & 0 \\ -\gamma_{01}/\det\gamma & \gamma_{00}/\det\gamma & 0 & 0 \\ 0 & 0 & (\tau_{11}\tau_{01}^2 + \tau_{22})/\det\tau & -\tau_{11}\tau_{01}/\det\tau \\ 0 & 0 & -\tau_{11}\tau_{01}/\det\tau & \tau_{11}/\det\tau \end{pmatrix}. \quad (4.68)$$

To obtain an expression of the form Eq. (4.20) we must address the function operator $(\det\gamma)^{-1} = (U_1U_7 - U_4^2)^{-1}$; clearly this operator does not have the desired form. In our analysis we replace the operator $(\det\gamma)^{-1}$ by the operator $(U_1U_7)^{-1}$, which does have the desired form. To justify this simplification we note that due to the leading order expressions $\mathring{U}_1, \mathring{U}_4, \mathring{U}_7$, the expansion

$$1/\det(\gamma)[U] = \frac{1}{U_1U_7} + J[U],$$

for a function operator $J[U]$ which is $O(t^{3-k^2})$, is valid near the singularity. It turns out that the contribution of $J[U]$ can be ignored (that is it is higher order) if $\mu_i < 2$ for $i = 1, \ldots, 6$.[3] We make this assumption now, and verify that it is satisfied in the analysis below. Notice that each μ_i in Eqs. (4.57)-(4.62) is bounded above by two for $k \in (0,1)$.

[3]This can be see by multiplying $\frac{1}{U_1U_7} + J[U]$ by any function operator of the form $\prod_{i=1}^{18} c(t,x)U_i^{p_i}$.

At this point we have written each $H_A[U]$ in the form $\sum_{j=1}^{l_1^{(A)}} \prod_{i=1}^{18} c_j(t,x) U_i^{p_i^j}$. We refrain from showing these here because the number of terms is quite long ((22, 29, 22, 6, 8, 6) respectively), and because the specific terms are not particularly interesting. The next step in our analysis is to sort the terms given the expansion $U = \mathring{U} + W$, and determine for each $A = 1, \ldots, 6$, the functions $h_0(t,x), h_1^1(t,x), h_1^2(t,x)$ and $h_2(t,x,W)$ introduced in Eq. (4.20). We describe this analysis here, and give a couple of examples below.

As a first step we evaluate each term at $U = \mathring{U}$, and by inspecting the exponent determine in which space $X_{\delta,\nu,q}$ this function takes values. Note that the Fuchsian principle part is $O(t^{\mu_A+\kappa_A})$. If $\nu \leq \mu_A + \kappa_A$, then we place the function in $h_0^{(A)}(t,x)$ and save the term for later analysis. This is what we might call a "singular" term. If, on the other hand it is not clear that $\nu \leq \mu_A + \kappa_A$, we place function in $h_2^{(A)}(t,x,W)$ (and hence in $\mathcal{F}(\mathring{U})[W]$) and record the upper bound $\mu_A < \nu - \kappa_A$. Since such terms are higher-order at leading order, and we assume $\mu_A > 0$, these terms play no further role in the analysis.

Next we analyze the linear portions of the "singular" function operators. These terms can be divided into "within-block" terms containing the fields $W_{3A-2}, W_{3A-1}, W_{3A}$ and "out of block" terms which are proportional to the remaining W-field components. The within-block terms with $O(1)$ coefficients contribute to the Fuchsian principle part, with the coefficients forming elements of $N_0(x)$. We check that the remaining linear within-block terms have coefficients which are $O(t^\epsilon)$ for $\epsilon > 0$, and hence go into $\mathcal{F}(\mathring{U})[W]$. Also contributing to $\mathcal{F}(\mathring{U})[W]$ are the linear terms with out of block W-field components. We record the space $X_{\delta,\nu,q}$ in which each such term takes values, as well as the bound $\mu_A + \kappa_A < \nu = \mu_B + \kappa_B + \rho$ for $B \neq A$, and thus $\mu_A < \mu_B + \kappa_B - \kappa_A + \rho$. The higher-order parts of the function operator (i.e.

those at least quadratic in the W-field components) are part of $\mathcal{F}(\mathring{U})[W]$ and provide no additional information. The inequalities recorded above in Eqs. (4.57)-(4.62) are the maximal bounds consistent with our sorting of terms.

Due to the large number of terms in the equation under this type of expansions, even in relatively simple cases such as the Gowdy spacetimes, this analysis is implemented in the computer algebra system Mathematica. We now provide a few examples of the analysis described above for terms in $H_1[U]$.

Examples:

1. Consider the term

$$T_1[U] := tF_0(t,x)U_2.$$

At leading order, that is with $U_2 = \mathring{U}_2 = -\gamma_* \kappa_1 t^{\kappa_1}$, with $\kappa_1 = 1/2(k^2-1)$, we have

$$T_1[\mathring{U}] = -\gamma_*(x)\kappa_1(x)t^{1+\kappa_1(x)}F_0(t,x).$$

Since $F_0(t,x) \in X_{\delta,\xi_0,\infty}$, we know that at leading order this function behaves like $\varphi_0(x)t^{\xi_0}$ for some smooth function $\varphi_0(x)$. We expect this term to be in the Fuchsian source since there is no way for such a term to cancel with derivative terms in the principle part (since it involves gauge source functions), and the areal gauge is obtained by the limit $\xi_0 \to \infty$ (and thus it makes no sense to have this term in the Fuchsian principle part). The Fuchsian PP which is $\mathcal{O}(\mu_1 + \kappa_1)$, and as a result we obtain the inequality

$$\boxed{1 + \xi_0 > \mu_1}.$$

The next term in $T_1(\mathring{U})[W]$, which is $tF_0(t,x)W_2$, is higher order still, and placing it in $\mathcal{F}(\mathring{U})[W]$ only carries the information that $1 + \xi_0 > 0$.

This is an example of a "higher-order" function operator, and how such operators determine the bounds on μ. In fact, bounds such as this one are observed in Eqs. (4.57)-(4.62) above.

2. Next consider the function operator

$$T_2[U] = \frac{3}{2}\frac{U_2^2}{U_1},$$

which also appears in $H_1[U]$. The leading order function is easily computed

$$T_2[\mathring{U}] = \frac{3}{2}\mathring{U}_2^2\mathring{U}_1^{-1} = \frac{3}{2}\kappa_1^2\gamma_* t^{\kappa_1}$$

Since $\kappa_1 < \kappa_1 + \mu_1$ under the assumption that $\mu_1 > 0$, this is "more singular" than the Fuchsian principle part. Hence, this function contributes to $h_0^{(1)}(t,x)$, and we consider higher-order parts of the function operator –i.e. those linear in W-fields.

The theory in Section C.3. shows that the function operators $U_2^2[W]$ and $U_1^{-1}[W]$ have the following expansions

$$\begin{aligned} U_1^{-1} &= \left(\mathring{U}_1^{-1} - \mathring{U}_1^{-2}W_1 + r_1(W)\right) \\ U_2^2 &= \left(\mathring{U}_2^2 + 2\mathring{U}_2 W_2 + r_2(W)\right) \end{aligned}$$

where $r_1(W) = \mathcal{O}(-\kappa_1 + 2\mu_1)$, and $r_2(W) = \mathcal{O}(2\kappa_1 + 2\mu_1)$. The linear part of the operator $T_2[U]$ is

$$-\frac{3}{2}\mathring{U}_2^2\mathring{U}_1^{-2}W_1 + 3\mathring{U}_1^{-1}\mathring{U}_2 W_2 = -\frac{3}{2}\kappa_1^2 W_1 + 3\kappa_1 W_2.$$

Comparing to Eq. (4.18), we see that $-\frac{3}{2}\kappa_1^2$ contributes to $h_0^{(1),1}(x)$ and $3\kappa_1$ to $h_0^{(1),2}(x)$. The next terms are all $\mathcal{O}(\kappa_1 + 2\mu_1)$, which is higher-order, again since $\mu_1 > 0$, so that the remaining parts $T_2[U]$ go into $h_2(t, x, W)$. Further, since $r_1(W)$ and $r_2(W)$ are Lipschitz operators (see Section C.3.), this property is achieved for the contributions to $\mathcal{F}(\mathring{U})[W]$. Note that this function operator does not constraint μ at all.

It remains to verify that the singular terms in $\sum_{j=0}^{n} tS^j(W)\partial_j\mathring{U}$ are canceled by corresponding singular terms in $h_{A,0}(t,x)$ possibly under additional restrictions on the asymptotic data, and to record the space in which the remaining terms live. We first investigate the spatial derivative terms. From Eq. (4.13) these are

$$2\beta(U)t\partial_1\mathring{U}_{3A-1} - \alpha(U)t\partial_1\mathring{U}_{3A}.$$

Since $\mathring{U}_{3A-2}$, which corresponds to the field itself, is $\mathcal{O}(\kappa_A)$, we have $\mathring{U}_{3A-1} = \mathcal{O}(\kappa_A)$ and $\mathring{U}_{3A} = \mathcal{O}(\kappa_A + 1)$ from Eq. (4.56). Above we computed that at leading order $\alpha_0(\mathring{U}) = 1$ and $\beta(\mathring{U}) = O(t)$. Thus each of the above spatial derivative terms is $\mathcal{O}(\kappa_A + 2)$. These terms provide a bound on μ:

$$\mu_A + \kappa_A < \kappa_A + 2 \Rightarrow \mu_A < 2$$

which is satisfied by the inequalities Eqs. (4.57)-(4.62) for $k \in (0, 1)$.

Next we verify that

$$D\mathring{U}_{3A-1} - U_{3A-1} + h_{A,0}(t,x) = 0.$$

This is satisfied identically for all A, except in the $A = 2$ block, where we require $\sigma_{**} = \frac{2(\tau_*\gamma_*' + k\gamma_*\tau_*')}{\tau_*(k^2-1)}$. This establishes Lemma 4.7. □

4.4.1.3. Existence and Uniqueness to the Evolution Equations

In this subsection we apply Theorem 2.28 to the quasilinear symmetric hyperbolic Fuchsian system found in Lemma 4.7. The following proposition is a result of this application.

Proposition 4.8. *There exists a unique solution* $U = \mathring{U} + W$ *to the system defined by Eq.* (4.12)*-Eq.* (4.14) *and Eq.* (4.40)*-Eq.* (4.45) *with* $\mathring{U}$ *given by Eq.* (4.56) *and Eq.* (4.50)*-Eq.* (4.55) *for* $W \in X_{\delta,\mu+\kappa,q}$ *for all* $q > n/2 + 1$ *provided: The asymptotic data* $\{k, \gamma_*, \sigma_{**}, \tau_*, \tau_{**}\}$ *satisfy the relations*

$$\sigma_{**} = \frac{2\,(\tau_* \gamma'_* + k\gamma_* \tau'_*)}{\tau_*(k^2 - 1)}, \quad k \in (0, 1)$$

The exponent vector μ *given by Eq.* (4.49) *satisfies*

$$\max\{(1 - k^2)/2, 1/2(3 + k^2) - \Lambda\} < \mu_1 < \mu_3, \tag{4.69}$$

$$0 < \mu_2 < \min\{\xi_1, \mu_1, \mu_4, \mu_6\}, \tag{4.70}$$

$$(1 - k^2)/2 < \mu_3 < \min\{1 + \xi_0, 2 - 2k\}, \tag{4.71}$$

$$0 < \mu_4 < \min\{1 + \xi_0, 2 - 2k\}, \tag{4.72}$$

$$2k < \mu_5 < 2, \tag{4.73}$$

$$0 < \mu_6 < \min\{1 + \xi_0, 2 - 2k\}. \tag{4.74}$$

Proof. We have shown in Lemma 4.7 that this system is a smooth quasilinear symmetric hyperbolic Fuchsian system; it remains to verify that the hypotheses of Theorem 2.28 are satisfied.

The System Satisfies the Smooth Commutator Conditions Definition 2.27

First note that the matrix $S^0(w)$ is diagonal, and thus in particular S^0_0 commutes with $\mathcal{R}[\mu]$. Further, from the structure of N_0 given in Lemma 4.7, and that of μ in Eq. (4.49) we see that N_0 commutes with $\mathcal{R}[\mu]$ as well.

Recall the structure of each block of the matrix-valued operator $S_1^0(\cdot)$from Eq. (4.63). Since $S_1^0(\cdot)$ is diagonal and has a target $X_{\delta,\zeta_\alpha,q}$ with $\zeta_\alpha = \min\{\mu_1, \mu_3\}$ the requisite property for $S_1^0(\cdot)$ holds provided $\mu_1, \mu_3 > 0$.

Next we consider the condition on $tS^1(\mathring{U} + W)$ block-wise. The argument is similar to that for the T^2-symmetric solutions in Section 3.4.. Each block of $tS^a(\mathring{U} + W)$ and $\mathcal{R}[\mu]$, denoted by the index A, have the form (c.f. Eq. (4.64) and Eq. (4.49))

$$(tS^1(\mathring{U} + W))^{(A)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2t\beta(W) & -t - t\alpha_1(W) \\ 0 & -t - t\alpha_1(W) & 0 \end{pmatrix},$$

and

$$(\mathcal{R}[\mu])^{(A)} = \begin{pmatrix} t^{-\mu_A-\kappa_A} & 0 & 0 \\ 0 & t^{-\mu_A-\kappa_A} & 0 \\ 0 & 0 & t^{-\mu_A-\kappa_A-(1-\epsilon)} \end{pmatrix}.$$

As a result, each block of $\mathcal{R}[\mu]tS^a(\mathring{U} + W)\mathcal{R}[-\mu]$ is equal to

$$(\mathcal{R}[\mu]tS^1(\mathring{U} + W)\mathcal{R}[-\mu])^{(A)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2t\beta(W) & -(1+\alpha_1(W))t^{2-\epsilon} \\ 0 & -(1+\alpha_1(W))t^{\epsilon} & 0 \end{pmatrix}.$$

It is clear that in order to have control in $B_{\delta',\xi,q}(\tilde{r})$ for some exponent scalar $\xi > 0$ it is necessary to assume $\epsilon > 0$. This is the reason for the slight loss of control over the spatial derivatives components. With this choice, and recalling properties of $\alpha_1(\cdot)$ and $\beta(\cdot)$ (see proof of Lemma 4.7) we find that the Condition 2.27 are satisfied.

The Product Compatibility Conditions and Higher-Order Source Conditions Hold Based on the structure of $\alpha_1(\cdot)$ and $\beta(\cdot)$ (c.f. proof of Lemma 4.7), observe that

$$\Delta\alpha_1(h) := \alpha_1(W) - \alpha_1(W+h) \in X_{\delta,\eta_\alpha,q} \quad \Delta\beta(h) := \beta(W) - \beta(W+h) \in X_{\delta,\eta_\beta,q}$$

with $\eta_\alpha = \gamma_0 + \min\{\mu_1, \mu_3\}$ and $\eta_\beta = 1 + \gamma_0 + \min\{\mu_2, \mu_3\}$. From the diagonality of $S_1^0(\cdot)$ it follows that Condition (i) of Definition 2.25 is satisfied provided $\mu_1, \mu_3 > 0$. To check Condition (ii), we compute block-wise

$$\left(\mathcal{R}[\mu]t(S^1(W) - S^1(W+h)\mathcal{R}[-\mu]\right)^{(A)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2t\Delta\beta(h) & -\Delta\alpha_1(h)t^{2-\epsilon} \\ 0 & -\Delta\alpha_1(h)t^{\epsilon} & 0 \end{pmatrix}.$$

The condition follows from form of η_α and η_β and the positivity of μ_1, μ_2, μ_3.

The higher-order source conditions Definition 2.26 follow from the form of $H_A[U]$ Eq. (4.20) and the results of Section C.3.3.

The Positivity Condition (iii) is Satisfied This is the positivity condition involving the matrix $\mathcal{N}$ and the exponent vector μ; it may be satisfied provided certain bounds on the components of μ are met. We state this as the following lemma.

Lemma 4.9. *The system in Lemma 4.7 satisfies Condition (iii) of Theorem 2.28 if the following inequalities on the exponent vector hold*

$$\begin{aligned}
\mu_1 >& \max\{1/2(1-k^2), 1/2(3+k^2)-\Lambda\},\\
\mu_2 >& 0,\\
\mu_3 >& 1/2(1-k^2),\\
\mu_4 >& 0,\\
\mu_5 >& 2k,\\
\mu_6 >& 0.
\end{aligned}$$

Proof of Lemma 4.9. The matrix S_0^0 is the identity, and hence $\mathcal{N} = N_0$. To make the analysis simpler we bring $\mathcal{N}$ into Jordan normal form, which we label $\underline{\mathcal{N}}$. Since N_0 has the block-diagonal structure of μ (Definition 2.8), it follows that $\underline{\mathcal{N}}$ also has the block-diagonal structure of μ. As a result, we may easily read off the inequalities obtained from the condition $\mu < -Re\{\lambda\}$, where λ is the vector of eigenvalues of $\underline{\mathcal{N}}$.

In the first block we have

$$(\underline{\mathcal{N}})^{(1)} = \begin{pmatrix} 1/2(1-k^2) & 0 & 0 \\ 0 & \Lambda - 1 - k^2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The positivity condition applied to the first component then yields $\mu_1 + 1/2(k^2-1) > -1/2(1-k^2)$, which implies $\mu_1 > 0$. Identical computations for the second and third components yield

$$\mu_1 > 1/2(3+k^2) - \Lambda, \quad \mu_1 > 1/2(1-k^2) + \epsilon.$$

We can take ϵ arbitrarily close to zero, yet for any choice of Λ the smallest lower bound on μ_1 is $1/2(1-k^2)$. Since we leave Λ arbitrary at this stage, the lower bound is as reported in Lemma 4.9. For the remaining blocks we simply list the block of $\underline{\mathcal{N}}$ and the resulting inequalities.

$$(\underline{\mathcal{N}})^{(2)} = \begin{pmatrix} 1/2(1-k^2) & 0 & 0 \\ 0 & -k^2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \mu_2 > \max\{-1/2(1+k^2)+\epsilon, 0\}.$$

$$(\underline{\mathcal{N}})^{(3)} = \begin{pmatrix} 1/2(1-k^2) & 1 & 0 \\ 0 & 1/2(1-k^2) & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mu_3 > \max\{1/2(1-k^2)+\epsilon, 0\}.$$

$$(\underline{\mathcal{N}})^{(4)} = \begin{pmatrix} k-1 & 1 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \mu_4 > \max\{0, k-1+\epsilon\}.$$

$$(\underline{\mathcal{N}})^{(5)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2k & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \mu_5 > \max\{2k, \epsilon\}.$$

$$(\underline{\mathcal{N}})^{(6)} = \begin{pmatrix} -1-k & 1 & 0 \\ 0 & -1-k & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \mu_6 > \max\{0, \epsilon-1-k\}.$$

Taking the limit $\epsilon \to 0$ concludes the proof of Lemma 4.9. □

To finish the proof of Proposition 4.8 we note that the inequality $1/2(1-k^2) < 2-2k$ is consistent with the condition $k \in (0,1)$. The choice of coefficient Λ is discussed after analyzing the constraint equations. □

4.4.2. Analysis of the Constraint Equations

In the Cauchy formulation of the Einstein equations one verifies that the constraint violation quantities are propagated by the evolution equations, so that if initial data is chosen such that the constraint equations are satisfied on the initial slice, then they are guaranteed to be satisfied in the domain of dependence of the initial data. In the Fuchsian formulation we work with in this paper, one can only guarantee, by an appropriate choice of leading order terms, that the constraint equations are satisfied asymptotically. The Fuchsian formulation of the constraint propagation system in Section 4.2.3. allows us to argue that if the constraints are satisfied asymptotically, then they are satisfied in a region $(0, \delta] \times T^n$ near the singularity.

We start by constructing a first order system for the constraint violation quantities $\mathcal{D}_i = \Gamma_i - \mathcal{F}_i$. Two of these are identically satisfied $\mathcal{D}_2 = \mathcal{D}_3 \equiv 0$, so we are left with a six-dimensional system for

$$(V_1, \ldots, V_6) = (\mathcal{D}_0, D\mathcal{D}_0, t\partial_x \mathcal{D}_0, \mathcal{D}_1, D\mathcal{D}_1, t\partial_x \mathcal{D}_1).$$

The computations in Section 4.2.3. show that the resulting first order system for the constraint propagation equations are of the form Eq. (4.23) and Eq. (4.24), where $A = 1, 2$ denotes the block of equations corresponding to $\mathcal{D}_0$ and $\mathcal{D}_1$ respectively.

In the section below we show, using the known solutions of the Einstein evolution equations, that under certain constraints on the asymptotic data the first order quantities $(V_1, \ldots, V_6)$ vanish asymptotically as $t \searrow 0$. As a result a zero leading order term $\mathring{V} = 0$, is consistent with the solutions to the evolution equations. In Section 4.4.2.2. we show that the first order system forms a linear symmetric hyperbolic Fuchsian system about this zero leading order term, and the Fuchsian theorem (Theorem 2.28) may be applied in order to conclude that $\mathcal{D} = 0$ is the unique solution to the constraint propagation equation in some space $X_{\delta,\eta,q}$.

4.4.2.1. The first order constraint violation quanities vanish asymptotically

Having found solutions to the Einstein evolution equations in the previous section, we can straightforwardly compute the constrain violation quantities $\mathcal{D}_i := \Gamma_i - \mathcal{F}_i$; we have

$$\mathcal{D}_0 = 1/t - F_0(t,x) + \mathcal{D}_0(g), \quad \text{and} \quad \mathcal{D}_1 = -F_1(t,x) + \mathcal{D}_1(g),$$

where $\mathcal{D}_0(g)$ and $\mathcal{D}_1(g)$ are nonlinear functions of the metric fields. Inserting the known expressions for the metric fields $g = \mathring{g} + \hat{g}$, we find that at leading order

$$\mathcal{D}_1(\mathring{g}) = -\sigma_{**}/\gamma_*.$$

Thus, in order for $\mathcal{D}_1$ vanish at leading order would require that $F_1(t,x)$ have an $O(1)$ term, in contradiction with the condition of asymptotically areal gauge that $F_1(t,x)$ vanish as $t \searrow 0$. It follows that in order to satisfy the constraints asymptotically in the asymptotically areal gauge, we must choose $\sigma_{**}(x) \equiv 0$. What then is the first non-vanishing term in the expansion for γ_{01}? To answer this question we write

$$\mathring{\gamma}_{01} = \gamma_{**}(x)t^{\lambda(x)},$$

for arbitrary $\gamma_{**}(x)$ and $\lambda(x)$. We also write the leading order behavior of the gauge source function $F_1(t,x)$ as

$$\mathring{F}_1(t,x) = \varphi_1(x)t^{\xi_1},$$

recalling that $F_1 \in X_{\delta,\xi_1,\infty}$. The asymptotic form of the constraint $\mathcal{D}_1 = 0$ then yields

$$\frac{\gamma_{**}}{2\gamma_*}(k^2 - 1 - 2\lambda)t^{\lambda - 1/2(1+k^2)} - \varphi_1(x)t^{\xi_1} = 0,$$

which implies that

$$\lambda = 1/2(k^2+1) + \xi_1 \quad \text{and} \quad \gamma_{**} = -\frac{\varphi_1\gamma_*}{1+\xi_1}.$$

With this form for $\mathring{\gamma}_{01}$ we find [4]

$$\begin{aligned}
\mathcal{D}_0 &= -t^{\xi_0}\varphi_0 + t^{1+\xi_1}\varphi_1'/(1+\xi_1) + \mathcal{O}(1+2\xi_1),\\
D\mathcal{D}_0 &= \mathcal{O}(\min\{\xi_0, 1+\xi_1\})\\
t\partial_x\mathcal{D}_0 &= \mathcal{O}(\min\{1+\xi_0, 2+\xi_1\}),
\end{aligned}$$

all of which vanish asymptotically for $\xi_0 > 0, \xi_1 > 0$, and

$$\begin{aligned}
\mathcal{D}_1 &= \mathcal{O}(\min\{\mu_1, \mu_2+\xi_1, \mu_3, \mu_4, \mu_6\}),\\
D\mathcal{D}_1 &= \mathcal{O}(2+3\xi_1)\\
t\partial_x\mathcal{D}_1 &= \mathcal{O}(3+3\xi_1).
\end{aligned}$$

[4]Recall that a function f is $\mathcal{O}(\mu)$ if $f(t) = O(t^\mu)$.

Since the first order fields for the constraint violation quantities vanish asymptotically given solutions to the evolution equations with the specified leading order terms and asymptotic data satisfying certain constraints, the leading order term $\mathring{V} = 0$ is consistent.

The updated leading order term for the shift γ_{01} modifies the constraint on the asymptotic data which we obtained in the analysis of the evolution equations, namely that

$$\sigma_{**} = \frac{2\,(\tau_* \gamma_*' + k\gamma_* \tau_*')}{\tau_*(k^2-1)}.$$

If we now impose $\sigma_{**} = 0$, we find the new constraint on the asymptotic data is

$$\gamma_*'/\gamma_* = -k\tau_*'/\tau_*.$$

4.4.2.2. Constraint propagation equation in the case of the Gowdy equations

Having obtained the conditions under which the constraint violation quantities vanish asymptotically, we now use the Fuchsian formulation of the constraint propagation equations, Section 4.2.3., to show that the $\mathcal{D} = 0$ is the unique solution in one of our weighted Sobolev spaces. In a first lemma we verify that for a zero leading order term the first order system for $(V_1, \ldots, V_6)$ is a smooth linear symmetric hyperbolic Fuchsian system. As is illustrated in Section 4.2.3. the main work is in analyzing H_1, H_2 and in particular in computing the limiting matrices $\{a^{(0)}, a^{(1)}, b, c, c^{(0)}, c^{(1)}\}$. In order to compute these matrices we use the updated information about the leading order term $\mathring{\gamma}_{01}$, as well as the leading order terms for the other metric fields. The matrices $c_i^j, c^{(0)}{}_i^j$, and $c^{(1)}{}_i^j$ depend on the coefficients ${C_{ij}}^k$, which as above are ${C_{00}}^0 = \Lambda(x)/t$ and all other coefficients vanishing.

Lemma 4.10. *The system Eq.* (4.23) *and Eq.* (4.24) *is a smooth linear symmetric hyperbolic Fuchsian system about* $\mathring{V} = 0$ *with parameters* $\{\delta, \eta, s\}$ *for* η *given by*

$$\eta = (\eta_1, \eta_1, \eta_1 + 1 - \epsilon, \eta_2, \eta_2, \eta_2 + 1 - \epsilon)$$

satisfying

$$|\eta_1 - \eta_2| < 1.$$

Further, $S_0^0 = \mathbb{I}_6$ *and the* N_0*-matrix is block-diagonal with blocks*

$$N_0{}^{(1)} = \begin{pmatrix} 0 & -1 & 0 \\ -1/2(1+k^2) - \Lambda & 1/2(3-k^2) - 1 - \Lambda & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$N_0{}^{(2)} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1/2(3-k^2) - 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Proof. We start by finding expressions for the S^0 and S^a matrices, and verifying that they satisfy the properties of Definition 2.7. Comparing Eq. (4.23) with Eq. (4.12) -Eq. (4.14) we see that S^0 and S^a are the same in both applications, and hence are given as in Lemma 4.7. We note however that in this application the matrices depend on the metric fields and not the unknown (the constrain violation quantity). It follows that S^0 and S^a satisfy the properties of a symmetric hyperbolic Fuchsian system.

As before, the bulk of analysis for the $f(W)$ term concerns the functionals $H_1[W]$ and $H_2[W]$. In order to facilitate this analysis we compute the relevant quantities from Section 4.2.3..

$$a^{(0)} = \begin{pmatrix} 1/2(k^2-3) & 0 \\ 0 & 1/2(k^2-3) \end{pmatrix},$$

$$a^{(1)} = \begin{pmatrix} 0 & 1/2(1-k^2) \\ 1/2(1-k^2) & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} 1/2(1+k^2) & 0 \\ 0 & 0 \end{pmatrix},$$

and further

$$t(g^{00})^{-1}A_i^{0j} - a^{(0)} = \begin{pmatrix} \mathcal{O}(2+\xi_1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(2+\xi_1) \end{pmatrix},$$

$$t(g^{00})^{-1}A_i^{1j} - a^{(1)} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(2+2\xi_1) \\ \mathcal{O}(2+\xi_1) & \mathcal{O}(1) \end{pmatrix},$$

$$t^2(g^{00})^{-1}B_i^j - b = \begin{pmatrix} \mathcal{O}(2) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(2-2k) \end{pmatrix}.$$

We also find

$$\frac{t^2}{g^{00}}\widetilde{C}_i^i = \begin{pmatrix} \Lambda + \mathcal{O}(2+\xi_1) & 0 \\ \mathcal{O}(1) & 0 \end{pmatrix},$$

and

$$\frac{t}{g^{00}}\widetilde{C^{(0)}}_i^{\,i} = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}, \quad \frac{t}{g^{00}}\widetilde{C^{(1)}}_i^{\,i} = \begin{pmatrix} \mathcal{O}(1+\xi_1) & 0 \\ \Lambda & 0 \end{pmatrix},$$

From these expressions and the definition of $H_A[W]$ (Eq. (4.24)), we write down the expressions for the non-trivial components of $f(W)$: $f(W)_2$ and $f(W)_5$. Generally we have

$$f(W)_{3A-1} = -W_{3A-1} + H_A[W],$$

with $H_A[W]$ given by Eq. (4.24). From the first row of the matrices above we compute

$$\begin{aligned} f(W)_2 =& - W_2 + H_1[W] \\ =& - (\frac{1}{2}(1+k^2) + \Lambda)W_1 + (\frac{1}{2}(3-k^2) - 1 - \Lambda)W_2 \\ &+ \frac{1}{2}(k^2-1)W_6 \\ &\mathcal{O}(\eta_2+1) + \mathcal{O}(\eta_1+2) \end{aligned}$$

The terms in the first line are apart of N_0. The term in the second line does not belong in the N_0 matrix since it would break the block-diagonal structure. As a result, we place it in $\mathcal{F}(\mathring{V})[W]$ and this imposes a constraint on the exponent vector η: $\eta_2+1-\epsilon > \eta_1$. The first term in the last line also imposes the condition $\eta_2+1 > \eta_1$, while the second term is clearly higher order.

We similarly compute

$$\begin{aligned} f(W)_5 =& - W_5 + H_2[W] \\ =& \left(\frac{1}{2}(3-k^2) - 1\right) W_5 \\ &(\frac{1}{2}(k^2-1) + \Lambda)W_3 \\ &+ \mathcal{O}(1+\eta_1) + \mathcal{O}(\eta_2+2-2k). \end{aligned}$$

Again the first line is apart of the N_0 matrix. From the term in the second line we obtain $\eta_1+1-\epsilon > \eta_2$. Another constraint on η is created by the first term in the third line. In order to place this term into $\mathcal{F}(\mathring{V})[W]$ we must choose $\eta_2 < 1+\eta_1$. The last set of terms of $\mathcal{O}(\eta_2 + 2 - 2k)$ are clearly higher order.

Combining the contributions to N_0 in the above two expressions with the usual terms -1 from Eq. (4.27), we find the blocks listed in the lemma. □

We now prove that this smooth linear symmetric hyperbolic Fuchsian system satisfies the conditions of Theorem 2.28.

Proposition 4.11. $V = 0$ *is the unique solution to the smooth linear symmetric hyperbolic Fuchsian system of Lemma 4.10 in* $X_{\delta,\eta,q}$ *for all* $q > n/2+1$ *and* η *satisfying*

$$\eta_1 > \max\{1/2(1+k^2)+\Lambda, 0\} \quad \eta_2 > 0$$

Proof. Since the system is linear homogeneous, we know that $V \equiv 0$ is a solution. The Fuchsian theorem (Theorem 2.28) tells us in which space we are able to guarantee the uniqueness of this solution. In Lemma 4.10 we have verified that $\mathcal{F}(0)[W]$ is a bounded operator in some space $X_{\delta,\nu,q}$ for some $\nu > \eta$. Since this operator is linear in W it also satisfies the higher-order source properties, Definition 2.26. The product compatibility conditions (Definition 2.25 do not apply in the case of a linear system, and the coefficients S^0 and S^a are easily shown to satisfy the smooth commutator conditions (Definition 2.27) for the exponent vector η given in Lemma 4.10. We note as in the case of the evolution equations that we must choose $\epsilon > 0$, although it can be arbitrarily close to zero. The main work in applying Theorem 2.28 is in checking Condition (iii). From the expressions for S_0^0 and N_0 in Lemma 4.10 we compute $\mathcal{N}$ and transform this into Jordan normal form. Because of the block-diagonal structure of the N_0 matrix, $\underline{\mathcal{N}}$ has the block-diagonal structure of the exponent vector η. We find

$$\underline{\mathcal{N}}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2(1+k^2) - \Lambda & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \underline{\mathcal{N}}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2(1-k^2) & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The inequalities given by $\eta > -Re\{\lambda\}$ are satisfied if the inequalities in the proposition statement hold. □

Note that the lower bound on η sets the largest space in which we may guarantee that $\mathcal{D} = 0$ is the unique solution to the constraint propagation system. In Section 4.4.2.1. we computed the constraint violation quantities from the metric fields and found

$$\mathcal{D}_0 \in X_{\delta,\rho_0,q}, \quad \mathcal{D}_1 \in X_{\delta,\rho_1,q},$$

where $\rho_0 = \min\{\xi_0, 1+\xi_1\}$, and $\rho_1 = \min\{\mu_1, \mu_2+\xi_1, \mu_3, \mu_4, \mu_6\}$. This is a measure of how much control we actually have over the constraint violation quantities. In order to guarantee that these constraints are propagated uniquely by the evolution system, we require our level of of control to be at least as great as that required for the solution to be unique (that specified by $X_{\delta,\eta,q}$). That is, we require $X_{\delta,\rho,q} \subset X_{\delta,\eta,q}$, or $\eta < \rho$. This imposes the conditions

$$\min\{\xi_0, 1+\xi_1\} > \max\{1/2(1+k^2)+\Lambda, 0\},$$

and hence gives the lower bounds on ξ_0, ξ_1

$$\xi_0 > 1/2(1+k^2)+\Lambda, \quad \xi_1 > 1/2(-1+k^2)+\Lambda. \tag{4.75}$$

4.4.3. Addition of Constraint Violation Quantity

The addition of $(\Lambda(x)/t)\mathcal{D}_0$ to the γ_{00} evolution equation in Section 4.4.1. is designed to improve the lower bound on μ_1

$$\max\{(1-k^2)/2, 1/2(3+k^2)-\Lambda\} < \mu_1.$$

The idea is to choose Λ in order to obtain the smallest lower bound for μ_1, corresponding to the largest space in which uniqueness of the solution can be guaranteed. It is clear that the best one can do independent of the choice of Λ is a lower bound of $(1-k^2)/2$. Choosing Λ such that $1/2(3+k^2)-\Lambda \leq (1-k^2)/2$, we find $\Lambda(x) \geq 1+k^2$.

Due to the modified constraint propagation system Section 4.2.3., the coefficient Λ also shows up in the lower bound for η_1 where, $\eta_1 > \max\{1/2(1+k^2)+\Lambda, 0\}$. From this inequality, it appears that in order to optimize the lower bound for η_1 we should choose $\Lambda \leq -1/2(1+k^2)$, which is clearly at odds with the optimum choice according to the evolution equations.

Unless we optimize the lower bound on μ_1 we are left with a severe restriction on the asymptotic data $k(x)$, since $\mu_1 < \mu_3 < 2-2k$. Therefore we choose

$$\Lambda(x) = 1+k^2,$$

and deal with the consequences of an non-optimal bound for η_1. The consequence is a stricter lower bound on the exponent vectors from the gauge source functions. From Eq. (4.75) we compute

$$\xi_0 > 3/2(1+k^2), \quad \xi_1 > 1/2(1+3k^2).$$

With this choice of Λ we can simplify the inequalities in Proposition 4.8. Due to the lower bound on ξ_0, we find $1 + \xi_0 > 2 - 2k$, so that these inequalities become

$$\begin{aligned}
(1-k^2)/2 &< \mu_1 < \mu_3, \\
0 &< \mu_2 < \min\{\xi_1, \mu_1, \mu_4, \mu_6\}, \\
(1-k^2)/2 &< \mu_3 < 2 - 2k, \\
0 &< \mu_4 < 2 - 2k, \\
2k &< \mu_5 < 2, \\
0 &< \mu_6 < 2 - 2k.
\end{aligned}$$

This concludes the proof of Theorem 4.4.

4.5. Solutions in $\mathcal{S}(\xi_0, \xi_1)$ Are AVTD

We prove Lemma 4.5. Fix an asymptotically areal gauge satisfying the inequalities on ξ_0, ξ_1 in Theorem 4.4, and a fix a choice of asymptotic data in $\mathcal{K}$. To proceed we drop the spatial derivative terms from the Einstein evolution equations Eq. (4.3), and multiply by $-2t^2(g^{00})^{-1}$ to eliminate the singular coefficient. These are the same manipulations as we have done in Section 4.4. for the Fuchsian analysis, except that in additiona we have essentially ignored the spatial derivative terms here. The result is a system of six, coupled, second-order, nonlinear ordinary differential equations of the form

$$D^2 g_{ij} - D g_{ij} + B_1^k(g)\mathcal{F}_k + B_2(g) = 0,$$

where $B_1^k(g)$ and $B_2(g)$ are nonlinear functions of the metric fields. To verify that these equations are asymptotically satisfied by the leading order terms Eqs. (4.32)-(4.37) we insert the leading order terms into the equation and evaluate the limit $t \searrow 0$. Labeling each ODE operator by $VTD_A(g)$, $A = 1, \ldots, 6$, we find

$$
\begin{aligned}
VTD_1(\mathring{g}) &= -\frac{1}{2}(k^2-1)\gamma_* t^{1/2(k^2+1)} F_0 + \mathcal{O}(\frac{1}{2}(3+k^2)) \\
VTD_2(\mathring{g}) &= -\frac{1}{2}(k^2-1)\gamma_* t^{1/2(k^2+1)} F_1 + \mathcal{O}(\min\{\frac{1}{2}(3+k^2), \frac{1}{2}(1+k^2)+\xi_1\}) \\
VTD_3(\mathring{g}) &= -\frac{1}{2}(k^2-1)\gamma_* t^{1/2(k^2+1)} F_0 + \mathcal{O}(\frac{1}{2}(3+k^2)+\xi_1) \\
VTD_4(\mathring{g}) &= (k-1)\tau_* t^{(2-k)} F_0 + \mathcal{O}(3-k+\xi_1) \\
VTD_5(\mathring{g}) &= (k-1)\tau_*\tau_{**} t^{(2-k)} F_0 + \mathcal{O}(3-k+\xi_1) \\
VTD_6(\mathring{g}) &= (k-1)\tau_*\tau_{**}^2 t^{(2-k)} F_0 + \mathcal{O}(2+k).
\end{aligned}
$$

Clearly each of the terms on the right hand side vanishes in the limit $t \searrow 0$. This shows that the leading order terms Eqs. (4.32)-(4.37) are VTD leading order terms in the sense that they satisfy (asymptotically) the VTD equations in the corresponding gauge. We have thus shown that the solutions obtained in Theorem 4.4 and the solutions to the VTD equations have the same leading order term, and moreover that the difference vanishes as $t \searrow 0$. Hence, we conclude that the solutions $\mathcal{S}(\xi_0, \xi_1)$ are AVTD solutions.

CHAPTER V

CONCLUSIONS AND DISCUSSION

The results presented in Chapters II -IV contribute both understanding of the singular behavior of cosmological solutions to the Einstein equations, and tools to aid in future investigations.

The families of AVTD solutions in the (half)-polarized T^2-symmetric and the Gowdy classes which we find in Chapter III and Chapter IV respectively, extend the knowledge of this type of behavior in the respective classes of spacetimes. In the (half)-polarized T^2-symmetric case we find a family of AVTD solutions with Sobolev-regularity. Before this result, all such AVTD solutions were known only to exist in the smooth class. These AVTD solutions in the larger and less regular function space add to the large amount of current research concerning "rough" solutions [28, 39, 52, 54, 55]. As mentioned in Section 1.3.2., the regularity of solutions is particularly relevant in studies of extendibility. We do *not* show that generic polarized T^2-symmetric solutions are AVTD. Such a result would be an important step in proving the restricted strong cosmic censorship conjecture within this space, but remains an open problem.

In the case of the Gowdy solutions, the results we present in Chapter IV corroborate evidence in [45] for the $U(1)$-symmetric solutions, that AVTD behavior is found in families of gauges. We are unable however to characterize the family of all such gauges or coordinate systems in which Gowdy solutions are AVTD. It is of particular interest to determine whether the Gowdy solutions exhibit AVTD behavior in constant mean curvature (CMC) coordinates. In a cosmological spacetime the

CMC foliation provides a global time coordinate which is invariantly defined by the geometry. These questions are under current investigation.

As mentioned above this dissertation also develops "tools" which we anticipate will be useful in future research. The most significant is the existence and uniqueness theorems for a class of quasilinear symmetric hyperbolic systems of Fuchsian type. Our results, along with those in Ames et al. [3, 4] are the first such theorems for quasilinear equations. Since the Einstein equations are generally quasilinear, this is an important step for studying more general classes of solutions. Since these Fuchsian theorems require the equations to be in hyperbolic form, we also study the Einstein equations in a class of gauges which guarantees this structure. In particular, we perform a general reduction of the equations in these gauges to a form suitable for checking and applying the Fuchsian theorems. Our reduction applies to classes of spacetimes in which the field variables depend only on time and one space coordinate, such as the T^2-symmetric spacetimes. The next step in this general theory is to extend this reduction to cases where the field variables may depend in general on all $n+1$ coordinates.

We expect these tools and techniques to be of particular use in obtaining smooth and less regular AVTD solutions in the polarized $U(1)$-symmetric class – in fact this is our primary motivation in developing them. As discussed in Section 1.4.3. the $U(1)$-symmetric spacetimes are much more varied and present additional difficulties not present in the T^2-symmetric spacetimes. The proof of smooth AVTD solutions in the polarized $U(1)$-symmetric class would complete the first two columns of Table 1.1 and provide a significant step towards proving restricted strong cosmic censorship within that class. The investigation of these results is in progress.

APPENDIX A

CONCEPTS IN PDE AND ANALYSIS

The concepts and results in this appendix are all standard in functional analysis and PDE theory. We summarize the relevant theorems there only for completeness, and reference. For more in-depth and comprehensive treatments see [22, 34, 47, 75, 86].

A.1. Distributions and Sobolev Spaces

A.1.1. Distributional Derivatives

We briefly recall the notion of a distribution so that we may introduce the idea of weak derivatives and Sobolev spaces. A more comprehensive treatment can be found for example in [22].

Let $\mathcal{T}(U)$ denote the set of smooth functions with compact support in $U \subset R^d$; this topological vector space is called the space of *test functions*. A distribution S is an element of the dual space $\mathcal{T}^*(U)$, that is a distribution is a map $\mathcal{T}(U) \to \mathbb{R}$, and acts on functions $\varphi \in \mathcal{T}(U)$ by

$$S(\varphi) := \int_U S\varphi dx.$$

The notation $S(\varphi) \equiv \langle S, \varphi \rangle$ is used when S is a continuous linear functional on $\mathcal{T}(U)$. Examples of distributions are the Dirac delta and the Heavyside step.

The derivative of a distribution in direction x^i, $\partial S/\partial x^i$ is defined by

$$\left\langle \partial S/\partial x^i, \varphi \right\rangle := \left\langle S, \partial \varphi/\partial x^i \right\rangle.$$

The motivation for this definition clearly comes from the case when $S = f$ is a $C^1(U)$ function, performing integration by parts, and noting the compact support of the test functions.

Definition A.1 (Distributional or weak derivatives)**.** *Let $f \in L^1_{loc}(U)$. A function $v \in L^1_{loc}(U)$ is called the* distributional *or* weak *derivative of f in the direction x^i if*

$$\int_U v(x)\varphi(x)dx = \int_U f(x)\partial\varphi(x)/\partial x^i dx.$$

Now suppose α is a multi-index. The function v is the α-th distributional or weak derivative of f, provided

$$\int_U v(x)\varphi(x)dx = \int_U f(x)D_\alpha\varphi(x)dx.$$

The notation $D_\alpha\varphi(x) := (\frac{\partial}{\partial x^1})^{\alpha_1}\cdots(\frac{\partial}{\partial x^d})^{\alpha_d}\varphi$. Often the same notation is used for the weak derivative of the distribution $v = D_\alpha f$.

A.1.2. Sobolev spaces and the Sobolev embedding theorem

Let U be an open set of $\mathbb{R}^d$.

Definition A.2 (Sobolev spaces)**.** *Let $q \in \mathbb{N}$ and $0 \leq p \leq \infty$. We define the Sobolev space $W^{q,p}(U)$ to be the set of functions $w \in L^p(U)$ such that all distributional derivatives Definition A.1 $D_\alpha w$ for $|\alpha| \leq q$ are also in $L^p(U)$. For this space we have the norm*

$$\|w\|_{W^{q,p}} := \left(\sum_{|\alpha|\leq q}\int_U |D_\alpha w|^p dx\right)^{1/p}.$$

A special case of the Sobolev spaces occurs for $p = 2$; we denote these spaces by $H^q(U)$.

Theorem A.3. *If $f \in H^q(U)$, then $f \in C^m(U)$ for all integers m such that $0 \le m < q - n/2$. Further,*

$$\|f\|_{C^m} \le \|f\|_{H^q}.$$

Note that C^m-norm is defined by $\|f\|_{C^m(U)} = \sum_{\alpha,|\alpha|=0}^{m} |\partial^\alpha f|$.

A.1.3. The $H^s(\mathbb{R}^n)$ Sobolev Spaces and Duality

This section comes from Appendix C of [3]. F. Beyer is the primary author of this appendix; editing by E. Ames, J. Isenberg, and P.G. LeFloch.

Following [22, Chapter VI] or [75], one defines the Sobolev space $H^s(\mathbb{R}^n)$ for any $s \in \mathbb{R}$ as the set of temperate distributions u such that $\widehat{u}(1+|\xi|^2)^{s/2} \in L^2(\mathbb{R}^n)$, where $\widehat{u} := \mathcal{F}u$ is the Fourier transform (in the sense of temperate distributions) of u. The norm defined by

$$\|u\|_s := \|\widehat{u}(\xi)(1+|\xi|^2)^{s/2}\|_{L^2_\xi(\mathbb{R}^n)}$$

turns this space into a Banach space. If $s = q$ for any non-negative integer q, then $H^s(\mathbb{R}^n)$ is equivalent to the standard ($p = 2$) Sobolev space $H^q(\mathbb{R}^n)$. For general $s \in \mathbb{R}$, the space $H^s(\mathbb{R}^n)$ is in fact a Hilbert space for the scalar product

$$\langle u, v \rangle_s := \int_{\mathbb{R}^n} \widehat{u}(\xi)(1+|\xi|^2)^{s/2}\widehat{v}(\xi)(1+|\xi|^2)^{s/2} d\xi.$$

Let $u \in H^{-s}(\mathbb{R}^n)$ and $v \in H^s(\mathbb{R}^n)$ for any $s \in \mathbb{R}$. Then the **dual pairing** between $H^s(\mathbb{R}^n)$ and $H^{-s}(\mathbb{R}^n)$,

$$(u, v) := \int_{\mathbb{R}^n} \widehat{u}(\xi)\widehat{v}(\xi) d\xi, \tag{A.1}$$

is well-defined, as a consequence of the inequality

$$|\,(u,v)\,| \leq \left| \int_{\mathbb{R}^n} \widehat{u}(\xi)(1+|\xi|^2)^{-s/2}\widehat{v}(\xi)(1+|\xi|^2)^{s/2}d\xi \right| \leq \|u\|_{-s}\|v\|_s. \tag{A.2}$$

By means of this pairing, we can identify $H^{-s}(\mathbb{R}^n)$ with $H^s(\mathbb{R}^n)^*$ (the dual space) as follows. For every $u \in H^{-s}(\mathbb{R}^n)$, the map $(u,\cdot) : H^s(\mathbb{R}^n) \to \mathbb{R}$ is a bounded linear functional, i.e., an element of $H^s(\mathbb{R}^n)^*$. Conversely, according to the Riesz representation theorem, there exists a unique element $w_\phi \in H^s(\mathbb{R}^n)$ for each element $\phi \in H^s(\mathbb{R}^n)^*$ such that

$$\phi(v) = \langle w_\phi, v\rangle_s$$

for all $v \in H^s(\mathbb{R}^n)$. The last expression can be written as

$$\langle w_\phi, v\rangle_s = \int_{\mathbb{R}^n} \widehat{w}_\phi(\xi)(1+|\xi|^2)^{s/2}\widehat{v}(\xi)(1+|\xi|^2)^{s/2}d\xi = \int_{\mathbb{R}^n} \widehat{v}_\phi(\xi)\widehat{v}(\xi)d\xi,$$

where $\widehat{v}_\phi := \widehat{w}_\phi(\xi)(1+|\xi|^2)^s$ is the Fourier transform of $v_\phi := \mathcal{F}^{-1}(\widehat{w}_\phi(\xi)(1+|\xi|^2)^s)$. We have $v_\phi \in H^{-s}(\mathbb{R}^n)$, since $\widehat{v}_\phi(1+|\xi|^2)^{-s/2} = \widehat{w}_\phi(\xi)(1+|\xi|^2)^{s/2} \in L^2(U)$. By means of the pairing above, we have thus constructed a unique element $v_\phi \in H^{-s}(\mathbb{R}^n)$ corresponding to each $\phi \in H^s(\mathbb{R}^n)^*$. In this sense, we can therefore identify $H^{-s}(\mathbb{R}^n)$ with $H^s(\mathbb{R}^n)^*$ for every $s \in \mathbb{R}$.

The following result concerns the relationship between Sobolev spaces of different indices.

Proposition A.4. *For every $s \in \mathbb{R}$ and $\sigma \geq 0$, the space $H^{s+\sigma}(\mathbb{R}^n)$ is a dense subset of $H^s(\mathbb{R}^n)$.*

Proof. We first show that $H^{s+\sigma}(\mathbb{R}^n)$ is indeed a subset of $H^s(\mathbb{R}^n)$ for $\sigma \geq 0$. Suppose that $u \in H^{s+\sigma}(\mathbb{R}^n)$. Calculating the $\|\cdot\|_s$ norm of u, we obtain

$$\|u\|_s^2 = \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2(1+|\xi|^2)^s d\xi \leq \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2(1+|\xi|^2)^{s+\sigma} d\xi = \|u\|_{s+\sigma}^2 < \infty,$$

from which it follows that $u \in H^s(\mathbb{R}^n)$. To check that $H^{s+\sigma}(\mathbb{R}^n)$ is a *dense* subset, it is sufficient to note (see, e.g., [22]) that $C_0^\infty(\mathbb{R}^n)$ (the space of smooth functions with compact support) is dense in both $H^s(\mathbb{R}^n)$ and $H^{s+\sigma}(\mathbb{R}^n)$. □

A.1.4. Convergence results in Sobolev spaces

One can use this dense inclusion property (Proposition A.5) together with the duality properties discussed above to derive certain convergence and closedness-type results for sequences in Sobolev spaces. We first discuss a result of this sort for Sobolev spaces on $\mathbb{R}^n$, and then do the same for Sobolev spaces on T^1.

Proposition A.5. *Choose $s, s_0 \in \mathbb{R}$ so that $0 \leq s_0 < s$. Let (w_m) be a bounded sequence in $H^s(\mathbb{R}^n)$ in the sense that there exists a constant $C > 0$ so that $\|w_m\|_s \leq C$, for all integer m. Moreover, suppose that (w_m) converges to some $w \in H^{s_0}(\mathbb{R}^n)$; i.e., $\|w_m - w\|_{s_0} \to 0$. Then, w is contained in $H^s(\mathbb{R}^n)$.*

Proof. The boundedness of the sequence implies the existence of a subsequence of (w_m) (which for simplicity we identify with (w_m)) which converges weakly. Hence, as a consequence of the Riesz Representation Theorem and the above dual pairing in Eq. (A.1), there exists an element $\widetilde{w} \in H^s(\mathbb{R}^n)$, so that, for every $Y \in H^{-s}(\mathbb{R}^n)$,

$$(Y, \widetilde{w} - w_m) \to 0 \qquad \text{(A.3)}$$

We wish to show that $w = \widetilde{w}$ and hence that $w \in H^s(\mathbb{R}^n)$. To do this, we consider an arbitrary $X \in H^{-s_0}(\mathbb{R}^n)$ and the dual pairing

$$|(X, \widetilde{w} - w)| \leq |(X, \widetilde{w} - w_m)| + |(X, w - w_m)|,$$

where $\widetilde{w} - w$ is considered as an element of $H^{-s_0}(\mathbb{R}^n)$, and where we have used the triangle inequality. Since $X \in H^{-s_0}(\mathbb{R}^n) \subset H^{-s}(\mathbb{R}^n)$ according to Proposition A.4, we can consider the first term on the right hand side as a pairing between $H^s(\mathbb{R}^n)$ and $H^{-s}(\mathbb{R}^n)$, and hence Eq. (A.3) implies that this term can be made arbitrarily small by choosing m sufficiently large. The second term is considered as a pairing between $H^{s_0}(\mathbb{R}^n)$ and $H^{-s_0}(\mathbb{R}^n)$ so that Eq. (A.2) yields

$$|(X, w - w_m)| \leq \|X\|_{-s_0} \|w - w_m\|_{s_0}.$$

Also this term can be made arbitrarily small by choosing m sufficiently large. Hence, we have found that $(X, \widetilde{w} - w) = 0$ for all $X \in H^{-s_0}(\mathbb{R}^n)$. Now, the Riesz representation theorem implies that for every $X \in H^{-s_0}(\mathbb{R}^n)$ there exists precisely one $\widetilde{X} \in H^{s_0}(\mathbb{R}^n)$ for which

$$0 = (X, \widetilde{w} - w) = \left\langle \widetilde{X}, \widetilde{w} - w \right\rangle_{H^{s_0}(\mathbb{R}^n)}.$$

In particular, we may choose $\widetilde{X} = \widetilde{w} - w$, which implies that $\widetilde{w} - w = 0$. $\square$

Corollary A.6. *Choose non-negative integers q and q_0 so that $q_0 < q$. Let (w_m) be a bounded sequence in $H^q(T^1)$, in the sense that there exists a constant $C > 0$ so that $\|w_m\|_{H^q(T^1)} \leq C$, for all integers m. Moreover, suppose that (w_m) converges to some $w \in H^{q_0}(T^1)$; i.e., $\|w_m - w\|_{H^{q_0}(T^1)} \to 0$. Then, w is contained in $H^q(T^1)$.*

Proof. We formulate the proof so that it can be easily generalized to general smooth orientable, connected compact Riemannian manifolds M in any dimension n. For this paper, the relevant special case is $M = T^1$. Let $((U_i, \phi_i))$ be a collection of coordinate charts, i.e., open subsets $U_i \subset M$ and homeomorphisms $\phi_i : V_i \to U_i$ where $V_i := \phi_i^{-1}(U_i)$ are open subset of $\mathbb{R}^n$, which cover M, i.e., $M = \bigcup_i U_i$. It follows from compactness that we can assume that there are N such coordinate charts. Let (τ_i) be a subordinate partition of unity. Then we find that (w_m) is a bounded sequence in $H^q(M)$ if and only if for all $i = 1, \ldots, N$, we have that $(w_m \circ \phi_i)$ is a bounded sequence in $H^q(V_i)$. Moreover, $\|w_m - w\|_{H^{q_0}(T^1)} \to 0$ for some $w \in H^{q_0}(M)$ if and only if for all $i = 1, \ldots, N$, we have that $\|w_m \circ \phi_i - w \circ \phi_i\|_{H^{q_0}(V_i)} \to 0$ (since $w \circ \phi_i \in H^{q_0}(V_i)$). Now, the Stein Extension Theorem (Theorem 5.24 in [1]) implies the existence of **total extension operators** E_i (Definition 5.17 in [1]), which are linear maps E_i from functions defined on V_i to functions defined on $\mathbb{R}^n$ with the following property: If $f \in H^r(V_i)$ for any non-negative integer r, then

1. $(E_i f)|_{V_i} = f$ almost everywhere,

2. $E_i f$ is in $H^r(\mathbb{R}^n)$, and there exists a constant $C > 0$, so that

$$\|E_i f\|_{H^r(\mathbb{R}^n)} \leq C \|f\|_{H^r(V_i)}.$$

Hence, we find that (w_m) is a bounded sequence in $H^q(M)$ if and only if for all $i = 1, \ldots, N$, we have that $(E_i(w_m \circ \phi_i))$ is a bounded sequence in $H^q(\mathbb{R}^n)$. Moreover, $\|w_m - w\|_{H^{q_0}(T^1)} \to 0$ for some $w \in H^{q_0}(M)$ if and only if for all $i = 1, \ldots, N$, we have that $\|E_i(w_m \circ \phi_i) - E_i(w \circ \phi_i)\|_{H^{q_0}(\mathbb{R}^n)} \to 0$ (since $E_i(w \circ \phi_i) \in H^{q_0}(\mathbb{R}^n)$). It follows from Proposition A.5, that $E_i(w \circ \phi_i) \in H^q(\mathbb{R}^n)$. Hence, $w \circ \phi_i \in H^q(V_i)$. Since this is true for all $i = 1, \ldots, N$, it follows that $w \in H^q(M)$. □

A.2. Fundamental Concepts and Theorems from Analysis

A.2.1. Banach Space

Most function spaces are infinite dimensional vector spaces with a structure encoded in the following definition.

Definition A.7. *A vector space X which is complete with respect to a norm $\|\cdot\|$ is called a Banach space. A Banach space is usually denoted $(X, \|\cdot\|)$, or just by X if the norm is clear in context.*

A.2.2. Frechet Derivative

It is important to extend the notion of a derivative from $\mathbb{R}^n$ to abstract Banach spaces.

Definition A.8 (Derivative between Banach spaces)**.** *Let X, Y be two Banach spaces, and U an open subset of X. The mapping $f : X \to Y$ is said to be differentiable at $x_0 \in U$ if there exists a continuous linear mapping Df of X into Y such that*

$$f(x_0 + h) - f(x_0) = Df|_{x_0} + R(h), \quad \textit{where } \|R(h)\|_Y = o(\|h\|_X),$$

for all h such that $x_0 + h \in U$. Recall $\|R(h)\|_Y = o(\|h\|_X)$ means that $\lim_{\|h\|_X \to 0} \|R(h)\|_Y / \|h\|_X = 0$.

Equivalently, there exists a continuous linear mapping Df satisfying

$$\|f(x_0 + h) - f(x_0) - Df_{x_0}\|_Y \leq C\|h\|_X$$

for a constant C independent of h.

An important result for us is the case where f can be considered a map from a "time interval" into a function space, such as a Sobolev space. The following result concerns the derivatives of such maps.

Theorem A.9. *Let $I = [a,b]$ be a bounded interval of $\mathbb{R}$, and let X be a Banach space. Suppose $\{f_n\}$ with $f_n : I \to X$ is a sequence of continuously differentiable functions. Further, assume that*

- *$\{f_n\}$ converges to f uniformly on I*
- *the sequence of derivatives f_n' converges uniformly on I.*

Then f is differentiable at each $t \in I$, and

$$f'(t) = \lim_{n\to\infty} f_n'(t) \quad \textit{for all} \quad t \in I.$$

This is a generalization of Theorem 5.11 on page 51 of [47], to Banach-space valued functions. Note however, that we do *not* establish that f' is continuous on I.

Proof. Let $\|\cdot\|$ denote the norm on X, and $|\cdot|$ denote the usual norm on $\mathbb{R}$. Further, let $g = \lim_{n\to\infty} f_n'$, and suppose Ω_t is an open neighborhood of $t \in I$. We wish to show that for any $s \in \Omega_t$

$$\|f(t) - f(s) - (t-s)g\| \leq C|t-s|,$$

for some positive constant C independent of s. We compute

$$\begin{aligned}\|f(t)-f(s)-(t-s)g\| &= \|f(t)-f(s)-(t-s)(g(t)-f_n'(t))-(t-s)f_n'\| \\ &\leq \|f(t)-f(s)-(t-s)f_n'\|+|t-s|\|g(t)-f_n'(t)\| \\ &\leq \|f(t)-f_n(t)-(f(s)-f_n(s))\| \\ &\quad +\|f_n(t)-f_n(s)-(t-s)f_n'(t)\|+C|t-s|\end{aligned}$$

where we have used that the sequence of derivatives f_n' converges uniformly at t. Since f_n' is the derivative of f_n, and using the uniform convergence of f_n to f on Ω_t the desired inequality is obtained. □

A.2.3. Hölder inequality

Let U be an open set in $\mathbb{R}^d$.

Lemma A.10. *Suppose* $1 \leq p, q \leq \infty$ *and* $1/p + 1/q = 1$. *Then if* $u \in L^p(U)$ *and* $v \in L^q(U)$, *we have*

$$\int_U \langle u, v\rangle\, dx \leq \|u\|_{L^p}\|v\|_{L^q}.$$

A.2.4. Moser estimate

Lemma A.11. *Let* f, g *be functions in* $L^\infty(T^n) \cap H^q(T^n)$. *Then,*

$$\|fg\|_{H^q} \leq C\left(\|f\|_{L^\infty}\|g\|_{H^q}+\|f\|_{H^q}\|g\|_{L^\infty}\right).$$

Further, for all multiindices α with $|\alpha| \leq q$ we have

$$\|D^\alpha(fg) - fD^\alpha g\|_{H^q} \leq C\left(\|f\|_{H^q}\|g\|_{L^\infty} + \|\nabla f\|_{L^\infty}\|g\|_{H^{q-1}}\right).$$

This is Proposition 3.7 of Chapter 13 in [86], and the proof is contained there.

Note that the hypothesis of Lemma A.11 hold if for example $f, g \in H^q(T^n)$ and $q > n/2$ by the Sobolev inequality. This yields the following useful estimate.

Corollary A.12. *Suppose that $f, g \in H^q(T^n)$ for $q > n/2$. Then,*

$$\|fg\|_{H^q} \leq C\|f\|_{H^q}\|g\|_{H^q}$$

for a constant C depending on n, q.

A.2.5. Banach fixed point theorem

Theorem A.13. *Let $(X, \|\cdot\|)$ be a Banach space, and let $B \subset X$ be a closed subset. Suppose $f : B \to X$ is a map such that $f(B) \subset B$ and*

$$\|f(x) - f(y)\| \leq \theta\|x - y\|$$

for all $x, y \in X$, with $0 \leq \theta < 1$. Then, f has a unique fixed point in B.

This is Theorem 4.7 of [47] for example.

A.3. Symmetric Hyperbolic Systems

In this section we collect some results on (non-singular) symmetric hyperbolic systems. These are adapted from [85] Chapter 16. Ringström, [75] provides a more rigorous presentation.

A.3.1. Comments on PDE

In PDE theory the focus is often on proving that a particular class of PDE is *well-posed*. By this, it is informally meant that:

- The PDE has a solution.
- The solution is unique.
- The solutions depend continuously on the data specified in the problem.

What is meant by a solution to a PDE? The ideal notion of a solution to a given PDE is an explicit functional form of the independent variables which has enough continuous derivatives in order to satisfy the equation. Better, the solution is smooth or analytic in some or all of its arguments. A solution having sufficient continuous derivatives, whether or not it can be written down explicitly, is known as a *classical* solution. It is known that for most PDE such classical solutions cannot be found. Further, for some problems such as studying the evolution of shocks, the solutions one is trying to understand are not even continuous. It is therefore desirable to consider weaker notions of a solution.

One of the most useful notions of a weak solution is a function which satisfies the equation in a distributional sense. Given a PDE one usually forms an integral version of it in which all the derivatives are transferred to act on smooth test functions. The function is then said to be a weak solution of the equation if this integral equation holds for all smooth test functions. Note that in this case the solution in general only needs to be locally integrable.

Often, proving the existence of solutions in a weak sense is a good place to start when proving the existence of solutions to PDE. This separates the questions of existence from regularity. To improve upon the weak solution, the next step is to increase the regularity assumptions on the data and coefficients and use these stronger hypothesis to prove that the solutions you have found are actually differentiable in a distributional sense. A solution which is both a weak solution and can be shown to have sufficient distributional derivatives to satisfy the equation (in a distributional sense) is called a *strong solution*.

A.3.2. Well-posedness of Symmetric Hyperbolic PDE

Consider partial differential systems of the form

$$S^0(t,x,u)\partial_t u + \sum_{a=1}^{n} S^a(t,x,u)\partial_a u + f(t,x,u) = 0, \tag{A.4}$$

where $S^j : [0,\delta] \times T^n \times U \to R^{d\times d}$, and $f : [0,\delta] \times T^n \times U \to R^d$, and where U is an open set of $\mathbb{R}^d$.

Definition A.14. *The equation Eq.* (A.4) *is called a* ***quasilinear symmetric hyperbolic system*** *if* S^0, S^a *are symmetric and bounded for every* (t,x,u) *in* $[0,\delta]\times T^n \times U$*, and if there exists a* $c_0 > 0$ *such that* $|S^0| \geq c_0$*, (that is* S^0 *is uniformly bounded from below). Unless specified otherwise we suppose* S^j, f *are smooth in* $u \in U$*, for* U *an open set of* $\mathbb{R}^d$*, and further that* $S^j, f \in C\left(I; H^q(T^n)\right)$ *for* $q > n/2+1$*, and* I *an open set of* $\mathbb{R}$*.*

We have the following results concerning the initial value problem consisting of Eq. (A.4) and the data prescribed at $t = 0$

$$\phi(x) \in H^q(T^n), \tag{A.5}$$

for $q > n/2 + 1$. Suppose I in Definition A.14 is an interval about $t = 0$. We have existence of unique solutions. We first give a result for linear systems in the case that the coefficients have finite regularity. This is based on Propositions 1.7 and 2.1 of [85], Chapter 16.

Proposition A.15 (Existence and uniqueness for linear systems)**.** *Suppose Eq.* (A.4) *is a linear symmetric hyperbolic system, meaning that S^0, S^a, and f are independent of u, and suppose that S^0, S^a, f are in $C(I; H^q)$ for $q > n/2 + 1$. Then there is a unique solution $u \in C(I; H^q)$ to the initial value problem Eq.* (A.4) *and Eq.* (A.5).

For quasilinear systems we cite the following result in the case that the coefficients depend smoothly on the arguments (t, x, u). This is based on Corollary 1.6 of Taylor [85], Chapter 16.

Proposition A.16. *Suppose Eq.* (A.4) *is a quasilinear symmetric hyperbolic system and that S^0, S^a, f are $C^\infty(I \times T^n)$ and also depend smoothly on u. Then there is a uniques solution $u \in C^\infty(I \times T^n)$.*

APPENDIX B

PROPERTIES OF THE SPACES $X_{\delta,\mu,Q}$

B.1. Relations Between Spaces $X_{\delta,\mu,q}$

We discuss the relationships between different $X_{\delta,\mu,q}$ spaces, when two parameters are fixed and the third is allowed to vary. Clearly, $X_{\tilde{\delta},\mu,q} \subset X_{\delta,\mu,q}$ for any $\tilde{\delta} \in (0, \delta]$. Next we prove embedding lemmas for the exponent vector, and regularity parameters.

Lemma B.1. *Fix a $\delta > 0$, a $q \in \mathbb{Z}^+$, and an exponent vector ν, and suppose $f \in X_{\delta,\nu,q}$. Then $f \in X_{\delta,\mu,q}$ for any $\mu < \nu$, and we have the estimate*

$$\|f\|_{\delta,\mu,q} \leq C\|f\|_{\delta,\nu,q}$$

for a constant C depending only on the difference $\nu - \mu$ and δ.

Proof. Since $f \in X_{\delta,\nu,q}$,

$$\|f\|_{\delta,\nu,q} = \sup_{t\in(0,\delta]} \|\mathcal{R}[\nu]f\|_{H^q(T^n)} \leq C < \infty.$$

Computing

$$\begin{aligned} \|f\|_{\delta,\mu,q} &= \sup_{t\in(0,\delta]} \|\mathcal{R}[\mu]f\|_{H^q(T^n)} \\ &= \sup_{t\in(0,\delta]} \|\mathcal{R}[\mu - \nu]\mathcal{R}[\nu]f\|_{H^q(T^n)}. \end{aligned}$$

Because $\mathcal{R}[\mu-\nu] = \text{Diag}\{t^{\nu_1-\mu_1}, \ldots, t^{\nu_d-\mu_d}\}$, and since μ, ν are smooth on T^n and thus obtain there maximum and minimum values,

$$\sup_{t\in(0,\delta]} \|\mathcal{R}[\mu-\nu]\mathcal{R}[\nu]f\|_{H^q(T^n)} \leq \sup_{t\in(0,\delta]} \|\mathcal{R}[\mu-\nu]\|_{L^\infty(T^n)}\|\mathcal{R}[\nu]f\|_{H^q(T^n)} \leq C\|f\|_{\delta,\nu,q},$$

for a constant C depending in general on δ, μ, and ν. This shows that $f \in X_{\delta,\mu,q}$, with the proclaimed estimate. □

Lemma B.2. *Fix a $\delta > 0$, exponent vector μ, and let $q \in \mathbb{Z}^+$. Then the following embedding holds*

$$X_{\delta,\mu,q} \subset X_{\delta,\mu,q-1} \subset \ldots \subset X_{\delta,\mu,0},$$

and we have the estimates

$$\|w\|_{\delta,\mu,q} \geq \|w\|_{\delta,\mu,q-1} \geq \ldots \geq \|w\|_{\delta,\mu,0},$$

for any $w \in X_{\delta,\mu,q}$.

Proof. Let $w \in X_{\delta,\mu,q}$, l be an integer in $[0,q]$, and α a multi-index. Then,

$$\begin{aligned}
\|w\|_{\delta,\mu,q} &= \sup_{t\in(0,\delta]} \left(\sum_{\alpha,|\alpha|=0}^{q} \int_{T^n} |\partial_x^\alpha \mathcal{R}[\mu]w|^2 dx\right)^{1/2} \\
&= \sup_{t\in(0,\delta]} \left(\sum_{\alpha,|\alpha|=0}^{l} \int_{T^n} |\partial_x^\alpha \mathcal{R}[\mu]w|^2 dx + \sum_{\alpha,|\alpha|=l+1}^{q} \int_{T^n} |\partial_x^\alpha \mathcal{R}[\mu]w|^2 dx\right)^{1/2} \\
&\geq \sup_{t\in(0,\delta]} \left(\sum_{\alpha,|\alpha|=0}^{l} \int_{T^n} |\partial_x^\alpha \mathcal{R}[\mu]w|^2 dx\right)^{1/2} \\
&= \|w\|_{\delta,\mu,l}
\end{aligned}$$

This shows that $w \in X_{\delta,\mu,l}$, and that $\|w\|_{\delta,\mu,l} \leq \|w\|_{\delta,\mu,q}$ for any $l \in [0,q]$. Similar arguments show that $\|w\|_{\delta,\mu,l} \leq \|w\|_{\delta,\mu,k}$ for all $l,k \in [0,q]$ such that $k \geq l$. □

B.2. Relation to Bounded Continuous Maps

B.2.1. Relations to Other Function Spaces

We can think of $w \in X_{\delta,\mu,q}$ as a map between Banach spaces $w : (0,\delta] \to H^q(T^n)$.

Lemma B.3. *Fix parameters $\delta > 0$, $q \in \mathbb{Z}^+$ and an exponent vector μ. If $f \in X_{\delta,\mu,q}$ then at each $t \in (0,\delta]$, $f \in H^q(T^n)$.*

The proof of this follows from the definition of the $\|\cdot\|_{\delta,\mu,q}$ norm and smoothness of $\mathcal{R}[\mu]$.

In the case $\mu = 0$, the space $X_{\delta,0,q}$ consists of maps w such that the norm

$$\|w\|_{\delta,0,q} = \sup_{t\in(0,\delta]} \|w\|_{H^q}$$

is finite. This is equivalent to the more familiar space $L^\infty((0,\delta];H^q)$. It follows that if ζ is a non-negative definite exponent vector and $f \in X_{\delta,\zeta,q}$ for some $\delta > 0$ and positive integer q, then $f \in L^\infty((0,\delta];H^q)$.

B.2.2. The Spaces $\widehat{X}_{\delta,\mu,q}$

In the section above we discuss conditions under which $f \in X_{\delta,\mu,q}$ is a bounded map between Banach spaces. We now extend this idea, by investigating under what conditions such maps are continuous. Define $\widehat{X}_{\delta,\mu,q}$ as the set of maps $f : (0,\delta] \to H^q(T^n)$ with the property that $\mathcal{R}[\mu]f$ is bounded and continuous; cf. Eq. (2.2). If we endow $\widehat{X}_{\delta,\mu,q}$ with the norm $\|\cdot\|_{\delta,\mu,q}$, then $\widehat{X}_{\delta,\mu,q}$ are Banach spaces. Note that if $f \in \widehat{X}_{\delta,\mu+\epsilon,q}$ for some $\epsilon > 0$, then $\mathcal{R}[\mu]f : (0,\delta] \to H^q(T^n)$ is uniformly continuous.

All functions in $X_{\delta,\mu,q}$ can be approximated by smooth functions according to the definition of these space Section 2.2.2.. Functions in $\widehat{X}_{\delta,\mu,q}$, however, can be approximated by a particularly useful sequence of smooth functions as follows; the following lemma is taken from Appendix A of [3]. We refer to that paper for the proof.

Lemma B.4. *Let $f \in \widehat{X}_{\delta,\mu,q}$; i.e., $\mathcal{R}[\mu]f : (0,\delta] \to H^q(T^1)$ is bounded and continuous. Let $\widehat{f}$ be defined as follows*

$$\widehat{f}(t) = \begin{cases} f(t), & t \in (0,\delta], \\ \mathcal{R}[\mu]^{-1}(t)\mathcal{R}[\mu](\delta)f(\delta), & t \in [\delta,\infty). \end{cases}$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth with $\phi(x) > 0$ for all $|x| < 1$ and $\phi(x) = 0$ for all $|x| \geq 1$, with $\int_{\mathbb{R}} \phi(x)dx = 1$. Let (α_i) be a sequence of positive numbers with limit 0. For any integers i, j, we set

$$(\mathcal{R}[\mu]f)_{i,j}(t,x) := \int_0^\infty \int_{T^1} (\mathcal{R}[\mu]\widehat{f})(s,y)\frac{1}{\alpha_i}\phi\left(\frac{x-y}{\alpha_i}\right)\frac{1}{\alpha_j}\phi\left(\frac{s-t}{\alpha_j}\right) dy\, ds. \qquad \text{(B.1)}$$

Then $(\mathcal{R}[\mu]f)_{i,j}$ has the following properties:

1. *$(\mathcal{R}[\mu]f)_{i,j} \in C^\infty((0,\delta] \times T^1)$ for all integers i, j.*
2. *The function*

$$f_{i,j} := \mathcal{R}[\mu]^{-1}(\mathcal{R}[\mu]f)_{i,j} \qquad \text{(B.2)}$$

   has the property that

$$f_{i,j} \in \widehat{X}_{\delta,\mu,q} \cap X_{\delta,\mu,q} \quad \textit{for all integers } i, j.$$

In particular, for any given integers i, j, one has

$$\|(\mathcal{R}[\mu]f)_{i,j}(t,\cdot)\|_{H^q(T^1)} \leq C\|f\|_{\delta,\mu,q}, \quad \textit{for all } t \in (0,\delta],$$

for a constant $C > 0$ independent of t (but possibly dependent on i, j).

3. $(\mathcal{R}[\mu]f)_{i,j}(t,x) \longrightarrow \mathcal{R}[\mu]f(t,x)$ *for $i, j \to \infty$ at a.e. $(t,x) \in (0,\delta] \times T^1$.*

4. *If f is such that $\mathcal{R}[\mu]f : (0,\delta] \to H^q(T^1)$ is a uniformly continuous map (e.g., if $f \in \widehat{X}_{\delta,\mu+\epsilon,q}$ for some $\epsilon > 0$), then*

$$\|f_{i,j} - f\|_{\delta,\mu,q} \to 0 \quad \textit{for } i, j \to \infty.$$

We can now use Lemma B.4 to relate the spaces $X_{\delta,\mu,q}$ and $\widehat{X}_{\delta,\mu,q}$. The following embedding is originally proved in [3].

Lemma B.5. *If we fix a constant $\delta > 0$, an exponent vector μ, and a non-negative integer q, then for all $\epsilon > 0$, one has*

$$\widehat{X}_{\delta,\mu+\epsilon,q} \subset X_{\delta,\mu,q} \subset \widehat{X}_{\delta,\mu,q}.$$

Proof. The inclusion $X_{\delta,\mu,q} \subset \widehat{X}_{\delta,\mu,q}$ follows easily from the fact that each element in $X_{\delta,\mu,q}$ is the limit of a Cauchy sequence in $(C^\infty((0,\delta] \times T^1), \|\cdot\|_{\delta,\mu,q})$, whose elements are in particular bounded continuous maps $(0,\delta] \to H^q(T^1)$, and the convergence is uniform in time.

To check the inclusion $\widehat{X}_{\delta,\mu+\epsilon,q} \subset X_{\delta,\mu,q}$, let a function f be given in $\widehat{X}_{\delta,\mu+\epsilon,q}$. Hence f satisfies the condition of the previous lemma, in particular that of Condition 4. It follows that $f \in X_{\delta,\mu,q}$. □

B.3. Derivatives of Functions in $X_{\delta,\mu,q}$

B.3.1. Time derivatives

We also wish to comment on time derivatives of functions in $X_{\delta,\mu,q}$ and $\widehat{X}_{\delta,\mu,q}$. Let $f \in \widehat{X}_{\delta,\mu,q}$. We say that f is differentiable in time t if the (bounded continuous) map $\mathcal{R}[\mu]f : (0,\delta] \to H^q(T^1)$ is differentiable in the sense of a map between Banach spaces (Frechet derivatives). Its time derivative (multiplied by t) $D(\mathcal{R}[\mu]f)$ can then be considered to be a map $(0,\delta] \to H^q(T^1)$, and we set $Df := \mathcal{R}[\mu]^{-1}(D(\mathcal{R}[\mu]f) - D\mathcal{R}[\mu]f)$. If this map is continuous, then we call f continuously differentiable in t. If this is the case for f and if in addition $\mathcal{R}[\mu]Df$ is bounded, then we have $Df \in \widehat{X}_{\delta,\mu,q}$.

Now, let $f \in \widehat{X}_{\delta,\mu,q}$ be continuously differentiable. Then Df is the **distributional time derivative** of f in the following sense. Let ϕ be any test function with the properties as in Section 2.4.4.. Choose $\epsilon > 0$. Then we clearly have that

$$\int_\epsilon^\delta \partial_t(t \langle \mathcal{R}[\mu]f, \phi\rangle_{L^2(T^1)})dt = -\epsilon \langle \mathcal{R}[\mu]f, \phi\rangle_{L^2(T^1)}\Big|_{t=\epsilon}.$$

Hence, the boundary term vanishes in the limit $\epsilon \to 0$. The following integrals are meaningful for $\epsilon = 0$, and hence we obtain

$$\begin{aligned}&\int_0^\delta \langle \mathcal{R}[\mu]Df, \phi\rangle_{L^2(T^1)}\, dt \\ &\quad = -\int_0^\delta \Big(\langle \mathcal{R}[\mu]f, D\phi\rangle_{L^2(T^1)} + \langle \mathcal{R}[\mu]f + D\mathcal{R}[\mu]f, \phi\rangle_{L^2(T^1)}\Big)\, dt.\end{aligned} \tag{B.3}$$

The reader should compare this with the expressions for weak solutions in Section 2.4..

B.3.2. Spatial derivatives

Next we prove a result concerning the spatial derivatives of a function in our weighted Sobolev spaces.

Lemma B.6. *Fix a $\delta > 0$, a $q \in \mathbb{Z}^+$, and an exponent vector μ. Suppose that $w \in X_{\delta,\mu,q}(T^n)$, and that $\|w\|_{\delta,\mu,q} \leq M$, for some $M \in \mathbb{R}^+$. Then, there exists real numbers $\epsilon, C > 0$ such that for any coordinate $x^a \in T^n$,*

$$\partial_a w \in X_{\delta,\mu-\epsilon,q-1}(T^n) \quad \textit{and} \quad \|\partial_a w\|_{\delta,\mu-\epsilon,q-1} \leq C\|w\|_{\delta,\mu,q} \leq CM.$$

Note that the slight decrease in the exponent vector is necessary in order to control the factors of $\log t$ which appear when commuting ∂_a with $\mathcal{R}[\mu]$.

Proof. Let α be a multi-index, and compute

$$\begin{aligned}
\|\partial_a w\|_{\delta,\mu-\epsilon,q-1} &= \sup_{t\in(0,\delta]} \left(\int_{T^n} \sum_{\alpha,|\alpha|=0}^{q-1} |\partial_x^\alpha \mathcal{R}[\mu-\epsilon]\partial_a w|^2 \right)^{1/2} \\
&= \sup_{t\in(0,\delta]} \left(\int_{T^n} \sum_{\alpha,|\alpha|=0}^{q-1} \Big\{ |\partial_x^\alpha \partial_a \mathcal{R}[\mu-\epsilon] w|^2 + |\partial_x^\alpha (\partial_a \mathcal{R}[\mu-\epsilon]) w|^2 \right. \\
&\qquad \left. - 2 \left\langle \partial_x^\alpha (\partial_a \mathcal{R}[\mu-\epsilon]) w, \partial_x^\alpha \partial_a \mathcal{R}[\mu-\epsilon] w \right\rangle \Big\} \right)^{1/2}.
\end{aligned}$$

Now, $\partial_a \mathcal{R}[\mu-\epsilon] = t^\epsilon \log t \text{Diag}\{\partial_a \mu\}.\mathcal{R}[\mu]$, which can be bounded by $C\mathcal{R}[\mu]$ in $(0,\delta] \times T^n$ since μ is smooth and the t^ϵ dominates the logarithm. To deal with the cross term we note that from the Cauchy inequality

$$2 \left\langle \partial_x^\alpha (\partial_a \mathcal{R}[\mu-\epsilon]) w, \partial_x^\alpha \partial_a \mathcal{R}[\mu-\epsilon] w \right\rangle \leq |\partial_x^\alpha (\partial_a \mathcal{R}[\mu-\epsilon]) w|^2 + |\partial_x^\alpha \partial_a \mathcal{R}[\mu-\epsilon] w|^2.$$

It follows that

$$
\begin{aligned}
&\|\partial_a w\|_{\delta,\mu-\epsilon,q-1} \\
&\qquad \leq C \sup_{t\in(0,\delta]} \left(\int_{T^n} \sum_{\alpha,|\alpha|=0}^{q-1} |\partial_x^\alpha \partial_a \mathcal{R}[\mu]w|^2 + |\partial_x^\alpha \mathcal{R}[\mu]w|^2 \right)^{1/2} \\
&\qquad \leq C \sup_{t\in(0,\delta]} \left(\int_{T^n} \sum_{\alpha,|\alpha|=0}^{q-1} |\partial_x^\alpha \partial_a \mathcal{R}[\mu]w|^2 + |\partial_x^\alpha \mathcal{R}[\mu]w|^2 + \sum_{\beta,|\beta|=q} |\partial_x^\beta \mathcal{R}[\mu]w|^2 \right)^{1/2},
\end{aligned}
$$

where $\sum_{\beta,|\beta|=q}$ is a sum over all multi-indices of order q such that ∂_x^β is not of the form $\partial_x^\alpha \partial_a$. This last expression is equal to $C\|w\|_{\delta,\mu,q}$, which proves the lemma. □

APPENDIX C

FUNCTION OPERATORS ON $X_{\delta,\mu,Q}$ SPACES

In this appendix we develop the relevant theory for function operators between the weighted Sobolev spaces $X_{\delta,\mu,q}$. The results presented here are particularly relevant in applications of the Fuchsian theorems Theorem 2.10 and Theorem 2.28.

C.1. Function Operator Basics

In Section 2.2.3. we introduce the notion of a function operator corresponding to a function of the type $f : (0, \delta] \times T^n \times \Omega \to \mathbb{R}^m$. There we introduce the notion of a bounded function operator, as well as the Lipschitz property (Definition 2.3) which is critical in proving the existence results Theorem 2.10 and Theorem 2.28. For the second Fuchsian theorem additional properties (c.f. Definition 2.25 and Definition 2.26) are required on the relevant function operators in the equations. In the sections below we verify that these properties hold for the class of function operators which we find in our applications.

While most of our analysis is concerned with treating functions of the "new unknown" $w \in B_{\delta,\mu,q}(s)$, we are also interested in the "expansion" of a function operator $f(u)(t, x)$ under the Fuchsian ansatz $u = u_0 + w$. For this Fuchsian reduction we would like to partition a given functional $f(t, x, u)$ into terms

$$f(t, x, u) = f_0(t, x) + f_1(t, x) \cdot w + f_2(t, x, w), \tag{C.1}$$

where f_0 is purely a function of the coordinates, $f_1(t, x)$ encodes the coefficients for the linear terms, and $f_2(t, x, w)$ contains the remaining, generally nonlinear, terms in the expanded functional. We then wish to know the exponents ν_0, ν_1, ν_2 associated to the spaces $f_0 \in X_{\delta,\nu_0,q}$, $f_1 \in X_{\delta,\nu_1,q}$, and $f_2 \in X_{\delta,\nu_2,q}$.

To begin we look at u as a function operator –one might think of this as the "fundamental Fuchsian function operator." Consider

$$w \mapsto u(w) = u_0 + w,$$

for $u_0 \in X_{\delta,\kappa,q}$, and $w \in B_{\delta,\mu,q}(s)$ for $\mu \geq \kappa$ (this is necessary in order for w to be considered a "remainder" with respect to u_0). It follows that $u(w)$ is a function operator $u : B_{\delta,\mu,q}(s) \to X_{\delta,\kappa,q}$. Further, it is clear that $u(w)$ satisfies the Lipschitz property.

C.2. Linear Function Operators

Let m, d be positive integers, and suppose $\Omega \subset \mathbb{R}^d$ is open. In this section we consider functions $L(t, x, u)$ defined by $L : (0, \delta] \times T^n \times \Omega \to \mathbb{R}^d$ and the corresponding function operators $L(u)(t, x)$, which are linear in $u \in \Omega$. Such function operators can be written

$$L(u) = A(t, x)u,$$

where $A(t, x)$ is a $\mathbb{R}^{d\times d}$-valued function. These operators have the expansion

$$L(w)(t, x) := L(u(t, x, w(t, x))) = A(t, x)u_0(t, x) + A(t, x)w(t, x). \tag{C.2}$$

The following lemma tells us about what function space we can expect for the target.

Lemma C.1. *Let w be a d-vector-valued function in $X_{\delta,\mu,q}$ for some exponent d-vector μ, a constant $\delta > 0$, and an integer $q > n/2$. Let A be a $d \times d$-matrix-valued function* *so that* $\mathcal{R}[\mu] \cdot A \cdot \mathcal{R}[-\mu]$ *is an element of* $X_{\delta,\zeta,q}$ *for an exponent d-vector* ζ*. Then, the d-vector-valued function* $A.w$ *is in* $X_{\delta,\zeta+\mu,q}$ *and*

$$\|A.w\|_{\delta,\zeta+\mu,q} \leq C \|\mathcal{R}[\mu] \cdot A \cdot \mathcal{R}[-\mu]\|_{\delta,\zeta,q} \|w\|_{\delta,\mu,q}, \tag{C.3}$$

for some constant $C > 0$ *depending only on* q *and* n*.*

Note that a similar theorem can be proved with w replaced by a $d \times d$ dimensional matrix in the space $X_{\delta,\mu,q}$. Also, in the case $d = 1$, the $\mathbb{R}$-valued functions $\mathcal{R}[\mu]$ and A trivially commute, and we have that for $A \in X_{\delta,\zeta,q}$,

$$\|Aw\|_{\delta,\zeta+\mu,q} \leq C \|A\|_{\delta,\zeta,q} \|w\|_{\delta,\mu,q}.$$

Proof of Lemma C.1. From the definition of the weighted Sobolev spaces, and given that $\mathcal{R}[\mu] \cdot A \cdot \mathcal{R}[-\mu] \in X_{\delta,\zeta,q}$, and $w \in X_{\delta,\mu,q}$, there exists a sequence of matrices $B_n \in X_{\delta,\zeta,q} \cap C^\infty((0,\delta] \times T^n)$ and a sequence of elements $w_n \in X_{\delta,\mu,q} \cap C^\infty((0,\delta] \times T^n)$ which converge to $\mathcal{R}[\mu] \cdot A \cdot \mathcal{R}[-\mu]$ and w in $X_{\delta,\zeta,q}$ and $X_{\delta,\mu,q}$ respectively.

To show that Aw is in $X_{\delta,\zeta+\mu,q}$, we show that the sequence of elements $(\mathcal{R}[-\mu] B_n \mathcal{R}[\mu])\, w_n$ converges to Aw in $X_{\delta,\mu+\zeta,q}$. While it follows from the definitions that $(\mathcal{R}[-\mu] B_n \mathcal{R}[\mu])\, w_n \in C^\infty((0,\delta] \times T^n)$, we show that $\mathcal{R}[-\mu] B_n \mathcal{R}[\mu] w_n \in X_{\delta,\mu+\zeta,q} \cap C^\infty((0,\delta] \times T^n)$. Due to Corollary A.12 we have

$$\begin{aligned} \|\mathcal{R}[-\mu] B_n \mathcal{R}[\mu] w_n\|_{\delta,\mu+\zeta,q} &= \sup_{t \in (0,\delta]} \|\mathcal{R}[\zeta] B_n \mathcal{R}[\mu] w_n\|_{H^q} \\ &\leq C \sup_{t \in (0,\delta]} \|\mathcal{R}[\zeta] B_n\|_{H^q} \sup_{t \in (0,\delta]} \|\mathcal{R}[\mu] w_n\|_{H^q} < \infty, \end{aligned} \tag{C.4}$$

as desired. Now consider

$$\begin{aligned}
&\|\mathcal{R}[\zeta]B_n\mathcal{R}[\mu]w_n - \mathcal{R}[\zeta]\mathcal{R}[\mu]A\mathcal{R}[-\mu]\mathcal{R}[\mu]w\|_{H^q} \\
&\quad = \|\mathcal{R}[\zeta]\left(B_n - \mathcal{R}[\mu]A\mathcal{R}[-\mu]\right)\mathcal{R}[\mu]w_n + \mathcal{R}[\zeta]\left(\mathcal{R}[\mu]A\mathcal{R}[-\mu]\right)\mathcal{R}[\mu]\left(w_n - w\right)\|_{H^q} \\
&\quad \leq C\|\mathcal{R}[\zeta](B_n - \mathcal{R}[\mu]A\mathcal{R}[-\mu])\|_{H^q}\|\mathcal{R}[\mu]w_n\|_{H^q} \\
&\qquad + C\|\mathcal{R}[\zeta]\mathcal{R}[\mu]A\mathcal{R}[-\mu]\|_{H^q}\|\mathcal{R}[\mu](w_n - w)\|_{H^q} \\
&\quad \leq C\left(\|B_n - \mathcal{R}[\mu]A\mathcal{R}[-\mu]\|_{\delta,\zeta,q}\|w_n\|_{\delta,\mu,q} + \|\mathcal{R}[\mu]A\mathcal{R}[-\mu]\|_{\delta,\zeta,q}\|w_n - w\|_{\delta,\mu,q}\right).
\end{aligned}$$

Since the right hand side vanishes in the $n \to \infty$ limit, we have shown that $\mathcal{R}[-\mu]B_n\mathcal{R}[\mu]w_n$ converges to Aw in $X_{\delta,\zeta+\mu,q}$ and that Aw is in $X_{\delta,\zeta+\mu,q}$. The estimate for Aw follows by taking the limit $n \to \infty$ of Eq. (C.4). □

We now show that these operators are Lipschitz.

Lemma C.2. *Let $L(w)$ be a linear function operator as in Eq.* (C.2)*, with A satisfying the properties of Lemma C.1, and choose an exponent vector μ, any exponent scalar $\gamma_0 \geq 0$, and positive real numbers $0 < \hat{s} \leq s$. Then $L(\cdot)$ satisfies the Lipschitz property, and for all $w \in B_{\delta,\mu,q}(s)$ and $h \in B_{\delta,\hat{\mu},q}(\hat{s})$ with $\hat{\mu} = \mu + \gamma_0$ we have that*

$$L(w) - L(w + h) \in X_{\delta,\zeta+\hat{\mu},q}.$$

Proof. 1. Let $w, \tilde{w} \in B_{\delta,\mu,q}(s)$. The proof of the Lipschitz property follows from $L(w) - L(\tilde{w}) = A(w - \tilde{w})$ and the estimate Eq. (C.3).

2. We have $L(w) - L(w + h) = -Ah$. Since $\hat{\mu}$ differs from μ by a scalar, and $\mathcal{R}[\mu]A\mathcal{R}[-\mu] \in X_{\delta,\zeta,q}$ by assumption, it follows from Lemma C.1 that $Ah \in X_{\delta,\zeta+\hat{\mu},q}$. □

C.3. Nonlinear Function Operators

In this section we verify the desired properties of the function operators in the case that these operators are nonlinear in the fields. Again this is relevant for applications of the Fuchsian theory. However, since in most cases the analysis can be performed separately for each component of the function operator we simplify the following analysis by considering only f of the form $f : (0, \delta] \times T^n \times \Omega \to \mathbb{R}$. The basic non-linearities we deal with are multiplications, positive powers, exponentials, and inverses. We first establish a general result, and later consider specific cases in more detail.

C.3.1. General Smooth $\mathbb{R}$-Valued Function Operators

To begin we establish the expansion of such operators under the Fuchsian ansatz.

Lemma C.3. *Let $f(t, x, u)$, $f : (0, \delta] \times T^n \times U \to \mathbb{R}$ be a function which is smooth in its arguments. Further suppose that $w \mapsto u$ is the basic Fuchsian function operator defined by $u(w) = u_0 + w$, for a given $u_0 \in X_{\delta,\kappa,q}$. Then the corresponding function operator $f(w)(t, x) = (f \circ u)(t, x) = f(t, x, u(w(t, x)))$ has an expansion of the form Eq.* (C.1)*.*

The proof of this lemma is based on Taylor's theorem.

Proof of Lemma C.3. At each $(t, x) \in (0, \delta] \times T^n$ and for all $u \in V \subset U$ we have

$$f(t, x, u) = g_0(t, x) + g_1(t, x)(u) + g_2(t, x)(u, u) + g_r(t, x, u)$$

where $g_1(t,x)(u)$ is the linear form $D_u f|_0$, and $g_2(t,x)(u,u)$ is the quadratic form $D_u^2 f|_0$, and where $\lim_{u\to 0, u\neq 0} g_r(t,x,u)/\|u\|^2 = 0$. We may write

$$g_1(t,x)(u) = \sum_i \lambda^i(t,x) u_i, \quad \text{and} \quad g_2(t,x)(u,v) = \sum_{i,j} \rho^{ji}(t,x) u_i v_j,$$

for some smooth functions $\lambda^i(t,x)$ and $\rho^{ji}(t,x)$. Expanding $u_i = u_{0i} + w_i$ we obtain the desired function operator for w contained in some set Ω. □

C.3.2. Some Basic Constructions

C.3.2.1. Products

Consider functions of the form $f(u) = u_1 u_2$, where $u_1, u_2 : (0,\delta] \times T^n \to \mathbb{R}$ are interpreted as any two components of the vector u. Clearly this has an expansion of the form Eq. (C.1) with $f_0 = u_{0,1}u_{0,2}$, $f_1^1 = u_{0,2}, f_1^2 = u_{0,1}$, and $f_2 = w_1 w_2$. To understand the function operator $f_2 = w_1 w_2$, we establish the following.

Lemma C.4. *Let $w_1 \in X_{\delta,\mu_1,q}$ and $w_2 \in X_{\delta,\mu_2,q}$ be two functions $(0,\delta] \times T^n \to \mathbb{R}$, for some constant $\delta > 0$, some smooth exponents μ_1 and μ_2, and an integer $q > n/2$. Then $w_1 \cdot w_2$ is in $X_{\delta,\mu_1+\mu_2,q}$ and, for some constant $C > 0$, depending only on n, q we have*

$$\|w_1 \cdot w_2\|_{\delta,\mu_1+\mu_2,q} \leq C \|w_1\|_{\delta,\mu_1,q} \cdot \|w_2\|_{\delta,\mu_2,q}.$$

Lemma C.5. *Let w_1, w_2 be as above, and define $g(w) = w_1 w_2$. Then $g(\cdot)$ satisfies the Lipschitz property.*

Proof of Lemma C.4. Since $w_1 \in X_{\delta,\mu_1,q}$ and $w_2 \in X_{\delta,\mu_2,q}$, there exist sequences $(w_{1,m}) \subset X_{\delta,\mu_1,q} \cap C^\infty((0,\delta] \times T^n)$ and $(w_{2,m}) \subset X_{\delta,\mu_2,q} \cap C^\infty((0,\delta] \times T^n)$ so that

$$\|w_{1,m} - w_1\|_{\delta,\mu_1,q} \to 0, \quad \|w_{2,m} - w_2\|_{\delta,\mu_2,q} \to 0.$$

We also know that $w_1(t)w_2(t) \in H^q$ for every $t \in (0,\delta]$ and $w_{1,m}w_{2,m} \in X_{\delta,\mu_1+\mu_2,q} \cap C^\infty((0,\delta] \times T^n)$ for every m. Moreover,

$$\begin{aligned}\|t^{-\mu_1-\mu_2}(w_{1,n}w_{2,n} - w_1w_2)\|_{H^q} &= \|t^{-\mu_1}(w_{1,n} - w_1)t^{-\mu_2}w_{2,n} + t^{-\mu_1}w_1t^{-\mu_2}(w_{2,n} - w_2)\|_{H^q} \\ &\le C\|t^{-\mu_1}(w_{1,n} - w_1)\|_{H^q}\|t^{-\mu_2}w_{2,n}\|_{H^q} + C\|t^{-\mu_1}w_1\|_{H^q}\|t^{-\mu_2}(w_{2,n} - w_2)\|_{H^q} \\ &\le C\|w_{1,n} - w_1\|_{\delta,\mu_1,q}\|w_{2,n}\|_{\delta,\mu_2,q} + \|w_1\|_{\delta,\mu_1,q}\|w_{2,n} - w_2\|_{\delta,\mu_2,q}\end{aligned}$$

This implies that $w_{1,m}w_{2,m}$ converges to w_1w_2 in $X_{\delta,\mu_1+\mu_2,q}$, and hence that $w_1w_2 \in X_{\delta,\mu_1+\mu_2,q}$. We have, for every $t \in (0,\delta]$, that

$$\|t^{-\mu_1-\mu_2}w_1(t)w_2(t)\|_{H^q} \le C\|t^{-\mu_1}w_1(t)\|_{H^q}\|t^{-\mu_2}w_2(t)\|_{H^q}$$

where the constant $C > 0$ does not depend on w_1, w_2 and t. This establishes the remaining estimate. □

Proof of Lemma C.5. This follows from straight-forward computation

$$\begin{aligned}\|\mathcal{R}[\mu_1 + \mu_2]\left(g(w) - g(\tilde{w})\right)\|_{H^q} &= \|t^{\mu_1+\mu_2}\left(w_1w_2 - \tilde{w}_1\tilde{w}_2\right)\|_{H^q} \\ &\le \|t^{\mu_1}w_1\|_{H^q}\|t^{\mu_2}(w_2 - \tilde{w}_2)\|_{H^q} \\ &\quad + \|t^{\mu_2}\tilde{w}_2\|_{H^q}\|t^{\mu_1}(w_1 - \tilde{w}_1)\|_{H^q}\end{aligned}$$

Taking the supremum for $t \in (0,\delta]$ establishes the estimate. □

C.3.2.2. Integer Powers

Consider the function operator $f(u) = u_i^p$ for any integer p with $p \geq 2$ or $p \leq -1$. For $p = 0$, the expansion is of course trivial and for $p = 1$, we simply find the fundamental Fuchsian function operator, (c.f. Section C.1.). Here u_i is a component of the $\mathbb{R}^d$-valued function u. However, to simplify the expressions below, we now use the expression u for a single component. Such function operators have the expansions

$$u^p = \sum_{k=0}^{p} \binom{p}{k} {u_0}^{p-k} w^k$$

in the case that $p > 0$, and

$$f(u) = u^p = \sum_{k=0}^{\infty} (-1)^k \binom{-p+k-1}{k} {u_0}^{p-k} w^k,$$

in the case that $p < 0$.

Let us focus on the case $p > 0$. We assume

$$u_0 \in X_{\delta,\kappa,q} \quad w \in X_{\delta,\kappa+\mu,q}.$$

In terms of the expansion Eq. (C.1), we have

$$f_0 = {u_0}^p \in X_{\delta,p\kappa,q}.$$

This follows from applying the product lemma (Lemma C.4) iteratively. We also have

$$f_1 w = p{u_0}^{p-1} w \in X_{\delta,p\kappa+\mu,q},$$

and

$$f_2[w] = \frac{p(p-1)}{2} {u_0}^{p-2} w^2 \in X_{\delta,p\kappa+2\mu,q}.$$

We have the following lemma.

Lemma C.6. *Suppose $\mu > 0$. Then the function operator $f(u) = u^p$, with $u = u_0 + w$ has the expansion above, is a well-defined function operator into $X_{\delta,p\kappa,q}$, and satisfies the Lipschitz property.*

The proof of this lemma follows from a repeated application of Lemma C.4.

C.3.2.3. Exponentials

Consider the function operators of the form $f(u) = \exp(ru_i)$ for a real number r. Again u_i represents a component of u, although in the rest of this discussion we let u denote the component. The expansion has the form $f(w) = \exp(ru_0)\exp(rw)$, and we can use Lemma C.3 (basically the Taylor expansion) to see that it takes the form Eq. (C.1).

Lemma C.7. *The functional $f(w) = exp(rw)$ is a well-defined operator $B_{\delta,\mu,p}(s) \cap X_{\delta,\mu,q} \to X_{\delta,0,q}$, for $\mu \geq 0$, and satisfies the Lipschitz property.*

The proof of this lemma is based on the following result (see [86] Ch. 13, Proposition 3.9).

Proposition C.8. *Let F be smooth, and F(0) = 0. Then for $w \in H^q$ with $q > n/2$*

$$\|F(w)\|_{H^q} \leq C\|w\|_{L^\infty} \left(1 + \|w\|_{H^q}\right),$$

for a constant depending on q, n.

Proof of Lemma C.7. Consider $h(w) = f(w) - 1 = \exp(rw) - 1$. Then the proposition and the Sobolev inequality implies

$$\|h(w)\|_{H^q} \leq C\|w\|_{H^q} \left(1 + \|w\|_{H^q}\right).$$

By the reverse triangle inequality we also have

$$\|f(w)\|_{H^q} \leq \|h(w)\|_{H^q} + \|1\|_{H^q}.$$

Therefore,

$$\|f(w)\|_{H^q} \leq C(q)\|w\|_{H^q} \left(1 + \|w\|_{H^q}\right) + C(n).$$

Taking the $\sup_{t \in (0,\delta]}$ shows that the operator is well-defined with target $X_{\delta,0,q}$.

The Lipschitz property follows from the Taylor expansion formula and Lipshitz property for products Lemma C.5. □

C.3.2.4. Inverses

Let $f(w)(t,x)$ have an expansion as in Eq. (C.1), and consider $f^{-1}(w)$. We have the following lemma, which is a consequence of the Taylor theorem.[1]

Lemma C.9. *Let $f(w)$ have an expansion as in Eq.* (C.1) *with $f_0(t,x) = 1$, and $f_2(w)(t,x) := f_2(t,x,w(t,x))$ a smooth function operator which is at least quadratic in w for all $w \in B_{\delta,\mu,q}(s)$. Then there exists a $0 < \hat{s} \leq s$ and a $\hat{f}(w)(t,x)$ such that for all $w \in B_{\delta,\mu,q}(\hat{s})$*

$$f^{-1}(w) = \frac{1}{1 + f_1^i(t,x)w_i + f_2(w)(t,x)} = 1 - f_1^i(t,x)w_i + \hat{f}(w)(t,x)$$

[1]See for example Corollary 8.17 [47] for the Taylor theorem for Banach spaces.

Function operators involving inverses which we are particularly interested in are the components of the inverse metric.

Lemma C.10 (Function operator properties of inverse metric)**.** *The inverse metric is a smooth function operator on $w \in B_{\delta,\mu,q}(s)$ for a sufficiently small $s > 0$, which can be expanded as*

$$g^{-1}(w) = y_0(t,x) + y_1^i(t,x)w_i + y_2(t,x,w),$$

where y_1 is a $\mathbb{R}^d$-valued function of t, x, and the sum over i is implied.

Proof. The inverse of a matrix can be written $g^{-1} = \frac{\mathrm{adj} g}{\det g}$, where adj$g$ is the adjugate of g which is made up from the cofactors of g. In terms of the first order fields u_i both $\det g$ and adj$g$ have the form $\sum_j \prod_i u_i^{p_i^j}$ for positive integers p_i^j. Under the expansion ansatz $u = u_0 + w$,

$$\det g = b_0(t,x) + b_1^i(t,x)w_i + b_2(t,x,w).$$

Since this is a smooth function operator of $w \in B_{\delta,\mu,q}(s)$ for a sufficiently small $s > 0$, and provided $b_1^i(t,x)/b_0(t,x)w_i$ and $b_2(t,x,w)/b_0(t,x)$ vanish near $t \searrow 0$, $1/\det g$ can be expanded in a similar form. Likewise, adj$g$, can be expanded, and the product of such operators has the form stated in the lemma for some scalar function $y_0(t,x)$ and function operator $y_2(t,x,w)$ and a vector valued function $y_1(t,x)$. □

Next we record the specific function expression for the inverse determinant of the metric in Chapter IV.

Lemma C.11. *The inverse determinant of the γ-block of the metric Eq.* (4.31) *can be written as the following expansion of function operators*

$$
\begin{aligned}
G(w) :=&(\det \gamma)^{-1} \\
=&O(t^{1-k^2})\left(1 - O(t^{1/2(1-k^2)})w_7 - O(t^{1/2(1-k^2)})w_1 - O(t^{(1-k^2)})w_1 w_7 + \ldots\right).
\end{aligned}
$$

The proof of this lemma follows from the fact that near $t \searrow 0$ the fields $U_1 U_7$ dominate the square of the shift U_4^2, and from the leading order expressions Eqs. (4.50)-(4.55).

Lemma C.12. *The inverse determinant of the τ-block of the metric Eq.* (4.31) *can be written as the following expansion of function operators*

$$
(\det \tau)^{-1} = O(t^{-2})\left(1 - O(t^{-1-k})w_{16} - O(t^{1+k})w_{10} - O(t^{-2})w_{10}w_{16} + \ldots\right).
$$

C.3.3. Higher-Order Properties for Simple Function Operators

In this section we establish the higher-order property Definition 2.24 for some of the simple function operators considered above. These results are essential for verifying the conditions of Definition 2.25 and Definition 2.26. We consider both the cases that $\hat{\mu} = \mu + \gamma_0$ and $\hat{\mu}$ is "nearly" a scalar, meaning certain components are allowed to differ my $\pm\epsilon$.

Lemma C.13. *The $\mathbb{R}$-valued function operator $F(w) := \lambda(t,x)w^i$ for $\lambda(t,x) \in X_{\delta,\xi,q}$, and w^i the i^{th} component of the vector w satisfies the higher order property with respect to:*

1. $\hat{\mu} = \mu + \gamma_0$ *for any* ξ

2. $\hat{\mu}$ is scalar for $\xi > 0$.

3. $\hat{\mu}$ is nearly scalar for ξ sufficiently large.

Proof. We suppose $F(w)$ occurs in the j^{th} component of the source. Computing $\Delta F_w(h)$ we find the higher order property requires

$$\hat{\mu}^j + \nu^j - \mu^j < \xi + \hat{\mu}^i.$$

For $\hat{\mu} = \mu + \gamma_0$, this inequality reduces to $\nu^j < \xi + \mu^i$ which must be satisfied if $F(w)$ appears in the source term. In the case that $\hat{\mu}$ is scalar the inequality becomes $\nu^j - \mu^j < \xi$, which can hold for $\xi > 0$. Finally in the nearly scalar case, such that $\hat{\mu}^i = \hat{\mu}^j + \epsilon$ we have $\nu^j - \mu^j < \xi + \epsilon$, which can be satisfied if $\xi + \epsilon > 0$. The Lipschitz property is straightforward. □

Lemma C.14. *The function operator $F(w) = \prod_{i=1}^{d} w_i^{p_i}$ satisfies the higher-order condition.*

Proof. The case of only one p_i non-zero is covered in Lemma C.13.

Now consider a function operator which depends on two of the components, $F_2(w) = w_1 w_2$. We compute

$$\Delta F_2(h) = h_1 w_2 + h_2(w_1 + h_1)$$

Let j denote the component of the source in which the function operator occurs. The higher-order property then yields

$$\hat{\mu}^j + \nu^j - \mu^j < \min\{\hat{\mu}^1 + \mu^2, \hat{\mu}^2 + \mu^1\}$$

For $\hat{\mu} = \mu + \gamma_0$, this inequality becomes $\nu^j < \mu^1 + \mu^2$, which holds by definition. In the case that $\hat{\mu}$ is scalar or nearly scalar the inequality reduces to $\nu^j - \mu^j < \min\{\mu^1 + \epsilon^1, \mu^2 + \epsilon^2\}$, for $\epsilon_1, \epsilon_2 \geq 0$. Clearly this holds provided $\min\{\mu^1 + \epsilon^1, \mu^2 + \epsilon^2\} > 0$. A similar argument holds for function operators with three fields, and arbitrary products follow from an application of Lemma C.19. The Lipschitz property is proved as in Lemma C.5. □

Lemma C.15. *Suppose $\mu \geq 0$. The function operator $F(w) = (exp \circ P_i)(w)$, where $P_i(w) = w_i$ is the projection onto the i^{th} component satisfies the higher-order condition.*

Proof. We compute using the Taylor expansion of $F(w)$,

$$\Delta F_w(h) = h_i + h_i(w_i + h_i) + \dots$$

and thus $\Delta F_w(h) \in X_{\delta,\hat{\mu}^i,q}$. It follows that $\Delta F_w(h) \in X_{\delta,\hat{\mu}^j+\nu^j-\mu^j,q}$ (or $X_{\delta,\hat{\mu}^j+\gamma,q}$ as a special case) if

$$\hat{\mu}^j + \nu^j - \mu^j = \hat{\mu}^j - \mu^j < \hat{\mu}^i. \tag{C.5}$$

Here we have used that $\nu^j = 0$ for the exponential function operator.

1. Case $\hat{\mu} = \mu + \gamma_0$: The inequality Eq. (C.5) becomes

$$\mu^j - \mu^j = 0 < \mu^i$$

   which is true since $\mu > 0$.

2. Case $\hat{\mu}$ is nearly scalar: We assume $\hat{\mu}^j = \hat{\mu}^i \pm \epsilon$ for some $\epsilon \geq 0$ (here equality corresponds to the scalar case). The inequality Eq. (C.5) becomes

$$\pm\epsilon < \mu^j,$$

which holds since ϵ may be taken arbitrarily small.

The Lipschitz property is proved as in Lemma C.7. □

C.3.4. Combinations of Function Operators

Lemma C.16 (Product of function operators)**.** *Let f and g be two $\mathbb{R}$-valued function operators satisfying the Lipschitz property and defined on $w \in B_{\delta,\mu,p}(s_0) \cap X_{\delta,\mu,q}$, where we have chosen s_0 and p to be such that both operators are defined on $B_{\delta,\mu,p}(s_0)$. Suppose that for all such w, $w \mapsto f(w) \in X_{\delta,\nu,q}$ and $w \mapsto g(w) \in X_{\delta,\eta,q}$ for exponent scalars $\nu, \eta : T^n \to \mathbb{R}$. Then the function operator $w \mapsto h(w) := f(w)g(w)$ is well-defined and satisfies the Lipschitz property with target $X_{\delta,\nu+\eta,q}$.*

Applying the above lemma iteratively to a function operator of the form $h(w) = (f(w))^k$ for some positive integer k, we find the following corollary.

Corollary C.17 (Positive powers of function operators)**.** *Let f be an $\mathbb{R}$-valued Lipschitz function operator mapping $B_{\delta,\mu,p}(s_0) \cap X_{\delta,\mu,q}$ to $X_{\delta,\nu,q}$. Then the function operator $w \mapsto h(w) := (f(w))^k$ for positive integer k is a well-defined Lipschitz operator taking values in $X_{\delta,k\nu,q}$.*

Lemma C.16 is proved in Appendix B of [3]. We now consider the higher-order property Definition 2.24.

Lemma C.18. *Let $F(w)$ be an $\mathbb{R}$-valued function operator from $X_{\delta,\mu,q}$ to $X_{\delta,\nu,q}$ satisfying the higher-order property, and let $\lambda(t,x)$ be an $\mathbb{R}$-valued function in $X_{\delta,\xi,q}$*

for some $q > n/2$. Then as long as $\xi \geq 0$, the function operator $\tilde{F}(w) := \lambda(t,x)F(x)$ satisfies the higher order property with respect to both $\hat{\mu} = \mu + \gamma_0$, and $\hat{\mu}$ scalar, or nearly scalar.

Proof. We compute $\Delta\tilde{F}_w(h) = \lambda(t,x)\Delta F_w(h)$. Since $\Delta F_w(h) \in X_{\delta,\hat{\mu}+\nu-\mu,q}$, we see that this is guaranteed to hold as long as $\xi \geq 0$. The Lipschitz property follows from straightforward computation. □

Lemma C.19. *Let $F : X_{\delta,\mu,q} \to X_{\delta,\nu_1,q}$ and $G : X_{\delta,\mu,q} \to X_{\delta,\nu_2,q}$ be two $\mathbb{R}$-valued function operators satisfying the higher-order property with respect to both $\hat{\mu} = \mu+\gamma_0$, and $\hat{\mu}$ scalar, or nearly scalar. If $\nu_1, \nu_2 \geq 0$, the product $H(w) := F(w)G(w)$ satisfies the higher order property with respect to both $\hat{\mu} = \mu + \gamma_0$, and $\hat{\mu}$ scalar, or nearly scalar.*

Proof. By computation we find $\Delta H_w(h) = \mathcal{O}(\hat{\mu}-\mu+\nu_1+\nu_2)$, from which the property follows. The Lipschitz property is proved as in Lemma C.16. □

C.4. Properties of the Inverse of S^0

We start by quoting a result from [60] on computing the inverse of a sum of matrices.

Lemma C.20 (Inverse of a sum of matrices)**.** *Let G and $G + H$ be non-singular matrices, and let H have positive rank r. Write $H = E_1 + \ldots + E_r$, where $E_i, i = 1, \ldots, r$ are rank one matrices, and suppose $C_{k+1} = G + E_1 + \ldots + E_k$ is non-singular for $k = 1, \ldots, r$. Define $C_1 = G$. Then,*

$$C_{k+1}^{-1} = C_k^{-1} - \nu_k C_k^{-1} E_k C_k^{-1}$$

where

$$\nu_k = (1 + tr(C_k^{-1}E_k))^{-1}$$

and in particular

$$(G + H)^{-1} = C_{r+1}^{-1} = C_r^{-1} - \nu_r C_r^{-1} E_r C_r^{-1} \tag{C.6}$$

We now use this lemma in proving desired properties of the inverse of the matrix-valued operator S^0. Since $S^0(w) = S_0^0 + S_1^0(w)$, where we can decompose $S_1^0(w) = \sum_{k=1}^r S_{1,k}^0(w)$ we write

$$\left(S^0(w)\right)^{-1} = \left(S_0^0\right)^{-1} + \Sigma_1^0(w).$$

The perturbation matrix $\Sigma_1^0(w)$ consists of a sum of terms each with products of the form $(S_0^0)^{-1} \cdot S_{1,k}^0(w) \cdot (S_0^0)^{-1}$. Note that in many of our applications S_1^0 is diagonal with every third entry non-zero. In these cases S_1^0 has rank $d/3$, and each $S_{1,k}^0(w)$ can be represented as a sparse matrix containing only the non-zero diagonal element.

Lemma C.21. *Suppose as in Definition 2.27 that* S_0^0 *shares the block diagonal structure of* μ*, and* $\mathcal{R}[\mu]S_1^0(w)\mathcal{R}[-\mu] \in B_{\delta,\xi,q}(r)$ *for some* $\xi > 0$*. Then the function operator*

$$w \mapsto \mathcal{R}[\mu]\left(S^0(w)\right)^{-1}\mathcal{R}[-\mu]$$

is a bounded *operator (Definition 2.4) of* $X_{\delta,\mu,q}$ *to* $X_{\delta,0,q}$ *for all* $w \in B_{\delta,\mu,p}(s_0) \cap B_{\delta,\mu,q}(s)$*. Further, the function operator*

$$w \mapsto \mathcal{R}[\mu]\Sigma_1^0(w)\mathcal{R}[-\mu]$$

is a bounded *operator of* $X_{\delta,\mu,q}$ *to* $X_{\delta,\underline{\xi},q}$ *for some exponent scalar* $\underline{\xi}$*, and for all* $w \in B_{\delta,\mu,p}(s_0) \cap B_{\delta,\mu,q}(s)$*. The exponent scalar* $\underline{\xi}$ *is less than or equal to the minimum of* ξ*.*

Proof. First note that from the assumption that $\mathcal{R}[\mu]S_1^0(w)\mathcal{R}[-\mu] \in B_{\delta,\xi,q}(r)$, and the splitting of $S_1^0(w)$ into rank one matrices consisting of the rows of $S_1^0(w)$, we find that for each k

$$\mathcal{R}[\mu]S_{1,k}^0(w)\mathcal{R}[-\mu] \in B_{\delta,\xi_k,q}(r),$$

for the exponent scalar ξ_k. The proof of Lemma C.21 is by induction on C_{k+1}^{-1} from Lemma C.20. Clearly, there exists an $r > 0$ such that $C_1^{-1} = (S_0^0)^{-1}$ is contained in $B_{\delta,0,q}(r)$. Next note that

$$C_2^{-1} = \left(S_0^0\right)^{-1} - \nu_1 \left(S_0^0\right)^{-1} S_{1,1}^0(w) \left(S_0^0\right)^{-1},$$

and thus,

$$\mathcal{R}[\mu]C_2^{-1}\mathcal{R}[-\mu] = \left(S_0^0\right)^{-1} - \nu_1 \left(S_0^0\right)^{-1} \mathcal{R}[\mu]S_{1,1}^0(w)\mathcal{R}[-\mu] \left(S_0^0\right)^{-1}.$$

Since ν_1 is a bounded function in $X_{\delta,0,q}$, we find that

$$\mathcal{R}[\mu]C_2^{-1}\mathcal{R}[-\mu] \in B_{\delta,0,q}(r)$$

for the same r as with $k = 0$, and

$$\nu_1 \left(S_0^0\right)^{-1} \mathcal{R}[\mu]S_{1,1}^0(w)\mathcal{R}[-\mu] \left(S_0^0\right)^{-1} \in B_{\delta,\xi_1,q}(\tilde{r})$$

for some $\tilde{r} > 0$. Now suppose that $\mathcal{R}[\mu]C_k^{-1}\mathcal{R}[-\mu] \in B_{\delta,0,q}(r)$ and compute

$$\begin{aligned}\mathcal{R}[\mu]C_{k+1}^{-1}\mathcal{R}[-\mu] =& \mathcal{R}[\mu]C_k^{-1}\mathcal{R}[-\mu] \\ & - \nu_k \left(\mathcal{R}[\mu]C_k^{-1}\mathcal{R}[-\mu]\right)\left(\mathcal{R}[\mu]S_{1,k}^0(w)\mathcal{R}[-\mu]\right)\left(\mathcal{R}[\mu]C_k^{-1}\mathcal{R}[-\mu]\right).\end{aligned}$$

From Lemma C.1, and the fact that $\nu_k \in X_{\delta,0,q}$ we find that the second part is contained in $B_{\delta,\xi_k,q}(\tilde{r}_k)$ for some $\tilde{r}_k$, while the first term is in $B_{\delta,0,q}(r)$ by assumption. Since $\xi_k > 0$, it follows that $\mathcal{R}[\mu]C_{k+1}^{-1}\mathcal{R}[-\mu] \in B_{\delta,0,q}(r)$. This shows that in particular $\mathcal{R}[\mu]C_r^{-1}\mathcal{R}[-\mu] = \mathcal{R}[\mu]\left(S^0(w)\right)^{-1}\mathcal{R}[-\mu]$ is contained in $B_{\delta,0,q}(r)$ for all w as in the lemma. To see the bounded property on $\mathcal{R}[\mu]\Sigma_1^0(w)\mathcal{R}[-\mu]$ we note that for each k, $\mathcal{R}[\mu]C_{k+1}^{-1}\mathcal{R}[-\mu] - \left(S_0^0\right)^{-1} \in B_{\delta,\xi_{min},q}(\tilde{r})$ for $\xi_{min} = \min_{i=1,\ldots,k}\{\xi_i\}$, and $\tilde{r} = \min_{i=1,\ldots,k}\{\tilde{r}_i\}$. □

Lemma C.22. *Let $q > n/2$. Suppose as in Definition 2.27 that S_0^0 shares the block diagonal structure of μ, $S_1^0(w) \in B_{\delta,\zeta,q}(r)$ for all $w \in B_{\delta,\mu,q}(s)$, and $\mathcal{R}[\mu]S_1^0(w)\mathcal{R}[-\mu] \in B_{\delta,\xi,q}(r)$ for some $\xi > 0$. Further suppose that for some exponent vector ξ, the function operator $h \mapsto \mathcal{R}[\mu]\left(S_1^0(w) - S_1^0(w+h)\right)\mathcal{R}[-\mu]$ is a bounded operator from $X_{\delta,\hat{\mu},q}$ to $X_{\delta,\xi+\hat{\mu}-\mu,q}$ for all $w \in B_{\delta,\mu,q}(s)$ and for $\hat{\mu} = \mu + \gamma$, where γ is any exponent scalar. Define*

$$\Delta\left(S^0(u_0+w)\right)^{-1}[h] := \left(S^0(u_0+w)\right)^{-1} - \left(S^0(u_0+w+h)\right)^{-1}$$

Then the function operator

$$h \mapsto \mathcal{R}[\mu]\Delta\left(S^0(u_0+w)\right)^{-1}[h]\mathcal{R}[-\mu] = \mathcal{R}[\mu]\Delta\Sigma_1^0(u_0+w)[h]\mathcal{R}[-\mu]$$

is a bounded operator $X_{\delta,\hat{\mu},q} \to X_{\delta,\underline{\xi}+\hat{\mu}-\mu,q}$ *for some exponent scalar* $\underline{\xi} > 0$ *for all* $w \in B_{\delta,\mu,q}(s)$.

Proof. The proof again follows from induction on C_{k+1}^{-1}. First note that $\Delta C_1^{-1} = \Delta\left(S_0^0\right)^{-1}[h] = 0$. Next, we find

$$\begin{aligned}\mathcal{R}[\mu]\big(C_2^{-1}(w) - C_2^{-1}(w+h)\big)\mathcal{R}[-\mu] \\ &= \mathcal{R}[\mu]\Big(-\nu_1(w)\left(S_0^0\right)^{-1} S_{1,1}^0(w)\left(S_0^0\right)^{-1} \\ &\quad + \nu_1(w+h)\left(S_0^0\right)^{-1} S_{1,1}^0(w+h)\left(S_0^0\right)^{-1}\Big)\mathcal{R}[-\mu] \\ &= \mathcal{R}[\mu]\left(S_0^0\right)^{-1}\left(S_{1,1}^0(w) - S_{1,1}^0(w+h)\right)\left(S_0^0\right)^{-1}\mathcal{R}[-\mu] + \dots\end{aligned}$$

since $\nu_k(\tilde{w}) = O(1)$ for all $\tilde{w} \in B_{\delta,\mu,q}(s)$ under the hypothesis that $w \mapsto S_1^0(w)$ is a bounded map into $B_{\delta,\zeta,q}(r)$ for some $\zeta > 0$. It follows from the remaining hypotheses that there exists an exponent scalar $\underline{\xi}_2$ such that $\mathcal{R}[\mu]\big(C_2^{-1}(w) - C_2^{-1}(w+h)\big)\mathcal{R}[-\mu] \in B_{\delta,\underline{\xi}_1+\hat{\mu}-\mu,q}(r_1)$.

Now suppose that $\mathcal{R}[\mu]\Delta C_{k+1}^{-1}[h]\mathcal{R}[-\mu] \in B_{\delta,\underline{\xi}_k+\hat{\mu}-\mu,q}(r_k)$. Compute

$$\begin{aligned}&\mathcal{R}[\mu]\Delta C_{k+2}^{-1}[h]\mathcal{R}[-\mu] \\ &\quad = \mathcal{R}[\mu]\Delta C_{k+1}^{-1}[h]\mathcal{R}[-\mu] \\ &\qquad - \mathcal{R}[\mu]\Big(C_{k+1}^{-1}(w)S_{1,k+1}^0(w)C_{k+1}^{-1}(w) \\ &\qquad - C_{k+1}^{-1}(w+h)S_{1,k+1}^0(w+h)C_{k+1}^{-1}(w+h)\Big)\mathcal{R}[-\mu] \\ &\quad = \mathcal{R}[\mu]\Delta C_{k+1}^{-1}[h]\mathcal{R}[-\mu] \\ &\qquad - \mathcal{R}[\mu]\Big(C_{k+1}^{-1}(w)\Delta S_{1,k+1}^0[h]C_{k+1}^{-1}(w) + C_{k+1}^{-1}(w+h)\Delta S_{1,k+1}^0[h]C_{k+1}^{-1}(w+h) \\ &\qquad + C_{k+1}^{-1}(w)S_{1,k+1}^0(w)\Delta C_{k+1}^{-1}[h] + \Delta C_{k+1}^{-1}[h]S_{1,k+1}^0(w)C_{k+1}^{-1}(w+h) \\ &\qquad - C_{k+1}^{-1}(w)\Delta S_{1,k+1}^0[h]C_{k+1}^{-1}(w+h)\Big)\mathcal{R}[-\mu].\end{aligned}$$

Inserting $\mathcal{R}[\mu]\mathcal{R}[-\mu]$ between each matrix multiplication, and using the induction hypothesis, the knowledge from Lemma C.21 that $\mathcal{R}[\mu]\Delta C_{k+1}^{-1}(w)\mathcal{R}[-\mu] \in X_{\delta,0,q}$, as well as assumed properties of $\mathcal{R}[\mu]S_1^0(w)\mathcal{R}[-\mu]$, $\mathcal{R}[\mu]\Delta S_1^0(w)[h]\mathcal{R}[-\mu]$, and that $\hat{\mu} - \mu = \gamma$ is an exponent scalar, we find that

$$\mathcal{R}[\mu]\Delta C_{k+2}^{-1}[h]\mathcal{R}[-\mu] \in X_{\delta,\underline{\xi}+\hat{\mu}-\mu,q}.$$

Further, we can use the Moser and Sobolev estimates in combination with the boundedness hypotheses to show that this function operator is bounded.

The conclusion of Lemma C.22 follows from setting $k+1 = r = rank(S_1^0)$. $\square$

REFERENCES CITED

[1] R. A. Adams. *Sobolev spaces.* Academic Press, 1975.

[2] E. Ames, F. Beyer, J. Isenberg, and P. G. LeFloch. AVTD Gowdy solutions in wave gauges. *In preparation.*

[3] E. Ames, F. Beyer, J. Isenberg, and P. G. LeFloch. Quasilinear Hyperbolic Fuchsian Systems and AVTD Behavior in T 2-Symmetric Vacuum Spacetimes. *Ann. Henri Poincare*, pages 1–79, Sept. 2013.

[4] E. Ames, F. Beyer, J. Isenberg, and P. G. LeFloch. Quasilinear Symmetric Hyperbolic Fuchsian Systems in Several Space Dimensions. *Contemporary Mathematics*, 591:19, May 2013.

[5] L. Andersson and A. D. Rendall. Quiescent Cosmological Singularities. *Comm. Math. Phys.*, 218(3):479–511, 2001.

[6] L. Andersson, H. van Elst, and C. Uggla. Gowdy phenomenology in scale-invariant variables. *Class. Quantum Grav.*, 21(3):S29–S57, 2003.

[7] R. A. Bartnik. Remarks on Cosmological Spacetimes and Constant Mean Curvature Surfaces. *Comm. Math. Phys.*, 117(4):615–624, 1988.

[8] R. A. Bartnik and J. Isenberg. The Constraint Equations. In *The Einstein Equations and Large-Scale Behavior of Gravitational Fields*, pages 1–38. Birkhäuser Basel, Basel, 2004.

[9] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz. Oscillatory approach to a singular point in the relativistic cosmology. *Adv. Phys.*, 19(80):525–573, 1970.

[10] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz. A general solution of the Einstein equations with a time singularity. *Adv. Phys.*, 31(6):639–667, 1982.

[11] B. K. Berger. Numerical Approaches to Spacetime Singularities. *arXiv*, gr-qc, Jan. 2002.

[12] B. K. Berger, P. T. Chruściel, J. Isenberg, and V. Moncrief. Global Foliations of Vacuum Spacetimes with T^2 Isometry. *Ann. Phys.*, gr-qc(1):117–148, 1997.

[13] B. K. Berger, D. Garfinkle, J. Isenberg, V. Moncrief, and M. Weaver. The Singularity in Generic Gravitational Collapse Is Spacelike, Local, and Oscillatory. *Mod. Phys. Lett. A*, pages 1565–1574, 1998.

[14] B. K. Berger, J. Isenberg, and M. Weaver. Oscillatory approach to the singularity in vacuum spacetimes with T^2 isometry. *Phys. Rev. D*, 64(8), 2001.

[15] B. K. Berger and V. Moncrief. Numerical investigation of cosmological singularities. *Phys. Rev. D*, 48:4676–4687, 1993.

[16] B. K. Berger and V. Moncrief. Evidence for an oscillatory singularity in generic U(1) symmetric cosmologies on T3 x R. *Phys. Rev. D*, 58(6):064023, 1998.

[17] B. K. Berger and V. Moncrief. Numerical evidence that the singularity in polarized U(1) symmetric cosmologies on $T^3 \times R$ is velocity dominated. *Phys. Rev. D*, 57(12):7235–7240, 1998.

[18] F. Beyer and P. G. LeFloch. Second-order hyperbolic Fuchsian systems and applications. *Classical And Quantum Gravity*, 27(24):245012, Nov. 2010.

[19] F. Beyer and P. G. LeFloch. Second-order hyperbolic Fuchsian systems. General theory. *arXiv*, gr-qc/1004.4885, 2010.

[20] A. Bove and H. Tahara. Cauchy problem for Fuchsian hyperbolic operatcrs, II. 1987.

[21] Y. Choquet-Bruhat. *General Relativity and the Einstein Equations*. Oxford mathematical monographs. Oxford University Press, Oxford, New York, 2008.

[22] Y. Choquet-Bruhat, C. Dewitt-Morette, and M. Dillard-Bleick. *Analysis, Manifolds and Physics*. North-Holland, revised edition edition, 1982.

[23] Y. Choquet-Bruhat and R. P. Geroch. Global aspects of the Cauchy problem in general relativity. *Comm. Math. Phys.*, 14(4):329–335, 1969.

[24] Y. Choquet-Bruhat and J. Isenberg. Half polarized $U(1)$-symmetric vacuum spacetimes with AVTD behavior. *J. Geom. Phys.*, 56(8):1199–1214, Nov. 2005.

[25] Y. Choquet-Bruhat, J. Isenberg, and V. Moncrief. Topologically general U(1) symmetric vacuum space-times with AVTD behavior. *Nuovo Cimento B*, gr-qc:625, Feb. 2005.

[26] P. T. Chruściel. On space-times with $U(1) \times U(1)$ symmetric compact Cauchy surfaces. *Ann. Phys.*, 202:100–150, 1990.

[27] P. T. Chruściel. *On Uniqueness in the Large of Solutions of Einsteins Equations (Strong Cosmic Censorship)*, volume 27 of *Proceedings of the Centre for Mathematics and its Applications*. The Australian National University, 1991.

[28] P. T. Chruściel. On maximal globally hyperbolic vacuum space-times. Dec. 2011.

[29] P. T. Chruściel and J. Isenberg. Nonisometric vacuum extensions of vacuum maximal globally hyperbolic spacetimes. *Phys. Rev. D*, 48(4):1616–1628, 1993.

[30] P. T. Chruściel, J. Isenberg, and V. Moncrief. Strong cosmic censorship in polarised Gowdy spacetimes. *Class. Quantum Grav.*, 7(10):1671–1680, 1990.

[31] C. M. Claudel and K. P. Newman. The Cauchy problem for quasi–linear hyperbolic evolution problems with a singularity in the time. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 454(1972):1073–1107, 1998.

[32] A. Clausen. *Singular behavior in T^2 symmetric spacetimes with cosmological constant*. PhD thesis, University of Oregon, 2007.

[33] D. M. Eardley, E. Liang, and R. K. Sachs. Velocity-Dominated Singularities in Irrotational Dust Cosmologies. *J. Math. Phys.*, 13(1):99, 1972.

[34] L. C. Evans. *Partial Differential Equations*. American Mathematical Society.

[35] Y. Fourès-Bruhat. Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. *Acta Math.*, 88(1):141–225, 1952.

[36] H. Friedrich. On the hyperbolicity of Einstein's and other gauge field equations. *Comm. Math. Phys.*, 100(4):525–543, 1985.

[37] D. Garfinkle. Simulations of Generic Singularities in Harmonic Coordinates. In *The conformal structure of space-time: Geometry, Analysis, Numerics*, pages 349–358. Springer, 2002.

[38] R. H. Gowdy. Vacuum spacetimes with two-parameter spacelike isometry groups and compact invariant hypersurfaces: Topologies and boundary conditions. *Ann. Phys.*, 83(1):203–241, 1974.

[39] N. Grubic and P. G. LeFloch. Weakly regular Einstein-Euler spacetimes with Gowdy symmetry. The global areal foliation. Dec. 2012.

[40] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*. Cambridge Monographs, 1973.

[41] J. M. Heinzle, C. Uggla, and W. C. Lim. Spike Oscillations. *arXiv.org*, June 2012.

[42] J. Isenberg. The Initial Value Problem in General Relativity. pages 1–24, Jan. 2013.

[43] J. Isenberg and S. Kichenassamy. Asymptotic behavior in polarized T^2-symmetric vacuum space–times. *J. Math. Phys.*, 40(1):340, 1999.

[44] J. Isenberg and V. Moncrief. Asymptotic behavior of the gravitational field and the nature of singularities in Gowdy spacetimes. *Annals of Physics*, 199(1):84–122, 1990.

[45] J. Isenberg and V. Moncrief. Asymptotic Behavior of Polarized and Half-Polarized U(1) symmetric Vacuum Spacetimes. *Class. Quantum Grav.*, 19:5361, Oct. 2002.

[46] J. Isenberg and M. Weaver. On the area of the symmetry orbits in T^2 symmetric spacetimes. *Class. Quantum Grav.*, gr-qc(16):3783–3796, 2003.

[47] J. Jost. *Postmodern Analysis*. Springer-Verlag, Berlin/Heidelberg, 2005.

[48] S. Kichenassamy. Fuchsian Equations in Sobolev Spaces and Blow-Up. *J. Differential Equations*, 125(1):299–327, 1996.

[49] S. Kichenassamy. *Nonlinear wave equations*. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, 1996.

[50] S. Kichenassamy. *Fuchsian reduction*. Applications to geometry, cosmology and mathematical physics. Birkhäuser, 2007.

[51] S. Kichenassamy and A. D. Rendall. Analytic description of singularities in Gowdy spacetimes. *Class. Quantum Grav.*, 15(5):1339–1355, 1999.

[52] S. Klainerman and I. Rodnianski. Rough solutions of the Einstein-vacuum equations. *Annals of mathematics*, 2005.

[53] J. M. Lee. *Introduction to Smooth Manifolds*. Springer, second edition, Aug. 2012.

[54] P. G. LeFloch. Einstein spacetimes with weak regularity. *arXiv.org*, Sept. 2010.

[55] P. G. LeFloch and J. Smulevici. Weakly regular T2 symmetric spacetimes. The global geometry of future developments. Dec. 2010.

[56] E. M. Lifshitz and I. M. Khalatnikov. Investigations in relativistic cosmology. *Adv. Phys.*, 12(46):185–249, 1963.

[57] W. C. Lim. *The Dynamics of Inhomogeneous Cosmologies*. PhD thesis, University of Waterloo, 2004.

[58] W. C. Lim, L. Andersson, D. Garfinkle, and F. Pretorius. Spikes in the mixmaster regime of G2 cosmologies. *Physical review D*, 79(12), June 2009.

[59] H. Lindblad and I. Rodnianski. Global existence for the Einstein vacuum equations in wave coordinates. *Communications in Mathematical Physics*, 2005.

[60] K. Miller. On the Inverse of the Sum of Matrices. *Mathematics Magazine*, 54(2):67–72, Mar. 1981.

[61] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation.* Freeman, 1973.

[62] V. Moncrief. Global properties of Gowdy spacetimes with $T^3 \times R$ topology. *Ann. Phys.*, 132(1):87–107, 1981.

[63] V. Moncrief and D. M. Eardley. The global existence problem and cosmic censorship in general relativity. *Gen. Rel. Grav.*, 13(9):887–892, 1981.

[64] P. S. Mostert. On a Compact Lie Group Acting on a Manifold. *Second Series*, 65(3):447–455, Jan. 1957.

[65] E. T. Newman, L. Tamburino, and T. Unti. Empty-Space Generalization of the Schwarzschild Metric. *J. Math. Phys.*, 4:915, 1963.

[66] R. Penrose. Gravitational Collapse: The Role of General Relativity. *Multiple values selected*, 1(7):252–276, 1969.

[67] A. D. Rendall. Fuchsian analysis of singularities in Gowdy spacetimes beyond analyticity. *Class. Quantum Grav.*, 17(16):3305–3316, 2000.

[68] A. D. Rendall and M. Weaver. Manufacture of Gowdy spacetimes with spikes. *Class. Quantum Grav.*, 18(15):2959–2975, 2001.

[69] H. Ringström. Curvature blow up in Bianchi VIII and IX vacuum spacetimes. *Classical And Quantum Gravity*, 2000.

[70] H. Ringström. The Bianchi IX Attractor. *Ann. Henri Poincare*, 2(3):405–500, 2001.

[71] H. Ringström. On a wave map equation arising in general relativity. *Comm. Pure Appl. Maths.*, 57(5):657–703, 2004.

[72] H. Ringström. Existence of an asymptotic velocity and implications for the asymptotic behavior in the direction of the singularity inT3-Gowdy. *Communications on Pure and Applied Mathematics*, 59(7):977–1041, 2006.

[73] H. Ringström. Strong cosmic censorship in the case of T3-Gowdy vacuum spacetimes. *Classical And Quantum Gravity*, 25(11):114010, May 2008.

[74] H. Ringström. Strong cosmic censorship in T3-Gowdy spacetimes. *Ann. Math.*, 170(3):1181–1240, 2009.

[75] H. Ringström. *The Cauchy Problem in General Relativity.* ESI Lectures in Mathematics and Physics. European Mathematical Society, Zürich, Switzerland, 2009.

[76] H. Ringström. Cosmic Censorship for Gowdy Spacetimes. *Living Reviews in Relativity*, 13, 2010.

[77] F. Ståhl. Fuchsian analysis of $S^2 \times S^1$ and S^3 Gowdy spacetimes. *Class. Quantum Grav.*, 19(17):4483–4504, 2002.

[78] H. Tahara. Singular hyperbolic systems, I Existence, uniqueness and differentiability. 1979.

[79] H. Tahara. Singular hyperbolic systems, III. On the Cauchy problem for Fuchsian hyperbolic partial differential equations. 1980.

[80] H. Tahara. Asymptotic expansions of solutions of Fuchsian hyperbolic partial differential equations. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 59(4):120–121, 1983.

[81] H. Tahara. Singular hyperbolic systems, V. Asymptotic expansions for Fuchsian hyperbolic partial differential equations. *Journal of the Mathematical Society of Japan*, 36(3):449–473, 1984.

[82] H. Tahara. On Fuchsian hyperbolic partial differential equations. *Differential Equations in Banach Spaces*, pages 243–253, 1986.

[83] H. Tahara. Singular hyperbolic systems, VI. Asymptotic analysis for Fuchsian hyperbolic equations in Gevrey classes. *Journal of the Mathematical Society of Japan*, 39(4):551–580, 1987.

[84] A. H. Taub. Empty Space-Times Admitting a Three Parameter Group of Motions. *Ann. Math.*, 53(3):472–490, 1951.

[85] M. E. Taylor. Nonlinear Hyperbolic Equations. In *Partial Differential Equations III - Nonlinear equations*. Springer, New York, NY, 2011.

[86] M. E. Taylor. *Partial Differential Equations III - Nonlinear equations*, volume 117 of *Applied Mathematical Sciences*. Springer, New York, NY, 2nd edition, 2011.

[87] J. Wainwright and G. F. R. Ellis, editors. *Dynamical Systems in Cosmology*. Cambridge University Press, 1997.

[88] R. M. Wald. *General relativity*. University of Chicago Press, 1984.