Interval values for strategic games in which players cooperate*

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Abstract

In this paper we propose a method to associate a coalitional interval game with each strategic game. The method is based on the lower and upper values of finite two-person zero-sum games. We axiomatically characterize this new method. As an intermediate step, we provide some axiomatic characterizations of the upper value of finite two-person zero-sum games.

1 Introduction

The problem of associating a coalitional game with every strategic game was already addressed by von Neumann and Morgenstern in their pioneer 1944 book. As they write in section 25.2.1, their purpose is "to determine everything that can be said about coalitions between players, compensations

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between partners in every coalition, mergers or fights between coalitions, etc., in terms of the characteristic function $v(S)$ alone". They define $v(S)$ using a zero-sum game between coalition S on one hand, and coalition $N \setminus S$ consisting of the other players on the other hand, and taking into account that the mixed extension of this game "has a well defined value" (see section 25.1.2).

The literature drifted away from the zero-sum game approach to this problem and toward a Nash-equilibrium approach. See, for example, Harsanyi (1963), Myerson (1991), and Bergantiños and García-Jurado (1995). Recently, the zero-sum game approach was revived in Carpente et al. (2005). In that paper, the same zero-sum games as in von Neumann and Morgenstern (1944) are used, but attention is limited to coordinated actions by coalitions of players rather than mixed strategies. They take a conservative or pessimistic point of view and define $v(S)$ to be the worth that the players in S can guarantee for themselves no matter what actions the players in the complementary coalition play. Hence, they use the lower value of the zero-sum game between coalition S and coalition $N \setminus S$. A more optimistic point of view would be to consider the worth of a coalition to be the amount that it can obtain for its members by correctly anticipating the actions of the complementary coalition and to use the upper value of the zero-sum game between coalition S and coalition $N \setminus S$.

In general, the pessimistic and optimistic methods lead to different predictions. Rather than choosing between them, in the current paper we simply acknowledge the fact that for every coalition S in a strategic game there is a reasonable *range* of worths that it can expect to obtain. This range is bounded from below by the pessimistic prediction obtained using the lower value of the associated zero-sum game and it is bounded from above by the optimistic prediction obtained using the upper value of that game. In doing so, we associate with each strategic game a *coalitional interval game*. A coalitional interval game is a pair $(N, \underline{\overline{v}})$, where N is the set of players, and $\underline{\overline{v}}$ is a correspondence that associates with every coalition $S \subset N$ an interval $\underline{\overline{v}}(S)$ that indicates that the worth of the coalition will be somewhere in this range.

Interval games were introduced and analyzed by Branzei et al. (2002) in the context of bankruptcy problems. In another paper, Branzei et al. (2003) propose two possible extensions of the Shapley value to the setting of interval bankruptcy games. We believe that in our setting of finding coalitional games to describe coalitions' expectations in strategic games, interval games are a very appropriate model because of the uncertainty that arises from not knowing the actions chosen by the complementary coalitions. Hence, it is very natural to represent the expectations of a coalition S as the interval defined by the lower and upper values of the zero-sum game played by S against $N \setminus S$. We formally introduce this method, which we call the lowper method, and characterize it axiomatically.

In order to understand the method based on the lower and the upper values of zero-sum games fully, we use results on the lower value as obtained in Carpente et al. (2005) and take some time to study the upper value of zero-sum games in the current paper. We combine the results for the lower and upper values to obtain axiomatic characterizations of what we call the lowper value: the interval defined by the lower and upper values. We then move on to the coalitional (interval) games associated with strategic games through zero-sum games between coalitions and their complements and the lower value, the upper value, and the lowper value. We prove that all these associated coalitional (interval) games are superadditive and show that, in fact, every superadditive coalitional game can be obtained as arising from a strategic game through either method. This extends a result in von Neumann and Morgenstern (1944) for coalitional games based on values of mixed extensions of the zero-sum games between coalitions and their complements. It shows that if we take seriously the idea that coalitional games describe coalitions' possibilities in strategic situations, then those coalitional games are going to be superadditive, and it also shows that every superadditive coalitional game can be interpreted as one that describes coalitions' possibilities in strategic situations.

The organization of the paper is as follows. In Section 2 we consider the lower and upper values of finite zero-sum two-player games. We recall axiomatizations of the lower value and provide ones of the upper value. In Section 3 we define the lowper value of finite zero-sum two-player games and provide two axiomatic characterizations of this evaluation correspondence. In Section 4 we consider coalitional (interval) games associated with strategic games. We address the superadditivity of the coalitional games based on the lower value, the upper value, and the lowper value, and show that all superadditive coalitional games can be obtained as arising from a strategic game through either method. In Section 5 we provide axiomatic characterizations of the lowper method, which associates a coalitional interval game with every strategic game. Finally, in Section 6 we provide some remarks on a possible extension of the current paper by applying Shapley values to the coalitional interval game that is obtained using the lowper method.

2 Values of finite two-person zero-sum games

We start by concentrating on finite two-person zero-sum games. We provide definitions of the lower and upper values of finite two-person zero-sum games. We also develop axiomatic characterizations of the upper value.

A finite two-person zero-sum game is modeled as a finite real matrix A. In this game, player 1 chooses a row and player 2 chooses a column and the number in the corresponding cell of the matrix is the amount that player 2 pays player 1 when these actions are chosen. Hence, if $A = [a_{ij}]_{i \in M,\ j \in N}$ is an $m \times n$ matrix, then the action sets of players 1 and 2, respectively, are $M = \{1, \ldots, m\}$ and $N = \{1, \ldots, n\}$, and if player 1 chooses action $i \in M$ and player 2 chooses action $j \in N$, then player 1's payoff is a_{ij} and player 2's payoff is $-a_{ij}$. We will refer to the finite two-person zero-sum game associated with matrix A as the matrix game A or simply the game A.

In the game A, player 1 wants to maximize a_{ij} and player 2 wants to minimize it. This leads to the definitions of the lower and upper values. The lower and upper values of A, $\underline{V}(A)$ and $\overline{V}(A)$ respectively, are defined as:

$$\underline{V}(A) := \max_{i \in M} \min_{j \in N} a_{ij}$$

$$\overline{V}(A) := \min_{j \in N} \max_{i \in M} a_{ij}.$$

Player 1 can make sure his payoff is not lower than the lower value by choosing an action $i \in M$ such that the minimum over all $j \in N$ of a_{ij} is $\underline{V}(A)$. Also, player 2 can make sure that he does not have to pay more than the upper value by choosing an action $j \in N$ such that the maximum over all $i \in M$ of a_{ij} is $\overline{V}(A)$. Of course, if player 1 can make sure he gets at least $\underline{V}(A)$ and player 2 can make sure player 1 gets no more than $\overline{V}(A)$, it follows immediately that $\underline{V}(A) \leq \overline{V}(A)$.

We denote the set of real matrices by $\mathcal{A}$. The *lower value function* $\underline{V} : \mathcal{A} \to \mathbb{R}$, associates with every matrix $A \in \mathcal{A}$ its lower value $\underline{V}(A)$. The *upper value function* $\overline{V} : \mathcal{A} \to \mathbb{R}$ is defined similarly. The lower and upper value functions are examples of *evaluation functions*, which we define as real-valued functions $f : \mathcal{A} \to \mathbb{R}$ that assign to every matrix $A \in \mathcal{A}$ a real number reflecting the evaluation of the finite two-person zero-sum game A from the point of view of player 1.

The lower value function was axiomatically characterized in Carpente et al. (2005). We state their characterization of the lower value function and provide an axiomatic characterization for the upper value function.

The following properties of an evaluation function $f : \mathcal{A} \to \mathbb{R}$ were used in Carpente et al. (2005).

Objectivity. For all $a \in \mathbb{R}$, $f([a]) = a$ [1].

Monotonicity. For all $A, B \in \mathcal{A}$, if $A \geq B$, then $f(A) \geq f(B)$.

Weak row dominance. The i^{th} row of the matrix A, denoted r_i, is *strongly dominated* if there exists another row r_k ($k \neq i$) in the matrix that is weakly larger than row r_i, i.e., $a_{kj} \geq a_{ij}$ for all $j \in N$. For all $A \in \mathcal{A}$, if row r is strongly dominated, then $f(A) = f(A \setminus r)$.[2]

Strong column dominance. The j^{th} column of the matrix A, denoted c_j, is *weakly dominated* if for all $i \in M$ there exists another

[1] Here, $[a]$ denotes the 1×1 matrix A with $a_{11} = a$.

[2] $A \setminus r$ denotes the matrix obtained from A by deleting row r. Similarly, we use the notation $A \setminus c$ when deleting a column c.

column c_k $(k \neq j)$ such that $a_{ik} \leq a_{ij}$. For all $A \in \mathcal{A}$, if column c is weakly dominated, then $f(A) = f(A \setminus c)$.

Objectivity states that in a trivial situation where both players have exactly one action available, player 1's evaluation is the payoff that player 1 receives when both players play their unique actions. Monotonicity states that a weak increase in player 1's payoff for every possible choice of actions by both players should not result in a decrease in player 1's evaluation. Weak row dominance expresses that player 1's evaluation should not change if he can no longer choose an action that is never better for him than some other action that is available to him. Strong column dominance states that player 1's evaluation does not change when player 2 can no longer use an action that is weakly dominated. An action for player 2 is weakly dominated when for every action by player 1, there is another action for player 2 that results in a weakly lower payoff for player 1 and therefore a weakly higher payoff for player 2. Note that this can be a different action of player 2 for every action by player 1.

Theorem 1 (Carpente et al. (2005)) *The lower value function* $\underline{V}$ *is the unique evaluation function that satisfies objectivity, monotonicity, weak row dominance, and strong column dominance.*

Analogous to Theorem 1, we can provide an axiomatic characterization of the upper value function. In addition to objectivity and monotonicity, this characterization uses the following two properties of an evaluation function $f : \mathcal{A} \to \mathbb{R}$.

Strong row dominance. The i^{th} row of the matrix A, denoted r_i, is *weakly dominated* if for every $j \in N$ there exists another row r_k $(k \neq i)$ such that $a_{kj} \geq a_{ij}$. For all $A \in \mathcal{A}$, if row r is weakly dominated, then $f(A) = f(A \setminus r)$.

Weak column dominance. The j^{th} column of the matrix A, denoted c_j, is *strongly dominated* if there exists another column c_k $(k \neq j)$ in the matrix that is weakly smaller than column c_j, i.e., $a_{ik} \leq a_{ij}$ for all $i \in M$. For all $A \in \mathcal{A}$, if column c is strongly dominated, then $f(A) = f(A \setminus c)$.

Note that strong row dominance implies weak row dominance. This holds because every row that is strongly dominated is also weakly dominated. So, if f satisfies strong row dominance and r is a strongly dominated row, then r is weakly dominated and can be eliminated without changing the evaluation. Analogously, strong column dominance implies weak column dominance.

Theorem 2 *The upper value function $\overline{V}$ is the unique evaluation function that satisfies objectivity, monotonicity, strong row dominance, and weak column dominance.*

Proof. *Existence.* It is easily seen that $\overline{V}$ satisfies *objectivity* and *monotonicity.* To see that $\overline{V}$ satisfies *weak column dominance*, note that $\max_{i\in M} a_{ik} \leq \max_{i\in M} a_{ij}$ if column c_j is strongly dominated by column c_k in matrix A. This means that player 2 does not need column c_j to reach the minimum of these expressions, which equals $\overline{V}(A)$. To see that $\overline{V}$ satisfies *strong row dominance*, note that if row r_i is weakly dominated in matrix A, then $\max_{k\in M} a_{kj} = \max_{k\in M\setminus i} a_{kj}$ for every $j \in N$. Hence, deleting row r_i does not change the upper value.

Uniqueness. To prove that there is no other evaluation function that satisfies objectivity, monotonicity, strong row dominance, and weak column dominance, let $f : \mathcal{A} \to \mathbb{R}$ be an evaluation function satisfying these properties and take a matrix $A = [a_{ij}]_{i\in M, j\in N} \in \mathcal{A}$. Suppose, without loss of generality, that $\overline{V}(A)$ is the element in the i^{th} row and the j^{th} column. We then have

$$f(A) \leq f(\begin{bmatrix} \max\{a_{11}, a_{1j}\} & \cdots & \max\{a_{1n}, a_{1j}\} \\ \vdots & \ddots & \vdots \\ \max\{a_{m1}, a_{mj}\} & \cdots & \max\{a_{mn}, a_{mj}\} \end{bmatrix}) =$$

$$f(\begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}) = f([a_{ij}]) = a_{ij} = \overline{V}(A).$$

Here, we have used *monotonicity* to create a matrix in which the j^{th} column strongly dominates all other columns so that we can then apply *weak column dominance* (repeatedly) to delete all the other columns. The $m \times 1$ matrix that is left consists of the j^{th} column of A and for this column we know that $a_{ij} \geq a_{kj}$ for all $k \in M$ because $\overline{V}(A) = a_{ij}$. So, in the remaining matrix all rows other than the i^{th} one are weakly dominated and can be eliminated by *strong row dominance*. *Objectivity* provides the last step.

To show that $f(A) \geq \overline{V}(A)$, we first add a row to the matrix A in which all elements are equal to $\overline{V}(A)$. Note that such a row is weakly dominated and its addition will not alter the valuation by *strong row dominance*. Using *monotonicity*, we make all rows weakly dominated by the newly added one, so that we can apply *strong row dominance* (repeatedly) to eliminate all but the newly added row. In the resulting $1 \times n$ matrix all elements are equal to $\overline{V}(A)$, so that all columns but one can be eliminated by *weak column dominance*. Application of *objectivity* finishes the following sequence.

$$f(A) = f(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \\ \overline{V}(A) & \cdots & \overline{V}(A) \end{bmatrix}) \geq$$

$$f(\begin{bmatrix} \min\{a_{11}, \overline{V}(A)\} & \cdots & \min\{a_{1n}, \overline{V}(A)\} \\ \vdots & \ddots & \vdots \\ \min\{a_{m1}, \overline{V}(A)\} & \cdots & \min\{a_{mn}, \overline{V}(A)\} \\ \overline{V}(A) & \cdots & \overline{V}(A) \end{bmatrix}) =$$

$$f([\overline{V}(A) \ldots \overline{V}(A)]) = f([\overline{V}(A)]) = \overline{V}(A).$$

We have shown that $f(A) = \overline{V}(A)$, which proves that the upper value $\overline{V}$ is the unique evaluation function that satisfies *objectivity*, *monotonicity*, *strong row dominance*, and *weak column dominance*. □

Carpente et al. (2005) show that in the axiomatization of the lower value provided in Theorem 1, monotonicity can be replaced by row elimination and column elimination. Row elimination states that if player 1 looses the ability to use one of his strategies, then his evaluation should not increase. Column elimination states that player 1's evaluation does not decrease and therefore player 2's evaluation does not increase when player 2 looses the ability to use one of his strategies.

Row elimination. For all $A \in \mathcal{A}$ and all rows r of A, $f(A) \geq f(A \backslash r)$.

Column elimination. For all $A \in \mathcal{A}$ and all columns c of A, $f(A) \leq f(A \setminus c)$.

Theorem 3 (Carpente et al. (2005)) *The lower value function $\underline{V}$ is the unique evaluation function that satisfies objectivity, row elimination, column elimination, weak row dominance, and strong column dominance.*

In the axiomatization of the upper value we can also replace monotonicity by row elimination and column elimination. This gives us the following axiomatization of the upper value.

Theorem 4 *The upper value function $\overline{V}$ is the unique evaluation function that satisfies objectivity, row elimination, column elimination, strong row dominance, and weak column dominance.*

Proof. *Existence.* We already established that $\overline{V}$ satisfies *objectivity*, *strong row dominance*, and *weak column dominance*. To see that it also satisfies *row elimination* and *column elimination*, it suffices to note that taking the maximum over a smaller set leads to a weakly smaller value and that taking the minimum over a smaller set leads to a weakly larger value.

Uniqueness. The proof of uniqueness is analogous to that in Theorem 2. Let $f : \mathcal{A} \to \mathbb{R}$ be an evaluation function that satisfies the five axioms listed in the theorem and let $A = [a_{ij}]_{i \in M, j \in N} \in \mathcal{A}$. Suppose, without loss of generality, that $\overline{V}(A)$ is the element in the i^{th} row and the j^{th} column. Then, applying *column elimination*, *strong row dominance*, and *objectivity*, successively, we obtain

$$f(A) \leq f(\begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}) = f([a_{ij}]) = a_{ij} = \overline{V}(A).$$

Using *strong row dominance*, *row elimination*, *weak column dominance*, and *objectivity*, we obtain

$$f(A) = f(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \\ \overline{V}(A) & \cdots & \overline{V}(A) \end{bmatrix}) \geq$$

$$f([\overline{V}(A) \dots \overline{V}(A)]) = f([\overline{V}(A)]) = \overline{V}(A).$$

This proves that $f(A) = \overline{V}(A)$. □

3 The lowper interval value

Our interest in values of two-person zero-sum games stems from our quest for methods to find coalitional games that appropriately describe coalitions' expectations in strategic games. We will discuss these methods extensively in the next section, but remark here that the lower value provides a pessimistic expectation, whereas the upper value provides an optimistic expectation. In the current section, rather than taking either an optimistic or a pessimistic point of view, we simply recognize that the expectations are bound by these two values and consider the correspondence that associates with each matrix game A the interval $[\underline{V}(A), \overline{V}(A)]$, with the lower value as its lower bound and the upper value as its upper bound. We call this correspondence *the lowper value* and denote it by $\underline{\overline{V}}$.

In the current section, we provide two axiomatizations of the lowper value. Both are inspired by the axiomatic characterizations of the lower and upper values discussed in the previous section.

The lowper value is an *evaluation correspondence*, which we define as an interval-valued correspondence f that assigns to every matrix $A \in \mathcal{A}$ a real interval $[\underline{f}(A), \overline{f}(A)]$, where $\underline{f}(A) \in \mathbb{R}$, $\overline{f}(A) \in \mathbb{R}$, and $\underline{f}(A) \leq \overline{f}(A)$. This interval reflects player 1's evaluation of the finite two-person zero-sum game A.

We define some properties of an evaluation correspondence f on $\mathcal{A}$.[3] To simplify our notations, we introduce the inequality between intervals defined by $[a, b] \geq [c, d]$ if and only if $a \geq c$ and $b \geq d$. Also, we define an addition of intervals by $[a, b] + [c, d] = [e, f]$ where $e = a + c$ and $f = b + d$.

Objectivity. For all $a \in \mathbb{R}$, $f([a]) = [a, a]$.

Monotonicity. For all $A, B \in \mathcal{A}$, if $A \geq B$, then $f(A) \geq f(B)$.

Weak row dominance. For all $A \in \mathcal{A}$, if row r is strongly dominated, then $f(A) = f(A \setminus r)$.

Weak column dominance. For all $A \in \mathcal{A}$, if column c is strongly dominated, then $f(A) = f(A \setminus c)$.

The lowper value satisfies all these four properties. Moreover, it is the maximal evaluation correspondence that satisfies the properties, in the way specified in the following theorem.

Theorem 5 *The lowper value function $\underline{\overline{V}}$ satisfies objectivity, monotonicity, weak row dominance, and weak column dominance. Moreover, if f is an evaluation correspondence that satisfies these four properties, then $f(A) \subset \underline{\overline{V}}(A)$ for each $A \in \mathcal{A}$.*

Proof. *Existence.* We already know that the lower value and the upper value both satisfy *objectivity*, *monotonicity*, *weak row dominance*, and *weak column dominance*.[4] The interval that the lowper value associates with each

[3]Note that we give these properties the same names as the corresponding properties for evaluation functions. This shouldn't be a source of confusion as it will always be clear from the context what version of the properties is intended.

[4]We remind the reader that strong row dominance implies weak row dominance and strong column dominance implies weak column dominance.

matrix game $A \in \mathcal{A}$ has as its lower bound the lower value and as its upper bound the upper value. From this it readily follows that the lowper value satisfies *objectivity*, *monotonicity*, *weak row dominance*, and *weak column dominance*.

Maximality. Let f be an evaluation correspondence that satisfies the four axioms listed in the theorem and let $A = [a_{ij}]_{i \in M, j \in N} \in \mathcal{A}$.

First we consider the lower bound of the interval $[\underline{f}(A), \overline{f}(A)]$. Suppose, without loss of generality, that $\underline{V}(A)$ is the element in the i^{th} row and the j^{th} column. First, we use *monotonicity* to make all the rows strongly dominated by the i^{th} row. Then, we apply *weak row dominance* (repeatedly) to delete all the other rows. We are then left with a $1 \times n$ matrix consisting of the i^{th} row of A. Because $\underline{V}(A) = a_{ij}$, we know that $a_{ik} \geq a_{ij}$ for all $k \in N$. Hence, in the $1 \times n$ matrix, all columns different from the j^{th} are strongly dominated and can be eliminated by *weak column dominance*. Then, we can apply *objectivity*, and obtain

$$\underline{f}(A) \geq \underline{f}(\begin{bmatrix} \min\{a_{11}, a_{i1}\} & \cdots & \min\{a_{1n}, a_{in}\} \\ \vdots & \ddots & \vdots \\ \min\{a_{m1}, a_{i1}\} & \cdots & \min\{a_{mn}, a_{in}\} \end{bmatrix}) =$$

$$\underline{f}([a_{i1} \dots a_{in}]) = \underline{f}([a_{ij}]) = a_{ij} = \underline{V}(A).$$

With respect to the upper bound of the interval $[\underline{f}(A), \overline{f}(A)]$, we can show that $\overline{f}(A) \leq \overline{V}(A)$ in a manner analogous to that followed in the proof of Theorem 2. We do not repeat this proof here, but refer the reader to the relevant part of the proof of Theorem 2.

Combining the two inequalities that we have shown, we obtain $f(A) \subset \overline{\underline{V}}(A)$ for each $A \in \mathcal{A}$. □

The characterization in Theorem 5 is not tight in the sense that it leaves room for a correspondence to associate with a matrix $A \in \mathcal{A}$ an interval that is strictly contained in the interval $[\underline{V}(A), \overline{V}(A)]$. In Theorem 6 we provide a tight axiomatization of the lowper value. To do this, we need two more properties of evaluation correspondences.

Strong row dominance in the upper bound. For all $A \in \mathcal{A}$, if row r is weakly dominated, then $\overline{f}(A) = \overline{f}(A \setminus r)$.

Strong column dominance in the lower bound. For all $A \in \mathcal{A}$, if column c is weakly dominated, then $\underline{f}(A) = \underline{f}(A \setminus c)$.

Theorem 6 *The lowper value function $\underline{\overline{V}}$ is the unique evaluation correspondence that satisfies objectivity, monotonicity, weak row dominance, weak column dominance, strong row dominance in the upper bound, and strong column dominance in the lower bound.*

Proof. *Existence.* We already know that the lowper value satisfies *objectivity*, *monotonicity*, *weak row dominance*, and *weak column dominance.* It readily follows from strong row dominance of the upper value and strong column dominance of the lower value that the lowper value also satisfies *strong row dominance in the upper bound*, and *strong column dominance in the lower bound.*

Uniqueness. To prove that there is no other evaluation correspondence that satisfies *objectivity*, *monotonicity*, *weak row dominance*, *weak column dominance*, *strong row dominance in the upper bound*, and *strong column dominance in the lower bound*, let f be an evaluation correspondence that satisfies these six properties and let $A = [a_{ij}]_{i \in M, j \in N} \in \mathcal{A}$. It readily follows from Theorem 5 that $f(A) \subset \underline{\overline{V}}(A)$, so the proof of uniqueness will be completed if we prove that $\underline{f}(A) \leq \underline{V}(A)$ and $\overline{f}(A) \geq \overline{V}(A)$.

We start with the lower bound of the interval. To show that $\underline{f}(A) \leq \underline{V}(A)$, we first add a column to the matrix A in which all elements are equal to $\underline{V}(A)$. Note that such a column is weakly dominated, so by *strong column dominance in the lower bound*, this addition will not alter the lower bound. Then, we apply *monotonicity* to make all columns weakly dominated by the newly added one, after which we use *strong column dominance in the lower bound* again (repeatedly) to eliminate all these other columns. We are then left with a $m \times 1$ matrix in which all elements are equal to $\underline{V}(A)$, in which all rows are strongly dominated so that we can eliminate all but one of them by *weak row dominance.* Then, we can apply *objectivity*, and obtain

$$\underline{f}(A) = \underline{f}(\begin{bmatrix} a_{11} & \cdots & a_{1n} & \underline{V}(A) \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & \underline{V}(A) \end{bmatrix}) \leq$$

$$\underline{f}(\begin{bmatrix} \max\{a_{11}, \underline{V}(A)\} & \cdots & \max\{a_{1n}, \underline{V}(A)\} & \underline{V}(A) \\ \vdots & \ddots & \vdots & \vdots \\ \max\{a_{m1}, \underline{V}(A)\} & \cdots & \max\{a_{mn}, \underline{V}(A)\} & \underline{V}(A) \end{bmatrix}) =$$

$$\underline{f}(\begin{bmatrix} \underline{V}(A) \\ \vdots \\ \underline{V}(A) \end{bmatrix}) = \underline{f}([\underline{V}(A)]) = \underline{V}(A).$$

We now consider the upper bound of the interval. To show that $\overline{f}(A) \geq \overline{V}(A)$, we first add a row to the matrix A in which all elements are equal to $\overline{V}(A)$. Note that such a row is weakly dominated and its addition will not alter the upper bound by *strong row dominance in the upper bound*. Using *monotonicity*, we make all rows weakly dominated by the newly added one, so that we can apply *strong row dominance in the upper bound* (repeatedly) to eliminate all but the newly added row. In the resulting $1 \times n$ matrix all elements are equal to $\overline{V}(A)$, so that all columns but one can be eliminated by *weak column dominance*. Application of *objectivity* finishes the following sequence.

$$\overline{f}(A) = \overline{f}(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \\ \overline{V}(A) & \cdots & \overline{V}(A) \end{bmatrix}) \geq$$

$$\overline{f}(\begin{bmatrix} \min\{a_{11}, \overline{V}(A)\} & \cdots & \min\{a_{1n}, \overline{V}(A)\} \\ \vdots & \ddots & \vdots \\ \min\{a_{m1}, \overline{V}(A)\} & \cdots & \min\{a_{mn}, \overline{V}(A)\} \\ \overline{V}(A) & \cdots & \overline{V}(A) \end{bmatrix}) =$$

$$\overline{f}([\overline{V}(A)\ldots\overline{V}(A)]) = \overline{f}([\overline{V}(A)]) = \overline{V}(A).$$

This completes the proof of the theorem. □

4 Coalitional interval games associated with strategic games

Methods to associate coalitional games with strategic games appear already in von Neumann and Morgenstern (1944), whose method is based on the value of matrix games, and in Carpente et al. (2005), who study a method that is based on the lower value function. As discussed in Carpente et al. (2005), the approach taken by von Neumann and Morgenstern (1944) requires mixing of coordinated strategies of coalitions. The lower-value based method requires only coordinated actions, but it takes a very conservative or pessimistic approach to the problem of finding the worths of coalitions in strategic games and looks at what the members of a coalition can *guarantee* themselves. A method based on the upper value would provide an optimistic approach and look at what the members of a coalition may obtain if they can react optimally to their opponents' actions. Alternatively, a method based on the upper value looks at what the other players cannot prevent the members of a coalition from getting.

We believe that this is a very natural situation for interval games to arise. In the current paper, rather than taking either an optimistic or a pessimistic point of view, we simply recognize that the expectations of coalitions are bound by the lower and upper values and associate with each strategic game an interval game. We start by providing the necessary definitions.

A *strategic game* $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N})$ consists of a set of players $N = \{1, \ldots, n\}$, and for every player $i \in N$ a set of actions X_i available to this player, and a payoff function $u_i : \prod_{j\in N} X_j \to \mathbb{R}$. In this paper we consider only finite strategic games, which are those games in which the actions sets $\{X_i\}_{i\in N}$ are all finite. The class of finite strategic games with player set N is denoted by Γ^N. We denote the class of all finite strategic games by Γ.

A *coalitional game* is a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function of the game, assigning to each coalition $S \subset N$ its worth $v(S)$. The worth $v(S)$ of a coalition S represents the benefits that this coalition can obtain for its members. By convention, $v(\emptyset) = 0$. From now on, we identify a coalitional game (N, v) with its characteristic function v. We denote the class of coalitional games with player set N by G^N and we use G to denote the class of all coalitional games. A coalitional game $v \in G^N$ is said to be *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all coalitions $S, T \subset N$ with $S \cap T = \emptyset$.

A *coalitional interval game* with player set N is a pair $(N, \underline{\overline{v}})$, where the correspondence $\underline{\overline{v}}$ associates an interval $\underline{\overline{v}}(S) = [\underline{v}(S), \overline{v}(S)]$ with each coalition $S \subset N$. The interpretation of this game is that a coalition S can obtain for its members a worth that is somewhere in the interval $\underline{\overline{v}}(S)$. We define $\underline{\overline{v}}(\emptyset) = [0, 0]$. Note that each coalitional game (N, v) can easily be modeled as an interval game where the interval associated with a coalition S consists of one point only, the point $v(S)$. We denote the class of coalitional interval games with player set N by IG^N and we use IG to denote the class of all coalitional interval games. A coalitional interval game $\underline{\overline{v}} \in IG^N$ is said to be *superadditive* if $\underline{\overline{v}}(S \cup T) \geq \underline{\overline{v}}(S) + \underline{\overline{v}}(T)$[5] for all coalitions $S, T \subset N$ with $S \cap T = \emptyset$.

Our method to associate coalitional (interval) games with strategic games is based on the following procedure that was proposed in von Neumann and Morgenstern (1944). Let $g \in \Gamma^N$ be a strategic game and take a non-empty coalition $S \subset N$, $S \neq N$. The two-person zero-sum game g_S is defined by

$$g_S = (\{S, N \setminus S\}, \{X_S, X_{N \setminus S}\}, \{u_S, -u_S\}),$$

where, for all $T \subset N$, $X_T = \prod_{i \in T} X_i$ and $u_T = \sum_{i \in T} u_i$. In this game, there are two players, coalition S and coalition $N \setminus S$. The actions available to each of these two coalitions are all the combinations of the actions available to its

[5] Addition of and inequality between intervals are as defined on page 11.

members in the game g. The payoff to coalition S is the sum of the payoffs of its members for every possible actions tuple, and the payoff to coalition $N \setminus S$ is the opposite of this. Note that the game g_S is a finite two-person zero-sum game. We denote by A_S the matrix of this game. Now, we can look at the lower or upper values of the game g_S as expectations of coalition S in the strategic game g. They represent the worth that the members of coalition S expect to be able to attain for themselves even if the players in $N \setminus S$ cooperate to keep the worth of coalition S as low as possible. For the grand coalition N, there are no players outside the coalition to try and keep the worth of N as low as possible and therefore the worth of the grand coalition is simply the maximum that its members can get.

Carpente et al. (2005) consider *the lower-value method* that associates with a strategic game $g \in \Gamma^N$ the coalitional game $\underline{v}_g \in G^N$ defined by

$$\underline{v}_g(S) = \underline{V}(A_S)$$

for all non-empty $S \subset N$, $S \neq N$, and $\underline{v}_g(N) = \max_{x \in X_N} u_N(x)$. This coalitional game represents a conservative or pessimistic point of view in that the worth of a coalition is defined as the amount that it can *guarantee* for its members by choosing an appropriate coordinated action. There exists an action $x_S \in X_S$ such that $u_S(x_S, x_{N \setminus S}) \geq \underline{v}_g(S)$ for all $x_{N \setminus S} \in X_{N \setminus S}$.

The upper-value method associates with a strategic game $g \in \Gamma^N$ the coalitional game $\overline{v}_g \in G^N$ defined by

$$\overline{v}_g(S) = \overline{V}(A_S)$$

for all non-empty $S \subset N$, $S \neq N$, and $\overline{v}_g(N) = \max_{x \in X_N} u_N(x)$. This coalitional game represents an optimistic point of view in that the worth of a coalition is defined as the amount that it can obtain for its members by correctly anticipating the coordinated action of its opponents. For every $x_{N \setminus S} \in X_{N \setminus S}$, there exists an action $x_S \in X_S$ such that $u_S(x_S, x_{N \setminus S}) \geq \overline{v}_g(S)$.

The lowper value method associates with a strategic game $g \in \Gamma^N$ the coalitional interval game $\overline{\underline{v}}_g \in IG^N$ defined by

$$\underline{\overline{v}}_g(S) = \underline{\overline{V}}(A_S) = [\underline{V}(A_S), \overline{V}(A_S)]$$

for all non-empty $S \subset N$, $S \neq N$, and

$$\underline{\overline{v}}_g(N) = [\max_{x \in X_N} u_N(x), \max_{x \in X_N} u_N(x)].$$

This coalitional interval game takes neither a pessimistic nor an optimistic point of view, but acknowledges that coalition S will obtain a worth somewhere between the pessimistic and the optimistic expectations. Note that for the grand coalition we obtain an interval that contains exactly one element, so that we obtain a class of coalitional interval games for which efficiency, for example, is well-defined.

We illustrate these three games in the following example.

Example 1 *Consider the following three-player strategic game g.*[6]

α_3	α_2	β_2
α_1	$(1,0,0)$	$(1,2,3)$
β_1	$(0,1,2)$	$(1,2,0)$

β_3	α_2	β_2
α_1	$(0,0,2)$	$(2,0,2)$
β_1	$(1,1,2)$	$(0,1,2)$

Consider the 2-player coalition $S = \{1,3\}$. The matrix of the 2-person zero-sum game associated with this coalition is

$$A_S = \begin{pmatrix} 1 & 4 \\ 2 & 1 \\ 2 & 4 \\ 3 & 2 \end{pmatrix},$$

where the columns correspond to the strategies α_2 and β_2 (from left to right) of player $2 \in N \setminus S$ and the rows are ordered as follows. The first row corresponds to the strategies (α_1, α_3) by the players in S, the second row to (β_1, α_3), the third row to (α_1, β_3), and the fourth row to (β_1, β_3). The lower

[6]Following the general standard, player 1 is the row player, player 2 the column player, and player 3 chooses the matrix to the left or that to the right.

value of this matrix is $\underline{V}(A_S) = 2$ *and its upper value equals* $\overline{V}(A_S) = 3$. *Hence, we find that* $\underline{v}_g(1,3) = 2$, $\overline{v}_g(1,3) = 3$ *and* $\overline{\underline{v}}_g(1,3) = [2,3]$.

Following the same procedure, we find the worths of the other coalitions to be $\underline{v}_g(1) = \underline{v}_g(2) = 0$, $\underline{v}_g(3) = 2$, $\underline{v}_g(1,2) = \underline{v}_g(2,3) = 2$, $\underline{v}_g(1,2,3) = 6$, *and* $\overline{v}_g(1) = 1$, $\overline{v}_g(2) = 0$, $\overline{v}_g(3) = 2$, $\overline{v}_g(1,2) = 2$, $\overline{v}_g(2,3) = 3$, $\overline{v}_g(1,2,3) = 6$. *The coalitional interval game* $\overline{\underline{v}}_g$ *associates the following intervals with the various coalitions:* $\overline{\underline{v}}_g(1) = [0,1]$, $\overline{\underline{v}}_g(2) = [0,0]$, $\overline{\underline{v}}_g(3) = \overline{\underline{v}}_g(1,2) = [2,2]$, $\overline{\underline{v}}_g(1,3) = \overline{\underline{v}}_g(2,3) = [2,3]$, *and* $\overline{\underline{v}}_g(1,2,3) = [6,6]$.

This example illustrates that in general the two coalitional games $\underline{v}_g$ and $\overline{v}_g$ are different, so that $\overline{\underline{v}}_g$ is a non-degenerate interval game.

The games $\underline{v}_g$, $\overline{v}_g$, and $\overline{\underline{v}}_g$ that we can obtain in the previous example are all superadditive. This is not a coincidence, but holds for all coalitional (interval) games derived in the described manner from strategic games using the lower, upper, or lowper value. For the games $\underline{v}_g$ this was shown in Carpente et al. (2005) and for the games $\overline{v}_g$ we show this in the following proposition. Note that superadditivity of the lowper value interval game $\overline{\underline{v}}_g$ readily follows from superadditivity of the lower value game $\underline{v}_g$ and the upper value game $\overline{v}_g$.

Proposition 1 *For every strategic game* $g \in \Gamma^N$, *the associated coalitional game* $\overline{v}_g$ *is superadditive.*

Proof. Let $g = (N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N}) \in \Gamma^N$ be a strategic game and take two non-empty coalitions $S, T \subset N$, such that $S \cap T = \emptyset$. Then

$$\overline{v}_g(S \cup T) = \min_{x_{-ST} \in X_{N \setminus (S \cup T)}} \max_{x_{ST} \in X_{S \cup T}} u_{S \cup T}(x_{ST}, x_{-ST}) =$$

$$\min_{x_{-ST} \in X_{N \setminus (S \cup T)}} \max_{x_S \in X_S} \max_{x_T \in X_T} u_{S \cup T}(x_S, x_T, x_{-ST}).$$

Let $y_S \in X_S$ and $y_T \in X_T$. For any $y_{-ST} \in X_{N \setminus (S \cup T)}$ we have that

$$u_{S \cup T}(y_S, y_T, y_{-ST}) \leq \max_{x_S \in X_S} \max_{x_T \in X_T} u_{S \cup T}(x_S, x_T, y_{-ST}).$$

Then, it holds that

$$\overline{v}_g(S \cup T) = \min_{x_{-ST} \in X_{N\setminus(S\cup T)}} \max_{x_S \in X_S} \max_{x_T \in X_T} u_{S\cup T}(x_S, x_T, x_{-ST}) \geq$$

$$\min_{x_{-ST} \in X_{N\setminus(S\cup T)}} u_{S\cup T}(y_S, y_T, x_{-ST}).$$

Using this, we derive that

$$\overline{v}_g(S \cup T) \geq \min_{x_{-ST} \in X_{N\setminus(S\cup T)}} u_{S\cup T}(y_S, y_T, x_{-ST}) \geq$$

$$\min_{x_{-ST} \in X_{N\setminus(S\cup T)}} u_S(y_S, y_T, x_{-ST}) + \min_{x_{-ST} \in X_{N\setminus(S\cup T)}} u_T(y_S, y_T, x_{-ST}) \geq$$

$$\min_{x_T \in X_T} \min_{x_{-ST} \in X_{N\setminus(S\cup T)}} u_S(y_S, x_T, x_{-ST}) +$$

$$\min_{x_S \in X_S} \min_{x_{-ST} \in X_{N\setminus(S\cup T)}} u_T(x_S, y_T, x_{-ST}) =$$

$$\min_{x_{-S} \in X_{N\setminus S}} u_S(y_S, x_{-S}) + \min_{x_{-T} \in X_{N\setminus T}} u_T(y_T, x_{-T}).$$

Since this holds for all $y_S \in X_S$ and $y_T \in X_T$, we can use this to derive that

$$\overline{v}_g(S \cup T) \geq \min_{x_{-S} \in X_{N\setminus S}} \max_{x_S \in X_S} u_S(x_S, x_{-S}) + \min_{x_{-T} \in X_{N\setminus T}} \max_{x_T \in X_T} u_T(x_T, x_{-T}) =$$

$$\overline{v}_g(S) + \overline{v}_g(T).$$

This proves that $\overline{v}_g$ is superadditive. □

In fact, every superadditive coalitional game (N, v) can be obtained as both a game $(N, \underline{v}_g)$ and a game $(N, \overline{v}_g)$. We prove this in the following proposition.

Proposition 2 *For every superadditive coalitional game (N, w), there exists a strategic game $g \in \Gamma^N$ such that $w = \underline{v}_g$ and $w = \overline{v}_g$.*

Proof. Let $(N, w) \in G^N$ be a superadditive coalitional game. Define the strategic game $g(w) = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}) \in \Gamma^N$ by $X_i = \{x_i \subset N \mid i \in x_i\}$ for each $i \in N$ and

$$u_i(x_N) = \begin{cases} \frac{1}{|x_i|} w(x_i) & \text{if } x_j = x_i \text{ for all } j \in x_i \\ w(i) & \text{otherwise.} \end{cases}$$

In this game, each player announces which coalition he wants to be a member of. The multi-player coalitions that are formed are the ones for which all of its members have indicated that they want to form exactly this coalition. The players who are not a member of such a multi-player coalition each form a singleton coalition. In each (one- or multi-player) coalition that is formed, the members of the coalition share the worth of the coalition equally. We will prove that $w = \underline{v}_{g(w)}$ and $w = \overline{v}_{g(w)}$. Take $S \subset N$, $S \notin \{\emptyset, N\}$. We denote the two-person zero-sum game used to determine the worth of coalition S in the strategic game $g(w)$ by $g_S(w)$. This is the game that is used to find both $\underline{v}_{g(w)}(S)$ and $\overline{v}_{g(w)}(S)$. Hence, $g_S(w) = (\{S, N\backslash S\}, \{X_S, X_{N\backslash S}\}, \{u_S, -u_S\})$. Superadditivity of w implies that for any partition $\mathcal{P}(\mathcal{S})$ of S it holds that $w(S) \geq \sum_{P \in \mathcal{P}(\mathcal{S})} w(P)$. Define the strategy profile y by

$$y_i = \begin{cases} S & \text{if } i \in S \\ \{i\} & \text{if } i \notin S. \end{cases}$$

Then we have

$$\max_{x_S \in X_S} u_S(x_S, y_{-S}) = u_S(y_S, y_{-S}) = w(S)$$

and $u_S(y_S, x_{-S}) = u_S(y_S, y_{-S})$ for all $x_{-S} \in X_{N\backslash S}$ so that

$$\min_{x_{-S} \in X_{N\backslash S}} u_S(y_S, x_{-S}) = u_S(y_S, y_{-S}) = w(S).$$

This proves that

$$\underline{v}_{g(w)}(S) = w(S) = \overline{v}_{g(w)}(S).$$

□

Von Neumann and Morgenstern (1944) showed that every superadditive coalitional game can be obtained as the game associated with a strategic game through values of matrix games.[7] Proposition 2 shows that the same is true if the worths of coalitions in strategic games are determined by the

[7]This appears in Section 57.3 in Chapter XI of von Neumann and Morgenstern (1944). In their proof, they use the strategic game $g(w)$ that appears in the proof of Proposition 2.

lower or upper value. Hence, the set of coalitional games associated with strategic games is exactly the same, independent of whether the value, the lower value, or the upper value is used to determine worths of coalitions in strategic games. However, these methods may each associate a different coalitional game with each strategic game, as is true for the strategic game in Example 1.

We have shown that for all strategic games, associated coalitional (interval) games based on the lower value, the upper value, or the lowper value are superadditive and that, in fact, every superadditive coalitional game can be obtained as arising from a strategic game through either method. This shows that if we take seriously the idea that coalitional games describe coalitions' possibilities in strategic situations, then those coalitional games are going to be superadditive, and it also shows that every superadditive coalitional game can be interpreted as one that describes coalitions' possibilities in strategic situations.

5 Axiomatizations of the lowper value method

In this section we concentrate on the lowper value method and study its properties. We provide two axiomatic characterizations for the lowper value method. Both characterizations are inspired by those of the lowper value function in Section 3.

A *method* is a function $\mu : \Gamma \to IG$ that associates a coalitional interval game $\mu(g) \in IG^N$ with every strategic game $g \in \Gamma^N$. Throughout the following, we will use the notation $\mu(g)(S) = [\underline{\mu}(g)(S), \overline{\mu}(g)(S)]$ for $g \in \Gamma^N$ and $S \subset N$. We denote the lowper value method by $\mu_{\underline{\overline{V}}}$, so that $\mu_{\underline{\overline{V}}}(g) = \underline{\overline{v}}_g$.

We now define several properties of methods. The first property is individual objectivity and it deals with strategic games in which there is a player i whose payoff does not depend on the actions chosen by the players. Individual objectivity states that in the coalitional interval game associated with such a strategic game, player i expects to get this stable payoff, so that the interval associated with the coalition consisting of this player contains only that payoff.

Individual objectivity. For every $g \in \Gamma^N$ and every player $i \in N$, if there exists a $c \in \mathbb{R}$ such that $u_i(x) = c$ for all $x \in X_N$, then $\mu(g)(i) = [c, c]$.

Monotonicity states, loosely speaking, that a (weak) increase in a player's payoff for all possible action tuples in a strategic game will weakly increase this player's expectations in the associated coalitional interval game.

Monotonicity. Given two strategic games $g = (N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N}) \in \Gamma^N$ and $g' = (N, \{X_i\}_{i \in N}, \{u'_i\}_{i \in N}) \in \Gamma^N$ such that $u_i \geq u'_i$ for some player $i \in N$, then for this player i it holds that $\mu(g)(i) \geq \mu(g')(i)$.

Irrelevance of strongly dominated actions states that if a player loses the ability to use an action that is weakly worse for him than another one of his actions, this does not affect the player's expectation in the associated coalitional interval game.

Irrelevance of strongly dominated actions. In a game $g \in \Gamma^N$, an action $x_i \in X_i$ of player i is strongly dominated if there exists an action $x'_i \in X_i$, $x'_i \neq x_i$, such that $u_i(x'_i, x_{N \setminus i}) \geq u_i(x_i, x_{N \setminus i})$ for all $x_{N \setminus i} \in X_{N \setminus i}$. For any $g \in \Gamma^N$ and player $i \in N$, if action $x_i \in X_i$ is strongly dominated, then $\mu(g)(i) = \mu(g')(i)$, where $g' \in \Gamma^N$ is the game obtained from g by deleting action x_i.

Irrelevance of strongly dominated threats deals with cross-player effects. If a player j has an action that is not necessary to keep a player i's payoff as low as possible, then the deletion of this action does not affect player i's expectations in the associated coalitional interval game.

Irrelevance of strongly dominated threats. In a game $g \in \Gamma^N$, an action $x_j \in X_j$ of player j is a strongly dominated threat to player $i \neq j$ if there exists an action $x'_j \in X_j$, $x'_j \neq x_j$, such that $u_i(x'_j, x_{N \setminus j}) \leq u_i(x_j, x_{N \setminus j})$ for all $x_{N \setminus j} \in X_{N \setminus j}$. For any $g \in \Gamma^N$ and players $i, j \in N$,

$i \neq j$, if action $x_j \in X_j$ is a strongly dominated threat to player i, then $\mu(g)(i) = \mu(g')(i)$, where $g' \in \Gamma^N$ is the game obtained from g by deleting action x_j.

The following property, merge invariance, has no equivalent in our characterizations of the lowper value. It deals with coalitions consisting of more than one player and states that the expectations of such coalitions are the same whether its members present themselves as a multi-player coalition or as a single player in a derived strategic game. We need some additional notation to be able to formally present these ideas. Let $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}) \in \Gamma^N$ and $S \subset N$, $S \neq \emptyset$. To study the opportunities of the members of S as a group, we introduce a new player $p(S)$ with action set $X_{p(S)} := X_S$ and payoff function $u_{p(S)} : \prod_{j\in(N\setminus S)\cup\{p(S)\}} X_j \to \mathbb{R}$ defined by $u_{p(S)}(x_{p(S)}, x_{N\setminus S}) = u_S(x_S, x_{N\setminus S})$ for all $x_{p(S)} = x_S \in X_S = X_{p(S)}$ and all $x_{N\setminus S} \in X_{N\setminus S}$. Denote $N(S) := (N\setminus S)\cup\{p(S)\}$. The game $g(S) \in \Gamma^{N(S)}$ is defined by $g(S) = (N(S), \{X_i\}_{i\in N(S)}, \{u_i\}_{i\in N(S)})$.[8]

Merge invariance. Let $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}) \in \Gamma^N$ and $S \subset N$, $S \neq \emptyset$. Then $\mu(g)(S) = \mu(g(S))(p(S))$, where $g(S)$ is the strategic game obtained from g by considering the coalition S as a single player $p(S)$.

The properties introduced above are all satisfied by the lowper value method. Also, among all methods satisfying the properties, the lowper value method associates the largest possible interval with each coalition of players.

Theorem 7 *The lowper value method $\mu_{\underline{V}}$ satisfies individual objectivity, monotonicity, irrelevance of strongly dominated actions, irrelevance of strongly dominated threats, and merge invariance. Moreover, if μ is a method satisfying these five properties, then $\mu(g)(S) \subset \mu_{\underline{V}}(g)(S)$ for each $g \in \Gamma^N$ and $S \subset N$.*

[8] Note the distinction between the game $g(S)$ defined here and the zero-sum two-player game g_S between S on the one hand and $N \setminus S$ on the other hand, which we defined in Section 4.

Proof. *Existence.* First, we show that $\mu_{\underline{\overline{V}}}$ satisfies the five properties. Let $g \in \Gamma^N$, $i \in N$, and $c \in \mathbb{R}$ be such that $u_i(x) = c$, for all $x \in X_N$. Then, in the matrix A_i of the game g_i, all entries are equal to c.[9] The lower value and the upper value of this matrix are equal to c. Hence, $\mu_{\underline{\overline{V}}}(g)(i) = [c, c]$, which shows that $\mu_{\underline{\overline{V}}}$ satisfies *individual objectivity*.

Now, let $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}) \in \Gamma^N$ and $g' = (N, \{X_i\}_{i\in N}, \{u'_i\}_{i\in N}) \in \Gamma^N$ be two strategic games such that $u_i \geq u'_i$ for player $i \in N$. Then, $A_i \geq A'_i$, where A_i denotes the matrix of the game g_i and A'_i denotes the matrix of the game g'_i. It now follows from monotonicity of the lower and upper values that $\mu_{\underline{\overline{V}}}(g)(i) \geq \mu_{\underline{\overline{V}}}(g')(i)$. This proves that $\mu_{\underline{\overline{V}}}$ satisfies *monotonicity.*

To see that $\mu_{\underline{\overline{V}}}$ satisfies *irrelevance of strongly dominated actions*, note that if action x_i of player i is strongly dominated in the game $g \in \Gamma^N$, then it corresponds to a strongly dominated row in the matrix A_i of the game g_i. Hence, by weak row dominance of the lowper value, it holds that $\mu_{\underline{\overline{V}}}(g)(i) = \mu_{\underline{\overline{V}}}(g')(i)$, where $g' \in \Gamma^N$ is the game that is obtained from g by deleting action x_i.

To see that $\mu_{\underline{\overline{V}}}$ satisfies *irrelevance of strongly dominated threats*, note that if action $x_j \in X_j$ of a player j is a strongly dominated threat to player $i \neq j$ in the game $g \in \Gamma^N$, then for all $x_{N\setminus i,j} \in X_{N\setminus i,j}$ action $(x_j, x_{N\setminus i,j})$ corresponds to a strongly dominated column in the matrix A_i of the game g_i. Hence, applying weak column dominance of the lowper value repeatedly, elimination of the columns corresponding to the actions $(x_j, x_{N\setminus i,j})$ for all $x_{N\setminus i,j} \in X_{N\setminus i,j}$ will not affect the lowper value. The matrix that is left after eliminating all these columns is that corresponding to the game $g' \in \Gamma^N$ that is obtained from g by deleting action x_j. For this game we thus have $\mu_{\underline{\overline{V}}}(g)(i) = \underline{\overline{v}}_g(i) = \underline{\overline{v}}_{g'}(i) = \mu_{\underline{\overline{V}}}(g')(i)$.

Merge invariance of $\mu_{\underline{\overline{V}}}$ follows straightforwardly by noting that the matrix A_S of the strategic game g_S derived from g and the matrix $A_{p(S)}$ of the strategic game $g(S)_{p(S)}$ derived from $g(S)$ are the same.

Maximality. We proceed by showing the second part of the statement

[9] To keep our notation as simple as possible, we use g_i instead of $g_{\{i\}}$.

in theorem. Let $\mu : \Gamma \to IG$ be a method satisfying *individual objectivity*, *monotonicity*, *irrelevance of strongly dominated actions*, *irrelevance of strongly dominated threats*, and *merge invariance*. Let $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}) \in \Gamma^N$ and fix a non-empty coalition $S \subset N$. We have to show that $\mu(g)(S) \subset \mu_{\underline{V}}(g)(S)$.

If $S = N$, then *merge invariance*, *irrelevance of strongly dominated actions*, and *individual objectivity* clearly imply that $\mu(g)(N) = \mu_{\underline{V}}(g)(N)$.

Assume now that $S \neq N$. Consider the game $g(S) = (N(S), \{X_i\}_{i\in N(S)}, \{u_i\}_{i\in N(S)})$ that is obtained from g by considering the coalition S as a single player $p(S)$. Because μ satisfies *merge invariance*, we know that

$$\mu(g)(S) = \mu(g(S))(p(S)).$$

As remarked before, the matrix $A_{p(S)}$ of the strategic game $g(S)_{p(S)}$ derived from $g(S)$, and the matrix A_S of the strategic game g_S derived from g are the same. Obviously, $\underline{V}(A_{p(S)}) = \underline{V}(A_S)$ and $\overline{V}(A_{p(S)}) = \overline{V}(A_S)$. We will show that $\underline{\mu}(g)(S) \geq \underline{V}(A_S)$ and $\overline{\mu}(g)(S) \leq \overline{V}(A_S)$.

Part I. $\underline{\mu}(g)(S) \geq \underline{V}(A_S)$.

Let $\underline{x} = (\underline{x}_i)_{i\in N} \in \prod_{i\in N} X_i$ be an action tuple such that the lower value of A_S is obtained in the row corresponding to action $\underline{x}_S$ for coalition S and the column corresponding to action $\underline{x}_{N\setminus S}$ for coalition $N \setminus S$. Define $\underline{x}_{p(S)} = \underline{x}_S \in X_{p(S)}$ to be the corresponding action of player $p(S)$.

Let g_1 be the game that is obtained from the game $g(S)$ by bounding the utility of player $p(S)$ from above by $\underline{V}(A_S)$, i.e.,

$$g_1 = (N(S), (X_i)_{i\in N(S)}, (u_i)_{i\in N\setminus S}, u'_{p(S)}),$$

where

$$u'_{p(S)}(x_{p(S)}, x_{N\setminus S}) = \min\{u_{p(S)}(x_{p(S)}, x_{N\setminus S}), \underline{V}(A_S)\}$$

for all $x_{N\setminus S} \in X_{N\setminus S}$ and all $x_{p(S)} \in X_{p(S)}$. Because μ satisfies *monotonicity*, we know that

$$\underline{\mu}(g(S))(p(S)) \geq \underline{\mu}(g_1)(p(S)).$$

Now, note that

$$\underline{V}(A_S) = \max_{x_{p(S)} \in X_{p(S)}} \min_{x_{N\setminus S} \in X_{N\setminus S}} u_{p(S)}(x_{p(S)}, x_{N\setminus S})$$

is obtained at $(\underline{x}_{p(S)}, \underline{x}_{N\setminus S})$, so that $\min_{x_{N\setminus S} \in X_{N\setminus S}} u_{p(S)}(\underline{x}_{p(S)}, x_{N\setminus S}) = \underline{V}(A_S)$ and $u_{p(S)}(\underline{x}_{p(S)}, x_{N\setminus S}) \geq \underline{V}(A_S)$ for all $x_{N\setminus S} \in X_{N\setminus S}$. Hence,

$$u'_{p(S)}(\underline{x}_{p(S)}, x_{N\setminus S}) = \underline{V}(A_S)$$

for all $x_{N\setminus S} \in X_{N\setminus S}$. Moreover, we have $u'_{p(S)}(x_{p(S)}, x_{N\setminus S}) \leq \underline{V}(A_S)$ for all $x_{N\setminus S} \in X_{N\setminus S}$ and all $x_{p(S)} \in X_{p(S)}$. Hence, every action $x_{p(S)} \in X_{p(S)}$, $x_{p(S)} \neq \underline{x}_{p(S)}$, is strongly dominated by action $\underline{x}_{p(S)}$. Because μ satisfies *irrelevance of strongly dominated actions*, we can eliminate all the actions $x_{p(S)} \neq \underline{x}_{p(S)}$ of player $p(S)$. Hence,

$$\underline{\mu}(g_1)(p(S)) = \underline{\mu}(g_2)(p(S)),$$

where g_2 is the game that is obtained from g_1 by deleting all actions of player $p(S)$ except action $\underline{x}_{p(S)}$.

In the game g_2, for every player $j \neq p(S)$ every action $x_j \in X_j \setminus \underline{x}_j$ is a strongly dominated threat to player $p(S)$, because $u'_{p(S)}(\underline{x}_{p(S)}, \underline{x}_{N\setminus S})$ $= \min_{x_{N\setminus S} \in X_{N\setminus S}} u'_{p(S)}(\underline{x}_{p(S)}, x_{N\setminus S})$. Since μ satisfies *irrelevance of strongly dominated threats*, we can eliminate all these strongly dominated threats to player $p(S)$. Hence,

$$\underline{\mu}(g_2)(p(S)) = \underline{\mu}(g_3)(p(S)),$$

where g_3 is the game that is obtained from g_2 by deleting all actions $x_j \in X_j \setminus \underline{x}_j$ for every player $j \in N \setminus S$.

In the game g_3 every player j has exactly one action, $\underline{x}_j$. Hence, for this game we can use *individual objectivity* of μ to derive that

$$\underline{\mu}(g_3)(p(S)) = u'_{p(S)}(\underline{x}).$$

We now have $\underline{\mu}(g)(S) = \underline{\mu}(g(S))(p(S)) \geq \underline{\mu}(g_1)(p(S)) = \underline{\mu}(g_2)(p(S)) = \underline{\mu}(g_3)(p(S)) = u'_{p(S)}(\underline{x}) = \underline{V}(A_S)$.

Part II. $\overline{\mu}(g)(S) \leq \overline{V}(A_S)$.

Now, let $\bar{x} = (\bar{x}_i)_{i \in N} \in \prod_{i \in N} X_i$ be an action tuple such that the upper value of A_S is obtained in the row corresponding to action $\bar{x}_S$ for coalition S and the column corresponding to action $\bar{x}_{N \setminus S}$ for coalition $N \setminus S$. Define $\bar{x}_{p(S)} = \bar{x}_S \in X_{p(S)}$ to be the corresponding action of player $p(S)$.

We will define a new game g_4 by deleting actions for players in $N \backslash S$. Without loss of generality, we assume that $N \backslash S = \{1, 2, \ldots, k\}$, where k denotes the number of players in $N \backslash S$.

We first define the game g_1^* that is obtained from the game $g(S)$ by changing the payoffs of player $p(S)$ to

$$u^1_{p(S)}(x_{p(S)}, x_1, (x_i)_{i \in \{2,3,\ldots,k\}}) =$$

$$\max\{u_{p(S)}(x_{p(S)}, x_1, (x_i)_{i \in \{2,3,\ldots,k\}}), u_{p(S)}(x_{p(S)}, \bar{x}_1, (x_i)_{i \in \{2,3,\ldots,k\}})\},$$

where $x_i \in X_i$ for all $i \in \{1, 2, \ldots, k\}$ and $x_{p(S)} \in X_{p(S)}$. Because μ satisfies *monotonicity*, we know that

$$\overline{\mu}(g(S))(p(S)) \leq \overline{\mu}(g_1^*)(p(S)).$$

In the game g_1^*, every action $x_1 \in X_1$, $x_1 \neq \bar{x}_1$, is a strongly dominated threat to player $p(S)$ because $u^1_{p(S)}(x_{p(S)}, \bar{x}_1, (x_i)_{i \in \{2,3,\ldots,k\}}) \leq u^1_{p(S)}(x_{p(S)}, x_1, (x_i)_{i \in \{2,3,\ldots,k\}})$ for all $x_i \in X_i$, $i \in \{2, \ldots, k\}$, and $x_{p(S)} \in X_{p(S)}$. Since μ satisfies *irrelevance of strongly dominated threats*, we can eliminate all these strongly dominated threats to player $p(S)$ from g_1^* without changing the worth of $p(S)$ in the image of the game under μ. Hence,

$$\overline{\mu}(g_1^*)(p(S)) = \overline{\mu}(g_1^{\#})(p(S)),$$

where $g_1^{\#}$ is the game that is obtained from g_1^* by deleting all actions $x_1 \in X_1 \backslash \bar{x}_1$. Note that for all $x_i \in X_i$, $i \in \{2, \ldots, k\}$, and $x_{p(S)} \in X_{p(S)}$

$$u^1_{p(S)}(x_{p(S)}, \bar{x}_1, (x_i)_{i \in \{2,\ldots,k\}}) = u_{p(S)}(x_{p(S)}, \bar{x}_1, (x_i)_{i \in \{2,\ldots,k\}}).$$

We proceed by induction. Let $2 \leq j \leq k$ and suppose that we have deleted all actions $x_i \in X_i \backslash \bar{x}_i$ for all players $i = 1, 2, \ldots, j-1$ and defined the corresponding games g_i^* and $g_i^{\#}$ with payoff functions $u^i_{p(S)}$ for player $p(S)$ such that each player $i = 1, 2, \ldots, j-1$ has only action $\bar{x}_i$ in the games $g_i^{\#}, g_{i+1}^{\#}, \ldots, g_{j-1}^{\#}$, $u_{p(S)}^{j-1}(x_{p(S)}, (\bar{x}_i)_{i\in\{1,\ldots,j-1\}}, (x_i)_{i\in\{j,\ldots,k\}}) = u_{p(S)}(x_{p(S)}, (\bar{x}_i)_{i\in\{1,\ldots,j-1\}}, (x_i)_{i\in\{j,\ldots,k\}})$ for all $x_i \in X_i$, $i \in \{j, \ldots, k\}$, and $x_{p(S)} \in X_{p(S)}$, and $\overline{\mu}(g(S))(p(S)) \leq \overline{\mu}(g_{j-1}^*)(p(S)) = \overline{\mu}(g_{j-1}^{\#})(p(S))$. To obtain the game g_j^* from the game $g_{j-1}^{\#}$, we change the payoffs of player $p(S)$ to

$$u_{p(S)}^{j}(x_{p(S)}, (\bar{x}_i)_{i\in\{1,\ldots,j-1\}}, x_j, (x_i)_{i\in\{j+1,\ldots,k\}}) =$$
$$\max\{u_{p(S)}^{j-1}(x_{p(S)}, (\bar{x}_i)_{i\in\{1,\ldots,j-1\}}, x_j, (x_i)_{i\in\{j+1,\ldots,k\}}),$$
$$u_{p(S)}^{j-1}(x_{p(S)}, (\bar{x}_i)_{i\in\{1,\ldots,j-1\}}, \bar{x}_j, (x_i)_{i\in\{j+1,\ldots,k\}})\},$$

where $x_i \in X_i$ for all $i \in \{j, \ldots, k\}$ and $x_{p(S)} \in X_{p(S)}$. Because μ satisfies *monotonicity*, we know that

$$\overline{\mu}(g_{j-1}^{\#})(p(S)) \leq \overline{\mu}(g_j^*)(p(S)).$$

In the game g_j^*, every action $x_j \in X_j$, $x_j \neq \bar{x}_j$, is a strongly dominated threat to player $p(S)$. Because μ satisfies *irrelevance of strongly dominated threats*, we can eliminate all these strongly dominated threats to player $p(S)$ from g_j^* without changing the worth of $p(S)$ in the image of the game under μ. Hence,

$$\overline{\mu}(g_j^*)(p(S)) = \overline{\mu}(g_j^{\#})(p(S)),$$

where $g_j^{\#}$ is the game that is obtained from g_j^* by deleting all actions $x_j \in X_j \backslash \bar{x}_j$. Also, for all $x_i \in X_i$, $i \in \{j+1, \ldots, k\}$, and $x_{p(S)} \in X_{p(S)}$

$$u_{p(S)}^{j}(x_{p(S)}, (\bar{x}_i)_{i\in\{1,\ldots,j\}}, (x_i)_{i\in\{j+1,\ldots,k\}})$$
$$= u_{p(S)}^{j-1}(x_{p(S)}, (\bar{x}_i)_{i\in\{1,\ldots,j\}}, (x_i)_{i\in\{j+1,\ldots,k\}})$$
$$= u_{p(S)}(x_{p(S)}, (\bar{x}_i)_{i\in\{1,\ldots,j\}}, (x_i)_{i\in\{j+1,\ldots,k\}}).$$

The game g_4 is the game $g_k^{\#}$ that emerges from the procedure described above. In this game, every player $i \in N\backslash S$ has only one action, namely $\bar{x}_i$. The payoff function of player $p(S)$ in the game g_4 is $u_{p(S)}$ because

$$u_{p(S)}^{k}(x_{p(S)}, \bar{x}_{N\backslash S}) = u_{p(S)}(x_{p(S)}, \bar{x}_{N\backslash S})$$

for all $x_{p(S)} \in X_{p(S)}$. Also,

$$\overline{\mu}(g(S))(p(S)) \leq \overline{\mu}(g_4)(p(S)).$$

Now, note that

$$\overline{V}(A_S) = \min_{x_{N \setminus S} \in X_{N \setminus S}} \max_{x_{p(S)} \in X_{p(S)}} u_{p(S)}(x_{p(S)}, x_{N \setminus S})$$

is obtained at $(\bar{x}_{p(S)}, \bar{x}_{N \setminus S})$, so that $\max_{x_{p(S)} \in X_{p(S)}} u_{p(S)}(x_{p(S)}, \bar{x}_{N \setminus S}) = \overline{V}(A_S) = u_{p(S)}(\bar{x}_{p(S)}, \bar{x}_{N \setminus S})$. This means that every action $x_{p(S)} \in X_{p(S)} \setminus \bar{x}_{p(S)}$ is strongly dominated by action $\bar{x}_{p(S)}$ in the game g_4. Because μ satisfies *irrelevance of strongly dominated actions*, we can eliminate all these strongly dominated actions of player $p(S)$. Hence,

$$\overline{\mu}(g_4)(p(S)) = \overline{\mu}(g_5)(p(S)),$$

where g_5 is the game that is obtained from g_4 by deleting all actions $x_{p(S)} \in X_{p(S)}$ except action $\bar{x}_{p(S)}$.

In the game g_5 every player j has exactly one action, $\bar{x}_j$. Hence, for this game we can use *individual objectivity* of μ to derive that

$$\overline{\mu}(g_5)(p(S)) = u_{p(S)}(\bar{x}).$$

We now have $\overline{\mu}(g)(S) = \overline{\mu}(g(S))(p(S)) \leq \overline{\mu}(g_4)(p(S)) = \overline{\mu}(g_5)(p(S)) = u_{p(S)}(\bar{x}) = \overline{V}(A_S)$.

This ends the proof. □

As we did for the lowper value function in Theorem 6, we can make the axiomatization of the lowper value method tight by adding two axioms that are stronger versions of irrelevance of strongly dominated actions and irrelevance of strongly dominated threats for the upper and lower bounds, respectively.

Irrelevance of weakly dominated actions in the upper bound. In a game $g \in \Gamma^N$, an action $x_i \in X_i$ of player i is weakly dominated if for every $x_{N\setminus i} \in X_{N\setminus i}$ there exists an action $x'_i \in X_i,\ x'_i \neq x_i$, such that $u_i(x'_i, x_{N\setminus i}) \geq u_i(x_i, x_{N\setminus i})$. For any $g \in \Gamma^N$ and player $i \in N$, if action $x_i \in X_i$ is weakly dominated then $\overline{\mu}(g)(i) = \overline{\mu}(g')(i)$, where $g' \in \Gamma^N$ is the game obtained from g by deleting action x_i.

Irrelevance of weakly dominated threats in the lower bound. In a game $g \in \Gamma^N$, an action $x_j \in X_j$ of player j is a weakly dominated threat to player $i \neq j$ if for every $x_{N\setminus j} \in X_{N\setminus j}$ there exists an action $x'_j \in X_j,\ x'_j \neq x_j$, such that $u_i(x'_j, x_{N\setminus j}) \leq u_i(x_j, x_{N\setminus j})$. For any $g \in \Gamma^N$ and players $i, j \in N,\ i \neq j$, if action $x_j \in X_j$ is a weakly dominated threat to player i, then $\underline{\mu}(g)(i) = \underline{\mu}(g')(i)$, where $g' \in \Gamma^N$ is the game obtained from g by deleting action x_j.

Theorem 8 *The lowper value method $\mu_{\underline{\overline{V}}}$ is the unique method that satisfies individual objectivity, monotonicity, irrelevance of strongly dominated actions, irrelevance of strongly dominated threats, irrelevance of weakly dominated actions in the upper bound, irrelevance of weakly dominated threats in the lower bound, and merge invariance.*

Proof. *Existence.* In light of Theorem 7, we only need to show that $\mu_{\underline{\overline{V}}}$ satisfies *irrelevance of weakly dominated actions in the upper bound* and *irrelevance of weakly dominated threats in the lower bound.* To see that $\mu_{\underline{\overline{V}}}$ satisfies *irrelevance of weakly dominated actions in the upper bound*, note that if action x_i of player i is weakly dominated in the game $g \in \Gamma^N$, then it corresponds to a weakly dominated row in the matrix A_i of the game g_i. Hence, by strong row dominance in the upper bound of the lowper value, it holds that $\overline{\mu}_{\underline{\overline{V}}}(g)(i) = \overline{\mu}_{\underline{\overline{V}}}(g')(i)$, where $g' \in \Gamma^N$ is the game that is obtained from g by deleting action x_i.

To see that $\mu_{\underline{\overline{V}}}$ satisfies *irrelevance of weakly dominated threats in the lower bound*, note that if action $x_j \in X_j$ of a player j is a weakly dominated threat to player $i \neq j$ in the game $g \in \Gamma^N$, then for all $x_{N\setminus i,j} \in X_{N\setminus i,j}$ action $(x_j, x_{N\setminus i,j})$ corresponds to a weakly dominated column in the matrix A_i of the game g_i. Hence, applying strong column dominance in the lower bound of the lowper value repeatedly, elimination of the columns corresponding to the actions $(x_j, x_{N\setminus i,j})$ will not affect the lower bound of the interval $\mu_{\overline{\underline{V}}}(g)(i)$. The matrix that is left after eliminating all these columns is that corresponding to the game $g' \in \Gamma^N$ that is obtained from g by deleting action x_j. For this game we thus have $\underline{\mu_{\overline{\underline{V}}}}(g)(i) = \underline{\mu_{\overline{\underline{V}}}}(g')(i)$.

Uniqueness. Let $\mu : \Gamma \to IG$ be a method satisfying the seven axioms mentioned in the statement of the theorem. Let $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}) \in \Gamma^N$ and fix a non-empty coalition $S \subset N$. It readily follows from Theorem 7 that $\mu(g)(S) \subset \mu_{\overline{\underline{V}}}(g)(S)$, so the proof of uniqueness will be completed if we prove that $\underline{\mu}(g)(S) \leq \underline{\mu_{\overline{\underline{V}}}}(g)(S)$ and $\overline{\mu}(g)(S) \geq \overline{\mu}_{\overline{\underline{V}}}(g)(S)$.

If $S = N$, then *merge invariance*, *irrelevance of strongly dominated actions*, and *individual objectivity* clearly imply that $\mu(g)(N) = \mu_{\overline{\underline{V}}}(g)(N)$.

Assume now that $S \neq N$. Consider the game $g(S) = (N(S), \{X_i\}_{i\in N(S)}, \{u_i\}_{i\in N(S)})$ that is obtained from g by considering the coalition S as a single player $p(S)$. Because μ satisfies *merge invariance*, we know that

$$\mu(g)(S) = \mu(g(S))(p(S)).$$

As remarked before, the matrix $A_{p(S)}$ of the strategic game $g(S)_{p(S)}$ derived from $g(S)$, and the matrix A_S of the strategic game g_S derived from g are the same. Obviously, $\underline{V}(A_{p(S)}) = \underline{V}(A_S)$ and $\overline{V}(A_{p(S)}) = \overline{V}(A_{p(S)})$. We will show that $\underline{\mu}(g)(S) \leq \underline{V}(A_S)$ and $\overline{\mu}(g)(S) \geq \overline{V}(A_S)$.

Part I. $\underline{\mu}(g)(S) \leq \underline{V}(A_S)$. This follows straightforwardly from Theorem 7 in Carpente et al. (2005).

Part II. $\overline{\mu}(g)(S) \geq \overline{V}(A_S)$.

Let $\bar{x} = (\bar{x}_i)_{i\in N} \in \prod_{i\in N} X_i$ be an action such that the upper value of A_S is obtained in the row corresponding to action $\bar{x}_S$ for coalition S and the column corresponding to action $\bar{x}_{N\setminus S}$ for coalition $N \setminus S$. Define $\bar{x}_{p(S)} = \bar{x}_S \in X_{p(S)}$ to be the corresponding action of player $p(S)$.

We define a new game g_1 by adding an action $x^*_{p(S)}$ for player $p(S)$ in the game $g(S)$. The payoff to player $p(S)$ is as in the game $g(S)$ when he plays an action $x_{p(S)} \in X_{p(S)}$ and it is equal to $\overline{V}(A_S)$ when he plays action $x^*_{p(S)}$. Since

$$\overline{V}(A_S) = \min_{x_{N\setminus S} \in X_{N\setminus S}} \max_{x_{p(S)} \in X_{p(S)}} u_{p(S)}(x_{p(S)}, x_{N\setminus S}),$$

it holds that $\max_{x_{p(S)} \in X_{p(S)}} u_{p(S)}(x_{p(S)}, x_{N\setminus S}) \geq \overline{V}(A_S) = u_{p(S)}(x^*_{p(S)}, x_{N\setminus S})$ for all $x_{N\setminus S} \in X_{N\setminus S}$. This means that action $x^*_{p(S)}$ is a weakly dominated action in the game g_1. Because μ satisfies *irrelevance of weakly dominated actions in the upper bound*, we can eliminate this weakly dominated action from g_1 without changing the upper bound of the interval $\mu(g_1)(p(S))$. Hence,

$$\overline{\mu}(g(S))(p(S)) = \overline{\mu}(g_1)(p(S)).$$

Let g_2 be the game that is obtained from the game g_1 by bounding the payoff of player $p(S)$ from above by $\overline{V}(A_S)$, i.e., the payoff function of player $p(S)$ is now

$$u'_{p(S)}(x_{p(S)}, x_{N\setminus S}) = \min\{u_{p(S)}(x_{p(S)}, x_{N\setminus S}), \overline{V}(A_S)\}$$

for all $x_{N\setminus S} \in X_{N\setminus S}$ and all $x_{p(S)} \in X_{p(S)} \cup \{x^*_{p(S)}\}$.

Because μ satisfies *monotonicity*, we know that

$$\overline{\mu}(g_1)(p(S)) \geq \overline{\mu}(g_2)(p(S)).$$

Obviously, in the game g_2 all actions $x_{p(S)} \in X_{p(S)}$ are strongly dominated by action $x^*_{p(S)}$. Because μ satisfies *irrelevance of strongly dominated actions*, we can eliminate all these strongly dominated actions without changing the worth of player $p(S)$ in the image of the game under μ. Hence,

$$\overline{\mu}(g_2)(p(S)) = \overline{\mu}(g_3)(p(S)),$$

where g_3 is the game that is obtained from g_2 by deleting all actions of player $p(S)$ except action $x^*_{p(S)}$.

In the game g_3, player $p(S)$ has only one action and the payoff to this player is $u_{p(S)}(x^*_{p(S)}, x_{N\setminus S}) = \overline{V}(A_S)$ for all $x_{N\setminus S} \in X_{N\setminus S}$. Hence, for every player $j \neq p(S)$ every action $x_j \in X_j$ is a strongly dominated threat to player $p(S)$ and all but one can be eliminated by *irrelevance of strongly dominated threats*. Therefore,

$$\overline{\mu}(g_3)(p(S)) = \overline{\mu}(g_4)(p(S)),$$

where g_4 is the game that is obtained from g_3 by deleting for every player $j \in N \backslash S$ all actions except one, an arbitrarily chosen $x_j^* \in X_j$.

In the game g_4 every player $i \in N(S)$ has exactly one action, x_i^*, and hence, for this game we can use *individual objectivity* of μ to derive that

$$\overline{\mu}(g_4)(p(S)) = u_{p(S)}(x^*) = \overline{V}(A_S).$$

We proved that $\overline{\mu}(g)(S) = \overline{\mu}(g(S))(p(S)) = \overline{\mu}(g_1)(p(S)) \geq \overline{\mu}(g_2)(p(S)) = \overline{\mu}(g_3)(p(S)) = \overline{\mu}(g_4)(p(S)) = \overline{V}(A_S)$.

This ends the proof. □

6 Concluding remarks

Carpente et al. (2004) define a valuation function that associates with every non-empty coalition of players in a strategic game a vector of payoffs for the members of the coalition. This vector of payoffs provides these players' valuations of cooperating in the coalition. The valuation function defined in Carpente et al. (2004) is based on the lower-value method of associating coalitional games with strategic games and the Shapley value for coalitional games. An axiomatic characterization of this so-called Shapley valuation was obtained.

We can extend the results in Carpente et al. (2004) to the setting of interval games as follows. Define a *valuation correspondence* as a map φ that associates with every game $g \in \Gamma^N$ and non-empty coalition $S \subset N$ a vector of closed intervals $\varphi(S,g) = (\varphi_i(S,g))_{i \in S}$, where $\varphi_i(S,g) = [\underline{\varphi}_i(S,g), \overline{\varphi}_i(S,g)] \subset \mathbb{R}$ provides an interval valuation for player i of cooperating in coalition S in game g, for each $i \in S$. The *Shapley valuation* *correspondence* $\overline{\underline{\Phi}}$ is given by $\overline{\underline{\Phi}}(S,g) = ([\Phi_i(S,\underline{v}_g), \Phi_i(S,\overline{v}_g)])_{i\in S}$, where $\underline{v}_g$ and $\overline{v}_g$ are the lower-value method and the upper-value method, respectively, and Φ denotes the Shapley value (cf. Shapley (1953)). The axiomatization of the lowper value method that we provided in Theorem 8 in the current paper can be used to find an axiomatic characterization of the Shapley valuation correspondence $\overline{\underline{\Phi}}$. The procedure followed to obtain this characterization is similar in flavor to that followed in Carpente et al. (2004). It involves adapting the properties in the current paper to the setting of valuation correspondences and introducing an additional property of balanced contributions. A valuation correspondence φ satisfies *balanced contributions* if for all $g \in \Gamma^N$ and non-empty $S \subset N$, and all $i,j \in S$, it holds that $\varphi_i(S,g) - \varphi_i(S\backslash\{j\},g) = \varphi_j(S,g) - \varphi_j(S\backslash\{i\},g)$.

Theorem 9 *The Shapley valuation correspondence* $\overline{\underline{\Phi}}$ *is the unique valuation correspondence satisfying individual objectivity, monotonicity, irrelevance of strongly dominated actions, irrelevance of strongly dominated threats, irrelevance of weakly dominated actions in the upper bound, irrelevance of weakly dominated threats in the lower bound, merge invariance, and balanced contributions.*

We leave the details of this theorem to the reader. They are pretty straightforward using the results in the current paper and the definitions and line of proof in Carpente et al. (2004) for inspiration.

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