THE SYNTHETIC PROJECTIVE
TREATMENT OF CONICS

A THESIS

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CHAPTER I

INTRODUCTION

The history of geometry may be divided roughly into four periods. The first period consists of the synthetic geometry of the Greeks, including not merely the geometry of Euclid but the work on conics by Appolonius, and the less formal contributions of many other writers. The birth of analytic geometry characterized the second period. In this period the synthetic geometry of Desargues, Kepler, Roberval, and other writers of the seventeenth century merged into the coordinate geometry of Descartes and Fermat. Calculus was applied to geometry in the third period. This period extended from about 1650 to 1800, and included the names of Cavalieri, Newton, Leibnitz, the Bernoullis, L'Hopital, Clairaut, Euler, Lagrange, and D'Alembert. The renaissance of pure geometry was in the fourth period. This period began with the nineteenth century and was characterized by the descriptive geometry of Monge, the projective geometry of Poncelet, the modern synthetic geometry of Steiner and Von Staudt, the modern analytic geometry of Plucker, the non-Euclidean hypotheses of Lobachevsky, Bolyai, and Riemann, and the laying of the logical foundations of geometry. It was a period of remarkable richness in the development of all phases of the science.

It is in this fourth period that projective geometry has had its development, even if its origin is more remote. The origin of any branch of science can always be traced far back in human history, and this fact is patent in the case of this phase of geometry.

Modern synthetic geometry was created by several investigators about the same time. It seemed to be the outgrowth of a desire for general methods which should serve as guides for students in learning theorems, corollaries, and problems. Synthetic geometry was first cultivated by Monge, Carnot, and Poncelet in France. It then bore rich fruits at the hands of Mobius and Steiner in Germany and Switzerland. Finally, it was developed to still higher perfection by Chasles in France, Von Staudt in Germany and Cremona in Italy. The recent contributions have naturally been of an advanced character, seeking to lay more strictly logical foundations for the science. In this line the American work by Professors Veblen and Young is noteworthy.

It is quite impossible to draw the line in the historical development between analytic geometry and synthetic geometry. According to Klein's definition of geometry, it is the study of the invariants of a configuration under a group of transformations. Analytic geometry uses a coordinate system in the study of the invariants. Synthetic geometry does not use a coordinate system. Metric geometry is the study of the invariants of a configuration under a group of motions. In metric geometry a figure can be moved from place to place without altering its shape or size. Projective geometry is the study of the invariants of a configuration under a group of projections. In projective geometry a figure can be projected from one place to another without altering it. Thus we have these four branches: synthetic projective geometry, synthetic metric geometry, analytic projective geometry and analytic metric geometry.

The purpose of this thesis is to investigate the properties of conics from a synthetic and projective standpoint. In this treatment special use has been made of Pascal's Theorem and its dual and the Principle of Duality.

I wish to express my very great appreciation to Professor C. B. Dansby for his sympathetic interest and aid during the preparation of this thesis.

CHAPTER II

POINT AND LINE CONICS

1. Definitions

A point conic is the locus of the points of intersection of corresponding lines of two projective, coplanar, non-concentric flat pencils.

The order of a point locus in a plane is the greatest number of its points less than an infinite number that can be on one straight line.

A line conic is the envelope of the lines joining corresponding points of two projective, coplanar, non-coaxial point ranges.

The class of a line envelope in a plane is the greatest number of its tangents less than an infinite number that can pass through one point.

2. Theorems on Determining Conics

Theorem I.- The point conic passes through the centers of two generating pencils.

Proof: The line $S\,S'$ of the pencil S corresponds to the tangent of the conic at S'. The tangent is regarded as a line of S'. The line $S'\,S$ of S' corresponds to the tangent of the conic at S. Hence the conic passes through S and S'.

Theorem I'.- The line conic is tangent to the bases of two generating ranges.

Proof: The point of intersection of the point ranges S and S' corresponds to the point of contact of the conic with S'. When the point of contact is on S the corresponding point on S' is any point on S', which is not one of the given points.

Theorem II.- Any two points on a conic may be used as centers of two projective pencils which generate the conic.

Proof: Consider two pencils S(a b c) and S'(a' b' c') with a b c ..... $\overline{\wedge}$ a' b' c' ..... These pencils generate a conic S S' A B C. Draw any other line d from S. Draw its corresponding line d' from S'. A, B, C, D are any four points on the conic. Draw A B and A C. A B meets S C in a point P. A B meets S D in a point L. S' B meets A C in a point Q. S' D meets A C in a point N. S' D meets S C in a point G. S D meets S' B in a point F.

S(a b c d) $\overline{\overline{\wedge}}$ (A B P L) and S'(a' b' c' d') $\overline{\overline{\wedge}}$ (A Q C N). Therefore point ranges A B P L and A Q C N are projective. Since the two point ranges have a self-corresponding point, they are perspective. If two projective point ranges have a self-corresponding point, then the point ranges are perspective. Therefore L N passes through the point of intersection M of the lines.

Consider two pencils with centers at B and C, any two points on the conic. B(A S S'D) $\overline{\overline{\wedge}}$ (L S F D) $\overline{\overline{\wedge}}$ (N G S' D) $\overline{\overline{\wedge}}$ C(A S S' D). Hence the pencils B(A S S' D) and C(A S S' D) are projective, coplanar, and non-concentric. Therefore the pencils generate a conic.

Theorem II'.- Any two tangents of a conic can be used as bases of projective ranges which generate the conic.

Proof: Consider two point ranges s and s' with A B C $\overline{\wedge}$ A' B' C'. These point ranges generate a conic. Select any other point D on s, and its corresponding point D' on s'. Any four lines on the conic are a, b, c, d. Join the points of intersection of a b and a c by a line. Line p joins a b and s c. The points a b and s d are joined by line, l. The points s' b and a c are joined by line, q. Line n joins s' d and a c. The points s' d and s c are joined by line, g. Line f joins the points s d and s' b.

(A B C D) $\overset{=}{\Lambda}$ B(a b p l) and (A' B' C' D') $\overset{=}{\Lambda}$ C(a q c n). Therefore B(a b p l) and C(a q c n) are projective pencils with a self-corresponding line. Hence the pencils are perspective. If two projective pencils have a self-corresponding line, then the pencils are perspective.

The line, m, joins the points b q and p c and passes through l n. Consider two point ranges with bases b and c, any two lines on the conic. Point range b(a s s' d) $\overline{\Lambda}$ pencil (l s f d) $\overline{\Lambda}$ pencil (n g s' d) $\overline{\Lambda}$ point range c(a s s' d).

Therefore the point ranges b(a s s' d) and c(a s s' d) are projective, coplanar and non-coaxial. Hence they generate a conic.

Theorem III.- The conic is determined by five of its points.

Proof: Use any two of the five points as centers of pencils and the other three points to determine a projectivity between the two pencils. For, two projective, coplanar, non-concentric flat pencils generate a conic. Hence five points determine a conic.

Theorem III'.- The conic is determined by five of its tangents.

Proof: Use any two tangents as bases of ranges and the other three to determine the projectivity between the two rays. For, two projective, coplanar, non-coaxial point ranges generate a conic. Therefore five tangents determine a conic.

3. Pascal's Theorem

The opposite sides of a simple hexagon inscribed in a conic intersect in three collinear points.

Proof: Consider the simple hexagon with vertices 1, 2, 3, 4, 5, 6, on the conic with the three pairs of opposite sides 1 2, 4 5 meeting at L; 2 3, 5 6 meeting at M and 3 4, 6 1 meeting at N. Prove that L, M and N are collinear.

Regard the conic generated by pencils whose centers are S and S'.

S(A B C D) $\overline{\overline{\wedge}}$ (A B P L) and S'(A B C D) $\overline{\overline{\wedge}}$ (A Q C N). But S(A B C D) $\overline{\wedge}$ S'(A B C D). For, by hypothesis the pencils generate a conic.

Cross ratio of S(A B C D) equals cross ratio of S'(A B C D). This is true because cross ratio of lines joining any four fixed points on a conic to a variable fifth point is constant.

Therefore (A B P L) = (A Q C N). For the theorem of Pappus states that cross ratio is invariant under projection and section.

Hence the point ranges A B P L and A Q C N are projective. Because any one to one correspondence that preserves cross ratio is a projectivity.

(A B P L) $\overline{\overline{\wedge}}$ (A Q C N). Two projective point ranges having a self-corresponding point are perspective. That is to say B Q, P C and L N pass through the same point or they are concurrent. But B Q and P C meet at M. Therefore L N passes through M. Hence L, M and N are collinear.

Converse of Pascal's Theorem.- If the opposite sides of a simple hexagon intersect in collinear points, the hexagon can be inscribed in the conic.

Proof: Consider a simple hexagon A B S' D S C, with the opposite sides intersecting at L, M and N. Consider pencils S(B C D) and S'(B C D) and a conic generated by them. Therefore the conic passes through five points.

Prove that the conic passes through the point A. S(A B C D) $\overline{\wedge}$ (A B P L). But (A B P L) $\overline{\overline{\wedge}}$ (A Q C N). M is the center of perspectivity. Therefore A B P L $\overline{\wedge}$ A Q C N $\overline{\wedge}$ S'(A B C D). Hence S(A B C D) $\overline{\wedge}$ S'(A B C D). But S(B C D) $\overline{\wedge}$ S'(B C D). For, they generate a conic. Therefore S A and S' A meet on the conic.

4. Brianchon's Theorem (Dual of Pascal's Theorem)

The lines joining opposite vertices of a simple hexagon circumscribed about a conic are concurrent.

Proof: Consider the simple hexagon circumscribed about a conic. The sides are tangent to the conic. Join the points of intersection of 2 3, 5 6 by line m. Join the points of intersection of 3 4, 6 1 by line n. Join the points of intersection of 1 2, 4 5 by line l. Prove that m, n and l are concurrent. Then join the points of intersection of 1 2, 5 6 by line p and 2 3, 6 1 by line q.

Regard the conic generated by point ranges s and s'. s(a b c d) $\overset{=}{\Lambda}$ (a b p l) and s'(a b c d) $\overset{=}{\Lambda}$ (a q c n). Since s(a b c d) and s'(a b c d) generate the conic, then s(a b c d) $\overline{\Lambda}$ s'(a b c d). Therefore (a b p l) $\overline{\Lambda}$ (a q c n). Then (a b p l) $\overset{=}{\Lambda}$ (a q c n). For, two projective pencils having a self-corresponding line are perspective. Hence b q, p c and l n are the points on a straight line (axis of perspectivity). But b q and p c are joined by line m (axis of perspectivity). Therefore l n is on m. Hence l, m and n are concurrent. The point is the Brianchon point.

Converse of Brianchon's Theorem.- If the lines joining opposite vertices of a simple hexagon are concurrent the hexagon can be circumscribed about a conic.

Proof: Consider the simple hexagon, with the lines joining opposite vertices being concurrent. Consider ranges s(b c d) and s'(b c d) and a conic generated by them. Therefore the conic is tangent to five lines.

Prove that the conic is tangent to line a. s(a b c d) $\overline{\Lambda}$ s'(a b p l). But s'(a b p l) $\overset{=}{\Lambda}$ (a q c n). Hence s'(a b p l) $\overline{\Lambda}$ (a q c n) $\overline{\Lambda}$ s'(a b c d). Therefore s(a b c d) $\overline{\Lambda}$ s'(a b c d). But s(b c d) $\overline{\Lambda}$ s'(b c d). For, they generate a conic. Therefore s a and s' a are joined by a line which is tangent to the conic, i.e. line a is tangent to the conic.

5. Limiting Forms of Pascal's Theorem and Its Dual

Inscribed Pentagon Theorem.- If a simple pentagon is inscribed in a conic, then two pairs of non-adjacent sides intersect in points collinear with the intersection of the remaining side and the tangent at the opposite vertex.

Proof: Let $A_1 \longrightarrow C_6$ $(A_1 \equiv C_6)$, then $l_{16} \longrightarrow$ tangent at $A_1 \equiv C_6$. Given the simple pentagon A_1, B_2, S'_3, D_4, S_5 inscribed in a conic. Prove that $A\ B_{12}$, $D\ S_{45}$; $B\ S'_{23}$, $S\ A_{56}$; $S'\ D_{34}$, $C\ A_{61}$ meet in collinear points L, M, N respectively. The lines 1 2, 4 5 meet at L; 2 3, 5 6 meet at M; and 3 4, 6 1 meet at N. The opposite sides of a simple hexagon inscribed in a conic intersect in three collinear points.

Circumscribed Pentagon Theorem.- If a simple pentagon is circumscribed about a conic, then two pairs of non-adjacent vertices are joined by lines concurrent with the line joining the remaining vertex to the point of contact on the opposite side.

Proof: Given the simple pentagon with sides $a_1 \equiv c_6$, b_2, s'_3, d_4, s_5 circumscribed about a conic. To prove that $a\ b_{12}$, $d\ s_{45}$; $b\ s'_{23}$, $s\ a_{56}$; $s'\ d_{34}$, $c\ a_{61}$ are concurrent. The line, l, joins 1, 2 and 4, 5. The line m joins 2, 3 and 5, 6. The line, n, joins 3, 4 and 6, 1. This is true by Brianchon's Theorem.

Inscribed Quadrangle Theorem.- If a simple quadrangle is inscribed in a conic then the two pairs of opposite sides intersect in points collinear with the point of intersection of tangents at pairs of opposite vertices.

Proof: Consider the simple quadrangle A B S' S inscribed in the conic with A B and S S'; A S and B S' pairs of opposite sides intersecting in L and M respectively. Regard tangents at opposite vertices S' and A intersecting at N. The tangents at opposite vertices B and S intersect at P. Draw line, l. Prove L, M, N and P are collinear.

First, consider the Pascal hexagon $A_1 \equiv 6$, B_2, $S'_3 \equiv 4$, S_5. L is the point of intersection of 1,2 and 4,5. M is the point of intersection of 2,3 and 5,6. N is the point of intersection of 3,4 and 6,1. Hence L,M and N are collinear. This is valid by Pascal's Theorem.

Then consider the Pascal hexagon A_2, $B_6 \equiv 1$, S'_5, $S_3 \equiv 4$. In this case P is the point of intersection of 3,4 and 6,1. L and M remain the same. Hence by Pascal's Theorem L, M and P are collinear. Therefore L, M, N and P are collinear.

Circumscribed Quadrilateral Theorem.- If a simple quadrilateral is circumscribed about a conic then the two pairs of opposite vertices are joined by lines concurrent with the lines joining the points of contact at pairs of opposite sides.

Proof: Consider the simple quadrilateral $a_1\ b_2\ s'_3\ s_5$ circumscribed about the conic with a b_{12} and $s'\ s_{35}$; a s_{15} and b s'_{23} pairs of opposite vertices joined by lines l and m respectively. Also consider the points of contact of opposite sides a_1 and s'_3 joined by line n. The points of contact of opposite sides b_2 and s_5 are joined by line p. Prove that l, m, n and p are concurrent.

First, consider the simple hexagon $a_1 \equiv c_6$, b_2, $s'_3 \equiv d_4$, s_5. The line l joins 1,2 and 4,5. The points 2,3 and 5,6 are joined by line m. The line n joins the points 3,4 and 6,1. Hence l,m and n are concurrent. This is true by Brianchon's Theorem.

Then consider the simple hexagon a_2, $b_6 \equiv 1$, s'_5, $s_3 \equiv 4$. In this hexagon line p joins the points 3,4 and 6,1. The lines l and m join 1,2 and 4,5; 2,3 and 5,6 respectively. Hence l, m, and p are concurrent lines. Therefore l, m, n and p are concurrent.

Inscribed Triangle Theorem.- If a triangle is inscribed in a conic, the tangents at the three vertices meet the opposite sides of the triangle in three collinear points.

Proof: Let vertices 1 and 2, 3 and 4, 5 and 6 coincide. Use Pascal's construction. The tangents are 1 and 2, 3 and 4, 5 and 6. The sides are 4 and 5, 2 and 3, 6 and 1. The lines 1,2 and 4,5 meet at L. The point M is the point of intersection of 2,3 and 5,6. The lines 3,4 and 6,1 meet at N. Hence L, M and N are collinear by Pascal's Theorem.

Circumscribed Triangle Theorem.- If a triangle is circumscribed about a conic, the points of contact on the three sides are joined to the opposite vertices of the triangle by three concurrent lines.

Proof: Let sides 1 and 2, 3 and 4, 5 and 6 coincide. The line l joins 1,2 and 4,5. The line m joins 2,3 and 5,6. Line n joins 3,4 and 6,1. Hence the lines joining each vertex to the contact point of the opposite side are concurrent. This is true by Brianchon's Theorem.

Theorem on Composite Conic.- If the odd numbered vertices of a simple hexagon are on one straight line and the even numbered are on another, then the opposite sides intersect in collinear points.

Proof: Consider the simple hexagon with vertices 1,2, 3, 4, 5, 6. Vertices 1, 3 and 5 are on one straight line. Vertices 2, 4 and 6 are on another straight line. Use the Pascal construction. Thus 1,2 and 4,5 meet at L; 2,3 and 5,6 meet at M; and 3,4 and 6,1 meet at N. Hence L, M and N are collinear by Pascal's Theorem.

Dual of Theorem on Composite Conic.- If the odd numbered sides of a simple hexagon pass through a point and the even numbered pass through another point, then the opposite vertices are joined by concurrent lines.

Proof: Consider the simple hexagon with sides a_1, b_2, s'_3, d_4, s_5, c_6. The line l joins 1,2 and 4,5. The points 2,3 and 5,6 are joined by line, m. The line n joins 3,4 and 6,1. Hence l, m and n are concurrent lines. This is valid by Brianchon's Theorem.

CHAPTER III

POLE AND POLAR RELATIONS

1. Definitions and Theorems

Construct the polar of a point with respect to a pair of lines. Given two lines a and b and a point P. Let A_1 B_1, A_2 B_2, A_3 B_3 ..... be points of a and b respectively collinear with P. Then A_1 B_2, A_2 B_1; A_2 B_3, A_3 B_2; A_1 B_3, A_3 B_1 intersect in points on a line p.

The line p is the polar line of P with respect to a and b.

Proof: The shaded triangles are perspective from a line since their corresponding sides intersect on a line. Therefore the triangles are perspective from a point by Desargues' Theorem. Two of the vertices of the triangle are joined by lines which go through a b. Hence p, which joins the third vertices, goes through a b.

Construct the pole of a line with respect to a pair of points. Given two points A and B and a line p. Let a_1, b_1; a_2, b_2; a_3, b_3 ..... be lines of A and B respectively concurrent with p. Then a_1 b_2, a_2 b_1; a_2 b_3, a_3 b_2; a_1 b_3, a_3 b_1 are joined by lines meeting in a point P.

The point P is the pole of line p with respect to A and B.

The proof is true by the dual of the proof of the polar of a point with respect to a pair of lines.

Construct the polar of a point with respect to a triangle. Given triangle A B C and point P. Let P A, P B, P C meet B C, C A, A B in A', B', C' respectively. Let A B and A' B' intersect in H. Let B C and B' C' intersect in K. Let C A and C' A' intersect in L. Then H, K and L lie on a line p.

The line p is the polar line of P with respect to triangle A B C.

Proof: Triangles A B C and A' B' C' are perspective from P. Hence they are perspective from a line. This is true by Desargues' Theorem. That is to say A B and A' B', B C and B' C', C A and C' A' meet in H, K and L on a line p.

Construct the pole of a line with respect to a triangle. Given triangle A B C and line p. Let p a, p b, p c be joined to b a, c a, a b by a', b', c' respectively. Let a b and a' b' be joined by h. Let b c and b' c' be joined by k. Let c a and c' a' be joined by l. Then h, k and l meet at a point P.

The point P is the pole of line p with respect to triangle A B C.

Proof: Triangles A B C and A' B' C' are perspective from p. Hence by Desargues' Theorem they are perspective from a point. That is to say a b and a' b', b c and b' c', c a and c' a' are joined by lines h, k and l which meet at P.

Construct the polar of a point with respect to a conic. Given a point P inside, outside or on a conic. Draw through P any two lines meeting the conic in the points K, M and N, L. Draw K N and L M intersecting at Q. Draw K L and M N intersecting at O.

The line O Q is the polar line of P with respect to the conic.

The inside of a conic is that portion of the plane from which no real tangent can be drawn to the conic.

The outside of a conic is the portion of the plane from which two real tangents can be drawn to the conic.

Comments: If the point P is inside the conic the polar line lies entirely outside the conic.

If the point P is outside the conic the polar line cuts across the conic.

If the point P is on the conic, the polar line is tangent to the conic.

Construct the pole of a line with respect to a conic. Given a line p tangent to the conic, cutting the conic, or entirely outside the conic. Let P_1 and P_2 be two points on p. Draw tangents to the conic from these two points. k and m are the tangents from P_1. Tangents from P_2 are n and l. Join k n and l m by line q. Join k l and m n by line o.

The point of intersection of o and q is the pole of line p with respect to the conic. Call this point P.

Theorem I.- If a complete quadrangle is inscribed in a conic, each side of the diagonal triangle is the polar line of the opposite vertex.

Proof: In the diagonal triangle O P Q by definition O Q is the polar line of P, O P is the polar line of Q and P Q is the polar line of O.

Theorem I'.- If a complete quadrilateral is circumscribed about a conic, each vertex of the diagonal triangle is the pole of the opposite side.

Proof: In the diagonal triangle o p q by definition o q is the pole of p, o p is the pole of q and p q is the pole of o.

A triangle is a self polar triangle with respect to a conic, if each side is a polar line of the opposite vertex.

Theorem II.- The diagonal triangle of a complete quadrangle inscribed in a conic is self polar with respect to the conic.

This theorem is proved by the definition of a self polar triangle and Theorem I.

Theorem III.- The polar line of a given point is unique. That is to say the polar line O Q of a point P is independent of the points K, M, N and L used to construct it.

Proof: Let N and L be fixed. Let K and M vary. Prove that

O Q does not change. A B Q O are four harmonic points by definition. K M and O Q meet at A. L N and O Q meet at B.

Pencil M (A B Q O) are harmonic lines, that is by projecting A B Q O from M. Hence P B L N are harmonic points. But P, N and L are fixed. Therefore B must be fixed.

Consider the simple quadrangle K L M N. The lines L M and K N; L K and M N; l and n; m and k intersect in collinear points by the inscribed quadrangle theorem.

Tangents through L and N meet at E. The point E is on O B Q and is fixed because L and N are fixed. Therefore the line has two fixed points E and B. Hence O Q is fixed as K and M vary. Therefore O Q is independent of the points K, M, N and L.

Theorem III'.- The pole of a given line is unique. That is to say the pole P of a line p is independent of the tangents k, m, n and l used to construct it.

Proof: Let tangents n and l be fixed. Let tangents k and m vary. Prove that P does not change. a b q o are four harmonic lines. The point k m and P are joined by a. The point l n and P are joined by b.

Point range m (a b q o) are harmonic points. Hence (p b l n) are harmonic lines. But p, n and l are fixed. Therefore b must be fixed.

Consider the simple quadrilateral k l m n. The points l m and k n; l k and m n; L and N; M and K are joined by concurrent lines by the circumscribed quadrilateral theorem.

Points of contact on l and n are joined by line e. The line e goes through the point o b q and is fixed because l and n are fixed. Therefore the point has two fixed lines e and b. Hence P is fixed as tangents k and m vary. Therefore P is independent of the lines k, m, n and l.

2. Properties

Harmonic Property.- The polar line, or part of the polar line of a point P with respect to a conic is the locus of the harmonic conjugate of P and the two points of intersection of the conic with a variable line through P.

Proof: In the figure for Theorem III, Section I, on uniqueness of a polar line, the points L, N, P and B form a harmonic set as previously proved. Hold K and M fixed. Let N and L vary and repeat the construction. Then B varies and A is fixed.

Comment: The locus is the whole polar line if point P is inside the conic.

The locus is a part of the polar line if point P is outside the conic.

The locus is part (a point) of the polar line if point P is on the conic.

Dual of Harmonic Property.- The pole of a line p with respect to a conic is the locus of the harmonic conjugate of p and the two tangents to the conic from a variable point on line p.

Proof: In the figure for Theorem III', Section I, on the uniqueness of a pole, the lines l, n, p and b form a harmonic set. Let k and m be fixed. Vary n and l, and repeat construction. Then b varies and a is fixed.

Symmetric Property.- If a point P lies on the polar line of a point Q, then Q lies on the polar line of P.

Proof: Consider a conic and a point P. Draw the polar line p of point P. On p take a point Q. Let P Q intersect the conic at R and S. (Suppose P Q meets the conic). By the harmonic property it follows that P Q R S are harmonic points. Hence Q is on the polar line of P by construction.

Consider the polar line q of point Q. Since P Q R S are harmonic points, q must pass through P.

Dual of Symmetric Property.- If a line, p, passes through the pole of a line q, then q passes through the pole of line p. This is true by the principle of duality.

Two points so situated that the polar line of each with respect to a conic passes through the other are called conjugate points with respect to the conic.

Two lines so situated that the pole of each with respect to a conic lies on the other line are called conjugate lines with respect to the conic.

Theorem IV.- Locus of all points conjugate to a given point P is the polar line of P.

The proof is true by the symmetric property.

Theorem IV'.- Locus of all lines conjugate to a given line p is the pole of p.

Projective Property.- If a point P moves along a straight line, q, the polar line p of P with respect to a conic turns about the pole, Q, and generates a pencil. The pencil is projective with the range generated by P on line q.

Proof: Consider only the complete quadrangles K L M N that have L and M fixed. Because Q and q are fixed by hypothesis. The point range P on q $\overline{\wedge}$ pencil M P. M P $\overline{\wedge}$ L K. L K $\overline{\wedge}$ point range O on q. Point range O $\overline{\wedge}$ pencil O Q. Therefore point range P on q is projective with pencil O Q. O is the point of intersection of K L and M N.

Dual of Projective Property.- If a line p forms a pencil of lines with center Q, the pole P of line p with respect to a conic moves along the polar line q, and generates a point range. The point range is projective with the pencil generated by p through Q.

Proof: Consider only the complete quadrilaterals k l m n that have l and m fixed. For q and Q are fixed by hypothesis. The pencil generated by p through Q $\overline{\wedge}$ the point range m p. Point range m p $\overline{\wedge}$ point range l k. Point range l k $\overline{\wedge}$ pencil o with center Q. Pencil o $\overline{\wedge}$ point range o q. Therefore pencil p with center Q is projective with point range o q.

CHAPTER IV

INVOLUTIONS

1. Involution on a Line

Six points on a line are in involution in case they can be grouped in pairs so that a complete quadrangle can be drawn, with a pair of opposite sides through each pair of points.

Construction: Given a line and two pairs of points A, A' and B, B' and point C to determine the sixth point C'. Draw a line through A and also one through A'. Then draw a line through C intersecting the line through A at point K, and the line through A' at M. Draw B K determining N on line A'M. Draw B'M determining L on A K. Then draw L N determining C'.

An involution is determined whenever enough is known to establish two determining triads of the projectivity. Consequently the following data are sufficient for this purpose: two pairs of corresponding elements; one self-corresponding element and a pair of corresponding elements; two self-corresponding elements.

Theorem I.- If two complete quadrangles have their sides in a one-to-one correspondence so that five pairs of corresponding sides intersect in points on a straight line, then the sixth pair also intersect in a point on the same straight line.

Given points A, A', B, B' and C on line 1. K L M N and K' L' M' N' are complete quadrangles.

Consider triangles K L M and K' L' M', which are perspective from line, 1. Therefore by Desargues' Theorem the triangles are perspective from a point. That is to say K K': L L', M M' pass through the same point. Call the point S.

Also consider triangles K M N and K' M' N', which are perspective from line, 1. Hence by Desargues' Theorem they are perspective from a point. That is to say K K', M M', N N' pass through the same point, S.

Therefore K K', L L', M M', N N' pass through the point S.

Hence the triangles L M N and L' M' N' are perspective from S. Thus by Desargues' Theorem the triangles L M N and L' M' N' are perspective from line, 1.

But by construction L M and L' M' meet at B' on 1. M N and M' N' meet at A' on 1. Therefore L N and L' N' must meet at a point C' on line 1.

Theorem II.- An involution on a line is a transformation of period two.

Proof: K L M N and K' L' M' N' are complete quadrangles. Consider an involution on a line, 1, determined by A, A', B, B'. Given a point C_1 construct the corresponding point C'_1. Then take C_2 where C'_1 is and construct the corresponding point C'_2. Prove that C'_2 is identical with C_1. True by Theorem I. Therefore an involution is of period two.

Theorem III. - An involution on a line is a projective transformation or a projectivity.

Proof: Consider an involution on a line, 1, determined by two pairs of points A, A' and B, B'. Given a point C construct C'. Regard only those complete quadrangles with L and M fixed.

Let C vary. The point range C on 1 is projective with the line pencil M C. Pencil M C is projective with the point range K on A.L. Point range K on A L is projective with the line pencil B K. Line pencil B K is projective with the point range N on A'. Point range N on A' is projective with line pencil L N. Line pencil L N is projective with the point range C'on 1.

Theorem IV.- A projective transformation of period two is an involution.

Proof: Consider a projectivity A B C..... $\overline{\wedge}$ A' B' C' ..... on a line. Suppose this projectivity is of period two, then A A', B B' ..... project into A' A, B' B .....

Consider the involution determined by the two pairs of points A A' and B B'. Hence an involution is a projective transformation of period two.

A point that corresponds to itself in an involution is a double or invariant point.

Theorem V.- If an involution on a line has two double points, they separate every pair of corresponding points harmonically.

Proof: Consider an involution on a line, l, determined by A, A', B, B' and having two double points. Prove that B B' separate the double points harmonically. Given point C construct the corresponding point C'. Denote the point of intersection of K M and L N by P. Hold A A', B B', L and M fixed. Let C vary.

Pencils M C and L C' are projective since an involution is projective. The pencils M C and L C' generate a conic. For, two projective, coplanar, non-concentric flat pencils generate a conic. The centers of the pencils are M and L.

Let F_1 and F_2 be the intersections of the conic with line, l. Prove that F_1 and F_2 are double points.

As C $\longrightarrow$ B', C' $\longrightarrow$ B, then L C' $\longrightarrow$ L B. Thus P $\longrightarrow$ L and L P $\longrightarrow$ L B. L B is tangent to the conic at L. As C $\longrightarrow$ B, C' $\longrightarrow$ B', then L C' $\longrightarrow$ L B'. Thus P $\longrightarrow$ M and M P $\longrightarrow$ M B. M B is tangent to the conic at M.

Prove that B B' separate F_1 and F_2 harmonically. Let L M be the polar line of B. If the point is outside of the conic its polar line with respect to the conic is the chord of contact of the tangent from the point to the conic.

B B' separate F_1 and F_2 harmonically. For, the polar line of a point B with respect to a conic is the locus of the harmonic conjugate of B and the intersection points of the conic with a variable line through B.

Steiner's construction for an involution on a line. Consider an involution on a line, l, determined by two pairs of points A A' and B B'.

Select any point S not on l. Draw the pencil S (A A' B B'). Draw any conic through S determining A_1 B_1 A'_1 B'_1. Draw the lines A_1 (B_1 B'_1 A'_1) and A'_1 (B'_1 B_1 A_1). The pencils with centers A_1 and A'_1 are projective and have a self-corresponding line (A_1 A'_1). Hence they are perspective. Determine the axis of perspectivity, u. Denote the intersections of u and the conic by H and K.

Given a point C on the line to construct the corresponding point C'. Draw S C determining C_1 on the conic. Draw A'_1 C_1 determining P on u. Draw A_2 P determining C'_1 on the conic. Draw S C'_1 determining C' on l. Draw A_1 C_1 and A'_1 C'_1. They should intersect on u.

2. <u>Involution on a Conic</u>

On a conic select four points A, A', B, B'. Draw the pencils A (A' B' B) and A' (A B B'). A B' and A' B determine M. A B and A' B' determine N. M and N determine the axis of perspectivity.

Given a point C on the conic to construct C'. Draw A' C determining P on u. Draw A P determining C' on the conic.

This correspondence is an involution on the conic.

The involution is determined by two pairs of points.

The correspondence is a projectivity for it is a one-to-one correspondence that preserves cross ratio.

Theorem VI.- The correspondence is of period two and is therefore an involution.

Proof: In the figure for the construction of an involution on a conic place C_1 where C' is and repeat the construction. Draw $A'\,C_1$ determining P' on u. Draw $A\,P'$ determining C'_1. Prove $C'_1 \equiv C$. $(M\,P\,P'\,N)$

$$(B\ C\ C'\ B') \underset{(A')}{\overline{\wedge}} (B'\ C'\ C'_1\ B) \underset{(A)}{\overline{\wedge}} (B_1\ C'_1\ C'\ B') \quad \therefore C \equiv C'_1$$

Theorem VII.- The lines joining corresponding points of an involution on a conic pass through a point O. The line joining any two points meets the lines joining the corresponding point in the polar line of O.

Proof: Consider a conic and on it an involution determined by two pairs of points A, A'; B, B'. Draw pencils A (A' B' B) and A' (A B B').

The points A, A', B, B' determine a complete quadrangle. Draw the axis of perspectivity, u, of the two pencils A (A' B' B) and A' (A B B'). Denote by O the intersection of A A' and B B'.

Consider any third point C and construct its corresponding point C'. B C and B' C' intersect in P on u. B C' and B' C intersect in P' on u. The axis, u, is the polar line of O.

Prove C C' passes through O. Use the symmetric property of pòle and polar relations. The polar line, u, of O passes through P by construction. Therefore the polar line of P passes through O. The polar line of P joins P' to the point when C C' intersects B B'. This point must be O. Hence C C' passes through O.

CHAPTER V

PROJECTIVITY

1. General Definitions

Any one-to-one correspondence between the elements of two one dimensional forms that preserves cross ratio is a projectivity.

A projectivity is a sequence of perspectivities.

A projectivity is also a projective transformation.

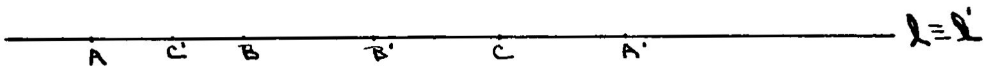
2. Projectivity on a Line

A projectivity on a line can be constructed by a sequence of perspectivities. Let lines 1 and 1' coincide. Draw any other line 1" distinct from 1. It is also distinct from 1'. Project A', B', C' on 1 into A", B", C" on 1" from any point P_1. Draw a line 1''' from any point Q on 1" through A. Take any point P_2 on the line A A" and project A", B", C" into A''', B''', C''' on 1'''. Draw B B''' and C C''' determining P_3. Draw P_3 A. Therefore by means of three perspectivities A', B', C' are projected into A, B, C on 1.

Steiner's Construction gives a second construction of a projectivity on a line. Consider a projectivity A B C ..... $\overline{\wedge}$ A' B' C' ..... on a line, 1. Given a point D on 1 to construct the corresponding point D'. First, select any point S not on 1. Draw S (A B C A' B' C'). Draw any conic through S determining points A_1, B_1, C_1, A'_1, B'_1, C'_1. Draw the lines C_1 (A'_1, B'_1 , C'_1) and C'_1 (A_1, B_1, C_1). The pencils with centers C_1 and C'_1 are projective and have a self-corresponding line. Hence the pencils are perspective. Denote the axis of perspectivity by u. Denote the intersections of the axis with the conic by H and K. Draw S D determining D_1. Draw C'_1 D_1 determining P on u. Draw C_1 P determining

D'_1. Draw $S\ D'_1$ determining D' on l.

A projectivity on a line may be defined in this way. Consider two point ranges on $l \equiv l'$ and $A\ B\ C\ \ldots\ldots \overline{\wedge}\ A'\ B'\ C'\ \ldots\ldots$



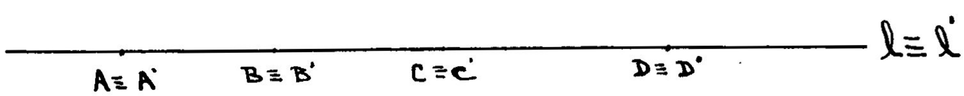
This is a projectivity on a line.

The projectivity between superposed forms is said to be hyperbolic when there are two self-corresponding (invariant) elements, parabolic when there is one, and elliptic when there is none.

Regard a projectivity on a line with one invariant point. Consider the flat pencil a, b, c, d with vertex at S_1 and the flat pencil a', b', c', d' with vertex at S_2. The pencils are perspective. The axis of perspectivity is r. Let line l pass through the intersection of $S_1\ S_2$ and r, i.e. P_2. P_1 is the intersection of r and l. Then $P_1 \equiv P_2$. Therefore $A\ B\ C\ \ldots\ldots \overline{\wedge}\ A'\ B'\ C'\ \ldots\ldots$ has one invariant point (P_1).

Consider a projectivity on a line with two invariant points. Consider two perspective flat pencils with centers S_1 and S_2. The axis of perspectivity is r. Let any line l cut these pencils in A, B, C, D, A', B', C', D'. Then point ranges A B C D ..... on l and A' B' C' D'.....on l are projective. This projectivity has two self-corresponding points P_1 and P_2. P_1 is the intersection of r and l. P_2 is the intersection of the line joining $S_1\ S_2$ with l. Therefore $A\ B\ C\ \ldots\ldots \overline{\wedge}\ A'\ B'\ C'\ \ldots\ldots$ has two invariant points. $A\ B\ C\ \ldots\ldots P_1\ \overline{\wedge}\ A'\ B'\ C'\ \ldots\ldots P_1$. $A\ B\ C\ \ldots\ldots P_2\ \overline{\wedge}\ A'\ B'\ C'\ \ldots\ldots P_2$.

Von Staudt's Theorem I.- If a projectivity on a line leaves three distinct points invariant, then it leaves every point invariant. In this case the projective transformation is called the identity transformation.



Proof: Consider $A\ B\ C\ \ldots\ldots \barwedge A'\ B'\ C'\ \ldots\ldots$ on $l \equiv l'$. Suppose $A \equiv A'$, $B \equiv B'$, $C \equiv C'$. Select any fourth point D on l. Find the corresponding point D' on l'. $(A\ B\ C\ D) = (A'\ B'\ C'\ D')$. Cross ratio is invariant under projection and section. By assumption $(A\ B\ C\ D) = (A\ B\ C\ D')$. Dedekind's Continuity Postulate states that if A, B, C and K (cross ratio) are given, then a point D is uniquely determined so that $(A\ B\ C\ D) = K$. Therefore $D \equiv D'$. In this projectivity on a line there are infinite invariant points.

Theorem II.- If a projectivity on a line has one pair of reciprocal corresponding points then every corresponding pair corresponds reciprocally and the projectivity is an involution.

Proof: Consider projectivity π on a line l. A and A' are reciprocal corresponding points i.e. $\pi(A) = A'$ and also $\pi(A') = A$.

Consider any other point P on l of the projectivity π, but not a double point of the projectivity. Let $\pi(P) = P'$. Prove that $\pi(P') = P$. That is to say prove P and P' reciprocal corresponding points in the projectivity π.

Consider the four points A, A', P, P'. $(A\ A'\ P\ P') = (A'\ A\ P'\ P)$. Interchanging the elements of a cross ratio in pairs leaves the value of the cross ratio unchanged.

Then there is a projectivity π', such that $A\ A'\ P\ P'\ \ldots\ldots \barwedge A'\ A\ P'\ P\ \ldots\ldots$ Any one-to-one correspondence between elements of two one dimensional forms that preserves cross ratio is a projectivity.

But in the projectivity π' there are three pairs of corresponding elements. They are $A\ A'$, $A'\ A$ and $P\ P'$. Therefore projectivity π' equals projectivity π. A projective transformation between two one dimensional forms is uniquely determined by three pairs of corresponding elements. But in the projectivity π', P' transforms into P. Thus every pair of corresponding points correspond reciprocally.

The projectivity π on l is an involution by definition. An involution on a line is determined by two pairs of corresponding points.

3. Projectivity on a Conic

Consider a conic and three pairs of points A, A'; B, B'; C, C' on it. Draw pencils C (A' B' C') and C' (A B C). C B' and C' B determine M. C A' and C' A determine N. M N determine u, axis of perspectivity.

Given D on the conic to construct D' on the conic. Draw C' D determining P on u. Draw C P determining D' on the conic.

This correspondence is called a projectivity on the conic.

This one-to-one correspondence of point on the conic is determined by three pairs of corresponding points.

Theorem III.- To any four points correspond four points with the same cross ratio.

Proof: Prove that (A B C D) = (A' B' C' D'). (A B C D) equals the cross ratio of the lines joining C' to A B C D. Then this cross ratio of the four lines from C' to A B C D is equal to the cross ratio of the four points on u. The four points are N, M, P and Q. (N M P Q) equals the cross ratio of four lines through C. That equals the cross ratio of the four points A' B' C' D'. That is a projectivity.

If the conic is proper the projectivity has 0, 1 or 2 double points.

CHAPTER VI

PENCIL OF CONICS AND ITS DUAL

1. Pencil of Conics

A pencil of conics is the single infinity of conics through four points, no three of which are collinear. These points are base points.

Desargues' Theorem I.- A pencil of conics cuts any line in the plane not through any of the four base points in pairs of points in an involution.

Proof: Consider four points K, L, M and N and the pencil of conics determined by them.

Consider a line 1 not through K, L, M or N. Select any point D on line 1. There will be one conic of the pencil through D for five points of a plane determine a conic. Let K L intersect line 1 at A. Let M N intersect 1 at A'. Let K N intersect 1 at B. Let L M intersect 1 at C'. Let K M intersect 1 at C. Let N L intersect 1 at C'.

Regard the involution on 1 determined by A A', B B' and D. Construct D' corresponding to D in the involution.

Prove that the conic through D passes through D'. Use the converse of Pascal's Theorem. Use hexagon D_1, D'_2, L_3, K_4, N_5, M_6. D_1 D'_2 meets K_4 N_5 at B. D'_2 L_3 meets N_5 M_6 at N'. L_3 K_4 meets M_6 D_1 at K'.

But the points B, N' and K' are collinear. Therefore the conic through D passes through D'.

Two conics of the pencil or none at all are tangent to any given line in a plane.

2. Range of Conics

A range of conics is all the conics tangent to four given lines, no three of which are concurrent. These lines are called base lines.

Dual of Desargues' Theorem.- The tangents from a point to all the conics of a range form pairs of lines in an involution. The proof is evident by duality.

Two conics of the range of conics or none at all pass through a given point.

CHAPTER VII

PROBLEMS IN CONSTRUCTION

1. Given Five of Its Points

Construct the conic given the points A_1, B_2, S'_3, D_4, S_5. Find C_6. Draw any line 1 through A_1. Draw A_1 B_2 and D_4 S_5 determining L. Draw S'_3 D_4 meeting 1 at N. Draw L N and B_2 S'_3 determining M. Draw M S_5 meeting L in the point required, C_6.

Proof: Since 1,2 and 4,5; 2,3 and 5,6 ; 3,4 and 6,1 meet in collinear points L, M, N respectively, the conic is the one required. This is true by the Converse of Pascal's Theorem.

2. Given Five of Its Tangents

Given the tangents a_1, b_2, s'_3, d_4, s_5. Find c_6. Select any point L on a_1. Join a_1 b_2 and d_4 s_5 by line 1. The point of intersection of s'_3 and d_4 is joined to L by line, n. Then 1 n and b_2 s'_3 determine line m. The points m s_5 and L are joined by the line required c_6.

Proof: Since 1,2 and 4,5; 2,3 and 5,6; 3,4 and 6,1 are joined by concurrent lines 1, m, n respectively, the conic is the one required. The Converse of Brianchon's Theorem verifies this.

3. Given Three of Its Points and Tangents at Two of Them

Construct the conic given the points $A_1 \equiv C_6$, B_2, $S'_3 \equiv D_4$ and tangents at A_1 and S'_3. Find S_5. Draw any line 1 through A_1. Draw tangents from A_1 and S'_3 meeting at point N. Draw B_2 S'_3 meeting 1 at M. Draw M N and A_1 B_2 meeting at L. Draw L S'_3 meeting 1 in a point S_5, which is the required point. Vary 1 and repeat the operation.

Proof: Since 1,2 and 4,5; 2,3 and 5,6; 3,4 and 6,1 meet in collinear points L, M, N respectively, the conic is the one required. This is true by the Converse of Pascal's Theorem.

4. Given Three of Its Tangents and Points of Contact of Two of Them

Construct the conic given the tangents $a_1 \equiv c_6$, b_2, $s'_3 \equiv d_4$ and the points of contact of a_1 and s'_3. Find s_5. Select any point L on a_1. Draw line n through the points of contact of a_1 and s'_3. Through b_2 s'_3 and L draw line m. The points mn and a_1 b_2 determine l. Join l s'_3 and L by a line s_5, which is the required line. Vary L and repeat the operation.

Proof: Since 1,2 and 4,5; 2,3 and 5,6; 3,4 and 6,1 are joined by concurrent lines l, m, n respectively, the conic is the one required. This is valid by the Converse of Brianchon's Theorem.

5. Given Four Points and a Tangent at One of Them.

Construct the conic given the four points $A_1 \equiv C_6$, B_2, S'_3, D_4 and a tangent at A_1. Find S_5. Draw any line l through A_1. Draw S'_3 D_4 meeting tangent A_1 C_6 at N. Draw B_2 S'_3 meeting l at M. Draw M N and A_1 B_2 determining L. Draw L D_4 meeting l at S_5.

Proof: Since 1,2 and 4,5; 2,3 and 5,6; 3,4 and 6,1 meet in collinear points L, M, N respectively, the conic is the required conic. The Converse of Pascal's Theorem verifies this.

6. Given Five Points with Two of Them at Infinity

Construct the conic given the points A_1, B_2 at infinity, S'_3, D_4 at infinity, and S_5. Find C_6. Draw any line l through A_1. Draw A_1 B_2 and D_4 S_5 determining L. Draw S'_3 D_4 meeting l at N. Draw L N and B_2 S'_3 meeting at M. Draw M S_5 meeting l at C_6. Vary l and repeat the operation.

Proof: Since 1,2 and 4,5; 2,3 and 5,6; 3,4 and 6,1 meet in collinear L, M, N respectively, the conic is the required conic. This is true by the Converse of Pascal's Theorem.

7. Given Two Finite Points, Two Points at Infinity and a Tangent at One of the Finite Points

Construct the conic given the points $A_1 \equiv C_6$, B_2 at infinity,

S'_3, D_4 at infinity and the tangent at A_1. Draw any line 1 through A_1. Draw S'_3 D_4 meeting tangent (1,6) at N. Draw B_2 S'_3 meeting 1 at M. Draw M N and A_1 B_2 determining L. Draw L D_4 meeting 1 at S_5. Vary 1 and repeat the operation.

Proof: Since 1,2 and 4,5; 2,3 and 5,6; 3,4 and 6,1 meet in collinear points L, M, N respectively, the conic is the one required. This is valid by the Converse of Pascal's Theorem.

8. Given the Asymptotes and a Finite Point

Construct the conic given the asymptotes and B_2 on the conic. Find S_5. Draw any line 1 through A_1. In this case draw 1 parallel to the direction of A_1. The lines tangent at A_1 and S'_3 are the asymptotes. The asymptotes intersect at N, i.e. A_1 C_6 and S'_3 D_4. Draw B_2 S'_3 meeting 1 at M. Draw M N and A_1 B_2 determining L. Draw L D_4 meeting 1 at S_5. Vary 1 and repeat the operation.

Proof: Since 1,2 and 4,5; 2,3 and 5,6; 3,4 and 6,1 meet in collinear points L, M, N respectively, the conic is the required conic. This is true by the Converse of Pascal's Theorem.

9. Given Four Tangents and a Line at Infinity

Construct the conic given the lines a_1, b_2, s'_3, d_4 and s_5 at infinity. Select any point L on a_1. Join $s'_3 d_4$ and L by line n. Through a_1 b_2 draw 1 parallel to d_4. Join b_2 s'_3 and 1 n by line m. The points m s_5 and L determine the line c_6. That is to say draw a line through L parallel to m. Vary L and repeat the operation.

Proof: Since 1,2 and 4,5; 2,3 and 5,6; 3,4 and 6,1 are joined by concurrent lines 1, m, n respectively, the conic is the required conic. This is valid by the Converse of Brianchon's Theorem.

10. Given Three Finite Lines, Two Identical Lines at Infinity and a Contact Point at Infinity

Construct the conic given the lines b_2, s'_3, d_4, $a_1 \equiv c_6$ at infinity and a contact point at infinity. Find s_5. Select any point L on a_1, i.e. indicate the direction of L at infinity on $a_1 \equiv c_6$. The point of contact on a_1 is fixed. Draw line m through b_2 s'_3 and parallel to the direction of L. The line through s'_3 d_4 parallel to the direction of the point of contact is n. Draw a line l through m n and parallel to b_2. The line through d_4 l and parallel to the direction of L is s_5.

Proof: Since 1,2 and 4,5; 2,3 and 5,6; 3,4 and 6,1 are joined by concurrent lines l, m, n respectively, the conic is the required conic. This is true by the Converse of Brianchon's Theorem.

11. Given Three Lines and Asymptotes

Construct the conic given the three lines $a_1 \equiv c_6$, b_2, $s'_3 \equiv d_4$; a_1 and s'_3 are asymptotes. The points of contact are on a_1 and s'_3. Find s_5. Select any point L on a_1. The line joining L and the point b_2 s'_3 is m. Through a_1 b_2 draw a line l parallel to m. Then join l s'_3 and L by a line s_5. Vary L and repeat the operation.

Proof: The conic is the required conic by the Converse of Brianchon's Theorem.

CHAPTER VIII

CONCLUSIONS

Pascal's Theorem marks the climax of the classical theory of projective geometry. Its importance in the synthetic treatment of conics can hardly be exaggerated. But it has enjoyed a popularity commensurate with if not exceeding its importance. Discovered by its precocious author at the age of sixteen, studied by many of the eminent fathers of projective geometry, this theorem caught the imagination of mathematicians to an astonishing degree. As the remarkable properties of the complete 6-point were unfolded, men called it in their enthusiasm the mystic hexagon. This is perhaps not surprising in view of the possibility of drawing with the aid of the theorem elements of a conic, such as a tangent at a point, tangents from a point, asymptotes, center, etc. The conic itself is represented only by a skeleton of five points. At one time however the theorem became almost a menace to mathematical progress. Investigators turned away from their search for new truths to devote themselves to finding new proofs of Pascal's Theorem.

It is a remarkable fact that while Pascal's Theorem was published in 1640 Brianchon's did not appear until 1806. Needless to say the principle of duality was unknown at the earlier date.

Since the Principle of Duality has been used to a great extent in this study it is fitting to give a few facts about it. Poncelet and Gergonne produced the Principle of Duality. It asserts that from any statement on theorems concerning the relative position of elements composing a geometrical configuration, another statement or theorem can be obtained by a simple interchange of elements of the configuration with their reciprocals.

This thesis has been prepared for the purpose of investigating the properties of conics through a special method, which is the synthetic and projective treatment.

The subject has been developed with due attention to the important and recognized principles. The difficulties of the subject were reduced almost half by the Principle of Duality and the valuable scheme from Pascal's Theorem.

The value of the application of the Principle of Duality has been illustrated in chapters II, III, VI and VII.

The investigation developed in chapters II through VII verifies that the properties of conics can be proved by the synthetic and projective method.

In chapter VII it was verified that it is necessary and sufficient to have five independent conditions given to construct a conic.

The symbols that are used in this thesis are defined in the Appendix.

The theorems that are used, but not proved in the thesis, are stated in the Appendix.

In the list of figures II, 1 means Chapter II, Section 1.

APPENDIX

The symbol $\overset{-}{\underset{\wedge}{}}$ expresses projective relationship.

The symbol $\overset{=}{\underset{\wedge}{}}$ represents perspective position.

The symbol $\equiv$ means identically equal.

The symbol ( ) represents cross ratio.

The symbol $\pi(A) = A'$ says that A and A' are reciprocal corresponding points in a projectivity.

The symbol $\longrightarrow$ is used for the word approaches. $A \longrightarrow C$, i.e. A approaches C.

Theorem of Pappus.- Cross ratio is invariant under projection and section.

Desargues' Theorem.- If two triangles are perspective from a point, then they are perspective from a line.

Converse of Desargues' Theorem.- If two triangles are perspective from a line, then they are perspective from a point.

Dedekind's Continuity Postulate.- If A, B, C and K (cross ratio) are given, then a point D is uniquely determined so that $(A\ B\ C\ D) = K$.

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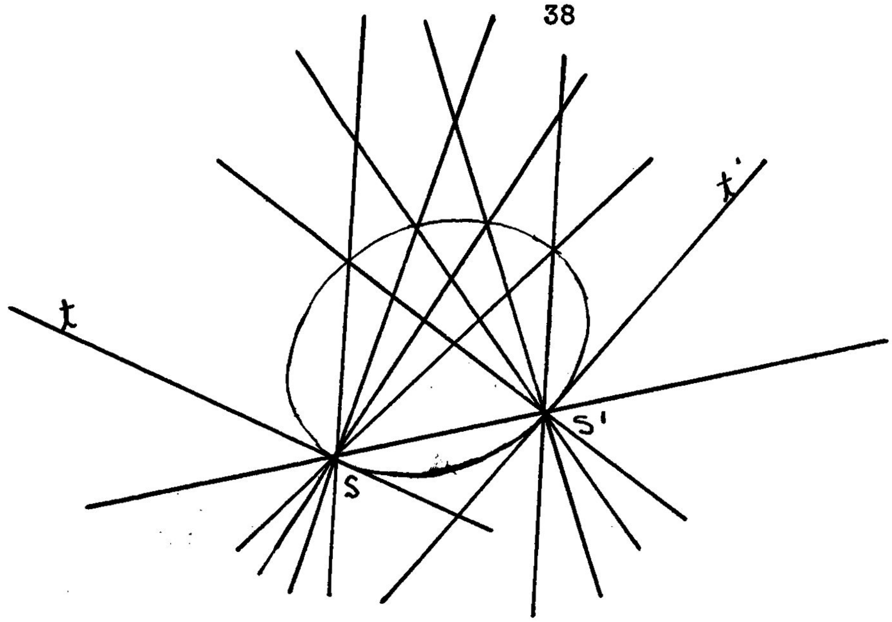

Figure 1.-Point Conic

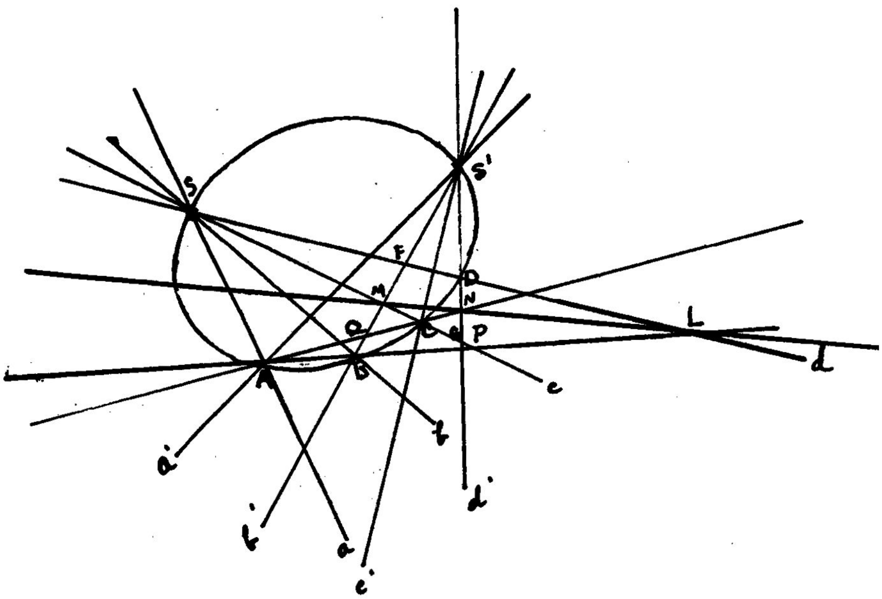

Figure 2.-Point Conic

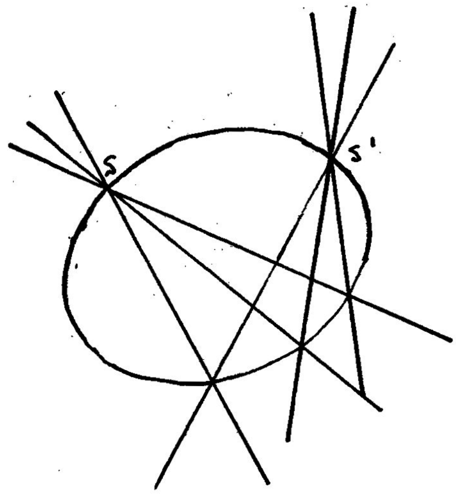

Figure 3.-Point Conic

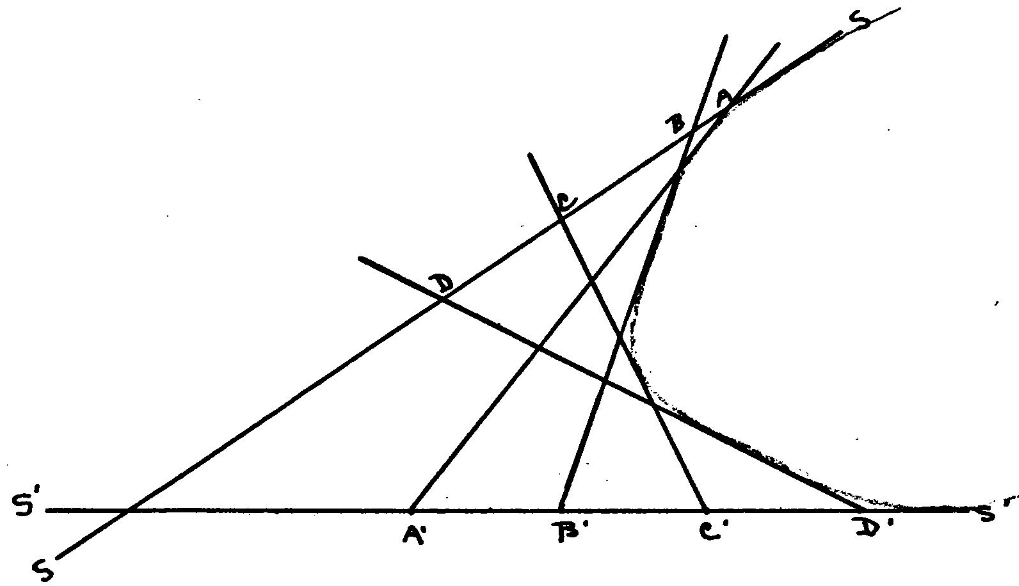

Figure 4.-Line Conic

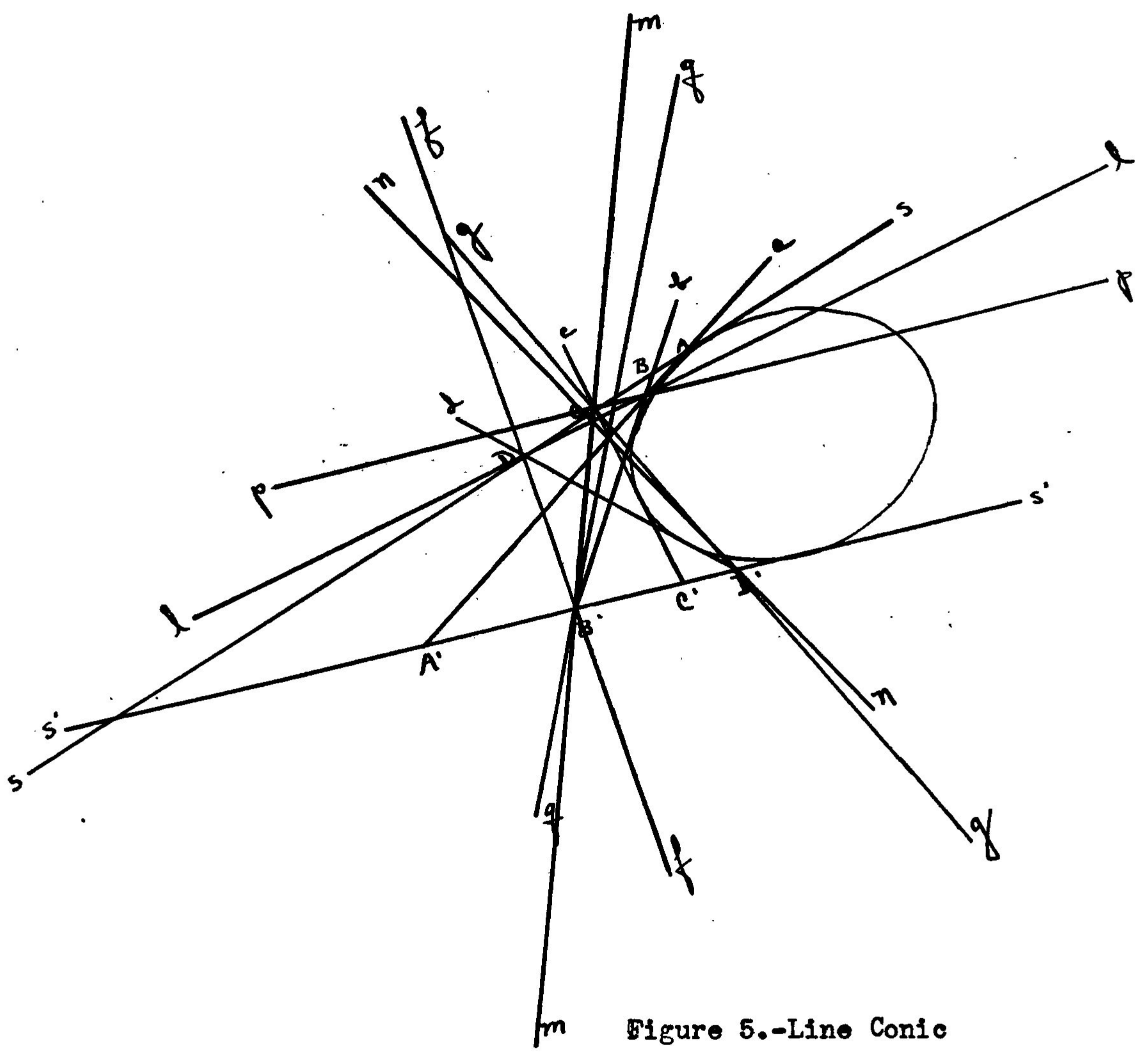

Figure 5.-Line Conic

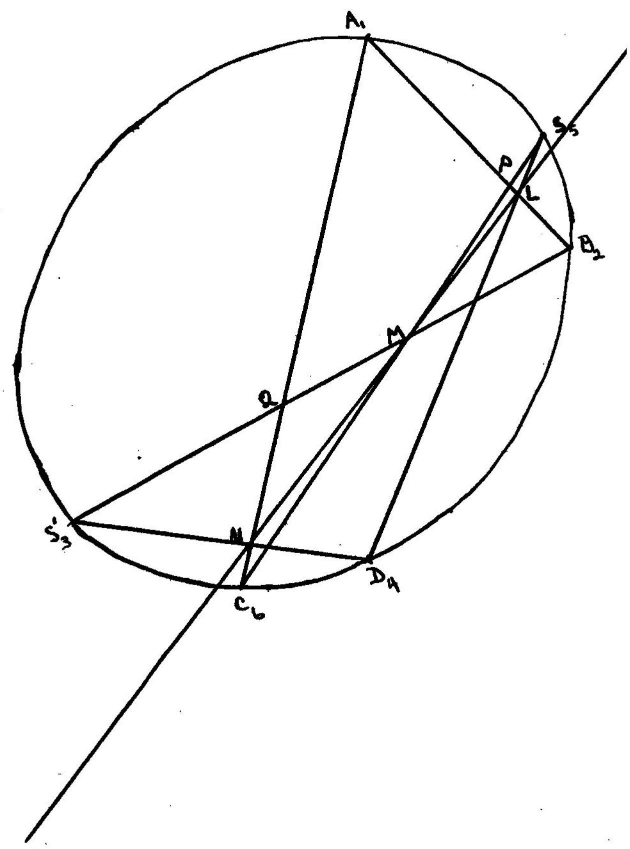

Figure 6.-Pascal's Hexagon

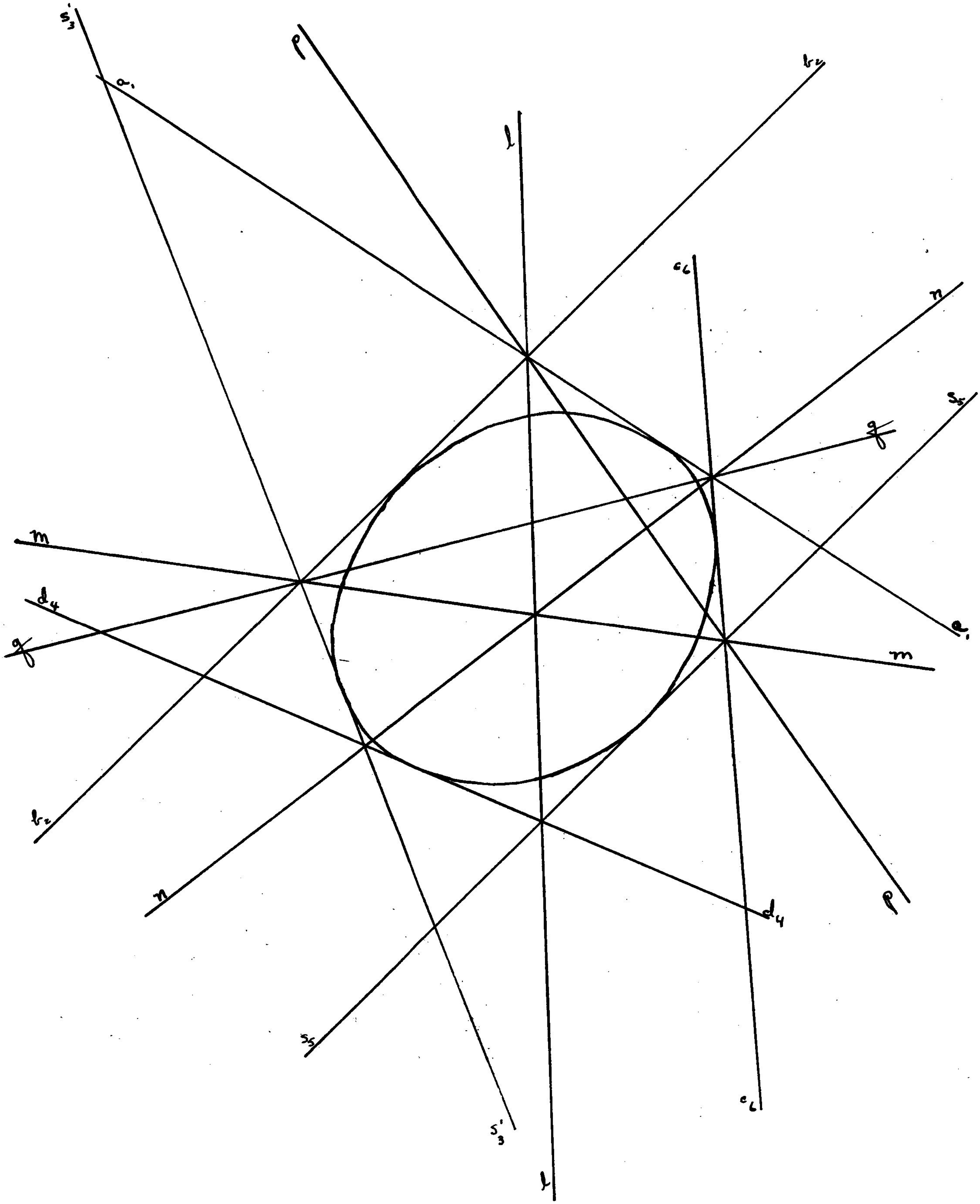

Figure 7.-Brianchon's Hexagon

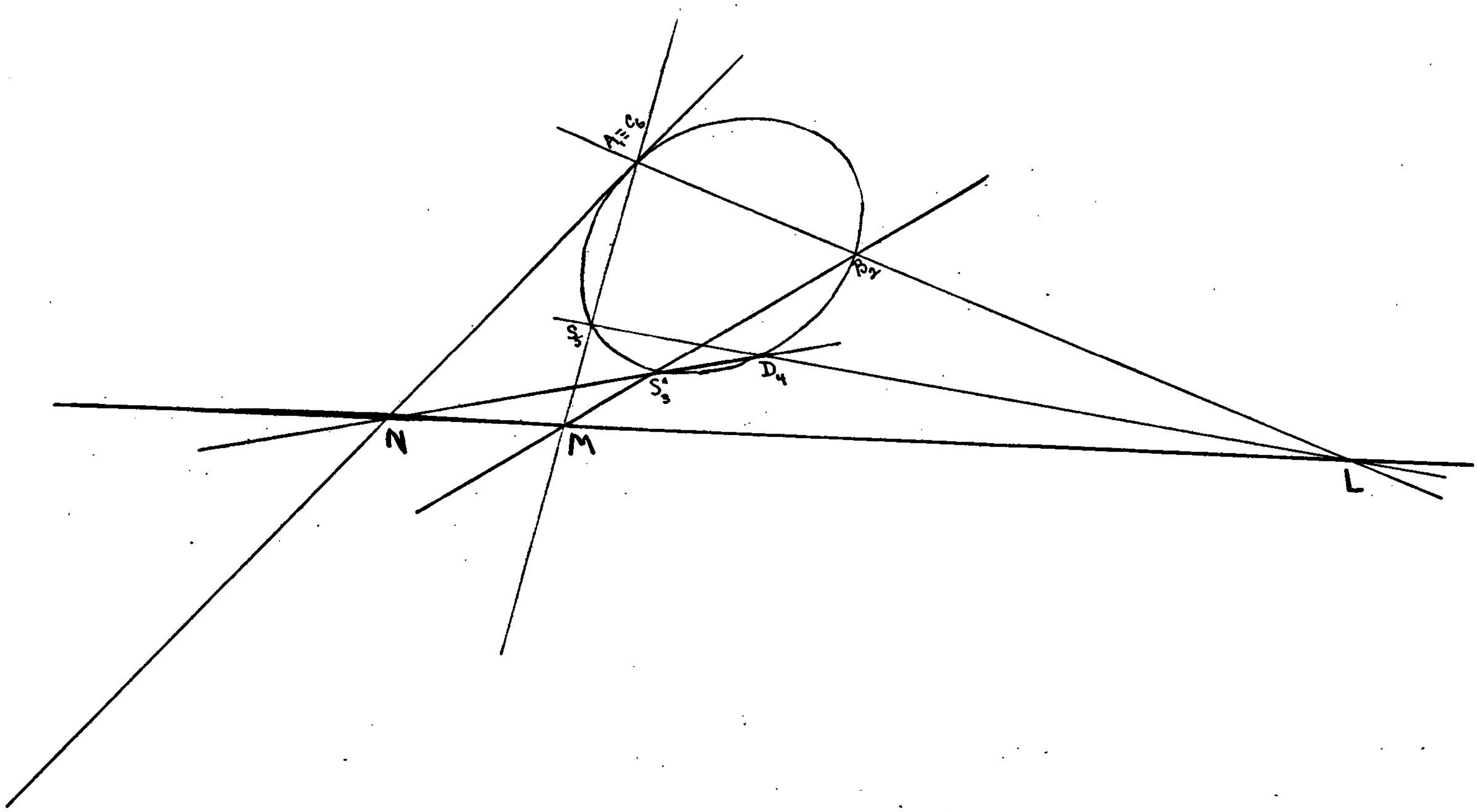

Figure 8.-Inscribed Pentagon

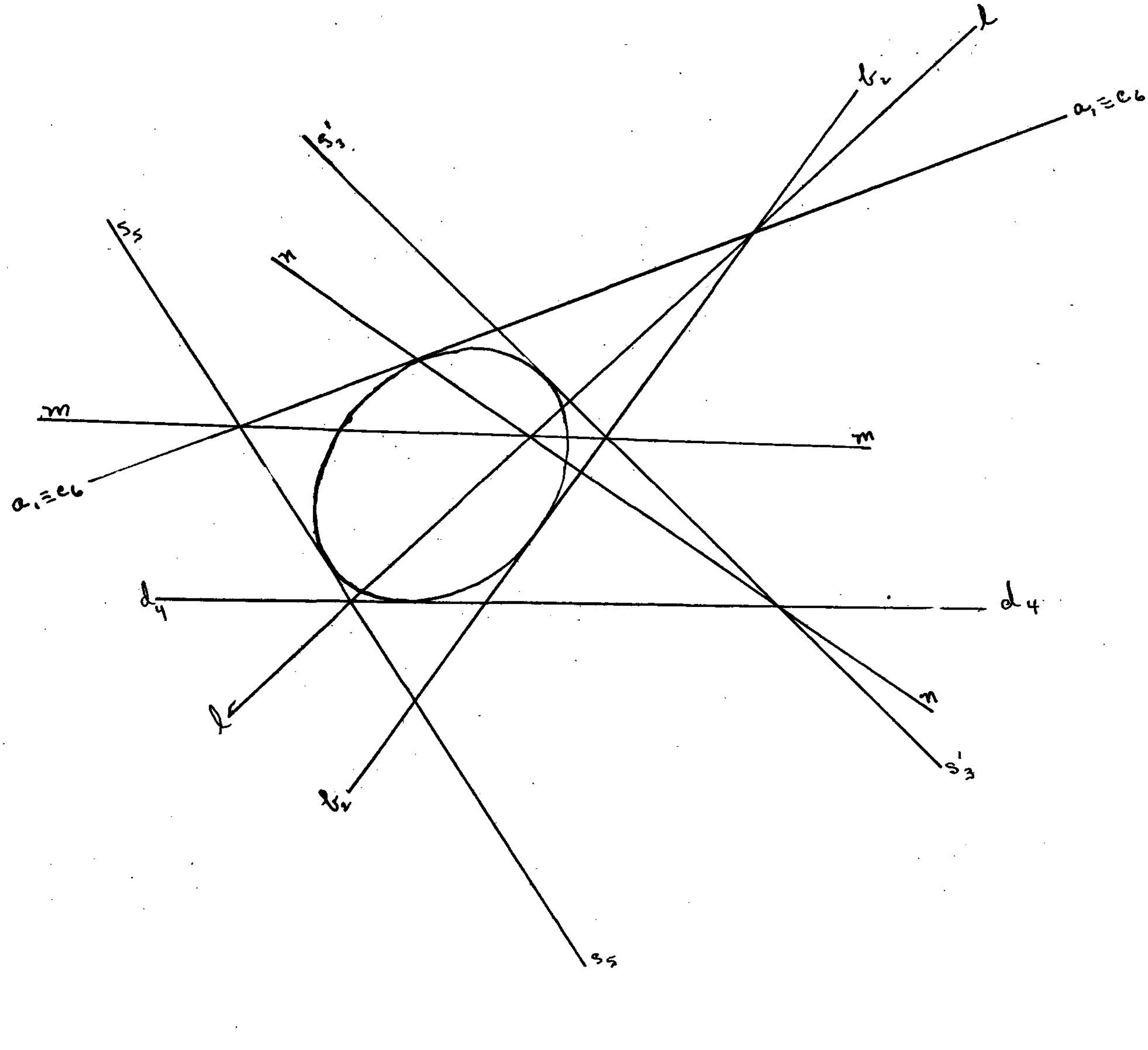

Figure 9.-Circumscribed Pentagon

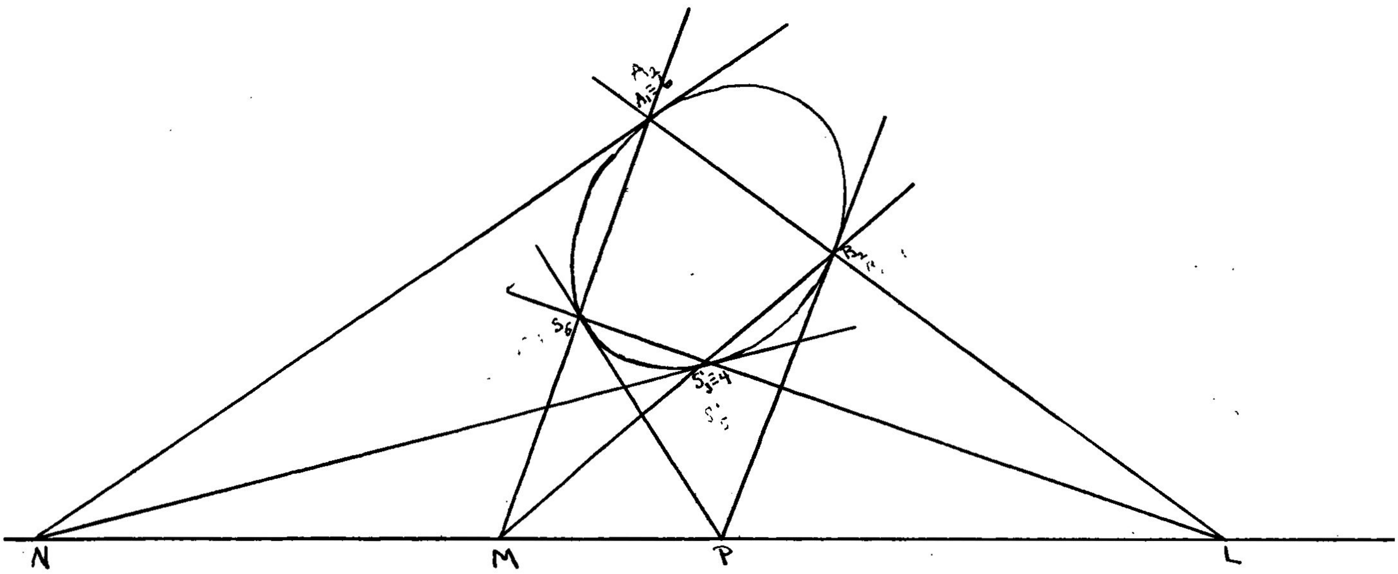

Figure 10.-Inscribed Quadrangle

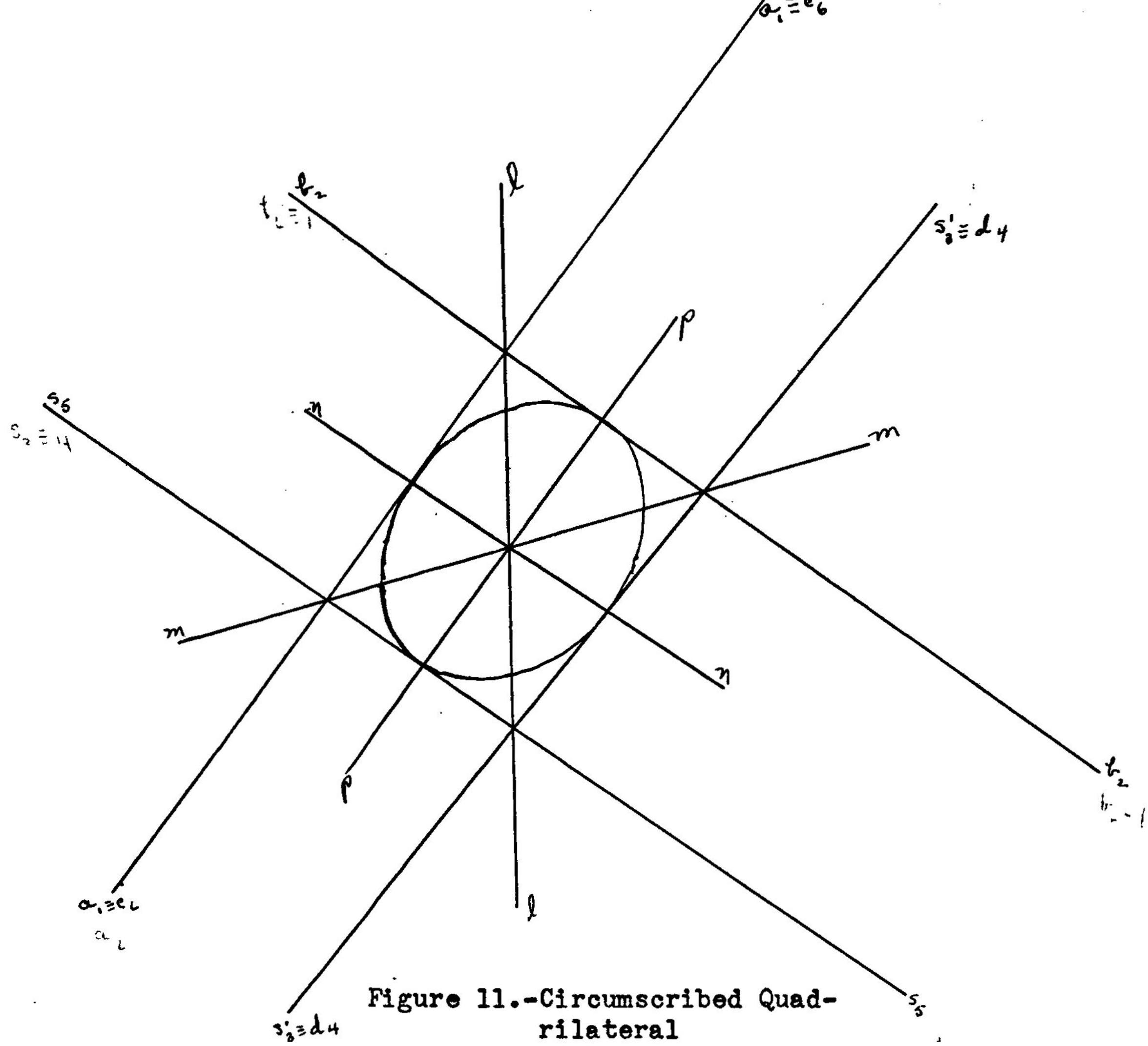

Figure 11.-Circumscribed Quadrilateral

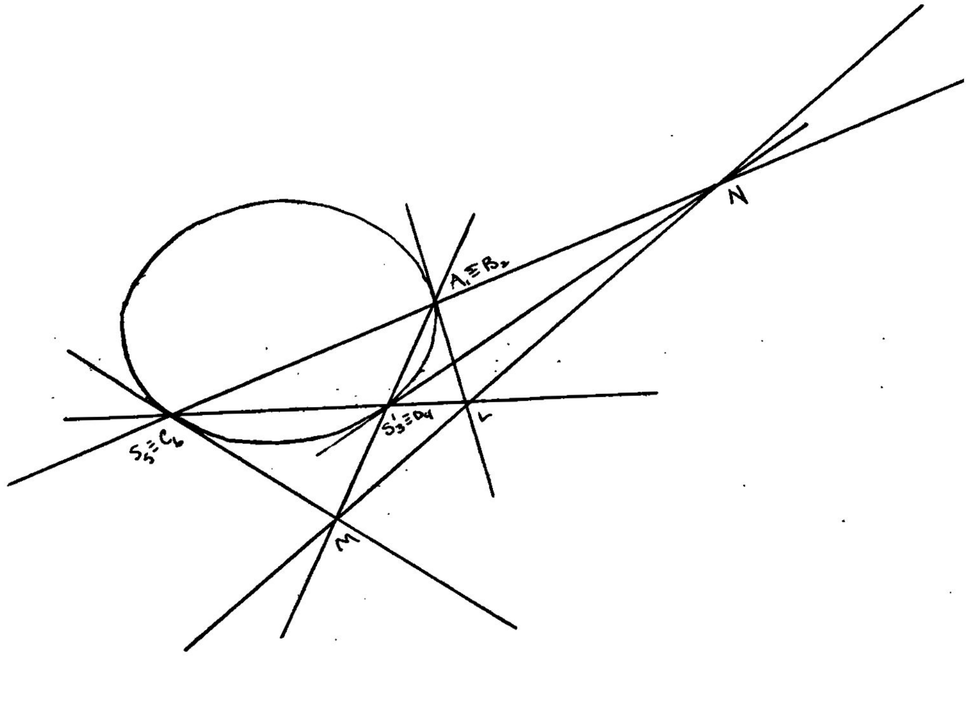

Figure 12.-Inscribed Triangle

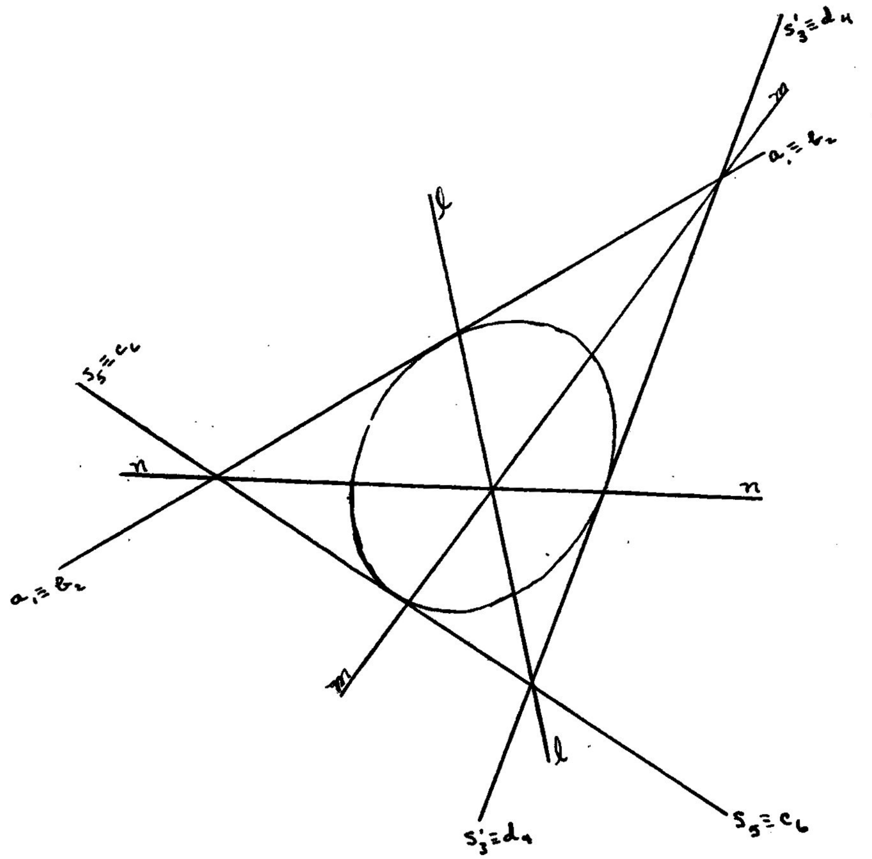

Figure 13.-Circumscribed Triangle

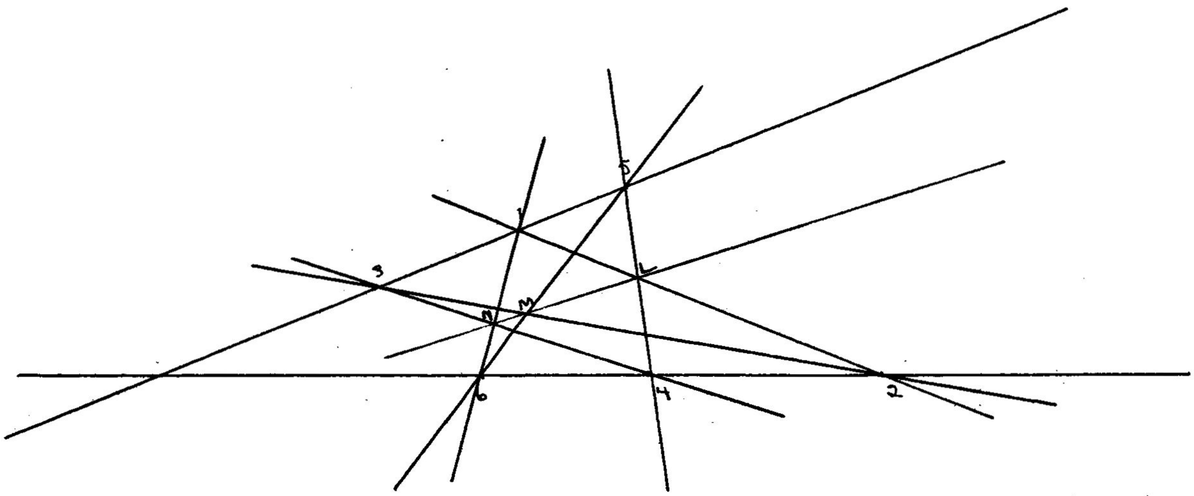

Figure 14.-Pascal's Composite Conic

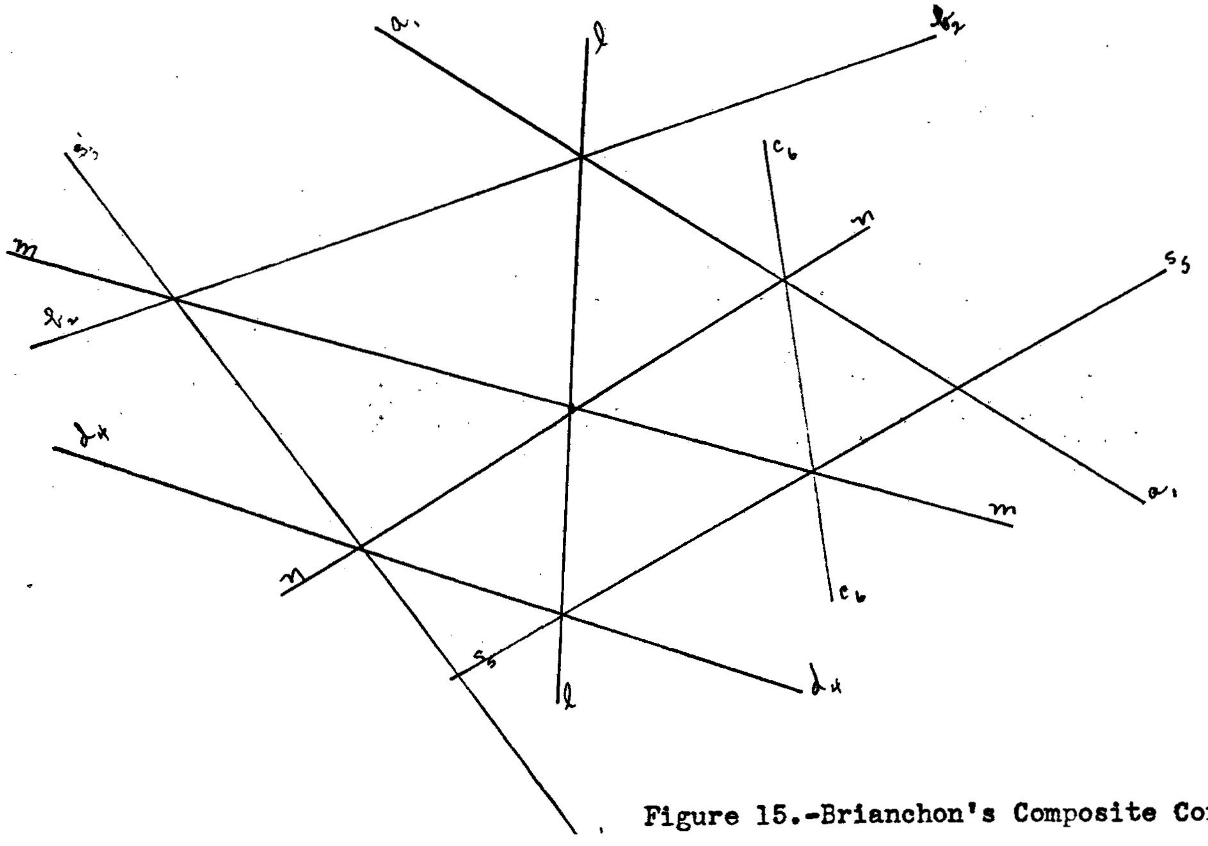

Figure 15.-Brianchon's Composite Conic

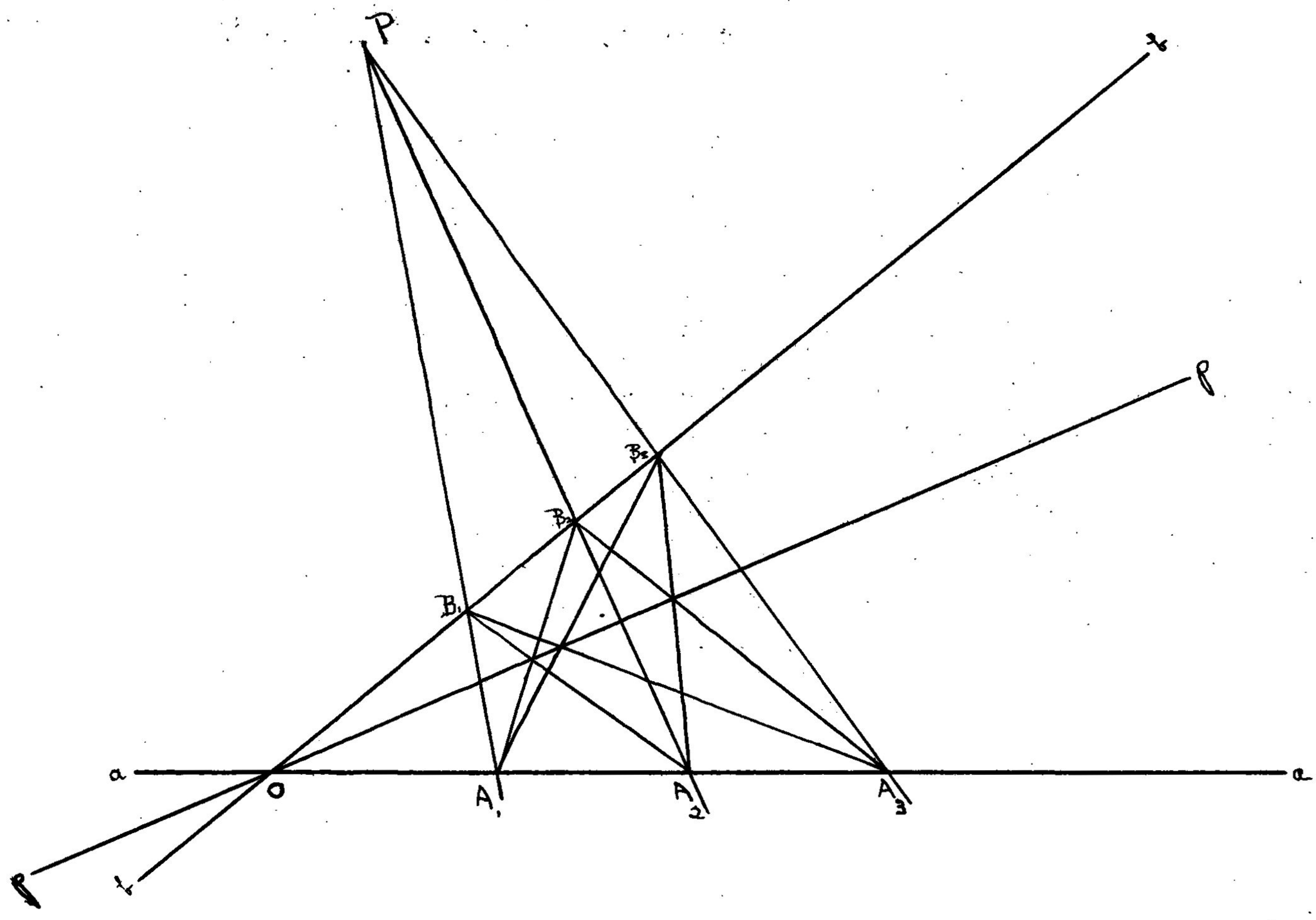

Figure 16.-Polar of a Point with Respect to a Pair of Lines

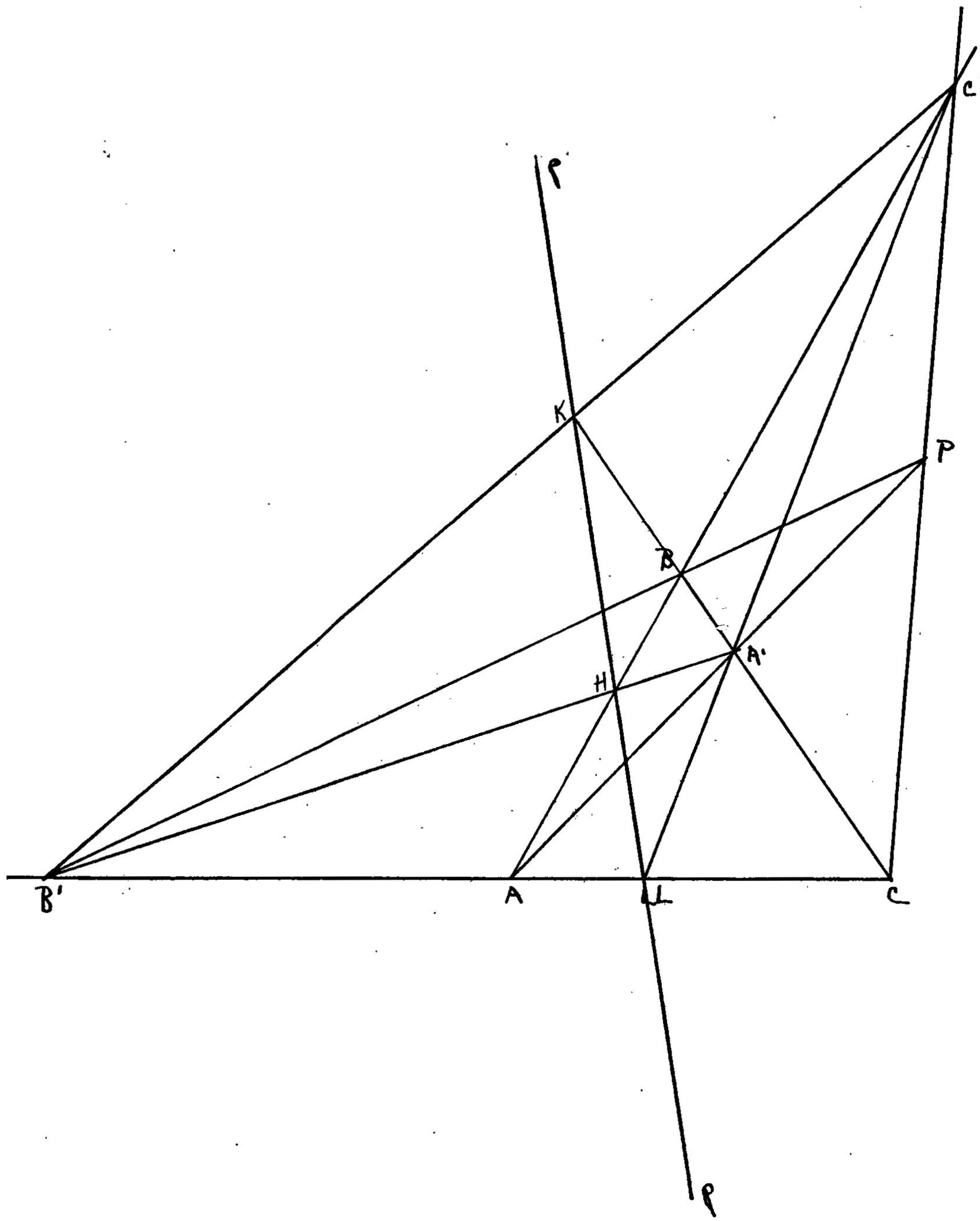

Figure 17.-Polar of a Point with Respect to a Triangle

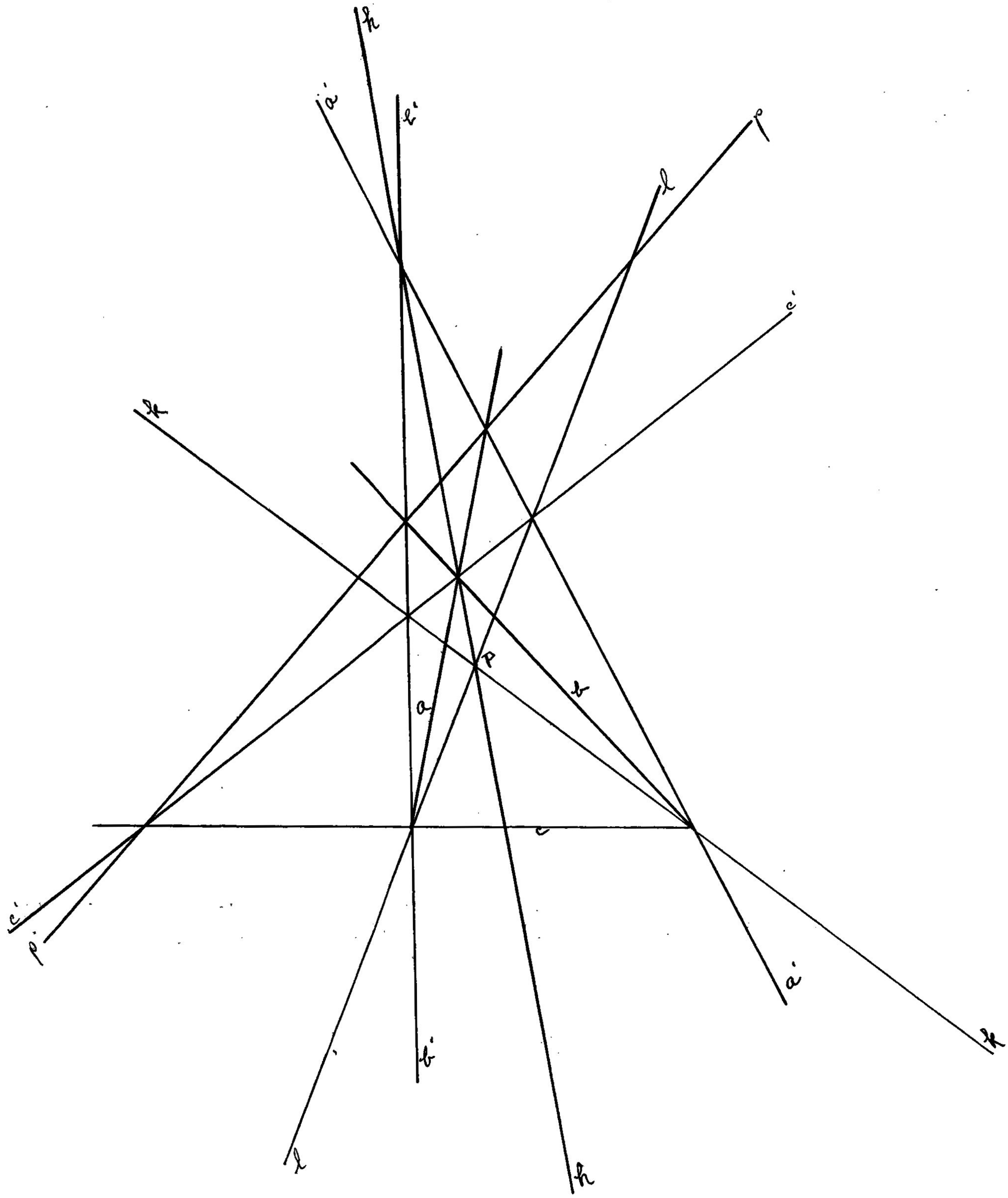

Figure 18.-Pole of a Line with Respect to a Triangle

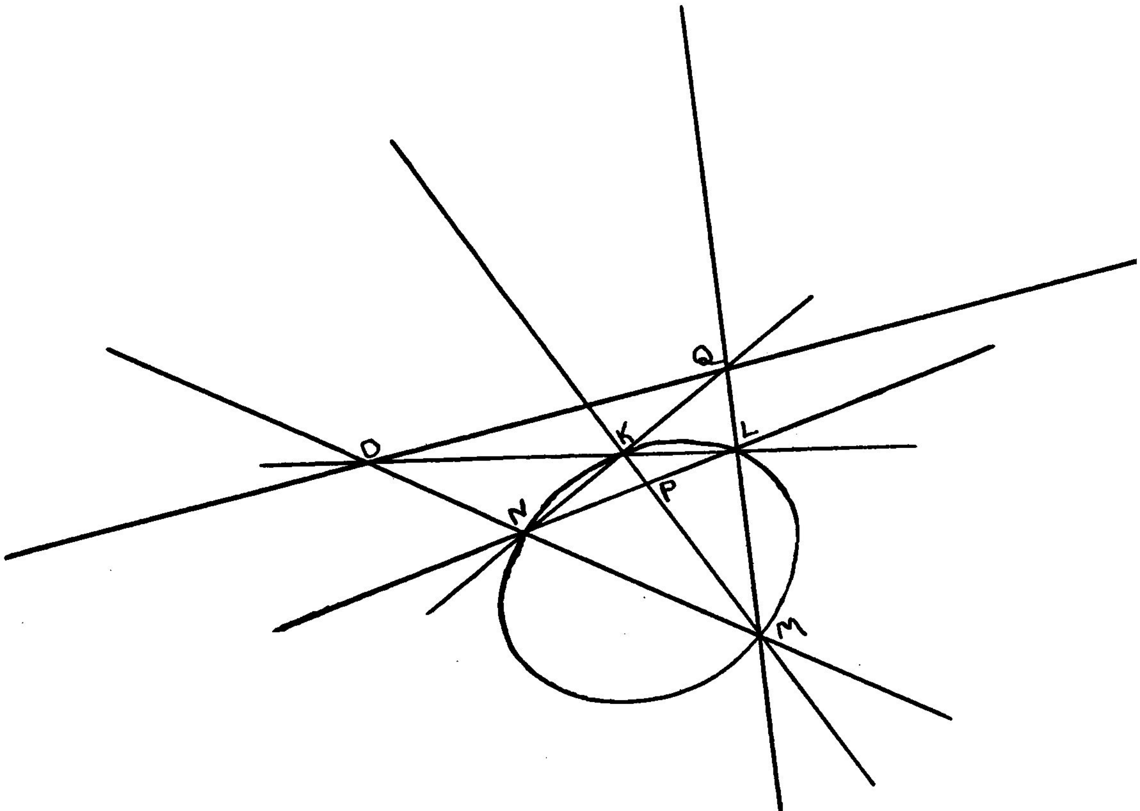

Figure 19.-Polar of a Point with Respect to a Conic

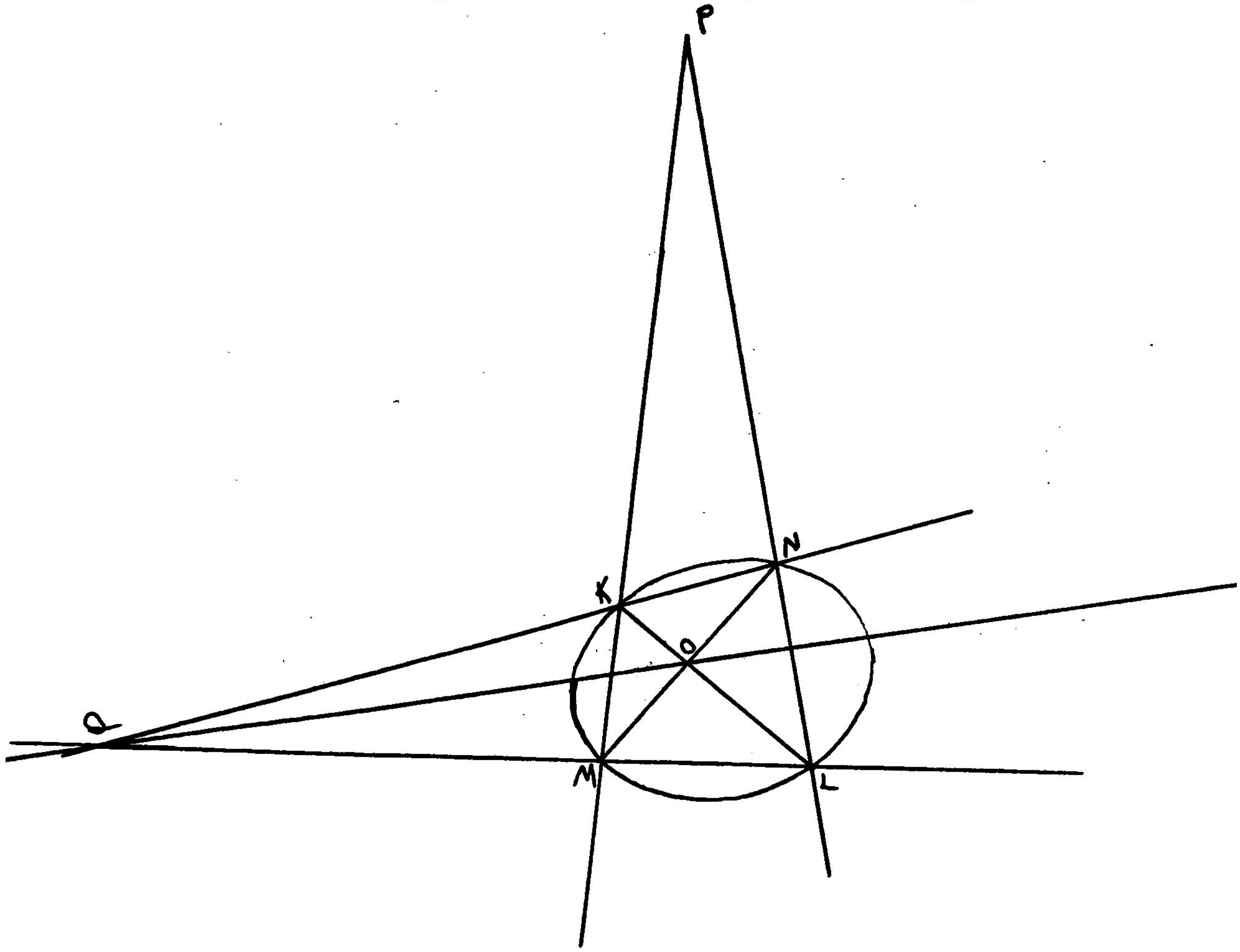

Figure 20.-Polar of a Point with Respect to a Conic

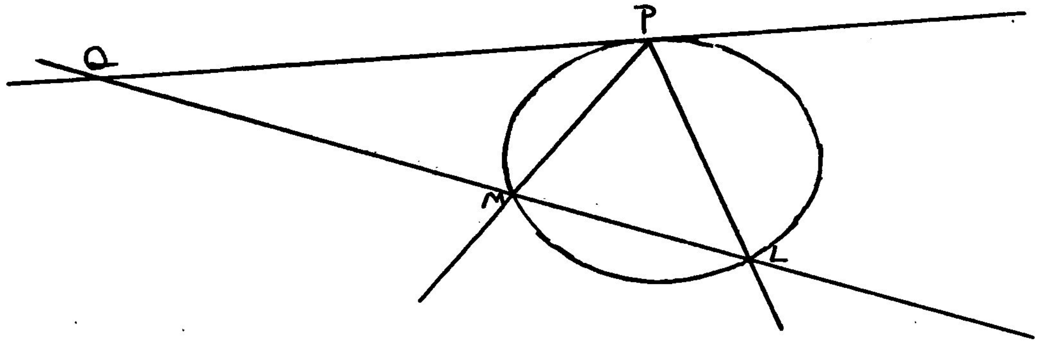

Figure 21.-Polar of a Point with Respect to a Conic

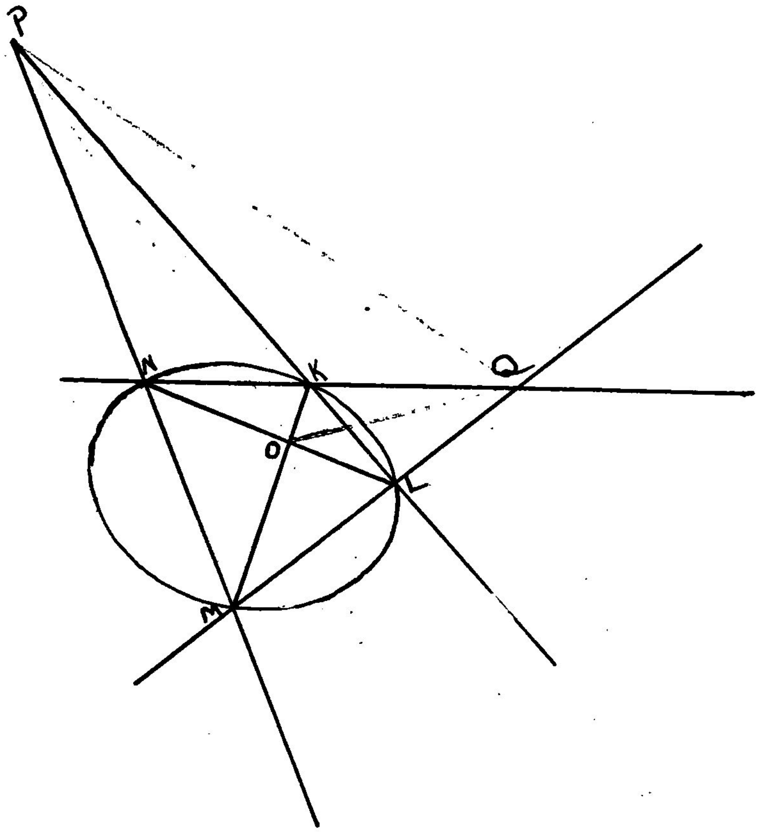

Figure 22.-Complete Quadrangle Inscribed in a Conic

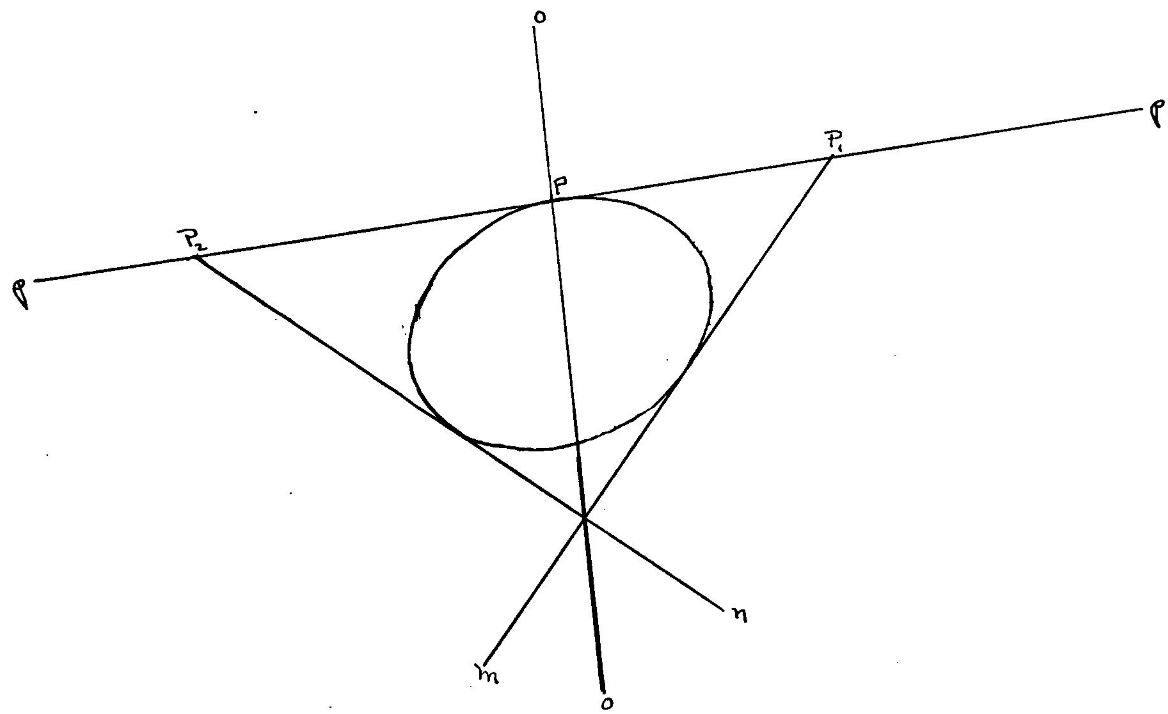

Figure 23.-Pole of a Line wi th Respect to a Conic

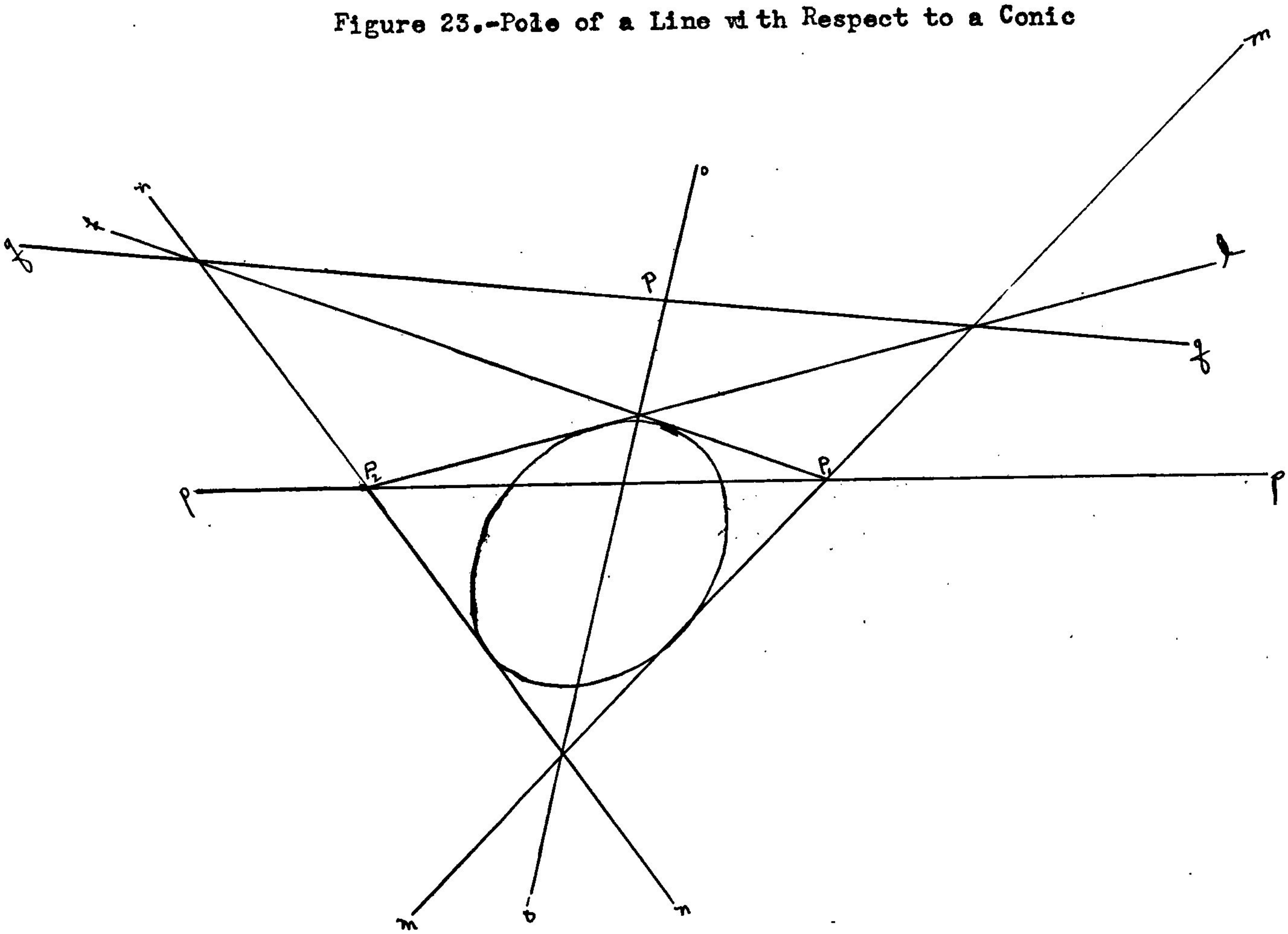

Figure 24.-Pole of a Line with Respect to a Conic

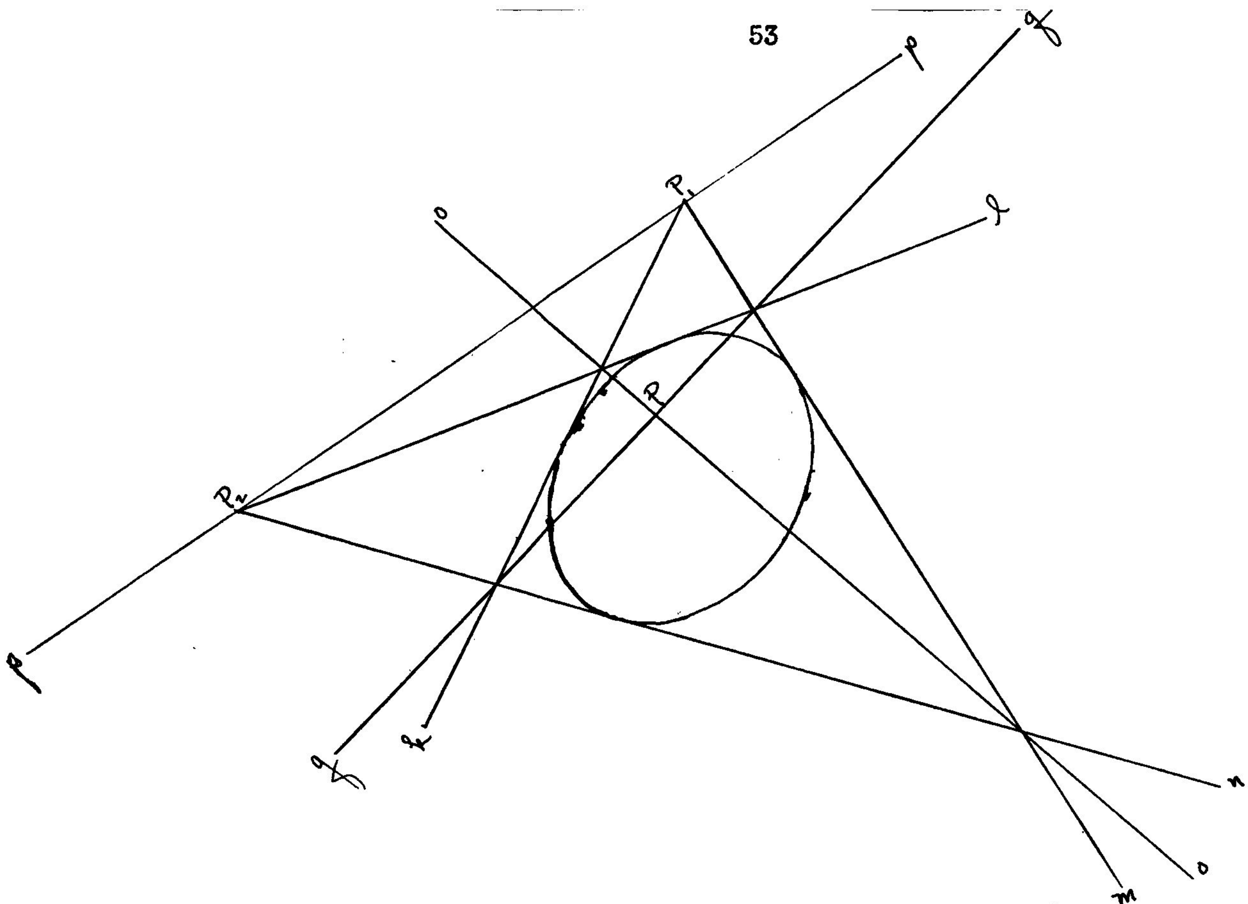

Figure 25.-Pole of a Line with Respect to a Conic

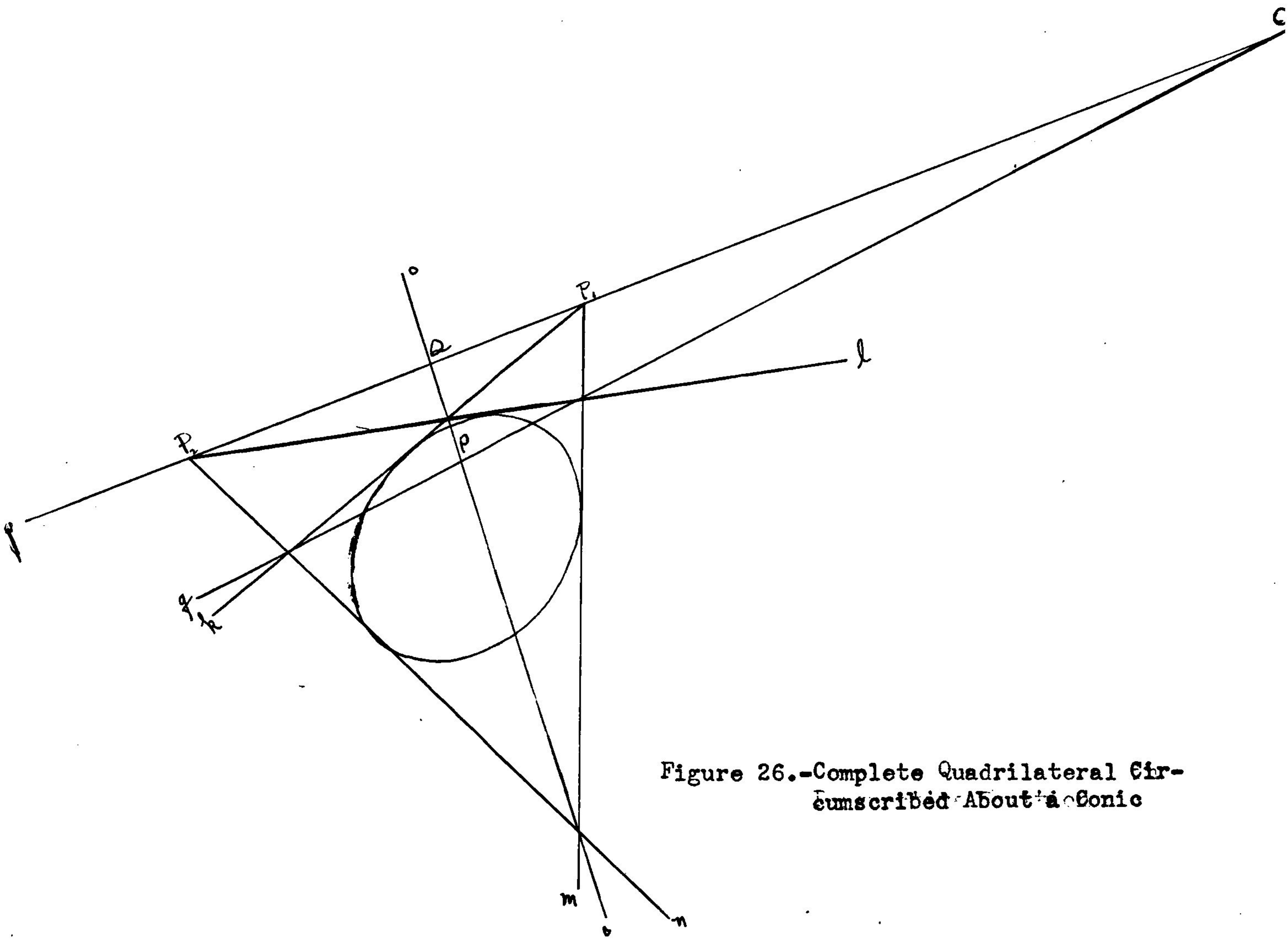

Figure 26.-Complete Quadrilateral Cir-
cumscribed About a Conic

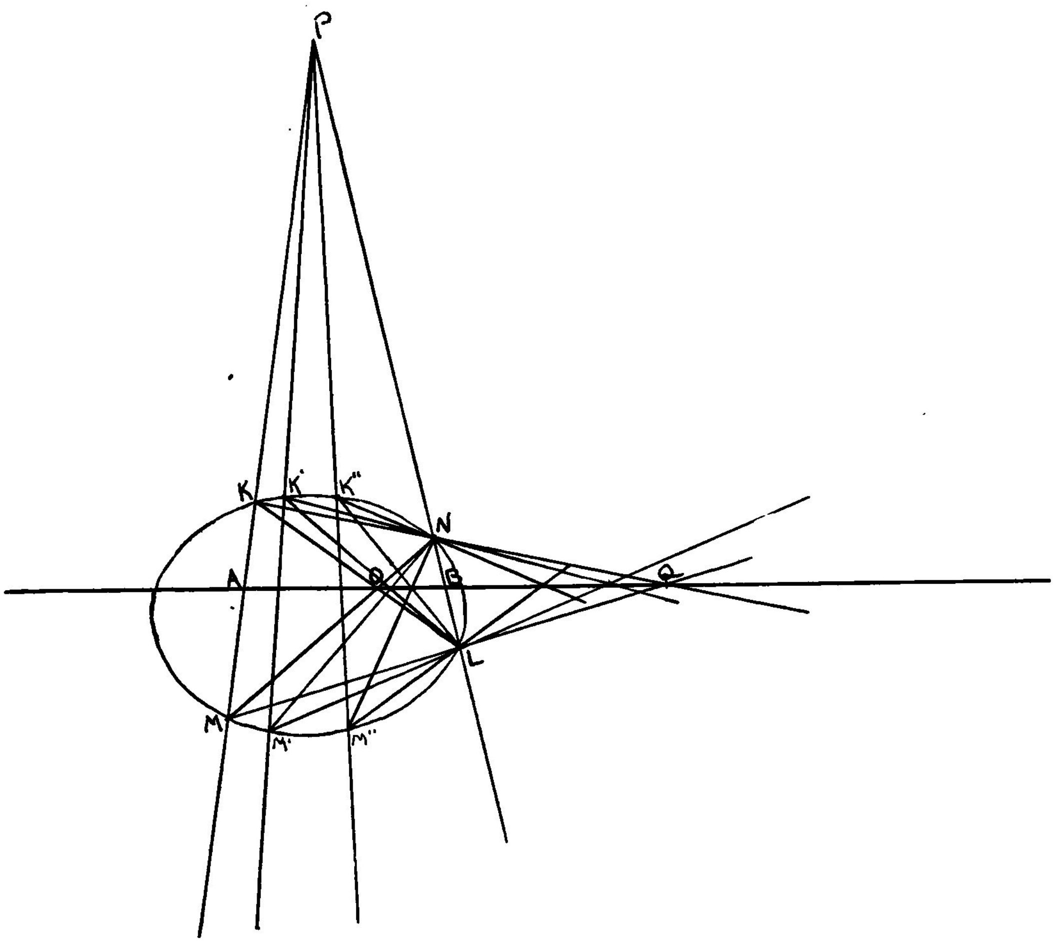

Figure 27.-Polar Line with Respect to a Conic

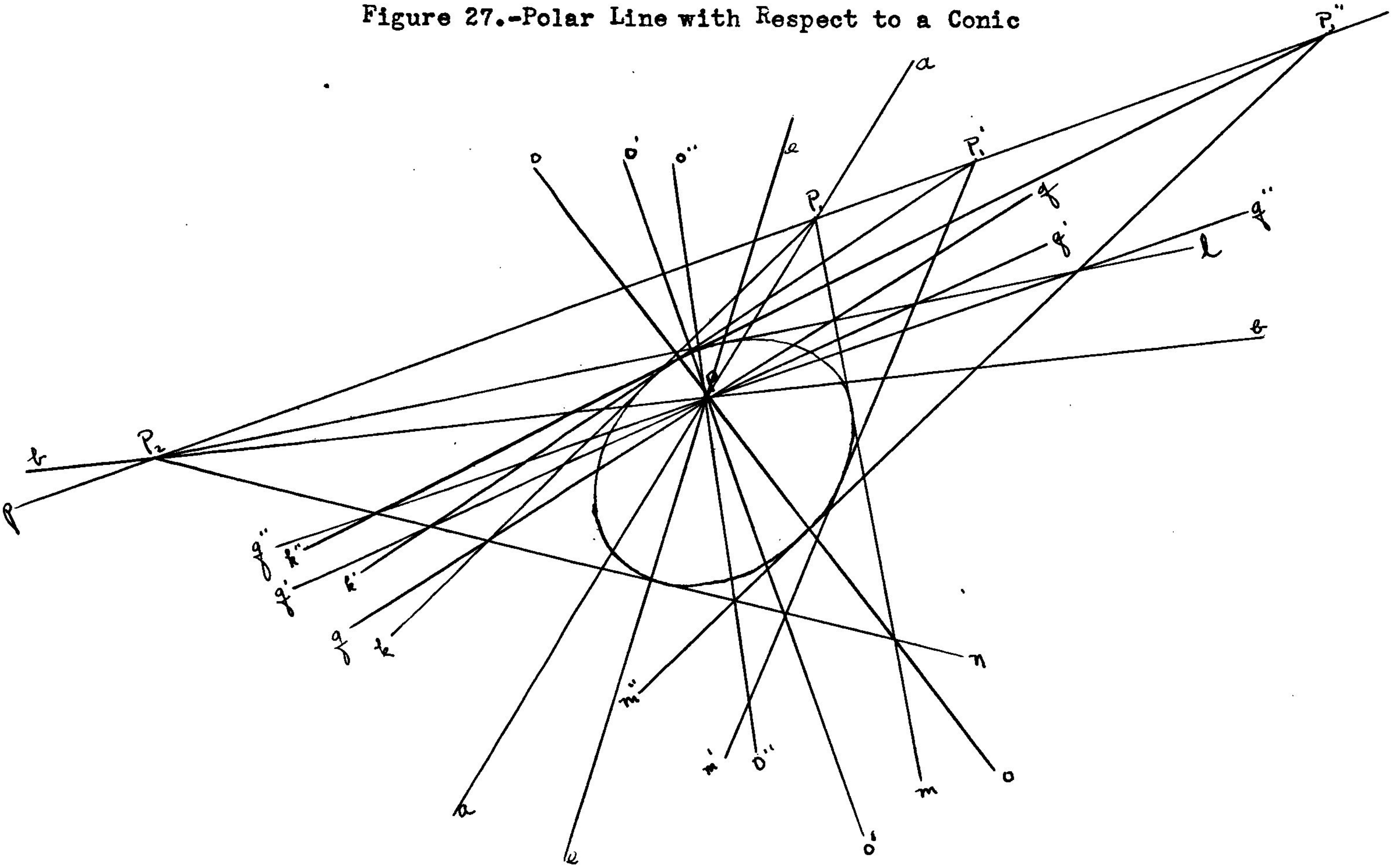

Figure 28.-Pole with Respect to a Conic

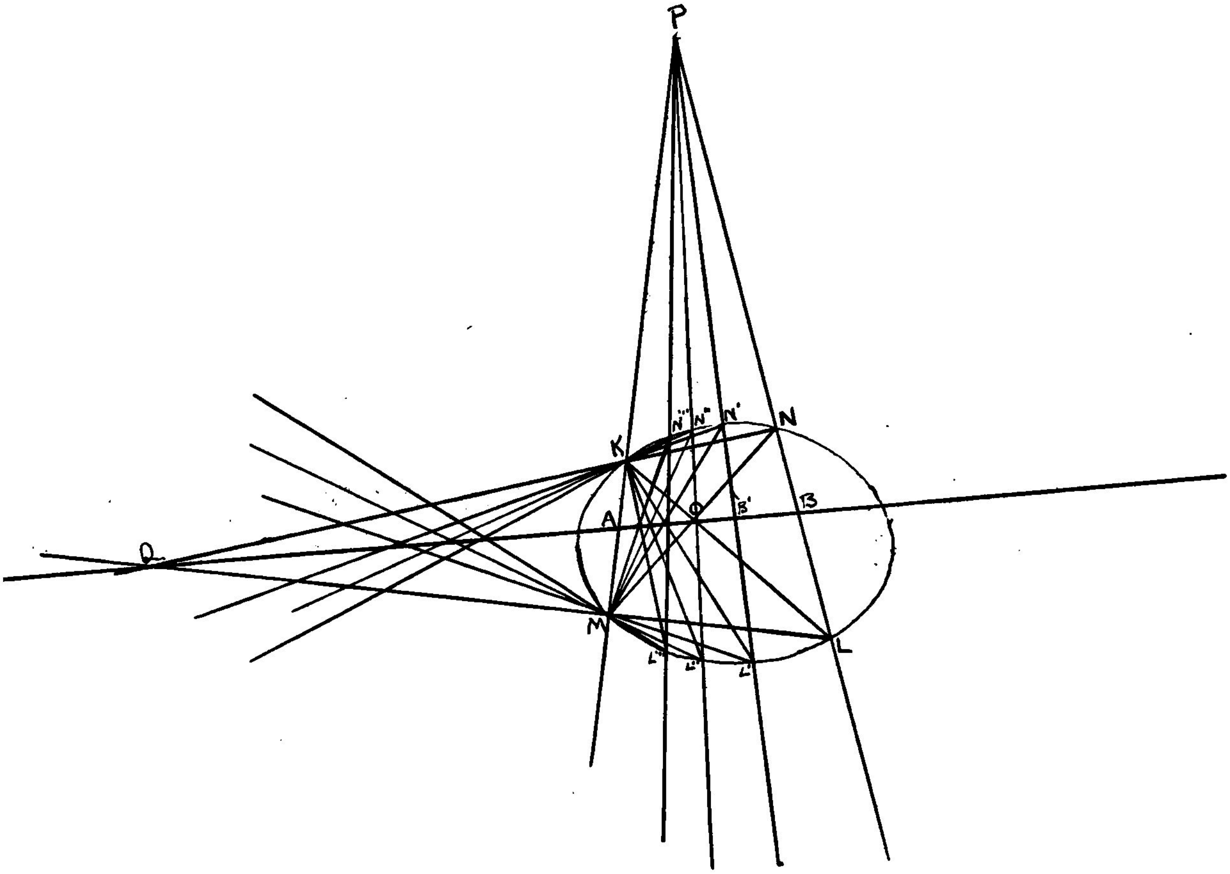

Figure 29.-Polar Line of a Conic

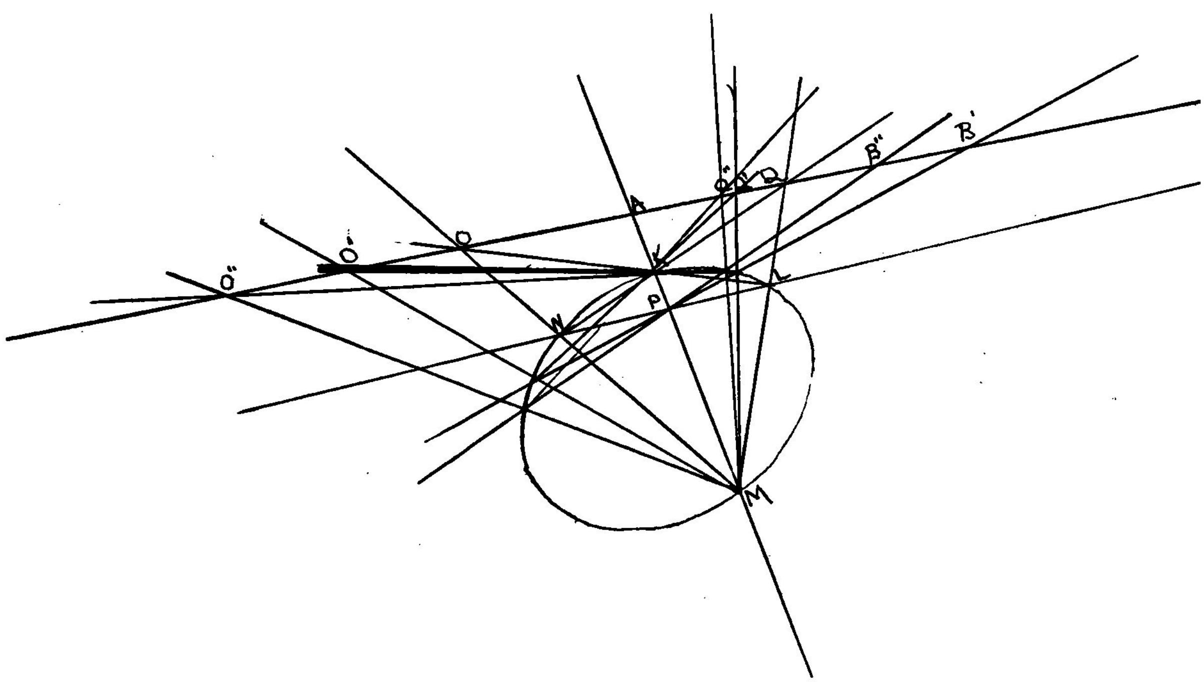

Figure 30.-Polar Line of a Conic

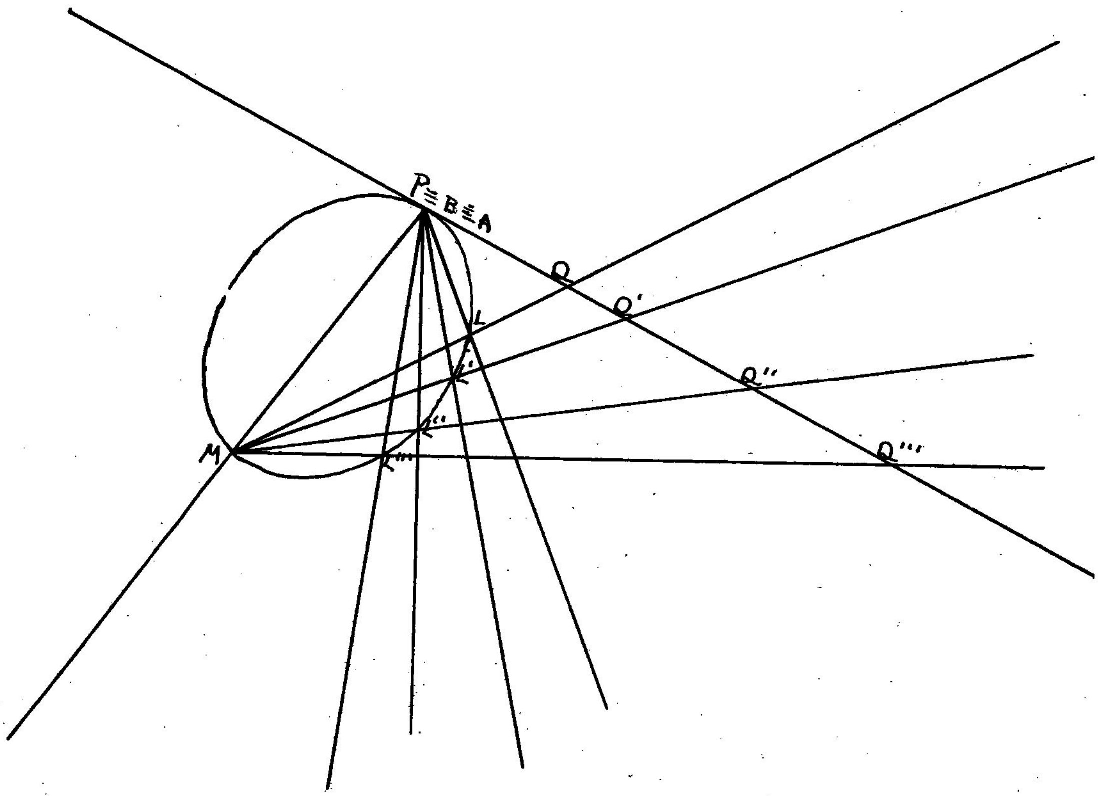

Figure 31.-Polar Line of a Conic

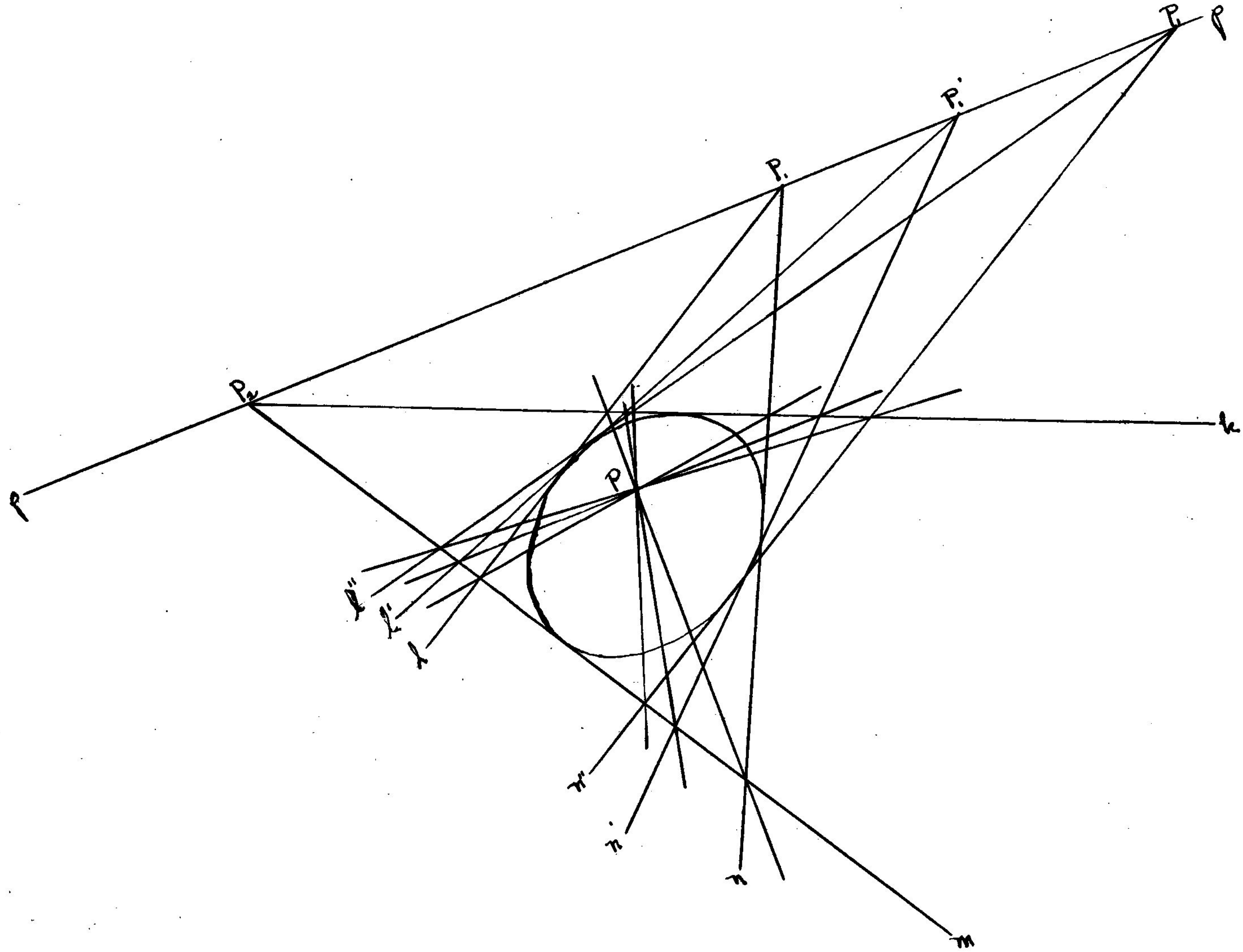

Figure 32.-Pole of a Conic

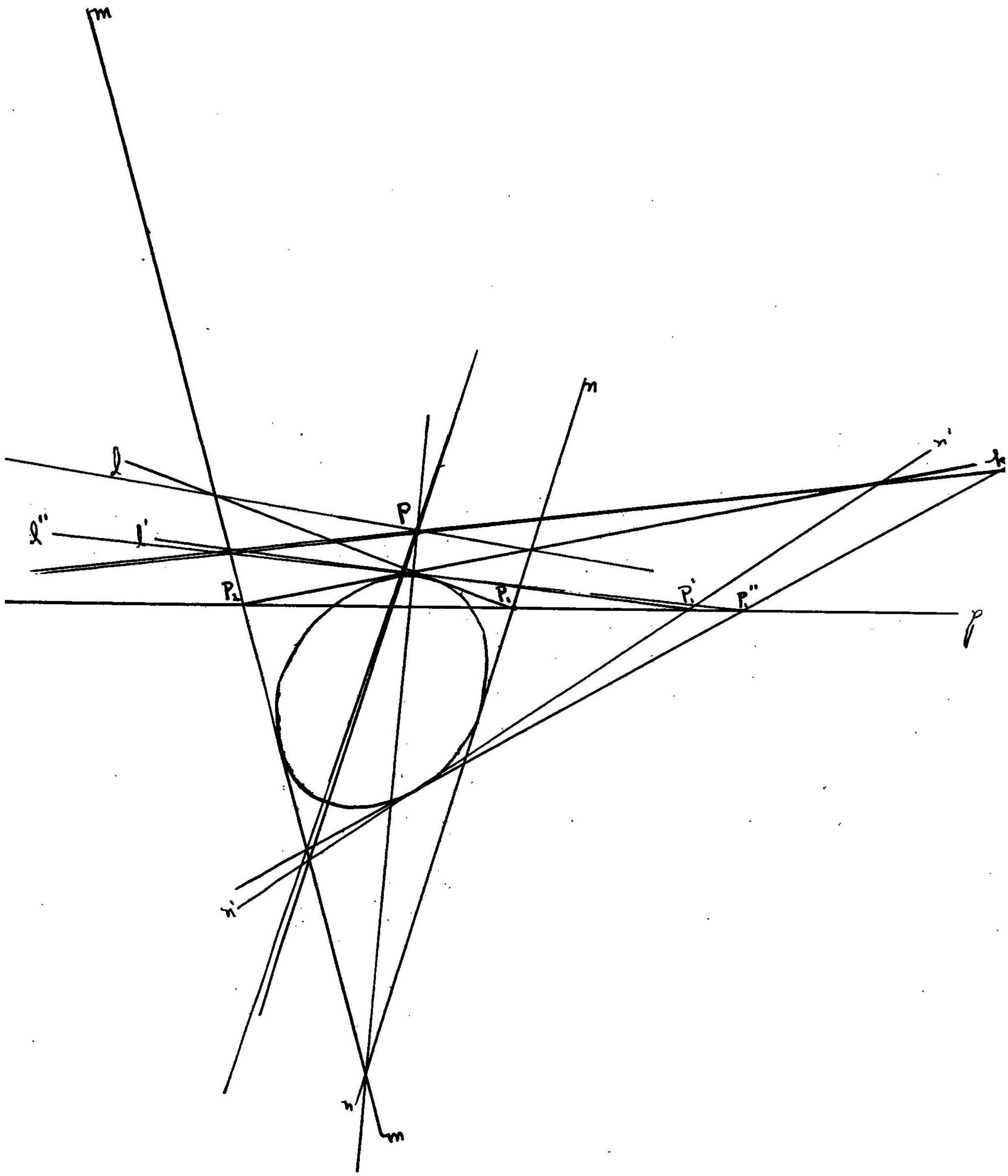

Figure 33.-Pole of a Conic

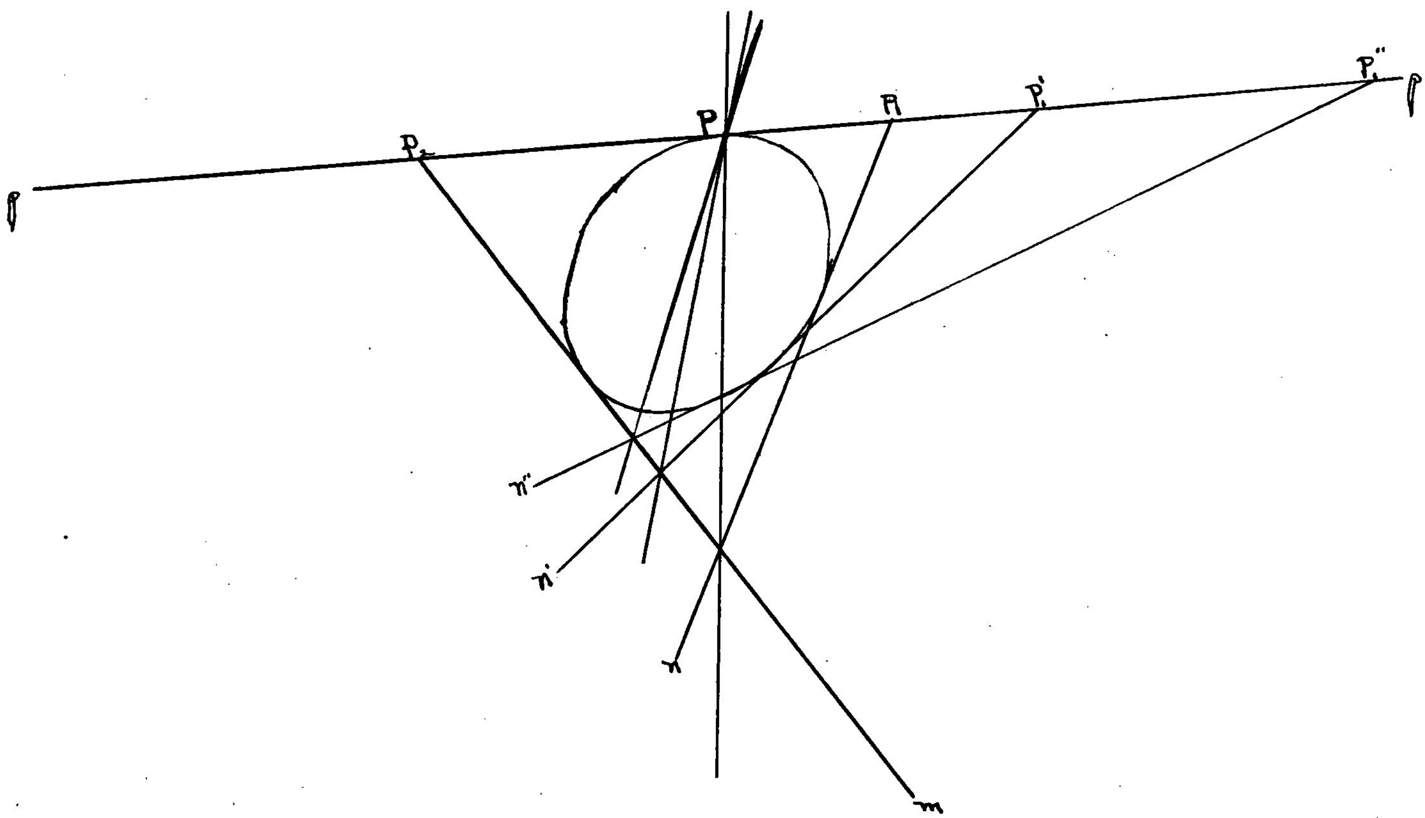

Figure 34.-Pole of a Conic

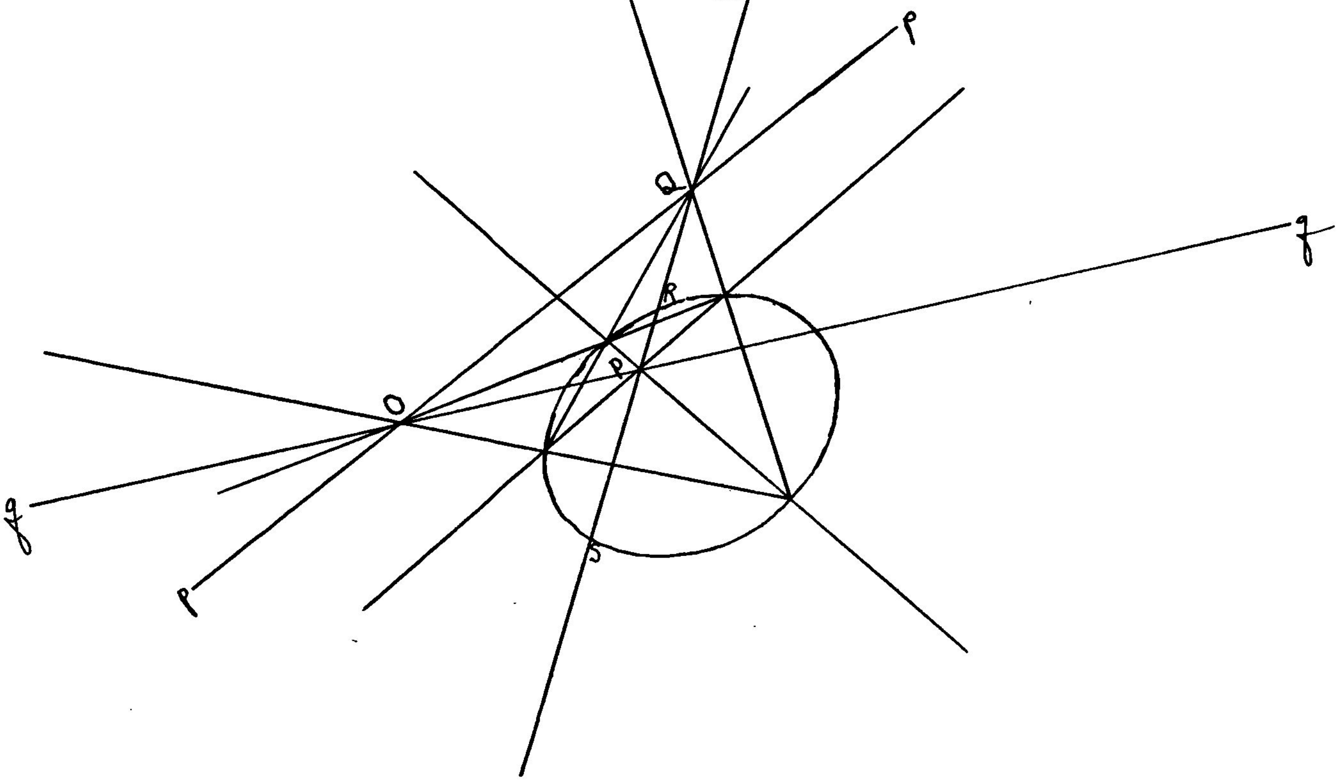

Figure 35.-Polar Line of a Conic

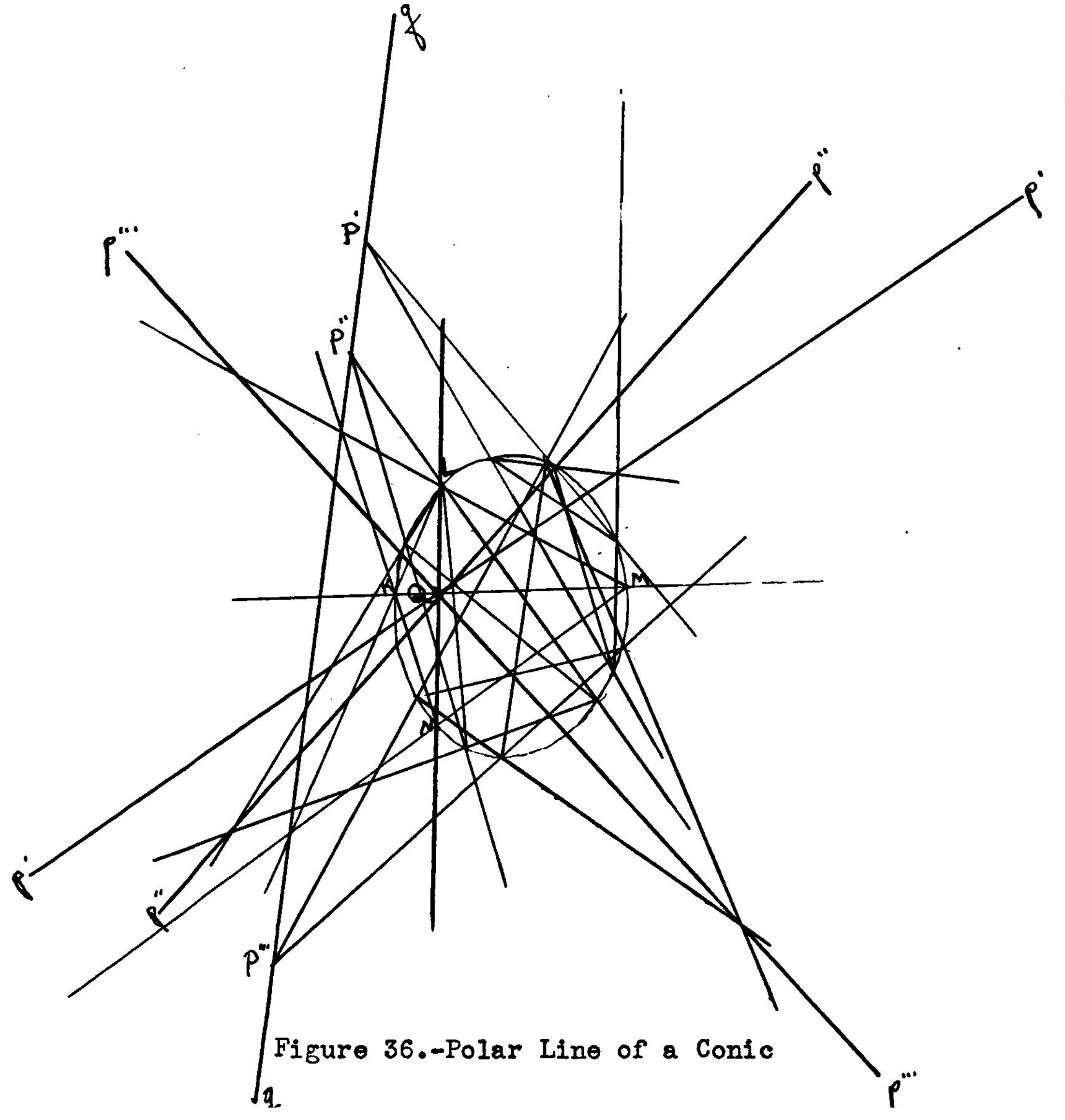

Figure 36.-Polar Line of a Conic

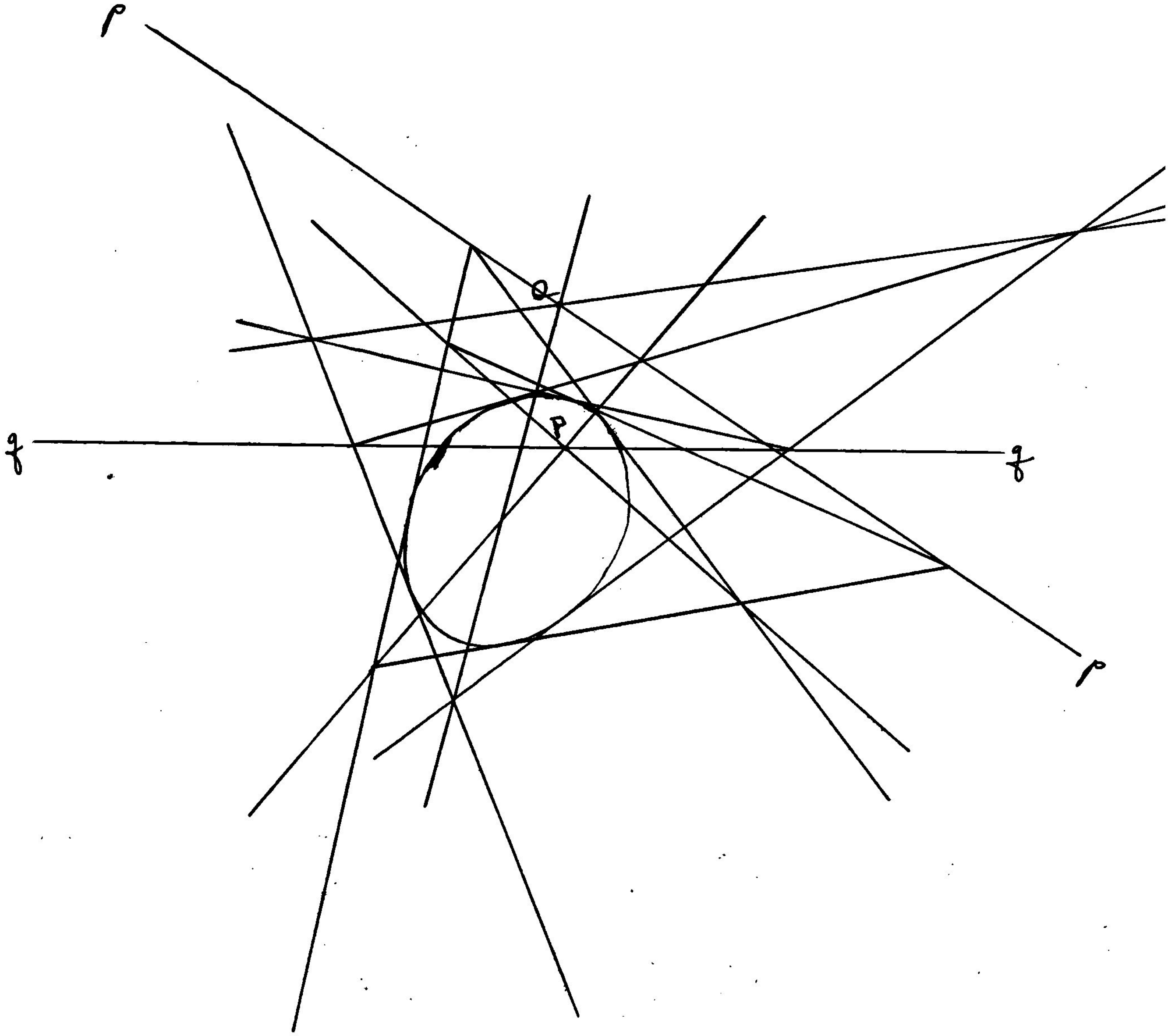

Figure 37.-Pole of a Conic

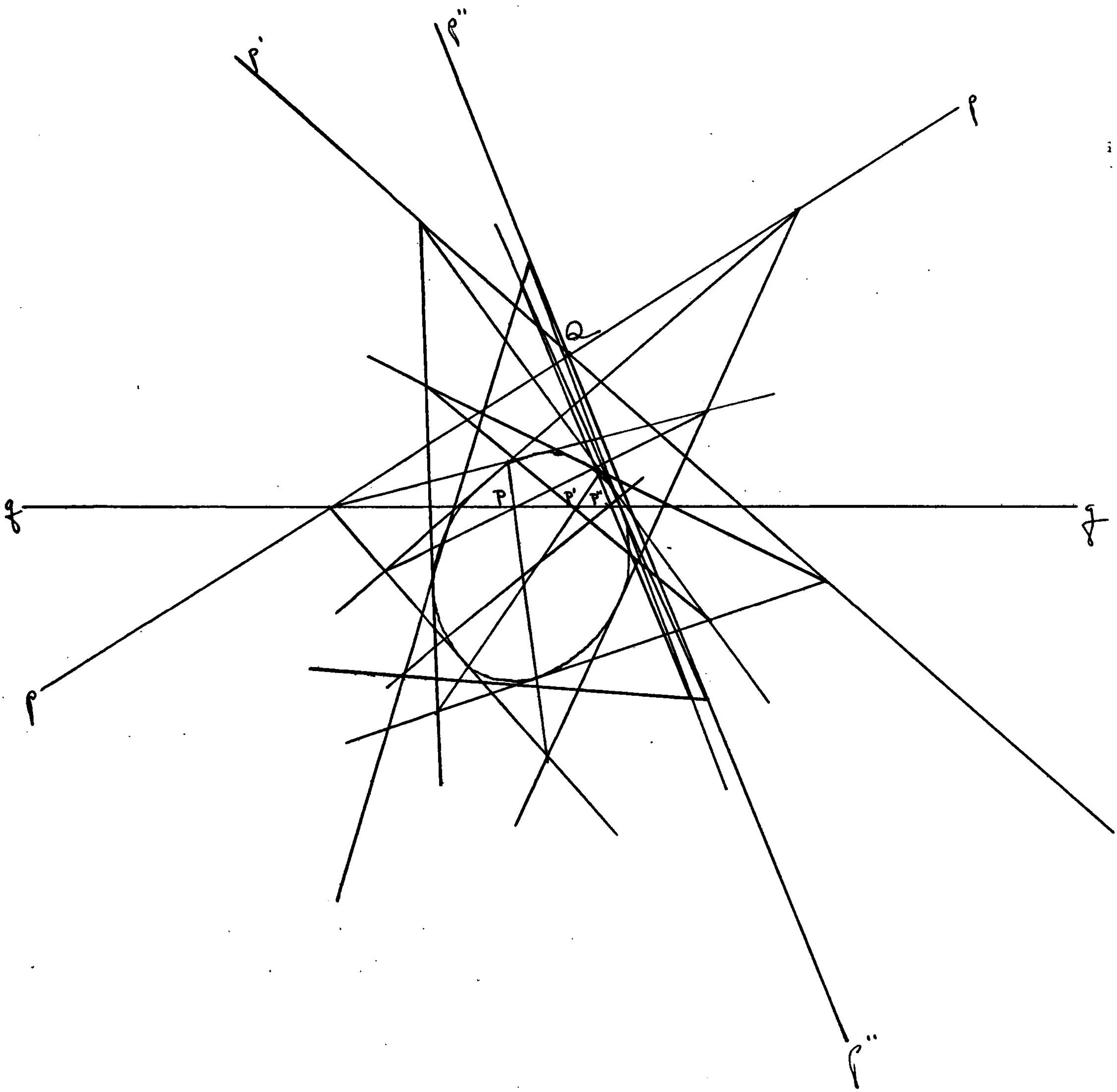

Figure 38.-Pole of a Conic

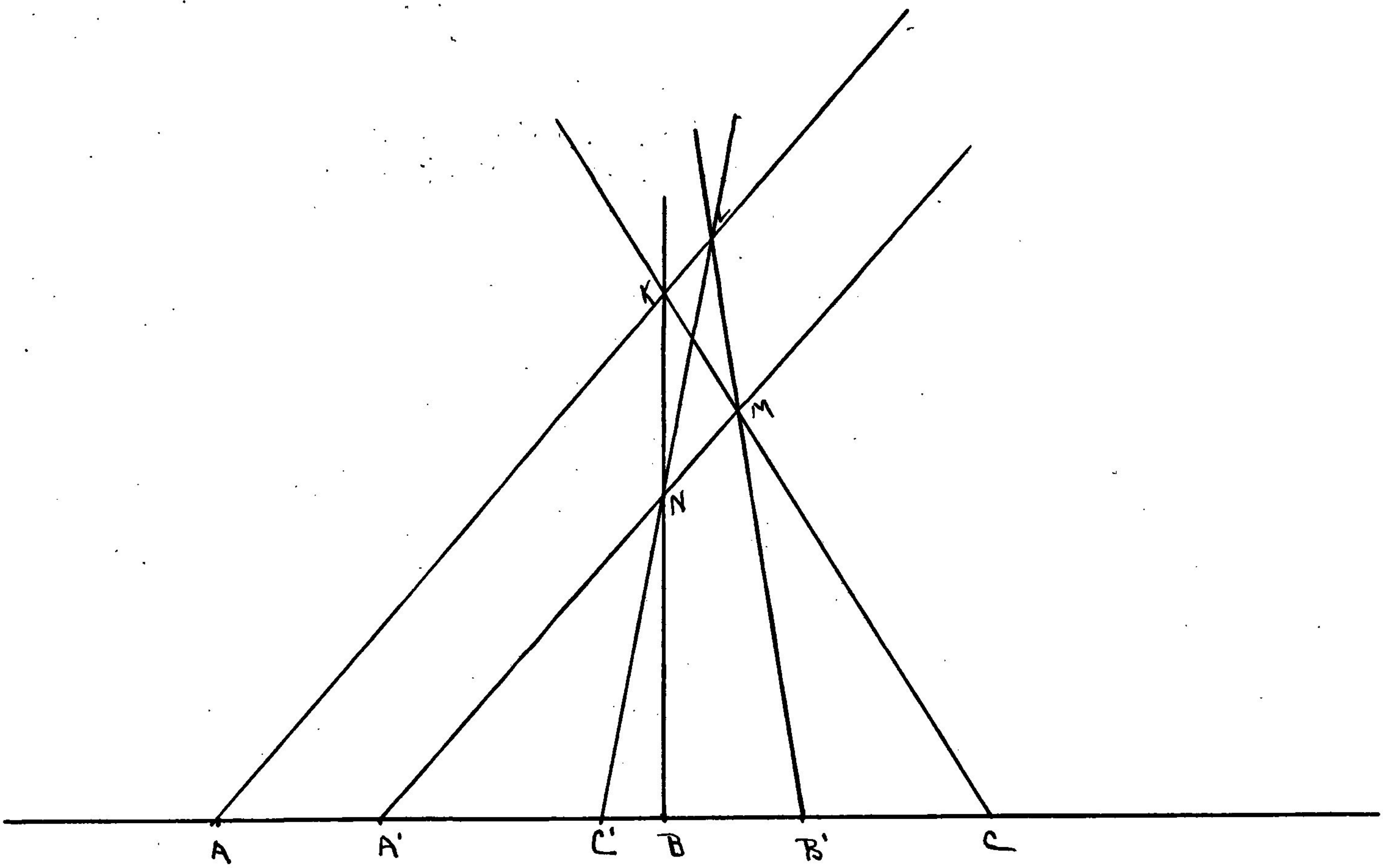

Figure 39.-Involution on a Line

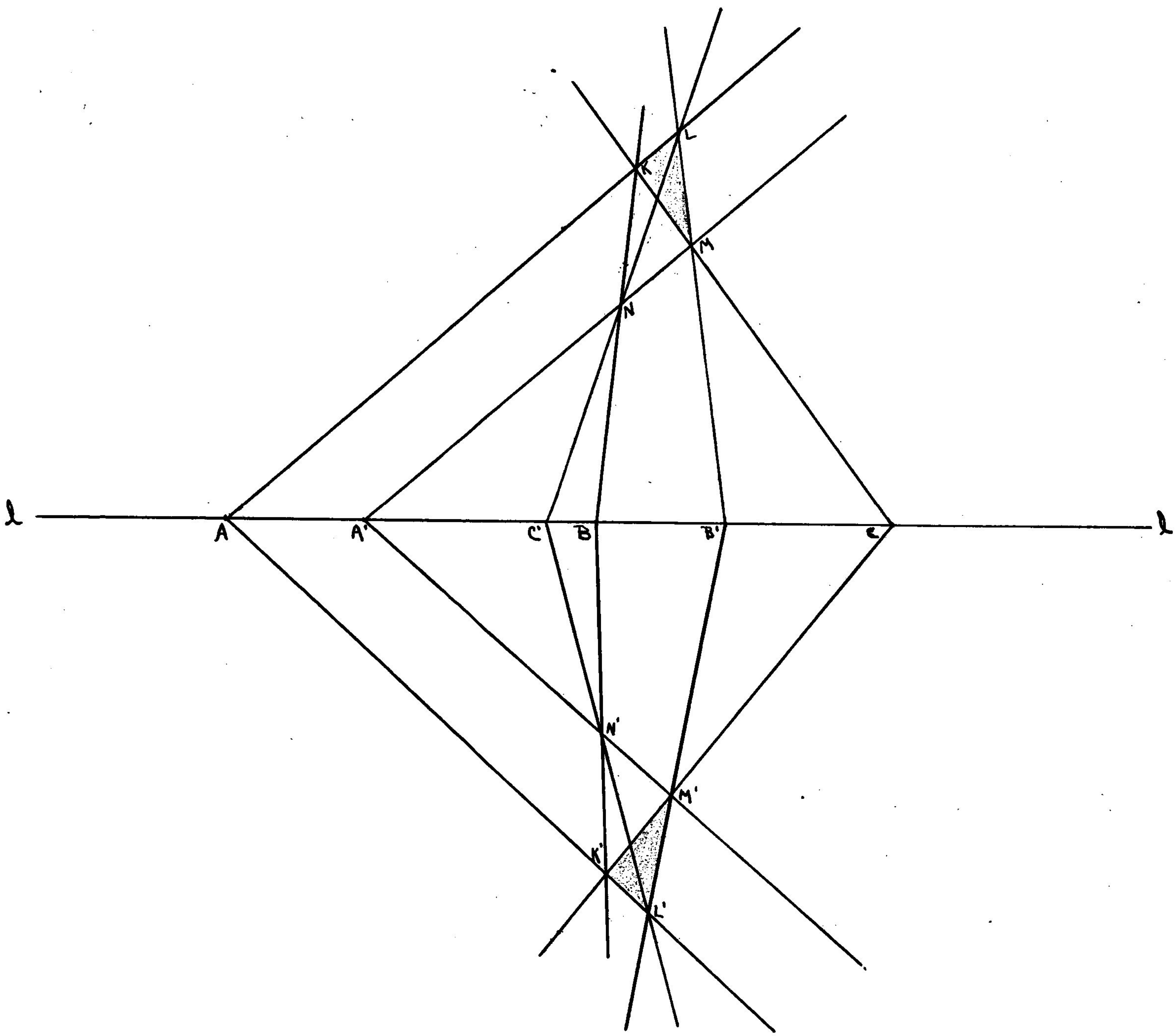

Figure 40.-Involution on a Line

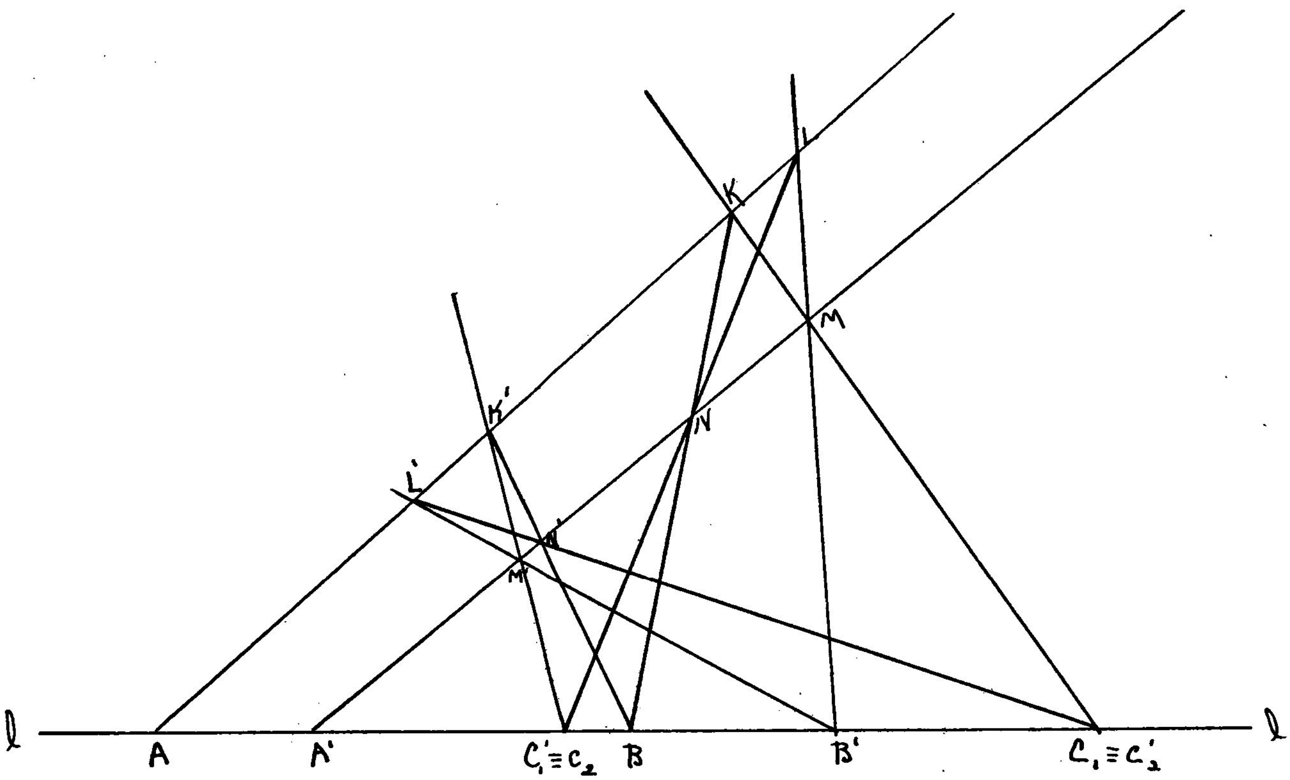

Figure 41.-Involution on a Line

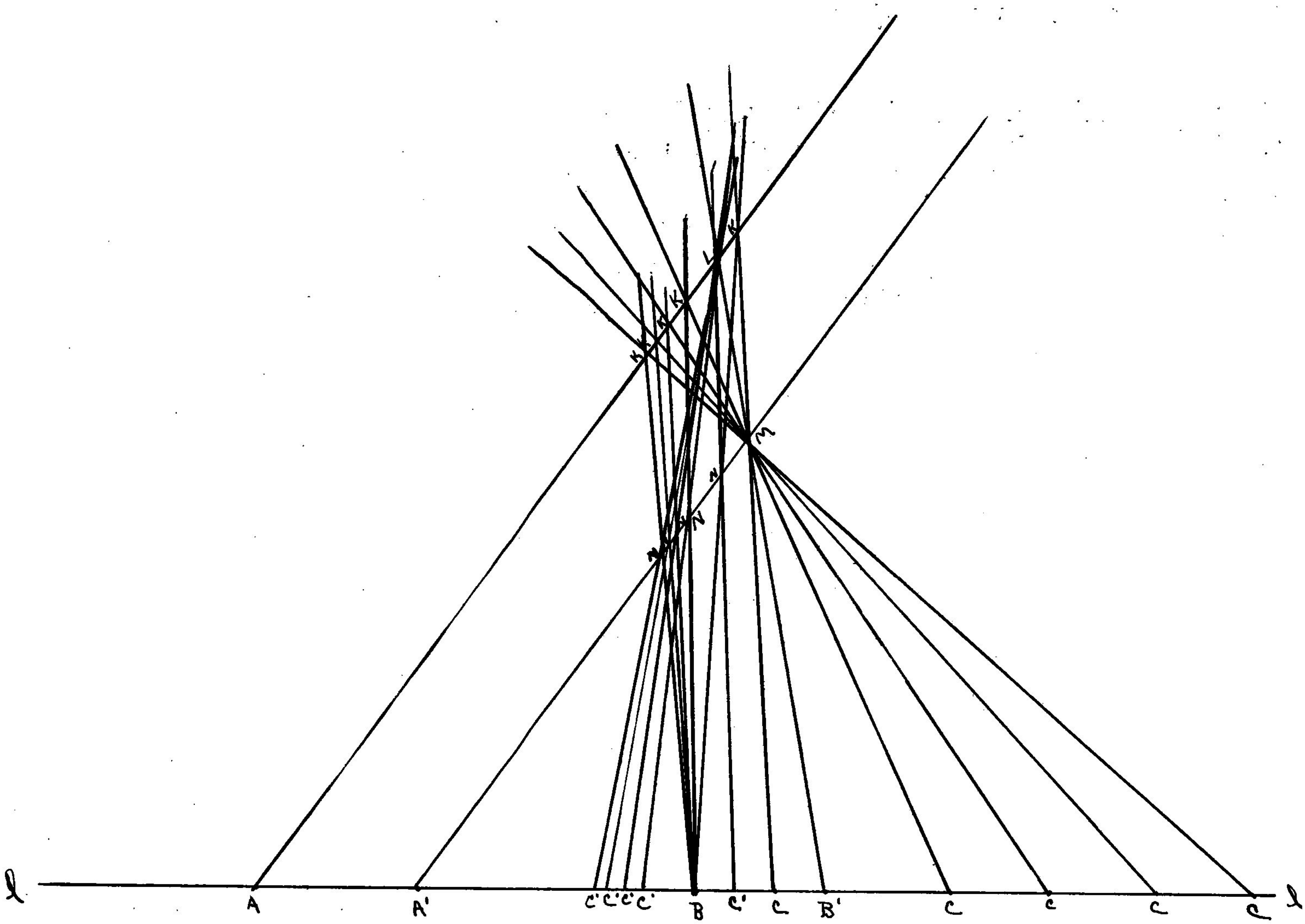

Figure 42.-Involution on a Line

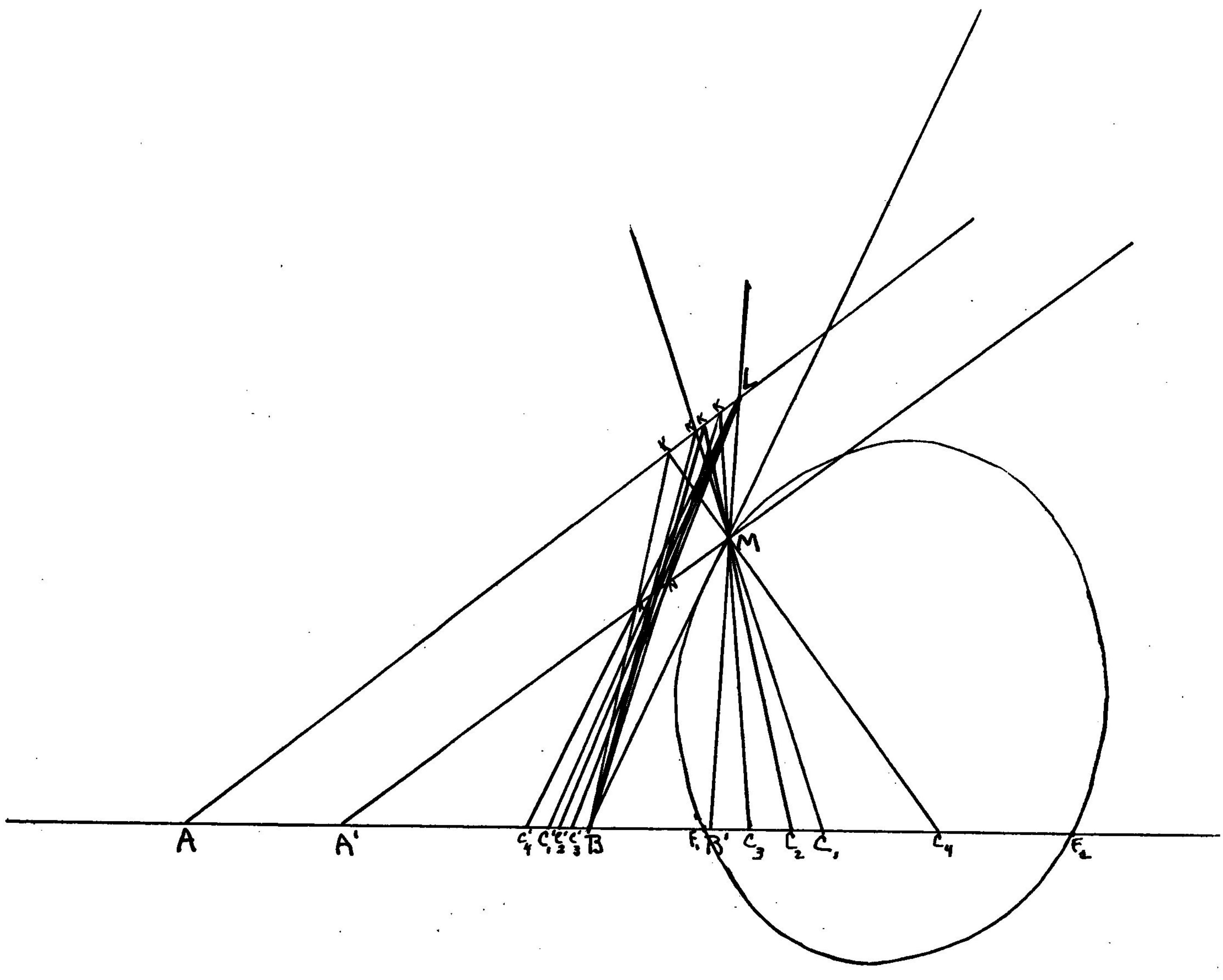

Figure 43.-Involution on a Line

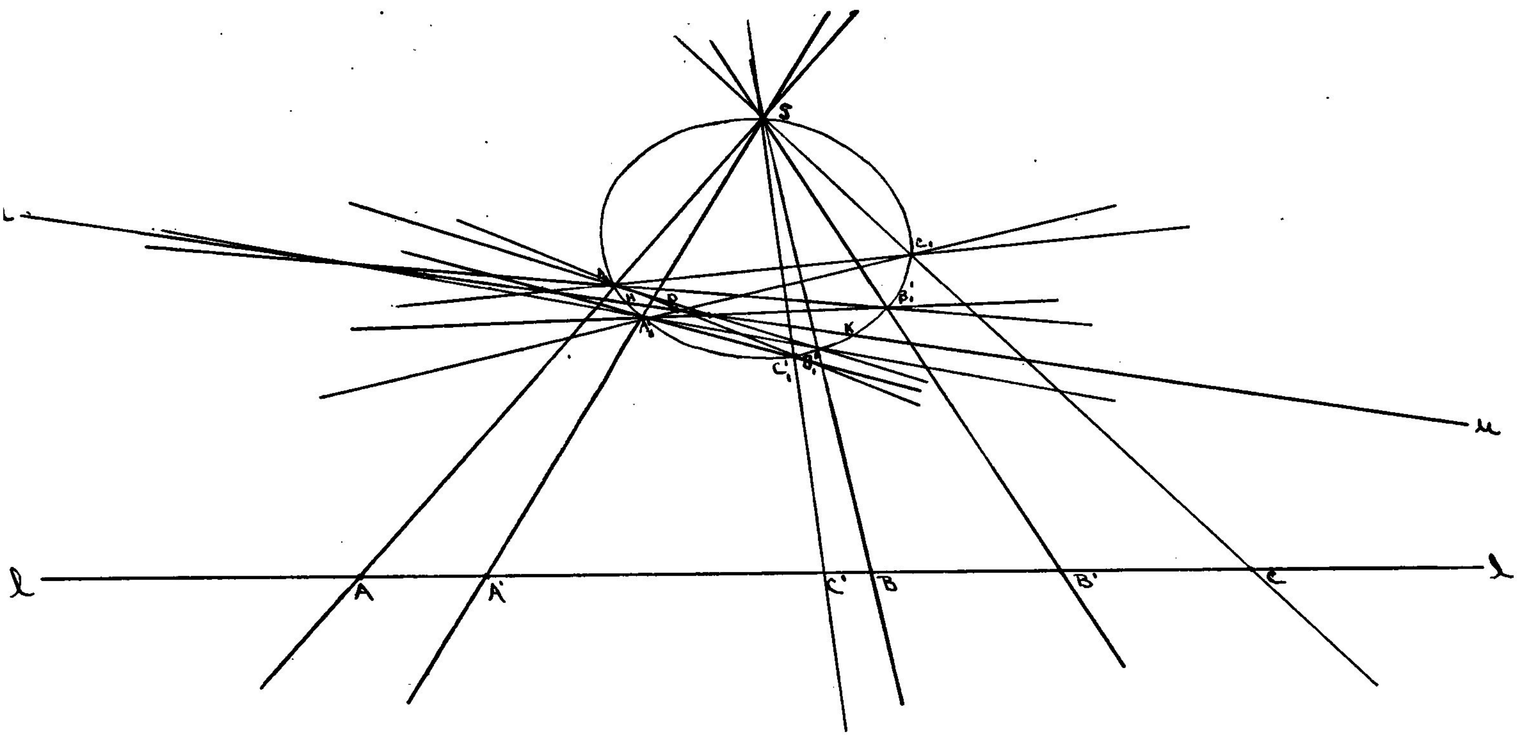

Figure 44.-Steiner's Construction of Involution on a Line

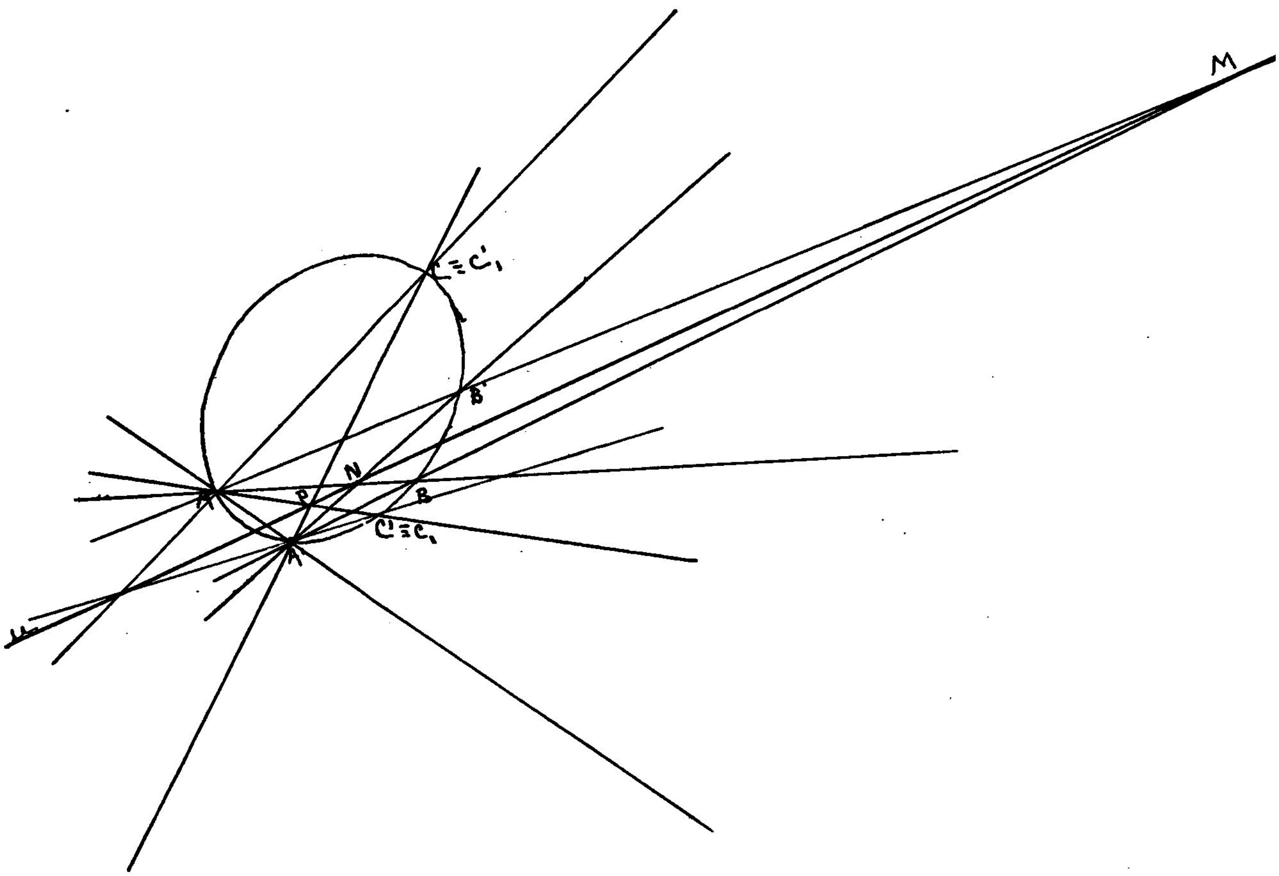

Figure 45.-Involution on a Conic

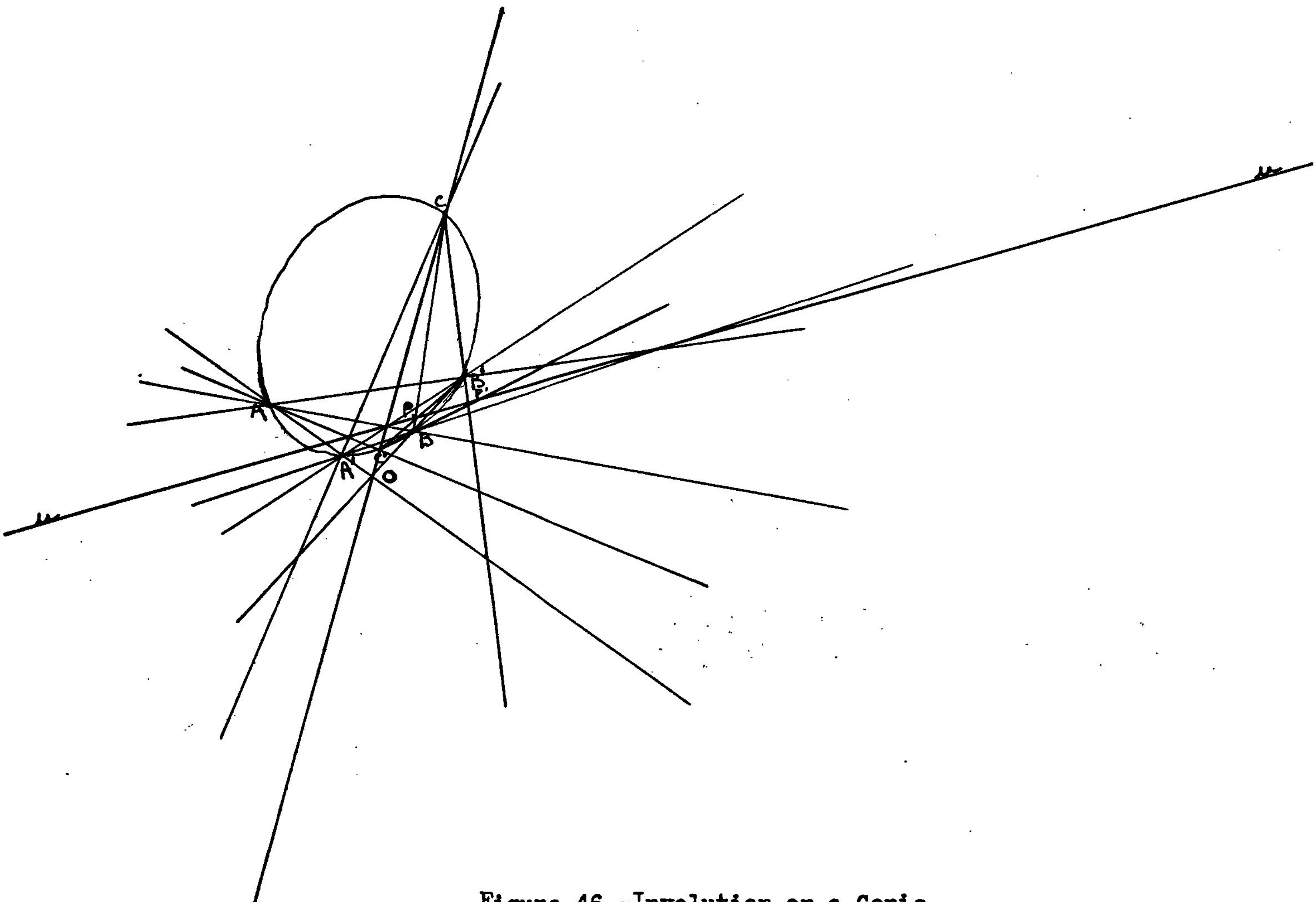

Figure 46.-Involution on a Conic

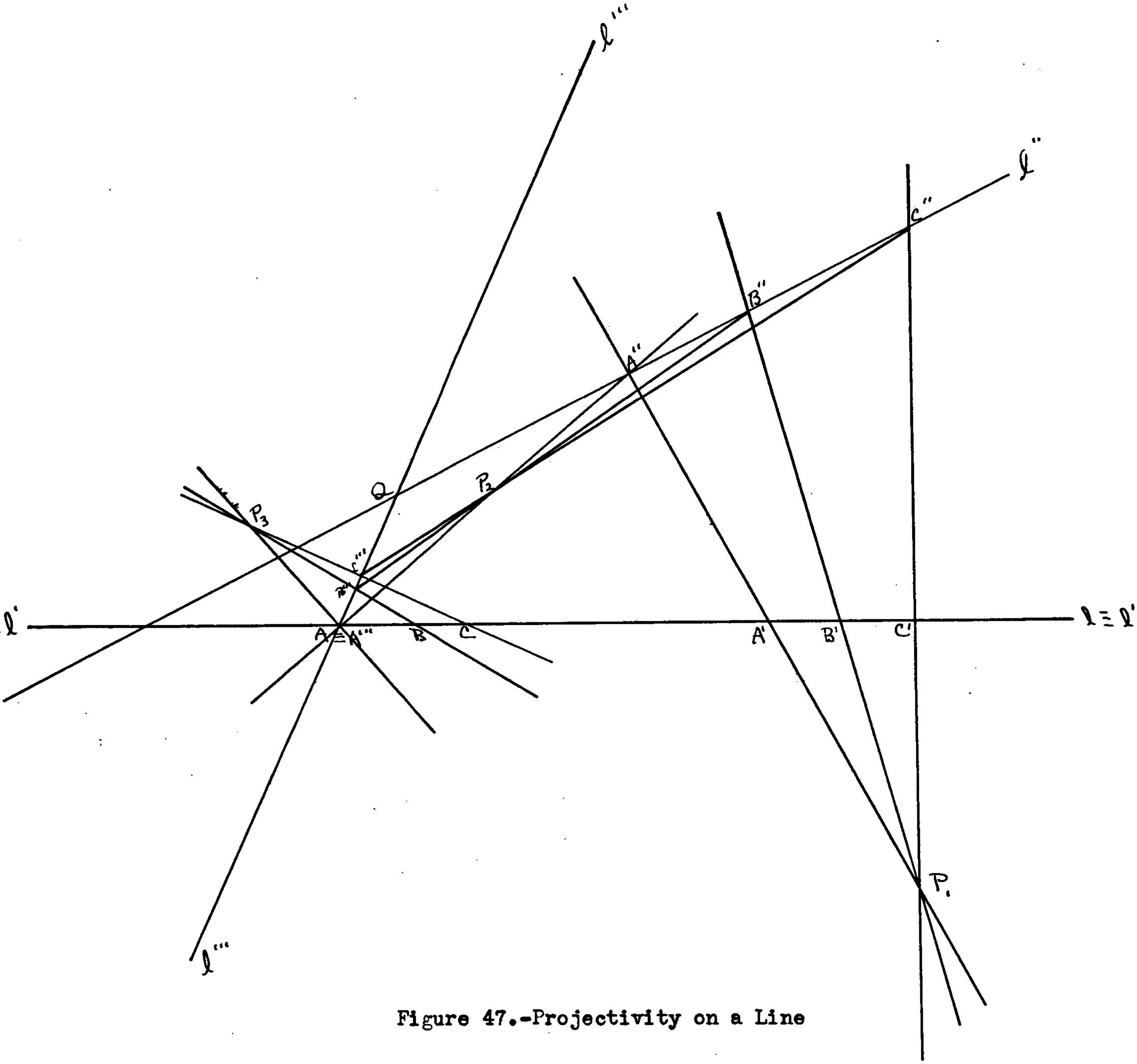

Figure 47.-Projectivity on a Line

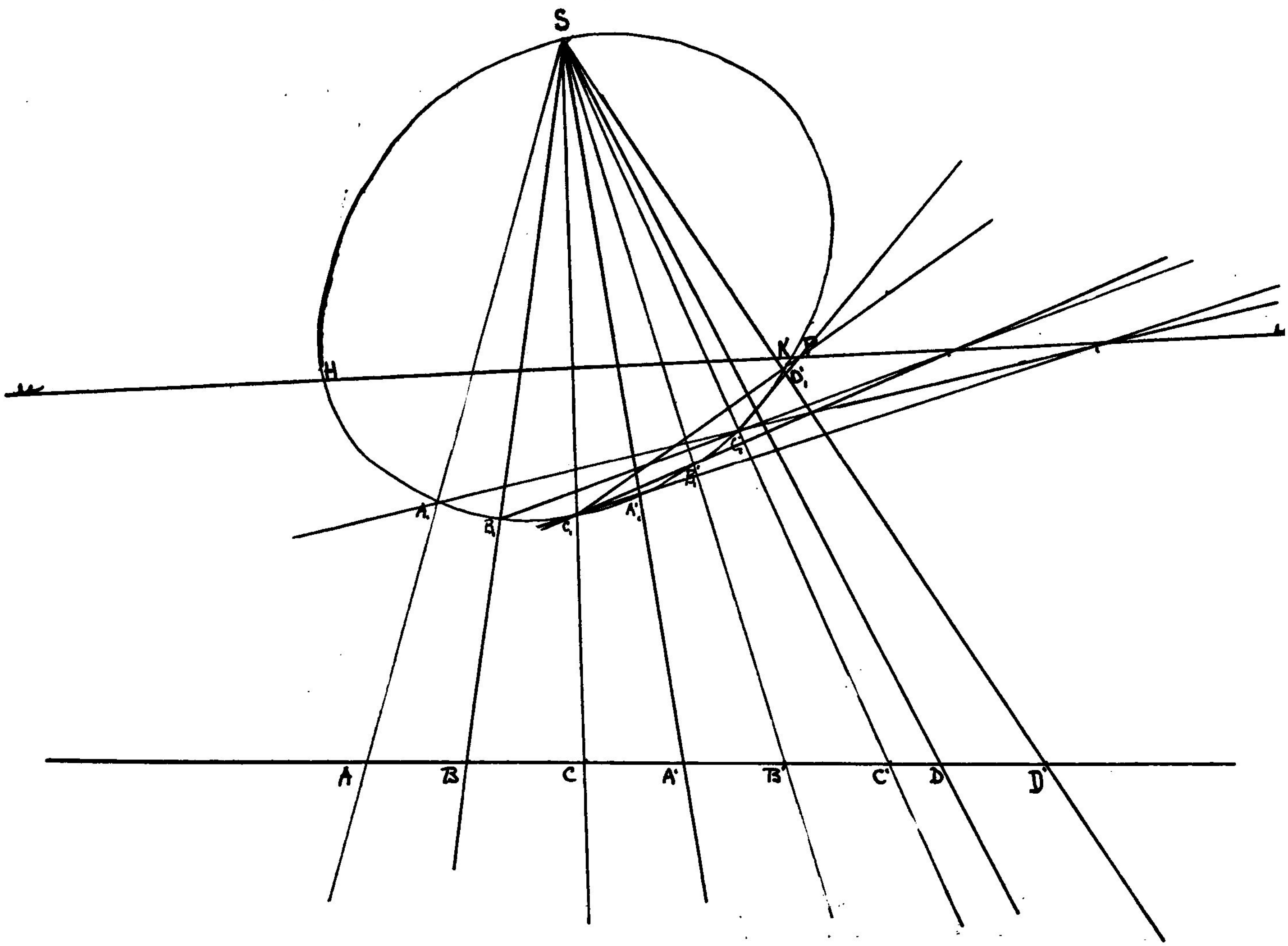

Figure 48.-Steiner's Construction of a Projectivity on a Line

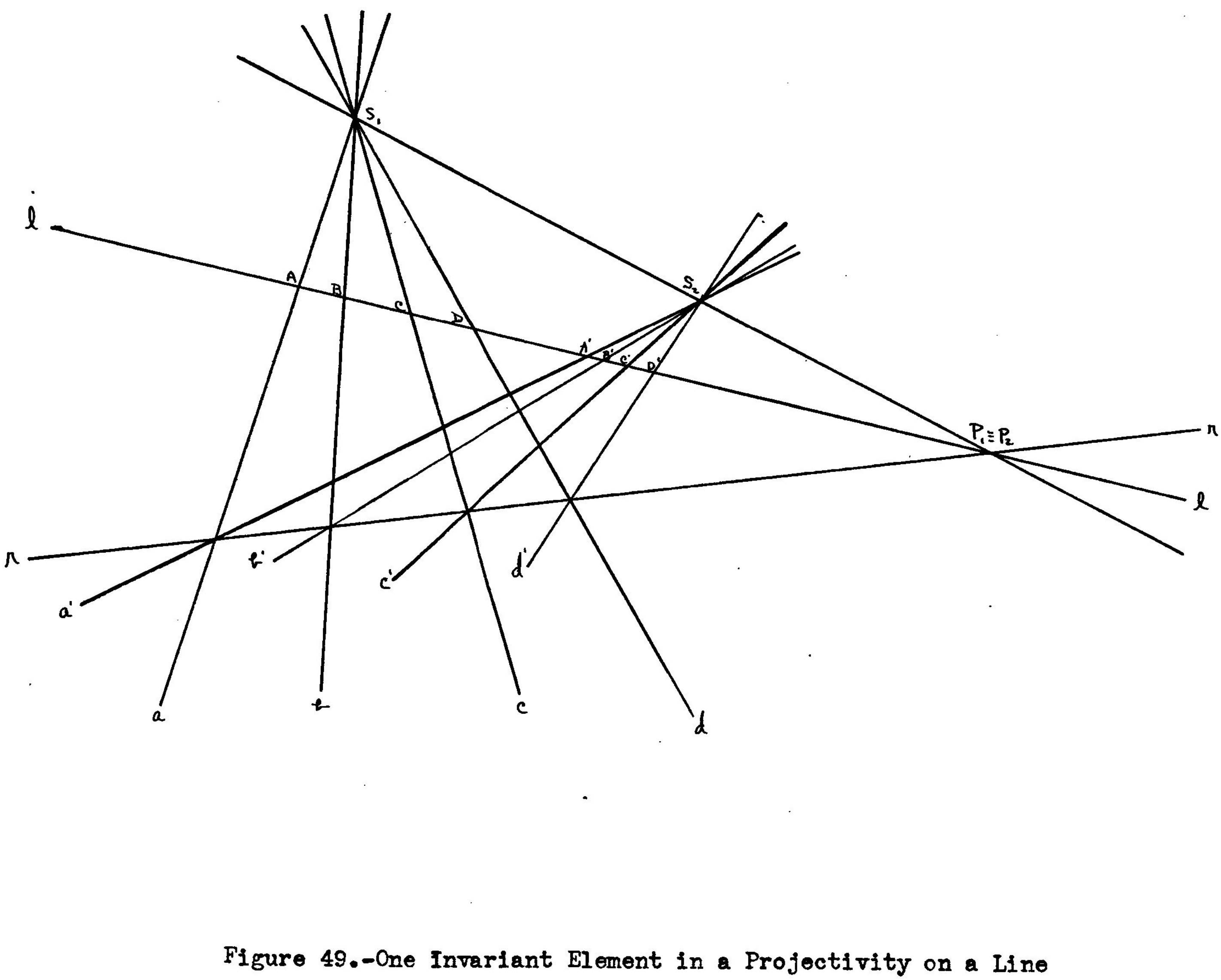

Figure 49.-One Invariant Element in a Projectivity on a Line

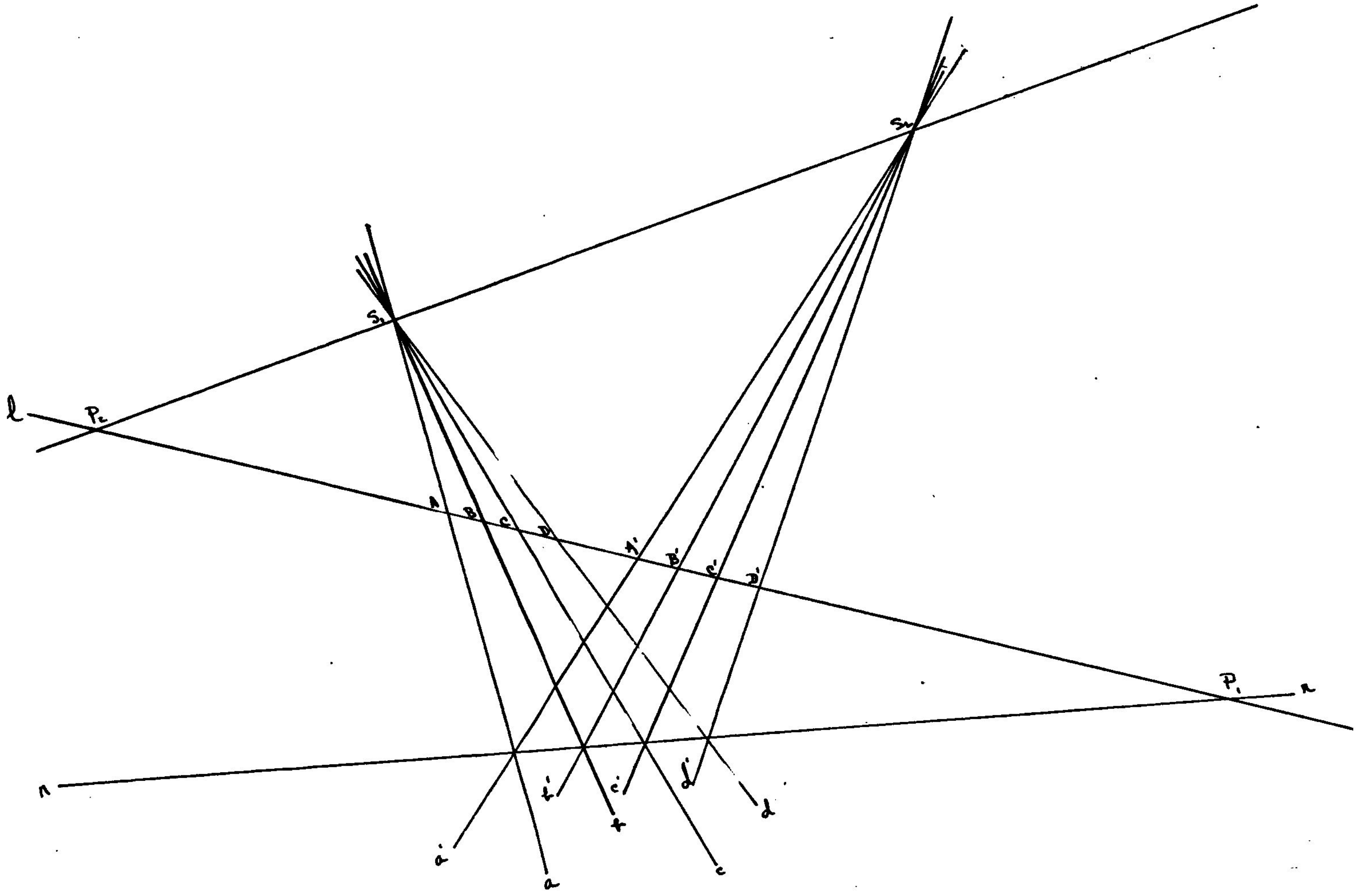

Figure 50.-Two Invariant Elements in a Projectivity on a Line

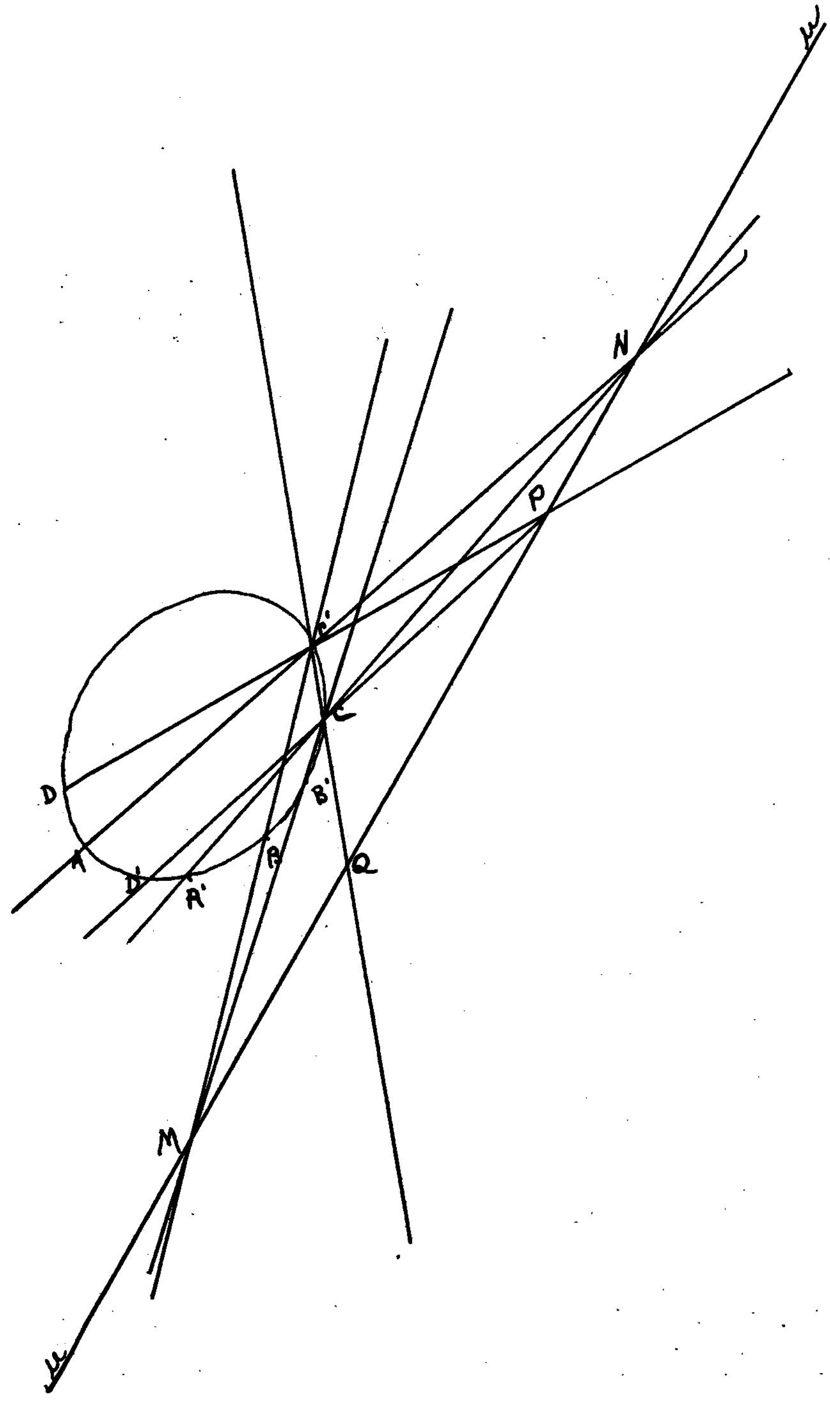

Figure 51.-Projectivity on a Conic

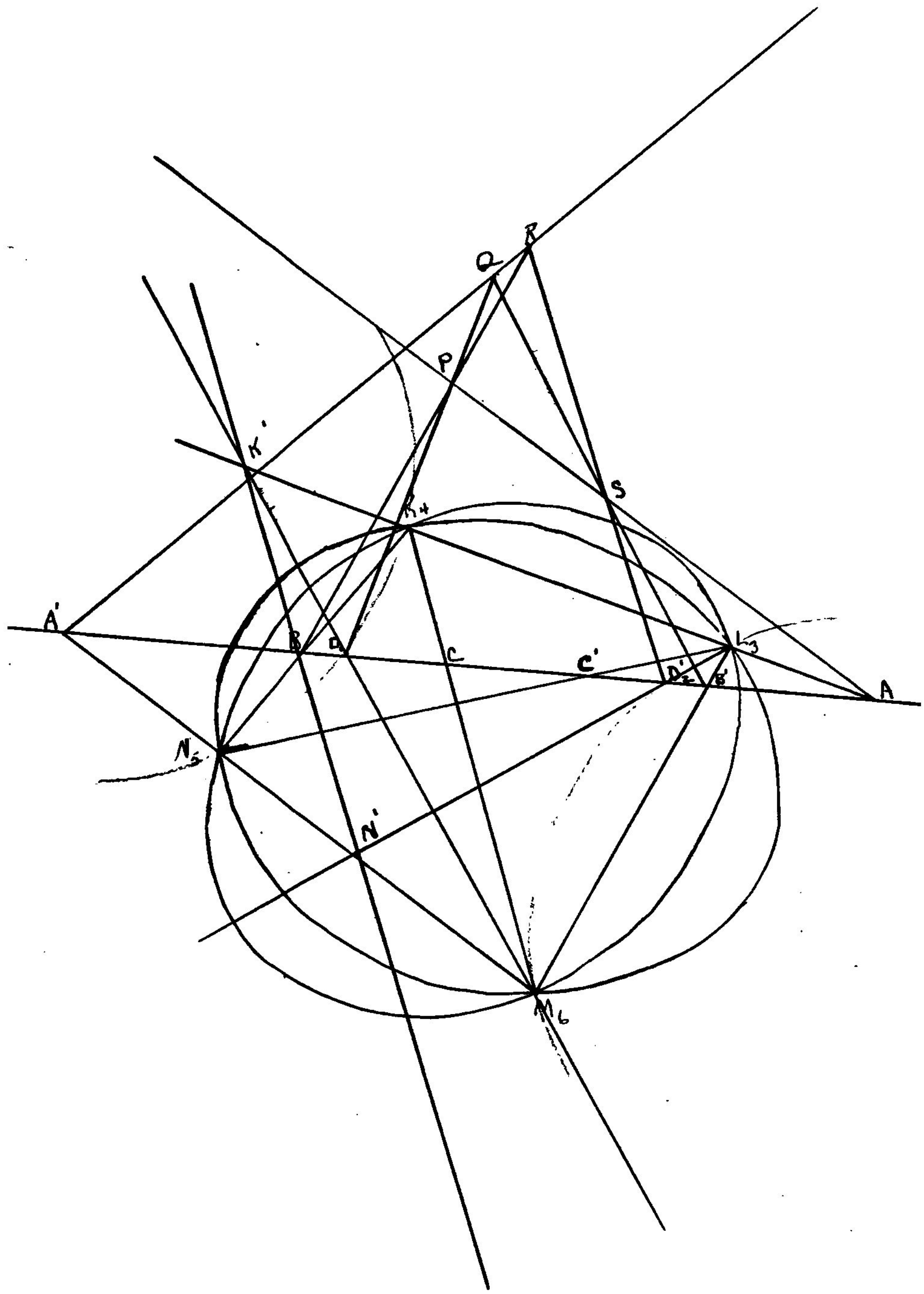

Figure 52.-Pencil of Conics

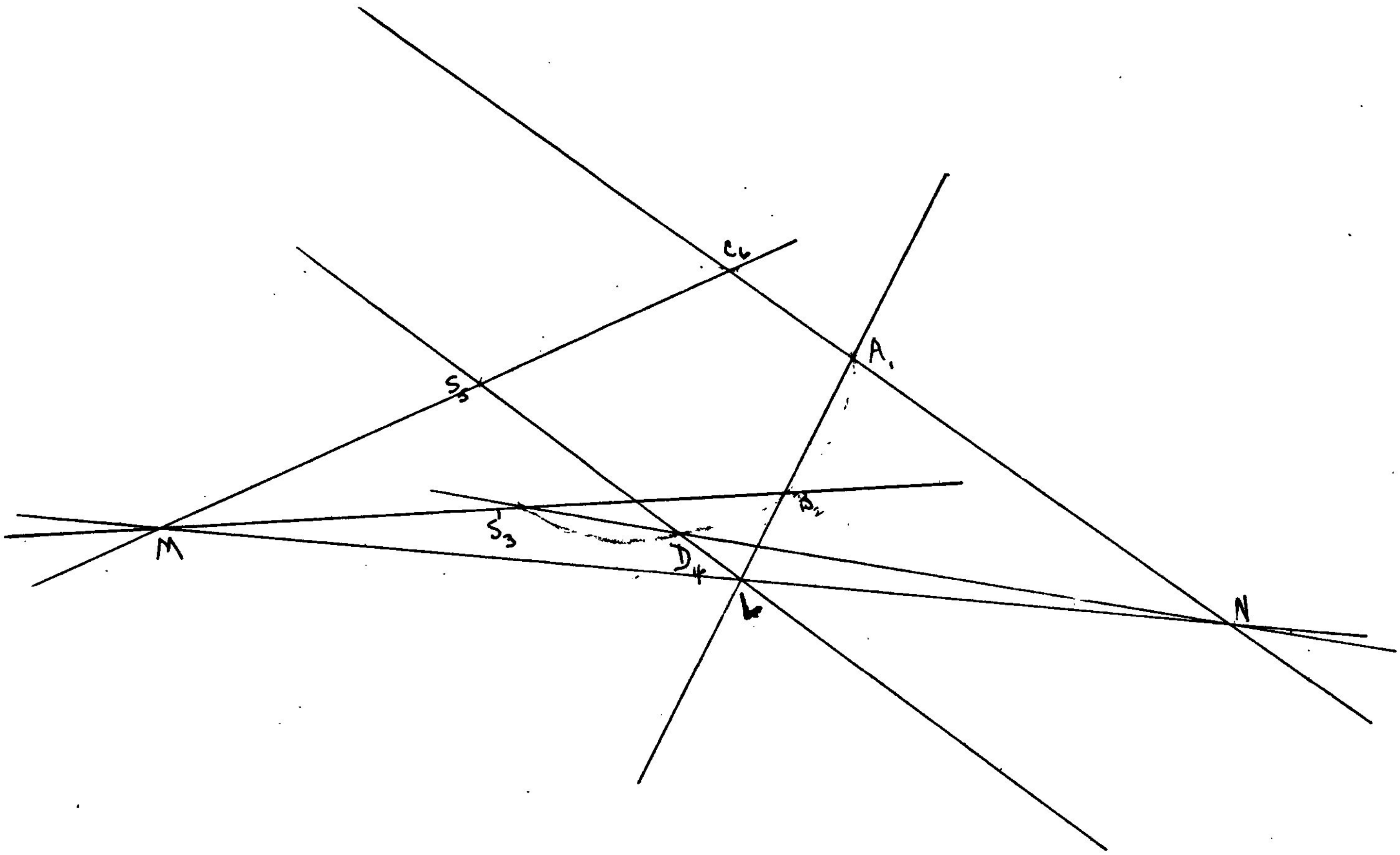

Figure 53.-Point Conic

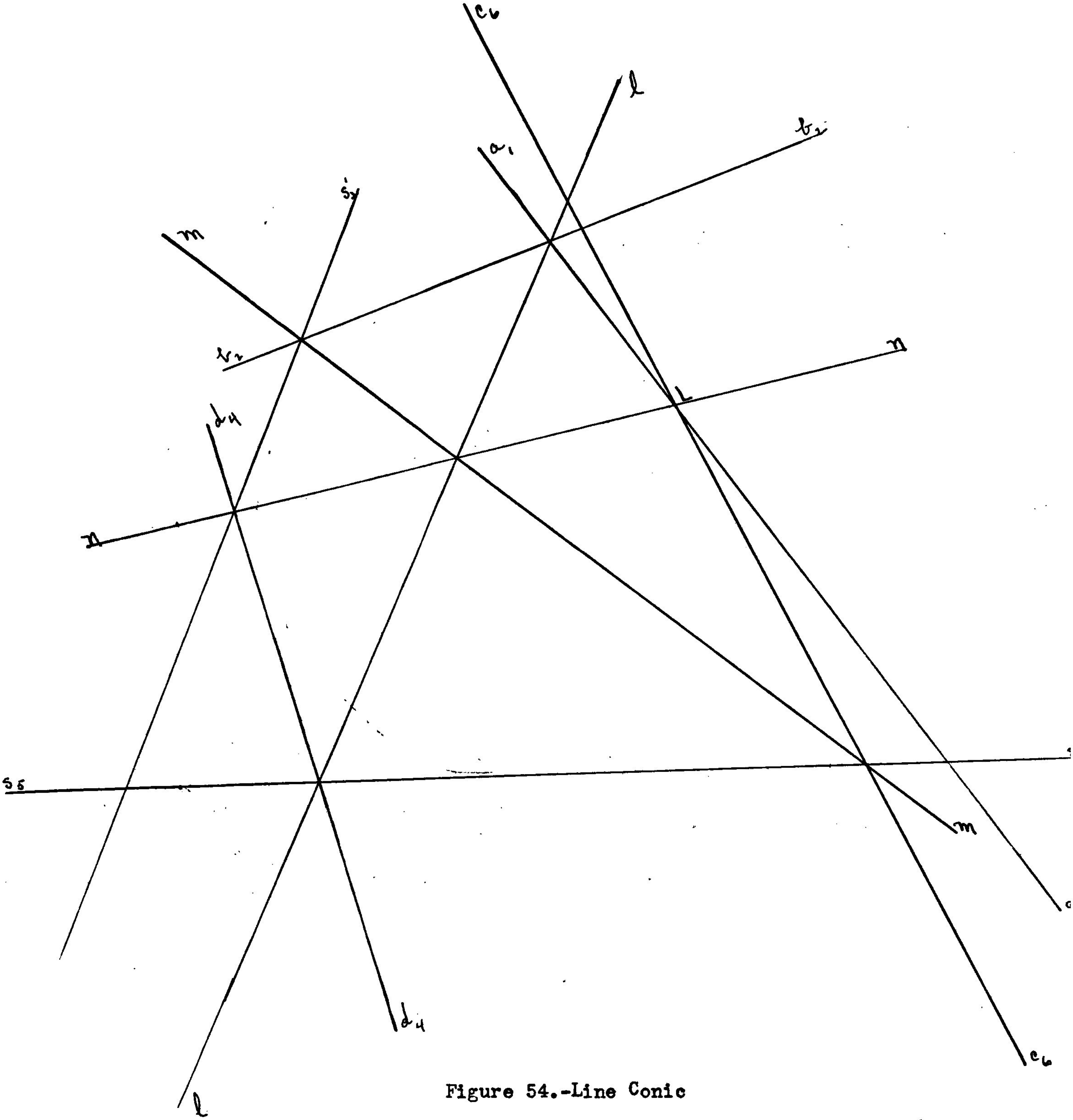

Figure 54.-Line Conic

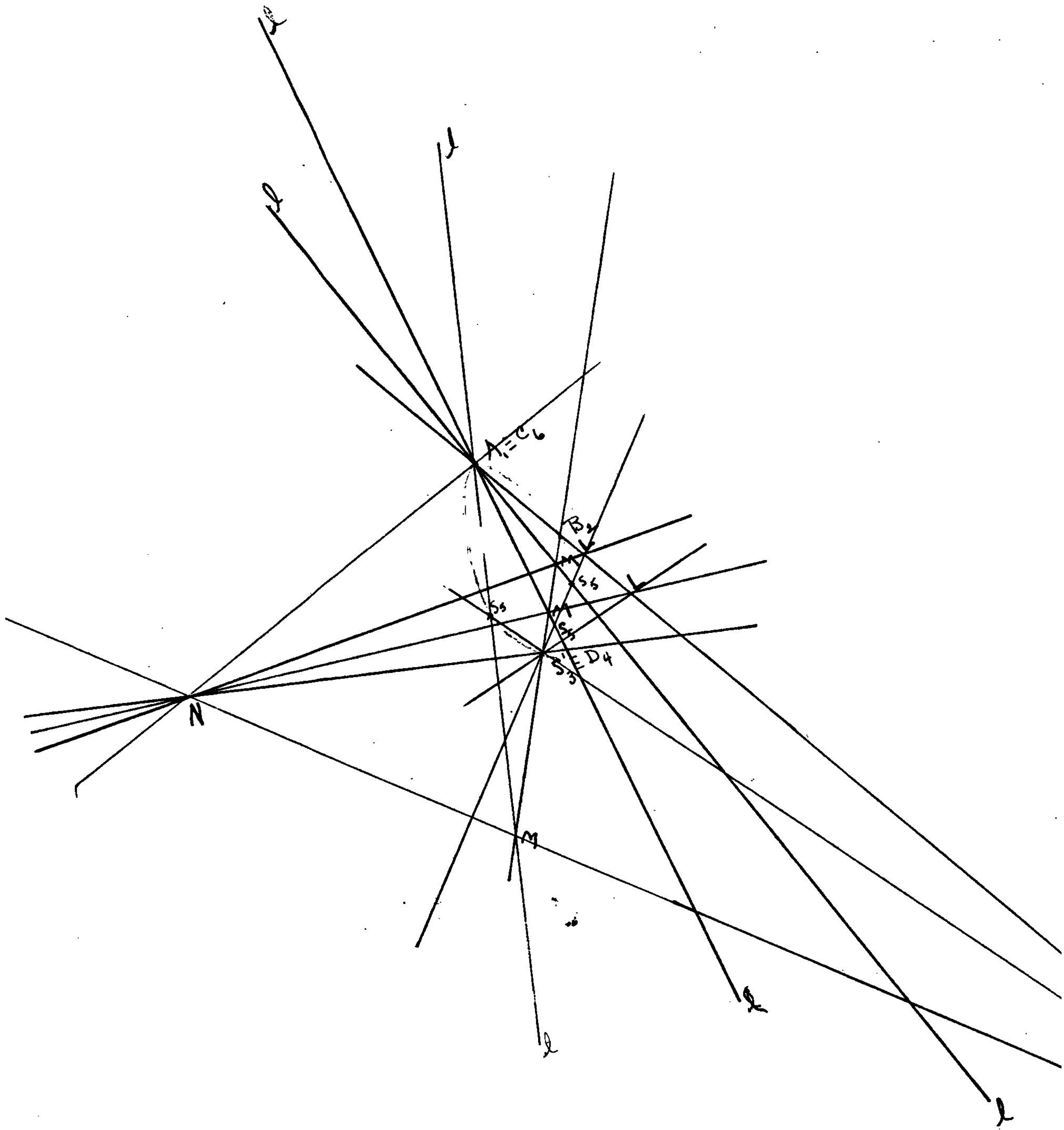

Figure 55.-Point Conic

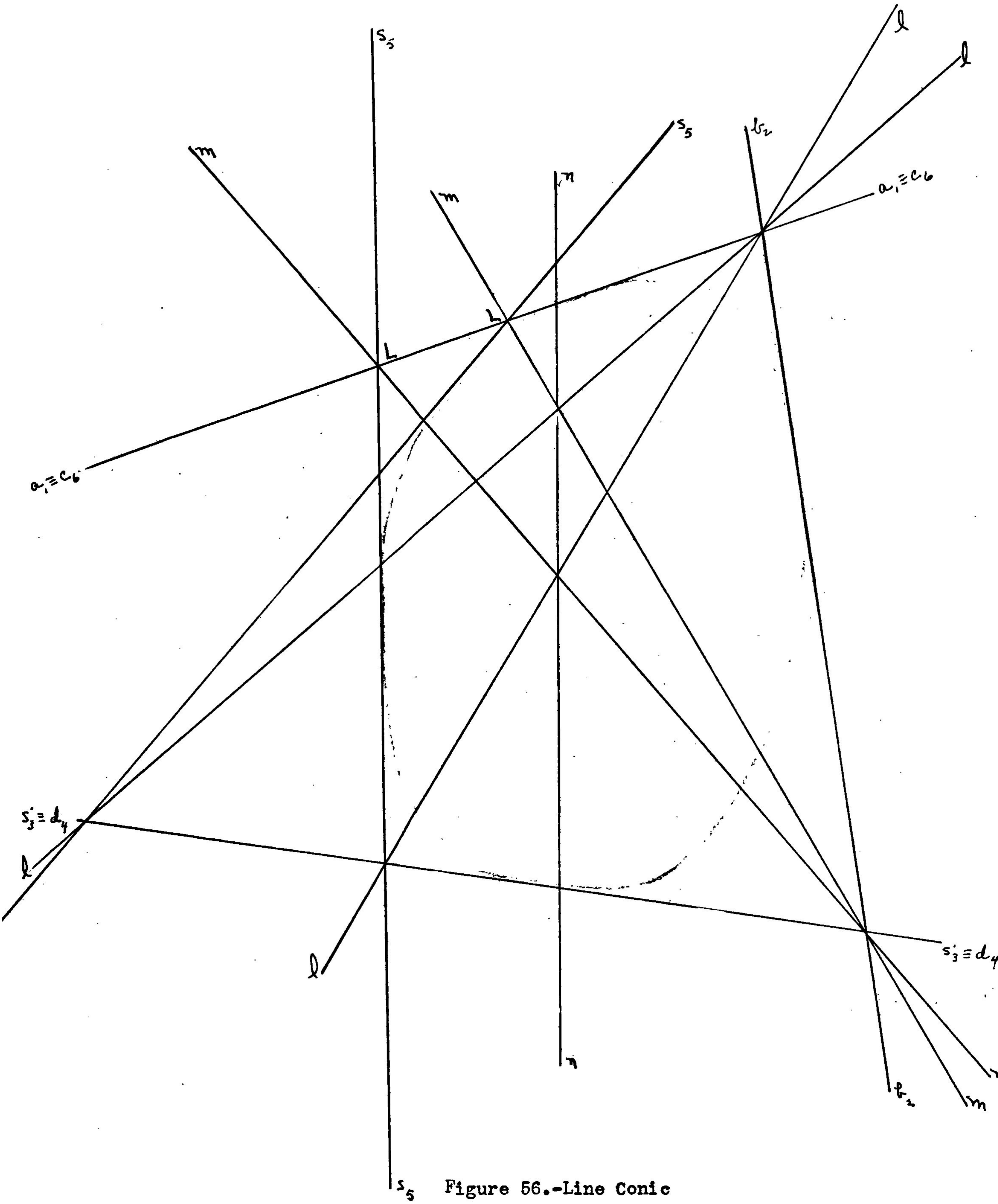

Figure 56.-Line Conic

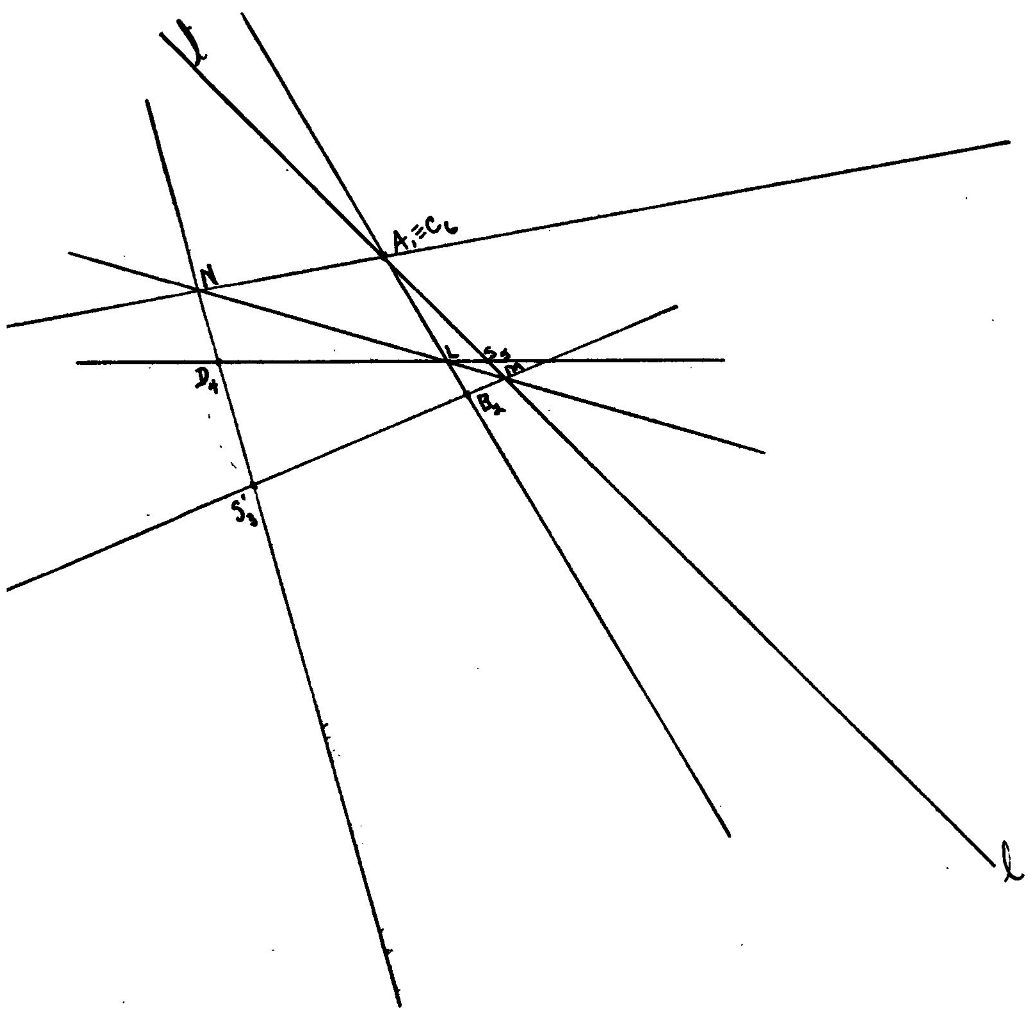

Figure 57,-Point Conic

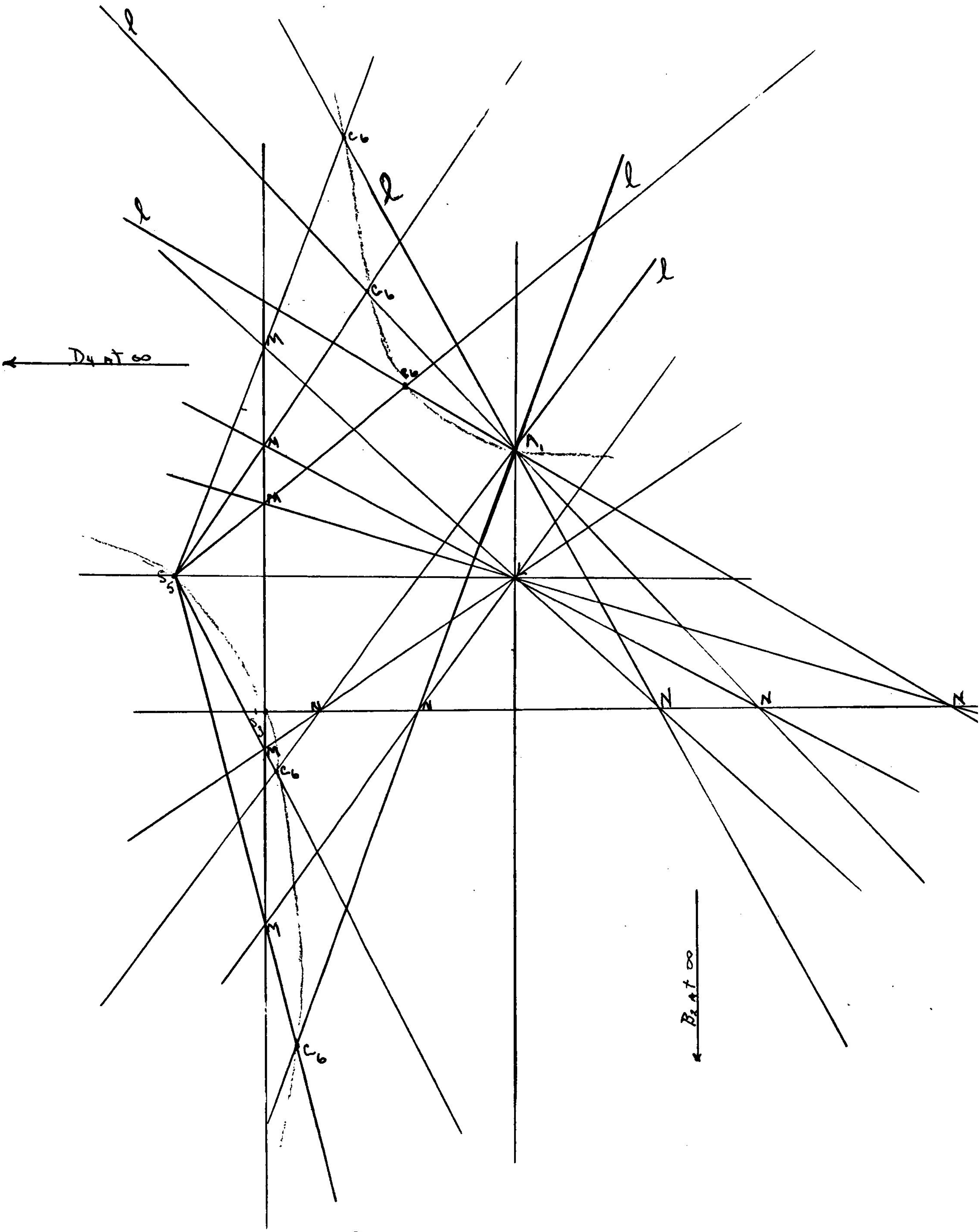

Figure 58.-Point Conic

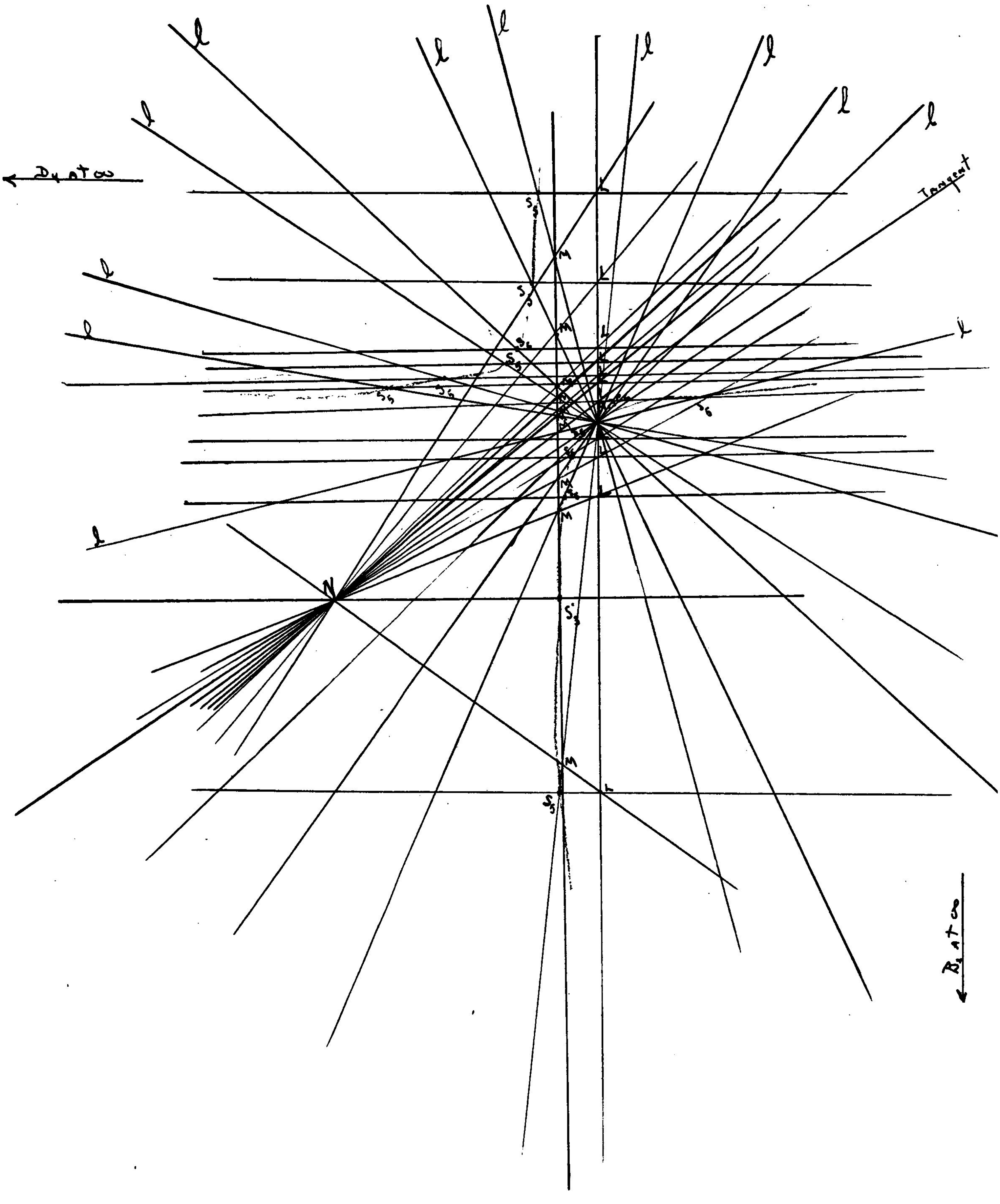

Figure 59.-Point Conic

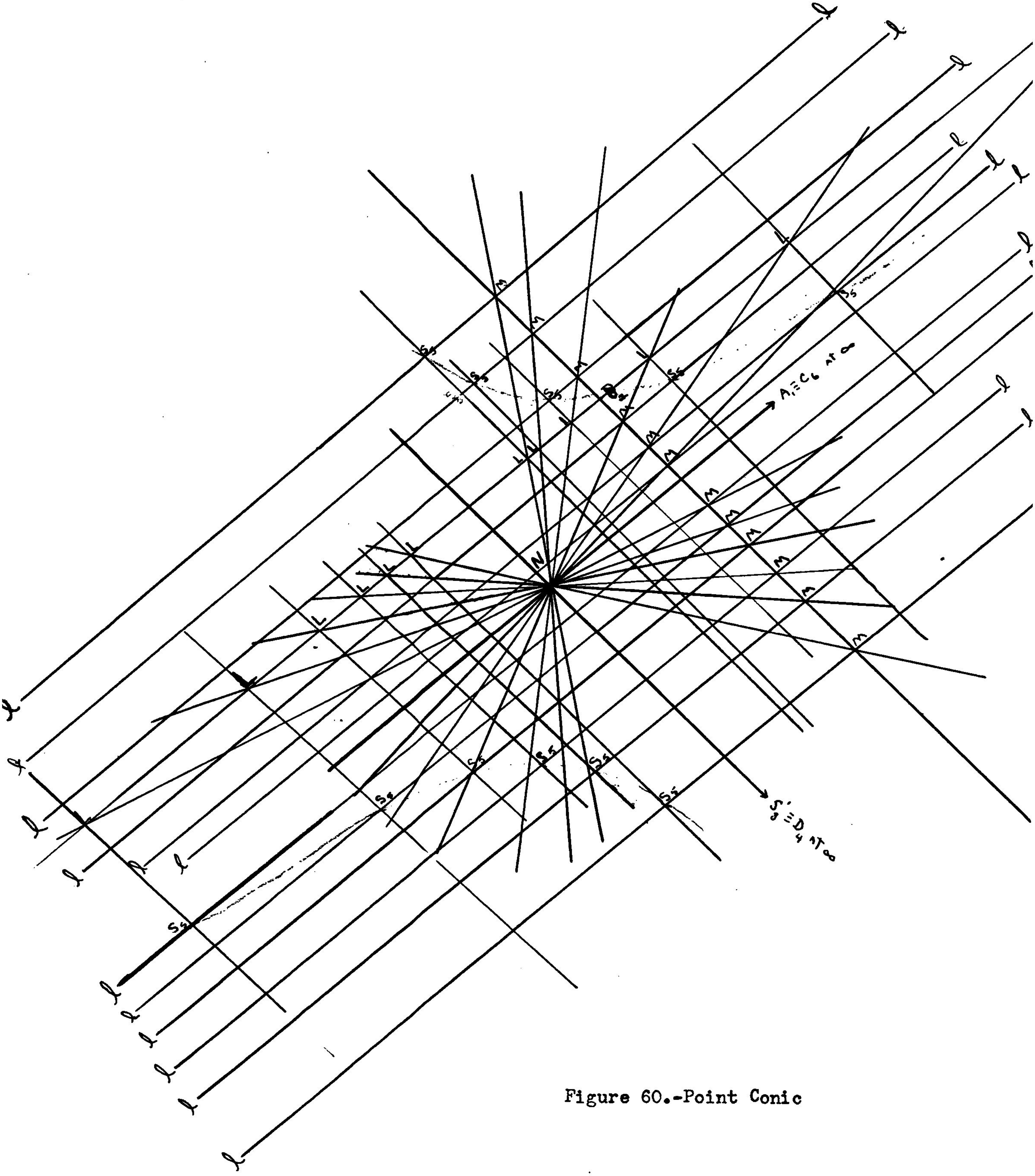

Figure 60.-Point Conic

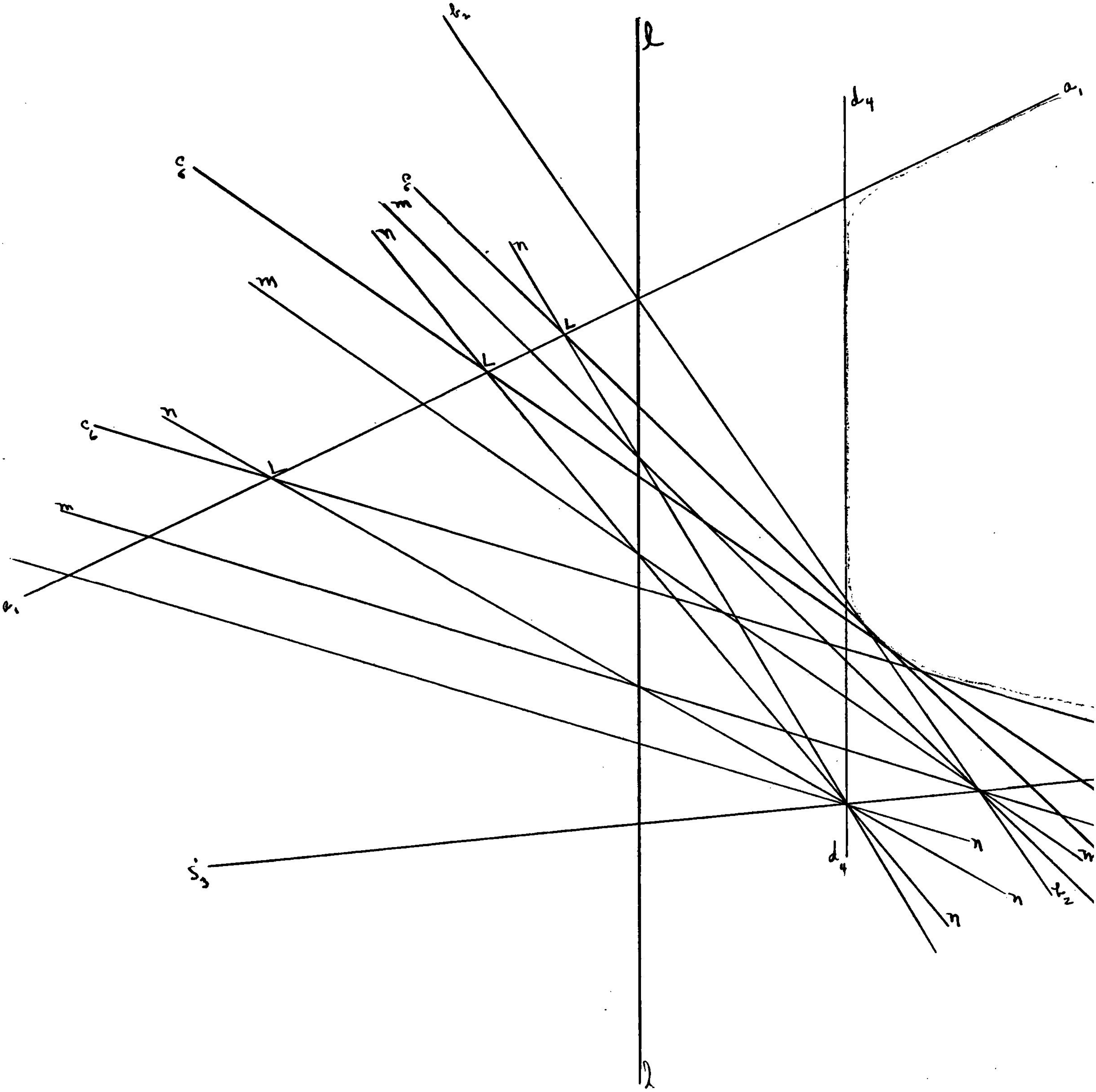

Figure 61.-Line Conic

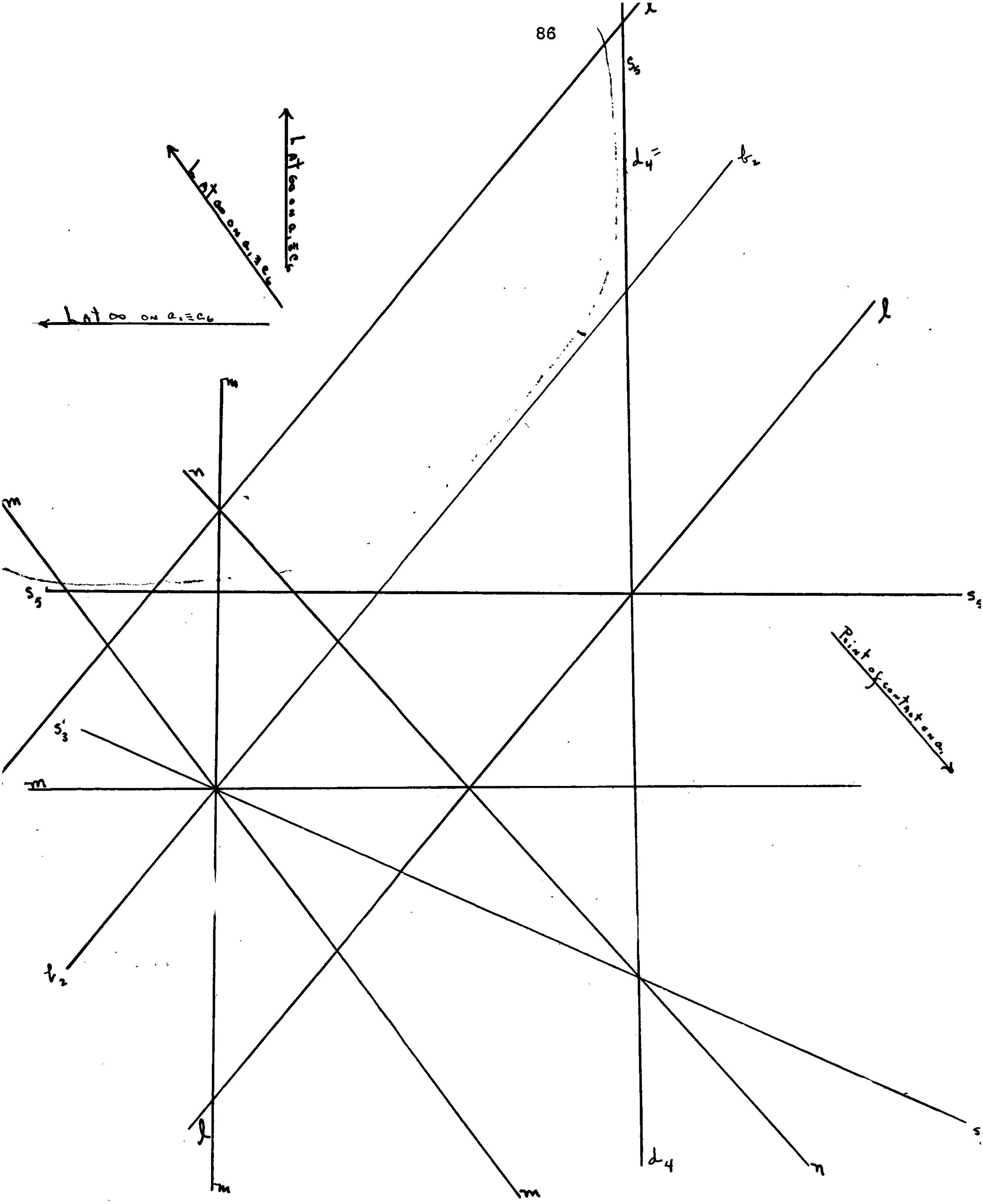

Figure 62.-Line Conic

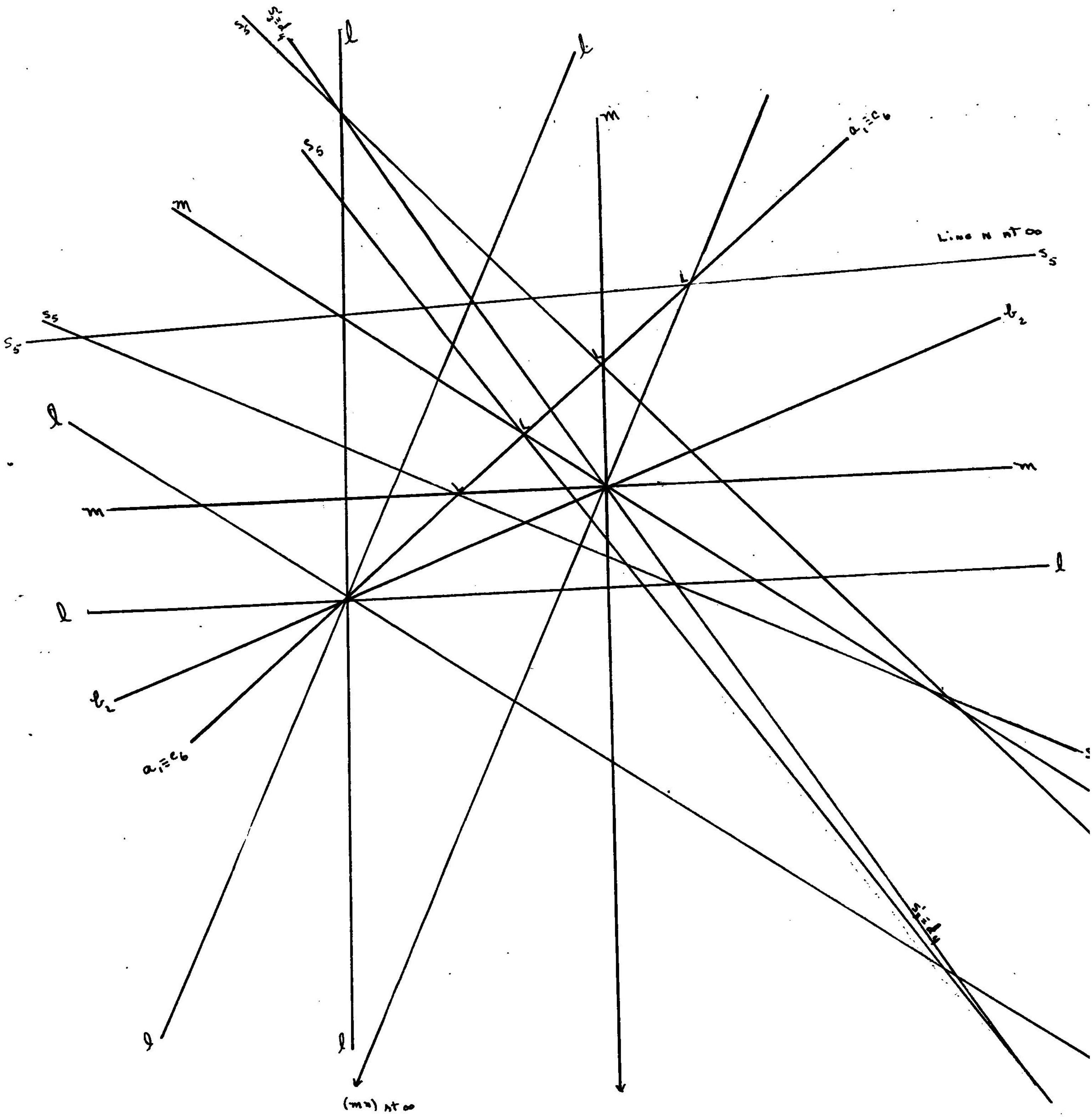

Figure 63.-Line Conic