SELECTED INTRODUCTORY CONCEPTS FROM COMBINATORIAL MATHEMATICS

A THESIS

SUBMITTED TO THE FACULTY OF ATLANTA UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF MASTER OF SCIENCE

BY

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ATLANTA, GEORGIA

AUGUST 1967

6463-13

ACKNOWLEDGEMENTS

The author wishes to acknowledge the valuable assistance and suggestions of Dr. L. K. Williams during the preparation of this thesis.

He also wishes to express his gratitude to his wife, Cindy, for her patient work in typing the initial drafts and the final copies.

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INTRODUCTION

Combinatorial mathematics, also known as combinatorial analysis or combinatorics, had its beginnings in ancient times. References can be found dating back to the Chinese Emperor Yu (c. 2200 B.C.). Permutations, an important part of this discipline, had a beginning in China around 1100 A.D.

In spite of these early beginnings, much of the material of combinatorial mathematics was merely recreational mathematics until fairly recent times, when the explosion of technical and scientific knowledge developed many useful and practical applications of the subject.

An exact definition of combinatorial mathematics seems to be impossible, as the subject matter itself, as well as applications of the same, is constantly increasing. It has been described as the analysis of complicated developments by means of 'a priori' consideration and collection of different combinations of terms which enter the coefficients. Or from another source one might find it described as a subject that is concerned with arrangements, operations, and selections within a finite or discrete system.

Combinatorial problems seem to automatically separate themselves into three main types, although there is some overlapping. For example, consider a basketball tournament with a given number of teams and a given number of courts. The question of whether it is possible to arrange a schedule so that no team plays two consecutive games is an existence problem. If it is determined that this is possible, then the question of how to go about determining the actual schedule is a construction or evaluation problem. It might be desirable in some instances to determine all possible such schedules. This is an enumeration problem.

The purpose of this paper is to examine some of the fundamental principles of combinatorial analysis and their applications to the resolution of existence problems, although enumeration problems will appear.

The theory of this phase of combinatorial analysis lends itself quite readily to development along several different lines. However, from evaluation of available literature, it appears to the writer that the most basic development, that is, that requiring the least amount of background material, is through the framework of modern algebra. Consequently, this is the method followed by the writer.

The only background material necessary for the reader is a familiarity with matrices and matrix manipulation, integral congruences from the theory of numbers, and the definitions of groups and fields.

CHAPTER I

FUNDAMENTALS

1. n-sets, generalized rule of sum, generalized rule of product.

It is assumed that the reader has a thorough knowledge of the following standardized concepts from set theory: set; subset; proper subset; null set; power set ($P(S)$); intersection; union; disjoint sets; partition; finite set; product set or cross product.

The following definitions are not so standardized. Let T_i and T_i' $(i = 1,2,\ldots,r)$, be two partitions of a set, M; i.e., $M = \cup T_i = \cup T_i'$. The partitions are ordered if equality of the partitions means $T_i = T_i'$ $(i = 1,2,\ldots,r)$ and unordered if equality of the partitions means each T_i is equal to some T_j'.

An <u>n-set</u> is a finite set with exactly n elements. By convention we take $n>0$. An <u>r-subset</u> of an n-set is a subset with exactly r elements. If S is an m-set, T an n-set, and $S \cap T = \emptyset$, then $S \cup T$ is an $(M + n)$-set. More generally, if T_i is an n_i-set $(i = 1,2,\ldots,r)$ and the T_i partition M, then M is an $(n_1 + n_2 + \ldots + n_r)$-set (generalized rule of sum).

Let $M(S,T,n)$ denote a set of ordered pairs, (s,t), where each $s \in S$ is paired with exactly n elements $t \in T$. Distinct elements of S need not be paired with elements of the same n-subsets of T. Obviously, T must contain at least n elements and $M(S,T,n) = S \times T$ if and only if T is an n-set. If S is an m-set, then $M(S,T,n)$ is an (mn)-set. More generally, if T_i is an n_i-set and $M_2 = M(T_1, T_2, n_2)$, $M_3 = M(M_2, T_3, n_3)$, . . ., $M_r = M(M_{r-1}, T_r, n_r)$, then M_r is an $(n_1 n_2 \ldots n_r)$-set (generalized rule of product).

These definitions are basic to the definitions, theorems, and corollaries appearing throughout the remainding of the paper.

2. Samples and permutations.

For any set, S, consider

$$(2.1) \qquad (a_1, a_2, \ldots, a_r)$$

an ordered r-tuple of elements of S, where the a_i, $i = 1,2,\ldots,r$, need not be distinct. We take the usual definition for equality of r-tuples, i.e., $(a_1, a_2, \ldots, a_r) = (b_1, b_2, \ldots, b_r)$ if and only if $a_i = b_i$ for $i = 1,2,\ldots,r$. We refer to (2.1) as an r-sample, and say it is of size r.

Theorem 2.1 Let S be an n-set. The number of r-samples of S is n^r.

Proof: This is nothing more than a special case of the generalized rule of product, where $T_1 = T_2 = \ldots = T_r = S$ and $n_1 = n_2 = \ldots = n_r = n$. #

In the preceding discussion, if we take the a_i of the n-sample to be distinct, the n-sample is called an n-permutation. If S is an m-set, then an n-permutation must have $n \leq m$, and an m-permutation is called a permutation of m elements, or simply a permutation.

Theorem 2.2 The number of r-permutations of n elements is

$$P(n,r) = n(n-1) \ . \ . \ . \ (n-r+1) \tag{2.2}$$

Again we have a special case of the generalized rule of product, where $T_1 = T_2 = \dots = T_r = S$ and $n_1 = n$, $n_2 = n-1$, ..., $n_r = n-r+1$. #

By (2.2), p(n,n) is the product of the first n positive integers, called <u>n-factorial</u> and written <u>n!</u>. Hence $P(n,n) = n! = n(n-1) \dots 1$.

The standard definitions of mapping, single valued mapping, image, one-to-one mapping and onto mapping are assumed in the following.

Let S be an n-set and consider the set, G(S), of all 1-1 mappings of S onto itself. Let f and g be in G(S). $f = g$ if $f(a) = g(a)$ for all $a \in S$. If f and g are any two elements of G(S), the mapping that maps $a \in S$ into $g(f(a))$ is a 1-1 mapping called the product of f and g. Thus G(S) is an algebraic system with a binary operation called product, and it may be readily verified that G(S) is a group.

Let S be an n-set, and represent the elements of S by 1,2,...,n. Then the <u>symmetric</u> <u>group</u> of <u>degree</u> <u>n</u> is G(S), and is denoted by S_n. If $f \in S_n$ such that i is mapped into f(i), i = 1,2,...n, then f is characterized by the permutation (f(1),f(2),...f(n)).

It can also be seen that each permutation of the n elements is in reality a 1-1 mapping of S onto S.

The number of elements in a group is called its order, therefore we may restate $P(n,n) = n! = n(n-1)\dots1$ as: the order of S_n is $n!$.

Examples (1) The number of 2-permutations of 4 elements is $P(4,2) = 4\cdot3 = 12$. If the elements are labeled a,b,c,d, the 2-permutations are:

$$(a,b),(a,c),(a,d),(b,a),(b,c),(b,d),$$
$$(c,a),(c,b),(c,d),(d,a),(d,b),(d,c).$$

(2) Consider the number of 4-letter words that can be constructed out of the 26 letters of the English alphabet.

(a) If repetition of letters is permitted, these are 4-samples, hence by Theorem 2.1, the number is 26^4.

(b) If repetition of letters is not permitted, these become 4-permutations, hence by Theorem 2.2, the number is $P(26,4) = 26\cdot25\cdot24\cdot23 = 358,773$. Of course, in both cases many of these "words" will be meaningless.

(3) S_{100} is of order $100! = (9.3326\dots)\cdot10^{157}$. The number of electrons in the universe has been estimated at merely $(136)\cdot2^{256}$.

(4) Let D be a matrix of p rows and q columns, and let the entries of D be the integers 0 and 1. D may be considered as an (pq)-sample of a 2-set, hence there are 2^{pq} different matrices.

3. Unordered selections, combinations, binomial coefficients.

Let S be a set and

(3.1) $$\{a_1, a_2, \dots, a_r\}$$

an unordered collection of r elements of S, not necessarily distinct. The number of times a given element appears in this collection is called the multiplicity of the element. Two such collections, $\{a_1, a_2, \ldots, a_r\}$ and $\{b_1, b_2, \ldots, b_r\}$ are equal provided the elements and their respective multiplicities are the same, regardless of order. This is an unordered selection of S of size r, and is referred to as an r-selection of S. Note that if each element of an r-selection is of multiplicity 1, the r-selection is then an r-subset of S. An r-subset of an n-set is called an r-combination of n elements.

You will recall that $P(n,n) = n!$. It is convenient to define

$$0! = 1. \tag{3.2}$$

Therefore for every positive integer n,

$$n! = n(n-1)!. \tag{3.3}$$

In the following definitions, n and r are positive integers.

$$\begin{aligned} C(n,r) &= \binom{n}{r} = \frac{n(n-1)\ldots(n-r+1)}{r!} \\ C(n,o) &= \binom{n}{o} = 1 \\ C(o,r) &= \binom{o}{r} = 0 \\ C(o,o) &= \binom{o}{o} = 1 \end{aligned} \tag{3.4}$$

Hence we have defined $C(n,r)$ for all non-negative integers n and r. Note that if $r > n$, then $C(n,r) = 0$. The numbers defined by (3.4) are the well-known binomial coefficients, and are of basic importance in enumeration problems.

Theorem 3.1 If S is an n-set, the number of r-subsets is $\binom{n}{r}$.

Proof: The number of r-permutations of n elements is P(n,r). However, each r-permutation may be ordered in r! ways. For combinations the order is disregarded, so the number of distinguishable arrangements is

$$\frac{P(n,r)}{r!} = \frac{n(n-1)\ldots(n-r+1)}{r!} = C(n,r) = \binom{n}{r}. \quad \# \tag{3.5}$$

Let S be an n-set and P(S) the set of subsets of S. Let T be the set of all n-samples obtained from the 2-set of 0 and 1. Then there is a natural 1-1 mapping of P(S) onto T.

Example Let $S = \{a,b,c\}$, a 3-set. Then P(S) is $\{\{a,b,c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a\}, \{b\}, \{c\}, \emptyset\}$ and T is $\{(1,1,1), (1,1,0), (1,0,1), (0,1,1), (1,0,0), (0,1,0), (0,0,1), (0,0,0)\}$.

Note that, while a subset is not ordered, we can use some scheme to order the elements of S and maintain this order in the subsets as $\{a,b,c\} = \{c,a,b\}$. Using Theorem 3.1 to count the elements in P(S) and Theorem 2.1 to count the elements in T and equating the counts we get

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n. \tag{3.6}$$

This is an elementary identity, but serves to illustrate an effective technique in combinatorial investigations.

Theorem 3.2 The number of r-selections of an n-set is

(3.7) $$\binom{n+r-1}{n-1} = \binom{n+r-1}{r}$$

Proof: Replace the n-set S by the n-set $S' = \{1,2,\ldots n\}$. Then every r-selection of S' can be represented in the form $\{a_1, a_2, \ldots a_r\}$ where $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_r$. Let S^* be the (n+r-1)-set of integers $1,2,\ldots,n+r-1$. Then $\{a_1 + 0,\ a_2 + 1, \ldots a_r + r-1\}$ is an r-subset of S^*, and establishes a 1-1 correspondence between r-selections of S' and r-subsets of S^* thus:

$$\{a_1, a_2, \ldots a_r\} \leftrightarrow \{a_1 + 0,\ a_2 + 1, \ldots a_r + r-1\}.$$

That is, we have simply developed a set of r-subsets that are in a 1-1 correspondence with a set of r-selections, hence, since by Theorem 3.1 the number of r-subsets of S^* is $\binom{n+r-1}{r}$, the number of r-selections of S is $\binom{n+r-1}{r}$. By expansion and simple algebra it can be readily determined that $\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$. #

Let an n-set, S, be partitioned by $T_1, T_2, \ldots, T_k$ into r_i-subsets T_i $(i = 1,2,\ldots k)$. Then $n = r_1 + r_2 + \ldots + r_k$ and we call the partition $S = T_1 \cup T_2 \cup \ldots \cup T_k$ an $(r_1, r_2, \ldots, r_k)$-partition of S.

Theorem 3.3 The number of ordered $(r_1, r_2, \ldots, r_k)$-partitions of an n-set is $\dfrac{n!}{r_1!\,r_2!\ldots r_k!}$.

Proof: The number of r_1-subsets of an n-set is $\binom{n}{r_1}$ by Theorem 3.1. Once we choose an r_1-subset, there are $n-r_1$ elements remaining, and the number of r_2-subsets of an $(n-r_1)$-set is $\binom{n-r_1}{r_2}$. Continuing this process, we have the number of partitions is:

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-r_2-\cdots-r_{k-1}}{r_k} = \frac{n!}{r_1!r_2!\cdots r_k!}$$

We can arrive at this directly by considering the proof as a direct application of Theorem 3.1 and the generalized rule of product. #

The numbers of the form $\dfrac{n!}{r_1!r_2!\cdots r_k!}$ are the multinomial coefficients. It follows directly from Theorem 3.3 that the number of ordered $(1,1,\ldots,1)$-partitions of an n-set is $n!$, and Theorem 3.3 is reduced to the number of permutations of an n-set. The number of ordered $(r,n-r)$-partitions of an n-set is $\dfrac{n!}{r!(n-r)!}$, whence Theorem 3.3 reduces to Theorem 3.1.

Examples (1) A bridge hand consists of a selection of 13 cards from a full deck of 52-cards. Since the order of the cards is of no importance, each hand is a 13-combination, and the number of possible different hands is $\binom{52}{13} =$ 635,013,559,600.

(2) At bridge, there are four players at a table, each receiving 13 cards. Hence a given situation at a bridge table is an ordered $(13,13,13,13)$-partition of a 52-set, and the number of different situations is $\dfrac{52!}{(13!)^4} = (5.3645\ldots)10^{28}$ by Theorem 3.3.

(3) A throw with a set of r dice may be considered as an r-selection of a 6-set, hence the number of distinct throws is $\binom{r+5}{5} = \binom{r+5}{r}$ by Theorem 3.2.

4. Binomial coefficients.

From section 3, it would appear that the binomial coefficients are integers, which indeed they are. Given any r successive positive integers, one of them must be a multiple of r, another of r-1, and so on, hence the product of any r successive positive integers is divisible by r!, hence $\binom{n}{r}$ is an integer.

Theorem 4.1 If p is a prime, then $\binom{p}{1}$, $\binom{p}{2}, \ldots, \binom{p}{p-1}$ are divisible by p.

Proof: Let p be a prime and r an integer such that $1 \leq r \leq p-1$. Then r! divides $p(p-1)\ldots(p-r+1)$. But r! is a prime to p, hence r! divides $(p-1)(p-2)\ldots(p-r+1)$, hence $\binom{p}{r} = p\,\frac{(p-1)(p-2)\ldots(p-r+1)}{r!}$ is divisible by p. #

Consider the well-known Pascal's Triangle for binomial coefficients:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}$$

If the arrows are considered as one-way paths, then each number of the triangle tells the number of one-way paths we can follow to get from the topmost 1 to that position in the triangle. This feature is an inherent property from the relation

$$(4.2) \qquad \binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} .$$

The symmetry and monotonicity of the horizontal rows are consequences of the following easily proved relationships:

(4.3) $$\binom{n}{r} = \binom{n}{n-r} \qquad (0 \leq r \leq n)$$

(4.4) $$\binom{2n}{0} < \binom{2n}{1} < \cdots < \binom{2n}{n}$$

(4.5) $$\binom{2n-1}{0} < \binom{2n-1}{1} < \cdots < \binom{2n-1}{n-1} = \binom{2n-1}{n}$$

If n is a positive integer,

(4.6) $$(x + y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{n} y^n.$$

To prove this, let A be an n-set of symbols, $A = \{(x+y)_1, (x+y)_2, \ldots (x+y)_n\}$. Then for $r > 0$ the coefficient of $x^{n-r} y^r$ in the expansion of $(x+y)^n$ is equal to the number of r-subsets of A, which by Theorem 3.1 is $\binom{n}{r}$.

(4.7) By setting $x = y = 1$ in (4.6), we have

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

(4.8) By setting $x = 1$, $y = -1$ in (4.6), we have

$$\binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n \binom{n}{n} = 0,$$

hence it can be seen that (4.6) is the source of many relationships among coefficients.

The following identities are typical of those that occur throughout this paper. They may be derived by elementary methods.

(4.9) $$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n},$$

(4.10) $$\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1},$$

(4.11) $$\sum_{k=1}^{n} k^2 \binom{n}{k} = n(n+1) \cdot 2^{n-2},$$

(4.12) $$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n}.$$

CHAPTER II

INCLUSION AND EXCLUSION

1. A fundamental formula.

Consider the following problem. How many integers between 1 and 6300 inclusive are divisible by neither 5 or 3? Since every fifth integer is divisible by 5 and every third integer by 3, the number divisible by 5 is $6300 \div 5 = 1260$, and by 3 is $6300 \div 3 = 2100$, hence $6300 - 2100 - 1260$ appears to be the answer. But we have subtracted numbers divisible by both 3 and 5 (15, 30, etc.) twice. Hence we must add to our result the number divisible by both 3 and 5, or by 15, which is $6300 \div 15 = 420$. Hence we have $6300 - 2100 - 1260 + 420 = 3360$.

This illustrates the general idea of the principle of inclusion and exclusion. Let A be an n-set and to each $a \in A$ assign a unique <u>weight</u>, $w(a)$, with $w(a)$ an element of some field, F. While F and $w(a)$ are arbitrary, a particular combinatorial problem often suggests a natural choice of F and $w(a)$.

Let P denote an N-set of properties,

(1.1) $\qquad P_1, P_2, \ldots, P_N$ connected with the elements of A,

and let

(1.2) $\qquad \{P_{i_1}, P_{i_2}, \ldots, P_{i_r}\}$ be an r-subset of P. Let

(1.3) $\qquad W(P_{i_1}, P_{i_2}, \ldots, P_{i_r})$ be the sum of the weights of those elements of A that satisfy each of the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_r}$. If there are no such elements, the expression is assigned the value zero. Now let

(1.4) $$W(r) = \sum W(P_{i_1}, P_{i_2}, \ldots, P_{i_r})$$ be the sum of the quantities (1.3) over all the r-subsets of P. Extend (1.4) to the case r = 0 and let W(0) equal the sum of the weights of the elements of A.

The necessary foundations are now laid for developing the basic inclusion and exclusion formula, which is simply the formula for finding the sum of the weights of the elements of S that satisfy exactly m of the properties (1.1). Denote this sum by E(m). This formula is postulated, and an intuitive explanation of it is given.

$$E(m) = W(m) - \binom{m+1}{m} W(m+1) + \binom{m+2}{m} W(m+2) - \ldots + (-1)^{N-m} \binom{N}{m} W(N).$$

Note that W(m) is the summation over all m-subsets of (1.1). Obviously it is possible that some elements of S might satisfy all of the properties in more than ome m-subset of P. Hence the weights of these elements are added more than once. To compensate for this $\binom{m+1}{m} W(m+1)$ is subtracted to eliminate duplication of weights of elements satisfying both m and m+1 properties. However, too much has now been subtracted for it might be possible for an element to satisfy more than one (m+1)-subset of P, hence its weight was subtracted more than once. Consequently it becomes necessary to add another sum. This process of addition and subtraction must be continued until one arrives at the sum of the weights of the elements which satisfy all of the properties (1.1), when it obviously ends. The following theorem shows that if an element of S satisfies fewer than m properties or more than m properties its weight is not included in the calculations.

Theorem 1.1 $E(m) = W(m) - \binom{m+1}{m} W(m+1) + \binom{m+2}{m} W(m+2) - \cdots + (-1)^{N-m}\binom{N}{m} W(N)$ is the sum of the weights of the elements of S that satisfy <u>exactly</u> m properties (1.1).

Proof: Let $a \in A$ and a of weight $w(a)$ satisfy exactly t of the properties (1.1). If $t < m$, then a contributes 0 to the right side of the equation. If $t = m$, then a contributes $w(a)$ to the right side of the equation.

If $t > m$, then a contributes

$$(1.5) \qquad \left[\binom{t}{m} - \binom{m+1}{m}\binom{t}{m+1} + \binom{m+2}{m}\binom{t}{m+2} - \cdots + (-1)^{t-m}\binom{t}{m}\binom{t}{t}\right] w(a)$$

to the right side of the equation. But if $m \leqslant k \leqslant t$, $\binom{k}{m}\binom{t}{k} = \binom{t}{m}\binom{t-m}{t-k}$, therefore (1.5) reduces to

$$(1.6) \qquad \binom{t}{m}\left[\binom{t-m}{t-m} - \binom{t-m}{t-(m+1)} + \binom{t-m}{t-(m+2)} - \cdots + (-1)^{t-m}\binom{t-m}{t-t}\right] w(a).$$

But by (4.8) of Chapter I, the bracketed expression of (1.6) is equal to zero. Hence if $t > m$, a contributes zero to the right side of the equation. This implies that the right side of the equation is the sum of the weights of the elements of A that satisfy exactly m of the properties (1.1). #

Theorem 1.2 Let E(0) denote the sum of the weights of elements of A that satisfy none of the properties (1.1). Then

$$E(0) = W(0) - W(1) + W(2) - \dots + (-1)^N W(N). \tag{1.7}$$

Proof: This is Theorem 1.1 where $m = 0$. #

If for each $a \in A$ we let $w(a) = 1$, a sum of weights is the number of entries in the sum. Theorem 1.2 specialized in this was has $W(0) = n$ and $E(0)$ as the number of elements of A satisfying none of the properties (1.1). Equation (1.7) specialized in this way is called the <u>sieve formula</u>.

2. Application to number theory.

If $x \geq 0$, let $[x]$ denote the greatest integer $\leq x$. Let (a,b) denote the g.c.d. of two integers a and b not both zero. Write a/b for "a divides b", and $a \not/ b$ for "a does not divide b".

Theorem 2.1 Let n be a positive integer and let $a_1, a_2, \dots, a_N$ be positive integers such that $(a_i, a_j) = 1$ for $i \neq j$. Then the number of integers k such that $0 < k \leq n$, $a_i \not/ k$ $(i = 1,2,\dots,N)$ is

$$n - \sum_{1 \leq i \leq N} \left[\frac{n}{a_i}\right] + \sum_{1 \leq i < j \leq N} \left[\frac{n}{a_i a_j}\right] - \dots + (-1)^N \left[\frac{n}{a_1, a_2, \dots, a_N}\right]. \tag{2.1}$$

Proof: Let A be the n-set of positive integers $1,2,\dots,n$ and let P_i be the property that an element of A is divisible by a_i $(i = 1,2,\dots,N)$. The a_i are relatively prime in pairs hence the expression $W(P_{i_1}, P_{i_2}, \dots, P_{i_r})$ in the sieve formula is the number of integers k such that $0 < k \leq n$, $a_{i_1} a_{i_2} \dots a_{i_r} / k$. But this number is $\left[\frac{n}{a_{i_1} a_{i_2} , , , a_{i_r}}\right]$. #

The Euler φ-function $\varphi(n)$ of the positive integer n is the number of integers k such that $0 < k \leq n$, $(k,n) = 1$.

Theorem 2.2 Let n be a positive integer. Then

(2.2) $$\varphi(n) = n \prod_p \left(1 - \frac{1}{p}\right).$$ The product extends over all prime divisors p of n.

Proof: In Theorem 2.1, replace a_i by p_i and suppose $p_1, p_2, \ldots, p_N$ are the prime divisors of n. Then (2.1) implies

(2.3) $$\varphi(n) = n - \sum_{1 \leq i \leq N} \frac{n}{p_i} + \sum_{1 \leq i < j \leq N} \frac{n}{p_i p_j} - \ldots + (-1)^N \frac{n}{p_1 p_2 \cdots p_N}.$$

But this is equivalent to (2.2). #

The Mobius function $\mu(n)$ of the positive integer n is defined by

$$\mu(1) = 1,$$

(2.4) $\mu(n) = 0$ if n is divisible by the square of a prime,

$\mu(p_1 p_2 \cdots p_k) = (-1)^k$ if the primes $p_1, p_2, \ldots, p_k$ are distinct.

This allows us to write (2.3) as

(2.5) $$\varphi(n) = n \sum_d \frac{\mu(d)}{d}$$ over all positive divisors d of n.

Let n be a positive integer. If the primes $\leq \sqrt{n}$ are known, then the primes $\leq n$ may be found. Write the sequence of integers

(2.6) $$2, 3, \ldots, n.$$

Strike out all numbers divisible by 2, then all numbers divisible by 3, then all numbers divisible by 5, and so on up to all numbers divisible by q where q is the largest prime $\leq \sqrt{n}$. The remaining numbers are primes $> \sqrt{n}$ and $\leq n$, for a remaining number cannot have a prime factor $\leq n$, nor can it be the product of two numbers $> \sqrt{n}$. This method for the construction of primes is called the <u>sieve</u> of <u>Eratosthanes</u>. To find the number of primes, p, such that $\sqrt{n} < p \leq n$, we can again use Theorem 2.1, but shall omit this application. (1)

3. Derangements.

Let

(3.1) $(a_1, a_2, \ldots, a_n)$ be a permutation of n elements labeled $1, 2, \ldots, n$. The permutation is a <u>derangement</u> if $a_i \neq i$, $(i = 1, 2, \ldots, n)$. Thus a derangement has no element in its natural position.

A problem by Montmort, known as "le problème des recontres" asks for the number of these derangements. Let D_n denote this number. We may evaluate D_n by the sieve formula. Let A be the set of n! permutations (3.1) and P_i the property that $a_i = i$ $(i = 1, 2, \ldots, n)$. Then

(3.2) $W(P_{i_1}, P_{i_2}, \ldots, P_{i_r}) = (n - r)!$ as r elements are fixed, hence n - r elements are being permuted. Also, as the number of r-subsets of an n-set is $\binom{n}{r}$, and $W(r) = \sum W(P_{i_1}, P_{i_2}, \ldots, P_{i_r})$ over all r-subsets, we have

(3.3) $$W(r) = \binom{n}{r} (n - r)! = \frac{n!}{r!} .$$

D_n is obviously the number of permutations that satisfy none of the properties, P_i $(i = 1, 2, \ldots, n)$. But this is the E(0) of Theorem 1.2, hence $D_n = E(0) = W(0) -$

$W(1) + W(2) - \ldots + (-1)^N W(N)$, or $D_n = \frac{n!}{0!} - \frac{n!}{1!} + \frac{n!}{2!} - \ldots + (-1)^n \frac{n!}{n!}$, or

(3.4) $$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^n \frac{1}{n!}\right).$$

4. The permanent.

Let S be a set. A rectangular array based on S is a configuration of m rows and n columns of the form

(4.1) $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The entry a_{ij} in row i and column j must be an element of S, but S need not be restricted in any way. a_{ij} is said to occupy the (i, j) position of A. A is referred to as an m by n array, or A is of size m by n. If m = n, A is a <u>square array</u>. If m - r rows and n - s columns of A are deleted, the result is an <u>r by s subarray of A</u>. Two m by n arrays are equal if corresponding entries are equal. A can be considered as an (mn)-sample of set S. A 1 by n array may be regarded as an n-sample.

We may replace (4.1) by

(4.2) $$A = [a_{ij}], \quad (i = 1,2,\ldots,m; \quad j = 1,2,\ldots,n).$$

Let e = min (m,n). Then those entries a_{ii} in position (i,i), i = 1,2,...,e, form the <u>main diagonal</u> of A. The <u>transpose</u> A^T of A is an n by m array obtained by reflecting A about its main diagonal. Thus a_{ji} will be in the (i,j) position of A $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,m)$. If $A = A^T$, A is symmetric.

The array is a matrix if the set S is a field. Addition and scalar multiplication of m by n matrices is defined in the usual way, and the set of all m by n matrices with elements in a field, F, is a vector space of dimension mn over F. An m by n matrix may be multiplied by an n by t matrix by the usual method, resulting in an m by t matrix.

Let $A = [a_{ij}]$ be a matrix of size m by n with $m \leq n$, and define the permanent of A by:

(4.3) $$\text{per}(A) = \sum a_{1i_1} a_{2i_2} \cdots a_{mi_m}.$$

This summation extends over all the m-permutations $(i_1, i_2, \ldots, i_m)$ of the integers $1,2,\ldots,n$. This scalar function of A is very frequently used throughout the literature of combinatorics in connection with certain enumeration problems. A few of the properties of it are stated herein, but no attempt is made to elaborate or explain these. (2)

(4.4) The per (A) remains invariant under arbitrary permutations of the rows and columns of A.

(4.5) Multiplication of a row of A by a scalar α in F replaces per (A) by $\alpha \cdot$ per (A).

(4.6) If A is a square matrix of order n, per (A) = per (A^T), and per (A) is the same as the determinant det (A) apart from a factor ± 1 preceding each product on the right side of (4.3).

(4.7) The multiplicative law for determinants,

$\det(A) \cdot \det(B) = \det(AB)$ is false for permanents.

(4.8) To evaluate per (A), let A be a matrix of size m by n with $m \leq n$. Let A_r denote a matrix obtained from A by replacing r columns of A by zeros. Let $S(A_r)$ denote the product of the row sums of A_r and let $\sum S(A_r)$ denote the sums of the $S(A_r)$ over all of the choices for A_r. Then per $(A) = \sum S(A_{n-m}) - \binom{n-m+1}{1} \sum S(A_{n-m+1})$

$+ \binom{n-m+2}{2} \sum S(A_{n-m+2}) - \cdots + (-1)^{m-1} \binom{n-1}{m-1} \sum S(A_{n-1})$.

If A is a square matrix of order n, then per (A) $= S(A) - \sum S(A_1) + \sum S(A_2) - \cdots + (-1)^{n-1} \sum S(A_{n-1})$.

This chapter is concluded with a few introductory remarks about matrices whose entries are the integers zero and one. These are called (0,1)-matrices, and as each (0,1)-matrix of size m by n can be considered as an mn-sample of the 2-set of integers zero and one, Theorem 2.1 of Chapter I states that there are 2^{mn} such m by n matrices.

Let I denote the identity matrix of order n and let J denote the matrix of order n with every entry equal to 1. Then it is clear that

(4.9) $\text{per}(J) = n!$

and for the purposes of this paper it is postulated that

(4.10) $\text{per}(J - I) = D_n$.

CHAPTER III

RECURRENCE RELATIONS

1. Elementary recurrences.

(1.1) $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ is a simple instance of a recurrence. From this one can evaluate the binomial coefficients for all non-negative integers n and r, as schematically illustrated by Pascal's triangle. Many different types of relationships are called recurrences, and no attempt is made here to formalize a definition for this concept. Generally, recurrences are relationships that are used to evaluate a quantity term by term from given initial values and previously computed values. This chapter treats only a few simple recurrences of special relationship to the general theme of this paper. For a much more sophisticated treatment, see the recent text by Riordan. (3)

Consider the set of all n-samples obtained from the 2-set of the integers 0 and 1. Let $f(n)$ denote the number of these that do not contain two successive 0's, and define $f(0) = 1$. It is apparent that $f(1) = 2$. If $n \geqslant 2$ the number of such samples with 1 as the first component is obviously $f(n-1)$ and the number with 0 as the first component is $f(n-2)$ since fixing 0 as the first component also fixes 1 as the second component. Hence from this interesting fact it is readily seen that

$$(1.2) \qquad f(n) = f(n - 1) + f(n - 2)$$

for all $n \geq 2$. Thus $f(n)$ is determined for all non-negative integers n. These numbers, $f(n)$, are called Fibonacci numbers and have many remarkable arithmetical and combinatorial properties.

Turning to the topic of derangements as introduced in the preceding chapter, consider these as a type of recurrence. Define $D_0 = 1$. It is apparent that $D_1 = 0$. Consider a derangement

$$(1.3) \qquad (a_1, a_2, \ldots, a_n)$$

of n elements labeled $1, 2, \ldots, n$ with $n \geq 2$. The first position may be occupied by any of the n integers except 1, hence by $n - 1$ different integers. Let the first entry of (1.3) be fixed with $a_1 = k$ $(k \neq 1)$. Then the derangements are of two types depending on whether or not 1 is in the kth position. If 1 is in the kth position, then the number of permutations is that of $n - 2$ elements with all elements displaced, or D_{n-2}. On the other hand, if 1 is not in the kth position, then the permutations permitted are those that involve the elements $1, 2, \ldots, k-1, k+2, \ldots, n$ in the positions 2 through n with 1 not in the kth position and every element out of its own position. But this is the same as the permutations of $n - 1$ elements labeled 2 through n with every element displaced. Hence the number of these is D_{n-1}. All of this implies

$$(1.4) \qquad D_n = (n - 1)(D_{n-1} + D_{n-2})$$

and this may be used to give a proof by induction of

(1.5) $$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^n \frac{1}{n!}\right).$$

(The proof is omitted here).

2. Ménage numbers.

Let U_n denote the number of permutations of n elements labeled $1,2,\dots,n$ such that integer i is in neither position i nor $i + 1$ for $i = 1,2,\dots,n - 1$, and n is in neither position n nor 1. In other words, U_n is the number of permutations that have no elements in the same position as $(1,2,\dots,n)$ and $(n,1,2,\dots,n-1)$; that is, the number of permutations that are discordant with these two permutations. These numbers, U_n , are called <u>ménage numbers</u> from the following "problème des ménages" formulated by Lucas.

In how many ways can n married couples be seated at a circular table, alternating men and women, so that no husband and wife are in adjacent seats?

If the wives are seated first, there are two different n-sets of chairs they may be seated in, since they will leave an empty seat between each two of them. For each n-set of chairs, the number of ways in which the wives can be seated is simply $P(n,n) = n!$. Hence the wives can be seated in $2n!$ ways. Then each husband is excluded from the two seats adjacent to his wife, but the number of ways of seating the husbands is independent of the seating arrangement of the wives. It should be clear that for any given arrangement of the wives, the number of arrangements for the husbands is simply U_n, hence if M denotes the total number of arrangements,

(2.1) $$M = 2n!U_n.$$

Thus to solve the "problème des ménages" it is only necessary to find U_n.

Theorem 2.1 The ménage numbers U_n are given by

(2.2) $$U_n = n! - \frac{2n}{2n-1}\binom{2n-1}{1}(n-1)! + \frac{2n}{2n-2}\binom{2n-2}{2}(n-2)! - \ldots + (-1)^n \frac{2n}{n}\binom{n}{n}0! \qquad (n > 1).$$

Proof: This proof is a recurrence argument by Kaplansky involving several lemmas.

Lemma 2.2 Let $f(n,k)$ denote the number of ways of selecting k objects, no two consecutive, from n objects arranged in a row. Then

(2.3) $$f(n, k) = \binom{n - k + 1}{k}$$

Proof: We have $f(n,1) = n = \binom{n}{1}$, and for $n > 1$, $f(n,n) = 0 = \binom{1}{n}$. If $1 < k < n$, we may split the selections into those that contain the first object and those that do not. The selections that include the first object cannot include the second and can obviously be enumerated by $f(n-2,k-1)$. The selections that do not include the first object are enumerated by $f(n-1,k)$. Hence we have

(2.4) $$f(n, k) = f(n - 1, k) + f(n - 2, k - 1).$$

It is now possible to prove (2.3) by induction. The induction hypothesis asserts

(2.5) $$f(n - 1, k) = \binom{n - k}{k}, \quad f(n - 2, k - 1) = \binom{n - k}{k - 1}.$$

But (2.4) and (2.5) imply that

$$(2.6) \qquad f(n, k) = \binom{n-k}{k} + \binom{n-k}{k-1}$$

and by simple algebra it is established that $\binom{n-k}{k} + \binom{n-k}{k-1} = \binom{n-k+1}{k}$, hence the lemma is proved.

Lemma 2.3 Let $g(n,k)$ denote the number of ways of selecting k objects, no two consecutive, from n objects arranged in a circle. Then

$$(2.7) \qquad g(n, k) = \frac{n}{n-k}\binom{n-k}{k} \qquad (n > k).$$

Proof: As before, split the selections into those that include the first object and those that do not. The selections that include the first object cannot include the second object or the last object, and by lemma 2.2 can be enumerated by $f(n-3,k-1)$. The selections that do not include the first object are enumerated by $f(n-1,k)$, hence

$$(2.8) \qquad g(n, k) = f(n-1, k) + f(n-3, k-1).$$

But then from lemma 2.2, $g(n,k) = \binom{n-k}{k} + \binom{n-k-1}{k-1}$, and again by using simple algebra we have $\binom{n-k}{k} + \binom{n-k-1}{k-1} = \frac{n}{n-k}\binom{n-k}{k}$, which proves lemma 2.3.

Returning again to permutations on the elements labeled $1,2,\ldots,n$, let P_i be the property that a permutation has i in position i $(i = 1,2,\ldots,n)$ and P_i' the property that the permutation has i in position $i + 1$, $(i = 1,2,\ldots,n-1)$ with P_n' the property that n is in position 1. List the 2n properties in a row.

$$(2.9) \qquad P_1, P_1', P_2, P_2', \ldots, P_n, P_n'.$$

Select k of these properties. How many of the permutations satisfy each of the k properties? If the properties are not compatible (for example, P_1 and P_n' are not compatible, for no permutation satisfies both of them) the number is zero. If the properties are compatible, then exactly k of the elements in all of the desired permutations are fixed. Hence there are only n-k elements that can be permuted and the desired number is $P(n-k,n-k) = (n-k)!$. Now let v_k denote the number of ways of selecting k compatible properties from the 2n properties (2.9).

It is now obvious that U_n is the number of permutations that satisfy none of the properties (2.9). Referring again to the sieve formula and Theorem 1.2 of Chapter II, we have $U_n = E(0) = W(0) - W(1) + W(2) - \dots + (-1)^N W(N)$, where $W(r) = v_r\,(n-r)!$. Hence $U_n = v_0\, n! - v_1 (n-1)! + v_2 (n-2)! - \dots + (-1)^n\, v_n \cdot 0!$.

It now remains to evaluate v_k. It is apparent that if the 2n properties (2.9) are arranged in a circle, the only properties that are not compatible are the consecutive ones, hence $v_k = g(2n,k) = \frac{2n}{2n-k}\binom{2n-k}{k}$ by lemma 2.3. In conclusion then, $U_n = v_0 n! - v_1(n-1)! + v_2(n-2)! - \dots + (-1)^n v_n 0! = \frac{2n}{2n-0}\binom{2n-0}{0} n! - \frac{2n}{2n-1}\binom{2n-1}{1}(n-1)! + \frac{2n}{2n-2}\binom{2n-2}{2}(n-2)! - \dots + (-1)^n \frac{2n}{2n-n}\binom{2n-n}{n} 0! = n! - \frac{2n}{2n-1}\binom{2n-1}{1}(n-1)! + \frac{2n}{2n-2}\binom{2n-2}{2}(n-2)! - \dots + (-1)^n \frac{2n}{n}\binom{n}{n} 0!$. #

3. Latin rectangles.

Let S be a set of n elements. A <u>Latin rectangle</u> based on S is an r by s rectangular array

(3.1) $$A = [a_{ij}], \quad (i = 1,2,\ldots,r; \quad j = 1,2,\ldots,s)$$

with each row of (3.1) an s-permutation of elements of S and each column of (3.1) an r-permutation of elements of S. This immediately restricts r and s so that $r \leqslant n$ and $s \leqslant n$. If the elements of S are labeled $1,2,\ldots,n$ and $s = n$, then each row of A contains a permutation of the elements $1,2,\ldots,n$. Also, from the definition, no element is repeated in a given column. If, in addition, the elements of the first row are written in standard order $1,2,\ldots,n$ (i.e., $a_{1i} = i$, $i = 1,2,\ldots,n$) the Latin rectangle is said to be <u>normalized</u>. Let $L(r,n)$ denote the number of r by n Latin rectangles and $K(r,n)$ denote the number of normalized r by n Latin rectangles. The number of Latin rectangles resulting from fixing the first row as a given permutation is the same, regardless of what permutation is used as the first row. Hence it is trivial that

(3.2) $$L(r, n) = n! \ K(r, n).$$

Consider now normalized 2 by n Latin rectangles. The condition that $a_{2i} \neq i$ $(i = 1,2,\ldots,n)$ is necessary from the definitions, hence each normalized 2 by n Latin rectangle can be considered as a derangement and consequently

(3.3) $$K(2, n) = D_n .$$

The ménage numbers U_n are the number of 3 by n Latin rectangles where the first two rows are fixed as

$$(3.4) \qquad \begin{matrix} 1 & 2 & 3 & \dots & n, \\ n & 1 & 2 & \dots & n-1 \end{matrix}$$

since the definitions require that $a_{3i} \neq i$, $a_{3i} \neq i - 1$, and $a_{31} \neq n$.

Riordan has developed the formula

$$(3.5) \qquad K(3, n) = \sum \binom{n}{k} D_{n-k} D_k \, U_{n-2k} \qquad (4)$$

where $m = \left[\frac{n}{2}\right]$ and $U_0 = 1$.

Enumeration of Latin rectangles of more than three lines has scarcely been touched. One formula states that if $r < (\log n)^{3/2}$ then $L(r,n) \sim (n!)^r \; e^{-\binom{r}{2}}$ and it has been established that this remains valid for $r < n^{1/3}$.

If $r = s = n$, the Latin rectangle becomes a <u>Latin square of order n</u>. It is mentioned in passing that a multiplication table of a finite group depicts a Latin square. From previous discussion we have

$$(3.6) \qquad L(n, n) = n! \cdot (n - 1)! \; l_n$$

where l_n is the number of Latin squares of order n with the first row and the first column in standard order (i.e., $a_{i1} = i$ and $a_{1j} = j$, $i = j = 1,2,\dots,n$). That the evaluation of l_n is not easy is obvious from the following table which displays all of the known values of l .

n	1	2	3	4	5	6	7
l_n	1	1	1	5	56	9408	16,942,080

CHAPTER IV

A THEOREM OF RAMSEY

1. A fundamental theorem.

This chapter is devoted to the statement, description and some applications of a very important combinatorial theorem. The theorem is called Ramsey's theorem after the English logician F. P. Ramsey.

The well-known pigeon-hole principle in mathematics asserts that if a set of sufficiently many elements is partitioned into not-too-many subsets, then at least one of the subsets must contain many of the elements. Ramsey's theorem may be considered as a profound generalization of this principle.

Unfortunately, a thorough discussion of the meaning of Ramsey's theorem, a complete proof of the theorem, and and adequate discussion of its applications proves to be too voluminous for the limited scope of this paper; in fact, these topics in themselves might well provide the basis for another such paper. However, the topic is of sufficient importance that some discussion of it seems advisable.

Let S be an n-set and let $P_r(S)$ be the set of all r-subsets of S. Let

$$P_r(S) = A_1 \cup A_2 \cup \dots \cup A_t \tag{1.1}$$

be an arbitrary ordered partition of $P_r(S)$ into t components $A_1, A_2, \ldots, A_t$. Let $q_1, q_2, \ldots, q_t$ be integers such that

(1.2) $$1 \leq r \leq q_1, q_2, \ldots, q_t .$$

If there exists a q_i-subset of S with all of its r-subsets in A_i, then that subset is called a $\underline{(q_i, A_i)}$-subset of S. Ramsey's theorem asserts the following.

Theorem 1.1 Let $q_1, q_2, \ldots, q_t$, and r be any given integers satisfying (1.2). Then there exists a minimal positive integer, $N(q_1, q_2, \ldots, q_t, r)$, such that for all $n \geqslant N(q_1, q_2, \ldots, q_t, r)$, if S is an n-set and (1.1) an arbitrary ordered partition of $P_r(S)$ into t components, then S contains a (q_i, A_i)-subset for some $i = 1, 2, \ldots, t$.

The complexity of the theorem makes it very difficult to state it in any simpler terms. However, several readings of the theorem itself and the material preceding the theorem should make the assertion of the theorem clear.

No attempt is made here to prove either Ramsey's theorem or any of the several following statements. These proofs are contained in a recent text by Ryser. (5)

(1.3) if $r = 1$, $N(q_1, q_2, \ldots, q_t, 1) = q_1 + q_2 + \ldots + q_t - t + 1$.

(1.4) $$N(q_1, r, r) = q_1$$

(1.5) $$N(r, q_2, r) = q_2$$

The integers $N(q_1, q_2, r)$ have deep combinatorial significance, but unfortunately no recurrence is known for these integers, and serious difficulties are encountered in their evaluation. The trivial values of (1.3), (1.4) and (1.5) are known but apart from these all known $N(q_1, q_2, 2)$ are contained in the following symmetric array for $N(q_1, q_2, 2)$.

(1.6)

		q_2		
		3	4	5
	3	6	9	14
q_1	4	9	18	
	5	14		

Even less is known for $t > 2$. In this case the main piece of information at present is

(1.7) $$N(3, 3, 3) = 17.$$

2. Applications.

a). Given six points in a plane, no three collinear, there are $C(6,2) = 15$ line segments connecting the points. Let each segment be colored either red or blue. All may be red, all blue or some red and some blue. By using the pigeon-hole principle and simple logic it can be readily determined that it is always possible to find a chromatic triangle; that is, a triangle connecting three of the points that has all three sides the same color.

Extending this idea, consider n points in general position in three-dimensional space. Two distinct points determine a line segment. Let each of these segments be colored either red or blue. The 2-subsets of points may be partitioned into the set A_1 of red segments and the set A_2 of blue segments. Now if q_1 and q_2 are integers such that $2 \leq q_1, q_2$ and if $n \geqslant N(q_1, q_2, 2)$, then Ramsey's theorem asserts that either there are q_1 points with all segments red or q_2 points with all segments blue. Moreover, $N(q_1, q_2, 2)$ is the minimal integer with this property.

b). A submatrix of order m of a matrix A of order n is called principal provided the submatrix is obtained from A by deleting n-m of its rows and the same n-m columns.

Theorem 2.1 Let m be an arbitrary positive integer. Then every (0,1)-matrix A of a sufficiently large order n contains a principal submatrix of order m of one of the following types:

$$(2.1)\qquad \begin{bmatrix} * & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & * \end{bmatrix} \quad \begin{bmatrix} * & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 1 & & & * \end{bmatrix}$$
$$\begin{bmatrix} * & & & 1 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & * \end{bmatrix} \quad \begin{bmatrix} * & & & 1 \\ & \cdot & & \\ & & \cdot & \\ 1 & & & * \end{bmatrix}$$

The asteriks on the main diagonal denote 0's and 1's, but the entries above and below the main diagonal are all 0's or all 1's as indicated in the upper-right and lower-left corners in (2.1).

Proof: Let the n-set S of Ramsey's theorem be the set of the n row vectors of $A = [a_{ij}]$. Denote row i of A by α_i. Let $i < j$, and associate with the row vectors α_i and α_j of A the vector (a_{ji}, a_{ij}). Now this vector is (0,0),(1,0),(0,1), or (1,1). Hence the 2-subsets of S are partitioned.

$$(2.2)\qquad P_2(S) = A_1 \cup A_2 \cup A_3 \cup A_4 .$$

Now suppose that

(2.3) $$n \geq N(m, m, m, m, 2).$$

Then by Ramsey's theorem there exists an m-subset of S with all of its 2-subsets in one of the four components of $P_{a}(S)$. But this implies the existence of a principal submatrix of one of the four types of (2.1). #

c). Theorem 2.2 Let m be an integer greater than or equal to three. Then there exists a minimal positive integer N_m such that the following proposition is valid for all integers $n \geq N_m$. If n points in the plane have no three points collinear, then m of the points are the vertices of a convex m-gon.

Lemma 2.3. If five points in the plane have no three points collinear, then four of the points are the vertices of a convex quadrilateral.

Proof: The five points determine ten line segments, and the perimeter of this configuration is a convex polygon. If this convex polygon is a pentagon or a quadrilateral, the lemma is trivial. Suppose that the convex polygon is a triangle. Then two of the five points are in the interior of the triangle. The two interior points determine a straight line, and two of the three points of the triangle lie on one side of this line. Then these two points of the triangle and the two interior points form a convex quadrilateral.

Lemma 2.4. If m points in the plane have no three points collinear and if all quadrilaterals formed from the m points are convex, then the m points are the vertices of a convex m-gon.

Proof: The m points determine $\frac{m(m-1)}{2}$ line segments, and the perimeter of this configuration is a convex q-gon. Let the consecutive vertices of the q-gon be labeled V_1, $V_2, \ldots, V_q$. If one of the points is within the q-gon, it must lie in one of the triangles $V_1V_2V_3$, $V_1V_3V_4$, ... , $V_1V_{q-1}V_q$. But this contradicts the assertion that all quadrilaterals formed from the m points are convex. Hence q = m and the m-gon is convex.

Theorem 2.2 is now an easy consequence of Ramsey's theorem. To prove this let $m \geq 4$ and let $n \geq N(5,m,4)$. Partition the 4-subsets of the n points into the concave and the convex quadrilaterals. Then by Ramsey's theorem there exists a 5-gon with all quadrilaterals concave or an m-gon with all quadrilaterals convex. By Lemma 2.3 the first alternative cannot occur, and by Lemma 2.4 the m-gon is convex. #

It has been shown that

(2.4) $$N_m \leq N(5, m, 4).$$

It is known that $N_3 = 3 = 2 + 1$, $N_4 = 5 = 2^2 + 1$, and it has been shown that $N_5 = 9 = 2^3 + 1$. This leads one to conjecture that

(2.5) $$N_m = 2^{m-2} + 1,$$

but the assertion (2.5) is an unsettled question.

CHAPTER V

SYSTEMS OF DISTINCT REPRESENTATIVES

1. A fundamental theorem.

Let S be an arbitrary set and P(S) the set of all subsets of S. Let

(1.1) $$D = (a_1, a_2, \ldots, a_m)$$

be an m-sample of S and let

(1.2) $$M(S) = (S_1, S_2, \ldots, S_m)$$

be an m-sample of P(S). Now suppose that the m elements of D are distinct and that $a_i \in S_i$ $(i = 1, 2, \ldots, m)$. Then the element a_i <u>represents</u> the set S_i, and the subsets $S_1, S_2, \ldots, S_m$ have a <u>system of distinct representatives</u> (SDR). D is an SDR for M(S). This definition requires that $i \neq j$ implies $a_i \neq a_j$, but S_i and S_j need not be distinct subsets of S.

Examples Let $S = \{1,2,3,4,5,6\}$. Let $S_1 = \{2,5\}$, $S_2 = \{2,5\}$, $S_3 = \{2,6\}$, $S_4 = \{1,2,3,4\}$, $S_5 = \{1,2,5\}$. Then $D = (2,5,6,3,1)$ is an SDR for $(S_1, S_2, S_3, S_4, S_5)$. If S_5 is replaced by $S_5' = \{2,5\}$, then the subsets have no SDR, for $S_1 \cup S_2 \cup S_5'$ is a 2-set and three elements are required to represent S_1, S_2, S_5'.

Theorem 1.1 (by P. Hall). The subsets $S_1, S_2, \ldots, S_m$ have an SDR if and only if the set $S_{i_1} \cup S_{i_2} \cup \ldots \cup S_{i_k}$ contains at least k elements. This must hold for

$k = 1,2,\ldots,m$ and for all k-combinations $\{i_1, i_2, \ldots, i_k\}$ of the integers $1,2,\ldots,m$.

From the definition and the preceding example the validity of the necessity of this theorem is immediately apparent.

The following theorem gives a refinement on the sufficiency of Theorem 1.1.

Theorem 1.2 Let the subsets $S_1, S_2, \ldots, S_m$ satisfy the necessary conditions for the existence of an SDR and let each of these subsets contain at least t elements. If $t \leq m$, then M(S) has at least t! SDR's. If $t > m$, then M(S) has at least $\frac{t!}{(t-m)!}$ SDR's.

Proof (by induction on m): Let $m = 1$. If $t \leq m$, $t = 1$ and M(S) has $1! = 1$ SDR. If $m = 1$ and $t > m$, M(S) obviously has t SDR's; but $t = \frac{t!}{(t-1)!} = \frac{t!}{(t-m)!}$.

For the induction hypothesis, take the statement of the theorem for all m'-samples of P(S) where $m' < m$, and prove the theorem for the m-sample $M(S) = (S_1, S_2, \ldots, S_m)$.

Case 1 Assume the set $S_{i_1} \cup S_{i_2} \cup \ldots \cup S_{i_k}$ contains at least $k + 1$ elements. This holds for $k = 1,2,\ldots,m-1$ and for all k-combinations $\{i_1, i_2, \ldots, i_k\}$ of the integers $1,2,\ldots,m$. Let a_i be a fixed element of S. Delete a_i whenever it appears in $S_2, S_3, \ldots, S_m$ and call the resulting sets $S_2', S_3', \ldots, S_m'$, respectively. The (m-1)-sample $M'(S) = (S_2', S_3', \ldots, S_m')$ satisfies the necessary condition for the existence of an SDR because the set $S_{i_1} \cup S_{i_2} \cup \ldots \cup S_{i_k}$ contains at least $k + 1$ elements. Now if $t \leq m$ then

$t - 1 \leq m - 1$ and by the induction hypothesis $M'(S)$ has at least $(t-1)!$ SDR's. Also, for $t > m$, then $t - 1 > m - 1$ and again by the induction hypotheses $M'(S)$ has at least $\frac{(t-1)!}{[(t-1) - (m-1)]!} = \frac{(t-1)!}{(t-m)!}$ SDR's. But taking any SDR for $M'(S)$ together with a_i gives an SDR for $M(S)$ in which a_i represents S_1. Hence for $t \leq m$ and a fixed a_i there are $(t-1)!$ SDR's for $M(S)$. But S_1 is a t-set and a sample is ordered hence there are $t \cdot (t-1)! = t!$ SDR's for $M(S)$. For $t > m$, using the same argument, there are $t \cdot \frac{(t-1)!}{(t-m)!} = \frac{t!}{(t-m)!}$ SDR's for $M(S)$.

Case 2 There exists a k-subset of S of the form $S_{i_1} \cup S_{i_2} \cup \dots \cup S_{i_k}$, where k is an integer such that $1 \leq k \leq m-1$ and $\{i_1, i_2, \dots, i_k\}$ is a certain k-combination of the integers $1, 2, \dots, m$. Renumber the subsets $S_1, S_2, \dots, S_m$ so that $S_{i_1} \cup S_{i_2} \cup \dots \cup S_{i_k}$ is $S_1 \cup S_2 \cup \dots \cup S_k$. If this k-subset exists, then of necessity $t \leq k$. Since $k \leq m-1$ the induction hypothesis implies the k-sample $(S_1, S_2, \dots, S_k)$ has at least $t!$ SDR's. Let $D^* = (a_1, a_2, \dots, a_k)$ denote one such SDR. Whenever the elements of D^* appear in the sets $S_{k+1}, S_{k+2}, \dots, S_m$, delete them and call the resulting sets $S^*_{k+1}, S^*_{k+2}, \dots, S^*_m$, respectively. The (m-k)-sample

(1.3) $$M^*(S) = (S^*_{k+1}, S^*_{k+2}, \dots, S^*_m)$$

must satisfy the necessary conditions for the existence of an SDR for if, say, $S^*_{k+1} \cup S^*_{k+2} \cup \dots \cup S^*_{k+k^*}$ contains fewer than k^* elements, then

(1.4) $$S_1 \cup S_2 \cup \dots \cup S_k \cup S_{k+1} \cup S_{k+2} \cup \dots \cup S_{k+k^*}$$

contains fewer than $k + k^*$ elements which contradicts the hypothesis of the theorem. Hence by the induction hypothesis $M^*(S)$ has at least one SDR. But as stated earlier, $(S_1, S_2, \ldots, S_k)$ has at least $t!$ SDR's. Consequently $M(S)$ has at least $t!$ SDR's, which proves Theorem 1.2 and also Theorem 1.1. #

2. Partitions.

Let

$$T = A_1 \cup A_2 \cup \ldots \cup A_m \tag{2.1}$$

and

$$T = B_1 \cup B_2 \cup \ldots \cup B_m \tag{2.2}$$

denote two partitions of a set T such that $A_i \neq \emptyset \neq B_j$ for $i, j = 1,2,\ldots,m$. Let E be an m-subset of T such that $A_i \cap E \neq \emptyset$, $B_j \cap E \neq \emptyset$, $i, j = 1,2,\ldots,m$. Then each of these intersections must be a 1-set and E is called a <u>system of common representatives</u> (SCR) for the partitions (2.1) and (2.2). Note that an SCR exists for these partitions if and only if there is a suitable renumbering of the components of either (2.1) or (2.2) such that

$$A_i \cap B_i \neq \emptyset \qquad (i = 1,2,\ldots,m). \tag{2.3}$$

SDR theory is used to obtain the following necessary and sufficient condition for the existence of an SCR.

Theorem 2.1 The partitions (2.1) and (2.2) have an SCR if and only if the set $A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k}$ contains at most k of the components $B_1, B_2, \ldots, B_m$. This must hold for $k = 1,2,\ldots,m$ and for all k-combinations $\{i_1, i_2, \ldots, i_k\}$ of the integers $1,2,\ldots,m$.

Proof: Again, the necessity of the theorem is apparent. To prove the sufficiency, let S be the m-set of elements $A_1, A_2, \ldots, A_m$ and let S_i be the set of all A_j such that $A_j \cap B_i \neq \emptyset$. Then $M(S) = (S_1, S_2, \ldots, S_m)$ is an m-sample of subsets of S. Further, M(S) satisfies the necessary condition for the existence of an SDR for if, say, $S_1 \cup S_2 \cup \ldots \cup S_{k+1}$ contains only k elements $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$, then $A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k}$ contains the k + 1 components $B_1, B_2, \ldots, B_{k+1}$, contrary to the hypothesis of the theorem. Hence by Theorem 1.1 there exists an SDR for M(S). Now renumber the components of (2.1) so that this SDR is $D = (A_1, A_2, \ldots, A_m)$. But then (2.3) is valid. #

Theorem 2.2 Let $T = A_1 \cup A_2 \cup \ldots \cup A_m$ and $T = B_1 \cup B_2 \cup \ldots \cup B_m$ denote two partitions of T, where each A_i and each B_j is an r-subset of T. Then the partitions have an SCR.

Proof: If each A_i is an r-subset of T, then $A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k}$ is an rk-subset of T. Each B_j is an r-subset, therefore $A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k}$ contains at most k of the components $B_1, B_2, \ldots, B_m$, and this must be true for k = 1,2,...,m and for all k-combinations $\{i_1, i_2, \ldots, i_k\}$ of the integers 1,2,...,m. Then Theorem 2.1 implies the partitions have an SCR. #

Applications a) Let A be the following r by m array of the integers 1,2,...,rm.

$$(2.4) \qquad A = \begin{bmatrix} 1 & 2 & \cdots & m \\ m+1 & m+2 & \cdots & 2m \\ 2m+1 & 2m+2 & \cdots & 3m \\ \vdots & \vdots & \vdots & \vdots \\ (r-1)m+1 & (r-1)m+2 & \cdots & rm \end{bmatrix}$$

Now let B be an r by m array of the integers $1,2,\ldots,rm$, but with the integers in arbitrary positions within B. Then Theorem 2.2 implies there exists a permutation of the columns of B such that corresponding columns of A and B each contain atleast one element in common.

b) This application requires an understanding of the elementary properties of cosets in the theory of groups.

Let G be a finite group and let H be a subgroup of G. Let $G = Hx_1 \cup Hx_2 \cup \ldots \cup Hx_m$ be a right coset decomposition for H and let $G = y_1H \cup y_2H \cup \ldots \cup y_m H$ be a left coset decomposition for H.

Then Theorem 2.2 implies there exists elements $z_1, z_2, \ldots, z_m$ in G such that $G = Hz_1 \cup Hz_2 \cup \ldots \cup Hz_m = z_1H \cup z_2H \cup \ldots \cup z_mH$.

3. Latin rectangles.

Let there be given an r by s Latin rectangle based on n elements labeled $1,2,\ldots,n$. The Latin rectangle may be <u>extended</u> to a Latin square of order n provided n-r rows and n-s columns can be adjoined to the Latin rectangle so that the resulting configuration is a Latin square of order n. This new configuration will contain the original Latin rectangle in the upper left corner.

Theorem 3.1 Let there be given an r by n Latin rectangle based on n elements labeled $1,2,\ldots,n$. Then the Latin rectangle may be extended to a Latin square of order n.

Proof (using SDR theory): Let S be the n-set of elements $1,2,\ldots,n$ and S_i be the set of all elements of S that do not appear in column i of the Latin rectangle. Then each S_i is an (n-r)-subset of S and $M(S) = (S_1, S_2, \ldots, S_n)$ is an n-sample of subsets of S. Let i be an element of S. Then i appears exactly r times in the Latin rectangle, once in each row. Also, the appearances are in distinct columns. Hence i is in exactly n-r of the sets $S_1, S_2, \ldots, S_n$. Now if $S_1 \cup S_2 \cup \ldots \cup S_k$ contains only k-1 elements, then these k-1 elements appear in the sets $S_1, S_2, \ldots, S_k$ no more than (n-r)(k-1) times. But this contradicts the fact that each of these sets is an (n-r)-subset of S. Hence M(S) satisfies the necessary condition for an SDR, and therefore has an SDR. Denote this SDR by $D = (i_1, i_2, \ldots, i_n)$. Since i_j is in S_j, $j = 1,2,\ldots,n$, i_j does not appear in column j of the Latin rectangle. Hence D may be adjoined to the r by n Latin rectangle to form an r + 1 by n Latin rectangle. Now repeat the entire process, and keep repeating it until it has been performed n-r times. The result is the required Latin square of order n. #

Theorem 3.2 There are at least $n!(n-1)! \ldots (n-r+1)!$ r by n Latin rectangles and hence at least $n!(n-1)! \ldots 1!$

n by n Latin squares.

Proof: There are n! Latin rectangles of size 1 by n. Theorems 3.1 and 1.2 imply each of these may be extended to a 2 by n Latin rectangle in (n-1)! ways. Hence there are n!(n-1)! Latin rectangles of size 2 by n. Repetition of the same argument proves the theorem. #

Let l_n denote the number of Latin squares of order n with the first row and first column in standard order. Then Theorem 3.2 asserts

$$l_n \geq (n-2)!(n-3)! \ldots 1! \tag{3.1}$$

The following table displays the values of l_n and $b_n = (n-2)!(n-3)! \ldots 1!$ for n = 3,4,5,6,7.

n	3	4	5	6	7
l_n	1	4	56	9408	16,942,080
b_n	1	2	12	288	34,560

4. Matrices of zeros and ones.

The (0,1)-matrices mentioned at the conclusion of Chapter II play a leading role in the development of many combinatorial topics. One of the chief reasons for this follows.

Let S be an n-set of elements $a_1, a_2, \ldots, a_n$ and let $M(S) = (S_1, S_2, \ldots, S_m)$ be an m-sample of subsets of S. Let $a_{ij} = 1$ if a_j is a member of S_i and let $a_{ij} = 0$ if a_j is not a member of S_i. Then

$$A = [a_{ij}] \qquad (i = 1,2,\ldots,m;\ \ j = 1,2,\ldots,n) \tag{4.1}$$

is a (0,1)-matrix of size m by n. This matrix is called the <u>incidence matrix</u> for the subsets $S_1, S_2, \ldots, S_m$ of the n-set S. The 1's in row i of A specify the elements that belong to S_i , and the 1's in column j of A specify the sets that contain a_j . Thus A contains a complete description of the subsets $S_1, S_2, \ldots, S_m$ of S. Also, given a (0,1)-matrix, A, of size m by n, and an arbitrary n-set, S, then there exists subsets $S_1, S_2, \ldots, S_m$ of S such that A is the incidence matrix for these subsets.

Thus the (0,1)-matrix A characterizes the subsets $S_1, S_2, \ldots, S_m$ of S. A choice of +1 and -1, or even of two distinct entries x and y would serve just as well as 0 and 1. However, the behavior of 0 and 1 under addition and multiplication makes them especially convenient as illustrated in the following theorem.

Theorem 4.1 Let $S_1, S_2, \ldots, S_m$ be subsets of an n-set and let $m \leq n$. Let A be the incidence matrix for these subsets. Then the number of SDR's for $M(S) = (S_1, S_2, \ldots, S_m)$ is per (A).

Proof: By definition, per $(A) = \sum a_{1i_1}, a_{2i_2}, \ldots, a_{mi_m}$ where the summation extends over all m-permutations $(i_1, i_2, \ldots, i_m)$ of $1, 2, \ldots, n$. Also, the definition of per (A) requires $m \leq n$, as does the hypothesis of this theorem. Note that for the incidence matrix A, each product in the summation must be 0 or 1. Note also that each product represents a possible SDR, since it contains m factors, no two from the same row or the same column. Hence if a product has the value zero, one of the factors a_{ij} in the product is not in set S_i; that is, S_i is not represented in that product hence the product does not represent an

SDR for $M(S) = (S_1, S_2, \ldots, S_m)$. If, on the other hand, a given product has the value 1, then that product represents an SDR for $M(S)$ for it indicates the existence of a selection of m distinct objects, one from each of the S_i. Since the per (A) is a summation over all m-permutations $(i_1, i_2, \ldots, i_m)$ of $1, 2, \ldots, n$, every possible SDR for $M(S)$ is considered and the summation represents the total number of SDR's. #

A <u>permutation</u> <u>matrix</u> P is a (0,1)-matrix of size m by n such that $PP^T = I$, where P^T denotes the transpose of P and I denotes the identity matrix of order m. This definition implies $m \leq n$. In a permutation matrix of order m all entries are 0 with the exception of exactly one entry in each row and each column, which are 1. If the elements and the subsets of S are now renumbered the incidence matrix A is replaced by an incidence matrix A' of the form

(4.2) $$A' = PAQ.$$

Here P is a permutation matrix of order m determined by the renumbering of the subsets, and Q is a permutation matrix of order n determined by the renumbering of the elements. Many investigations involving the (0,1)-matrix A deal with functions like per (A) that remain invariant under arbitrary permutations of the rows and columns of A, and such functions are of interest in combinatorics because they do not depend on the particular labeling of the elements and subsets of S.

Example Let $S = \{a, b, c, d, e\}$, $S_1 = \{a, c\}$,

$S_2 = \{a,b,d\}$, $S_3 = \{b,c,d,e\}$, and $S_4 = \{a,c,e\}$. Let $M(S) = (S_1,S_2,S_3,S_4)$ and note that $S_i \subseteq S$, $i = 1,2,3,4$. Following the method of section four, with $a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = d$ and $a_5 = e$, the incidence matrix for the given subsets of the 5-set S is

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} .$$

If the columns are labeled from left to right as a,b,c,d,e and the rows are labeled from top to bottom as S_1,S_2,S_3,S_4 the composition of the given subsets is immediately apparent from the appearance of the incidence matrix.

Now renumber the subsets and the elements so that $S_1 = \{b,c,d,e\}$, $S_2 = \{a,c,e\}$, $S_3 = \{a,c\}$, $S_4 = \{a,b,d\}$, $a_1 = b$, $a_2 = d$, $a_3 = c$, $a_4 = e$, and $a_5 = a$. The incidence matrix for this new numbering is:

$$A' = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} .$$

Note that the renumbering of the subsets and elements is nothing more than a permutation of the subsets and a permutation of the elements. Hence (S_1,S_2,S_3,S_4) became (S_3,S_4,S_1,S_2). This makes apparent the permutation matrix

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} .$$

Also, $(a_1, a_2, a_3, a_4, a_5)$ became $(a_5, a_1, a_3, a_2, a_4)$ which gives the permutation matrix

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} .$$

Note now that $A' = PAQ$.

5. Term rank.

A <u>line</u> of a matrix designates either a row or a column of the matrix. The <u>trace</u> of a matrix is the sum of the entries on the main diagonal of the matrix. Let A be a (0,1)-matrix. The <u>term rank</u> of A is the maximal number of 1's in A with no two 1's on a line. Thus the term rank of A is the maximal trace of A under arbitrary permutations of rows and columns of A. The term rank provides a convenient generalization of the SDR concept for the subsets $S_1, S_2, \ldots, S_m$ of an n-set S, for if A is the incidence matrix for these subsets, then the subsets have an SDR if and only if the term rank of A equals m.

Theorem 5.1 Let A be a (0,1)-matrix of size m by n. The minimal number of lines in A that contain all of the 1's in A is equal to the term rank of A.

Proof: Let ρ' be the minimal number of lines in A that contain all of the 1's in A and let ρ be the term rank of A. Then the theorem states that $\rho' = \rho$.

No line can contain two of the 1's that account for the ρ 1's of the term rank. Hence

(5.1) $$\rho' \geqslant \rho \quad .$$

Let the minimal covering of 1's by ρ' lines consist of e rows and f columns, where $e + f = \rho'$. Both ρ and ρ' are invariant under permutations of rows and columns of A. Hence these e rows and f columns may be taken as the initial rows and columns of the matrix, which can be written in the form

(5.2) $$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where A_1 is of size e by f. A_2 is of term rank e, for it may be regarded as an incidence matrix for subsets $S_1, S_2, \ldots, S_e$ of the (n-f)-set of the integers f + 1, f + 2, ..., n. These subsets must satisfy the necessary condition for the existence of an SDR, for if not, certain of the e rows can be replaced by fewer columns and retain the covering of 1's in A. Hence this covering will be accomplished with fewer than e + f lines which contradicts the minimality of ρ'. The transpose A_3^T of A_3 may be regarded as an incidence matrix for subsets, and it can similarly be shown that A_3 is of term rank f. Hence

(5.3) $$\rho \geqslant e + f = \rho' .$$

Hence from (5.1) and (5.3), $\rho = \rho'$. #

Theorem 5.1 can be immediately generalized. Let A be a matrix of size m by n with elements in a field F. The minimal number of lines in A that contain all of the nonzero entries in A is equal to the maximal number of nonzero entries in A with no two nonzero entries on a line.

Theorem 5.2 Let A be a matrix of size m by n. Let the entries of A be nonnegative reals and let $m \leqslant n$. Let each row sum of A equal m' and let each column sum of A equal n'. Then

$$A = c_1 P_1 + c_2 P_2 + \ldots + c_t P_t , \tag{5.4}$$

where in (5.4) each P_i is a permutation matrix and each c_i is a nonnegative real.

Proof: If A is not a square matrix, we replace A by

$$A' = \begin{bmatrix} A \\ \frac{m'}{n'} J \end{bmatrix} \tag{5.5}$$

where J is a matrix of 1's of size $n' - m'$ by n'. The matrix A' is of order n, and the entries of A' are nonnegative reals. Each row and column sum of A' is equal to m'. If A' is not the zero matrix, A' has n positive entries with no two on a line. For if A' did not have n such entries, then by the remarks following Theorem 5.1 we could cover the positive entries in A' with e rows and f columns, where $e + f < n$. But then $m' n \leqslant m \ (e + f) < m'n$, and this is a contradiction. Now let P be the permutation matrix of order n with 1's in the same positions occupied by the n positive entries of A'. Let c_1 be the smallest of these n entries. Then $A' - c_1 P_1'$ is a matrix whose entries are nonnegative reals. Also, $A' - c_1 P_1'$ has each row and column sum equal to the nonnegative real $m' - c_1$. But at least one more zero entry appears in $A' - c_1 P_1'$ than in A'. Hence we may now work on $A' - c_1 P_1'$, and we may repeat the argument until $A' = c_1 P_1' + c_2 P_2' + \dots + c_t P_t'$. But this gives us a decomposition of the form (5.2) for the matrix A. #

Theorem 5.3 Let A be a (0,1)-matrix of order n such that each row and column sum of A is equal to the positive integer k. Then

(5.6) $$A = P_1 + P_2 + \dots + P_k ,$$

where the P_i are permutations matrices.

Proof: This follows from the proof of Theorem 5.2. Each $c_j = 1$ and the process terminates in $t = k$ steps. #

Theorem 5.3 gives an affirmative answer to the following problem. A dance is attended by n boys and n girls. Each boy has been previously introduced to exactly k girls and each girl has been previously introduced to exactly k boys. No one desires to make further introductions. Can the boys and girls be paired so that no further introductions are necessary? Let $A = [a_{ij}]$ be the (0,1)-matrix defined by $a_{ij} = 1$ if the boy j has been previously introduced to girl i and 0 otherwise. Then A satisfies the requirements of Theorem 5.3, and the permutation matrix P_1 of (5.6) gives the desired pairing of boys and girls.

A matrix A of order n is called <u>doubly stochastic</u> provided its entries are nonnegative reals and its row and column sums are equal to 1. These matrices have been extensively studied in their own right because of their importance in the theory of transition probabilities. Theorem 5.2 implies the following.

Theorem 5.4 Let A be a doubly stochastic matrix of order n. Then

(5.7) $$A = c_1 P_1 + c_2 P_2 + \dots + c_t P_t ,$$

where the P_i are permutation matrices and the c_i are positive reals such that

(5.8) $$c_1 + c_2 + \dots + c_t = 1.$$

Let A be doubly stochastic. The entries of A are nonnegative reals so per (A) cannot exceed the product of the row sums of A. But since each row sum of A is 1, we have

(5.9) $$\operatorname{per}(A) \leq 1.$$

Equality holds in (5.9) if and only if the doubly stochastic A is a permutation matrix. By Theorem 5.4 it is clear that if A is doubly stochastic, then per (A) > 0. But if A is doubly stochastic of order n, then the determination of the minimal value of per (A) is a difficult unsolved problem. A conjecture of van der Waerden asserts

(5.10) $$\operatorname{per}(A) \geq \frac{n!}{n^n} .$$

Equality holds in (5.10) if $A = n^{-1} J$, where J is the matrix of 1's of order n. In fact this may be the only case of equality. The following conjecture is a generalization of (5.10). If A and B are doubly stochastic, then

(5.11) $$\operatorname{per}(AB) \leq \operatorname{per}(A), \operatorname{per}(B).$$

The special case $B = n^{-1} J$ of (5.11) is equivalent to (5.10).

EPILOGUE

The writer has attempted to present as much introductory material pertaining to combinatorial mathematics in general and existence problems in particular as a paper of such limited scope permits.

Much more development along these same lines is possible, some of which is contained in the work by Ryser(6) which has served as the basis for this paper.

For a much more elementary treatment of many of these topics, Niven's Mathematics of Choice (7) is recommended. For a much more advanced treatment, development upon different lines, and extensions to construction and enumeration problems, the most popular works seem to be those of Riordan (8) and MacMahon (9), both of which require more extensive background knowledge than the aforementioned two books.

LIST OF REFERENCES

1. RYSER, HERBERT JOHN. Combinatorial Mathematics. John Wiley & Sons, Inc., 1963, p. 22.

2. Ibid., pp. 25 - 28.

3. Ibid., Reference 4, p. 37.

4. Ibid, Reference 4, p. 37.

5. Ibid., pp. 39 - 43.

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ABSTRACT

MATHEMATICS

BERGMAN, RONALD C. B.S., Indiana University of Pennsylvania, 1962.

Selected Introductory Concepts from Combinatorial Mathematics

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Master of Science degree conferred August 3, 1967

Thesis dated August, 1967

This paper is concerned with the development of that part of combinatorial mathematics that deals with existence-type problems. This development is accomplished through the framework of modern algebra. Beginning with such elementary topics as sets, permutations, and combinations the paper goes on to the principle of inclusion and exclusion, recurrence relations, the elegant Theorem of Ramsey, and an introduction to systems of distinct representatives.

In addition to the treatment of combinatorial mathematics as a mathematical system in itself, a few of the multitudinous applications of this theory are presented. These include applications and relationships to the theory of numbers, matrices, group and field theory, and conbimatorial-type problems which occur in every day life.