Cyclic Interference Alignment and Cancellation in 3-User X-Networks with Minimal Backhaul

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Abstract—We consider the problem of cyclic interference alignment (IA) on the 3-user X-network and show that it is infeasible to exactly achieve the upper bound of $\frac{K^2}{2K-1} = \frac{9}{5}$ degrees of freedom for the lower bound of $n = 5$ signalling dimensions and $K = 3$ user-pairs. We observe that this infeasibility goes beyond the problem of common eigenvectors in invariant subspaces within spatial IA. In order to gain non-asymptotic feasibility with minimal intervention, we first investigate a communication strategy that enables IA by feedforwarding a subset of messages with minimal rate. In a second step, we replace the proposed feedforward strategy by an analogous cyclic interference alignment and cancellation scheme with a backhaul network on the receiver side and also by its dual cyclic interference neutralization scheme with a backhaul network on the transmitter side.

I. Introduction

In the seminal work [1], the basic principle of interference alignment (IA) and its consequences for a K-user interference channel is introduced. Besides the thorough discussion of the spatial IA scheme, those authors also briefly described IA in terms of a propagation delay based channel for an illustrative example. The idea of IA by propagation delay is further expanded in [2], [3]. In [4], the given example is revisited by formulating a cyclic channel model that operates on cyclically shifted polynomials. The mathematical framework is inspired by the representation of algebraically formulated cyclic codes using polynomial rings as in [5]. The proposed cyclic polynomial channel model (CPCM) is closely related to the linear deterministic channel model (LDCM) introduced in [6]. Especially in the case of multiple users exceeding two transmitter-receiver pairs, capacity results in the LDCM are mainly provided for symmetric channel gains, e.g., as in [7]. A significant property of the LDCM is the linear down-shift of finite dimensional coding vectors. But this property involves to track a number of side-effects when considering general asymmetric channels. It is quite challenging to derive optimal communication schemes in closed-form since the number of parameters involved increases exponentially with the total number of users K. In contrast to such linear shifts, we observed that the use of cyclic shifts is quite beneficial to
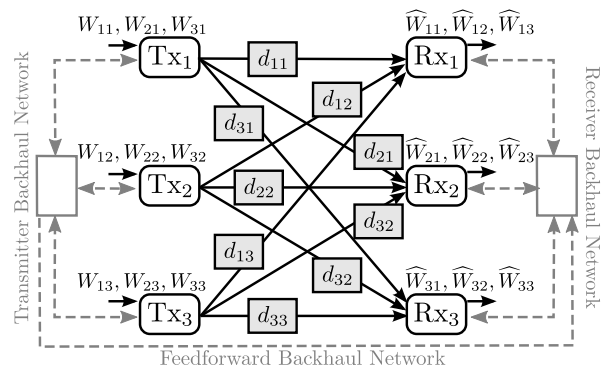
Fig. 1. The fully-connected 3-user X-network with 3 transmitters $\mathrm{Tx}_1, \mathrm{Tx}_2$, Tx_3, 3 receivers $\mathrm{Rx}_1, \mathrm{Rx}_2, \mathrm{Rx}_3$, 9 independent messages W_{ji} and 9 corresponding estimated messages $\widehat{W}_{ji}$. The influence of the channel between transmitter Tx_i and receiver Rx_j is parameterized by d_{ji} as indicated by solid black arrows. The interference-free backhaul networks for IA, IAC and IN are depicted by dashed grey arrows.

derive closed-form solutions with a notable lower complexity, even for arbitrary channel symmetry. The CPCM also applies to line-of-sight channels [8] with limited multipath.

The combination of interference alignment and cancellation (IAC) is initially introduced in [9] and also applied in [10]. Therein, a backhaul network (BHN) provides a limited exchange of decoded messages at the receiver-side to support the cancellation of known interference by the aid of other receivers. Yet another related approach important for this work is to inhibit interference by *Interference Neutralization* (IN) [11], a communication scheme cancelling interference 'over the air' by aligning complementary versions of the same message within the same signalling space.

In this work, we first address an open problem stated in the discussion of [2, Sect. V]: Is perfect IA by propagation delay feasible on X-networks with more than 2 users? We show that perfect cyclic IA in a K-user X-network for $K \geq 3$ users is actually overconstrained and hence infeasible. We observe that this property goes even beyond the closely related problem of common invariant subspaces in spatial IA [12]. In order to tackle this infeasibility, we first analyze a simple feedforward scheme achieving $\frac{9}{5}$ degrees of freedom (DoF) within 5 dimensions and only one message over the interference-free feedforward BHN. Our second step leads us to a related cyclic

IAC scheme on a receiver-sided BHN. It achieves the $\frac{9}{5}$ DoF of perfect IA with a minimum of only 2 messages over the BHN. We observe a duality between IAC on the receiver-sided BHN and IN on the dual transmitter-sided BHNs. In all three cases, these BHNs provide a minimal number of cognitive messages for specific sets of users only. This insight is related to the observations in [13] for a 2-user interference channel.

The CPCM of the K-user X- network is presented in Sec. II. The infeasibility of perfect cyclic IA is shown in Sec. III. Cyclic IA with a feedforward BHN, the cyclic IAC scheme and the cyclic IN scheme are presented in Sec. IV.

II. System Model

We adapt the notation from [4] to the given problem. A K-*user* X-*network* describes a wireless channel with K transmitters and K receivers, as depicted in Fig. 1 for $K = 3$. The set of user indices is $\mathcal{K} := \{1, 2, \ldots, K\}$. The K-user X-network is physically equivalent to the K-user interference channel [1], [4]. But in the K-user X-network, there are K^2 unicast messages with exactly one dedicated message from each transmitter to each receiver. A dedicated message from transmitter Tx_i to receiver Rx_j, $i, j \in \mathcal{K}$, is denoted by W_{ji}.

We consider polynomial rings $\mathbb{F}(x)$ modulo $x^n - 1$ with the indeterminate x. The channel access at each Tx_i and Rx_j is partitioned into $n \in \mathbb{N}$ equally sized dimensions, each normalized to length one. A single dimension in the period of n dimensions is addressed by one of the *offsets* $x^0, x^1, \ldots, x^{n-1}$. A transmitter Tx_i allocates coded messages to each coefficient of a polynomial in x. A message W_{ji} can be represented by a coefficient in a binary field $\mathbb{F}_2$ (or a binary extension field of finite degree), for instance. Altogether, the transmitted signal from Tx_i is a polynomial with messages $W_{ji} \in \mathbb{F}$ for each receiver Rx_j:

$$v_i(x) \equiv \sum_{j \in \mathcal{K}} W_{ji} x^{p_{ji}} \mod (x^n - 1). \tag{1}$$

The parameters $p_{ji} \in \mathbb{N}$ allocate the message W_{ji} to a particular offset. The influence of the channel is modelled by cyclic shifts of the transmitted polynomials. A cyclic shift of k offset positions in $v(x)$ is expressed by the multiplication $x^k v(x)$ modulo $x^n - 1$. E.g., to shift $u(x) = W x^l$ by k positions, we compute $x^k u(x) \equiv W x^{l+k} \mod (x^n - 1)$ within the period of n dimensions. An arbitrary shift between a pair $(\text{Rx}_j, \text{Tx}_i)$ is denoted by $d_{ji} \in \mathcal{D} = \{x^k | k \in \mathbb{N}\}$. The channel matrix is defined by $\boldsymbol{D} = (d_{ji})_{1 \le j \le K, 1 \le i \le K}$ and it is assumed to be fully known to all users and constant over time. The received signal $r_j(x)$ at Rx_j is a superposition of shifted signals $u_i(x)$:

$$r_j(x) \equiv \sum_{i \in \mathcal{K}} d_{ji} v_i(x) \mod (x^n - 1). \tag{2}$$

In the CPCM, the achieved DoF measure the total number of dedicated messages M received interference-free per n dimensions [4]:

$$\text{DoF} = \frac{M}{n}. \tag{3}$$

The total number of the DoF is upper bounded for m_{ji} submessages per transmitter-receiver pair $(\text{Rx}_j, \text{Tx}_i)$ by:

$$\text{DoF} \le \frac{\sum_{j=1}^{K_{\text{R}}} \sum_{i=1}^{K_{\text{T}}} m_{ji}}{\max\limits_{j, i \in \mathcal{K}} \left(\sum_{k=1}^{K_{\text{T}}} m_{jk} + \sum_{l=1}^{K_{\text{R}}} m_{li} - m_{ji} \right)}. \tag{4}$$

as shown in [4, Sec. VI]. The numerator represents the total number of messages M and the denominator is the lower bound on the number of dimensions n. This upper bound for the CPCM is basically analogous to the bound in [12, Thm. 1] but applied to the CPCM-version of a multi-user X- network. The bound on the number of DoF is maximal if the $\max$-term in (4) is the same for all m_{ji}. In the symmetric case, i.e., with $m_{ji} = L$ submessages, for $L \in \mathbb{N}$ and all $i, j \in \mathcal{K}$, the upper bound yields its maximum $\frac{K^2}{2K-1}$ DoF, e. g., $\frac{9}{5}$ DoF for $K = 3$.

Note that the cyclic shifts can simply be unrolled over time such that they model a propagation delay scenario as in [4].

III. Cyclic IA on the 3-user X- Network

The case for $K = 2$ with $m_{ji} = 1$ for all $i, j \in \mathcal{K}$, is already solved in our previous work [4, Sec. III]. Here, each transmitter intends to convey $K = 3$ dedicated messages, one to each receiver. We have $M = K^2 = 9$ unicast messages in total. Each receiver must decode three dedicated messages, while it must cope with two interfering messages per transmitter.

A. Separability Conditions

To successfully decode, each dedicated message must be received interference-free. A message is called interference-free, if three types of *separability conditions* [4] hold for a proper choice of each transmission parameter p_{ji}. Including all $3^2 = 9$ messages, the *intra-user interference conditions* at Tx_i for pair-wise distinct p_{ii}, p_{ji}, p_{ki} are:

$$x^{p_{ji}} \not\equiv x^{p_{ki}} \mod (x^n - 1), \tag{5}$$
$$x^{p_{ii}} \not\equiv x^{p_{ji}} \mod (x^n - 1), \tag{6}$$
$$x^{p_{ii}} \not\equiv x^{p_{ki}} \mod (x^n - 1). \tag{7}$$

The *multiple-access interference conditions* at Rx_i are:

$$d_{ij} x^{p_{ij}} \not\equiv d_{ik} x^{p_{ik}} \mod (x^n - 1), \tag{8}$$
$$d_{ii} x^{p_{ii}} \not\equiv d_{ij} x^{p_{ij}} \mod (x^n - 1), \tag{9}$$
$$d_{ii} x^{p_{ii}} \not\equiv d_{ik} x^{p_{ik}} \mod (x^n - 1). \tag{10}$$

And the *inter-user interference conditions* at Rx_i are:

$$d_{ii} x^{p_{ii}} \not\equiv d_{ij} x^{p_{kj}} \mod (x^n - 1), \tag{11}$$
$$d_{ii} x^{p_{ii}} \not\equiv d_{ij} x^{p_{jj}} \mod (x^n - 1), \tag{12}$$
$$d_{ii} x^{p_{ii}} \not\equiv d_{ik} x^{p_{jk}} \mod (x^n - 1), \tag{13}$$
$$d_{ii} x^{p_{ii}} \not\equiv d_{ik} x^{p_{kk}} \mod (x^n - 1), \tag{14}$$
$$d_{ij} x^{p_{ij}} \not\equiv d_{ii} x^{p_{ji}} \mod (x^n - 1), \tag{15}$$
$$d_{ij} x^{p_{ij}} \not\equiv d_{ii} x^{p_{ki}} \mod (x^n - 1), \tag{16}$$
$$d_{ij} x^{p_{ij}} \not\equiv d_{ik} x^{p_{jk}} \mod (x^n - 1), \tag{17}$$
$$d_{ij} x^{p_{ij}} \not\equiv d_{ik} x^{p_{kk}} \mod (x^n - 1), \tag{18}$$

for distinct indices $i, j, k \in \mathcal{K}$, respectively. By a circular relabelling of indices, analogous conditions are expressed for Tx_j, Tx_k, Rx_j and Rx_k.

B. Infeasibility Problem of Perfect Cyclic IA

Rx_i receives six interfering signals in total: $d_{ii}x^{p_{ji}}$, $d_{ii}x^{p_{ki}}$, $d_{ij}x^{p_{kj}}$, $d_{ij}x^{p_{jj}}$, $d_{ik}x^{p_{jk}}$, $d_{ik}x^{p_{kk}}$. Two interfering signals from the same transmitter can not be aligned due to the intra-user interference conditions (5) to (7). As 3 dimensions are reserved for dedicated signals and at least 2 must be reserved for interference, we demand $n \geq 5$. Perfect cyclic IA is optimal and requires exactly $n = 5$. In the following, we omit that all congruences are reduced modulo $x^5 - 1$ for brevity. With perfect cyclic IA, three interference signals, i.e., one from each transmitter, must be aligned to a single dimension reserved for interference only. A potential IA scheme at Rx_i is constructed by choosing one element from each of these three sets as implied by curly brackets:

$$\left\{\begin{array}{c} d_{ii}x^{p_{ji}} \\ d_{ii}x^{p_{ki}} \end{array}\right\} \equiv \left\{\begin{array}{c} d_{ij}x^{p_{jj}} \\ d_{ij}x^{p_{kj}} \end{array}\right\} \equiv \left\{\begin{array}{c} d_{ik}x^{p_{jk}} \\ d_{ik}x^{p_{kk}} \end{array}\right\}. \tag{19}$$

For instance, by taking the elements in the first row, we obtain: $d_{ii}x^{p_{ji}} \equiv d_{ij}x^{p_{jj}} \equiv d_{ik}x^{p_{jk}}$ for a fully symmetric perfect alignment scheme similar to [12, Sec. V-C]. IA in one interference dimension directly implicates the complementary alignment of the other interference dimension at the same receiver, e.g., the elements of the second row for the example given above: $d_{ii}x^{p_{ki}} \equiv d_{ij}x^{p_{kj}} \equiv d_{ik}x^{p_{kk}}$.

For notational convenience, we denote auxiliary submatrices of $\boldsymbol{D}$ by:

$$\boldsymbol{D}_{i,k,j,l} = \left(\begin{array}{cc} d_{ij} & d_{il} \\ d_{kj} & d_{kl} \end{array}\right).$$

Note that the determinant implies the following symmetries:

$$\det(\boldsymbol{D}_{i,k,j,l}) \equiv \det(\boldsymbol{D}_{k,i,l,j}) \equiv$$
$$-\det(\boldsymbol{D}_{k,i,j,l}) \equiv -\det(\boldsymbol{D}_{i,k,l,j}) \bmod (x^n - 1).$$

Theorem 1. *Perfect cyclic IA is infeasible on the 3-user X-network with $m_{ij} = 1$, for all $i, j \in \mathcal{K}$, and $n = 5$.*

Proof: Each $\boldsymbol{D}_{i,k,j,l}$ corresponds to a subordinate 2×2 X-channel matrix with distinct transmitters Tx_j, Tx_l, and distinct receivers Rx_i, Rx_k. Note that for the 2×2 X-channel, a non-zero determinant of the channel matrix is necessary to perform IA, as we have already shown in [4, Thm. 1 (a)].

Firstly, we assume that $\det(\boldsymbol{D}_{i,j,i,j}) \equiv 0$ holds w.l.o.g. On the one hand, we could align $d_{ii}x^{p_{ji}} \equiv d_{ij}x^{p_{jj}}$ and the above assumption implies $d_{jj}x^{p_{jj}} \equiv d_{ji}x^{p_{ji}}$. But this contradicts the multiple-access interference conditions. In analogy, aligning $d_{jj}x^{p_{ij}} \equiv d_{ji}x^{p_{ii}}$ implies that $d_{ii}x^{p_{ii}} \equiv d_{ij}x^{p_{ij}}$, yielding another violation. Hence we need $\det(\boldsymbol{D}_{i,j,i,j}) \not\equiv 0$ for these schemes.

Contrariwise, aligning $d_{ii}x^{p_{ki}} \equiv d_{ij}x^{p_{kj}}$ with the initial assumption $\det(\boldsymbol{D}_{i,j,i,j}) \equiv 0$ implies $d_{ji}x^{p_{ki}} \equiv d_{jj}x^{p_{kj}}$. This is not a contradiction so far. Beyond that, the above assumption is even necessary if both of these alignments are used.

However, considering the complementary alignment at Rx_i to $d_{ii}x^{p_{ki}} \equiv d_{ij}x^{p_{kj}}$ as given by (19), provides the first alignment scheme $d_{ii}x^{p_{ji}} \equiv d_{ij}x^{p_{jj}}$, demanding $\det(\boldsymbol{D}_{i,j,i,j}) \not\equiv 0$. Thus, the separability conditions are violated for both

$\det(\boldsymbol{D}_{i,j,i,j}) \equiv 0$ and $\det(\boldsymbol{D}_{i,j,i,j}) \not\equiv 0$ leaving no feasible $\boldsymbol{D}$. This conflict carries over to all minors of $\boldsymbol{D}$ analogously. ∎

Thm. 1 entails that perfect cyclic IA is also infeasible for the K-user X-network with $K \geq 3$ users since there are $\binom{K}{3}$ embedded 3-user X-networks.

Note that this problem does not exclude non-perfect cyclic IA schemes with $n > 2K - 1$ dimensions. But non-perfect schemes do not achieve the upper bound exactly.

Interestingly, the well-known problem of common eigenvectors in invariant subspaces for perfect spatial IA, as discussed and proven in [12, Sec. V-C] for X-networks, is recognizable. But this problem is only a subordinary part of the infeasibility problem presented in Thm. 1 above. To briefly elaborate this, we consider a symmetric perfect cyclic IA scheme which is analogous to the spatial IA scheme in [12, Eqns. (10)-(12)]:

$$d_{ii}x^{p_{ji}} \equiv d_{ij}x^{p_{jj}} \equiv d_{ik}x^{p_{jk}}, \tag{20}$$
$$d_{ii}x^{p_{ki}} \equiv d_{ij}x^{p_{kj}} \equiv d_{ik}x^{p_{kk}}, \tag{21}$$

for pair-wise distinct indices $i, j, k \in \mathcal{K}$. Due to symmetry, the alignment of (20) at receiver Rx_k corresponds to:

$$d_{kk}x^{p_{jk}} \equiv d_{kj}x^{p_{jj}} \equiv d_{ki}x^{p_{ji}}, \tag{22}$$

by a simple relabelling of indices. With (20), (22), we obtain:

$$d_{ii}x^{p_{ji}} \equiv d_{ij}x^{p_{jj}},$$
$$d_{ki}x^{p_{ji}} \equiv d_{kj}x^{p_{jj}},$$
$$\Rightarrow d_{ii}d_{ij}^{-1}d_{kj}d_{ki}^{-1}x^{p_{ji}} \equiv x^{p_{ji}}. \tag{23}$$

An analogous formulation arises in [12, Eq. (15)] with corresponding diagonal MIMO channel matrices and leads to the problem of common eigenvectors in perfect spatial IA. But in the case of cyclic IA, the result of (23) only implies the constraint that $\det(\boldsymbol{D}_{i,k,j,i}) \equiv 0$ is needed.

IV. 3-USER X-NETWORKS WITH MINIMAL BACKHAUL

Since perfect cyclic IA is already shown to be an over-constrained problem, our approach is to relax the restraining conditions by providing a limited number of messages over a BHN to achieve sufficient feasibility.

A. Cyclic IA with Minimal Feedforward Backhaul Networks

A first and very intuitive approach is to include a *feedforward backhaul network* (FF-BHN) between a minimal subset of transmitters and receivers. A single FF-link $\theta_{\text{FF},ji}$ between Tx_i and Rx_j with rate $\Theta_{\text{FF},ji} = 1$ simply bypasses the channel d_{ji} for a single message so that the actual transmission of a forwarded message may be omitted. The FF-BHN is also shown at the bottom of Fig. 1. Note that this approach corresponds to using one cognitive receiver Rx_j knowing W_{ji}.

We propose the following alignment scheme and prove its optimality w. r. t. the minimal necessary sum-rate Θ_{FF}:

At Rx_i, interference from Rx_j and Rx_k is perfectly aligned within two dimensions:

$$d_{ii}x^{p_{ji}} \equiv d_{ij}x^{p_{kj}} \equiv d_{ik}x^{p_{jk}}, \tag{24}$$
$$d_{ii}x^{p_{ki}} \equiv d_{ij}x^{p_{jj}} \equiv d_{ik}x^{p_{kk}}. \tag{25}$$

The dedicated and interfering signals at Rx_j are aligned by:

$$d_{ji}x^{p_{ki}} \equiv d_{jj}x^{p_{kj}} \equiv d_{jk}x^{p_{ik}}, \tag{26}$$

$$d_{ji}x^{p_{ii}} \equiv d_{jk}x^{p_{kk}}, \tag{27}$$

$$d_{jj}x^{p_{ij}} \equiv d_{jk}x^{p_{jk}}, \tag{28}$$

and similarly, we use the following cyclic IA scheme at Rx_k:

$$d_{ki}x^{p_{ii}} \equiv d_{kj}x^{p_{jj}} \equiv d_{kk}x^{p_{ik}}, \tag{29}$$

$$d_{kj}x^{p_{ij}} \equiv d_{ki}x^{p_{ji}}, \tag{30}$$

$$d_{ki}x^{p_{ki}} \equiv d_{kk}x^{p_{jk}}. \tag{31}$$

A relabelling of indices is not permitted in this asymmetric scheme. Note that (28) and (31) explicitly violate the separability conditions. Independent of channel matrix $\boldsymbol{D}$, the dedicated messages W_{jk} and W_{ki} can not be decoded yet.

Theorem 2. *The upper bound of $\frac{9}{5}$ DoF for $n = 5$ on the 3-user X-network is achievable by cyclic IA with feedforwarded messages and rate $\Theta_{\text{FF}} \geq 1$.*

For the considered scheme in (24) to (31), we assume that:

(i) $d_{ij}d_{ki}d_{jk} \equiv d_{ji}d_{ik}d_{kj}$,
(ii) $d_{ii}d_{jk}d_{kj} \equiv d_{jj}d_{ik}d_{ki} \equiv d_{kk}d_{ij}d_{ji}$,
(iii) $\det(\boldsymbol{D}_{i,k,i,k}) \not\equiv 0$,
(iv) $\det(\boldsymbol{D}_{i,j,i,j}) \not\equiv 0$,
(v) $\det(\boldsymbol{D}_{i,j,i,k}) \not\equiv 0$,
(vi) $\det(\boldsymbol{D}_{i,j,j,k}) \not\equiv 0$,
(vii) $\det(\boldsymbol{D}_{k,j,k,i}) \not\equiv 0$,
(viii) $\det(\boldsymbol{D}_{k,j,k,j}) \not\equiv 0$,
(ix) $d_{ii}d_{jj}d_{kk} \not\equiv d_{ij}d_{jk}d_{ki} \equiv d_{ji}d_{kj}d_{ik}$,
(x) $d_{kk}d_{ii}d_{jj}d_{kk} \not\equiv d_{jk}d_{kj}d_{ik}d_{ki}$,
    $d_{ii}d_{ii}d_{jj}d_{kk} \not\equiv d_{ij}d_{ji}d_{ik}d_{ki}$,
    $d_{jj}d_{ii}d_{jj}d_{kk} \not\equiv d_{ij}d_{ji}d_{jk}d_{kj}$,

hold for distinct indices $i, j, k \in \mathcal{K}$.

Proof: (a) *Necessity of $\Theta_{\text{FF}} \geq 1$:*
Since $\Theta_{\text{FF}} = 0$ would correspond to using no FF-link at all, the necessity of $\Theta_{\text{FF}} \geq 1$ follows from Thm. 1 evidently.

(b) *Necessity of constraints (i) to (x) for the given scheme*:
First, we consider the constraints (i), (ii) that are implied by the cyclic IA scheme given in (24) to (31). We depict how the parameters are interlinked by the adjacency graph shown in Fig. 2. Other potentially valid alignment schemes will imply a different set of constraints and hence another adjacency graph.

Constraint (i) is obtained by substituting parameters p_{jj}, p_{kk}, p_{ii} in (25), (26), (29).

One part of constraint (ii) yields from substituting p_{jk}, p_{ji}, p_{ij} in (24), (28), (27). Another part from (ii) yields from substituting p_{jj}, p_{kj}, p_{ik} in (25), (29), (26), and the remaining part from substituting p_{kj}, p_{ki}, p_{jk} in (26), (31), (24).

Now, we consider the impact of the separability conditions. Note that the feasibility of (28) and (31) is provided by the FF-BHN, as we will show in part (b), i.e., these particular violations are excluded. Constraint (iii) is derived from (10) and (29). Constraint (iv) is derived by substituting p_{ki}, p_{ji}, p_{kj} with (5), (24), (26).

The complete proof of the remaining constraints (v) to (x) w.r.t. all separability conditions at each receiver and some
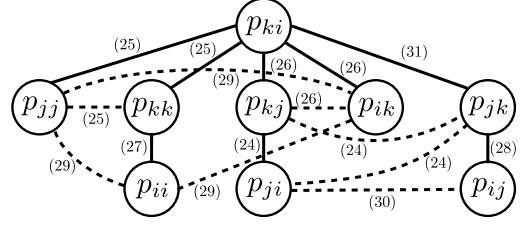

Fig. 2. Adjacency graph of the cyclic IA scheme in (24) to (31). Solid lines indicate the assignments for a fixed p_{ki} given in the proof of Thm. 2, and dashed lines the remaining conditions that must be checked for feasibility.

examples are provided in the extended version [14] of this paper. The three constraints of (x) are equivalent due to (ii).

(b) *Sufficiency of cyclic IA with $\Theta_{\text{FF}} = 1$:*
It suffices to bypass the transmission of the dedicated message W_{jk} over d_{ji} through a single FF-link $\theta_{\text{FF},jk}$ with sum-rate $\Theta_{FF} = 1$. Then, Tx_k may omit the transmission of W_{jk} over $\boldsymbol{D}$, and Rx_j can still decode W_{jk} from the FF-BHN. As W_{jk} is not transmitted over $\boldsymbol{D}$ at all, Rx_k can also decode W_{ki} interference-free.

To show that the proposed IA scheme with FF is feasible now, all nine transmission parameters must be resolved. As indicated by the adjacency graph in Fig. 2, we fix the topmost parameter p_{ki}, w.l.o.g. With (25), we obtain p_{jj}, p_{kk}, (26) provides p_{kj}, p_{ik}, and (31) yields p_{jk}. With (27), p_{ii} yields from p_{kk}. With (24), p_{ji} yields from p_{kj}. And with (28), p_{ij} yields from p_{jk}.

A valid matrix, normalized w.r.t. the main diagonal is, e.g.:

$$\boldsymbol{D} = \begin{pmatrix} 1 & x^4 & x^2 \\ x^4 & 1 & x^2 \\ x & x & 1 \end{pmatrix},$$

as all constraints (i) $\equiv x^2$, (ii) $\equiv x^3$, (iii) $\equiv 1-x^3$, (iv) $\equiv 1-x^3$, (v) $\equiv x^2-1$, (vi) $\equiv x^1-x^2$, (vii) $\equiv x^4-x^3$, (viii) $\equiv 1-x^3$, (ix) $1 \not\equiv x^2$, (x) $1 \not\equiv x^1$ for $i = 1$, $j = 2$, $k = 3$, are fulfilled. A valid set of transmission parameters satisfying all conditions with a fixed $p_{31} = 4$ yields the following vector $\boldsymbol{p}$:

$$\boldsymbol{p} = (p_{11}, p_{21}, p_{31}, p_{12}, p_{22}, p_{32}, p_{13}, p_{23}, p_{33})$$
$$= (\ \ 0, \quad 2, \quad 4, \quad 2, \quad 0, \quad 3, \quad 1, \quad 0, \quad 2). \tag{32}$$

W_{23} is not transmitted over $\boldsymbol{D}$, but forwarded over the FF-BHN. Altogether, a number of $\frac{9}{5}$ DoF is achieved by cyclic IA with $\Theta_{\text{FF}} = 1$. ∎

Note that a delayed transmission on the FF-BHN only delays the decoding time, but it does not affect the feasibility.

We would like to emphasize, that the given constraints on $\boldsymbol{D}$ are profoundly interdependent with the proposed IA scheme. Nonetheless, the analysis of comparable valid cyclic IA schemes can be performed analogously to Thm. 2.

B. Cyclic IAC over Minimal Receiver Backhaul Networks

Now, instead of a FF-BHN, we consider a *receiver backhaul network* (Rx-BHN). The Rx-BHN only permits that receivers may exchange messages to resolve leaking interference in order to satisfy all separability conditions. The Rx-BHN is

depicted in Fig. 1 on the right-hand side. A single link with rate $\Theta_{\mathrm{R},ij}$ in the Rx-BHN from Rx_i to Rx_j is denoted by $\theta_{\mathrm{R},ji}$.

Similar to the previous section, our aim is to characterize the minimal sum-rate Θ_{R} on the Rx-BHN, that is necessary to achieve the upper bound of $\frac{9}{5}$ DoF on the 3-user X-network.

Lemma 3. *The upper bound of $\frac{9}{5}$ DoF for $n = 5$ on the 3-user X-network is achievable by cyclic IAC with $\Theta_{\mathrm{R}} \geq 2$ for the conditions given in Thm. 2.*

Proof: (a) *Necessity of $\Theta_{\mathrm{R}} \geq 2$:*
As cyclic IA without cancellation is precluded by Thm. 1 for $n = 5$, it follows that $\Theta_{\mathrm{R}} > 0$. In contrast to Thm. 2, no message can be neglected and bypassed so that all must be sent over the channel D. Thus the case $\Theta_{\mathrm{R}} = 1$ demands that the interference at two receivers, say, Rx_i and Rx_j, must be perfectly aligned and only one interfering signal may leak at the remaining receiver Rx_k. However, simultaneous perfect cyclic IA at two receivers is already precluded by Thm. 1.

(b) *Sufficiency of Cyclic IAC for $\Theta_{\mathrm{R}} = 2$:*
We consider the same cyclic IA scheme as provided in (24) to (31) subject to constraints (i) to (x) of Thm. 2. Note that message W_{jk} is indeed transmitted here. But now, the leaking interference in (28) and (31) is resolved by $\Theta_{\mathrm{R}} = 2$ messages over the Rx-BHN. In particular W_{ij} is conveyed over $\theta_{\mathrm{R},ji}$, so that W_{ij} can be cancelled from the aligned $W_{ij} + W_{jk}$ to decode the dedicated message W_{jk} at Rx_j. In a subsequent step, W_{jk} is conveyed over $\theta_{\mathrm{R},kj}$, so that W_{jk} is cancelled from the aligned $W_{jk} + W_{ki}$ to decode the dedicated message W_{ki} at Rx_k. ∎

A delayed Rx-BHN transmission does not affect feasibility as long as the backhaul messages W_{ij} and W_{jk} for cancellation adhere to the proposed sequence.

C. Cyclic IN over Minimal Transmitter Backhaul Networks

We now consider the reversed case: Transmitters are connected via a *transmitter backhaul network* (Tx-BHN) instead. A backhaul link from Tx_i to Tx_j is described by $\theta_{\mathrm{T},ji}$, correspondingly. The sum-rate over the Tx-BHN is denoted by Θ_{T}. The Tx-BHN is depicted in Fig. 1 on the left-hand side.

Lemma 4. *The upper bound of $\frac{9}{5}$ DoF for $n = 5$ on the 3-user X-network is achievable by cyclic IN with $\Theta_{\mathrm{T}} \geq 2$ for the conditions given in Thm. 2.*

Proof: This scheme is a dual to Lem. 3 for the Rx-BHN as considered above, so that the necessity of $\Theta_{\mathrm{T}} \geq 2$ is analogous. Again, we use the alignment scheme of (24) to (31) subject to constraints (i) to (x) of Thm. 2. But in contrast to Lem. 3, W_{ij} is firstly conveyed over $\theta_{\mathrm{T},kj}$ with $\Theta_{\mathrm{T},kj} = 1$ and then the combined message $W_{ij} - W_{jk}$ is conveyed over $\theta_{\mathrm{T},ik}$ with $\Theta_{\mathrm{T},ik} = 1$. Tx_k transmits the superposition $W_{jk} - W_{ij}$ instead of W_{jk} only, and Tx_i transmits the superposition $W_{ij} - W_{ki} - W_{jk}$ instead of W_{ki} only. This change does not only maintain the decodability of the dedicated signals received at Rx_i or Rx_j, but rather neutralizes the previously leaking interference observed at Rx_j and Rx_k. ∎

In contrast to cyclic IAC over the Rx-BHN, the exchange of signals over the Tx-BHN must be performed at any time before the actual transmission. This relationship also endorses a related IAC-IN duality property reported in [13] for a linear deterministic 2-user interference channel with cooperation.

D. Combined Cyclic IAC and IN over Minimal Tx/Rx-BHNs

If transmitters and receivers are each connected to a disjoint Tx-BHN and Rx-BHN, the IAC and IN schemes of Sec. IV-B and IV-C can be combined. The sum-rate over both BHNs is denoted by $\Theta_{\mathrm{TR}} = \Theta_{\mathrm{R}} + \Theta_{\mathrm{T}}$.

Corollary 5. *The upper bound of $\frac{9}{5}$ DoF for $n = 5$ on the 3-user X-network is achievable by combined cyclic IAC and cyclic IN with $\Theta_{\mathrm{TR}} \geq 2$ for the conditions given in Thm. 2.*

Proof: Using (24) to (31), W_{ij} is provided over $\theta_{\mathrm{T},kj}$ and $W_{jk} - W_{ij}$ replaces W_{ij} at Tx_k so that W_{ij} is neutralized at Rx_j. Rx_k receives $W_{ki} + W_{jk} - W_{ij}$. Then, $W_{jk} + W_{ji}$ is provided over $\theta_{\mathrm{R},kj}$. Rx_k decodes $(W_{ki} + W_{jk} - W_{ij}) - (W_{jk} + W_{ji}) + (W_{ji} + W_{ij}) = W_{ki}$ using its interfered signal and (30). ∎

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