Exact Solution for a Boson-Fermion Model and application to ultra-cold atoms

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1 Introduction

The progress on ultra-cold atoms experiments allowing to tune fermionic systems through a Feshbach resonance where itinerant fermionic atoms may form tightly bound pairs ("bosonic" molecules) [1,2] has lead to a renewed theoretical interest in boson-fermion models. We study a 1D boson fermion resonance model describing itinerant spin-$1/2$ fermions and itinerant scalar bosons coupled through a local interaction of strength g which describes the binding of a pair of opposite spin fermions to form a scalar boson, and the reverse process. The model also includes a detuning term characterized by a detuning parameter ν. It is found that the model has an exact solution by Bethe Ansatz.

2 Model

Hamiltonian given by

$$
\begin{aligned}
H = & -\frac{1}{2m}\int dx \psi^+_\sigma(x)\partial_x^2\psi_\sigma(x) - \\
& -\frac{1}{2m_b}\int dx\phi^+(x)\partial_x^2\phi(x) + \nu\int dx\phi^+(x)\phi(x) + \\
& +g\int dx[\psi^+_\uparrow(x)\psi^+_\downarrow(x)\phi(x) + \phi^+(x)\psi_\downarrow(x)\psi_\uparrow(x)]
\end{aligned}
$$

where m and m_b are the fermion and boson masses, respectively, σ are the fermion spin-1/2 indices, g is the strength of the boson-fermion coupling and ν is a detuning parameter. The field operators satisfy canonical commutation/anti-commutation relations. Importantly, neither the number of fermions nor the number of bosons are conserved. The total particle number operator $N = N_f + 2N_b$ where $N_f = \sum_a \int dx\psi^+_a(x)\psi_a(x)$ and $N_b = \int dx\phi^+(x)\phi(x)$ are the total number of fermions and bosons, respectively, is conserved.

3 Bethe Ansatz

Two-particle wave function:

$$
\begin{aligned}
|F_2\rangle = & \int dx_1 \int dx_2 F_{a_1a_2}(x_1,x_2)\psi^+_{a_1}(x_1)\psi^+_{a_2}(x_2)|0\rangle + \\
& + \int dx G(x)\phi^+(x)|0\rangle
\end{aligned}
$$

Coupled Schödinger equations:

$$
\begin{cases}
\frac{1}{2m}(\partial^2_{x_1} + \partial^2_{x_2})F_{a_1a_2}(x_1,x_2) + \\
+\frac{g}{2}(\delta_{a_1\uparrow}\delta_{a_2\downarrow} - \delta_{a_1\downarrow}\delta_{a_2\uparrow})\delta(x_1-x_2)G(x_1) = EF_{a_1a_2}(x_1,x_2) \\
-\frac{1}{2m}\partial_x^2 G(x) + g(F_{\uparrow\downarrow}(x,x) - F_{\downarrow\uparrow}(x,x)) = (E-\nu)G(x)
\end{cases}
$$

Ansatz:

$$F_{a_1a_2}(x_1,x_2) = \mathcal{A}e^{i(k_1x_1+k_2x_2)}[A_{a_1a_2}\theta(x_2-x_1) + B_{a_1a_2}\theta(x_1-x_2)]$$

where $\mathcal{A}$ is the antisymmetrizer operator. Energy eigenvalue:

$$E = -\frac{1}{2m}(k_1^2 + k_2^2)$$

Two-particle S-matrix: $B_{a_1a_2} = S^{b_1b_2}_{a_1a_2}(k_1,k_2)A_{b_1b_2}$

$$S^{b_1,b_2}_{a_1,a_2}(k_1,k_2) = \frac{\alpha(k_1,k_2)\mathbf{1}^{b_1,b_2}_{a_1,a_2} + ic\mathbf{P}^{b_1,b_2}_{a_1,a_2}}{\alpha(k_1,k_2) + ic}$$

where $\mathbf{1}$ is the unit operator, $\mathbf{P}$ the spin exchange operator, $c = \frac{g^2}{2}$ and $\alpha(k_1,k_2) = (k_1^3 - k_2^3)/m^2 + \nu(k_1 - k_2)$. Bosonic wave function:

$$G(x) = \frac{2}{img}(k_1 - k_2)\left(1 - S^{12}_s\right)A_s e^{i(k_1+k_2)x}$$

where A_s is the fermionic singlet spin amplitude and S^{12}_s is the "singlet part" of the S-matrix. The N particle wavefunction is constructed as follows:

$$
\begin{aligned}
|F_N\rangle = & \left[\int d\bar{x}F^{N;0}_{a_1\ldots a_N}(\bar{x})\prod_{i=1}^{N}\psi^+_{a_i}(x_i) + \right. \\
& \left. + \int d\bar{x}dy F^{N-2;1}_{a_1\ldots a_{N-2}}(\bar{x};y)\prod_{i=1}^{N-2}\psi^+_{a_i}(x_i)\phi^+(y) + \ldots\right]|0\rangle
\end{aligned}
$$

The usual Bethe Ansatz is used to construted the purely fermionic wave function$F^{N;0}$. It can be shown that Yang-Baxter equations [3] are satisfied. The system is defined in a ring of finite length L and periodic boundary conditions (PBC) must be imposed to the wave function $|F_N\rangle$. The use of PBC leads to the Bethe equations:

$$e^{ik_jL} = \prod_{\gamma=1}^{M}\frac{\Lambda_\gamma - \alpha(k_j) - ic/2}{\Lambda_\gamma - \alpha(k_j) + ic/2}$$

$$\prod_{\delta=1(\delta\neq\gamma)}^{M}\frac{\Lambda_\gamma - \Lambda_\delta - ic}{\Lambda_\gamma - \Lambda_\delta + ic} = \prod_{j=1}^{N}\frac{\Lambda_\gamma - \alpha(k_j) - ic/2}{\Lambda_\gamma - \alpha(k_j) + ic/2}$$

where $M = N_\downarrow$. In the thermodynamic limit, density of solutions are used and the integral equations are:

$$
\begin{aligned}
\rho(k) &= \frac{1}{2\pi} + \alpha'(k)\int_{D_\Lambda}\sigma(\Lambda)K_1(\alpha(k) - \Lambda)d\Lambda \\
\sigma(\Lambda) &= \int_{D_k}\rho(k)K_1(\alpha(k) - \Lambda)dk - \int_{D_\Lambda}\sigma(\Lambda')K_2(\Lambda - \Lambda')d\Lambda'
\end{aligned}
$$

where $K_n(x) = \frac{1}{\pi}\frac{nc/2}{(nc/2)^2 + x^2}$. The integration limits satisfy the conditions:

$$\int_{D_k}\rho(k)dk = \frac{N}{L} \quad \int_{D_\Lambda}\sigma(\Lambda)d\Lambda = \frac{M}{L}$$

Energy per site: $E/L = -(1/2m)\int_{D_k}k^2\rho(k)dk$. Real k's and real Λ's solutions correspond to unbound fermions. Bound states of two fermions correspond to k-strings solutions. For two bound fermions $k^\pm = q \pm i\xi$ with q and ξ real and $\xi > 0$. When dealing with unbound and bound states we apply Bethe Ansatz for Composites [4] and obtain the following Bethe equations:

$$e^{ik_jL} = \prod_{\gamma=1}^{M}\frac{\Lambda_\gamma - \alpha(k_j) - ic/2}{\Lambda_\gamma - \alpha(k_j) + ic/2}\prod_{l=1}^{N^b}\frac{\phi(q_l) - \alpha(k_j) - ic/2}{\phi(q_l) - \alpha(k_j) + ic/2}$$

$$\prod_{\delta=1(\delta\neq\gamma)}^{M}\frac{\Lambda_\gamma - \Lambda_\delta - ic}{\Lambda_\gamma - \Lambda_\delta + ic} = \prod_{j=1}^{N^u}\frac{\Lambda_\gamma - \alpha(k_j) - ic/2}{\Lambda_\gamma - \alpha(k_j) + ic/2}$$

$$e^{2iq_lL} = \prod_{j=1}^{N^u}\frac{\alpha(k_j) - \phi(q_l) - ic/2}{\alpha(k_j) - \phi(q_l) + ic/2}\prod_{n=1(n\neq l)}^{N^b}\frac{\phi(q_n) - \phi(q_l) - ic}{\phi(q_n) - \phi(q_l) + ic}$$

where $N = N^u + 2N^b$ is the total number of particles, N^u the number of unbound fermions, N^b the number of bound fermion pairs, and $M = N^u_\downarrow$. Integral equations in the thermodynamic limit:

$$
\begin{aligned}
\rho_u(k) &= \frac{1}{2\pi} + \alpha'(k)\int_{D_\Lambda}\sigma(\Lambda)K_1(\alpha(k) - \Lambda)d\Lambda + \alpha'(k)\int_{D_q}\rho_b(q)K_1(\alpha(k) - \phi(q))dq \\
\sigma(\Lambda) &= \int_{D_k}\rho_u(k)K_1(\alpha(k) - \Lambda) - \int_{D_\Lambda}\sigma(\Lambda')K_2(\Lambda - \Lambda')d\Lambda' \\
\rho_b(q) &= \frac{1}{\pi} + \phi'(q)\int_{D_k}\rho_u(k)K_1(\alpha(k) - \phi(q))dk + \phi'(q)\int_{D_q}\rho_b(q')K_2(\phi(q) - \phi(q'))dq'
\end{aligned}
$$

where $\phi(q) = \frac{1}{m^2}(q^2 - 3\xi^2(q) + m\nu)q$. The integration limits satisfy:

$$\int_{D_k}\rho_u(k)dk = \frac{N^u}{L}, \quad \int_{D_\Lambda}\sigma(\Lambda)d\Lambda = \frac{M}{L}, \quad \int_{D_q}\rho_b(q)dk = \frac{N^b}{L}$$

Energy per site: $E/L = -(1/2m)\int_{D_k}k^2\rho_u(k)dk - (1/m)\int_{D_q}(q^2 - \xi^2(q))\rho_b(q)dq$.

4 Results

Ground state results by solving numerically the Bethe Ansatz equations are shown in Figures 1, 2, and 3.

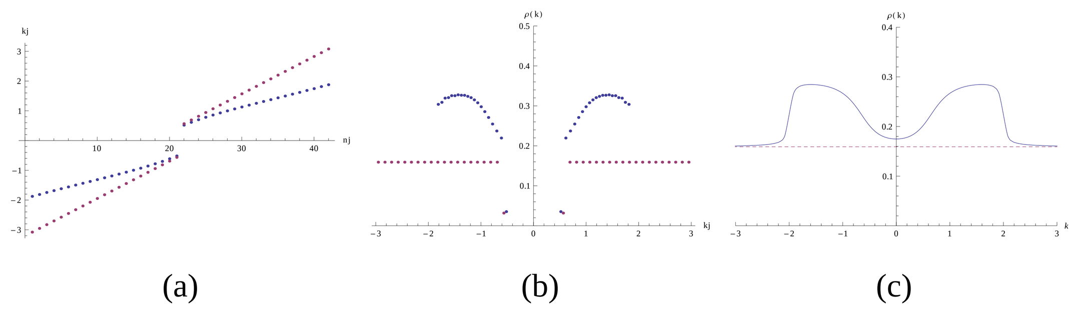

Fig. 1: (a)(b): $c = 1$ $\nu = 2$ ($L = 50$, $N = 42$, $N_u = 42$, $N_b = 0$, $M = 0$), (c): $c = 1$, $\nu = 2$, $n = 0.78$.

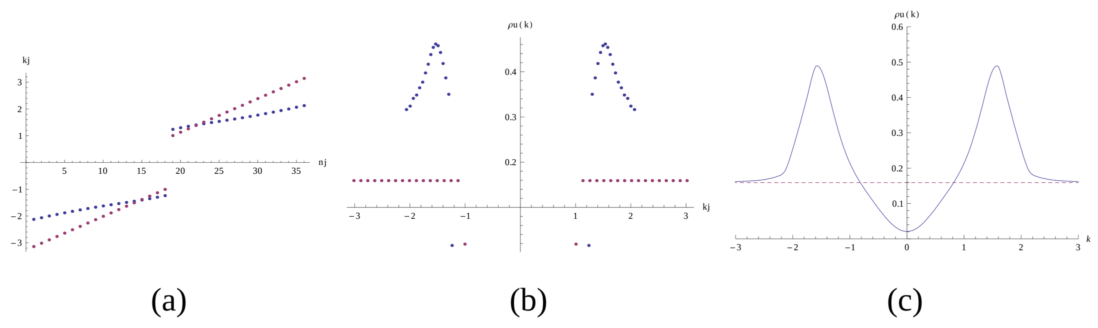

Fig. 2: (a)(b): $c = 1$ $\nu = -2$ ($L = 50$, $N = 40$, $N_u = 36$, $N_b = 2$, $M = 0$), (c): $c = 1$, $\nu = -2$, $n = 0.77$.

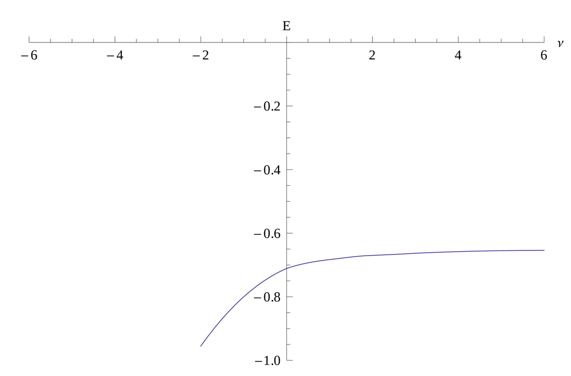

Fig. 3: Ground State energy as function of the detuning parameter ν ($c = 1$, $n = 0.80$).

5 Conclusions

We find that the model supports fermion bound pairs. For sufficiently large values of ν the ground state consists of purely unbound fermions forming a Fermi liquid. As one decreases the detuning parameter the system goes through a Feshbach resonance and the ground state becomes unstable with respect to the formation of bound fermion pairs. In this case unbound fermions and bound fermions pairs coexist.

6 Acknowledgements

One of us (L.M.M.) is especially grateful to J. L. dos Santos for stimulating and illuminating discussions. This work was partially supported by FCT (Portugal) through the Grant SFRH/BSAB/600/2006.

7 References

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