Annales Mathematicae et Informaticae
40 *(2012) pp. 105–112*
http://ami.ektf.hu

The signed Roman domatic number of a graph

Seyed Mahmoud Sheikholeslami[a], Lutz Volkmann[b]

[a]Research Group of Processing and Communication
and
Department of Mathematics
Azarbaijan Shahid Madani University
Tabriz, I.R. Iran
s.m.sheikholeslami@azaruniv.edu

[b]Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen, Germany
volkm@math2.rwth-aachen.de

Submitted March 30, 2012 — Accepted November 28, 2012

Abstract

A *signed Roman dominating function* (SRDF) on a graph G is a function $f\colon V(G) \to \{-1,1,2\}$ such that $\sum_{u\in N[v]} f(u) \geq 1$ for every $v \in V(G)$, and every vertex $u \in V(G)$ for which $f(u) = -1$ is adjacent to at least one vertex w for which $f(w) = 2$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(G)$, is called a *signed Roman dominating family* (of functions) on G. The maximum number of functions in a signed Roman dominating family on G is the *signed Roman domatic number* of G, denoted by $d_{sR}(G)$. In this paper we initiate the study of signed Roman domatic number in graphs and we present some sharp bounds for $d_{sR}(G)$. In addition, we determine the signed Roman domatic number of some graphs.

Keywords: signed Roman dominating function, signed Roman domination number, signed Roman domatic number

MSC: 05C69

1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A graph G is *k-regular* if $d(v) = k$ for each vertex v of G. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. A *tree* is an acyclic connected graph. The complement of a graph G is denoted by $\overline{G}$. A *cactus graph* is a connected graph in which any two cycles have at most one vertex in common. We write K_n for the *complete graph* of order n and C_n for a *cycle* of length n.

A *Roman dominating function* (RDF) on a graph $G = (V, E)$ is defined in [6, 8] as a function $f\colon V \to \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The *weight* of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of an RDF on G. The Roman domination number has been studied by several authors (see for example [2, 3, 4]). A set $\{f_1, f_2, \ldots, f_d\}$ of distinct Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \le 2$ for each $v \in V(G)$, is called a *Roman dominating family* (of functions) on G. The maximum number of functions in a Roman dominating family (RD family) on G is the *Roman domatic number* of G, denoted by $d_R(G)$. The Roman domatic number was introduced by Sheikholeslami and Volkmann [7] and has been studied by several authors (see for example [5]).

A *signed Roman dominating function* (SRDF) on a graph $G = (V, E)$ is defined in [1] as a function $f\colon V \to \{-1, 1, 2\}$ such that $\sum_{u\in N[v]} f(u) \ge 1$ for each $v \in V(G)$, and such that every vertex $u \in V(G)$ for which $f(u) = -1$ is adjacent to at least one vertex w for which $f(w) = 2$. The *weight* of an SRDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The *signed Roman domination number* of a graph G, denoted by $\gamma_{sR}(G)$, equals the minimum weight of an SRDF on G. A $\gamma_{sR}(G)$-*function* is a signed Roman dominating function of G with weight $\gamma_{sR}(G)$. A signed Roman dominating function $f\colon V \to \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) (or (V_{-1}^f, V_1^f, V_2^f) to refer f) of V, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \le 1$ for each $v \in V(G)$, is called a *signed Roman dominating family* (of functions) on G. The maximum number of functions in a signed Roman dominating family (SRD family) on G is the *signed Roman domatic number* of G, denoted by $d_{sR}(G)$. The signed Roman domatic number is well-defined and

$$d_{sR}(G) \ge 1 \tag{1.1}$$

for all graphs G since the set consisting of the SRDF with constant value 1 forms an SRD family on G. If $G_1, G_2, \ldots, G_k$ are the connected components of G, then obviously $d_{sR}(G) = \min\{d_{sR}(G_i) \mid 1 \le i \le k\}$. Hence, we only consider connected graphs.

Our purpose in this paper is to initiate the study of signed Roman domatic number in graphs. We first study basic properties and bounds for the signed Roman domatic number of a graph. In addition, we determine the signed Roman domatic number of some classes of graphs.

We make use of the following results in this paper.

Proposition A ([1]). *If K_n is the complete graph of order $n \ge 1$, then $\gamma_{sR}(K_n) = 1$, unless $n = 3$ in which case $\gamma_{sR}(K_n) = 2$.*

Proposition B ([1]).

1. *For $n \ge 3$, $\gamma_{sR}(C_n) = \lceil \frac{2n}{3} \rceil$,*
2. *For $n \ge 2$, $\gamma_{sR}(P_n) = \lfloor \frac{2n}{3} \rfloor$.*

Proposition C ([1]). *Let G be a graph of order $n \ge 1$. Then $\gamma_{sR}(G) = n$ if and only if $G = \overline{K_n}$.*

Proposition D ([1]). *If G is a δ-regular graph of order n with $\delta \ge 1$, then $\gamma_{sR}(G) \ge \lceil n/(\delta + 1) \rceil$.*

2. Properties of the signed Roman domatic number

In this section we present basic properties of $d_{sR}(G)$ and sharp bounds on the signed Roman domatic number of a graph.

Theorem 2.1. *For every graph G,*

$$d_{sR}(G) \le \delta(G) + 1.$$

Moreover, if $d_{sR}(G) = \delta(G) + 1$, then for each SRD family $\{f_1, f_2, \ldots, f_d\}$ on G with $d = d_{sR}(G)$ and each vertex v of minimum degree, $\sum_{u \in N[v]} f_i(u) = 1$ for each function f_i and $\sum_{i=1}^{d} f_i(u) = 1$ for all $u \in N[v]$.

Proof. If $d_{sR}(G) = 1$, the result is immediate. Let now $d_{sR}(G) \ge 2$ and let $\{f_1, f_2, \ldots, f_d\}$ be an SRD family on G such that $d = d_{sR}(G)$. Assume that v is a vertex of minimum degree $\delta(G)$. We have

$$d \le \sum_{i=1}^{d} \sum_{u \in N[v]} f_i(u) = \sum_{u \in N[v]} \sum_{i=1}^{d} f_i(u) \le \sum_{u \in N[v]} 1 = \delta(G) + 1.$$

Thus $d_{sR}(G) \le \delta(G) + 1$.

If $d_{sR}(G) = \delta + 1$, then the two inequalities occurring in the proof become equalities. Hence for the SRD family $\{f_1, f_2, \ldots, f_d\}$ on G and for each vertex v of minimum degree, $\sum_{u \in N[v]} f_i(u) = 1$ for each function f_i and $\sum_{i=1}^{d} f_i(u) = 1$ for all $u \in N[v]$. □

The next results are immediate consequences of Proposition C and Theorem 2.1.

Corollary 2.2. *For* $n \geq 1$, $d_{sR}(\overline{K_n}) = 1$.

Corollary 2.3. *For any tree* T *of* $n \geq 3$, $d_{sR}(T) \leq 2$. *The bound is sharp for a double star obtained from two vertex disjoint stars* $K_{1,3}$ *by connecting their centers.*

Problem 2.4. *Characterize all trees* T *for which* $d_{sR}(T) = 2$.

Corollary 2.5. *For* $n \geq 2$, $d_{sR}(K_{1,n}) = 1$.

Proof. It follows from Theorem 2.1 that $d_{sR}(K_{1,n}) \leq 2$. Suppose to the contrary that $d_{sR}(K_{1,n}) = 2$ and assume that $\{f_1, f_2\}$ is an SRD family on $K_{1,n}$. Let $V(K_{1,n}) = \{v, u_1, \ldots, u_n\}$ and $E(K_{1,n}) = \{vu_i \mid 1 \leq i \leq n\}$. Theorem 2.1 implies that $f_1(v) + f_2(v) = 1$. Since $f_j(x) \in \{-1, 1, 2\}$ for each j and each vertex x, we deduce that $f_1(v) = -1$ and $f_2(v) = 2$ or $f_1(v) = 2$ and $f_2(v) = -1$. Assume, without loss of generality, that $f_1(v) = -1$ and $f_2(v) = 2$. By Theorem 2.1, we must have $f_2(u_i) + f_2(v) = 1$ for each $1 \leq i \leq n$ and therefore $f_2(u_i) = -1$ for each $1 \leq i \leq n$. Since $n \geq 2$, we obtain the contradiction $1 \leq \sum_{x \in N[v]} f_2(x) = 2 - n \leq 0$. Thus $d_{sR}(K_{1,n}) = 1$. □

Theorem 2.6. *If* G *is a graph of order* n, *then*

$$\gamma_{sR}(G) \cdot d_{sR}(G) \leq n.$$

Moreover, if $\gamma_{sR}(G) \cdot d_{sR}(G) = n$, *then for each SRD family* $\{f_1, f_2, \ldots, f_d\}$ *on* G *with* $d = d_{sR}(G)$, *each function* f_i *is a* $\gamma_{sR}(G)$*-function and* $\sum_{i=1}^{d} f_i(v) = 1$ *for all* $v \in V$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be an SRD family on G such that $d = d_{sR}(G)$ and let $v \in V$. Then

$$d \cdot \gamma_{sR}(G) = \sum_{i=1}^{d} \gamma_{sR}(G) \leq \sum_{i=1}^{d} \sum_{v \in V} f_i(v) = \sum_{v \in V} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V} 1 = n.$$

If $\gamma_{sR}(G) \cdot d_{sR}(G) = n$, then the two inequalities occurring in the proof become equalities. Hence for the SRD family $\{f_1, f_2, \ldots, f_d\}$ on G and for each i, $\sum_{v \in V} f_i(v) = \gamma_{sR}(G)$. Thus each function f_i is a $\gamma_{sR}(G)$-function, and $\sum_{i=1}^{d} f_i(v) = 1$ for all $v \in V$. □

The next two results are immediate consequences of Propositions B, C and Theorem 2.6.

Corollary 2.7. *For* $n \geq 3$, $d_{sR}(C_n) = 1$.

Corollary 2.8. *Let* G *be a graph of order* $n \geq 1$. *Then* $\gamma_{sR}(G) = n$ *and* $d_{sR}(G) = 1$ *if and only if* $G = \overline{K_n}$.

Corollary 2.9. *For* $n \geq 1$, $d_{sR}(P_n) = 1$, *unless* $n = 2$ *in which case* $d_{sR}(P_n) = 2$.

Proof. If follows from Proposition B and Theorem 2.6 that $d_{sR}(P_n) = 1$, unless $n = 2$ or $n = 4$. Let $P_n := v_1v_2\dots v_n$. First let $n = 2$. Define the functions $f_i\colon \{v_1, v_2\} \to \{-1, 1, 2\}$ for $i = 1, 2$ by $f_1(v_1) = 2, f_1(v_2) = -1, f_2(v_1) = -1$ and $f_2(v_2) = 2$. Obviously f_1 and f_2 are signed Roman dominating functions of P_2 and $\{f_1, f_2\}$ is a signed Roman dominating family on P_2. Hence $d_{sR}(P_2) \geq 2$. Therefore $d_{sR}(P_2) = 2$ by Theorem 2.1.

Now let $n = 4$. It follows from Theorem 2.1 that $d_{sR}(P_4) \leq 2$. Suppose to the contrary that $d_{sR}(P_4) = 2$ and let $\{f_1, f_2\}$ be a signed Roman dominating family on P_4. By Theorem 2.1, we must have $f_i(v_1) + f_i(v_2) = 1$ for $i = 1, 2$ and $f_1(v_2) + f_2(v_2) = 1$. By Theorem 2.1, $f_1(v_1) + f_2(v_1) = 1$. Similarly, we have $f_1(v_4) + f_2(v_4) = 1$. Thus $f_1(v_i) + f_2(v_i) = 1$ for $1 \leq i \leq 4$. Since $f_1(v_i), f_2(v_i) \in \{-1, 1, 2\}$ and $f_1(v_i) + f_2(v_i) = 1$, we deduce that $f_1(v_i) = -1, f_2(v_i) = 2$ or $f_1(v_i) = 2, f_2(v_i) = -1$ for $1 \leq i \leq 4$. Assume, without loss of generality, that $f_1(v_1) = 2$ and $f_2(v_1) = -1$. Since $f_i(v_1) + f_i(v_2) = 1$ for $i = 1, 2$, we must have $f_1(v_2) = -1$ and $f_2(v_2) = 2$. If $f_1(v_3) = -1$, then we have $\sum_{u\in N[v_2]} f_1(u) \leq 0$ which is a contradiction. Thus, $f_1(v_3) = 2$ and hence $f_2(v_3) = -1$ which implies that $\sum_{u\in N[v_2]} f_2(u) \leq 0$ which is a contradiction again. Therefore $d_{sR}(P_4) = 1$ and the proof is complete. □

Theorem 2.10. *If K_n is the complete graph of order $n \geq 1$, then $d_{sR}(K_n) = n$, unless $n = 3$ in which case $d_{sR}(K_n) = 1$.*

Proof. If $n = 3$, the the result follows from Proposition A and Theorem 2.6. Now let $n \neq 3$ and let $V(K_n) = \{v_0, v_1, \dots, v_{n-1}\}$ be the vertex set of K_n. Consider two cases.

Case 1. Assume that n is even. Define the functions $f_1, f_2, \dots, f_n$ as follows. $f_1(v_{n-1}) = 2$, $f_1(v_i) = -1$ if $0 \leq i \leq \frac{n-2}{2}$ and $f_1(v_i) = 1$ if $\frac{n}{2} \leq i \leq n-2$, and for $2 \leq j \leq q$ and $0 \leq i \leq n-1$,

$$f_j(v_i) = f_{j-1}(v_{i+j-1}),$$

where the sum is taken modulo n. It is easy to see that f_j is a signed Roman dominating function of K_n of weight 1 and for each $1 \leq j \leq n$ and $\{f_1, f_2, \dots, f_n\}$ is a signed Roman dominating family on K_n. Hence $d_{sR}(K_n) \geq n$. Therefore $d_{sR}(K_n) = n$ by Proposition A and Theorem 2.6.

Case 2. Assume that n is odd. Define the functions $f_1, f_2, \dots, f_n$ as follows. $f_1(v_{n-1}) = f(v_{n-2}) = 2$, $f_1(v_i) = -1$ if $0 \leq i \leq \frac{n-1}{2}$ and $f_1(v_i) = 1$ if $\frac{n+1}{2} \leq i \leq n-3$, and for $2 \leq j \leq q$ and $0 \leq i \leq n-1$,

$$f_j(v_i) = f_{j-1}(v_{i+j-1}),$$

where the sum is taken modulo n. It is easy to see that f_j is a signed Roman dominating function of K_n of weight 1, for each $1 \leq j \leq n$ and $\{f_1, f_2, \dots, f_n\}$ is a signed Roman dominating family on K_n. Hence $d_{sR}(K_n) \geq n$. Therefore $d_{sR}(K_n) = n$ by Proposition A and Theorem 2.6. □

For some regular graphs we will improve the upper bound given in Theorem 2.1.

Theorem 2.11. *Let G be a δ-regular graph of order n such that $\delta \geq 1$. If $n \not\equiv 0 \pmod{(\delta+1)}$, then $d_{sR}(G) \leq \delta$.*

Proof. Since $n \not\equiv 0 \pmod{(\delta+1)}$, we deduce that $n = p(\delta+1)+r$ with integers $p \geq 1$ and $1 \leq r \leq \delta$. Let $\{f_1, f_2, \ldots, f_d\}$ be an SRD family on G such that $d = d_{sR}(G)$. It follows that

$$\sum_{i=1}^{d} \omega(f_i) = \sum_{i=1}^{d} \sum_{v \in V} f_i(v) = \sum_{v \in V} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V} 1 = n.$$

Proposition D implies $\omega(f_i) \geq \gamma_{sR}(G) \geq p+1$ for each $i \in \{1, 2, \ldots, d\}$. If we suppose to the contrary that $d \geq \delta + 1$, then the above inequality chain leads to the contradiction

$$n \geq \sum_{i=1}^{d} \omega(f_i) \geq d(p+1) \geq (\delta+1)(p+1) = p(\delta+1)+\delta+1 > n.$$

Thus $d \leq \delta$, and the proof is complete. □

Theorem 2.10 demonstrates that Theorem 2.11 is not valid in general when $n \equiv 0 \pmod{(\delta+1)}$.

Theorem 2.12. *If G is a graph of order $n \geq 1$, then*

$$\gamma_{sR}(G) + d_{sR}(G) \leq n + 1 \tag{2.1}$$

with equality if and only if $G \simeq \overline{K_n}$ or $G \simeq K_n$ ($n \neq 3$).

Proof. It follows from Theorem 2.6 that

$$\gamma_{sR}(G) + d_{sR}(G) \leq \frac{n}{d_{sR}(G)} + d_{sR}(G). \tag{2.2}$$

According to Theorem 2.1, we have $1 \leq d_{sR}(G) \leq n$. Using these bounds, and the fact that the function $g(x) = x + n/x$ is decreasing for $1 \leq x \leq \sqrt{n}$ and increasing for $\sqrt{n} \leq x \leq n$, the last inequality leads to the desired bound immediately.

If $G \simeq K_n$ ($n \neq 3$) then it follows from Proposition A and Theorem 2.10 that $\gamma_{sR}(G) + d_{sR}(G) = n+1$. If $G \simeq \overline{K_n}$, then it follows from Proposition C and Corollary 2.2 that $\gamma_{sR}(G) + d_{sR}(G) = n+1$.

Conversely, let equality hold in (2.1). It follows from (2.2) that

$$n+1 = \gamma_{sR}(G) + d_{sR}(G) \leq \frac{n}{d_{sR}(G)} + d_{sR}(G) \leq n+1,$$

which implies that $\gamma_{sR}(G) = \frac{n}{d_{sR}(G)}$ and $d_{sR}(G) = 1$ or $d_{sR}(G) = n$. If $d_{sR}(G) = n$, then $\delta(G) = n-1$ by Theorem 2.1 and hence G is a complete graph K_n. Since also $\gamma_{sR}(G) = 1$, we deduce that $n \neq 3$ and hence $G \simeq K_n$ ($n \neq 3$). If $d_{sR}(G) = 1$, then $\gamma_{sR}(G) = n$, and it follows from Proposition C that $G \simeq \overline{K_n}$. This completes the proof. □

As an application of Theorems 2.1 and 2.11, we will prove the following Nordhaus-Gaddum type result.

Theorem 2.13. *For every graph G of order n,*

$$d_{sR}(G) + d_{sR}(\overline{G}) \le n + 1. \tag{2.3}$$

Furthermore, $d_{sR}(G) + d_{sR}(\overline{G}) = n + 1$ if and only if $n \neq 3$ and $G \simeq K_n$ or $G \simeq \overline{K_n}$.

Proof. It follows from Theorem 2.1 that

$$\begin{aligned} d_{sR}(G) + d_{sR}(\overline{G}) &\le (\delta(G) + 1) + (\delta(\overline{G}) + 1) \\ &= (\delta(G) + 1) + (n - \Delta(G) - 1 + 1) \le n + 1. \end{aligned}$$

If G is not regular, then $\Delta(G) - \delta(G) \ge 1$, and hence the above inequality chain implies the better bound $d_{sR}(G) + d_{sR}(\overline{G}) \le n$.

If $n \neq 3$ and $G \simeq K_n$ or $G \simeq \overline{K_n}$, then Corollary 2.2 and Theorem 2.10 lead to $d_{sR}(G) + d_{sR}(\overline{G}) = n + 1$.

Conversely, assume that $d_{sR}(G) + d_{sR}(\overline{G}) = n + 1$. Then G is δ-regular and thus $\overline{G}$ is $(n - \delta - 1)$-regular. If $\delta = 0$ or $\delta = n - 1$, then $G \simeq K_n$ or $G \simeq \overline{K_n}$, and we obtain the desired result.

Next assume that $1 \le \delta \le n - 2$ and $1 \le \delta(\overline{G}) = n - \delta - 1 \le n - 2$. We assume, without loss of generality, that $\delta \le (n-1)/2$. If $n \not\equiv 0 \pmod{(\delta+1)}$, then it follows from Theorems 2.1 and 2.11 that

$$\begin{aligned} d_{sR}(G) + d_{sR}(\overline{G}) &\le \delta(G) + (\delta(\overline{G}) + 1) \\ &= \delta(G) + (n - \delta(G) - 1 + 1) = n, \end{aligned}$$

a contradiction. Next assume that $n \equiv 0 \pmod{(\delta + 1)}$. Then $n = p(\delta + 1)$ with an integer $p \ge 2$. If $n \not\equiv 0 \pmod{(n - \delta)}$, then it follows from Theorems 2.1 and 2.11 that

$$\begin{aligned} d_{sR}(G) + d_{sR}(\overline{G}) &\le (\delta(G) + 1) + \delta(\overline{G}) \\ &= \delta(G) + 1 + (n - \delta(G) - 1) = n, \end{aligned}$$

a contradiction. Therefore assume that $n \equiv 0 \pmod{(n - \delta)}$. Then $n = q(n - \delta)$ with an integer $q \ge 2$. Since $\delta \le (n - 1)/2$, this leads to the contradiction

$$n = q(n - \delta) \ge \left(n - \frac{n-1}{2}\right) = \frac{q(n+1)}{2} \ge n + 1,$$

and the proof is complete. □

The next result is a generalization of Corollary 2.3.

Theorem 2.14. *If G is a connected cactus graph, then $d_{sR}(G) \le 2$.*

Proof. Let $d = d_{sR}(G)$. If $\delta(G) \leq 1$, then Theorem 2.1 implies the desired bound $d \leq 2$ immediately.

It remains the case that $\delta(G) = 2$. If G is a cycle, then the result follows from Corollary 2.7. Otherwise, the cactus graph G contains a cycle $v_1v_2 \dots v_tv_1$ as an end block with exactly one cut vertex, say v_1. Applying Theorem 2.1, we see that $d \leq 3$. Suppose to the contrary that $d = 3$. Let $\{f_1, f_2, f_3\}$ be a signed Roman dominating family on G.

Claim. If $f_i(v_j) = 2$ for $1 \leq i \leq 3$ and $2 \leq j \leq t$, then $d \leq 2$.

Proof of claim. Assume, without loss of generality, that $f_1(v_2) = 2$. Because of $f_1(v_2) + f_2(v_2) + f_3(v_2) \leq 1$, we deduce that $f_2(v_2) = f_3(v_2) = -1$. Since f_i is a signed Roman dominating function, we see that $f_i(v_1) = 2$ or $f_i(v_3) = 2$ for $2 \leq i \leq 3$. Assume, without loss of generality, that $f_2(v_1) = 2$. It follows as above that $f_1(v_1) = f_3(v_1) = -1$. Hence we obtain the contradiction $1 \leq \sum_{x\in N[v_2]} f_3(x) = -2 + f_3(v_3) \leq 0$, and the claim is proved.

Thus we assume that $f_i(v_j) \leq 1$ for $1 \leq i \leq 3$ and $2 \leq j \leq t$. If $t \geq 4$, then we conclude that $f_i(v_3) = 1$ for $1 \leq i \leq 3$, a contradiction to $f_1(v_3)+f_2(v_3)+f_3(v_3) \leq 1$. Finally, assume that $t = 3$. If $f_i(v_1) \leq 1$ for $1 \leq i \leq 3$, then $f_i(v_2) = 1$ for $1 \leq i \leq 3$, a contradiction. Now assume, without loss of generality, that $f_1(v_1) = 2$. This implies that $f_2(v_1) = f_3(v_1) = -1$ and therefore $f_2(v_2) = f_3(v_2) = f_2(v_3) = f_3(v_3) = 1$. This leads to $f_1(v_2) = f_1(v_3) = -1$. Thus we obtain the contradiction $1 \leq \sum_{x\in N[v_2]} f_1(x) = f_1(v_1) + f_1(v_2) + f_1(v_3) = 0$, and the proof is complete. □

References

[1] Ahangar, H. A., Henning, M. A., Zhao, Y., Löwenstein, C., Samodivkin, V., Signed Roman domination in graphs, *J. Comb. Optim.*, (to appear).

[2] Chambers, E. W., Kinnersley, B., Prince, N., West, D. B., Extremal problems for Roman domination, *SIAM J. Discrete Math.*, 23 (2009) 1575–1586.

[3] Cockayne, E. J., Dreyer Jr., P. M., Hedetniemi, S. M., Hedetniemi, S. T., On Roman domination in graphs, *Discrete Math.*, 278 (2004) 11–22.

[4] Favaron, O., Karami, H., Sheikholeslami, S. M., On the Roman domination number in graphs, *Discrete Math.*, 309 (2009) 3447–3451.

[5] Kazemi, A. P., Sheikholeslami, S. M., Volkmann, L., Roman (k, k)-domatic number of a graph, *Ann. Math. Inform.*, 38 (2011) 45–57.

[6] ReVelle, C. S., Rosing, K. E., Defendens imperium romanum: a classical problem in military strategy, *Amer. Math. Monthly*, 107 (2000) 585–594.

[7] Sheikholeslami, S. M., Volkmann, L., Roman domatic number of a graph, *Appl. Math. Letters*, 23 (2010) 1295–1300.

[8] Stewart, I., Defend the Roman Empire, *Sci. Amer.*, 281 (1999) 136–139.

[9] West, D.B, Introduction to Graph Theory, Prentice-Hall, Inc, 2000.