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Zeidis, Igor; Zimmermann, Klaus:

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Original published in:	ZAMM : journal of applied mathematics and mechanics. - Weinheim [u.a.] : Wiley-VCH. - 99 (2019), 12, art. e201900173, 22 pp.
Original published:	2019-11-12
ISSN:	1521-4001
DOI:	10.1002/zamm.201900173
[*Visited:*	2020-06-08]

Received: 27 June 2019 | Revised: 17 October 2019 | Accepted: 26 October 2019
DOI: 10.1002/zamm.201900173

ZAMM

EDITOR'S CHOICE

Dynamics of a four-wheeled mobile robot with Mecanum wheels

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Funding information
Deutsche Forschungsgemeinschaft, Grant/Award Number: ZI 540-19/2; European Social Fund, Grant/Award Number: 2011 FGR 0127

The paper deals with the dynamics of a mobile robot with four Mecanum wheels. For such a system the kinematical rolling conditions lead to non-holonomic constraints. From the framework of non-holonomic mechanics Chaplygin's equation is used to obtain the exact equation of motion for the robot. Solving the constraint equations for a part of generalized velocities by using a pseudoinverse matrix the mechanical system is transformed to another system that is not equivalent to the original system. Limiting the consideration to certain special types of motions, e.g., translational motion of the robot or its rotation relative to the center of mass, and impose appropriate constraints on the torques applied to the wheels, the solution obtained by means of the pseudoinverse matrix will coincide with the exact solution. In these cases, the constraints imposed on the system become holonomic constraints, which justifies using Lagrange's equations of the second kind. Holonomic character of the constraints is a sufficient condition for applicability of Lagrange's equations of the second kind but it is not a necessary condition. Using the methods of non-holonomic mechanics a greather class of trajectories can be achieved.

KEYWORDS
Chaplygin's equation, Mecanum wheels, mobile robots, non-holonomic constraints

1 | INTRODUCTION

This paper relates to mechanics of wheeled locomotion. Beside biological inspired forms of locomotion like crawling, flying or swimming, this form of locomotion with a special propelling device is still in the focus of research. The demand for mobile platforms working in complex environments or personal robots with a high maneuverability for handicapped people lead to new kinds of wheels especially in the past fifty years. Starting with the first patent of J. Grabowiecki in 1919 in the U.S.A. [1], engineers developed wheels which cannot move only in the direction of the wheels plane, but also perpendicular to this plane, e.g. omnidirectional wheels. A key issue of an efficient application of these wheels and an optimal control of the whole mobile system is the understanding of the physical interaction between the wheels and the environment. For this reason, mechanics of wheeled locomotion draws attention of both mechanical engineers, see, e.g., [2–6] and control engineers, e.g., in [7–11]. In a number of studies e.g. [12–17] the motion of systems with so-called Mecanum wheels, as a special class of omnidirectional wheels, was investigated. The authors of the mentioned papers mainly use methods of analytical mechanics to obtain the equations of motion. In many cases the Lagrange equation of second kind is the selected tool, which is correct applicable for holonomic systems.

In this article we consider the classical kinematic constraint, involving point contact and rolling without slipping. There are more complex models of rolling bodies on the surface, taking into account the contact spot, the distribution of normal pressure

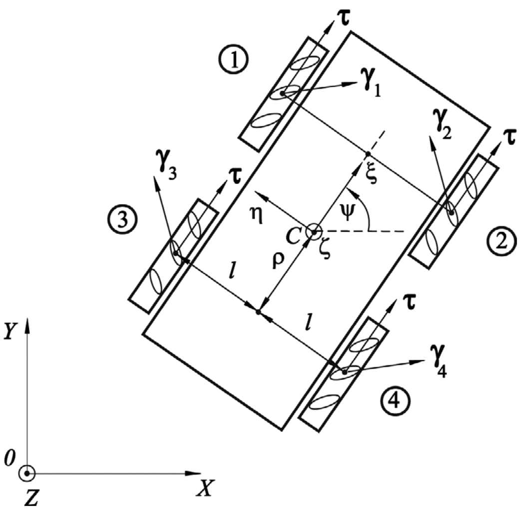

FIGURE 1 Four wheeled mobile robot with Mecanum wheels

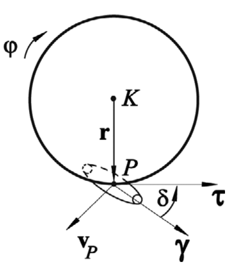

FIGURE 2 Model of a Mecanum wheel

forces on the contact spot and rolling resistance. Such studies are contained in [18–20], and others. Taking these phenomena into account is useful in solving a number of practical problems related to the dynamics of Mecanum wheels. The purpose of the present work was to analyze the method widely used in robotics related to the use of a pseudoinverse matrix and the subsequent compilation of Lagrange's equations of the second kind. We called this method approximate. These equations are compared with equations obtained using non-holonomic mechanics methods. Non-holonomic mechanics methods we called exact.

A Mecanum wheel is a wheel with rollers attached to its circumference. Each roller rotates about an axis that forms an angle of 45 degrees with the plane of the disk. Such a design provides additional kinematic advantages for the Mecanum wheels in comparison with the conventional wheels and leads to non-holonomic constraints. Thus, a full non-holonomic approach is used describing the dynamics of a mobile robot with four Mecanum wheels.

2 | FORMULATION OF THE MECHANICAL PROBLEM AND ITS MATHEMATICAL MODEL

The dynamics of a four-wheeled robot with Mecanum wheels arranged on two parallel axles (Figure 1) are studied. The robot moves so that all its wheels have permanent contact with a plane. The body of the robot has a mass of m_0, its center of mass lies on the longitudinal axis of symmetry of the body. The distance from the center of mass C of the robot to each of its wheel axles is ρ, the distance between the centers of the wheels is $2l$. The coordinates of the center of mass in a fixed coordinate system XOY are x_c, y_c , the angle formed by the longitudinal axis of symmetry of the body with axis OX is ψ, each wheel has a mass of m_1. The angles of rotation of the wheels relative to the axes that are perpendicular to the planes of the respective wheels and pass through their centers are φ_i, and the torques applied to the wheels are M_i $(i = 1, \dots, 4)$.

2.1 | Model of a Mecanum wheel

A Mecanum wheel is a wheel with rollers fixed on its outer rim. The axis of each of the rollers forms the same angle δ ($0° < \delta \leq 90°$) with the plane of the wheel. As a rule, the angle δ is equal to 45°. Each roller may rotate freely about its axis, while the wheel may roll on the roller. We will model a Mecanum wheel by a thin disk of radius R ; the velocity $\boldsymbol{V}_P$ of the point of contact P of the disk with the supporting plane is orthogonal to the axis of the roller, see Figure 2. The rollers are densely attached to the circumference of the wheel (disk). The dimensions of the rollers are much less than the diameter of the disk and are comparable with the disk's thickness. Within the framework of these assumptions we can use a model of the Mecanum wheel in which the rollers have infinitesimal dimensions.

Let $\boldsymbol{\gamma}$ be the unit vector of the roller's axis. The wheels move without slip, which implies the constraint

$$\boldsymbol{V}_P \cdot \boldsymbol{\gamma} = 0\,. \tag{1}$$

If $\boldsymbol{V}_K$ is the velocity of the wheel's center K, then

$$\boldsymbol{V}_P = \boldsymbol{V}_K + \boldsymbol{\omega} \times \boldsymbol{r}, \tag{2}$$

where $\boldsymbol{\omega}$ is the angular velocity of the wheel and $\boldsymbol{r} = \overrightarrow{KP}$. Let φ be the angle of rotation of the wheel about the axis that is perpendicular to the wheel's plane and passes through its center. Then expression (1) can be represented as follows:

$$(\boldsymbol{V}_K - R\,\dot{\varphi}\,\boldsymbol{\tau}) \cdot \boldsymbol{\gamma} = 0\,, \tag{3}$$

where $\boldsymbol{\tau}$ is the unit vector tangent to the wheel's rim at the point of contact. From expression (3) we find

$$\boldsymbol{V}_K \cdot \boldsymbol{\gamma} = R\,\dot{\varphi} \cos\delta. \tag{4}$$

2.2 | Kinematic constraint equations

In what follows, we assume that $\delta = \pi/4$. Let $\boldsymbol{V}_C$ denote the velocity of the center of mass of the mobile robot and let $\overrightarrow{CK_i} = \boldsymbol{r}_i$. Then

$$\boldsymbol{V}_{K_i} = \boldsymbol{V}_C + \boldsymbol{\Omega} \times \boldsymbol{r}_i, \quad i = 1, \dots, 4, \tag{5}$$

where $\boldsymbol{V}_{K_i}$ is the velocity of the center of the respective wheel and $\boldsymbol{\Omega}$ is the angular velocity of the body of the robot. Then the kinematic constraint that follows from (4) and represents the condition for rolling without slip along the roller's axis can be written as

$$(\boldsymbol{V}_C + \boldsymbol{\Omega} \times \boldsymbol{r}_i) \cdot \boldsymbol{\gamma}_i = \frac{R}{\sqrt{2}}\,\dot{\varphi}_i, \quad i = 1, \dots, 4. \tag{6}$$

Here, $\boldsymbol{\gamma}_i$ is the unit vector that points along the axis of the roller of the respective wheel that has contact with the plane at the current time instant. Since

$$(\boldsymbol{\Omega} \times \boldsymbol{r}_i) \cdot \boldsymbol{\gamma}_i = (\boldsymbol{r}_i \times \boldsymbol{\gamma}_i) \cdot \boldsymbol{\Omega}, \quad i = 1, \dots, 4 \tag{7}$$

constraint equation (6) can be rewritten as follows:

$$\boldsymbol{V}_C \cdot \boldsymbol{\gamma}_i + (\boldsymbol{r}_i \times \boldsymbol{\gamma}_i) \cdot \boldsymbol{\Omega} = \frac{R}{\sqrt{2}}\,\dot{\varphi}_i, \quad i = 1, \dots, 4. \tag{8}$$

Introduce a robot-attached coordinate system $\xi\eta\zeta$ with origin at the canter of mass C of the cart. We point axis $C\xi$ along the longitudinal symmetry axis of the cart, axis $C\eta$ along the lateral symmetry axis, and axis $C\zeta$ vertically upward. Denote by $V_{C\xi}$ and $V_{C\eta}$ the projections of the velocity of the center of mass onto the movable axes $C\xi$ and $C\eta$, respectively, and represent expression (8) as follows:

$$V_{C\xi}\gamma_{i\xi} + V_{C\eta}\gamma_{i\eta} + (r_{i\xi}\gamma_{i\eta} - r_{i\eta}\gamma_{i\xi})\dot{\psi} = \frac{R}{\sqrt{2}}\,\dot{\varphi}_i, \\ i = 1, \dots, 4. \tag{9}$$

Since

$$\begin{aligned} &\gamma_{1\xi} = \gamma_{4\xi} = \frac{1}{\sqrt{2}}, \quad \gamma_{1\eta} = \gamma_{4\eta} = -\frac{1}{\sqrt{2}}, \\ &\gamma_{2\xi} = \gamma_{3\xi} = \frac{1}{\sqrt{2}}, \quad \gamma_{2\eta} = \gamma_{3\eta} = \frac{1}{\sqrt{2}}, \\ &r_{1\xi} = r_{2\xi} = \rho, \qquad r_{1\eta} = r_{3\eta} = l, \\ &r_{3\xi} = r_{4\xi} = -\rho, \qquad r_{2\eta} = r_{4\eta} = -l \end{aligned} \tag{10}$$

the constraint equations become

$$\begin{aligned} V_{C\xi} - V_{C\eta} - (\rho + l)\dot{\psi} &= R\dot{\varphi}_1, \\ V_{C\xi} + V_{C\eta} + (\rho + l)\dot{\psi} &= R\dot{\varphi}_2, \\ V_{C\xi} + V_{C\eta} - (\rho + l)\dot{\psi} &= R\dot{\varphi}_3, \\ V_{C\xi} - V_{C\eta} + (\rho + l)\dot{\psi} &= R\dot{\varphi}_4. \end{aligned} \tag{11}$$

The four Equations (11) can also be represented equivalently in form of the following four equations:

$$\begin{aligned} V_{C\xi} &= \frac{R}{2}(\dot{\varphi}_1 + \dot{\varphi}_2), \\ V_{C\eta} &= \frac{R}{2}(\dot{\varphi}_3 - \dot{\varphi}_1), \\ \dot{\psi} &= \frac{R}{2(\rho + l)}(\dot{\varphi}_2 - \dot{\varphi}_3), \\ \dot{\varphi}_1 + \dot{\varphi}_2 &= \dot{\varphi}_3 + \dot{\varphi}_4. \end{aligned} \tag{12}$$

The character of constraints is essential for deriving the dynamic equations. If the constraints imposed on a mechanical system restrict its position, then these constraints are called holonomic (geometrical) constraints. For the systems subject to holonomic constraints, Lagrange's equations of the second kind can be used. The constraints that restrict the velocities but do not restrict the coordinates are called non-holonomic constraints. Such constraints cannot be integrated and reduced to geometrical constraints. Non-holonomic constraints require different methods for deriving the dynamic equations for the mechanical system. The answer to the question of whether the system of equations that defines constraints is integrable (holonomic) is given by Frobenius theorem.[21]

The quantities $V_{C\xi}$ and $V_{C\psi}$ are related to the components $\dot{x}_c$ and $\dot{y}_c$ of the velocity vector of the center of mass in the fixed reference frame by

$$\begin{aligned} V_{C\xi} &= \dot{x}_c \cos\psi + \dot{y}_c \sin\psi, \\ V_{C\eta} &= -\dot{x}_c \sin\psi + \dot{y}_c \cos\psi. \end{aligned} \tag{13}$$

Then

$$\begin{aligned} \dot{x}_c &= V_{C\xi} \cos\psi - V_{C\eta} \sin\psi, \\ \dot{y}_c &= V_{C\xi} \sin\psi + V_{C\eta} \cos\psi. \end{aligned} \tag{14}$$

Taking into account the first two equations of (12), we can represent (14) as follows:

$$\begin{aligned} \dot{x}_c &= \frac{R}{\sqrt{2}} \cos\left(\psi - \frac{\pi}{4}\right)\dot{\varphi}_1 + \frac{R}{2}(\cos\psi\,\dot{\varphi}_2 - \sin\psi\,\dot{\varphi}_3), \\ \dot{y}_c &= \frac{R}{\sqrt{2}} \sin\left(\psi - \frac{\pi}{4}\right)\dot{\varphi}_1 + \frac{R}{2}(\sin\psi\,\dot{\varphi}_2 + \cos\psi\,\dot{\varphi}_3). \end{aligned} \tag{15}$$

The other two equations (12) can be integrated to obtain

$$\psi = \frac{R}{2(\rho + l)}(\varphi_2 - \varphi_3) + C_1,$$
$$\varphi_4 = \varphi_1 + \varphi_2 - \varphi_3 + C_2, \tag{16}$$

where C_1, and C_2 are constants.

Therefore, the system under consideration has two holonomic constraints that allow two generalized coordinates ψ and φ_4 to be eliminated. As to the constraints of (15), they, with reference to (16), can be represented by

$$\dot{x}_c = a_1\dot{\varphi}_1 + a_2\dot{\varphi}_2 + a_3\dot{\varphi}_3,$$
$$\dot{y}_c = b_1\dot{\varphi}_1 + b_2\dot{\varphi}_2 + b_3\dot{\varphi}_3. \tag{17}$$

Here

$$\begin{aligned} a_1 &= \frac{R}{\sqrt{2}}\cos\left(\psi - \frac{\pi}{4}\right), & b_1 &= \frac{R}{\sqrt{2}}\sin\left(\psi - \frac{\pi}{4}\right), \\ a_2 &= \frac{R}{2}\cos\psi, & b_2 &= \frac{R}{2}\sin\psi, \\ a_3 &= -\frac{R}{2}\sin\psi, & b_3 &= \frac{R}{2}\cos\psi. \end{aligned} \tag{18}$$

The fact that constraints (17) are non-holonomic (non-integrable) implies that not all of the six skew-symmetric quantities

$$\begin{gathered} \alpha_{ij} = \frac{\partial a_i}{\partial \varphi_j} - \frac{\partial a_j}{\partial \varphi_i}, \quad \alpha_{ij} = -\alpha_{ji}, \\ \beta_{ij} = \frac{\partial b_i}{\partial \varphi_j} - \frac{\partial b_j}{\partial \varphi_i}, \quad \beta_{ij} = -\beta_{ji}, \\ (i, j) = (1,2), (1,3), (2,3). \end{gathered} \tag{19}$$

are equal to zero. For the case under consideration we have

$$\begin{aligned} \alpha_{12} &= -\frac{R^2}{2\sqrt{2}}\sin\left(\psi - \frac{\pi}{4}\right), & \alpha_{13} &= \frac{R^2}{2\sqrt{2}}\sin\left(\psi - \frac{\pi}{4}\right), & \alpha_{23} &= \frac{R^2}{2\sqrt{2}}\sin\left(\psi + \frac{\pi}{4}\right), \\ \beta_{12} &= \frac{R^2}{2\sqrt{2}}\sin\left(\psi + \frac{\pi}{4}\right), & \beta_{13} &= -\frac{R^2}{2\sqrt{2}}\sin\left(\psi + \frac{\pi}{4}\right), & \beta_{23} &= \frac{R^2}{2\sqrt{2}}\sin\left(\psi - \frac{\pi}{4}\right), \end{aligned} \tag{20}$$

which implies that Equations (15) correspond to non-holonomic constraints.

3 | DYNAMIC EQUATIONS OF MOTION OF THE MECHANICAL SYSTEM

Let us assume that the configuration of a mechanical system is defined by n generalized coordinates, q_i, $i = 1, \dots, n$ of which $n - m$ coordinates can be expressed in terms of the remaining m coordinates by using the non-holonomic constraint equations. Then the non-holonomic constraint equations can be represented as follows:

$$\dot{q}_{m+k} = \sum_{s=1}^{m} b_{s,m+k}\,\dot{q}_s, \quad k = 1, \dots, n - m, \tag{21}$$

Let the coefficients $b_{s,m+k}$ in these equations be functions of only the independent coordinates $q_1, \dots, q_m$. Since the constraints are independent of time (i.e. scleronomic), the kinetic energy of the system is expressed by a quadratic form of the generalized velocities. Let the coefficients of this quadratic form depend only on the coordinates $q_1, \dots, q_m$. Such a mechanical system is called Chaplygin system, since they can be described by Chaplygin's equations for non-holonomic systems. The equations of motion for such a mechanical system can be represented as follows[22–25]:

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot q_s}\right) - \frac{\partial T^*}{\partial q_s} + P_s = Q_s, \quad s = 1, \dots, m, \tag{22}$$

where T^* is the kinetic energy of the system in which the dependent generalized velocities $\dot q_{m+1}, \dots, \dot q_n$ have been expressed in terms of the independent velocities $\dot q_1, \dots, \dot q_m$ by using expressions (21), and Q_s are the generalized forces. The additional terms P_s that are accounted by non-holonomic constraints and distinguish these equations from Lagrange's equations of the second kind are given by

$$\begin{gathered} P_s = \sum_{k=1}^{n-m} \frac{\partial T}{\partial \dot q_{m+k}} \sum_{r=1}^{m} \left(\frac{\partial b_{r,m+k}}{\partial q_s} - \frac{\partial b_{s,m+k}}{\partial q_r}\right)\dot q_r, \\ s = 1, \dots, m, \end{gathered} \tag{23}$$

where T is the expression for the kinetic energy of the system in which the dependent generalized velocities have not been expressed in terms of the independent velocities according to (21). Expression (23) implies that the additional terms P_s vanish if the constraints are holonomic. In this case,

$$\begin{gathered} \frac{\partial b_{r,m+k}}{\partial q_s} - \frac{\partial b_{s,m+k}}{\partial q_r} = 0\,, \\ r,\, s = 1, \dots, m, \quad k = 1, \dots, n-m. \end{gathered} \tag{24}$$

However, the terms P_s may vanish even if the constraints are non-holonomic. In this case, not all of expressions (24) are equal to zero, but the sum of these expressions multiplied by $\partial T/\partial \dot q_{m+k}$ vanishes. The respective example is given below.

Chaplygin system allow the dynamic equations of motion to be separated from non-integrable constraint equations,[26] and therefore, the dynamic equations form a closed system of equation. In addition, these equations have a form of Lagrange's equations of the second kind with additional terms due to non-holonomic constraints. These equations are convenient; in particular, they allow identifying special cases where equations of motion for non-holonomic systems coincide in form with Lagrange's equations of the second kind.

The configuration of the system under consideration is defined by seven generalized coordinates: x_c, y_c, ψ, φ_1, φ_2, φ_3, φ_4. Since we have four constraint equations, the system has three degrees of freedom. Notice that the coefficients a_i, b_i, $i = 1, \dots, m$ in equations of non-holonomic constraints (17) depend only on m independent generalized coordinates φ_s, $s = 1, \dots, m$ (only on φ_2, φ_3, in our case) and are independent of x_c and y_c. For the mechanical system under consideration, Chaplygin's equations are given by

$$\begin{gathered} \frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot\varphi_s}\right) - \frac{\partial T^*}{\partial \varphi_s} + P_s = Q_s, \quad s = 1, \dots, m, \\ P_s = \frac{\partial T}{\partial \dot x_c}\sum_{k=1}^{m} \alpha_{ks}\dot\varphi_k + \frac{\partial T}{\partial \dot y_c}\sum_{k=1}^{m} \beta_{ks}\dot\varphi_k = \sum_{k=1}^{m}\left(\frac{\partial T}{\partial \dot x_c}\alpha_{ks} + \frac{\partial T}{\partial \dot y_c}\beta_{ks}\right)\dot\varphi_k. \end{gathered} \tag{25}$$

Here, T^* is the function of the kinetic energy of the system from which the velocities x_c, y_c have been eliminated using Equations (17) for non-holonomic constrains, T is the kinetic energy of the unconstrained system, Q_s are the generalized forces, and m is the number of independent generalized coordinates. The additional (as compared with Lagrange's equations of the second kind) terms vanish, if all quantities α_{ks} and β_{ks} are zero, i.e, if the constraints are holonomic. However, as follows from (25), the terms may vanish also for nonzero α_{ks} and β_{ks}, if the respective sums are equal to zero, i.e., for some cases of non-holonomic constrains. Then Chaplygin's equations coincide in form with Lagrange's equations of the second kind.

To derive the dynamic equations we, first of all, calculate the kinetic energy T for the system under consideration. The total kinetic energy is the sum of the kinetic energy T_0 of the translational motion,

$$T_0 = \frac{1}{2}(m_0 + 4m_1)\vec V_C^2 \tag{26}$$

and the kinetic energy T_1 of the rotational motion,

$$T_1 = \frac{1}{2}\Big(J_0 + 4\big(J_2 + m_1(\rho^2 + l^2)\big)\Big)\dot{\psi}^2 + \frac{J_1}{2}(\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}_3^2 + \dot{\varphi}_4^2), \tag{27}$$

where J_0 is the moment of inertia of the body about the center of mass C, J_1 is the moment of inertia of the wheel about the axis that is perpendicular to the plane of the wheel and passes through its center of mass, and J_2 is the moment of inertia of the wheel about the vertical axis passing through the center of mass of the wheel. Since

$$\vec{V}_C^2 = V_{C\xi}^2 + V_{C\eta}^2 = \dot{x}_c^2 + \dot{y}_c^2, \tag{28}$$

the kinetic energy T of the mechanical system can be represented as

$$T = \frac{1}{2}\big(m_s(\dot{x}_c^2 + \dot{y}_c^2) + J_C\dot{\psi}^2 + J_1(\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}_3^2 + \dot{\varphi}_4^2)\big), \tag{29}$$

where $m_s = m_0 + 4m_1$ is the total mass of the mechanical system and $J_C = J_0 + 4(J_2 + m_1(\rho^2 + l^2))$ is the moment of inertia of the system about the vertical axis passing through the center of mass.

We will call the equations of motion for deriving which the non-holonomic character of constraints have been taken into account the exact equations, while the equations derived by another technique that is frequently used in robotics will be called the approximate equations.

3.1 | An approximate technique for deriving the equations of motion

In this section, we will analyze in detail the technique for compiling dynamic equations of a robot with the Mecanum wheels, which is widely used in robotics.

Consider kinematic constraints (11) as a system of four linear equations for three unknowns, $V_{C\xi}$, V_{C_η}, and $\dot{\psi}$. This system is overdetermined and does not have a solution for arbitrary values of $\dot{\varphi}_1$, $\dot{\varphi}_2$, $\dot{\varphi}_3$, and $\dot{\varphi}_4$. This system may have a solution only if the equations are linearly dependent. The compatibility condition is given by

$$\dot{\varphi}_1 + \dot{\varphi}_2 = \dot{\varphi}_3 + \dot{\varphi}_4. \tag{30}$$

In a number of studies on robotics, e.g., [27–29] the authors, all following and referencing,[14] proceed as follows.

Represent the equations of non-holonomic kinematic constraints (11) in matrix form

$$\dot{\boldsymbol{\varphi}} = \boldsymbol{J}\,\boldsymbol{V}. \tag{31}$$

Here, vector $\dot{\boldsymbol{\varphi}}$ has a dimension of 4×1, matrix $\boldsymbol{J}$ a dimension of 4×3, and vector $\boldsymbol{V}$ a dimension of 3×1

$$\dot{\boldsymbol{\varphi}} = \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\varphi}_3 \\ \dot{\varphi}_4 \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} V_{C\xi} \\ V_{C\eta} \\ (\rho + l)\dot{\psi} \end{pmatrix}, \quad \boldsymbol{J} = \frac{1}{R}\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}. \tag{32}$$

Premultiply relation (31) by the transpose $\boldsymbol{J}^T$ of the matrix $\boldsymbol{J}$ to obtain

$$\boldsymbol{J}^T\dot{\boldsymbol{\varphi}} = \boldsymbol{J}^T\boldsymbol{J}\,\boldsymbol{V}. \tag{33}$$

The 3×3 matrix $\boldsymbol{J}^T\boldsymbol{J}$ has the inverse $(\boldsymbol{J}^T\boldsymbol{J})^{-1}$. Then from (33) we find

$$\boldsymbol{V} = \boldsymbol{J}^+\dot{\boldsymbol{\varphi}}, \quad \boldsymbol{J}^+ = (\boldsymbol{J}^T\boldsymbol{J})^{-1}\boldsymbol{J}^T. \tag{34}$$

The 3×4 matrix $\boldsymbol{J}^+$ is called the pseudoinverse of the matrix $\boldsymbol{J}$

$$\boldsymbol{J}^+ = \frac{R}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}. \tag{35}$$

The values

$$V_{C\xi} = \frac{R}{4}(\dot{\varphi}_1 + \dot{\varphi}_2 + \dot{\varphi}_3 + \dot{\varphi}_4),$$
$$V_{C\eta} = \frac{R}{4}(-\dot{\varphi}_1 + \dot{\varphi}_2 + \dot{\varphi}_3 - \dot{\varphi}_4), \tag{36}$$
$$\dot{\psi} = \frac{R}{4(\rho + l)}(-\dot{\varphi}_1 + \dot{\varphi}_2 - \dot{\varphi}_3 + \dot{\varphi}_4)$$

found in such a way do not satisfy system (31) for arbitrary $\dot{\varphi}_1$, $\dot{\varphi}_2$, $\dot{\varphi}_3$, $\dot{\varphi}_4$. However, of all possible triples of quantities $V_{C\xi}$, $V_{C\eta}$, $\dot{\psi}$, the triple of (36) provides a minimum for the sum of the squared discrepancies, i.e., the sum of the squared differences of the left-hand and right-hand sides of the equations of system (31).[30] Of course, if the compatibility relations (30) hold, then the exact solution of the system of linear equations (11) coincides with the solution constructed by using the pseudoinverse matrix:

$$V_{C\xi} = \frac{R}{2}(\dot{\varphi}_1 + \dot{\varphi}_2),$$
$$V_{C\eta} = \frac{R}{4}(-\dot{\varphi}_1 + \dot{\varphi}_2), \tag{37}$$
$$\dot{\psi} = \frac{R}{2(\rho + l)}(\dot{\varphi}_2 - \dot{\varphi}_3).$$

Consider then our mechanical system as a system subject to three constrains (36). In this case, the configuration of the system will be characterized, as previously, by seven generalized coordinates x_c, y_c, ψ, φ_1, φ_2, φ_3, and φ_4, but now we have three constraint equations and, hence, the mechanical system has four degrees of freedom. Then the authors of the cited papers use Lagrange's equations of the second kind. Setting aside for a while the issue of applicability of Lagrange's equations of the second kind to systems with constraints in the form of (36) let us write down these equations. The kinetic energy T^* obtained by substituting expressions (36) into (29), taking into account (28), is given by

$$T^* = \frac{A}{2}\left(\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}_3^2 + \dot{\varphi}_4^2\right) - B(\dot{\varphi}_1\dot{\varphi}_2 - \dot{\varphi}_1\dot{\varphi}_3 - \dot{\varphi}_2\dot{\varphi}_4 + \dot{\varphi}_3\dot{\varphi}_4) + C(\dot{\varphi}_1\dot{\varphi}_4 + \dot{\varphi}_2\dot{\varphi}_3). \tag{38}$$

Here

$$A = \frac{m_s R^2}{8} + \frac{J_C R^2}{16(\rho + l)^2} + J_1, \quad B = \frac{J_C R^2}{16(\rho + l)^2}, \quad C = \frac{m_s R^2}{8} - \frac{J_C R^2}{16(\rho + l)^2}. \tag{39}$$

The respective Lagrange's equations have the form

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{\varphi}_s}\right) - \frac{\partial T^*}{\partial \varphi_s} = Q_s, \quad s = 1, \dots, 4. \tag{40}$$

Substitute expression (38) for the kinetic energy and into (40) to obtain

$$A\ddot{\varphi}_1 - B(\ddot{\varphi}_2 - \ddot{\varphi}_3) + C\ddot{\varphi}_4 = M_1,$$
$$A\ddot{\varphi}_2 - B(\ddot{\varphi}_1 - \ddot{\varphi}_4) + C\ddot{\varphi}_3 = M_2, \tag{41}$$
$$A\ddot{\varphi}_3 + B(\ddot{\varphi}_1 - \ddot{\varphi}_4) + C\ddot{\varphi}_2 = M_3,$$
$$A\ddot{\varphi}_4 + B(\ddot{\varphi}_2 - \ddot{\varphi}_3) + C\ddot{\varphi}_1 = M_4.$$

Equations (41) are a system of linear equations for the angular accelerations of rotation of the wheels $\ddot{\varphi}_s$, $s = 1, \dots, 4$. By solving these equations we obtain

$$\ddot{\varphi}_1 = A_1 M_1 + B_1(M_2 - M_3) - C_1 M_4,$$
$$\ddot{\varphi}_2 = A_1 M_2 + B_1(M_1 - M_4) - C_1 M_3, \tag{42}$$
$$\ddot{\varphi}_3 = A_1 M_3 + B_1(M_4 - M_1) - C_1 M_2,$$
$$\ddot{\varphi}_4 = A_1 M_4 + B_1(M_3 - M_2) - C_1 M_1,$$

here

$$A_1 = \frac{A(A-C)-2B^2}{(A+C)(A-2B-C)(A+2B-C)},$$
$$B_1 = \frac{B}{(A-2B-C)(A+2B-C)}, \tag{43}$$
$$C_1 = \frac{C(A-C)+2B^2}{(A+C)(A-2B-C)(A+2B-C)}.$$

For $M_s(t)$, $s = 1, \dots, 4$ given as a function of time, the system of Equations (42), subject to respective initial conditions, can be readily integrated. Using the constraint Equations (36), we can find the projections $V_{C\xi}$, $V_{C\eta}$ of the velocity vector of the center of mass

$$V_{C\xi} = \frac{R}{4(A+C)} \int_0^t (M_1 + M_2 + M_3 + M_4) d\tau = \frac{R}{4(mR^2 + 4J_1)} \int_0^t (M_1 + M_2 + M_3 + M_4) d\tau,$$
$$V_{C\eta} = -\frac{R}{4(A+C)} \int_0^t (M_1 - M_2 - M_3 + M_4) d\tau = -\frac{R}{mR^2 + 4J_1} \int_0^t (M_1 - M_2 - M_3 + M_4) d\tau \tag{44}$$

and the angular velocity of the body $\dot{\psi}$

$$\dot{\psi} = -\frac{R}{4(\rho + l)(A+2B-C)} \int_0^t (M_1 - M_2 + M_3 - M_4) d\tau = -\frac{R(\rho + l)}{J_C R^2 + 4J_1(\rho + l)^2} \int_0^t (M_1 - M_2 + M_3 - M_4) d\tau. \tag{45}$$

Then, having determined the angle ψ, we can readily find the coordinates x_c, and y_c of the center of mass from expressions (14). Let us revisit the constraint equations (36). Let us have a mechanical system subjected to such constraints. The third equation can be integrated to obtain the holonomic constraint

$$\psi = \frac{R}{4(\rho + l)}(-\varphi_1 + \varphi_2 - \varphi_3 + \varphi_4) + C_3, \tag{46}$$

where C_3 is a constant.

Therefore, our system has one holonomic constraint. Using (14), we rewrite the remaining two constraint Equations (36) as follows:

$$\dot{x}_c = a_1\dot{\varphi}_1 + a_2\dot{\varphi}_2 + a_3\dot{\varphi}_3 + a_4\dot{\varphi}_4,$$
$$\dot{y}_c = b_1\dot{\varphi}_1 + b_2\dot{\varphi}_2 + b_3\dot{\varphi}_3 + b_4\dot{\varphi}_4, \tag{47}$$

where

$$a_1 = a_4 = \frac{R}{2\sqrt{2}} \cos\left(\psi - \frac{\pi}{4}\right), \quad a_2 = a_3 = \frac{R}{2\sqrt{2}} \cos\left(\psi + \frac{\pi}{4}\right),$$
$$b_1 = b_4 = -\frac{R}{2\sqrt{2}} \cos\left(\psi + \frac{\pi}{4}\right), \quad b_2 = b_3 = \frac{R}{2\sqrt{2}} \cos\left(\psi - \frac{\pi}{4}\right). \tag{48}$$

For this case, non-holonomicity (non-integrability) of constraints (47) implies that some of the twelve skew-symmetric quantities

$$\alpha_{ij} = \frac{\partial a_i}{\partial \varphi_j} - \frac{\partial a_j}{\partial \varphi_i}, \quad \alpha_{ij} = -\alpha_{ji},$$
$$\beta_{ij} = \frac{\partial b_i}{\partial \varphi_j} - \frac{\partial b_j}{\partial \varphi_i}, \quad \beta_{ij} = -\beta_{ji}, \tag{49}$$
$$(i, j) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4).$$

are not equal to zero. In fact,

$$\begin{aligned}
&\alpha_{12} = \alpha_{34} = -\frac{R^2}{8(\rho+l)}\sin\psi, && \alpha_{13} = \alpha_{24} = -\frac{R^2}{8(\rho+l)}\cos\psi,\\
&\alpha_{14} = -\frac{R^2}{4\sqrt{2}(\rho+l)}\sin\left(\psi-\frac{\pi}{4}\right), && \alpha_{23} = \frac{R^2}{4\sqrt{2}(\rho+l)}\sin\left(\psi+\frac{\pi}{4}\right),\\
&\beta_{12} = \beta_{34} = \frac{R^2}{8(\rho+l)}\cos\psi, && \beta_{13} = \beta_{24} = -\frac{R^2}{8(\rho+l)}\sin\psi,\\
&\beta_{14} = \frac{R^2}{8(\rho+l)}\sin\left(\psi+\frac{\pi}{4}\right), && \beta_{23} = \frac{R^2}{8(\rho+l)}\sin\left(\psi-\frac{\pi}{4}\right).
\end{aligned} \tag{50}$$

Therefore, relations (36) are equations of non-holonomic constraints. The coefficients a_i and b_i, $i = 1, \dots, 4$ depend only on φ_1, φ_2, φ_3, φ_4 and, hence the system under consideration is a Chaplygin system. The additional terms due to nonholonomic constraints have the form

$$P_s = \frac{\partial T}{\partial \dot{x}_c}\sum_{k=1}^{4}\alpha_{ks}\dot{\varphi}_k + \frac{\partial T}{\partial \dot{y}_c}\sum_{k=1}^{4}\beta_{ks}\dot{\varphi}_k = \sum_{k=1}^{4}\left(\frac{\partial T}{\partial \dot{x}_c}\alpha_{ks} + \frac{\partial T}{\partial \dot{y}_c}\beta_{ks}\right)\dot{\varphi}_k, \tag{51}$$

where the kinetic energy T is defined by expression (29).

Using relations (50) and (14), we find

$$\begin{aligned}
P_1 &= \frac{m_s R^2}{8(\rho+l)}\left((\dot{x}_c\sin\psi - \dot{y}_c\cos\psi)\dot{\varphi}_2 + (\dot{x}_c\cos\psi + \dot{y}_c\sin\psi)\dot{\varphi}_3 + \sqrt{2}\left(\dot{x}_c\sin\left(\psi-\frac{\pi}{4}\right) - \dot{y}_c\sin\left(\psi+\frac{\pi}{4}\right)\right)\dot{\varphi}_4\right)\\
&= \frac{m_s R^2}{8(\rho+l)}\left(-V_{C\eta}\dot{\varphi}_2 + V_{C\xi}\dot{\varphi}_3 - (V_{C\eta}+V_{C\xi})\dot{\varphi}_4\right).
\end{aligned} \tag{52}$$

Then, using the constraint equations (36) we obtain

$$P_1 = \frac{m_s R^2}{8(\rho+l)}\cdot\frac{R}{4}(\dot{\varphi}_2+\dot{\varphi}_3)(\dot{\varphi}_1-\dot{\varphi}_2+\dot{\varphi}_3-\dot{\varphi}_4) = -\frac{m_s R^2}{8}(\dot{\varphi}_2+\dot{\varphi}_3)\dot{\psi}. \tag{53}$$

Similarly,

$$P_2 = P_3 = \frac{m_s R^2}{8}(\dot{\varphi}_1+\dot{\varphi}_4)\dot{\psi}, \quad P_4 = P_1. \tag{54}$$

Finally, the system of dynamic equations, with non-holonomic constraints being taken into account, becomes

$$\begin{aligned}
A\ddot{\varphi}_1 - B(\ddot{\varphi}_2-\ddot{\varphi}_3) + C\ddot{\varphi}_4 - (B+C)(\dot{\varphi}_2+\dot{\varphi}_3)\dot{\psi} &= M_1,\\
A\ddot{\varphi}_2 - B(\ddot{\varphi}_1-\ddot{\varphi}_4) + C\ddot{\varphi}_3 + (B+C)(\dot{\varphi}_1+\dot{\varphi}_4)\dot{\psi} &= M_2,\\
A\ddot{\varphi}_3 + B(\ddot{\varphi}_1-\ddot{\varphi}_4) + C\ddot{\varphi}_2 + (B+C)(\dot{\varphi}_1+\dot{\varphi}_4)\dot{\psi} &= M_3,\\
A\ddot{\varphi}_4 + B(\ddot{\varphi}_2-\ddot{\varphi}_3) + C\ddot{\varphi}_1 - (B+C)(\dot{\varphi}_2+\dot{\varphi}_3)\dot{\psi} &= M_4.
\end{aligned} \tag{55}$$

Unlike system (41), due to the additional terms, the system of Equations (55) is a set of nonlinear differential equations for the angular velocities of rotation of the wheels. By solving these equations with respect to the highest derivative we obtain

$$\begin{aligned}
\ddot{\varphi}_1 &= k_1(\dot{\varphi}_2+\dot{\varphi}_3)(-\dot{\varphi}_1+\dot{\varphi}_2-\dot{\varphi}_3+\dot{\varphi}_4) + A_1M_1 + B_1(M_2-M_3) - C_1M_4,\\
\ddot{\varphi}_2 &= k_1(\dot{\varphi}_1+\dot{\varphi}_4)(\dot{\varphi}_1-\dot{\varphi}_2+\dot{\varphi}_3-\dot{\varphi}_4) + A_1M_2 + B_1(M_1-M_4) - C_1M_3,\\
\ddot{\varphi}_3 &= k_1(\dot{\varphi}_1+\dot{\varphi}_4)(\dot{\varphi}_1-\dot{\varphi}_2+\dot{\varphi}_3-\dot{\varphi}_4) + A_1M_3 + B_1(M_4-M_1) - C_1M_2,\\
\ddot{\varphi}_4 &= k_1(\dot{\varphi}_2+\dot{\varphi}_3)(-\dot{\varphi}_1+\dot{\varphi}_2-\dot{\varphi}_3+\dot{\varphi}_4) + A_1M_4 + B_1(M_3-M_2) - C_1M_1,
\end{aligned} \tag{56}$$

where

$$k_1 = \frac{R(B+C)}{4(\rho+l)(A+C)}. \tag{57}$$

The angular velocity of rotation of the body $\dot{\psi}$ is

$$\dot{\psi} = \frac{R}{4(\rho+l)}(-\dot{\varphi}_1 + \dot{\varphi}_2 - \dot{\varphi}_3 + \dot{\varphi}_4). \tag{58}$$

From system (56) it follows that the expression for the angular velocity coincides with expression (45), obtained without taking into account the non-holonomic nature of the constraints.

Thus, it is shown that if we use the approximate equations of kinematic constraints (36) obtained using the pseudo-inverse matrix, they still remain non-holonomic and the Lagrange's equations of the second kind are generally unapplicable to such systems.

3.2 | Exact equations of motion for a non-holonomic system

Consider again the exact system subject to constraints (15) and (16). For this mechanical system, Chaplygin's equations (25) are given by

$$\begin{gathered}
\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{\varphi}_s}\right) - \frac{\partial T^*}{\partial \varphi_s} + P_s = Q_s, \quad s = 1,2,3, \\
P_s = \frac{\partial T}{\partial \dot{x}_c}\sum_{k=1}^{3}\alpha_{ks}\dot{\varphi}_k + \frac{\partial T}{\partial \dot{y}_c}\sum_{k=1}^{3}\beta_{ks}\dot{\varphi}_k = \sum_{k=1}^{3}\left(\frac{\partial T}{\partial \dot{x}_c}\alpha_{ks} + \frac{\partial T}{\partial \dot{y}_c}\beta_{ks}\right)\dot{\varphi}_k.
\end{gathered} \tag{59}$$

Calculate the kinetic energy T^* of the system using expression (29) and constraint equations (11). Since

$$\begin{aligned}
\dot{x}_c^2 + \dot{y}_c^2 &= \frac{R^2}{4}\left(2\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}_3^2 + 2\dot{\varphi}_1\dot{\varphi}_2 - 2\dot{\varphi}_1\dot{\varphi}_3\right), \\
\dot{\psi}^2 &= \frac{R^2}{4(\rho+l)^2}\left(\dot{\varphi}_2^2 + \dot{\varphi}_3^2 - 2\dot{\varphi}_2\dot{\varphi}_3\right),
\end{aligned} \tag{60}$$

we obtain

$$\begin{aligned}
T^* = \frac{1}{2}\Big(&\frac{m_s R^2}{4}\left(2\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}_3^2 + 2\dot{\varphi}_1\dot{\varphi}_2 - 2\dot{\varphi}_1\dot{\varphi}_3\right) + J_1\left(2\dot{\varphi}_1^2 + 2\dot{\varphi}_2^2 + 2\dot{\varphi}_3^2 + 2\dot{\varphi}_1\dot{\varphi}_2 - 2\dot{\varphi}_1\dot{\varphi}_3 - 2\dot{\varphi}_2\dot{\varphi}_3\right) \\
&+ \frac{J_C R^2}{4(\rho+l)^2}\left(\dot{\varphi}_1^2 + \dot{\varphi}_3^2 - 2\dot{\varphi}_2\dot{\varphi}_3\right)\Big).
\end{aligned} \tag{61}$$

The additional terms due to nonholonomic constraints are defined by

$$\begin{aligned}
P_1 &= -\frac{R(B+C)}{(\rho+l)}(\dot{\varphi}_2 + \dot{\varphi}_3)(\dot{\varphi}_2 - \dot{\varphi}_3), \\
P_2 &= \frac{R(B+C)}{(\rho+l)}(\dot{\varphi}_1 - \dot{\varphi}_3)(\dot{\varphi}_2 - \dot{\varphi}_3), \\
P_3 &= \frac{R(B+C)}{(\rho+l)}(\dot{\varphi}_1 + \dot{\varphi}_2)(\dot{\varphi}_2 - \dot{\varphi}_3).
\end{aligned} \tag{62}$$

Define now the generalized forces Q_s, $s = 1,2,3$. Using the first expression of (16), we find

$$Q_1 = M_1 + M_4, Q_2 = M_2 + M_4, Q_3 = M_3 - M_4. \tag{63}$$

Finally, the dynamic equations become

$$
\begin{aligned}
(A+C)(2\ddot{\varphi}_1+\ddot{\varphi}_2-\ddot{\varphi}_3)+P_1 &= M_1+M_4\ ,\\
(A+C)\ddot{\varphi}_1+2(A+B)\ddot{\varphi}_2-(A+2B-C)\ddot{\varphi}_3+P_2 &= M_2+M_4,\\
-(A+C)\ddot{\varphi}_1-(A+2B-C)\ddot{\varphi}_2+2(A+B)\ddot{\varphi}_3+P_3 &= M_3-M_4.
\end{aligned}
\tag{64}
$$

By solving these equations with respect to the highest derivative we obtain

$$
\begin{aligned}
\ddot{\varphi}_1 &= k_2(\dot{\varphi}_2+\dot{\varphi}_3)(\dot{\varphi}_2-\dot{\varphi}_3)+A_2M_1-\frac{(A_2-C_2)}{2}(M_2-M_3)+C_2M_4,\\
\ddot{\varphi}_2 &= k_2(\dot{\varphi}_3-2\dot{\varphi}_1-\dot{\varphi}_2)(\dot{\varphi}_2-\dot{\varphi}_3)+A_2M_2-\frac{(A_2-C_2)}{2}(M_1-M_4)+C_2M_3,\\
\ddot{\varphi}_3 &= k_2(\dot{\varphi}_3-2\dot{\varphi}_1-\dot{\varphi}_2)(\dot{\varphi}_2-\dot{\varphi}_3)+A_2M_3+\frac{(A_2-C_2)}{2}(M_1-M_4)+C_2M_2,
\end{aligned}
\tag{65}
$$

here

$$
\begin{gathered}
k_2=\frac{R(B+C)}{2(\rho+l)(A+C)}\ ,\\
A_2=\frac{3A+4B-C}{4(A+C)(A+2B-C)},\qquad C_2=\frac{A+4B-3C}{4(A+C)(A+2B-C)}.
\end{gathered}
\tag{66}
$$

For given torques M_i ($i=1,\dots,4$) and initial conditions, we determine the angular velocities $\dot{\varphi}_1$, $\dot{\varphi}_2$, $\dot{\varphi}_3$ by integrating the system of differential equations (65). Then we use the constraint equations (12) to find

$$
\begin{aligned}
\dot{\varphi}_4 &= \dot{\varphi}_1+\dot{\varphi}_2-\dot{\varphi}_3,\\
\dot{\psi} &= \frac{R}{2(\rho+l)}(\dot{\varphi}_2-\dot{\varphi}_3).
\end{aligned}
\tag{67}
$$

From the system of the Equations (65) follows that expression for $\dot{\psi}$ in case of the exact equations coincides with expression (45) too.

Then we use expressions (15) to calculate the velocities $\dot{x}_c$, $\dot{y}_c$, and the coordinates of the system's center of mass C.

4 | COMPARISON OF THE EXACT AND APPROXIMATE TECHNIQUES

We will find out how the exact and approximate solutions relate to each other. Notice first of all that if the system of dynamic equations derived by the approximate method is subject to additional compatibility conditions (30) for constraint equations, then, as mentioned previously, the constraint equations obtained by means of the pseudoinverse matrix coincide with the exact constraint equations. Since these constraints remain non-holonomic, the exact system of the Equations (64) or (65) should be compared with the system of the Equations (55) or (56). Relation (30) imposes a constraint on the torques applied to the wheels (these torques appear on the right-hand side of system (55)):

$$
M_1+M_2=M_3+M_4. \tag{68}
$$

With this constraint, four equations of the systems (55), (56) subject to respective initial conditions are equivalent to three equations of the systems (64), (65) combined with the relation (30). In fact, by adding the first two equations of the system (55) and subtracting from the resulting sum the sum of the remaining two equations we obtain

$$
(A-2B-C)(\ddot{\varphi}_1+\ddot{\varphi}_2-\ddot{\varphi}_3-\ddot{\varphi}_4)=M_1+M_2-M_3-M_4, \tag{69}
$$

where $A-2B-C=J_1$. This implies for the systems (64), (65) that relations (30) hold if and only if (68) holds.

Subject to the condition of (68), the system of dynamic equations (55), (56) with the constraint equations (36) is equivalent to the system of dynamic equations (64), (65) with the constraint equations (12).

Compare now the systems of Equations (41) and (64). For these systems to be equivalent, it is required, apart from relation (68), that the additional terms due to nonholonomic constraints vanish. As follows from Equations (64) this is the case if

$$\dot{\varphi}_2 - \dot{\varphi}_3 = 0 \tag{70}$$

or

$$\dot{\varphi}_1 = -\dot{\varphi}_2 = \dot{\varphi}_3. \tag{71}$$

For the case (70), as follows from the third relation of (12), the body moves translationally

$$\dot{\psi} = 0, \quad \psi = \psi_0. \tag{72}$$

Such a motion subjects the torques to the additional constraints:

$$M_4 = M_1, \quad M_3 = M_2. \tag{73}$$

Then the system of equations together with the equations of holonomic constraints becomes

$$\begin{gathered}(A+C)\ddot{\varphi}_1 = M_1, \quad (A+C)\ddot{\varphi}_2 = M_2,\\ \dot{\varphi}_3 = \dot{\varphi}_2, \quad \dot{\varphi}_4 = \dot{\varphi}_1, \quad \psi = \psi_0,\\ \dot{x}_c = \frac{R}{\sqrt{2}}\left(\cos\left(\psi_0 - \frac{\pi}{4}\right)\dot{\varphi}_1 - \sin\left(\psi_0 - \frac{\pi}{4}\right)\dot{\varphi}_2\right),\\ \dot{y}_c = \frac{R}{\sqrt{2}}\left(\sin\left(\psi_0 - \frac{\pi}{4}\right)\dot{\varphi}_1 + \cos\left(\psi_0 - \frac{\pi}{4}\right)\dot{\varphi}_2\right).\end{gathered} \tag{74}$$

where $A + C = mR^2/4 + J_1$.

For the case (71), where $\dot{\varphi}_3 = -\dot{\varphi}_2 = \dot{\varphi}_1$, the respective motion is a rotation of the body about the center of mass that remains fixed. In fact, using relations (37) we find that $V_{C\xi} = V_{C\eta} = 0$, and then from (14) conclude that $\dot{x}_c = \dot{y}_c = 0$. The additional constraint for this case is as follows:

$$M_4 = -M_3 = M_2 = -M_1. \tag{75}$$

The system of dynamic equations together with the equations of holonomic constraints becomes

$$\begin{gathered}(A+2B-C)\ddot{\varphi}_1 = M_1,\\ \dot{\varphi}_2 = -\dot{\varphi}_1, \quad \dot{\varphi}_3 = \dot{\varphi}_1, \quad \dot{\varphi}_4 = -\dot{\varphi}_1,\\ \dot{\psi} = -\frac{R}{\rho + l}\dot{\varphi}_1, \qquad \dot{x}_c = \dot{y}_c = 0.\end{gathered} \tag{76}$$

where $A + 2B - C = J_C/(4(\rho + l)^2) + J_1$.

As a result, we can draw the following conclusions:

1. If the torques applied to the wheels satisfy relation (68), i.e., $M_1 + M_2 = M_3 + M_4$, then for appropriate initial conditions, the exact system of Equations (64) or (65) is equivalent to the approximate system (55) or (56), where the kinematic relations are obtained by using the pseudoinverse matrix but the non-holonomic constraints (36) are taken into account.

2. If the torques are subjected to relation (73), i.e., $M_4 = M_1$, $M_3 = M_2$, (in this case, relation (68) is satisfied automatically), then the terms accounted for by the non-holonomic constraints disappear, the constraints become holonomic and Lagrange's equations of the second kind become applicable. Then the systems of Equations (42) and (56) coincide and are equivalent to the exact system (65)). In this case, the robot's body moves translationally ($\psi = \psi_0$).

3. If the conditions imposed on the torques have the form of (75), then relation (68) again is satisfied automatically, since in this case, $M_4 = -M_3 = M_2 = -M_1$. Then the terms accounted for by non-holonomic constraints are absent, and the motion of the robot is its rotation about the center of mass ($\dot{x}_c = \dot{y}_c = 0$).

Let us find out, whether the translational motion of the robot's body or its rotation about the center of mass are possible for the exact and approximate models if relation (68) for the torques does not hold and, hence, the exact and approximate systems of equations are not equivalent to each other.

For translational motions ($\dot{\psi} = 0$), the terms accounted for by non-holonomic constraints disappear and the approximate systems of Equations (42) and (56) coincide. For this case, the third kinematic constraint of (36) implies

$$\dot{\varphi}_1 + \dot{\varphi}_3 = \dot{\varphi}_2 + \dot{\varphi}_4. \tag{77}$$

Then, from the system of Equations (56) we obtain

$$(A_1 - 2B_1 + C_1)(M_1 - M_2 + M_3 - M_4) = 0. \tag{78}$$

Since $A_1 - 2B_1 + C_1 = 1/(A + 2B - C) = 1/(J_1 + J_C R^2/(\rho + l)^2) \neq 0$, we find

$$M_1 + M_3 = M_2 + M_4 \tag{79}$$

In this case, the angular accelerations of the wheels are expressed by

$$\begin{aligned}
\ddot{\varphi}_1 &= \frac{1}{A+C}\left(M_1 + \frac{(B+C)(M_2 - M_3)}{A - 2B - C}\right),\\
\ddot{\varphi}_2 &= \frac{(A-B)M_2 - (B+C)M_3}{(A+C)(A-2B-C)},\\
\ddot{\varphi}_3 &= \frac{(A-B)M_3 - (B+C)M_2}{(A+C)(A-2B-C)},\\
\ddot{\varphi}_4 &= \frac{1}{A+C}\left(M_1 - \frac{(A-B)(M_2 - M_3)}{A - 2B - C}\right).
\end{aligned} \tag{80}$$

In accordance with relations (36), the value of the velocity of the center of mass is defined by

$$V_{C\xi} = \frac{R}{2}\left(\dot{\varphi}_1 + \dot{\varphi}_3\right), \quad V_{C\eta} = \frac{R}{2}\left(\dot{\varphi}_2 - \dot{\varphi}_1\right). \tag{81}$$

Solve Equations (80) subject to zero initial conditions to find

$$\begin{aligned}
V_{C\xi} &= \frac{R}{2(A+C)}\int_0^t \left(M_1(\tau) + M_3(\tau)\right)d\tau,\\
V_{C\eta} &= \frac{R}{2(A+C)}\int_0^t \left(M_2(\tau) - M_1(\tau)\right)d\tau.
\end{aligned} \tag{82}$$

We will find a condition, subject to which the system governed by the exact system of Equations (65) moves translationally. From the third and fourth kinematic relations (12) we find

$$\dot{\varphi}_3 = \dot{\varphi}_2, \quad \dot{\varphi}_4 = \dot{\varphi}_1. \tag{83}$$

Then, the system (65) implies the relation

$$(A_2 - C_2)(M_1 - M_2 + M_3 - M_4) = 0. \tag{84}$$

Since $A_2 - C_2 = 1/(2(A + 2B - C)) = 1/(2J_1 + J_C R^2/(2(\rho + l)^2)) \neq 0$, the condition for the translational motion of the robot's body coincides with relation (79). In this case, since $A_2 + C_2 = A_1 - C_1 = 1/(A + C)$, we obtain

$$\begin{aligned}
\ddot{\varphi}_1 = \ddot{\varphi}_4 &= \frac{1}{2(A+C)}(2M_1 - M_2 + M_3),\\
\ddot{\varphi}_2 = \ddot{\varphi}_3 &= \frac{1}{2(A+C)}(M_2 + M_3)
\end{aligned} \tag{85}$$

and the velocity of the center of mass in accordance with relations (12) is expressed by

$$V_{C\xi} = \frac{R}{2}\left(\dot{\varphi}_1 + \dot{\varphi}_2\right), \quad V_{C\eta} = \frac{R}{2}\left(\dot{\varphi}_2 - \dot{\varphi}_1\right). \tag{86}$$

Solve Equations (85) subject to zero initial conditions to find the same expression (82).

Thus, the conditions of translational motion for the exact and approximate models coincide. The velocities of the center of mass and the trajectory of the motion of the center of mass are the same. However, the angular accelerations and the angular velocities of the wheels are different.

Consider now the rotation of the robot about the center of mass. In this case, $\dot{x}_c = \dot{y}_c = 0$ or $V_{C\xi} = V_{C\eta} = 0$, and the terms accounted for by nonholonomic constraints disappear from the systems of Equations (42) and (56).

For the approximate model, the kinematic relations (36) imply

$$\begin{gathered} \dot{\varphi}_2 + \dot{\varphi}_3 = 0, \quad \dot{\varphi}_1 + \dot{\varphi}_4 = 0, \\ \dot{\psi} = \frac{R}{2(\rho + l)}(\dot{\varphi}_2 - \dot{\varphi}_1). \end{gathered} \tag{87}$$

Then from the equations of motion (42) we conclude that

$$M_2 + M_3 = 0, \quad M_1 + M_4 = 0 \tag{88}$$

and, hence,

$$\begin{aligned} \ddot{\varphi}_1 &= -\ddot{\varphi}_4 = \frac{(A - C)M_1 + 2BM_2}{(A - 2B - C)(A + 2B - C)}, \\ \ddot{\varphi}_2 &= -\ddot{\varphi}_3 = \frac{2BM_1 + (A - C)M_2}{(A - 2B - C)(A + 2B - C)}. \end{aligned} \tag{89}$$

The angular velocity of the rotation about the center of mass subject to zero initial conditions is

$$\dot{\psi} = \frac{R}{2(\rho + l)(A + 2B - C)} \int_0^t (M_2 - M_1)d\tau. \tag{90}$$

For the exact model, from the kinematic relations (12) we find

$$\dot{\varphi}_1 = -\dot{\varphi}_2 = \dot{\varphi}_3 = -\dot{\varphi}_4, \quad \dot{\psi} = -\frac{R}{\rho + l}\dot{\varphi}_1 \tag{91}$$

and, taking into account system (65), we again arrive at relations (88).

For this case,

$$\ddot{\varphi}_1 = -\ddot{\varphi}_2 = \ddot{\varphi}_3 = -\ddot{\varphi}_4 = -\frac{M_2 - M_1}{2(A + 2B - C)} \tag{92}$$

and the angular velocity of the rotation of the body about the center of mass coincide with (90).

As it was the case for the translational motion, the conditions for the torques, subject to which the body of the robot rotates about the center of mass, coincide for the exact and approximate models. However, the angular accelerations of the wheels and the angular velocity of the rotation of the body about the center of mass are different for these models. Of course, the results of calculations according to the exact and approximate models coincide if the condition of (30) is imposed.

Therefore, the exact equations coincide with the approximate equations if the torques applied to the wheels satisfy appropriate relations and the character of the motion is determined in advance.

In the studies on robotics listed previously, only the translational motion of the body, with $\psi = 0$ and its rotation about the center of mass with $\dot{x}_c = \dot{y}_c = 0$, when the angular velocities satisfy the condition, (30) are considered. For these cases, the constraint equations obtained by means of the pseudoinverse matrix coincide with the exact constraint equations and, in addition, the constraints become holonomic, which enables Lagrange's equations of the second kind to be applied.

For these cases, all constraints on the torques presented above are valid. For this reason, the results obtained by using the pseudoinverse matrix and Lagrange's equations of the second kind appear to be correct.

4.1 | Calculations according to the exact and approximate models

Consider a particular case where all torques M_i $(i = 1, \dots, 4)$ applied to the wheels are constant. Let us find the trajectory of the system for this case on the basis of the approximate model. Solve Equations (44) subject to zero initial conditions to find

$$V_{C\xi} = M_\xi t, \quad V_{C\eta} = M_\eta t, \quad \dot{\psi} = M_\psi t, \tag{93}$$

where

$$\begin{aligned} M_\xi &= \frac{R}{4(A+C)}(M_1 + M_2 + M_3 + M_4), \\ M_\eta &= -\frac{R}{4(A+C)}(M_1 - M_2 - M_3 + M_4), \\ M_\psi &= -\frac{R}{4(A+C)(A+2B-C)}(M_1 - M_2 + M_3 - M_4). \end{aligned} \tag{94}$$

Since Equations (93) subject to zero initial conditions imply

$$\psi = \frac{1}{2} M_\psi t^2 \tag{95}$$

from Equations (14) we obtain

$$\begin{aligned} \dot{x}_c &= M_\xi t \cos\frac{M_\psi t^2}{2} - M_\eta t \sin\frac{M_\psi t^2}{2}, \\ \dot{y}_c &= M_\xi t \sin\frac{M_\psi t^2}{2} + M_\eta t \cos\frac{M_\psi t^2}{2}. \end{aligned} \tag{96}$$

Integrating these relations for $x_c(0) = y_c(0) = 0$ yields

$$\begin{aligned} x_c &= \frac{M_\xi}{M_\psi} \sin\frac{M_\psi t^2}{2} + \frac{M_\eta}{M_\psi} \cos\frac{M_\psi t^2}{2} - \frac{M_\eta}{M_\psi}, \\ y_c &= -\frac{M_\xi}{M_\psi} \cos\frac{M_\psi t^2}{2} + \frac{M_\eta}{M_\psi} \sin\frac{M_\psi t^2}{2} + \frac{M_\xi}{M_\psi}. \end{aligned} \tag{97}$$

The trajectory of the center of mass in this case is a circumference

$$\left(x_c + \frac{M_\eta}{M_\psi}\right)^2 + \left(y_c - \frac{M_\xi}{M_\psi}\right)^2 = \frac{M_\xi^2 + M_\eta^2}{M_\psi^2}. \tag{98}$$

Therefore, for any constant torques M_i $(i = 1, \dots, 4)$ and zero initial conditions, the trajectory according to the approximate model is a circumference of radius $\sqrt{M_\xi^2 + M_\eta^2}/|M_\psi|$ centered at the point with coordinates $(-M_\eta/M_\psi, M_\xi/M_\psi)$. Equation (93) implies

$$\vec{V}_C^2 = V_{C\xi}^2 + V_{C\eta}^2 = (M_\xi^2 + M_\eta^2)\, t^2, \tag{99}$$

This means that the center of mass of the system moves along a circumference with a velocity that is increasing in magnitude: $|\vec{V}_C| = \sqrt{M_\xi^2 + M_\eta^2}\, t$. The quantity M_ξ vanishes if $M_1 + M_3 = M_2 + M_4$. In this case, $\dot{\psi} = 0$, and the circumference degenerates into a straight line

$$y_c = \frac{M_\eta}{M_\xi} x_c = \frac{M_2 - M_1}{M_1 + M_3} x_c. \tag{100}$$

FIGURE 3 4WD Arduino compatible Mecanum Robot Kit from www.robotshop.com (Photo courtesy of RobotShop inc.)

Parameters of the robot for calculations correspond to the prototype shown in Figure 3. The values of the parameters defined as follows:

$$\begin{gathered} m_0 = 3.1\,kg, \quad m_1 = 0.35\,kg, \quad R = 0.05\,m, \quad \rho = l = 0.15\,m, \\ J_0 = \ 0.032\,kgm^2, \qquad J_1 = 6.25 \cdot 10^{-4}\,kgm^2, \quad J_2 = 3.13 \cdot 10^{-4}\,kgm^2. \end{gathered} \tag{101}$$

Figures 4–7 plot the computational results for constant all torques M_i $(i = 1, \dots, 4)$ and for zero initial conditions.

Figure 4 plots the results of the numerical integration of the nonlinear differential equations that govern the exact nonholonomic model. Let's set arbitrary constant torques applied to the wheels in the form:

$$M_1 = 0.05\,\mathrm{Nm}, \quad M_2 = 0.25\,Nm, \quad M_3 = -0.05\,Nm, \quad M_4 = 0.20\,Nm. \tag{102}$$

For arbitrary constant torques applied to the wheels the trajectory of the center of mass is a spiral converging to a focus-type singular point (Figure 4a). The dependence of coordinates x_c and y_c on time t is presented on a Figure 4b. On the Figure 4c are shown dependence of the robot's rotation angle ψ on time t, and on a Figure 4d a time dependence of angular velocities of wheels ω_i $(i = 1, \dots, 4)$ on time t.

Figure 5 shows the solution of the linear system that corresponds to the approximate model, the nonholonomic constraints are not being taken into account. For this case, the trajectory of the center of mass for arbitrary constant torques applied to the wheels (102) is a circumference (Figure 5a). The dependencies of coordinates x_c and y_c and angular velocities of wheels ω_i $(i = 1, \dots, 4)$ on time t are presented on Figures 5b, and 5c, respectively. As it was already noted, the dependence robot's body angular rotation ψ on time t coincides with Figure 4c.

Figure 6 presents the solution of the system of equations corresponding to the exact model in the case of the translational motion $(M_1 + M_3 = M_2 + M_4)$. The constant torques applied to the wheels are

$$M_1 = 0.02\,Nm, \quad M_2 = 0.11\,Nm, \quad M_3 = 0.04\,Nm, \quad M_4 = -0.05\,Nm. \tag{103}$$

For this case, the trajectory of the center of mass is a straight line (Figure 6a). The dependencies angular velocities of wheels ω_i $(i = 1, \dots, 4)$ on time t are presented on a Figure 6b.

Figure 7 presents the solution of the system of equations corresponding to the exact model in the case of the rotation of the robot about the center of mass $(M_2 + M_3 = M_1 + M_4 = 0)$. The constant torques applied to the wheels are

$$M_1 = 0.07\,Nm, \quad M_2 = -0.03\,Nm, \quad M_3 = 0.03\,Nm, \quad M_4 = -0.07\,Nm. \tag{104}$$

In this case the robot rotates about the center of mass. On the Figure 7a are shown dependence of the robot's body rotation angle ψ on time t, and on a Figure 7b a time dependence of angular velocities of wheels ω_i $(i = 1, \dots, 4)$ on time t.

The experiments with the prototype, qualitatively confirms the calculations on the basis of the exact nonholonomic model.

5 | A REMARK ABOUT LAGRANGE'S EQUATIONS OF THE SECOND KIND AND NON-HOLONOMIC CONSTRAINTS

The holonomic nature of the constraints imposed on a mechanical system is a sufficient condition subject to which Lagrange's equation of the second kind can be applied. However, it is not a necessary condition. As has already been mentioned, the additional terms in Chaplygin's equations may vanish even if not all coefficients α_{sk}, β_{sk}, are equal to zero. In this case, despite

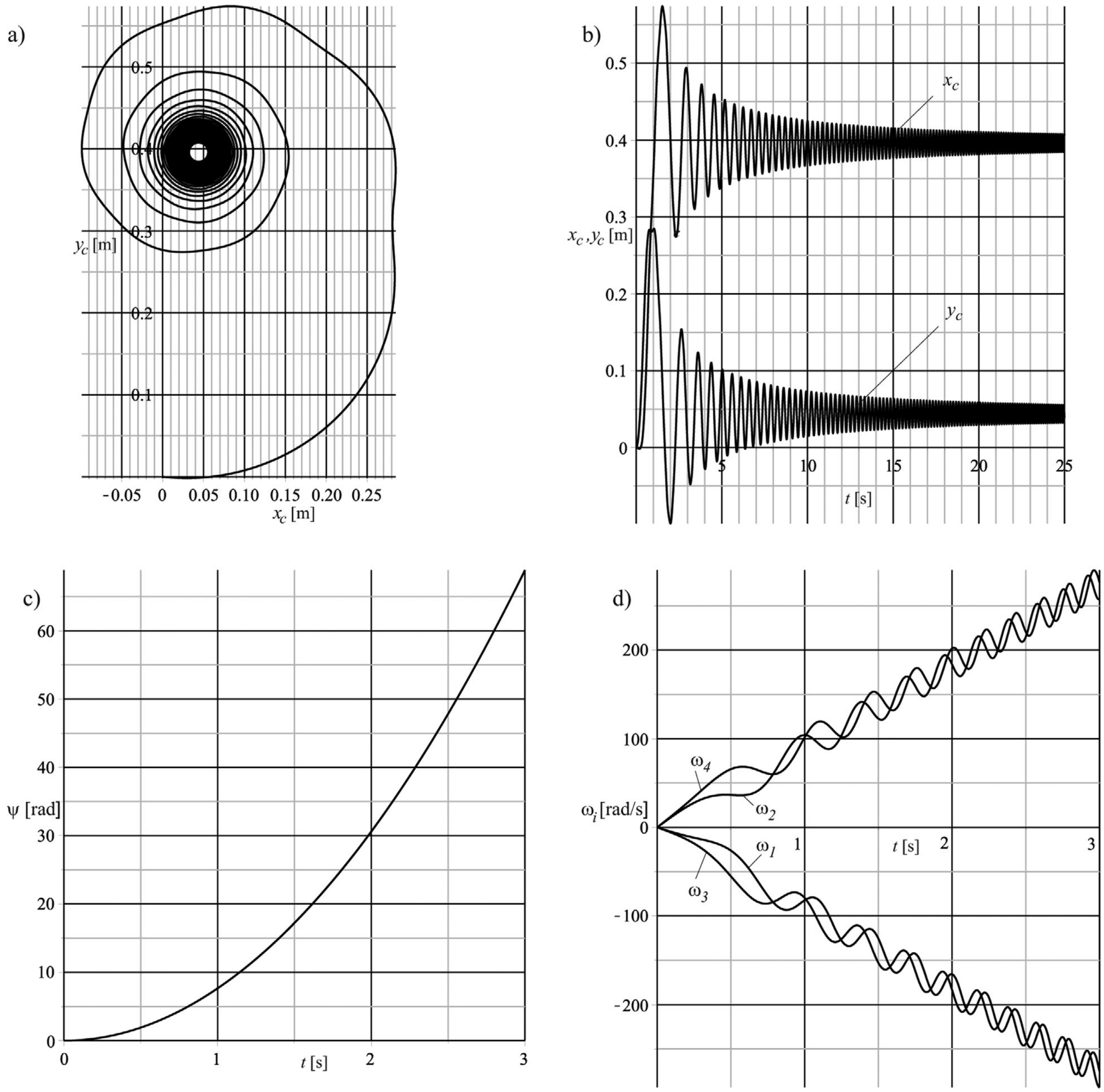

FIGURE 4 Exact non-holonomic model with arbitrary constant torques; a) the trajectory $y_C(x_C)$; b) coordinates x_C and y_C vs. time t; c) rotation angle ψ vs. time t; d) angular velocities ω_i of the wheels vs. time t

the constraints are nonholonomic, equations of motion coincide with Lagrange's equation of the second kind. Such cases are known but occur seldom.[31]

As an example, consider the rolling of a wheel pair along a plane. Both wheels are conventional, have the same mass m_1 and the same radius R. The wheels are set on the common axle that has a mass of m_0, and a length of $2l$ and can freely rotate about this axle. Let x_c, y_c be the coordinates of the axle midpoint C and let φ_1, φ_2 denote the angles of rotation of the wheels. The conditions for this system to roll without slip can be represented as follows[32]:

$$\begin{aligned} \dot{x}_c \cos\psi + \dot{y}_c \sin\psi - l\dot{\psi} &= R\dot{\varphi}_1, \\ \dot{x}_c \cos\psi + \dot{y}_c \sin\psi + l\dot{\psi} &= R\dot{\varphi}_2, \\ -\dot{x}_c \sin\psi + \dot{y}_c \cos\psi &= 0. \end{aligned} \tag{105}$$

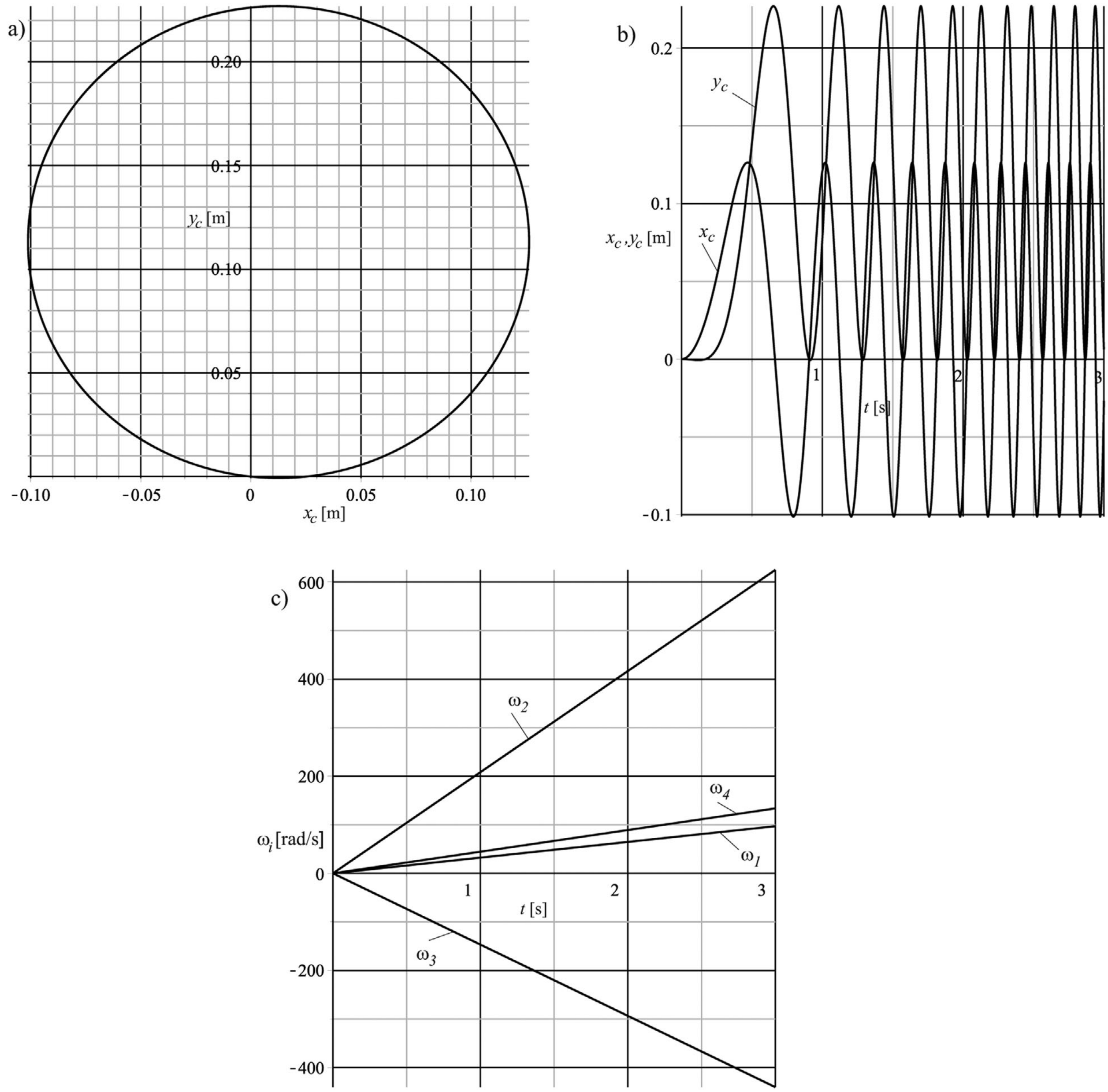

FIGURE 5 Approximate holonomic model with arbitrary constant torques; a) the trajectory $y_C(x_C)$; b) coordinates x_C and y_C vs. time t; c) angular velocities ω_i of the wheels vs. time t

The configuration of the system is characterized by five generalized coordinates x_c, y_c, ψ, φ_1, φ_2; the system is subject to three constraints (105) and, hence, has two degrees of freedom. The constraint equations (105) can be represented as follows:

$$\begin{aligned} \dot{x}_c &= \frac{R}{2}(\dot{\varphi}_1 + \dot{\varphi}_2)\cos\psi, \\ \dot{y}_c &= -\frac{R}{2}(\dot{\varphi}_1 + \dot{\varphi}_2)\sin\psi, \\ \dot{\psi} &= \frac{R}{2l}(\dot{\varphi}_2 - \dot{\varphi}_1). \end{aligned} \tag{106}$$

The third equation of (106) characterizes the holonomic constraint

$$\psi = \frac{R}{2l}(\varphi_2 - \varphi_1) + C_5, \tag{107}$$

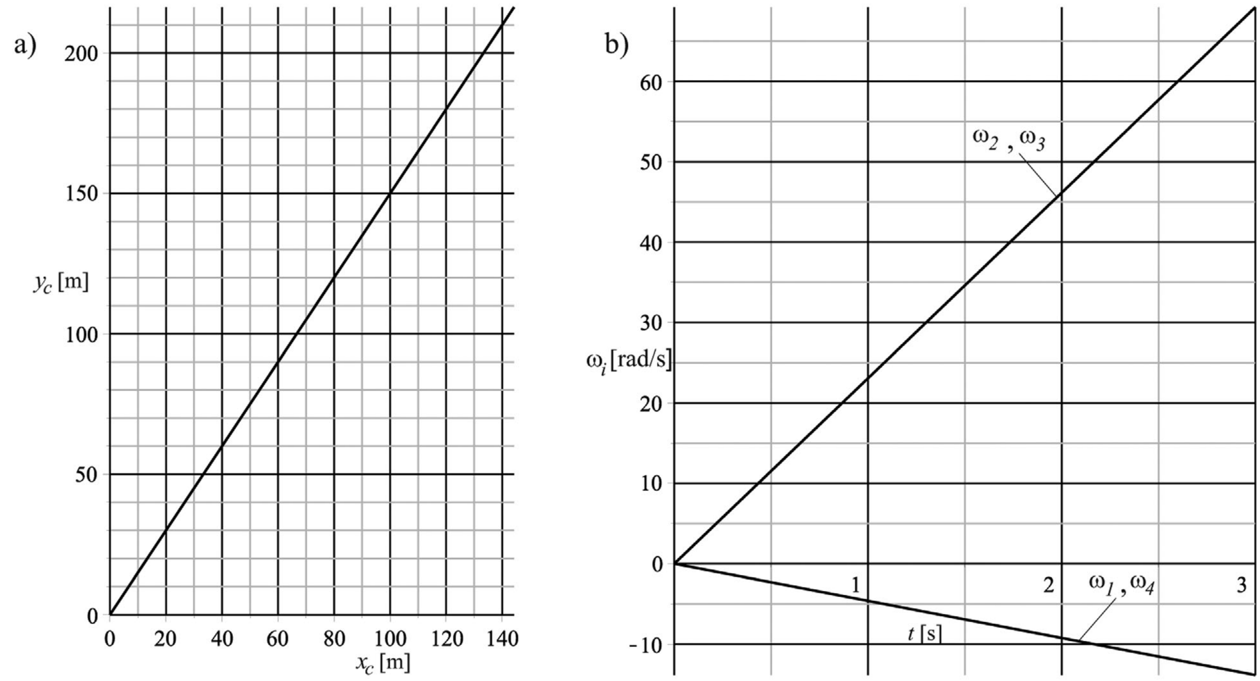

FIGURE 6 Translational motion with the condition $M_1 + M_3 = M_2 + M_4$; a) the trajectory $y_C(x_C)$; b) angular velocities ω_i of the wheels vs. time t

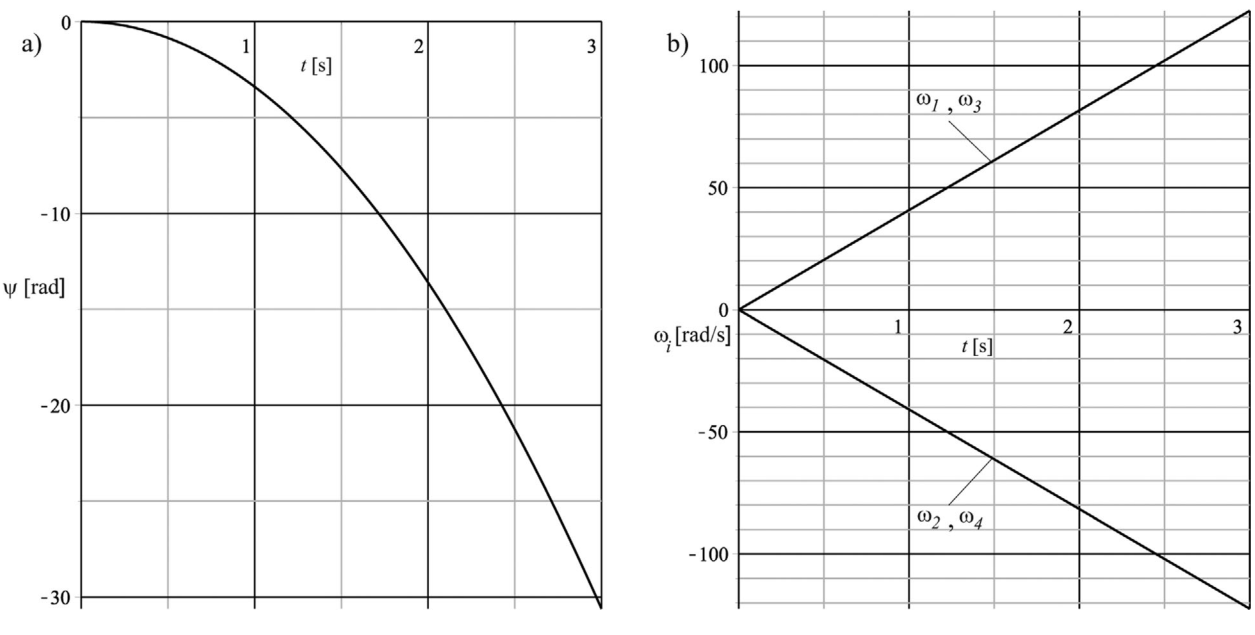

FIGURE 7 Rotational motion about the center of mass with $M_2 + M_3 = M_1 + M_4 = 0$; a) rotation angle ψ vs. time t; b) angular velocities ω_i of the wheels vs. time t

where C_5 is a constant. The kinetic energy T of this system is defined by

$$T = \frac{1}{2}\left(m_s\left(\dot{x}_c^2 + \dot{x}_c^2\right) + J_C\dot{\psi}^2 + J_1\left(\dot{\varphi}_1^2 + \dot{\varphi}_2^2\right)\right), \tag{108}$$

where $m_s = m_0 + 2m_1$ is the total mass of the mechanical system, $J_C = J_0 + 2(J_2 + m_1 l^2)$, and J_0 is the moment of inertia of the axle about its midpoint, J_1 and J_2 are the moments of inertia of the wheels.

The quantities

$$\alpha_{12} = -\frac{R^2}{4l}\sin\psi = -\alpha_{21}, \qquad \beta_{12} = -\frac{R^2}{4l}\cos\psi = -\beta_{21} \tag{109}$$

are not equal to zero and, hence, the first two constraints of (106) are non-holonomic.

However, the additional terms in Chaplygin's equations vanish. In fact,

$$\begin{aligned} P_1 &= \frac{\partial T}{\partial \dot{x}_c}\alpha_{21}\dot{\varphi}_2 + \frac{\partial T}{\partial \dot{y}_c}\beta_{21}\dot{\varphi}_2 = m_s(\dot{x}_c\alpha_{21} + \dot{y}_c\beta_{21})\dot{\varphi}_2 \\ &= \frac{m_s R^3}{8l}(\dot{\varphi}_1 + \dot{\varphi}_2)(\cos\psi\sin\psi - \sin\psi\cos\psi)\dot{\varphi}_2 = 0. \end{aligned} \tag{110}$$

Similarly, $P_2 = 0$.

Therefore, for the case under consideration, the equations of motion coincide with Lagrange's equations of the second kind, although the constraints imposed on the system are non-holonomic.

6 | CONCLUSION AND THE FUTURE WORK

The condition subject to which a robot with four Mecanum wheels moves without slip leads to non-holonomic constraints. To describe the dynamics of such a system one should use equations of motion that are appropriate for mechanical systems with non-holonomic constraints, for example, Chaplygin's equations, Voronets's equations, Appel's equations, Lagrange's equations with multipliers (Lagrange's equations of the first kind), etc. Lagrange's equations of the second kind do not apply to non-holonomic systems in the general case. Apparently, for Chaplygin systems, Chaplygin's equations should be preferred, since in this case, the dynamic equations form a closed system with respect to the generalized velocities treated as independent variables. The holonomic character of the constraints is a sufficient condition for applicability of Lagrange's equations of the second kind but it is not a necessary condition. Therefore, the additional terms that distinguish Chaplygin's equations from Lagrange's equations of the second kind may vanish for some systems with non-holonomic constraints. However, such occurrences are rather rare. In particular, this is not the case for a robot with four Mecanum wheels. In the general case, solving the constraint equations for a part of the generalized velocities by using the pseudoinverse matrix reduces the mechanical system under consideration to a system that is not equivalent to the original system, because the number of degrees of freedom of the reduced system is larger than the number of degrees of freedom of the original system. However, if we confine our consideration to certain special types of motions, e.g., translational motion of the robot or its rotation relative to the center of mass, and impose appropriate constraints on the torques applied to the wheels, the solution obtained by means of the pseudoinverse matrix will coincide with the exact solution. In these cases, the constraints imposed on the system become holonomic constraints, which justifies using Lagrange's equations of the second kind. It is just the motions and constraints that are considered in the papers on robotics cited above, however, it is not stated explicitly. In the general case, the mathematical methods of non-holonomic mechanics should be used.

Subsequent studies are expected to evaluate the effect of the finite linear dimensions of the rollers and the associated body vibrations.

ACKNOWLEDGEMENT

We would like to thank Prof. N.N. Bolotnik for his critical remarks. We also thank Prof. J. Steigenberger for his permanent interest in our work. Thank you to Prof. P. Maisser, who showed one of the authors the way to analytical mechanics. This study was partly supported by the Deutsche Forschungsgemeinschaft (DFG) (project ZI 540-19/2) and by the Development Bank of Thuringia and the Thuringian Ministry of Economic Affairs with funds of the European Social Fund (ESF) under grant 2011 FGR 0127.

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How to cite this article: I. Zeidis, K. Zimmermann. Dynamics of a four-wheeled mobile robot with Mecanum wheels. *Z Angew Math Mech*. 2019;99:e201900173. https://doi.org/10.1002/zamm.201900173