Technische Universität Ilmenau
Institut für Mathematik
Fachgebiet Diskrete Mathematik und Algebra

Dissertation

zur Erlangung des akademischen Grades Dr. rer. nat.

Rooted Structures in Graphs

A Project on Hadwiger's Conjecture, Rooted Minors, and Tutte Cycles

---

vorgelegt der Fakultät für Mathematik und Naturwissenschaften
der Technischen Universität Ilmenau von

Samuel Mohr

1. Gutachter: Prof. Dr. Matthias Kriesell
2. Gutachter: Prof. Dr. Henning Bruhn-Fujimoto
3. Gutachter: Dr. Nicolas Trotignon

Tag der Einreichung: 3. April 2020
Tag der wissenschaftlichen Aussprache: 24. August 2020

urn:nbn:de:gbv:ilm1-2020000294

Dedication

To T.

Abstract

Rooted Structures in Graphs

Dissertation by Samuel Mohr

Hugo Hadwiger conjectured in 1943 that each graph G with chromatic number $\chi(G)$ *has a clique minor on* $\chi(G)$ *vertices. Hadwiger's Conjecture is known to be true for graphs with* $\chi(G) \leq 6$, *with* $\chi(G) = 5$ *and* $\chi(G) = 6$ *being merely equivalent to the Four-Colour-Theorem. Even Paul Erdős (1980) stated that it is "one of the deepest unsolved problems in graph theory", thus reinforcing the extreme nature and difficulty of this conjecture.*

One promising approach to tackle Hadwiger's Conjecture is to bound the number of colourings of considered graphs and, in particular, to consider uniquely optimally colourable graphs. Matthias Kriesell started considerations on graphs with a Kempe colouring, that is graphs with a proper colouring such that the subgraph induced by the union of any two colour classes is connected. Evidently, the optimal colouring of a uniquely colourable graph is a Kempe colouring.

A transversal of a graph's Kempe colouring is a vertex set containing exactly one vertex from each colour class. For each transversal T of a Kempe colouring, there is a system of edge-disjoint paths between all vertices from T. The question arises, whether it is possible to show the existence of a clique minor of the same size as T such that each bag contains one vertex from T. We call a minor with the aforementioned property a rooted minor. An affirmative answer to this question, which was conjectured by Matthias Kriesell (2017), would imply Hadwiger's Conjecture for uniquely colourable graphs. Beyond that, there are more reasons to study this conjecture of Kriesell. On one hand, there are graphs with a Kempe colouring using far more colours than their chromatic number. On the other hand, many proofs of partial results on Hadwiger's Conjecture seem to leave no freedom for prescribing vertices in the clique minor at the expense of forcing any pair of colour classes to be connected.

The question is known to be true for $|T| \leq 4$ by a result of Fabila-Monroy and Wood. In this thesis, we confirm it for line graphs and for $|T| = 5$ if T induces a connected subgraph. For certain graph classes, it emerges that a relaxation of the problem holds. But in general, it turns out that this relaxation is false. Thus, it is not sufficient to force any two transversal vertices to be connected by a 2-coloured path in order to obtain a rooted minor.

Based on rooted minors and results built up from connectivity conditions, the question arose whether it is possible to reduce graphs, in which a certain set of vertices is highly connected, to some basic structure representing the connectivity. This question is investigated in this thesis and answered with two best-possible theorems. The first one elucidates that if a set of vertices, let us call it X, cannot be separated with less than c vertices ($c \leq 3$), then there exists a c-connected topological minor rooted at X. In case of $c = 4$, a second result ensures that it is at least possible to obtain a 4-connected rooted minor, whereas larger connectivities do not lead to highly connected minors in general.

These theorems are applied in this thesis to achieve results about subgraphs containing the set X by deriving them from various results regarding spanning subgraphs. For instance, a theorem of Barnette, stating that a 3-connected planar graph contains a spanning tree of maximum degree at most 3, can easily translated; i.e. there exists a tree with maximum degree at most 3 containing X. Since we cannot guarantee 4-connected topological minors rooted at X, more effort is needed to establish cycles spanning X in planar graphs. To escape this barrier, the theory of Tutte paths is adapted.

In 1956, William Tutte broke the long-standing open problem whether 4-connected planar graphs are hamiltonian and, thereby, extended a theorem of Whitney stating that 4-connected triangulations are hamiltonian. The underlying idea is to prove that planar graphs contain a cycle such that components outside the cycle have a limited number of attaching points to the cycle. In particular, this number of shared attachment vertices is bounded by 3, immediately implying Tutte's result.

By a construction of Moon and Moser it is known that there exist 3-connected planar graphs and a positive constant c such that the length of a longest cycle (circumference) is $c \cdot |V(G)|^{\log_3 2}$ *of such a graph* G*. Several years later, this formula was confirmed to be a lower bound by Chen and Yu.* We study a slight weakening of the 4-connectedness of planar graphs. A 3-connected graph is essentially 4-connected if all separators of order 3 only split single vertices from the remaining graph. If we consider essentially 4-connected planar graphs instead of 3-connected ones, then the minimum circumference in this graph class is described by a linear function in the number of vertices, as Jackson and Wormald showed first. In this thesis, we increase the previously best known coefficient of the linear bound, which was published by Fabrici, Harant, and Jendroľ. Moreover, the best-possible bound of $\frac{2}{3}(n + 4)$ for essentially 4-connected triangulations on n vertices is proved.

Finally, the thesis explores the class of 1-planar graphs and studies their longest cycles. *A graph is 1-planar if it has an embedding into the plane such that each edge is crossed at most once. This class, in contrast to planar graphs, differs with regard to many aspects. The decision problem of 1-planarity is* $\mathcal{NP}$*-complete and there is more than one way to construe maximal 1-planar graphs.* In this thesis, an analogue to Whitney's theorem is proved; i.e., that 3-connected maximal 1-planar graphs have a sublinear circumference just as their planar relatives, and that 4-connected maximal 1-planar graphs are hamiltonian. In the non-maximal case, Tutte's theorem cannot be translated directly; we will construct a family of 5-connected 1-planar graphs in the thesis that is far from being hamiltonian. Whether each 6-connected 1-planar graph is hamiltonian remains open. Some ideas on how an approach to this question using an extended Tutte theory could look like is finally touched upon in the outlook.

Zusammenfassung

Gewurzelte Strukturen in Graphen

Dissertation von Samuel Mohr

Der Schweizer Mathematiker Hugo Hadwiger stellte 1943 die Vermutung auf, dass jeder Graph G mit chromatischer Zahl $\chi(G)$ einen vollständigen Minoren auf $\chi(G)$ Knoten besitzt. Diese Vermutung ist für alle Graphen mit $\chi(G) \leq 6$ wahr, wobei für die Fälle $\chi(G) = 5$ und $\chi(G) = 6$ nur die Äquivalenz zum Vierfarbensatz bekannt ist. Paul Erdős (1980) nannte es eines der weitreichendsten ungelösten Probleme der Graphentheorie und bekräftigte die fehlende Greifbarkeit und Schwere dieser Vermutung.

Ein vielversprechender Ansatz zu neuen Erkenntnissen über Hadwigers Vermutung ist eine Beschränkung der Anzahl der Färbungen. Dabei kann insbesondere die Untersuchung eindeutig färbbarer Graphen helfen. Um darüber Ergebnisse zu erzielen, startete Matthias Kriesell erste Untersuchungen von Graphen mit einer Kempe-Färbung. Das sind Graphen mit einer speziellen Färbung, bei denen die Vereinigung je zweier Farbklassen einen zusammenhängenden Untergraphen induziert. Offensichtlich ist die optimale Färbung eines eindeutigen Graphens auch eine Kempe-Färbung.

Eine Transversale einer Kempe-Färbung eines Graphen ist eine Knotenmenge, die genau einen Knoten aus jeder Farbklasse enthält. Für jede Transversale T einer Kempe-Färbung gibt es ein System von kanten-disjunkten Wegen zwischen allen Knoten aus T. Es stellt sich die Frage, ob es möglich ist, die Existenz eines vollständigen Minoren derselben Größe wie T zu gewährleisten, sodass jede Tasche genau einen Knoten aus T enthält. Wir nennen einen Minor mit den oben beschriebenen Eigenschaften einen gewurzelten Minor. Eine positive Beantwortung dieser Frage, deren Bejahung bereits von Matthias Kriesell (2017) vermutet wurde, würde Hadwigers Vermutung für eindeutig färbbare Graphen bestätigen. Es gibt aber noch weitere Gründe, diese Vermutung zu untersuchen. Einerseits gibt es Graphen mit einer Kempe-Färbung, die wesentlich mehr Farben als die chromatische Zahl enthält. Andererseits lassen viele Beweise bekannter Teilresultate zu Hadwigers Vermutung keinen Spielraum, um Knoten in den Taschen der Minoren vorzuschreiben, auch nicht unter der Forderung, dass je zwei Farbklassen zusammenhängend sind.

Die Frage lässt sich positiv für $|T| \leq 4$ durch ein Resultat von Fabila-Monroy und Wood beantworten. In dieser Dissertation wird sie zudem für Kantengraphen und für $|T| = 5$ verifiziert, sofern T einen zusammenhängenden Teilgraphen induziert. Für bestimmte Graphenklassen wird sich zeigen, dass eine Verallgemeinerung des Problems gilt. Jedoch stellt sich heraus, dass die Gültigkeit der Verallgemeinerung im Allgemeinen nicht bestätigt werden kann. Somit ist es nicht ausreichend, zu fordern, dass je zwei Transversalknoten auf einem zweigefärbten Weg liegen, um einen gewurzelten Minor zu erhalten.

Basierend auf gewurzelten Minoren und verschiedenen Sätzen, die auf Zusammenhangsbedingungen aufbauen, stellt sich die Frage, ob es möglich ist, Graphen, in denen eine bestimmte Menge an Knoten hoch zusammenhängend ist, auf gewisse einfachere Strukturen zu reduzieren, die lediglich den Zusammenhang repräsentieren. Diese Frage wird in der vorliegenden Arbeit untersucht und mit zwei bestmöglichen Theoremen beantwortet. Das erste Theorem erläutert, dass, sofern eine Knotenmenge, welche wir X nennen, sich nicht durch weniger als c Knoten trennen lässt ($c \leq 3$), ein c-fach zusammenhängender topologischer Minor existiert, welcher in X verwurzelt ist. Im Fall von $c = 4$ besagt ein zweites Ergebnis, dass es in diesem Fall zumindest möglich ist, einen 4-fach zusammenhängenden gewurzelten Minoren zu erhalten. Für größere c ist es im Allgemeinen nicht möglich, einen hoch zusammenhängenden Minor zu ermitteln.

Diese neuen Ergebnisse werden im folgenden Verlauf der Dissertation angewendet, um Untergraphen, welche eine Menge X enthalten, sicherzustellen , wobei die Resultate von bekannten Theoremen über spannende Untergraphen abgeleitet werden. So lässt sich beispielsweise ein Theorem von Barnette, welches besagt, dass 3-fach zusammenhängende planare Graphen einen Spannbaum vom Maximalgrad höchstens 3 besitzen, einfach übertragen. Folglich gibt es einen Baum vom Maximalgrad höchstens 3, der X enthält. Da wir keinen 4-fach zusammenhängenden topologischen Minor gewurzelt auf X garantieren können, wird mehr Aufwand nötig sein, um Kreise durch alle Knoten aus X eines planaren Graphen zu erhalten. Um dieses Problem zu lösen, wird die Theorie der Tutte-Wege angepasst.

Im Jahr 1956 konnte William Tutte das lange offene Problem, ob 4-fach zusammenhängende planare Graphen hamiltonsch sind, lösen und somit ein Theorem von Whitney erweitern, welches besagt, dass 4-fach zusammenhängende Triangulationen hamiltonsch sind. Die zugrunde liegende Idee ist, zu zeigen, dass planare Graphen einen Kreis enthalten, sodass jede Komponente außerhalb des Kreises eine beschränkte Anzahl an Berührungspunkten an dem Kreis besitzt. Im Konkreten ist gezeigt worden, dass die Zahl der gemeinsamen Berührungsknoten durch 3 beschränkt ist, woraus sofort Tuttes Ergebnis folgt.

Durch eine Konstruktion von Moon und Moser ist bekannt, dass es eine positive Konstante c und 3-fach zusammenhängende planare Graphen gibt, sodass die Länge eines längsten Kreises (der Umfang) eines Graphen G durch $c \cdot |V(G)|^{\log_3 2}$ *beschrieben wird. Einige Jahre später konnte der Term durch Chen und Yu als untere Schranke bestätigt werden.* In dieser Arbeit wird eine leichte Abschwächung des 4-Zusammenhangs planarer Graphen untersucht. Wenn der 4-Zusammenhang durch wesentlichen 4-Zusammenhang ersetzt wird, das heißt, dass die betrachteten Graphen Separatoren der Größe 3 besitzen dürfen, diese aber nur einzelne Knoten vom restlichen Graphen trennen, so ist durch Jackson und Wormald zuerst gezeigt worden, dass der kurzmöglichste Umfang in dieser Graphenklasse durch eine lineare Funktion in der Anzahl an Knoten beschrieben wird. Diese Arbeit stellt eine Verbesserung des zuletzt bekannten Koeffizienten der linearen Schranke vor, welcher von Fabrici, Harant und Jendroľ veröffentlicht wurde. Im Falle von wesentlich 4-fach zusammenhängenden Triangulationen auf n Knoten wird die bestmögliche Schranke von $\frac{2}{3}(n+4)$ bewiesen.

Abschließend wird in der Dissertation die Klasse der 1-planaren Graphen untersucht und ihre längsten Kreise studiert. *Ein Graph ist 1-planar, wenn er eine Einbettung in die Ebene besitzt,*

sodass jede Kante höchstens einmal gekreuzt wird. Diese Klasse unterscheidet sich in vielerlei Hinsicht zu den planaren Graphen. Das Entscheidungsproblem, ob ein Graph 1-planar ist, ist $\mathcal{NP}$-vollständig und es gibt verschiedene Ansätze, einen 1-planaren Graphen als maximal anzusehen. In der vorliegenden Arbeit wird ein Analogon zu Whitneys Theorem bewiesen. Es wird gezeigt, dass 3-fach zusammenhängende maximal 1-planare Graphen einen sublinearen Umfang wie ihre planaren Verwandten haben können, und dass 4-fach zusammenhängende maximal 1-planare Graphen hamiltonsch sind. Im nicht-maximalen Fall kann Tuttes Theorem nicht direkt übertragen werden. Dies zeigt die Konstruktion einer Familie von 5-fach zusammenhängenden 1-planaren Graphen in dieser Arbeit, welche bei weitem nicht hamiltonsch sind. Ob 6-fach zusammenhängende 1-planare Graphen hamiltonsch sind, bleibt offen. Einige Ideen, wie eine Herangehensweise an diese Frage durch eine Erweiterung der Tutte Theorie aussehen könnte, werden abschließend kurz angerissen.

Acknowledgement

First and foremost, I would like to thank Matthias Kriesell for his incredible support and inexhaustible guidance during my doctoral time in Ilmenau. Long discussions at the black board, valuable suggestions of promising research questions about minors and his patience made this thesis a great experience. It was also a great pleasure of mine to be provided with a research position funded by the German Research Foundation (DFG).

I would like to express my gratitude to Jochen Harant for his enthusiastic assistance throughout my entire time in Ilmenau: He introduced me to Graph Theory and had a very important role in my development as a researcher. I have always appreciated his encouragement along this journey, his shared wisdom on questions far beyond mathematics, and eventually his help in avoiding typical pitfalls.

My deepest thanks to my colleagues Thomas Böhme, Thomas Hotz, Olaf Parczyk, Yury Person, Thomas Schweser, Michael Stiebitz, and Carsten Trunk who supported my doctorate in numerous manners.

Further thanks to my coauthors and research collaborators Igor Fabrici, Tomáš Madaras, Jens Schmidt, Roman Soták, and Carol Zamfirescu for spending time together at conferences and workshops and for many emails concerning drafts and revisions. I enjoyed our research a lot and hope for further good collaborations.

I would especially like to show appreciation to Jonis Alkewitz, Jacob Cooper, Thomas Schweser, and Teresa Tichai for their genuine interest in this thesis and helpful suggestions regarding its final draft.

Thanks to the Ilmenau University of Technology, all lecturers and students, I look back to many enjoyable years in Ilmenau with pleasure.

Finally, I would like to extend my deepest gratitude to all my family members and dear friends who have been constant support during these years. One day I hope to give back as much as you have given to me.

Table of Contents

Abstracts i

1 Introduction 1

1.1 Historical Context 1

1.1.1 Tutte Paths 2

1.1.2 Hadwiger's Conjecture 3

1.2 Structure of the Dissertation 4

1.3 Some Preliminaries 5

2 Extended Overview of the Project and Its Results 9

2.1 Hadwiger's Conjecture 9

2.1.1 Proving the Special Cases 9

2.1.2 Density Arguments 11

2.1.3 Random Graphs 13

2.2 Graphs with Few Colourings 14

2.2.1 Uniquely Colourable Graphs 15

2.2.2 Kempe Colourings 18

2.2.3 Other Concepts of Graphs with Few Colourings 21

2.3 Rooted Minors 22

2.3.1 Kriesell's Conjecture 24

2.3.2 Open Questions on Rooted Minors 26

2.4 Half-Rooted Minors 26

2.5 Tutte Cycles 30

2.5.1 Essentially 4-connected Planar Graphs 32

2.5.2 Spectrum 34

2.6 Tutte Cycles in Non-planar Graphs 36

2.6.1 1-planar Graphs 37

2.6.2 An Approach to a New Theory 39

3 A Construction of Uniquely Colourable Graphs with Equal Colour Class Sizes **43**

4 Rooted Complete Minors in Line Graphs With a Kempe Coloring **49**

5 Kempe Chains and Rooted Minors **56**

5.1 Introduction 57

5.2 Kempe Chains and Rooted K_7-minors 58

5.3 Unicyclically Arranged Kempe Chains 60

5.4 The Graphs $Z(G)$ for $|V(G)| \leq 6$ 62

5.5 Connected Transversals of 5-colorings 64

5.6 Concluding Remarks 66

6 Rooted Minors and Locally Spanning Subgraphs **68**

6.1 Introduction and Main Result 69

6.2 Proof of the Main Result 72

6.3 Locally Spanning Subgraphs 77

7 Longer Cycles in Essentially 4-Connected Planar Graphs **84**

8 Circumference of Essentially 4-connected Planar Triangulations **92**

9 Long Cycles and Spanning Subgraphs of Locally Maximal 1-planar Graphs **103**

9.1 Introduction and Results 104

9.2 Proofs of the Theorems 107

9.3 Overview 115

10 Outlook **116**

Bibliography **119**

Primary Sources 119

Publications and Preprints 124

Index **125**

1 Introduction

"One of the deepest unsolved problems in graph theory is the following conjecture due to Hadwiger [...]." These words were chosen by Béla Bollobás, Paul Allen Catlin, and Paul Erdős in 1980 to begin their paper on Hadwiger's Conjecture in random graphs [BCE80]. The object of interest is a conjecture postulated by Hugo Hadwiger in 1943 [Had43]. It states that for all $k \in \mathbb{N}$, a graph can be coloured with $k-1$ colours or contains a clique minor of size k. Before diving into the details, allow me to take you briefly back in history to the early days of Graph Theory.

1.1 Historical Context [Wil13]

It all began in the early 18$^{\text{th}}$ century in Königsberg, East Prussia, where a challenge arose to find a walk crossing each of the seven bridges over the Pregel river exactly once and then ending back at its starting point. Leonhard Euler finally proved in 1741 that this problem has no solution [Eul41]. In these early days, the challenge was to develop the right tools to attack and solve these kinds of problems.

Probably the most famous problem in graph theory is the *Four-Colour-Conjecture*. It was in 1852, when Francis Guthrie first mentioned it during his task to colour the map of the counties of England. He noticed that it could be done with four different colours in all considered cases. The conjecture states that every map, i.e. graph that can be drawn onto the Euclidean plane without crossing edges, can be coloured properly with four colours. It was obvious that it could not be done with three colours in general, but there had been no example where five colours were necessary.

The first known written reference of this conjecture is a letter from Augustus De Morgan, a professor of mathematics from London, to his colleague William Rowan Hamilton. He explained that the problem had been devised by F. Guthrie (one of the brothers Francis or Frederick), a student of his at the time, and lamented that he was presently unable to resolve it.

In 1878, Arthur Cayley presented this conjecture to the London Mathematical Society and made it well-known [Cay79]. He noticed that it was enough to consider cubic maps. Almost every famous mathematician from those times worked on this problem, which led to a series of failed attempts.

The first incorrect proof of this conjecture, due to Alfred Bray Kempe in 1879, was based on an innovative idea [Kem79]: It considers so-called unavoidable sets, i.e. sets of configurations such that each minimal counterexample to the conjecture, should it exist, must necessarily contain at least one of its elements. If it is possible to find a solution to reduce each of those unavoidable configurations, then this will lead to a smaller counterexample, contradicting the assumption. Nonetheless, Kempe's proof was falsified by Percy J. Heawood showing a mistake in one of the reductions [Hea98]. At the very least, he managed to modify Kempe's proof, thereby obtaining the Five-Colour-Theorem. In 1880, Peter Guthrie Tait associated this conjecture with the Hamiltonicity of cubic 3-connected planar graphs. His attempt to prove the Four-Colour-Conjecture [Tai80] was refuted by Julius Petersen [Pet98] some years later and was finally disproved by William T. Tutte [Tut46] in 1946.

Several decades later, in the 60s and 70s of the last century, Heinrich Heesch continued the study of unavoidable sets [Hee69]. He developed computer aided methods for a proof of the reducibility of all configurations. But the lack of accessible computing power at the time stopped him succeeding in his work. Based on his ideas, Kenneth I. Appel and Wolfgang Haken [AH89] from the University of Illinois eventually gave the first proof in 1976 by reducing a set of exactly 1973 unavoidable configurations, and converted the conjecture into what we know today as Four-Colour-Theorem. In 1996, further simplifications were proved by Neil Robertson, Daniel P. Sanders, Paul D. Seymour, and Robin Thomas [Rob+96] and the number of configurations was lowered to 633.

1.1.1 Tutte Paths

Up to the present day, there is the unpleasant flavour of the computer-aided proofs of the Four-Colour-Theorem. It obfuscates the real reason as to why the Four-Colour-Theorem holds and does not reveal the main attribute of planar graphs leading to the 4-colourability.

On the other hand, let us consider the idea behind Peter G. Tait's approach [Tai80] from 1880 to prove the Four-Colour-Theorem, which is suggestive of providing the reason that four colours suffice. To this end, take a planar graph G and a plane embedding. We can assume that this embedding is a triangulation of the plane, otherwise insert edges until it is triangulated. The dual graph of G, i.e. the graph with the face set of G as the vertex set and its adjacency relation, is a 3-connected, cubic, planar graph. Assume that — and conjectured by Tait — this graph has a Hamiltonian cycle, namely a cycle containing all vertices of the dual graph exactly once. This cycle separates the plane into an interior and an exterior and, therefore, separates the vertex set of G into two sets. The subgraph of G induced by each of the two sets is a tree and consequently 2-colourable. By merging both colourings, we obtain a 4-colouring of the graph G. The key observation is that planar graphs have planar dual graphs and that cycles in a plane separate the plane.

Tait's conjecture, claiming the situation above, turned out to be false. In 1946, William T. Tutte presented a cubic 3-connected planar graph on 45 vertices without a Hamiltonian cycle [Tut46]. Subsequently, the question arose which planar graphs are Hamiltonian, since “little was known about conditions for the existence of a Hamiltonian cycle in a planar graph” until 1956, as Tutte emphasises in his paper [Tut56]. In that year, Tutte proved that 4-connected planar graphs are HAMILTONian with an inventive idea [Tut56]. He also was facing the problem that it was unpromising to perform induction on the vertex set of 4-connected planar graphs, since removing vertices may decrease connectivity. The idea to overcome these issues were TUTTE paths. We define TUTTE paths later; for now it is sufficient to know that these are paths in graphs such that all components of the graph without that path have a very small number of attachments on the path. Since all planar graphs contain TUTTE paths, it is concluded that 4-connected planar graphs are HAMILTONian.

His ideas had a big impact on the following research: they were adopted in new elegant proofs of slightly stronger results [San97; Tho83], were generalised to obtain results for graphs embedded on higher surfaces [TY94; TYZ05], and were used to prove results related to HAMILTONicity [FHJ16; JW90]. Recently, it was shown that TUTTE paths can be computed in quadratic time [SS18], thus, implying some algorithmic aspects to all results obtained from TUTTE cycles. The theory of TUTTE paths and cycles will be the subject of Sections 2.4 to 2.6 and Chapters 6 to 9.

1.1.2 Hadwiger's Conjecture

Another attempt to understand the Four-Colour-Theorem was to "pare down its hypotheses to a minimum core, in the hope of hitting the essentials; throw away planarity and impose some weaker condition", as described by PAUL D. SEYMOUR in his survey on HADWIGER's Conjecture [Sey16].

It is well-known that planar graphs are precisely characterised by KLAUS WAGNER [Wag37] as those graphs containing neither a K_5- nor $K_{3,3}$-minor. Thus, the Four-Colour-Theorem states that all graphs without a K_5- and $K_{3,3}$-minor are 4-colourable. It is self-explanatory to exclude K_5 as a minor since K_5 itself is not a 4-colourable graph, but why is $K_{3,3}$ also excluded?

In 1943, HUGO HADWIGER tried to answer this question [Had43]. Since he was not able to prove the Four-Colour-Conjecture, he postulated his famous conjecture:

Conjecture 1 (HADWIGER, 1943 [Had43])**.** *Each graph without a clique minor of size $k \in \mathbb{N}$ can be coloured with $k-1$ colours.* ⋄

H. HADWIGER was able to prove his conjecture for values $k = 1, 2, 3, 4$. The case $k = 5$ would imply the Four-Colour-Conjecture, however, for $k > 4$, he noticed:

> *„Es scheint auch hier eine eigentümliche Regellosigkeit wirksam zu werden, welche eine gesetzmässige Erfassung der kombinatorisch-topologischen Möglichkeiten stark behindert."*
>
> HUGO HADWIGER, 1943 [Had43]

"A peculiar irregularity seems to occur, which strongly impedes the gathering of the combinatorial-topological possibilities into rules." This shows that H. HADWIGER already suspected that his conjecture would be very tough. And indeed, no proof of the case $k = 5$ exists to this present day. Though his approach to prove the Four-Colour-Conjecture has not been successful, Hadwiger's Conjecture remains a tantalising conjecture for graph theorists and offers a far-reaching generalisation of the Four-Colour-Theorem.

By an old result of Klaus Wagner [Wag37], it is elementary to show the equivalence of the Four-Colour-Conjecture and the case $k=5$ of Hadwiger's Conjecture. The only proof towards Hadwiger's Conjecture by Neil Robertson, Paul D. Seymour, and Robin Thomas in 1993 [RST93] shows that every minimal counterexample to Hadwiger's Conjecture for the case $k=6$ consists of a planar graph with one additional apex vertex; therefore, such graphs are 5-colourable assuming the Four-Colour-Theorem holds. Consequently, the case $k=6$ is also merely equivalent to the Four-Colour-Theorem.

This lack of results is another indicator that Hadwiger's Conjecture is tenacious. Furthermore, many strengthenings of this conjecture turned out to fail. For example, György Hajós conjectured that every graph with chromatic number k contains a subdivision of a complete graph on k vertices, which was disproved by Paul A. Catlin [Cat79]. Moreover, Hadwiger's Conjecture fails for infinite graphs [Zyp12] and a list colouring version does not hold as well [BJW11; Voi93]. Hadwiger's Conjecture will be the main topic of Sections 2.1 to 2.4 and Chapters 4 and 5.

1.2 Structure of the Dissertation

Based on the research project "Complete Minors in Graphs with Few Colourings" funded by the German Research Foundation (DFG), the goal of this thesis is a contribution to the collection of results on Hadwiger's Conjecture. Focusing on graphs with few colourings obliges one to study the structure of graphs. The main structural concept used in this thesis is that of rooted minors. This concept will be refreshed from literature and expounded upon in such a way that demonstrates its strength. In connection with rooted minors, this thesis will also focus heavily on two other important structural concepts in graph theory: Tutte paths and colourings of graphs.

This thesis is organised as follows. Chapter 2 starts with an overview of partial results on Hadwiger's Conjecture. Then, each new result of this thesis is motivated by an individual opening question and relevant, state-of-the-art results are discussed. We distill each problem to its 'core', and straightforward conclusions are made. Moreover, interesting initial approaches are stated and small results are mentioned that are not fully developed to be published in journal papers. Open problems complete these investigations.

Chapters 4 to 9 contain new results reported in the style of journal publications. Within these chapters, all stated results are developed in close collaboration with the co-authors listed at the beginning of each chapter. To conclude this thesis, Chapter 10 restates remaining open problems and questions tied to this thesis and motivates possible future research.

1.3 Some Preliminaries

We give a brief summary of basic definitions in graph theory used in this thesis. For a complete overview, we refer to [Die17] and [BM08]. Definitions used in a more specific context will be introduced when their need has come.

By $\mathbb{N}$ we denote the set of *natural numbers excluding zero*. Given an arbitrary set A, the *power set* $\mathfrak{P}(A)$ is the set of all subsets of A. The *size* or *order* of A is the number of elements in A and denoted by $|A|$. A subset $B \in \mathfrak{P}(A)$ is called a *k-subset* of A if $|B| = k$ for an $k \in \mathbb{N}$. The set of all k-subsets of A is denoted by $\mathfrak{P}_k(A)$. A *partition* of A is a set $\mathcal{A} = \{A_i : i \in \{1, \ldots, k\}\} \subseteq \mathfrak{P}(A)$ of pairwise disjoint, non-empty subsets of A such that its union $\bigcup_{i \in \{1,\ldots,k\}} A_i$ is A.

Since we will define an *inclusion relation* $\subseteq$ on graphs, it is justified to say, that a *maximal (with respect to a certain property)* object K is understood to be *inclusion-maximal*, i.e. K has this property and each object P such that $K \subseteq P, K \neq P$ does not have this property.

Graphs

All graphs are simple, undirected, and finite if not explicitly declared otherwise. Thus, a *graph* is a pair $G = (V(G), E(G))$ on a finite set $V(G)$, called the *vertex set*, and the *edge set* $E(G) \subseteq \mathfrak{P}_2(V(G))$. For an edge $e := \{x, y\} \in E(G)$, we usually write $e = xy$. A *subgraph* of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; we write $H \subseteq G$. A subgraph H is *induced* if $E(H) = \mathfrak{P}_2(V(H)) \cap E(G)$. To avoid unnecessary ambiguities, we always assume that $V(G) \cap E(G) = \emptyset$.

Let G be a graph and $x \in V(G)$. The *neighbourhood* $N_G(x)$ is defined by $N_G(x) := \{y \in V(G) : xy \in E(G)\}$. An edge $e \in E(G)$ is *incident* to x if $x \in e$. The *degree* $d_G(x)$ is the number of edges incident with x, and we call x *adjacent* to y if $xy \in E(G)$.

Let G be a graph, $X \subseteq V(G)$, and $E \subseteq E(G)$. We denote by $G[X]$, $G - X$, and $G - E$ the subgraph of G induced by X, induced by $V(G) \setminus X$, and the graph obtained from G by removing all edges from E, respectively. Abusing notation, we write $G - x$ and $G - e$ instead of $G - \{x\}$ and $G - \{e\}$ for $x \in V(G), e \in E(G)$, respectively.

The *complement* $\overline{G}$ of G is the graph on the same vertex set and $E(\overline{G}) := \mathfrak{P}_2(V(G)) \setminus E(G)$.

Let H_1 and H_2 be two graphs. We say that H_1 and H_2 are *isomorphic* if there exists a bijection $\varphi : V(H_1) \to V(H_2)$ such that $xy \in E(H_1)$ if and only if $\varphi(x)\varphi(y) \in E(H_2)$. The *union* $H_1 \cup H_2$ is the graph G with $V(G) := V(H_1) \cup V(H_2)$ and $E(G) := E(H_1) \cup E(H_2)$. The *join* $H_1 + H_2$ is the graph G with $V(G) := V(H_1) \cup V(H_2)$ and $E(G) := E(H_1) \cup E(H_2) \cup \{xy : x \in V(H_1), y \in V(H_2), x \neq y\}$.

Vertex sets

Let X be a finite set. The graph G with $V(G) = X$ and $E(G) = \mathfrak{P}_2(X)$ is the *complete graph* on the vertex set X and denoted by K_X. By K_k with $k \in \mathbb{N}$, we denote a complete graph on an unspecified k-element vertex set.

A *path* P of length $k \in \mathbb{N} \cup \{0\}$ is a graph on a vertex set $\{v_0, v_1, \ldots, v_k\}$ with $E(P) = \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\}$. We say that P starts in v_0 and ends in v_k and call it a v_0,v_k-path. A *cycle* can be defined in the same vein by identifying v_0 and v_k. For a cycle C, an edge $e \in \mathfrak{P}_2(V(C)) \setminus E(C)$ is a *chord* of C.

Let G be a graph and $X \subseteq V(G)$. The set X is an *anticlique* (also *independent set*) if all vertices of X are pairwise non-adjacent in G, i.e. $G[X]$ is an edgeless graph. A *clique* is a set $X \subseteq V(G)$ such that $G[X]$ is a complete graph. With the *independence number* α and the *clique number* $\omega(G)$ we denote the number of vertices in a largest independent set and largest clique, respectively.

A graph G is called *apex* if it contains a vertex $x \in V(G)$ adjacent to all vertices in $V(G)\setminus\{x\}$. Such a vertex x is an *apex vertex*.

Let $\mathcal{A} \subseteq \mathfrak{P}(V(G))$ be a set of vertex subsets and $T \subseteq V(G)$. We say that T is a *transversal* of $\mathcal{A}$ if $|A \cap T| = 1$ for all $A \in \mathcal{A}$. In other words, T *traverses* $\mathcal{A}$.

Colouring

A *k-colouring* of a graph G with $k \in \mathbb{N}$ is a partition $\mathcal{C}$ of the vertex set $V(G)$ into $k' \leq k$ non-empty sets $A_1, \ldots, A_{k'}$. The colouring $\mathcal{C}$ is called *proper* if each set is an anticlique of G that is there are no two adjacent vertices of G in the same *colour class* $A \in \mathcal{C}$. If a colouring is mentioned, we actually mean a proper colouring in this entire thesis. The *chromatic number* $\chi(G)$ is the minimum k such that there is a proper k-colouring of G. A colouring $\mathcal{C}$ of G is *optimal* if $\mathcal{C}$ is a proper k-colouring with $k = \chi(G)$.

A graph G is *k-critical* for $k \in \mathbb{N}$ if $\chi(G) = k$ and $\chi(H) < k$ for each proper subgraph H of G.

A graph G is called *perfect* if $\chi(H) = \omega(H)$ for all induced subgraphs H of G. This means that for all subgraphs H of G the necessary number of colours to colour its largest clique is sufficient to colour the whole graph. The well-known Weak Perfect Graph Theorem proved by L. Lovász states that a graph is perfect if and only if its complement is perfect [Lov72]. One property of perfect graphs is that for these graphs Hadwiger's Conjecture holds.

Connectedness

A *component of* G is a maximal connected subgraph, i.e. between each pair x, y of vertices there exists a path from x to y in G. A set $S \subseteq V(G)$ is a *separator* if $G-S$ has more components than G. If S is a minimal separator of a connected graph G and $G-S$ separates G in components $H_1, H_2, \ldots$, then $\{G[V(H_1) \cup S], G[V(H_2) \cup S], \ldots\}$ is a *separation of* G. If $A, B \subseteq V(G)$ and $S \subseteq V(G)$ such that there is no path from A to B in $G-S$, then S *separates* A and B. In this case, G is *disconnected*.

If a graph G on at least $k+1$ vertices is connected and has no separator S of size smaller than k for a $k \in \mathbb{N}$, then we say that G is *k-connected*. A well-known result by K. Menger is the following:

Theorem 1.1 (Menger, 1927 [Men27]). *Let G be a graph and $A, B \subseteq V(G)$, then the minimum size of a set S separating A and B is equal to the maximum number of internally disjoint paths from A to B.* ⋄

A graph G is *k-edge-connected* with $k \in \mathbb{N}$ if there is no set $E \subseteq E(G)$ with $|E| < k$ such that the graph $G' := G-E$ (obtained from G by removing all edges E) is disconnected. If we allow G' to be disconnected but at most one component contains cycles, then G is *cyclically k-edge-connected*.

Let H_1 and H_2 be two graphs on different vertex sets and C_1 and C_2 be cliques in H_1 and H_2 of equal size, respectively. The graph G obtained from H_1 and H_2 by identifying both graphs at C_1 and C_2 is a *clique-sum* of H_1 and H_2 at C_1 and C_2. Let $k := |V(C_1)|$, then G is a *k-clique-sum*.

Embeddings

All graphs can be represented by *drawings* in the Euclidean plane, such that vertices are distinct points and edges are arcs, i.e. non-self-intersecting continuous curves. A graph G is *planar* if there exists a drawing of G such that two arcs only meet at end vertices.

Without further explicit reference we use the *Jordan curve theorem*, which states that each closed polygonal curve splits the Euclidean plane in an exterior and an interior. An easy corollary asserts that every planar 2-connected graph partitions the Euclidean plane into arcwise connected regions called *faces* and each face has a cycle, called *facial cycle*, as a boundary. There is exactly one unbounded face, which we call the *outer face*.

A planar graph G is *maximal planar* or a *triangulation* if all faces of G are a triangle, i.e. each facial cycles consists of three edges.

For a connected planar graph G with n vertices, m edges, and f faces, *Euler's formula*

$$2 = n - m + f$$

holds.

The *planar dual* of a planar graph G with face set F is a graph H on the vertex set $V(H) := F$ and $e = \alpha\beta \in E(H)$ if and only if the faces α and β of G are incident to a common edge in G.

It is also possible to embed graphs on *higher surfaces*, such as the projective plane, and define a *genus* of a graph. For more details, we refer to contemporary text books of graph theory.

Hamiltonicity

Let G be a graph. A cycle C of G is a *Hamiltonian cycle* if C contains all vertices of G. A graph containing a Hamiltonian cycle is a *Hamiltonian graph*. If for each two distinct vertices $x, y \in V(G)$, there is a x,y-path in G containing all vertices of G, then we say that G is *Hamilton-connected*.

Minors

Let G and M be graphs and $(V_t)_{t\in V(M)}$ be a family of pairwise disjoint, non-empty subsets of $V(G)$ such that $G[V_t]$ is connected for all $t \in V(M)$. We say that G contains an *M-certificate* $c=(V_t)_{t\in V(M)}$ if and only if there is an edge $xy \in E(G)$ connecting $x \in V_t$ and $y \in V_s$ for all $st \in E(M)$. The set V_t, $t \in V(M)$ is the *branch set* or *bag of t* in c. If G contains an M-certificate, then M is a *minor* of G and we write $M \prec G$.

It is also common to define a minor of a graph G as a sequence of edge and vertex deletions and contractions of edges in G. It should be mentioned here that a graph M' isomorphic to the outcome of such a sequence is a minor of G if and only if there is an M'-certificate in G. Deleting all vertices not covered by branch sets and contracting each branch set to a single vertex results in a graph isomorphic to the minor.

The graph S is a *subdivision* of M if there is an injection $\beta : V(M) \to V(S)$ and for each edges $e = xy \in E(M)$ we choose a path P_e in S such that $V(S) = \bigcup_{e\in E(M)} V(P_e)$, $V(P_e) \cap \{\beta(v) : v \in V(M)\} = \{\beta(x), \beta(y)\}$ for $e \in E(M)$, and $V(P_e) \cap V(P_{e'}) = \beta(e \cap e')$ for $e \neq e'$. If G has a subgraph isomorphic to a subdivision of M, then M is a *topological minor* of G.

Linkage

Let G be a graph and $\{x_1, y_1, \dots, x_k, y_k\} \subseteq V(G)$ a set of $2k$ vertices of G for a $k \in \mathbb{N}$. An *$(x_1y_1, \dots, x_ky_k)$-linkage* is a system of k vertex-disjoint paths $P_1, \dots, P_k$ such that P_i is a x_i,y_i-path.

Extended Overview of the Project and Its Results | 2

Within this chapter, a thematic classification of the results developed in this thesis is provided and initial approaches to pursuing questions are elaborated upon. We start with an overview of HADWIGER's Conjecture in Section 2.1. One component of this conjecture is colourings; thus, different approaches for characterising graphs with few colourings are presented in Section 2.2. Rooted minors as the main tool of this thesis are introduced in Section 2.3, in which we further depict the results of two publications (Chapters 4 and 5). Motivated by "local connectivities", we refine the definitions of rooted minors to consider "half-rooted minors" and study the connectedness of a vertex subset in Section 2.4 (and Chapter 6). The essential tool of the last two Sections 2.5 and 2.6 are Tutte cycles. New applications and their capabilities are investigated; those are in relation to the publications of Chapters 7 to 9.

2.1 Hadwiger's Conjecture

This brief overview of HADWIGER's Conjecture does not raise a claim to be complete. It is rather focused on topics concerning this thesis. For a more detailed survey, we refer to SEYMOUR's [Sey16] and TOFT's excellent surveys [Tof96].

An equivalent formulation of HADWIGER's Conjecture (Conjecture 1) is

$$K_t \not\preceq G \Longrightarrow \chi(G) \leq t-1 \tag{H(t)}$$

for all $t \in \mathbb{N}$, which is more convenient to verify in the special cases with $t \leq 6$.

2.1.1 Proving the Special Cases

Proof of H(t) for $t \in \{1, 2, 3, 4\}$. Let G be a graph. If $t = 1$ and $K_t \not\preceq G$, then G contains no vertices, hence, G is the empty graph and 0-colourable. If $t = 2$ and $K_t \not\preceq G$, then G contains no edges, hence, G is an edgeless graph and we can colour all vertices with one colour. If $t = 3$ and $K_t \not\preceq G$, then G contains no cycles, hence, G is a forest and consequently 2-colourable.

Therefore, let $t = 4$ and $K_t \not\preceq G$. We follow the idea of H. HADWIGER in his paper about the conjecture [Had43]. We proceed by induction on the number of vertices and the proof consists of two assertions:

(i) Every minimal graph G with $K_3 \prec G$ is 3-colourable.

(ii) Let G be a K_4-minor-free graph and H an induced subgraph of G such that H is minimal with $K_3 \prec H$. Then G contains a 3-colouring extending a 3-colouring of H.

A minimal graph G with $K_3 \prec G$ is connected and all vertices have degree 2. Thus, G is a cycle and is 3-colourable.

Let G be a graph on n vertices and assume that the assertions hold for all graphs with less than n vertices. If G is a cycle, then $H = G$ and we are done. Otherwise, G contains an induced cycle C, which we assume to be coloured. For each subgraph of H, the vertices of H incident in G with some edge not belonging to H are the *vertices of attachment* of H in G. Let H be a minimal subgraph of G such that each vertex of attachment of H in G is a vertex of C and H is not a proper subgraph of C.

We observe that H is not an edge (not isomorphic to K_2) since C has no chord in G. Thus $V(H) \setminus V(C) \neq \emptyset$. Furthermore, $|V(C) \cap V(H)| \leq 2$ since otherwise there is a K_4-certificate $(V_i)_{i \in \{1,2,3,4\}}$ in G with $V_1 \cup V_2 \cup V_3 = V(C)$, each $V_i, i \in \{1,2,3\}$ contains one vertex of attachment of H, and $V_4 = V(H) \setminus V(C)$.

If $|V(C) \cap V(H)| = 1$, we can colour $G - (V(H) \setminus V(C))$ and H with three colours such that the colouring of the former graph extends a colouring of C and the colouring of the latter graph coincides the first colouring in $V(C) \cap V(H)$.

Consequently, $\{b_1, b_2\} := V(C) \cap V(H)$ and there is a separation of C in C_1 and C_2 such that $V(C_1) \cap V(C_2) = \{b_1, b_2\}$. Assume that $|V(C_2)| \geq 3$, otherwise change the roles of C_1 and C_2. Then $H \cup C_1$ has less than n vertices and contains a cycle D with $C_1 \subseteq D$. We can colour D such that it extends the colouring of C_1 and by the induction hypothesis, there is a 3-colouring of $H \cup C_1$. Moreover, there is a colouring of $G - (V(H) \setminus V(C))$ with three colours extending the colouring of C. We can merge both colourings, which completes the proof of $H(4)$. ■

For $t \in \{5, 6\}$, there is no known proof of H(t) without assuming the validity of the Four-Colour-Theorem. We prove the equivalence between both cases and the Four-Colour-Theorem.

Proof of the Equivalence of H(t) and the Four-Colour-Theorem for $t \in \{5, 6\}$.
Assume that $H(6)$ holds and let G be a graph without a K_5-minor. Let G' be the graph obtained from G by adding a new apex vertex v', i.e. a new vertex adjacent to all other vertices of G. It is straightforward to check that G' is K_6-minor-free, hence, there is a colouring $\mathcal{C}$ of G' with at most five colours. Since $N_{G'}(v') = V(G') \setminus \{v'\}$, there is a colour class $A \in \mathcal{C}$ such that $A = \{v'\}$. Thus, $\mathcal{C} \setminus A$ is a k-colouring of G with $k \leq 4$, and $H(5)$ holds.

If G is a planar graph, then by Wagner's result [Wag37] $K_5 \not\prec G$ and G is 4-colourable by $H(5)$. This proves that $H(t), t \in \{5, 6\}$ implies the Four-Colour-Theorem.

Assume that the Four-Colour-Theorem holds: let G be a K_5-minor-free graph. We may assume that G is a maximal graph with $K_5 \not\prec G$, otherwise repeatedly add edges. In [Wag60], Klaus Wagner proved the equivalence between $H(5)$ and the Four-Colour-Theorem by using his theorem in [Wag37]. It says that G is obtained from 2- and 3-clique-sums of maximal planar graphs and the 8-vertex Wagner graph, i.e. the graph obtained from C_8 by inserting edges for each pair $x, y \in V(C_8)$ of distance 4 in C_8. Since all these graphs are 4-colourable, we can obtain a colouring of G by merging all the colourings and identifying colour classes along the cliques.

Suppose that $H(6)$ fails and let G be a minimal K_6-minor-free counterexample to Hadwiger's Conjecture. By a result of N. Robertson, P. Seymour, and R. Thomas [RST93], G is a graph consisting of a planar graph and an additional apex vertex v. By the Four-Colour-Theorem, $G - v$ is 4-colourable; hence, we obtain a 5-colouring of G, contradicting the assumption. ■

We can summarise:

Theorem 2.1. *Hadwiger's Conjecture H(t) holds for $t \in \{1, 2, \dots, 6\}$.* ⋄

For $t \geq 7$, Hadwiger's Conjecture is widely open. For $t \in \{7, 8\}$, we have the following partial results:

Theorem 2.2 (Albar, Gonçalves [AG18]; Kawarabayashi, Toft [KT05]).

(i) *Every K_7-minor-free graph is 8-colourable.*

(ii) *Every K_8-minor-free graph is 10-colourable.*

(iii) *Every graph G with $\chi(G) = 7$ contains a K_7-minor or a $K_{4,4}$-minor.*

(iv) *Every graph G with $\chi(G) = 7$ contains a K_7-minor or a $K_{3,5}$-minor.* ⋄

If we restrict to subclasses of graphs, it might be easier to obtain some affirmative results. For example, Theorem 2.3 below verifies Hadwiger's Conjecture for line graphs. A superclass of line graphs are claw-free graphs, namely graphs without $K_{1,3}$ as an induced subgraph. For claw-free graphs, we have nearly the full picture.

Theorem 2.3 (Reed, Seymour [RS04]; Fradkin [Fra12]).

(i) *Hadwiger's Conjecture holds for line graphs.*

(ii) *Hadwiger's Conjecture holds for claw-free graphs with independence number at least 3.* ⋄

2.1.2 Density Arguments

Recall Hadwiger's Conjecture for the case $t = 3$. A graph without a K_3-minor is a graph without a cycle and, subsequently, a forest. Such a graph has a nice property: there is a vertex of degree 1 and each subgraph is a forest. Hence, we can colour a K_3-minor-free graph with two colours inductively by removing a vertex x of degree at most 1, colouring the remaining graph, and choosing a colour for x distinct from its neighbour's colour (if x has a neighbour).

This leads us to the concept of *degeneracy*:

Definition 2.4. A graph G is *k-degenerate* for $k \in \mathbb{N} \cup \{0\}$ if every non-empty subgraph has a vertex of degree at most k. ⋄

As demonstrated for the 1-degenerate forests, all k-degenerate graphs can easily be coloured with the same sequential colouring algorithm:

Proposition 2.5. *A k-degenerate graph is $(k+1)$-colourable.* ⋄

Using Proposition 2.5, we obtain another proof of H(t) for $t \in \{2,3,4\}$. A graph without K_t-minor, $t \in \{2,3,4\}$, is $(t-2)$-degenerate and, therefore, $(t-1)$-colourable. This proof of the case $t=4$ was given by Dirac [Dir52].

Bounding the average degree is a natural way to bound degeneracy. The *average degree* $\mathrm{av}(G)$ of a graph G is calculated by $2\,|E(G)|/|V(G)|$. If we were able to bound the average degree for K_k-minor-free graphs, we could prove some bound on the chromatic number of K_k-minor-free graphs. This idea was initially followed by W. Mader, bounding the average degree by the number of edges.

Theorem 2.6 (Mader [Mad68])**.** *For every integer $p \in \{2,3,\dots,7\}$, each K_p-minor-free graph on $n \geq p$ vertices has at most $(p-2)\,n - \binom{p-1}{2}$ edges.* ⋄

It follows immediately that a K_p-minor-free graph on $n \geq p$ vertices has $\mathrm{av}(G) \leq 2p-4$ and is $(2p-4)$-degenerate, hence, it is $(2p-3)$-colourable.

This bound is best-possible. For $p \in \{2,3,\dots,7\}$ take a complete graph K_{p-2} and join it to $n-(p-2)$ further vertices, i.e. $G = K_{p-2} + \overline{K_{n-(p-2)}}$. This graph attains the number of edges stated in Theorem 2.6 and is K_p-minor-free. If the assertion of Mader's theorem could be continued for larger p, then this would prove Hadwiger's Conjecture within a factor of 2.

However, the situation changes for $p \geq 8$. The *complete multipartite graph* $K_{2,2,2,2,2}$ is K_8-minor-free and has $(p-2)\,n - \binom{p-1}{2} + 1$ edges. Taking 5-clique-sums of copies of $K_{2,2,2,2,2}$ produces counterexamples exceeding $(p-2)\,n - \binom{p-1}{2} = 6\,n - 10$ by any constant. The silver lining is that there are no further counterexamples apart from those, as proved by Jørgensen in [Jør94].

The question arises whether it is possible to generalise Mader's theorem, maybe by characterising the counterexamples as done in the case $p=8$. The bad news is that for large p, a random graph on n vertices must have at least $\Omega(pn\sqrt{\log p})$ edges to contain a K_p-minor; this was shown by several people, e.g. [BCE80; Kos84].

Nevertheless, the next step to $p=9$ was taken by Song and Thomas [ST06]:

Theorem 2.7 (Song, Thomas [ST06])**.** *Every graph on $n \geq 9$ vertices and at least $7n-27$ edges either has a K_9-minor, is isomorphic to $K_{2,2,2,3,3}$, or is built of clique-sums of size 6 from copies of $K_{1,2,2,2,2,2}$.* ⋄

We observe that the number of examples not fitting into the scheme of Mader's theorem (Theorem 2.6) seems to be increasing. On the other hand, assuming $n \geq 13$, a 7-connected graph with at least $7n-27$ edges cannot be obtained from 6-clique-sums and, therefore, contains a K_9-minor by Theorem 2.7.

The continuation of this observation is conjectured by R. Thomas and P. Seymour:

Conjecture 2 (Thomas, Seymour – see e.g. [ST06]). *For every $p \geq 1$ there exists a constant $N = N(p)$ such that every $(p-2)$-connected graph on $n \geq N$ vertices and at least $(p-2)\,n - \binom{p-1}{2} + 1$ edges has a K_p-minor.* ⋄

Let us drop the condition on connectedness and consider reasonably large p. Which average degree forces a graph to contain a certain K_k-minor? By Theorem 2.7 and Conjecture 2 above, it does not seem to be promising to hope for a linear bound in k. Indeed, the answer was given independently by A. Kostochka and A. Thomason in the 80's:

Theorem 2.8 (Kostochka [Kos84], Thomason [Tho84]). *A graph of average degree r contains a K_k-minor with $k = \Theta(r/\sqrt{\log r})$.* ⋄

It was shown that this is best possible up to constants; the examples are random graphs with an appropriate probability for the edges. For the sake of completeness, let us consider these examples and how Hadwiger's Conjecture behaves with random graphs.

2.1.3 Random Graphs

The probably most-known model for random graphs is $\mathcal{G}(n,p)$. A definition can, for instance, be found in [BCE80].

Definition 2.9. For $n \in \mathbb{N}$, let $p(n) \in [0,1]$ be fixed. We say that $\mathcal{G}(n,p)$ is the discrete probability space consisting of all graphs with vertex set $\{1, 2, \dots, n\}$ and each pair of vertices is joined by an edge with probability p independently from all the other edges. ⋄

Bollobás, Catlin, and Erdős studied the size of a largest clique minor in $\mathcal{G}(n,p)$ with constant $p \in [0,1]$:

Theorem 2.10 (Bollobás, Catlin, Erdős [BCE80]). *For a graph G on n vertices, let $c(G)$ be the largest integer such that $K_{c(G)} \prec G$. The probability that for $G \in \mathcal{G}(n,p)$*

$$\frac{n}{\sqrt{\log_b n} + 4} \leq c(G) \leq \frac{n}{\sqrt{\log_b n} - 1}$$

with $b := 1 - 1/p$ holds tends to 1 with $n \to \infty$. ⋄

Theorem 2.10 verifies that Theorem 2.8 is best possible up to constants.

The first study on the chromatic number of $\mathcal{G}(n,p)$ was done by Grimmett and McDiarmid [GM75]. They bounded the independence number and clique number of $\mathcal{G}(n,p)$ and derived a bound on the chromatic number:

Theorem 2.11 (Grimmett, McDiarmid [GM75]). *For almost every graph $G \in \mathcal{G}(n,p)$ the chromatic number $\chi(G)$ is at least*

$$\frac{1}{2}\log(1-\tfrac{1}{p})\,\frac{n}{\log n}.$$ ⋄

Combining Theorems 2.10 and 2.11, we see that there exists $N \in \mathbb{N}$ such that for all graphs $G \in \mathcal{G}(n,p)$ with $n \geq N$, $c(G) \geq \chi(G)$ with high probability, which implies the assertion of Hadwiger's Conjecture for many graphs.

Theorem 2.10 is further developed: a strong result about K_k-minor in sparse random graphs was published in [FKO08].

Theorem 2.12 (Fountoulakis, Kühn, Osthus [FKO08]). *For a graph G, let $c(G)$ be the largest integer such that $K_{c(G)} \prec G$. Let $\varepsilon > 0$, then there exists a constant $C = C(\varepsilon)$ such that for all $p : \mathbb{N} \to [0,1]$ with $C/n < p < 1 - \varepsilon$, it holds asymptotically almost surely*

$$c(G) = (1 \pm \varepsilon)\frac{n}{\sqrt{\log_b(np)}}$$

with $b := 1 - 1/p$ and $G \in \mathcal{G}(n,p)$. ◇

This result about sparse graphs does not help to obtain results about Hadwiger's Conjecture. The considered graphs are too sparse to obtain reasonable results about the chromatic number and link them to Theorem 2.12. But the techniques of probability theory can be used to obtain results about large clique minors, for instance the following theorem. For a graph G, the *girth* is the length of a shortest cycle in G.

Theorem 2.13 (Kühn, Osthus [KO03]). *For all odd integers $g \geq 3$ there exists $c = c(g) > 0$ such that every graph of minimum degree r and girth at least g contains a K_t-minor for some $t \geq cr^{(g+1)/4}/\sqrt{\log r}$.* ◇

Theorem 2.13 implies the following corollary:

Corollary 2.14 (Kühn, Osthus [KO03]).

(i) *Hadwiger's Conjecture is true for C_4-free graphs of sufficiently large chromatic number.*

(ii) *Hadwiger's Conjecture is true for graphs of girth at least 19.* ◇

We finish our investigations on random graphs here and move on to graphs with few colourings.

2.2 Graphs with Few Colourings

In the short overview about Hadwiger's Conjecture, we have seen that all considerations meant to bound the number of colour classes or the size of the clique minors have been unrewarding in proving some of the cases H(t) for $t > 6$. The approach to uniformly bound the order of all colour classes, e.g. assuming $\alpha(G) = 2$ in Seymour's Conjecture (Conjecture 4), has also not been crowned with success. Instead, Matthias Kriesell suggested in [Kri17] to bound the *number of colourings*, in particular to consider uniquely optimally colourable graphs.

In this section, we focus on graphs with few colourings, investigate some of their simple properties, and consider minor-related conclusions supporting HADWIGER's Conjecture. We casually say — and this does not get to the heart of the matter from every point of view — that a graph has *few colourings* if it is impossible to obtain a second optimal colouring from a given one by recolouring only few vertices.

For instance assume that G is a k-critical graph and let $e = xy \in E(G)$. Then $G - e$ is $(k-1)$-colourable. Let $\mathcal{C}'$ be a $(k-1)$-colouring of $G - e$. It is easy to see that $\mathcal{C}_1 := \{A \setminus \{x\} : A \in \mathcal{C}'\} \cup \{\{x\}\}$ and $\mathcal{C}_2 := \{A \setminus \{y\} : A \in \mathcal{C}'\} \cup \{\{y\}\}$ are two k-colourings of G and that $\mathcal{C}_2$ can be derived from $\mathcal{C}_1$ by recolouring only two vertices, precisely x and y. The underlying idea of the research on graphs with few colourings is that more structure is expected to be present, which helps with finding a desired minor. This leads us to the first question, which will be considered in this section.

Question 1. How can we obtain large clique minors in graphs with few colourings? $\diamond$

2.2.1 Uniquely Colourable Graphs

We call a graph G *uniquely k-colourable* if $\chi(G) = k$ and for any two optimal colourings $\mathcal{C}$ and $\mathcal{C}'$ of G, we have $\mathcal{C} = \mathcal{C}'$. This means that there is only one optimal colouring of the graph. We start by proving some simple properties of uniquely colourable graphs.

Lemma 2.15. *Let G be a uniquely colourable graph with an optimal colouring $\mathcal{C}$. For any two distinct colour classes $A, B \in \mathcal{C}$, the subgraph of G induced by $A \cup B$ is connected.* $\diamond$

Proof. Suppose to the contrary that there is a graph G with unique colouring $\mathcal{C}$ and there are $A, B \in \mathcal{C}$, $A \neq B$, such that $G[A \cup B]$ has at least two components. Let H be such a component and consider the colouring $\tilde{\mathcal{C}}$ with $\tilde{\mathcal{C}} = (\mathcal{C} \setminus \{A, B\}) \cup \{(A \setminus V(H)) \cup (B \cap V(H))\} \cup \{(B \setminus V(H)) \cup (A \cap V(H))\}$. Then $\tilde{\mathcal{C}}$ is an optimal colouring of G distinct from $\mathcal{C}$, a contradiction. $\square$

The colouring $\tilde{\mathcal{C}}$ of G in the proof of Lemma 2.15 is obtained from the colouring $\mathcal{C}$ by changing colours in the component H of $G[A \cup B]$. We say that this operation is a *KEMPE change along H* and this is a simple operation for obtaining new colourings from given ones.

An easy consequence of Lemma 2.15 is the following lemma:

Lemma 2.16. *Let G be a uniquely colourable graph with colouring $\mathcal{C}$. Then:*

(i) *For each vertex $x \in V(G)$ and colour class $A \in \mathcal{C}$ such that $x \notin A$, there is a neighbour $y \in N_G(x)$ of colour A, i.e. $y \in A$.*

(ii) *G is connected.*

(iii) *The minimum degree of G is at least $|\mathcal{C}| - 1$.* $\diamond$

Moreover, Lemma 2.17 holds:

Lemma 2.17. *Let G be a uniquely k-colourable graph with $k \geq 2$. Then G is $(k-1)$-connected.* $\diamond$

Proof. Suppose to the contrary that there is a non-complete graph G (otherwise G is $(k–1)$-connected) with a unique k-colouring $\mathcal{C}$ and for two non-adjacent vertices x, y, there is $S \subseteq V(G)$ with $|S| \leq k-2$ separating x and y. But then there are distinct $A, B \in \mathcal{C}$ with $A \cap S = \emptyset = B \cap S$, and $(G-S)[A \cup B] = G[A \cup B]$ is connected. Since x and y have neighbours in $A \cup B$, they cannot be separated by S, a contradiction. □

It is easy to see that the complete graph K_k on k vertices is uniquely k-colourable and we can obtain a family of uniquely k-colourable graphs by consecutively adding a vertex and connecting it to all vertices except those of one colour class. This raises the question whether all uniquely k-colourable graphs contain K_k as a subgraph, therefore, trivially verifying Hadwiger's Conjecture for uniquely colourable graphs.

This question whether a uniquely k-colourable graph always contains K_k as a subgraph was first disproved by Harary, Hedetniemi, and Robinson [HHR69]. They found a uniquely 3-colourable graph F without triangles. For $k \geq 4$, a uniquely k-colourable graph without K_k is $F + K_{k-3}$, where $F + K_{k-3}$ is the complete join of F and K_{k-3}.

We are interested in constructions of uniquely k-colourable graphs such that the colour classes have "nearly the same size". The reason is that — later in Section 2.3 — we want to choose one vertex from each colour class and find a certificate containing those chosen vertices in the bags. Small colour classes do not give many possibilities to choose a vertex in a small colour class, which might simplify the problem since we can reduce it to considerations of graphs with one colour less.

Even though properties of uniquely colourable graphs have been widely studied [AMS01; CG69; EHK98; HHR69; Neš73; Xu90], it seems that there has not been much investigation regarding same-sized colour classes. Therefore, in this thesis a new construction of uniquely k-colourable graphs with the additional property that these graph are K_k-free is provided. Given a uniquely k-colourable graph H without K_k and equal colour class sizes, this construction leads to a uniquely $(k+1)$-colourable graph G without K_{k+1} as subgraph and all colour classes have equal size. This construction is published in S. Mohr: A construction of uniquely colourable graphs [Moh]; the entire paper can be found in Chapter 3. In what follows below, we consider the construction and analyse it with respect to Hadwiger's Conjecture.

Construction

Let H be a k-colourable graph with a proper k-colouring $\mathcal{C} = \{A_1, \dots, A_k\}$. We obtain a new graph $\nu(H) := G$ with a proper $(k+1)$-colouring $\mathcal{C}'$ from H by taking k further copies of H and adding edges such that the following holds:

$$\begin{aligned} V(G) &:= V(H) \cup \{v^p : v \in V(H), p = 1, \dots, k\}, \\ E(G) &:= E(H) \cup \{v^p u^p : vu \in E(H), p = 1, \dots, k\} \\ &\qquad \cup \{vu^p, uv^p : vu \in E(H), p = 1, \dots, k\} \\ &\qquad \cup \{v^p v^q : v \in V(H), v \in A_p, q \in \{1, \dots, k\} \setminus \{p\}\}, \end{aligned}$$

and

$$\mathcal{C}' := \{A'_i : i = 1, \ldots, k\} \cup \{\{v^p : v \in A_p, p = 1, \ldots, k\}\}$$

with $A'_i := \{v, v^p : v \in A_i, p \in \{1, \ldots, k\} \setminus \{i\}\}$.

Theorem 2.18. *For $k \geq 3$, let H be a uniquely k-colourable graph without K_k, then $\nu(H)$ as constructed on page 16 is uniquely $(k+1)$-colourable without K_{k+1}. Furthermore, $\omega(\nu(H)) = \omega(H) + 1$ and if all colour classes of H have equal size, so have all those of $\nu(H)$.* ⋄

For the proof, we refer to the proofs of Theorem 3.1 and Proposition 3.2.

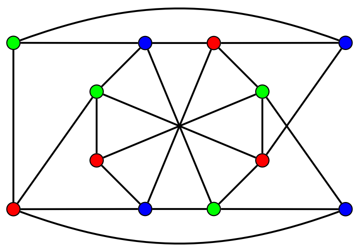


Figure 2.1: The graph G_3: a uniquely 3-colourable triangle-free graph on 12 vertices and equal colour class sizes.

Starting with the graph G_3 illustrated in Figure 2.1, let $G_{i+1} := \nu(G_i)$ be iteratively defined for $i \geq 3$. By Theorem 2.18, we obtain a sequence $G_3, G_4, G_5, \ldots$ of uniquely colourable graphs with equal colour class sizes. G_i is i-colourable and contains K_{i-1} as a subgraph.

Proposition 2.19. *Each graph of the sequence $G_3, G_4, G_5, \ldots$ fulfils the assertion of* Hadwiger*'s Conjecture.* ⋄

Proof. A uniquely 3-colourable graph is 2-connected by Lemma 2.17, hence, it has a cycle and $K_3 \prec G_3$.

We proceed by induction. Let $i \geq 4$ and $c = (V_\ell)_{\ell \in \{1,2,\ldots,i-1\}}$ be a K_{i-1}-certificate of G_{i-1}. Take an arbitrary transversal T of c and $x \in V(G) \setminus V(G_{i-1})$. Since $|T| = i - 1$ and G_i is $(i-1)$-connected by Lemma 2.17, there is a system of $(i-1)$ paths connecting x with T by Menger's Theorem. Let P be the vertices of all these paths and let $V_i := V(P) \setminus (V_1 \cup V_2 \cup \cdots \cup V_{i-1})$. Then $c' = (V_\ell)_{\ell \in \{1,2,\ldots,i\}}$ is a K_i-certificate in G_i and G_i contains a K_i-minor. ■

We have seen that Hadwiger's Conjecture holds for the graphs obtained from our construction. Obviously, the following conjecture inspired by Hadwiger's Conjecture immediately suggests itself.

Conjecture 3. *Every uniquely k-colourable graph has a K_k-minor.*

A first step to an answer of Conjecture 3 was made by M. Kriesell:

Theorem 2.20 (Kriesell [Kri17] and [Kri20a])**.** *For $k \leq 10$, each uniquely k-colourable graph has a K_k-minor.*

Conjecture 3 is still widely open, but the situation changes if we restrict our consideration to graphs without antitriangles, i.e. graphs with independence number at most 2. The related conjecture is the following Conjecture 4 due to Paul D. Seymour extending Hadwiger's Conjecture for $\alpha(G) = 2$ (see e.g. [Bla07]):

Conjecture 4 (Seymour)**.** *Let G be a graph with $\alpha(G) \leq 2$, then G contains a K_k-minor with $k := \lceil \frac{|V(G)|}{2} \rceil$. Furthermore, the K_k-certificate can be chosen such that each branch sets consists of one or two vertices.* ⋄

As a first impression, the consideration of graphs with independence number at most 2 seems to be rather restrictive. Quite the contrary, the complements of these graphs are triangle-free graphs and one notices a big interest and research on triangle-free graphs. Moreover, P. Seymour thinks that this class of graphs is an "excellent place to look for a counterexample to Hadwiger's Conjecture" [Sey16].

In the setting of uniquely colourable graphs, Seymour's Conjecture 4 is true. Actually, Kriesell proved the following stronger result:

Theorem 2.21 (Kriesell [Kri17])**.** *Let G be a uniquely colourable graph with $\alpha(G) \leq 2$, then G contains a K_k-minor with $k := \lceil \frac{|V(G)|}{2} \rceil$. Furthermore, for each transversal T of the unique colouring of G the K_k-certificate can be chosen such that the branch sets are traversed by T and each consists of one or two vertices.* ⋄

In the following, we turn our attention back to Conjecture 3 and study Kriesell's proofs of Theorem 2.20. He works with Kempe colourings, which are introduced in the next section. We will begin by restricting to antitriangle-free graphs to get familiar with the new concepts.

2.2.2 Kempe Colourings

The restriction to uniquely colourable graphs is too rigorous provided that in most situations we only need a certain property of the unique colourings. We would like to shift away from unique colourings and restrict our requirements to 'simpler' colourings. This is the point where Kempe colourings come into play. To understand the idea, consider the property of the (unique) colouring of a uniquely colourable graph that the subgraph of G induced by any two distinct colour classes is connected (Lemma 2.15). Based on this property, we define a Kempe colouring as follows:

Definition 2.22. A colouring $\mathcal{C}$ of a graph G is a *Kempe colouring* if for each pair of distinct colour classes $A, B \in \mathcal{C}$, the induced subgraph $G[A \cup B]$ is connected. ⋄

A Kempe colouring is a colouring that is fixed under Kempe change operations, i.e. these do not alter the colouring. We can consider this as a generalisation of an optimal colouring of a uniquely colourable graph.

Example 2.23. For integers $k, \ell \geq 3$, let $K = \{1, 2, \ldots, k\}$, $L = \{1, 2, \ldots, \ell\}$, and G be the graph with vertex set $V(G) := K \times L$ and edge set $E(G) := \{(i,j)(i',j') : (i,j),(i',j') \in V(G), i \neq i', j \neq j'\}$. Then $\mathcal{C} := \{K \times \{j\} : j \in L\}$ is a Kempe colouring of G. ⋄

By choosing $k=3$ in Example 2.23, the resulting graph is 3-colourable for all $\ell \geq 3$ but $\mathcal{C}$ is a Kempe ℓ-colouring. This shows that there are graphs with a Kempe colouring using significantly more colours than an optimal colouring. Hence, all following results about clique minors of the same size as a Kempe colouring are stronger results compared to those targeting the chromatic number as in Hadwiger's Conjecture.

Since the definition of Kempe colourings is based on the property of uniquely colourable graphs, which is most used in the proofs of Lemmas 2.16 and 2.17, these results can easily be extended to Lemma 2.24

Lemma 2.24. *Let G be a graph with Kempe colouring $\mathcal{C}$ of size k. Then:*

(i) $\delta(G) \geq k-1$,

(ii) *G is $(k-1)$-connected,*

(iii) $|E(G)| \geq (k-1)\,|V(G)| - \binom{k}{2}$. ⋄

We can ask whether Theorem 2.21 about antitriangle-free graphs can be generalised to Kempe colourings. Kriesell mentioned that he was not able to do so [Kri17]. Thus, the following problem is still open:

Problem 1 (Kriesell [Kri17])**.** Let G be a graph with $\alpha(G) \leq 2$ and $\mathcal{C}$ be a Kempe colouring of size k. Does G contain a K_k-minor? ⋄

To solve Problem 1, one has to find exactly as many branch sets as there are colour classes in the Kempe colouring. If we merely demand the number of branch sets to be half the size of the Kempe colouring, the problem turns out to be very easy:

Proposition 2.25. *Let G be a graph with $\alpha(G) \leq 2$ and $\mathcal{C}$ be a Kempe colouring of size k. Then G contains a K_ℓ-minor with $\ell \geq (k-1)/2$.* ⋄

Proof. Let G be a graph with $\alpha(G) \leq 2$ and $\mathcal{C}$ be a Kempe colouring of size k. Take a partition of $\mathcal{C}$ into pairs $(A_j, B_j)_{j=1,2,\ldots,\lfloor k/2 \rfloor}$ (drop one colour class if k is odd). Then for all $j \in \{1, 2, \ldots, \lfloor \frac{k}{2} \rfloor\}$ the branch sets $\mathfrak{K}_j := A_j \cup B_j$ are non-empty, connected, and adjacent. □

How can we do better than Proposition 2.25? To examine this, let us loosen Problem 1 and assume that we are satisfied if we can get three branch sets out of four colour classes. The following problem describes this setup. It is still open, but I would like to share a so far unsuccessful idea.

Problem 2. Let G be a graph with $\alpha(G) \leq 2$ and $\mathcal{C}$ be a Kempe colouring of size k. Does G contain a K_ℓ-minor with $\ell \geq \frac{3}{4}k$? ⋄

Approach to a proof. Let G be a graph with $\alpha(G) \leq 2$ and $\mathcal{C}$ be a KEMPE colouring of size k. We successively take colour classes from $\mathcal{C}$ and construct a set of branch sets. In each step, the number of taken colour classes is smaller or equal to $4/3$ times the number of produced branch sets.

As a first step, we observe that a colour class $A \in \mathcal{C}$ of size 1 consists of one vertex that is apex. Consequently, remove A from $\mathcal{C}$ and append it to the set of branch sets. We continue with a graph such that all colour classes have size 2 and each pair of two colour classes is connected either by a path on three edges or a cycle of length 4. In both cases, we take a suitable partition of $\mathcal{C}$ into sets $(A_j, B_j)_j$ of size 2 (assuming that $|\mathcal{C}|$ is even) and choose for each pair (A_j, B_j) a perfect matching in $G[A_j \cup B_j]$. Let $\mathcal{M}$ be the set of all these matching edges.

We will consider an auxiliary graph Q with vertex set $V(Q) := \mathcal{M}$. For two distinct edges $e, f \in \mathcal{M}$ we call the pair (e, f) *conflicting* if they are not connected, i.e. there is no edge connecting an end vertex of e with an end vertex of f. The edges of Q are all conflicting pairs of matching edges. An edge $e \in \mathcal{M}$ is called *conflict-free*, if there is no edge $f \in \mathcal{M} \setminus \{e\}$ such that e, f is conflicting, i.e. e is an isolated vertex in Q. Note that the two matching edges from one colour pair (A_j, B_j) cannot be conflicting.

If there is a colour pair (A_j, B_j) such that both corresponding matching edges e_j, f_j from $\mathcal{M}$ are conflict-free, then $V(e_j)$ and $V(f_j)$ are candidates for branch sets since they are connected to all other matching edges. Remove the corresponding colour classes and add $V(e_j)$ and $V(f_j)$ to the set of branching sets. Hence, it remains a set of matching edges $\mathcal{M}'$ and for each edge $e \in \mathcal{M}'$ there is an edge $f \in \mathcal{M}'$ with the end vertices in the same colour classes and at least one of e, f is not conflict-free.

Assume that there is such an edge $e \in \mathcal{M}'$ that has exactly one conflicting edge $g \in \mathcal{M}'$. Let $f, h \in \mathcal{M}'$ be the matching edge with vertices in the same colour class as e and g, respectively. Then e, f, g, h are distinct and there is no edge between e and g in G. Thus, because of the KEMPE colouring, all possible edges between f and g, h and all possible edges between h and e, f exist in $E(G)$. Assume that the end vertices of f are x, y and those of g, h are a_g, b_g and a_h, b_h, respectively, such that $a_g, a_h \in A$ for some $A \in \mathfrak{C}$. Then, $V(e)$, $\{x, a_g, a_h\}$, and $\{y, b_g, b_h\}$ induce connected subgraphs and are adjacent to all remaining matching edges. Thus, these three sets are branch sets obtained from four colour classes.

Any isolated vertex of Q can be considered as a branch set adjacent to all matching edges. Hence, we can assume that Q has minimum degree 2. The graph Q is triangle-free; furthermore, for each matching edge $e \in V(Q)$, the neighbourhood $N_Q(e)$ is not only independent in Q, the vertices from the set $X := \bigcup_{f \in N_Q(e)}$ induces a clique in G because $\alpha(G) \leq 2$. These ideas might help to complete the proof. However, we cannot reveal the prefect matching since this might destroy adjacencies to the already considered branch sets. Moreover, it is unclear how to choose a perfect matching to avoid bad situations. ⌟

Getting back to the general case (drop the assumption $\alpha(G) \leq 2$), one can ask how to improve the results that graph with KEMPE colourings of size k have K_k-minors for $k \geq 11$. For $k \leq 10$, KRIESELL actually proved the following stronger version of Theorem 2.20. We recall that a KEMPE colouring might have significantly more colours than an optimal colouring.

Theorem 2.26 (KRIESELL [Kri17] and [Kri20a])**.** *For $k \leq 10$, each graph with an arbitrary KEMPE colouring of size k has a K_k-minor.* ⋄

Let us investigate how the proof works. A first observation is that if Theorem 2.26 holds for an integer $k \in \mathbb{N}$, then it holds for graphs with a Kempe k'-colouring for all $k' < k$. The second proof showing the result for $k \leq 10$ [Kri20a] uses Song's and Thomas's result about minors in graphs with a certain edge density (Theorem 2.7).

Let G be a graph with a Kempe 10-colouring $\mathcal{C}$ and let G' be obtained from G by removing the vertices of two colours $A, B \in \mathcal{C}$. By choosing two suitable vertices in $x, y \in V(G')$, the graph $G' + xy$ has enough edges (by Lemma 2.24) to fulfil the hypothesis of Theorem 2.7. Since G' is 7-connected (again by Lemma 2.24), $G' + xy$ cannot be obtained from clique-sums and — assuming the cases with a small number of vertices can be solved — $G' + xy$ contains a K_9-minor. Since we have not used the vertices from $A \cup B$, it is possible using them to extend the K_9-minor to a K_{10}-minor of G.

This proof heavily depends on Song and Thomas's result (Theorem 2.7). A new contribution to this problem could well extend Theorem 2.26. Therefore, Problem 3 concerning Conjecture 2 is stated here:

Problem 3. Let $p \in \mathbb{N}$. How can K_p-minor-free graphs on n vertices and at least $(p-2)\,n - \binom{p-1}{2} + 1$ be characterised? ⋄

Problem 3 is known for $p \in \{1, 2, \dots, 9\}$, see page 12.

The first proof of Theorem 2.26 by Kriesell for $k \leq 6$ [Kri17] uses a result about rooted K_4-minors. Before going into the details, one also observes that Theorem 2.21 is a "rooted minors version". We postpone the considerations of this kind of minor to Section 2.3 and continue our studies of graphs with few colourings.

2.2.3 Other Concepts of Graphs with Few Colourings

We have seen some results about uniquely colourable graphs, namely graphs with only one optimal colouring. The proceeding step is to ask for graphs with exactly two optimal proper colourings. This question is discussed in [Kri20b], where a full characterisation of the maximal k-colourable graphs with more than one k-colouring is given. These graphs can be represented by matrices.

For $p \in \mathbb{N}$, let $A \in \mathbb{N}^{p \times p}$ be a matrix of positive integers. We construct a graph G_A obtained from A as follows:

$$\begin{aligned} V(G_A) &:= \{(z,t) : z \in \{1,2,\dots,p\}^2, t \in \{1,2,\dots,A(z)\}\}, \\ E(G_A) &:= \{(w,s)(z,t) : (w,s),(z,t) \in V(G_A), w_1 \neq z_1, w_2 \neq z_2\}. \end{aligned}$$

We notice that G_A is the graph with $A(z)$ vertices for each element z of A and two vertices of G_A are adjacent if they differ in all coordinates. It is easy to see that

$$\mathcal{C}_i := \{\{(z,t) \in V(G_A) : z_i = \ell\} : \ell \in \{1,2,\dots,p\}\}$$

for $i \in \{1, 2\}$ are two p-colourings of G_A that colour G_A row-wise and column-wise, respectively. By a straightforward argument we get $\chi(G_A) = p$ and $\mathcal{C}_1$ and $\mathcal{C}_2$ are the only optimal colourings of G_A (note that $A(z) \geq 1$ for all z). We can say that G_A belongs to the class of graphs with few colourings.

Assume first that A is constant, i.e. $A \in \{k\}^{p\times p}$ for a $k \in \mathbb{N}$. C. Brosse studied K_s-certificates in G_A such that each branch set consists of at most two vertices.

Proposition 2.27 (Brosse [Bro17]). *Let $p, k \in \mathbb{N}$ and G_A be obtained from $A \in \{k\}^{p\times p}$. Then there exists a K_s-certificates in G_A with $s = k \cdot \binom{p}{2} + \lfloor \frac{(k-1)\,n}{2} \rfloor + n$ such that each branch set has size at most 2.* ⋄

As a consequence, there are large clique minors in G_A, in particular $K_s \prec G_A$, and Hadwiger's Conjecture holds for these graphs. However, there is an easy observation concluding this fact even for non-constant matrices:

Let a be the maximum element of A. It is obvious that G_A is a subgraph of $G_{A'}$ with $A' \in \{a\}^{p\times p}$, the constant matrix with elements a. The graph $G_{A'}$ is isomorphic to $(\overline{K_p \times K_p})[\overline{K_a}]$, where $H_1 \times H_2$ and $H_1[H_2]$ denote the Cartesian and the lexicographic product of two graphs H_1 and H_2, respectively.

Ravindra and Parthasarathy proved that the lexicographic product of two perfect graphs is perfect [RP77] and the Cartesian product of two graphs is perfect if and only if it does not contain an induced odd cycle. Since it is well-known that complements (Weak Perfect Graph Theorem [Lov72]) and induced subgraphs of perfect graphs are also perfect, we conclude that G_A is perfect and therefore fulfils the assertion of Hadwiger's Conjecture.

Moreover, the definition of G_A can easily be generalised by using higher dimensional matrices; therefore obtaining graphs with a given number of optimal colourings. The results of Proposition 2.27 can easily be translated, rising hope that Hadwiger's Conjecture will also hold for these graphs. Nevertheless, arguing by graph products fails as $K_p \times K_p \times K_p$ is not perfect (for $p \geq 3$). We skip further details here and move on to "rooted minors".

2.3 Rooted Minors

We have investigated uniquely colourable graphs and graphs admitting a Kempe colouring in the last section. One property of all these graphs is that if the graph is k-colourable and $x_1, x_2, \ldots, x_k$ are vertices having different colours, then there exists a system of edge-disjoint x_i,x_j-paths ($i \neq j$ from $\{1, \ldots, k\}$), namely the paths using only vertices of the two colours corresponding to x_i and x_j. This so-called *clique immersion* of order k at $x_1, x_2, \ldots, x_k$ provides structure to find a K_k-minor and the question arises whether there exists a K_k-certificate such that $x_1, x_2, \ldots, x_k$ belong to different branch sets.

This has been answered affirmatively if one forbids antitriangles for uniquely colourable graphs, as we have seen in Theorem 2.21. In this section, we further develop our survey on minors so that we can force the branch sets to contain some predefined vertices. Minors having such properties are *rooted minors*; we investigate this concept and start with formal definitions.

Definition of Rooted Minors

Rooted minors were firstly mentioned in ROBERTSON's and SEYMOUR's first paper of the series "Graph minors". It is well-known that we can consider trees to be *rooted*, meaning that we can emphasise one vertex and call it *root of the tree*. As a generalisation, we say that a graph is a *rooted graph* G if one vertex $v \in V(G)$ is distinguished, and define the root $\rho(G) := v$.

Definition 2.28 (ROBERTSON, SEYMOUR [RS83])**.** Let G, H be rooted graphs. Then H is a *rooted minor* if there exists an H-certificate $c = (V_v)_{v \in V(H)}$ in G such that $\rho(G) \in V_{\rho(H)}$. ⋄

Several papers later, ROBERTSON and SEYMOUR introduced rooted bipartite minors:

Definition 2.29 (ROBERTSON, SEYMOUR [RS90])**.** Let G be a graph and k, ℓ integers. G has a *rooted $K_{k,\ell}$-minor* if for any ℓ distinct vertices $v_1, v_2, \ldots, v_\ell \in V(G)$, there is a $K_{k,\ell}$-certificate $c = (V_v)_{v \in V(K_{k,\ell})}$ in G such that for each $x \in B$, with B being the second colour class of $K_{k,\ell}$ of order ℓ, there is $1 \leq i \leq \ell$ with $v_i \in V_x$. ⋄

This leads us to the most general version that is stated for example in WOLLAN's Ph.D. thesis about "Extremal Functions for Graph Linkages and Rooted Minors" [Wol05].

Definition 2.30 (WOLLAN [Wol05; Wol08])**.** Let G, H be graphs, $X \subseteq V(G)$ with $|X| = |V(H)|$, and $\pi : X \to V(H)$ be a bijection. Then the pair (G, X) contains a *π-rooted minor* if there exists an H-certificate $c = (V_v)_{v \in V(H)}$ in G such that $x \in V_{\pi(x)}$ for all $x \in X$. ⋄

Using this definition, we can reword Theorem 2.21:

Theorem 2.21 (KRIESELL [Kri17])**.** *Let G be a uniquely colourable graph with $\alpha(G) \leq 2$ and T be a transversal of the unique colouring of G, furthermore let $\pi : T \to V(K_k)$ be an arbitrary bijection. Then (G, T) contains a π-rooted K_k-minor with $k := |T| \geq \lceil \frac{|V(G)|}{2} \rceil$. Furthermore, the branch sets consists of one or two vertices.* ⋄

In a multitude of cases, we are interested in K_k-minors. In this case, the choice of π is irrelevant. Therefore, we can simplify Definition 2.30 in the case that we are considering K_k-minors, as is done by R. FABILA-MONROY and D. WOOD:

Definition 2.31 (FABILA-MONROY, WOOD [FW13])**.** Let G be a graph, $X \subseteq V(G)$ with $k := |X|$. Then G contains a *K_k-minor rooted at X* if there exists an K_k-certificate $c = (V_v)_{v \in V(K_k)}$ in G such that $|V_v \cap X| = 1$ for all $v \in K_k$. ⋄

Motivated by Theorems 2.21 and 2.26, KRIESELL postulated the following conjecture about rooted minors in graphs admitting a KEMPE colouring, which serves as opening question for this section:

Conjecture 5 (KRIESELL [Kri17])**.** *Let G be a graph, $\mathcal{C}$ be its KEMPE colouring of size k and T a transversal of $\mathcal{C}$, then G contains a K_k-minor rooted at T.* ⋄

This conjecture is the subject of the following research and results.

2.3.1 Kriesell's Conjecture

Kriesell's conjecture (Conjecture 5) is known to be true for $k \leq 4$:

Proposition 2.32. *For $k \leq 4$, let G be a graph, $\mathcal{C}$ be a Kempe colouring of size k, and T be a transversal of $\mathcal{C}$, then G contains a K_k-minor rooted at T.* ⋄

To prove this result, we use a theorem of Fabila-Monroy and Wood [FW13]. This theorem was already used by Kriesell to prove the first cases ($k \leq 6$) of Theorem 2.26. We will see its varied usability a couple of times in this thesis.

Theorem 2.33 (Fabila-Monroy, Wood [FW13, Theorem 8])**.** *Let G be a 3-connected graph and $a, b, c, d \in V(G)$ distinct vertices. Then G contains a K_4-minor rooted at $\{a, b, c, d\}$ if and only if G contains an (ab, cd)-linkage, an (ac, bd)-linkage, and an (ad, bc)-linkage.* ⋄

Proof of Proposition 2.32. Let G be a graph, $\mathcal{C}$ be a Kempe colouring, and T be a transversal of $\mathcal{C}$. If $k := |\mathcal{C}| < 4$, we add $4 - k$ vertices $x_1, \dots, x_{4-k}$ to T and fully connected them to G and among each other. Hence, we can assume that $k = 4$. By Lemma 2.17, G is 3-connected, and for distinct $a, b \in T$, there exists a 2-coloured path in the subgraph of G induced by the two colour classes omitting vertices of the other two colours. Thus, all the linkages exist in G and the assertion follows from Theorem 2.33. ∎

As we have seen in the proof of Proposition 2.32, if there is a graph G with a Kempe colouring $\mathcal{C}$ such that there is $A \in \mathcal{C}$ with $|A| = 1$, then we know that this vertex $x \in A$ is an apex vertex of G and $x \in T$. Thus, we can consider $G - x$, $\mathcal{C} \setminus A$, and $T \setminus \{x\}$ to use inductive arguments in order to obtain an $K_{|\mathcal{C}|-1}$-minor rooted at $T \setminus \{x\}$, which easily extends to a rooted $K_{|\mathcal{C}|}$-minor.

Assuming that there is $A \in \mathcal{C}$ with $|A|$ very small, it might be possible to contract A to a single vertex by using only a few other vertices of the graph. This might not disturb the Kempe colouring too much. For these reasons, there is an interest in graphs with equal sized colour classes and, in particular, uniquely colourable graphs with equal sized colour classes in their unique colourings considered in Section 2.2.1. And indeed, the sequence $G_3, G_4, G_5, \dots$ of graphs considered in Proposition 2.19 also fulfils Kriesell's conjecture; hence, we can extend Proposition 2.19:

Proposition 2.34. *For each graph of $G_3, G_4, G_5, \dots$, let T_i be an arbitrary transversal of the unique colouring $\mathcal{C}_i$ of G_i, $i \geq 3$. Then G_i contains a K_i-minor rooted at T_i for all $i \geq 3$.* ⋄

Proof. We proceed by induction. Since G_3 is 2-connected (see Lemma 2.17), there exists a cycle in G_3 containing all three vertices from T_3; this proves that G_3 contains a K_3-minor rooted at T_3.

Assume that Proposition 2.34 holds for G_i and arbitrary T_i, $i \geq 3$. Let $A_{i+1} \in \mathcal{C}_{i+1}$ be the colour class with $V(G_i) \cap A_{i+1} = \emptyset$ and $t_{i+1} \in T_{i+1} \cap A_{i+1}$ the explicitly defined transversal vertex of A_{i+1}. For each $t \in T_{i+1} \setminus (\{t_{i+1}\} \cup V(G_i))$, there is a vertex $v_t \in V(G_i)$ and $p \in \{1, 2, \dots, i\}$ such that

$t = v_t^p$. Let $w_t \in V(G_i)$ be a neighbour of v_t and we can assume that w_t and $w_{t'}$ are distinct vertices for distinct $t, t' \in T_{i+1} \setminus (\{t_{i+1}\} \cup V(G_i))$.

Let T' be the set containing t if $t \in (T_{i+1} \setminus \{t_{i+1}\}) \cap V(G_i)$ and v_t if $t \in (T_{i+1} \setminus \{t_{i+1}\}) \setminus V(G_i)$. Then T' is a transversal of the colouring $\{A \cap V(H) : A \in \mathcal{C}_{i+1} \setminus \{A_{i+1}\}\}$ in G_i. By the induction hypothesis, G_i contains a K_i-minor rooted at T'. Denote by $c = (V_\ell)_{\ell \in \{1,2,\dots,i\}}$ the K_i-certificate in G_i.

For each $t \in T_{i+1} \setminus (\{t_{i+1}\} \cup V(G_i))$, there is an index ℓ such that $v_t \in V_\ell$. Let $V'_\ell := V_\ell \cup \{t = v_t^p, w_t^p\}$ and $V'_\ell := V_\ell$ if $V_\ell \cap T_{i+1}$ is not empty. We obtain a K_i-certificate $c' = (V'_\ell)_{\ell \in \{1,2,\dots,i\}}$ in G_{i+1} rooted at $T_{i+1} \setminus \{t_{i+1}\}$

There exists $p \in \{1, 2, \dots, i\}$ and $B := \{v^p : v \in V(G_i)\} \subseteq V(G_{i+1})$ such that $|B \cap (T_{i+1} \setminus \{t_{i+1}\})| \leq 1$ and $N_{G_{i+1}}(t_{i+1}) \cap B \neq \emptyset$. Set $V'_{i+1} := (B \setminus \bigcup_{\ell \in \{1,\dots,i\}} V'_\ell) \cup \{t_{i+1}\}$. By the assumptions above, $G_{i+1}[V'_{i+1}]$ is connected and $V(G_i) \subseteq N_{G_{i+1}}(B)$. Since each bag of c' contains a vertex of $V(G_i)$, $c'' = (V'_\ell)_{\ell \in \{1,2,\dots,i+1\}}$ is a demanded K_{i+1}-certificate, proving that G_{i+1} has a K_{i+1}-minor rooted at T_{i+1}. ■

The next question is whether there is an infinite class of graphs fulfilling Conjecture 5. One affirmative answer is given in M. Kriesell and S. Mohr: Rooted complete minors in line graphs with a Kempe coloring, *Graphs and Combinatorics* 35.**2** (2019) [KM19]; the entire paper can be found in Chapter 4. The following theorem holds (Theorem 4.1):

Theorem 2.35. *For every transversal of every Kempe colouring of the line graph $L(H)$ of any graph H there exists a complete minor in $L(H)$ traversed by T.* ◇

Recall Proposition 2.32, which verifies Kriesell's conjecture for $k \leq 4$. This is proved by using Theorem 2.33. As a consequence, a generalisation of this theorem to rooted K_5-minors would probably lead to new affirmations of Conjecture 5. However, there are no obvious ways to prove such a generalisation and David R. Wood (personal communication, 2019, [Woo19]) does also consider this as an interesting open problem. On the other hand, to prove Proposition 2.32, we just used the Kempe colouring to ensure 3-connectedness and to be able to apply Theorem 2.33. Hence, it is natural to ask for a relaxation of the hypotheses of Conjecture 5. This has been done in M. Kriesell and S. Mohr: Kempe chains and rooted minors [KM]; the paper can be found in Chapter 5. We will now state the problem and related results.

In all of the following, let G be a graph and $\mathcal{C}$ be one of its k-colourings. Furthermore, let T be an arbitrary transversal of $\mathcal{C}$. For each pair of distinct vertices $x, y \in T$, assume that there is a connected component, i.e. a Kempe chain, of $G[A \cup B]$ containing both vertices x and y, where $A, B \in \mathcal{C}, x \in A, y \in B$. We can ask the following question:

Question 2. For which integers k and all choices of $G, \mathcal{C}, T$ as above does G have a K_k-minor rooted at T? ◇

Since graphs with Kempe colourings fulfil the hypotheses of Question 2, an affirmative answer of Question 2 for all k would imply Conjecture 5.

Question 2 can be slightly weakened. Let K_T be the complete graph on vertex set T. Then Question 2 asks for a π-rooted K_T-minor with π being the identity on T. Let H be the graph on vertex set T and let $x, y \in T$ be joined by an edge in H if and only if there is a Kempe chain of $G[A \cup B]$ containing both vertices x and y, where $A, B \in \mathcal{C}, x \in A, y \in B$. Instead of asking for k in Question 2, we ask for graphs H such that for all choices of $G, \mathcal{C}, T$ with H defined as above, there exists a π-rooted H-minor with π being the identity on T. The complete survey on this problem can be found in Chapter 5.

It is concluded that Question 2 holds for $k \leq 4$ and fails for $k \geq 7$, hence, this approach is not successful to attack Conjecture 5. However, a tiny but nonetheless difficult positive answer for the case $k = 5$ in Conjecture 5 is the following proposition:

Proposition 2.36. *For $k = 5$, let G be a graph, $\mathcal{C}$ be a Kempe colouring of size k, and T be a transversal of $\mathcal{C}$ such that $G[T]$ is connected. Then G contains a K_k-minor rooted at T.* ⋄

2.3.2 Open Questions on Rooted Minors

We conclude this section by stating some open problems arising in this thesis.

First of all, it would be great to solve Kriesell's conjecture (Conjecture 5):

Problem 4. How may we prove that for each graph G with a Kempe colouring $\mathcal{C}$ of size k and for every transversal T of $\mathcal{C}$, that G contains a K_k-minor rooted at T? ⋄

To solve this problem, it might help to find a generalisation of Fabila-Monroy's and Wood's theorem about rooted K_4-minors (Theorem 2.33); however, this is very likely not possible for large k.

Problem 5.

(i) What is the full characterisation of all graphs and all tuples of vertices $\{a, b, c, d, e\}$ of G such that there is no K_5-minor rooted at $\{a, b, c, d, e\}$?

(ii) What are sufficient conditions for a graph G and a fixed vertex set $T \subseteq V(G)$ of k vertices in order to force a K_k-minor rooted at these vertices? ⋄

Problem 5 (i) is a question by David R. Wood (personal communication, 2019, [Woo19]).

In Question 2 it is asked for an integer k. It has been shown that the question fails for $k \geq 7$ and affirmative answers for k' imply Question 2 for all $k \leq k'$. Thus, it is self-explanatory to ask:

Problem 6. For which integers $k' \leq 6$ does Question 2 hold for all $k \leq k'$? ⋄

2.4 Half-Rooted Minors

Remember the following setting as in the previous section: We have a graph G with a k-colouring $\mathcal{C}$ and a transversal T of $\mathcal{C}$ such that each pair of vertices $x, y \in T$ belongs to the same Kempe chain. It is obvious that such a graph G might have isolated vertices, separating vertices, or even vertices of degree 2 that cannot be removed without disconnecting a Kempe chain of two transversal vertices. The graph in Figure 2.2 is an example.

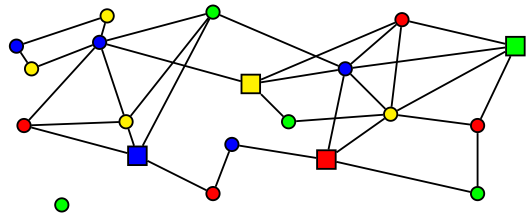


Figure 2.2: A graph with a 4-colouring such that each pair of transversal vertices (represented by squares) belong to one Kempe chain.

On the other hand, the vertices in T are connected in a "nice way". In particular, there exist all three linkages between the four transversal vertices; these linkages are 2-coloured paths. It might be of interest to apply some theorems to this graph, e.g. the result of Fabila-Monroy and Wood about rooted K_4-minors (Theorem 2.33), that depend on a certain connectedness. In the paper T. Böhme, J. Harant, M. Kriesell, S. Mohr, and J. M. Schmidt: Rooted Minors and Locally Spanning Subgraphs [Böh+] the following question is considered:

Question 3 (not precisely)**.** Given k, does there exist an integers $\ell(k)$ that fulfils: For each graph G and $X \subseteq V(G)$ such that there is no separator S in G with $|S| \leq \ell(k)$ separating vertices of X, G has a k-connected minor (or topological minor) that "contains X"? ⋄

A minor "containing X" should first be precisely defined. Let G be a graph and $X \subseteq V(G)$. A first idea is to use Definition 2.30 about π-rooted minors but it is very likely that all k-connected minors have more vertices than $|X|$ and the definition is not suitable. We have to extend the definition about π-rooted minors (Definition 2.30) and adapt it to our circumstances.

Definition 2.37. Let G, H be graphs, $X \subseteq V(G)$ with $|X| \leq |V(H)|$, and $\pi : X \to V(H)$ be an injection. Then the pair (G, X) contains a *π-rooted minor* if there exists an H-certificate $c = (V_v)_{v \in V(H)}$ in G such that $x \in V_{\pi(x)}$ for all $x \in X$. ⋄

This definition differs from Definition 2.30 in that $|X|$ is allowed to be strictly smaller than $|V(H)|$. Hence, π cannot be a bijection and we just require π to be an injection. We say that G contains an *k-connected X-minor* if there exists a k-connected graph H and an injection $\pi : X \to V(H)$ such that G has a π-rooted H-minor. In the same vein, we can define a topological X-minor:

Definition 2.38. Let G, H be graphs, $X \subseteq V(G)$ with $|X| \leq |V(H)|$, and $\pi : X \to V(H)$ be an injection. Then the pair (G, X) contains a *π-rooted topological minor* if G contains a subgraph M isomorphic to a subdivision of H with $\beta : V(H) \to V(M) \subseteq V(G)$ such that $\beta \circ \pi = \mathrm{id}_X$. ⋄

A set $S \subset V(G)$ is an *X-separator* of G if at least two components of $G - S$ contain a vertex of X. We define $\kappa_G(X)$ to be the maximum integer less than or equal to $|X| - 1$ such that the cardinality of each X-separator $S \subset V(G)$ — if any exists — is at least $\kappa_G(X)$. It follows that $\kappa_G(X) = |X| - 1$ if $G[X]$ is complete; however, if X is a proper subset of $V(G)$, then the converse need not to be true. If $\kappa_G(V(G)) \geq k$ for a graph G, then G is k-connected, and a $V(G)$-separator of G is a separator of G in the usual sense. This terminology corresponds to the commonly used definition of connectedness.

Now, we can precisely phrase Question 3:

Question 3 (precisely)**.** Given k, does there exist an integer $g(k)$ such that for each graph G and $X \subseteq V(G)$ with $\kappa_G(X) \geq g(k)$, that G has a k-connected X-minor (or topological X-minor)? $\diamond$

This question is investigated in the publication covered in Chapter 6:

Theorem 2.39 (see Theorem 6.2)**.** *Let $k \in \{1, 2, 3, 4\}$, G be a graph, and $X \subseteq V(G)$ such that $\kappa_G(X) \geq k$. Then:*

(i) *G has a k-connected X-minor.*

(ii) *If $k \leq 3$, then G has a k-connected topological X-minor.* $\diamond$

The proof can be found in Section 6.2 on page 72. This theorem is best possible in the sense that there are graphs with arbitrarily high $\kappa_G(X)$ without 4-connected topological X-minors (Observation 6.4), without 6-connected X-minors (Observation 6.5), and there are graphs with $\kappa_G(X) = 6$ that do not contain a 5-connected X-minor (Observation 6.3).

Since we are interested in the vertices of a graph H corresponding to X, it is convenient to assume that $X \subseteq V(H)$ and π is the identity on X. This simplification is practised in Chapter 6 and a suitable definition of these X-minors and edge contraction is provided there.

Theorem 2.39 has some immediate consequences. For example, the following theorem can be translated to the subsequent corollary.

Theorem 2.40 (Barnette [Bar66])**.** *If G is a 3-connected planar graph, then G has a spanning tree of maximum degree 3.* $\diamond$

Corollary 2.41. *If G is a planar graph, $X \subseteq V(G)$, and $\kappa_G(X) \geq 3$, then G contains a tree T containing all vertices of X and T has maximum degree 3.* $\diamond$

Proof. By Theorem 2.39, G has a topological X-minor H containing X (due to our assumption that $X \subseteq V(H)$). We apply Theorem 2.40 to H and obtain a tree T'. A subdivision of T' fulfilling the properties can be found in G. $\square$

As in the proof of Corollary 2.41, many results about 3-connected graph can be translated to graphs with $\kappa_G(X) = 3$. Since a graph G with a set $X \subseteq V(G)$ such that $\kappa_G(X) = 3$ contains a 3-connected topological minor, other results about subgraphs in 3-connected planar and non-planar graphs containing X can be obtained; this is also done in Theorem 6.8.

In the following, we will focus on structural properties of these graphs. By a result of TUTTE, we know that for all 3-connected graphs G there exists a sequence $H_0, H_1, \dots, H_n = G$ of 3-connected graphs starting with $H_0 = K_4$ and H_i can be obtained from H_{i+1} by contracting a suitable edge [Tut61]. The following proposition generalises this to graphs G with a vertex set X such that $\kappa_G(X) = 3$.

Proposition 2.42. *A graph G and $X \subseteq V(G)$ fulfils $\kappa_G(X) \geq 3$ if and only if it contains a subgraph H with $X \subseteq V(H)$, $d_H(x) \geq 3$ for $x \in X$, and there exists a sequence $H_0, H_1, \dots, H_n$, $H = H_n$ of graphs with the following properties:*

(i) *H_0 is a subdivision of K_4.*

(ii) *H_{i+1} is a graph such that each pair $x, y \in V(H_{i+1})$ with $d_{H_{i+1}}(x), d_{H_{i+1}}(y) \geq 3$ does not have two or more connecting paths with only internal vertices of degree 2. Let $P = x_0 x_1 \dots x_\ell$ be a path of length $\ell \in \mathbb{N}$ in H_{i+1} such that $d_{H_{i+1}}(x_0), d_{H_{i+1}}(x_\ell) \geq 3$ and $d_{H_{i+1}}(x_1) = \dots = d_{H_{i+1}}(x_{\ell-1}) = 2$. Then H_i is obtained from H_{i+1} by contracting P to a new vertex v_P of degree at least 3.* ◇

Proof. Let G be a graph with $X \subseteq V(G)$ fulfilling $\kappa_G(X) = 3$. By Theorem 2.39 (ii), G has a 3-connected topological X-minor M. Hence, there is a subgraph H of G obtained from a subdivision of M and $X \subseteq V(H)$. By the aforementioned result of TUTTE [Tut61], there is a sequence $M_0, M_1, \dots, M_n$ such that M_0 is isomorphic to K_4, $M_n = M$, M_{i+1} has an edge xy with $d_{M_{i+1}}(x), d_{M_{i+1}}(y) \geq 3$, and M_i is obtained from M_{i+1} by contracting xy. This complies with a demanded sequence starting with H_0 as a subdivision of M_0 and H_i is a subdivision of M_i for $i \in \{1, 2, \dots, n\}$. Since M_i has no parallel edges, the sequence has the claimed properties and ends in $H_n = H$.

Assume that for G and X there exists a subgraph H obtained from such a sequence as described in the assertion of Proposition 2.42. To finish the proof it is sufficient to show that $\kappa_H(X') \geq 3$ with $X' := \{v \in V(H) : d_H(v) \geq 3\}$. Therefore, let $X'_i := \{v \in V(H_i) : d_{H_i}(v) \geq 3\}$ and we show that $\kappa_{H_{i+1}}(X'_{i+1}) \geq 3$ if $\kappa_{H_i}(X'_i) \geq 3$. Suppose that this is not the case and let i the smallest index with $\kappa_{H_i}(X'_i) \leq 2$. It is obvious that $i \geq 1$.

Then there is a minimal separator S with $|S| \leq 2$ and two components C_1 and C_2 of $H_i - S$ each containing vertices from X'_i. Since $s \in S$ has a neighbour in C_1 and C_2, we can assume that S is chosen to minimize $|V(C_2)|$ and, therefore, $S \subseteq X'_i$. Thus, the path P is contained in one component plus S and we may assume that $V(P) \subseteq V(C_2) \cup S$. If C_2 comprises the whole path P, then S separates v_P and C_1, a contradiction. If C_2 contains a vertex $v \in X'_i$ such that $d_{H_{i-1}}(v) \geq 3$, then $(S \setminus \{x_0, x_n\}) \cup \{v_P\}$ is a separator of size at most 2 and separates v from C_1, a contraction. If $V(C_2) \cap X'_i \subseteq \{x_0, x_n\}$, then the end vertex of P in C_2 has degree 2, a contraction. Consequently, there is $v \in (X'_i \cap V(C_2)) \setminus V(P)$ such that $d_{H_{i-1}}(v) = 2$. There are two paths P_1 and P_2 from v to x_0 and x_n, respectively, and both paths have only inner vertices of degree 2. In H_{i-1}, there are two paths from v to v_P, a contradiction to the definition of H_{i-1}. We conclude that there is no X-separator. ■

We have seen that a graph G and $X \subseteq V(G)$ fulfilling $\kappa_G(X) = 3$ comprises a subgraph implementing the connectedness of X. Since there is no guaranty for a 4-connected topological minor if $\kappa_G(X) = 4$, it is not obvious what part of the graph forces $\kappa_G(X) = 4$. A step towards some knowledge about these graphs is Theorem 2.39 (i) implying that these graphs have a 4-connected X-minor. One can think of this as that in these graphs, each vertex from X can be grouped with some further vertices to obtain bags such that there are some possibilities to walk along the bags between two vertices of X.

The situation changes rapidly if $\kappa_G(X) = 5$. Apart from trivial conclusions of the smaller cases — G has a 4-connected X-minor — we do not know much. It is even hopeless to expect a 5-connected X-minor in this case as demonstrated in Observation 6.3.

Moreover, the following question about cycles in graphs with $\kappa_G(X) = 4$ turns out to be challenging.

Question 4. Let G be a planar graph, $X \subseteq V(G)$, and $\kappa_G(X) \geq 4$. Does G contain a cycle passing all vertices of X? ⋄

As a first approach, we apply Theorem 2.39 on G and obtain an X-minor H of G and an H-certificate $(V_x)_{x \in V(H)}$ such that H is 4-connected. By a famous result of W. Tutte — see also the forthcoming Theorem 2.44 — H is Hamiltonian and contains a cycle C through all vertices of H. For each edge xy in C, we choose an edge $e(xy) \in E(G)$ such that the edge $e(xy)$ connects a vertex in V_x with a vertex in V_y. Since $G[V_x]$ is connected for all $x \in V(H)$, the edges $e(xy)$ and $e(xz)$ for $xy, xz \in E(C)$ can be connected by a path in $G[V_x]$. In this manner, we obtain a cycle C' in G.

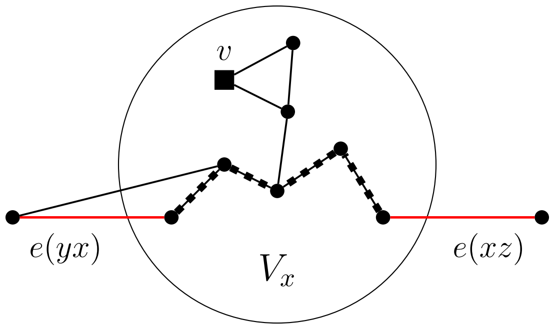


Figure 2.3: Cycle (red and dashed) cannot be extended in $G[V_x]$ to contain $v \in X$.

In general, it is not possible to route the cycle C' from $e(xy)$ to $e(xz)$ for $xy, xz \in E(C)$ through the unique vertex $v \in X \cap V_x$ (see Figure 2.3). Consequently, we need to deeper look into the proof of Tutte to solve Question 4.

In the next section, the proof of Tutte and related questions will be covered.

2.5 Tutte Cycles

Tutte cycles evolved as a strong tool to prove results about long cycles in graphs. The original and long-standing question was to prove that 4-connected planar graphs are Hamiltonian. In 1931, Whitney [Whi31] published a partial result; he was able to show that 4-connected triangulations of the plane, i.e. 4-connected maximal planar graphs, are Hamiltonian.

The big issue is how to use inductive proof methods on 4-connected graphs. We still seem to be unable to do induction on the number of vertices of 4-connected graphs. After removing a vertex of such a graph, it will hardly be fixable to remain 4-connected and one likely cannot apply the induction hypotheses.

HASSLER WHITNEY [Whi31] has overcome this problem: He proved a lemma that a cycle of arbitrary length with some minor further conditions in a 4-connected triangulations can be rerouted to contain the vertices of the cycle and all vertices in its interior. Note that the new cycle skips all vertices in the exterior. This lemma, which implies immediately the HAMILTONicity of 4-connected triangulations by applying the lemma to a facial cycle, was proved by induction on the length of the initial cycle. Using this approach, WHITNEY bypassed induction on the number of vertices of the graph and succeeded in showing the assertion. He explained in his publication [Whi31] that it "seemed to be the first case when a large class of planar graphs has been shown to have this property [HAMILTONicity]".

WILLIAM T. TUTTE [Tut56] finally came up with the appropriate idea to show that arbitrary 4-connected planar graphs are HAMILTONian. The key component is TUTTE cycles. W. TUTTE was able to verify that each planar graph contains a TUTTE cycle through two edges incident with the same face. Performing induction on the number of vertices of planar graphs came along without serious difficulties. Before getting further into the history of TUTTE cycles, let us consider their definition from TUTTE's paper in 1977 [Tut77].

Definition 2.43 (TUTTE [Tut77])**.** Let G be a graph, H be a subgraph of G and C be a cycle of G.

(i) The vertices of H incident in G with some edge not belonging to H are the *vertices of attachment* of H in G.

(ii) The *bridges of C in G* are all minimal subgraphs H of G such that each vertex of attachment of H in G is a vertex of C and H is not a proper subgraph of C.

(iii) We call C a *TUTTE cycle* if all bridges of C in G have at most three vertices of attachment and at most two vertices of attachment if the bridge contains an edge incident with the outer face.

(iv) We call a path P in G a *TUTTE path* if all bridges of P in G have the same properties as bridges of TUTTE cycles. ◇

Note that subgraphs of a graph G isomorphic to K_2 are bridges of a cycle C in G if and only if both end vertices of the edge but not the edge itself belong to C. We say that these bridges are *trivial bridges*.

TUTTE proved the following theorem.

Theorem 2.44 (TUTTE [Tut56])**.** *Let G be a planar graph and $e, e' \in E(G)$ two edges simultaneously contained in at least one cycle and incident with the outer face. Then G contains a TUTTE cycle.* ◇

The immediate consequence is that 4-connected planar graphs are HAMILTONian. Note that the vertices of attachment of a non-trivial bridge compose a separator in the graph contradicting the 4-connectedness. Based on this initial publication [Tut56] from 1956, a series of papers have evolved showing slightly stronger results and more artistical proofs. Even TUTTE published several years later a second paper [Tut77], in which he developed the theory of bridges from scratch and gave a short and simple proof of his Theorem 2.44.

TUTTE's classical result was generalised by CARSTEN THOMASSEN [Tho83], implying that 4-connected planar graphs are even HAMILTON-connected.

Theorem 2.45 (THOMASSEN [Tho83])**.** *Let G be a 2-connected planar graph and $e \in E(G)$, $v \in V(G)$ be an edge and a vertex incident with the outer face, respectively. Then G contains a TUTTE path from an arbitrary vertex $u \in V(G)$ through e and ending in v.* $\diamond$

The strongest version among them on 4-connected planar graphs is the theorem due to DANIEL P. SANDERS [San97]:

Theorem 2.46 (SANDERS [San97])**.** *Let G be a 2-connected planar graph and $e \in E(G)$ an edge incident with the outer face. Then G contains a TUTTE path between two arbitrary vertices $u, v \in V(G)$ through e.* $\diamond$

In a recent publication, SCHMID and SCHMIDT investigated algorithmic aspects of TUTTE cycles [SS18]. They concluded that they can be computed in $\mathcal{O}(n^2)$ for planar graphs on n vertices.

TUTTE paths and cycles are strong tools to prove HAMILTONicity of 4-connected planar graphs. However, there are mainly two ways of extending these theories. One of them is to translate the theory and results to other graph classes than planar graphs. The second one is to derive results from TUTTE cycles for subclasses of planar graphs. For the remainder of this section, the latter question will be considered. An outlook on what can be done with TUTTE paths in non-planar graphs will be presented in Section 2.6.

2.5.1 Essentially 4-connected Planar Graphs

Let G be a graph. We recall that the circumference circ(G) of G is the length of a longest cycle of G. By Theorem 2.44, we know that each 4-connected planar graph G is HAMILTONian, hence circ$(G) = |V(G)|$. It is natural to ask for the circumference of 3-connected planar graphs:

Example 2.47. Let G be an arbitrary 3-connected maximal planar graph. We consider the following construction: For each face α of G, we add a new vertex v_α to G and connect v_α to the three vertices incident with α by three edges. It is easy to see that the new obtained graph G' is still 3-connected and a triangulation of the plane.

Let C' be a longest cycle of G. By the construction of G', it is impossible that two new vertices are consecutive on C'. Hence, C' has at most twice as many edges as a longest cycle C in G. On the other hand, a longest cycle C of G splits the plane into an exterior and an interior. Replacing the edges of C alternatingly with the vertex of G' in the incident face on the exterior and interior, we obtain a cycle of G' with exactly twice the number of edges. ⋄

This Example 2.47 shows that for every $\varepsilon > 0$ there exists a 3-connected maximal planar graph G such that $\operatorname{circ}(G) \leq \varepsilon \cdot |V(G)|$. The repeated application of the construction in Example 2.47 was already observed by MOON and MOSER [MM63]. They constructed infinitely many maximal planar graphs G with $\operatorname{circ}(G) \leq 9|V(G)|^{\log_3 2}$ and conjectured that this bound is of right magnitude, i.e. there is a constant c such that each 3-connected planar graph G has $\operatorname{circ}(G) \geq c \cdot |V(G)|^{\log_3 2}$. This was later approved by CHEN and YU [CY02].

We observe that the circumference of 3-connected planar graphs is bounded by a sublinear function whereas for 4-connected planar graphs it is the identity function by TUTTE's Theorem 2.44. It is natural to ask how a slight weakening of the connectedness condition of planar graphs affects the circumference and whether there is such a graph class with a linear bounding function.

Definition 2.48. Let S be a minimal separator of a graph G. We say that S is a *trivial separator* if at most one component of $G - S$ contains edges.

A 3-connected graph is called *essentially 4-connected* if all 3-separators are trivial. ⋄

Each 4-connected graph is essentially 4-connected and the class of (3-connected) essentially 4-connected planar graphs can be considered as a graph class between 3-connected and 4-connected planar graphs with respect to a chain of inclusions. Let G be a 4-connected maximal planar graph on n vertices and G' be the graph obtained from the construction in Example 2.47. Then G' has $n + (2n - 4)$ vertices by EULER's formula and $\operatorname{circ}(G') = 2n$ since G was HAMILTONian. Furthermore, G' is an essentially 4-connected maximal planar graph. We remark:

Observation 2.49. There are essentially 4-connected maximal planar graphs G such that $\operatorname{circ}(G) = \frac{2}{3}(|V(G)| + 4)$. ⋄

BILL JACKSON and NICHOLAS C. WORMALD proved a first linear lower bound on the circumference of essentially 4-connected planar graphs [JW92].

Theorem 2.50. *For any essentially 4-connected planar graph G on n vertices,* $\operatorname{circ}(G) \geq \frac{2n+4}{5}$. ⋄

They did not explicitly postulate the following conjecture but since the publication of their first paper there has been put quite some effort into approaching this best-possible (see Observation 2.49) bound.

Conjecture 6 (see Conjecture 12 in Chapter 8)**.** *For every essentially 4-connected planar graph G on $n \geq 8$ vertices,* $\operatorname{circ}(G) \geq \frac{2}{3}(n + 4)$. ⋄

A next step was taken by FABRICI, HARANT, and JENDROL [FHJ16] as they showed that $\operatorname{circ}(G) \geq \frac{1}{2}(n + 4)$ for each graph G fulfilling the condition of Conjecture 6.

In I. Fabrici, J. Harant, S. Mohr, and J. M. Schmidt: Longer cycles in essentially 4-connected planar graphs, *Discussiones Mathematicae Graph Theory* 40.**1** (2020) [Fab+20b] — see Chapter 7 for the entire paper — the following theorem was proved and the factor $\frac{1}{2}$ was pushed to $\frac{3}{5}$.

Theorem 2.51 (see Theorem 7.1)**.** *For any essentially 4-connected planar graph G on n vertices,* $\operatorname{circ}(G) \geq \frac{3}{5}(n+2)$. ⋄

The complete proof can be found on page 85; here, I like to briefly present the main idea. Let G be an essentially 4-connected planar graph G and C be a Tutte cycle of G. Assume that there is a non-trivial bridge H of C in G, then H is isomorphic to $K_{1,3}$ since each non-trivial bridge has three vertices of attachment and there is at most one *inner vertex*, i.e. $|V(H) - V(C)| = 1$. This is due to the fact that otherwise the three vertices of attachment of H are a non-trivial 3-separator.

Thus, the number of vertices aside a Tutte path is strongly related to the number of faces in the following subgraph of G: Let G' be the graph obtained from G by removing all chords of C, i.e. by removing all edges in $E(G) \setminus E(C)$ that connect vertices of C. Using a charging-discharging argument, we conclude that there exists a Tutte cycle in G of desired length. For this, we charge all faces of G' adjacent to at most one vertex not belonging to C. After two rechargings, we can bound the number of edges in C and thereby the length of a longest cycle in G.

Doing the same procedure with a slightly more careful case distinguishing [Fab+20c], it is even possible to prove $\operatorname{circ}(G) \geq \frac{5}{8}(n+2)$.

The idea in the proof of Theorem 2.51 is to ignore chords of the Tutte cycle because one does not have much control over them. The situation changes completely if we assume that G is a maximal planar graph. In this case, it is rather useful to remove all vertices of G not belonging to a Tutte cycle C of G and keep the chords. The obtained graph is still maximal planar and the essentially 4-connectedness forces some structure. A complete analysis is done in I. Fabrici, J. Harant, S. Mohr, and J. M. Schmidt: Circumference of essentially 4-connected planar triangulations [Fab+], which can be found in Chapter 8. It was possible to verify Conjecture 6 for maximal planar graphs, see Theorem 8.2.

2.5.2 Spectrum

Recall Thomassen's result about Tutte paths (Theorem 2.45).

Example 2.52. Let G be a 4-connected planar graph and $x \in V(G)$, we prove that G contains a cycle through all vertices but x. To this extent, let $y \in N_G(x)$ and $v \in N_G(y) \setminus \{x\}$ such that vxy is not a facial triangle. By Theorem 2.45, G contains a Tutte path P from v to x using the edge xy. This edge will be the last edge of P. We can remove this edge and add the edge vy to obtain the desired cycle. ⋄

Thomas and Yu [TY94] extended the observation in Example 2.52 and confirmed a conjecture of Michael D. Plummer [Plu75] that every 4-connected planar graph contains a cycle through all but two vertices. Continuing in this manner, it is natural to ask the following question:

Question 5. Let G be a 4-connected planar graph and k an integer with $3 \leq k \leq |V(G)|$, does there exist a cycle of length k in G? $\diamond$

We define the spectrum of a graph G:

Definition 2.53. Let G be a graph on n vertices. The *spectrum* of a graph G is the maximal set $S \subseteq \{3, 4, \ldots, n\}$ of integers such that G has a cycle of all lengths $k \in S$.

If G has cycles of all lengths k for $3 \leq k \leq n$, i.e. $S = \{3, 4, \ldots, n\}$, then G is called *pancyclic*. $\diamond$

Thus, we can write the following conjecture about Question 5, which was already addressed by J A. Bondy:

Conjecture 7 (Bondy [Bon71])**.** *A planar Hamiltonian graph in which every vertex has the valency at least 4 is pancyclic.* $\diamond$

By the results in [Tho83] and [TY94], Question 5 has been answered affirmatively in case of $k \in \{|V(G)|, |V(G)|-1, |V(G)|-2\}$. A simple counting argument shows that a planar graph with minimum degree at least 4 always contains a triangle.

Assume that G is an arbitrary essentially 4-edge-connected cubic planar graph with girth at least 5. Then the line graph $L(G)$ of G is 4-regular, 4-connected, and planar. As a cycle of length 4 in $L(G)$ would correspond to a 4-cycle in G, $L(G)$ has no cycle of length 4. An example for such a graph is the dodecahedron, which refutes Conjecture 7. Beside of 4-cycles, no 4-connected planar graph without a cycle of length k for $3 \leq k \leq |V(G)|, k \neq 4$ is known.

Furthermore, by results of Chen, Fan, and Yu [CFY04] we know that using Tutte paths one cannot get to cycles of length below $n-3$ in 4-connected planar graphs on n vertices. They further proved that — assuming n is large enough — each of those graphs on n vertices contains cycles of length k for $k \in \{n, n-1, \ldots, n-6\}$ by using Tutte paths and contractible edges. Cui, Hu, and Wang [CHW09] showed that there is also always a cycle of length $n-7$.

We can summarise this in the following widely open conjecture due to Joseph Malkevitch:

Conjecture 8 (Malkevitch [Mal88])**.** *Every 4-connected planar graph with a cycle of length 4 is pancyclic.* $\diamond$

Together with Tomáš Madaras and Roman Soták in Košice (personal communication, 2017, [MS17]), we asked whether it is possible to obtain some results about the middle part of the spectrum. A possible draft of this problem is the following.

Proposition 2.54. *Let G be a 4-connected planar graph on n vertices. Then there is an integer k with $\frac{1}{3}n \leq k \leq \frac{2}{3}n$ such that G contains a cycle of length k.* $\diamond$

Proof. By Theorem 2.44, G contains a HAMILTONian cycle C that separates the plane into an interior and an exterior. Let G' be the graph obtained from G by removing all chords of C laying in the exterior. By EULER's formula, the number f of faces of G is

$$f = |E(G)| - n + 2 \geq \tfrac{1}{2}4\,n - n + 2 = n + 2.$$

Thus, we can assume without loss of generality that G' has at least $\frac{n}{2} + 1$ faces.

Let H be the *weak dual* of G', that is the dual of G' without the outer face. The graph H has at least $\frac{n}{2}$ vertices and is a tree. We assign the following weight function $w : V(H) \cup E(H) \to \mathbb{Z}$ to H:

$$\begin{aligned} w(\alpha) &:= \text{size of face } \alpha \text{ in } G, && \text{for } \alpha \in V(H), \\ w(e) &:= -2, && \text{for } e \in E(H). \end{aligned}$$

Let T be a subtree of H and take all edges of G incident with a vertex of T but not corresponding to the dual edges of T. It is easy to see that these edges represent a cycle in G and the weight $w(T) := \sum_{v \in V(T)} w(v) + \sum_{e \in E(T)} w(e)$ of T is the length of this cycle.

Thus, we transformed the problem of finding a cycle to the question for subtrees of specific weights. Assume first that H contains a vertex v of weight $\frac{1}{3}n \leq w(v) \leq \frac{2}{3}n$. Then $T := H[\{v\}]$ is the desired subtree. Since $w(H) = n$ and H has at least $\frac{n}{2}$ vertices, $w(v) \leq \frac{1}{3}n$ for all vertices $v \in V(H)$. We build at tree T starting with an arbitrary vertex $v \in V(H)$ and add iteratively vertices to T until the weight of T exceeds $\frac{n}{3}$. Then T has the desired weight and Proposition 2.54 is proved. ∎

Originating this idea, SOLOMON LO [Lo19] approached the problem of finding subtrees of weighted trees having some specific weight around the half of the total weight. He proved a rather technical theorem I will skip here and concentrate on its corollary.

Corollary 2.55 (LO [Lo19])**.** *Let G be a planar HAMILTONian graph with $\delta(G) \geq 4$, then G contains a cycle of length k for each $k \in \{\lfloor \frac{|V(G)|}{2} \rfloor, \lceil \frac{|V(G)|}{2} \rceil, \lceil \frac{|V(G)|}{2} \rceil + 1, \lceil \frac{|V(G)|}{2} \rceil + 2, \lceil \frac{|V(G)|}{2} \rceil + 3\}$.* ◇

This is a first result about the middle part of spectrum of 4-connected planar graphs. We conclude that we know that 4-connected planar graphs on n vertices — assuming n is large enough — have cycles of lengths $3, 5, \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, \frac{n}{2}+3, n{-}7, n{-}6, \dots, n{-}1, n$. One cannot guarantee that there are cycles of length 4. We like to end this section with the following problem to solve Conjecture 8:

Problem 7. Does each 4-connected planar graph G have cycles of length k for every $k \in \{3, 5, 6, \dots, |V(G)|\}$, i.e. is G almost pancyclic (cycles of length 4 may be missing)? ◇

2.6 Tutte Cycles in Non-planar Graphs

TUTTE paths have proved being a strong tool to obtain results on cycles and paths in planar graph. There were a some attempts to generalise TUTTE paths on graphs of higher genus.

For instance, Thomas and Yu [TY94] investigated Tutte paths in the projective plane. They were able to modify such graphs to obtain planar ones and apply the Tutte theory.

In the next Section 2.6.1, we first consider another possibility to use Tutte path on a graph class by reducing problems to associated planar graphs. In Section 2.6.2, a first idea about a Tutte theory on non-planar graphs is developed.

2.6.1 1-planar Graphs

In this section, we consider 1-planar graphs and obtain results about long cycles in this graph class by applying Tutte's theorem (Theorem 2.44) to certain related graphs. First, let us introduce 1-planar graphs as done in [Moh19].

All graphs can be represented by *drawings* in the plane, such that vertices are distinct points and edges are arcs, i.e. non-self-intersecting continuous Jordan curves. A graph G is *planar* if there exists a drawing of G such that two arcs only meet at end vertices. There are several approaches to generalise the concept of planarity. One of them is to allow a given constant number of crossings for each edge in a drawing. It is easy to see that a drawing can be changed locally to a different drawing with fewer crossings if two edges with a shared end vertex cross or if two edges cross several times. Thus, we can restrict our considerations to drawings with the property that if two edges cross, then they do so exactly once and their four end vertices are mutually distinct.

Definition 2.56. A graph G is *1-planar* if there exists a drawing such that each edge is crossed at most once by another edge. ⋄

This class of 1-planar graphs was introduced by Gerhard Ringel [Rin65] in connection with the simultaneous vertex-face colouring of plane graphs; properties of 1-planar graphs have been widely studied since then.

A graph G from a family $\mathcal{G}$ of graphs is *maximal* if $G + uv \notin \mathcal{G}$ for any two non-adjacent vertices $u, v \in V(G)$. In this sense, a graph is maximal 1-planar if it is 1-planar but each of the graphs $G + uv$ for non-adjacent vertices $u, v \in V(G)$ is not.

The *circumference* of a graph G, i.e. the number of vertices of a longest cycle of G is denoted by $\operatorname{circ}(G)$. If $\operatorname{circ}(G) = n$ for a graph G on n vertices, then G is *Hamiltonian* and a longest cycle of G is a *Hamiltonian cycle*.

Recall from Section 2.5.1 that there are infinitely many maximal planar graphs G with $\operatorname{circ}(G) \leq 9|V(G)|^{\log_3 2}$ by a result of Moon and Moser [MM63] and that this bound is of right magnitude as proved by Chen and Yu [CY02]. We are interested in the circumference of 3-connected maximal 1-planar graphs. In [HMS12], the question remained open whether every maximal 1-planar graph is Hamiltonian. Moreover, the question has arisen whether such a construction as the one of Moon and Moser is also possible in the class of 3-connected maximal 1-planar graphs.

The length of cycles in the class of 1-planar graphs is investigated in I. Fabrici, J. Harant, T. Madaras, S. Mohr, R. Soták, and C. T. Zamfirescu: Long cycles and spanning subgraphs of locally maximal 1-planar graphs, *Journal of Graph Theory* 95.1 (2020) [Fab+20a];

the entire paper can be found in Chapter 9. With the Theorems 9.1 and 9.5, we get the following Theorem 2.57 providing the sufficient tools for giving an answer to both questions (see also [Moh19]).

Theorem 2.57.

(i) *If H is a maximal planar graph on $n \geq 3$ vertices, then there is a 3-connected maximal 1-planar graph G on $7n - 12$ vertices such that $\operatorname{circ}(G) \leq 4 \cdot \operatorname{circ}(H)$.*

(ii) *Each 3-connected maximal 1-planar graph has a spanning 3-connected planar subgraph.*

◇

The proof of Theorem 2.57 (ii) is done by carefully removing some edges from crossings; the 3-connectedness is preserved by the fact that the graph is maximal 1-planar (see Theorem 9.5). For Theorem 2.57 (i) we refer to the proof of Theorem 9.1.

Given a 3-connected maximal planar graph H with $\operatorname{circ}(H) \leq 9|V(H)|^{\log_3 2}$, e.g. H is a graph as constructed by Moon and Moser in [MM63]. For the graph G obtained from H in Theorem 2.57 (i), $\operatorname{circ}(G) \leq c' \cdot |V(G)|^{\log_3 2}$ holds for a suitable constant c'. Hence, the circumference of these graphs is still sublinear in the same magnitude.

On the other hand, let G be a 3-connected maximal 1-planar graph on n vertices and H be a planar 3-connected spanning subgraph which must exist by Theorem 2.57 (ii). Using Chen and Yu's result, H has a cycle of order at least $c \cdot n^{\log_3 2}$. But this is also a cycle of G and immediately Corollary 2.58 follows:

Corollary 2.58. *There are positive constants c and c' such that each 3-connected maximal 1-planar graph G has $\operatorname{circ}(G) \geq c \cdot |V(G)|^{\log_3 2}$ and an infinitely family of maximal 1-planar graphs G with circumference $\operatorname{circ}(G) \leq c' \cdot |V(G)|^{\log_3 2}$ exists.* ◇

In the 4-connected case, Theorems 9.2 and 9.3 lead to the forthcoming Theorem 2.59, showing that there is a difference between the classes of planar and 1-planar graphs.

Theorem 2.59.

(i) *Each 4-connected maximal 1-planar graph is Hamiltonian.*

(ii) *There are infinitely many non-Hamiltonian 5-connected 1-planar graphs.* ◇

The proof of Theorem 2.59 (i) is a byproduct of stronger claims (Theorems 9.3 and 9.4) that are discussed in Chapter 9. Since the main idea is the same I like to present it here.

Let G be a 4-connected maximal 1-planar graph and assume that G is embedded in the plane. If $uv, xy \in E(G)$ are two crossing edges, then we can insert a new vertex z at the crossing point and replace the edges uv, xy by uz, vz, xz, yz. Since G is maximal 1-planar, it is an easy observation that for crossing edges $uv, xy \in E(G)$ all remaining four edges ux, uy, vx, vy are present in G. Thus, the vertex z cannot strongly influence the connectedness of G, and indeed, the new obtained graph is still 4-connected, see Lemma 9.6. In this way, we can construct a planar graph G' and apply Tutte's theorem, e.g. Theorem 2.44, to obtain a Hamiltonian cycle of G', which can be traced to a Hamiltonian cycle of G.

2.6.2 An Approach to a New Theory

It is known that a 1-planar graph on n vertices has at most $4n - 8$ edges [PT97]; hence, it is 7-degenerate. As a consequence, a 1-planar graph can be at most 7-connected. Figure 2.4 (right picture) presents a 7-connected 1-planar graph obtained from the 4-regular 4-connected planar graph on the left in Figure 2.4 by inserting two crossing edges in each 4-face.

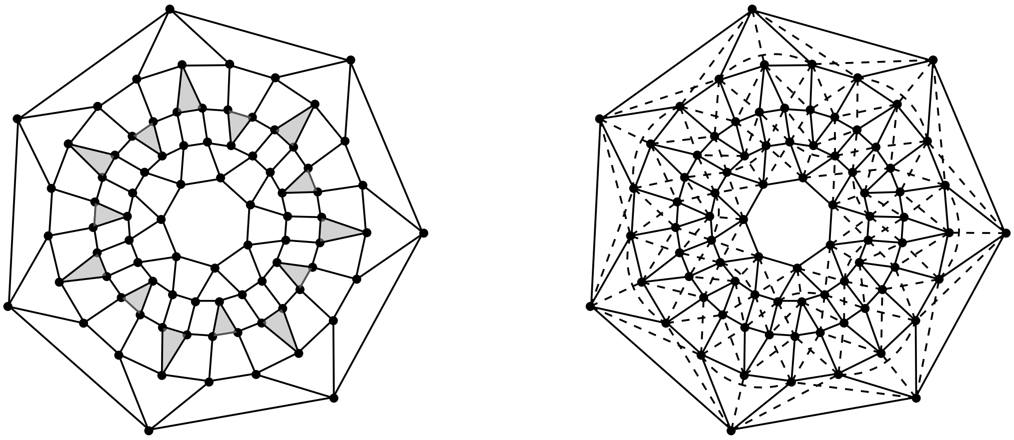


Figure 2.4: From left to right: to a 4-regular planar graph with each vertex incident to at least three 4-faces we insert two crossing edges into all 4-faces to obtain a 7-connected 1-planar graph.

In the previous section, we saw a possibility to reduce the HAMILTONian problem of maximal 1-planar graphs to the class of planar graphs and apply TUTTE theory. The paper about 1-planar graphs answers many questions as one can see in Table 9.1. But the following problem has remained open:

Problem 8. Is every 6-connected (not maximal) 1-planar graph HAMILTONian? ⋄

The following Figure 2.5 presents a 7-connected 1-planar graph embedded in the plane. If we remove an edge of each crossing, the resulting graph is not even 4-connected. Furthermore, inserting a vertex to each crossing of the 6-connected 1-planar graph without the dashed edge drawn in Figure 2.5 produces a non-4-connected graph, too.

Thus, it might be very likely that it is not possible to obtain a 4-connected planar graph from a 6-connected or even 7-connected 1-planar graph with the ideas of Chapter 9. Consequently, new ideas need to be developed to tackle Problem 8. In the following, we like to sketch one possible approach and discuss its weaknesses.

We start by recalling Definition 2.43 that defines *bridges* of a cycle C. This concept is used by W. TUTTE [Tut56] in his proof of the HAMILTONicity of 4-connected planar graphs. It is obvious that we can replace the cycle C in Definition 2.43 by any subgraph F. Given a graph G and a subgraph F of G, we define in the same manner the *F-bridges* as the minimal subgraphs H of G such that each vertex of attachment of H in G is a vertex of F and H is not a proper subgraph of F.

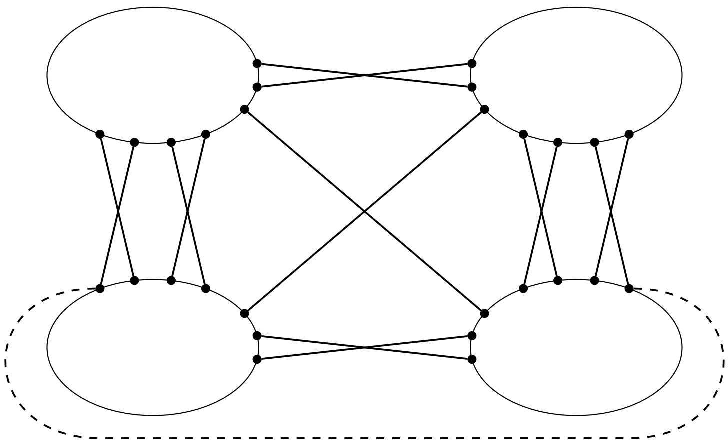


Figure 2.5: A 7-connected 1-planar graph without a planar 4-connected subgraph that can be obtained by removing crossing edges. The circles are filled by copies of the right graph in Figure 2.4.

For 2-connected planar graphs, TUTTE showed that one can find a path P such that all P-bridges do not have to many vertices of attachment. If the graph is in addition 4-connected, all P-bridges can only be edges, i.e. trivial bridges and the graph is HAMILTONian.

Lemma 2.60 (TUTTE [Tut56]). *Let G be a 2-connected planar graph with outer cycle C and v, u and e be two distinct vertices and a edge of C, respectively. Then G has a path P from v to u containing e such that*

(i) *Each P-bridge has at most three vertices of attachment.*

(ii) *Each P-bridge containing an edge of C has at most two vertices of attachment.* ⋄

To affirmatively answer Problem 8, it might be a promising approach to develop an analogous theory for 1-planar graphs. One component of the theory — as seen in Lemma 2.60 — are facial cycles, the observation that a cycle separates the plane into an exterior and an interior, and the concept of bridges allowing one to deal with induction without concerns about keeping the connectedness.

In 1-planar graphs there might be no facial cycle of the outer face; without even knowing what a face of a 1-planar graph could be. Moreover, a cycle can still be crossed by an edge connecting some subgraph in the interior with a subgraph in the exterior of a cycle. Therefore, we start with a possible definition of faces in 1-planar graphs. On that basis, we introduce a new concept of a way how to bound faces in 1-planar graphs. This will involve a path system of two parallel paths separating an interior from an exterior.

All graphs are assumed to be drawn in the EUCLIDean plane in a way such that edges are allowed to be crossed, and if an edge e is crossed, then it is crossed by exactly one edge f and the end vertices of e, f are distinct. Given a face α of a planar graph, there is not necessarily a facial cycle but instead, we can define a *facial walk* composed of the facial cycles of the blocks and the edges incident only with the face α.

For a 1-planar graph G we define an associated planar graph $G^\times$ as follows (see for example [HMS12]):

Definition 2.61. Let G be drawn in the plane. Then, $G^\times$ is the plane graph obtained from G by turning all crossings to new 4-valent vertices. If uv and xy are two crossing edges of G, then let c be the vertex of $G^\times$ corresponding to the *crossing point* of uv and xy. The edges uc and vc are the *half-edges* of uv. ⋄

Let α be a face of the planar graph $G^\times$ such that ucx is a subpath of the facial cycle of α in $G^\times$. If ux is an edge of G and ux is crossed by another edge in G, then it is possible to redraw the edge ux in G such that ux lies in a region of G corresponding to the face α of $G^\times$. It follows that ux is not crossed by another edge in G anymore. Thus, in the following we will assume that 1-planar graphs are embedded such that if uv and xy are crossing edges of G, then the edge xu (if exists) is not crossed by another edge in G.

Hereafter, if we mention a face of a 1-planar graph, we actually mean a face of $G^\times$. And an edge of G is incident with a face if the edge itself or one of its half-edges is incident with the face in $G^\times$. Note that this goes along with planar graphs, since for a planar graph G, it is $G = G^\times$.

To define the path system presenting facial boundaries, we need a few definitions.

Definition 2.62. Let G be a 1-planar graph. We call two edges uv and xy of G *parallel* if u, v, x, y are pairwise distinct and all possible six edges between these vertices can be inserted into G such that the embedding of the graph remains 1-planar and there is exactly one crossing between these six edges. ⋄

We remark that two crossed edges are always parallel as explained above.

Definition 2.63. Let G be a 1-planar graph and α be a face of G. Then there is a facial walk of α in $G^\times$ which corresponds to vertices, crossing points, non-crossing edges, and half-edges of G. We call all these objects together the *facial stripe* $\mathfrak{S}(\alpha)$ *of* α. All vertices of G in $\mathfrak{S}(\alpha)$, i.e. all vertices incident with α, are the *facial vertices* $V(\mathfrak{S}(\alpha))$ *of* α.

Let H be the graph $G^\times - V(\mathfrak{S}(\alpha))$ and β be the face of H with $\alpha \subset \beta$. Then, for each remaining component there is a facial walk of β in H which corresponds to vertices, crossing points, non-crossing edges, and half-edges of G. We call all these objects together the *outer stripe* $\mathfrak{O}(\alpha)$ *of* α. All vertices of G in $\mathfrak{O}(\alpha)$ are the *outer stripe vertices* $V(\mathfrak{O}(\alpha))$ *of* α.

The facial stripe of the outer face is also called the *boundary stripe* of G. The union of facial stripe and outer stripe of a face α is called the *α-band.* ⋄

Note that the facial stripe might not be a cycle of the graph G even if G is highly connected. The following Lemma 2.64 shows that for a face α of G the α-band separates the face from the remaining graph in a way as facial cycles of planar graphs also do. We omit the proof of the lemma here.

Lemma 2.64. *Let G be a graph and α be a face of G. Take an arbitrary 1-planar graph H and embed it in α. Insert edges to connect H to G such that the embedding remains 1-planar. Then all vertices of attachment of H in the new graph are vertices of the α-band.* ⋄

We have found useful analogues for faces and facial boundary cycles in 1-planar graph. The Definition 2.65 explains how to describe the face-incident vertices and edges in TUTTE's theorem.

Definition 2.65. Let G be a 1-planar graph with outer face α.

(i) Two vertices u and u' of G are *close* if either the edge uu' is non-crossed in G or an edge uu' can be added to G such that uu' is a non-crossing edge.

(ii) Two vertices u and u' of G are called a *band pair of* α if u and v are close and $u \in V(\mathfrak{S}(\alpha))$, i.e. u is a facial vertex of α, and $u' \in V(\mathfrak{O}(\alpha))$, i.e. u' is a outer stripe vertex of α.

(iii) Two edges e and e' of G are called *parallel edges along* α if e and e' are parallel and either

- e is an edge of the facial stripe and $V(e') \subseteq V(\mathfrak{O}(\alpha))$ or
- e and e' cross and each of them has one end vertex in $V(\mathfrak{S}(\alpha))$ and one end vertex in $V(\mathfrak{O}(\alpha))$. ◇

We are now prepared to define a bridge theory for 1-planar graphs. We state the following Conjecture 9, which implies an affirmative answer to Problem 8. It might be promising to follow the same pattern as in the proof of Theorem 2.44 by C. THOMASSEN [Tho83]. However, Conjecture 9 seems not to be obvious and easy to prove and there have appeared some difficulties while working on this conjecture. At least, the conjecture holds for 4-connected planar graphs which can be seen by applying TUTTE's theorem on the graph without outer facial cycle. Therefore, it remains a conjecture and maybe time will turn it into a theorem.

Conjecture 9. *Let G be a 4-connected 1-planar graph with outer face α and let u, u' be a band pair of vertices with $u \in V(\mathfrak{S}(\alpha))$ and $u' \in V(\mathfrak{O}(\alpha))$. Furthermore, let v, v' be another pairwise distinct band pair and let e, e' be a pair of parallel edges along α.*

Then G contains two vertex-disjoint paths connecting $U = \{u, u'\}$ with $V = \{v, v'\}$ starting with edges $uc, u'c'$ and each path contains one edge of $\{e, e'\}$. Let P be the union of these two graphs, then

(i) *each P-bridge has at most five vertices of attachment,*

(ii) *each P-bridge containing an edge of the α-band has at most four vertices of attachment, and*

(iii) *the edges $uc, u'c'$ are parallel along α or $G[u, u', c, c']$ is complete.* ◇

submitted to: *Discrete Applied Mathematics*.
S. Mohr, A construction of uniquely colourable graphs.
Available at arXiv: 2001.08801.

A Construction of Uniquely Colourable Graphs with Equal Colour Class Sizes

3

Samuel Mohr[1]

Ilmenau University of Technology, Department of Mathematics, Weimarer Straße 25, 98693 Ilmenau, Germany

A *uniquely k-colourable* graph is a graph with exactly one partition of the vertex set into at most k colour classes. Here, we investigate some constructions of uniquely k-colourable graphs and give a construction of K_k-free uniquely k-colourable graphs with equal colour class sizes.

AMS classification: 05c15.

Keywords: Uniquely colourable graphs, proper vertex colouring, critical chromatic number.

We use standard terminology from graph theory and consider *simple*, *finite* graphs G with vertex set $V(G)$ and edge set $E(G)$. A *k-colouring* of a graph G with $k \in \mathbb{N}$ is a partition $\mathcal{C}$ of the vertex set $V(G)$ into $k' \leq k$ non-empty sets $A_1, \dots, A_{k'}$. The colouring $\mathcal{C}$ is called *proper* if each set is an independent set of G, that means that there are no two adjacent vertices of G in the same *colour class* $A \in \mathcal{C}$. The *chromatic number* $\chi(G)$ is the minimum k such that there is a proper k-colouring of G.

We call a graph G *uniquely k-colourable* if $\chi(G) = k$ and for any two proper k-colourings $\mathcal{C}$ and $\mathcal{C}'$ of G, we have $\mathcal{C} = \mathcal{C}'$. It is easy to see that the complete graph K_k on k vertices is uniquely k-colourable and we can obtain a family of uniquely k-colourable graphs by consecutively adding a vertex and join it to all vertices except those of one colour class. This raises the question if all uniquely k-colourable graphs contain K_k as a subgraph.

[1] Gefördert durch die Deutsche Forschungsgemeinschaft (DFG) – 327533333.

The properties of uniquely colourable graphs have been widely studied, for example in [AMS01; BS76; CG69; EHK98; HHR69; Neš73; Xu90]. One such property — can be found in [CG69] — is that the union of any two distinct colour classes induces a connected graph. Assume to the contrary that there is a graph G with unique colouring $\mathcal{C}$ and there are $A, B \in \mathcal{C}$, $A \neq B$, such that $G[A \cup B]$ has at least two components. Let H be such a component and consider the colouring $\tilde{\mathcal{C}}$ with $\tilde{\mathcal{C}} = (\mathcal{C} \setminus \{A, B\}) \cup \{(A \setminus V(H)) \cup (B \cap V(H))\} \cup \{(B \setminus V(H)) \cup (A \cap V(H))\}$. Then $\tilde{\mathcal{C}}$ is a proper colouring of G distinct from $\mathcal{C}$, a contradiction. We say $\tilde{\mathcal{C}}$ is obtained from $\mathcal{C}$ by a Kempe *change along* H.

This implies that in a uniquely k-colourable graphs every vertex has a neighbour in every other colour class. Hence, it is connected and has minimum degree at least $k - 1$. Furthermore, a uniquely k-colourable graphs is $(k-1)$-connected. To see this, assume that there is a non-complete graph G with a unique k-colouring $\mathcal{C}$ and for two non-adjacent vertices x, y, there is a separator S with $|S| \leq k - 2$. But then there are distinct $A, B \in \mathcal{C}$ with $A \cap S = \emptyset = B \cap S$ and $(G - S)[A \cup B] = G[A \cup B]$ is connected. Since x and y have neighbours in $A \cup B$, they cannot be separated by S, a contradiction.

This question whether a uniquely k-colourable graphs always contains K_k as a subgraph was first disproved by Harary, Hedetniemi, and Robinson [HHR69]. They presented a uniquely 3-colourable graph F without triangles. For $k \geq 4$, a uniquely k-colourable graph is $F + K_{k-3}$, where $G_1 + G_2$ is the complete join of the two graphs G_1 and G_2.

Several years later, Xu [Xu90] proved that the number of edges of a uniquely k-colourable graph on n vertices is at least $(k-1)\,n - \binom{k}{2}$ and that this is best possible. He further conjectured that uniquely k-colourable graphs with exactly this number of edges have K_k as a subgraph [Xu90]. This conjecture was disproved by Akbari, Mirrokni, and Sadjad [AMS01]. They constructed a K_3-free uniquely 3-colourable graph G on 24 vertices and 45 edges. For the cases of $k \geq 4$, again $G + K_{k-3}$ disproves the conjecture.

We are interested in constructions of uniquely k-colourable graphs such that the colour classes have "nearly the same size". One useful concept for this is the critical chromatic number introduced by Komlós [Kom00] in the context of bounds on a Tiling Turán number. Given a k-colourable graph H on h vertices, let $\sigma(H)$ be the smallest possible size of a colour class in any proper k-colouring of H. Then the *critical chromatic number* is defined by

$$\chi_{cr}(H) = (\chi(H) - 1) \cdot \frac{h}{h - \sigma(H)}.$$

The critical chromatic number fulfils $\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$ and equality holds if and only if in every k-colouring of H the colour classes have the same size.

All constructions above have critical chromatic number close to $\chi(G) - 1 = k - 1$.

In the following, we give a new construction of uniquely k-colourable graphs. Given a uniquely k-colourable graph H without K_k and $\chi_{cr}(H) = \chi(H)$, this construction leads to a uniquely $(k+1)$-colourable graph G without K_{k+1} and $\chi_{cr}(G) = \chi(G)$. We further compare this construction with a result of Nešetřil [Neš73] and a probabilistic proof for the existence of uniquely colourable graphs by Bollobás and Sauer in [BS76].

Construction

Let H be a k-colourable graph with a proper k-colouring $\mathcal{C} = \{A_1, \ldots, A_k\}$. Then $G = \nu(H)$ with a proper $(k+1)$-colouring $\mathcal{C}'$ is obtained by

$$\begin{aligned} V(G) &= V(H) \cup \{v^p : v \in V(H), p = 1, \ldots, k\}, \\ E(G) &= E(H) \cup \{v^p u^p : vu \in E(H), p = 1, \ldots, k\} \\ &\qquad \cup \{vu^p, uv^p : vu \in E(H), p = 1, \ldots, k\} \\ &\qquad \cup \{v^p v^q : v \in V(H), v \in A_p, q \in \{1, \ldots, k\} \setminus \{p\}\} \end{aligned}$$

and

$$\mathcal{C}' = \{A_i' : i = 1, \ldots, k\} \cup \{\{v^p : v \in A_p, p = 1, \ldots, k\}\}$$

with $A_i' = \{v, v^p : v \in A_i, p \in \{1, \ldots, k\} \setminus \{i\}\}$.

Theorem 3.1. *Let H be a uniquely k-colourable graph with $k \geq 3$, then $\nu(H)$ is uniquely $(k+1)$-colourable.* $\diamond$

Proof. First, it is straightforward to check that $\mathcal{C}'$ is a proper colouring with $k+1$ colours. Therefore, assume that there is another colouring $\tilde{\mathcal{C}}$ with $k+1$ colours.

Fix a colour $D \in \tilde{\mathcal{C}}$. For $v \in V(H) \cap D$ consider the vertices v^q for all $q = 1, \ldots, k$. They do not have the same colour as they are not all pairwise non-adjacent. Thus, there is an index $p_v \in \{1, \ldots, k\}$ such that $v^{p_v} \notin D$. Let $v(D)$ be v if $v \notin D$ and v^{p_v} otherwise, and $X(D) = \{v(D) : v \in V(H)\}$. By the construction, $G[X(D)]$ is isomorphic to H and misses the colour D; hence it is coloured with k colours. By the hypothesis, this is the only colouring of $G[X(D)]$ and if $v(D), w(D)$ with $v, w \in V(H)$ have a common colour, then $v(D'), w(D')$ are coloured uniformly for all $D' \in \tilde{\mathcal{C}}$.

Assume first that $V(H) \cap D \neq \emptyset$ for all $D \in \tilde{\mathcal{C}}$. If there were $v, w \in V(H) \cap D$ for some $D \in \tilde{\mathcal{C}}$ such that $v(D) \in A, w(D) \in B$ for distinct $A, B \in \tilde{\mathcal{C}}$, then $v(C), w(C) \in D$ for $C \in \tilde{\mathcal{C}} \setminus \{A, B, D\}$ would have different colours, contradiction. Hence, for each $D \in \tilde{\mathcal{C}}$ there is a colour class $A^D \in \tilde{\mathcal{C}}$ such that $v(D) \in A^D$ for all $v \in V(H) \cap D$.

If there was $D \in \tilde{\mathcal{C}}$ such that $V(H) \cap A^D \neq \emptyset$, then for each $B \in \tilde{\mathcal{C}} \setminus \{A, D\}$, the induced subgraph $G[X(D) \cap (A^D \cup B)]$ would be connected. As $v(D) \in A^D$ for all $v \in V(H) \cap D$, the set $V(H) \cap (D \cup A^D)$ is independent and there is a vertex $w \in V(H) \cap B$ with $N_H(w) \cap D \neq \emptyset \neq N_H(w) \cap A^D$. This vertex w and, therefore, all w^i with $i \in \{1, \ldots, k\}$ have neighbours in all colours except B. Hence, $\{w, w^i : i \in \{1, \ldots, k\}\} \subseteq B$, contradiction.

Consequently, the vertices in $V(H)$ are coloured by k colours. Then, because of the uniquely k-colourability of H, $\{A \cap V(H) : A \in \tilde{\mathcal{C}}, A \cap V(H) \neq \emptyset\} = \mathcal{C}$. Let $D \in \tilde{\mathcal{C}}$ be the colour set with $D \cap V(H) = \emptyset$. Let $v \in V(H)$ with $v \in A \in \tilde{\mathcal{C}}$ and consider the vertices v^q, $q = 1, \ldots, k$. Since $G[(V(H) \setminus \{v\}) \cup \{v^q\}]$ is isomorphic to H and $v^p v^q \in E(G)$ for some $p \in \{1, \ldots, k\}$ and all $q \in \{1, \ldots, k\} \setminus \{p\}$, either $v^p \in A$ and $v^q \in D$ for $q \in \{1, \ldots, k\} \setminus \{p\}$, or $v^p \in D$ and $v^q \in A$. In the first case, let $w \in V(H)$ be a neighbour of v with $w \in B \in \tilde{\mathcal{C}}$ and choose $s, t \in \{1, \ldots, k\} \setminus \{p\}$ such that $s \neq t$ and $w^s w^t \in E(G)$; this is possible since $k \geq 3$. Since $G[(V(H) \setminus \{w\}) \cup \{w^s\}]$ and $G[V(H) \setminus \{w\} \cup \{w^t\}]$ are isomorphic to H, $w^s \in B, w^t \in D$ or vice versa. This is a contradiction as $v^s w^s, v^t w^t \in E(G)$ and $v^s, v^t \in D$. Therefore, we can conclude that $v^p \in D$ and $v^q \in A$, and, with the arbitrary choice of v, it follows $\tilde{\mathcal{C}} = \mathcal{C}'$; and Theorem 3.1 is proved. ■

Proposition 3.2. *The construction $G = \nu(H)$ with a k-colourable graph H on n vertices has the following properties:*

a) *G is uniquely $(k+1)$-colourable if $k \geq 3$ and H uniquely k-colourable,*

b) *$\omega(G) = \omega(H) + 1$,*

c) *$\chi_{cr}(G) = \chi(G)$ if $\chi_{cr}(H) = \chi(H)$,*

d) *$|E(G)| = (3k+1)\,|E(H)| + (k-1)\,n$ and $|V(G)| = (k+1)\,n$,*

e) *The minimum degree of G is $2\,\delta(H) + 1$,*

f) *H is an induced subgraph of G.* ⋄

Proof. By Theorem 3.1 we showed *a)* and it is easy to see *f)* from the construction. Simply counting vertices and edges leads to *d)*.

To show *b)*, let C be a maximum clique of G.

Assume that there are distinct $r, s, t \in \{1, \ldots, k\}$ and $u, v, w \in V(H)$ such that $u^r, v^s, w^t \in C$. Since these three vertices belong to a triangle, it is $u = v = w$, but these vertices can only induce a path. Hence, assume next that there are distinct $r, s \in \{1, \ldots, k\}$ and distinct $v, w \in V(H)$ with $v^r, w^r, w^s \in C$. But then $v^r w^s \notin E(G)$.

Thus, assume that $C \subseteq V(H) \cup \{v^p : v \in V(H)\}$ for some fixed $p \in \{1, \ldots, k\}$. Then there is no $v \in V(H)$ with $v, v^p \in C$ because $vv^p \notin E(G)$. But then, $\{v : v \in C \text{ or } v^p \in C\}$ is a clique of the same size.

Hence, there are adjacent $v^p, v^q \in C$ and $C \setminus \{v^p, v^q\} \subseteq V(H)$. Then $C \setminus \{v^p, v^q\} \cup \{v\}$ is a clique of size $\omega(G) - 1$ of H. On the other hand, replacing v from a clique of H with distinct and adjacent v^p, v^q, we get a clique of G; and *b)* is proved.

If $\chi_{cr}(H) = \chi(H)$, then each colour set of H has size $s = \frac{|V(H)|}{\chi(H)}$. By the construction of the colouring, each colour set of G has size $k \cdot s$. Since $\mathcal{C}'$ is the only colouring, we get $\chi_{cr}(G) = \chi(G)$ and we have proved *c)*.

By the construction, we obtain the following degree function for $v \in V(G)$, which shows *e)*.

$$d_G(x) = \begin{cases} (k+1)\,d_H(v), & \text{if } x = v \in V(H), \\ 2\,d_H(v) + k - 1, & \text{if } x = v^p \text{ and } v \in A_p, \\ 2\,d_H(v) + 1, & \text{if } x = v^p \text{ and } v \notin A_p. \end{cases}$$ ∎

Small triangle-free uniquely 3-colourable graphs

Using some computer calculation, Figure 3.1 shows a graph on 12 vertices and 22 solid drawn edges. Adding at most one of the dashed edges, we obtain a list of three non-isomorphic uniquely 3-colourable triangle-free graphs on 12 vertices with critical chromatic number 3.

Corollary 3.3. *For all $k \geq 3$ there are uniquely k-colourable K_k-free graphs on $2 \cdot k!$ vertices with critical chromatic number k.* ⋄

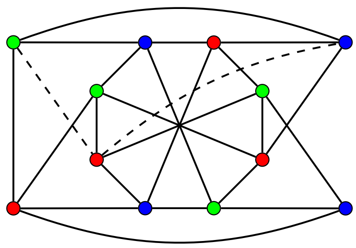


Figure 3.1: Non-isomorphic uniquely 3-colourable triangle-free graphs on 12 vertices with critical chromatic number 3

Proof. For $k = 3$, it is straightforward to check that the graphs in Figure 3.1 are uniquely 3-colourable, triangle-free, have 12 vertices and critical chromatic number 3.

For $k > 3$, iteratively apply the construction ν to one of the graphs in Figure 3.1 to obtain a k-colourable graph. All demanded properties follow from Proposition 3.2. □

Probably the graphs in Figure 3.1 are the only non-isomorphic uniquely 3-colourable triangle-free graphs with critical chromatic number 3 on 12 vertices. But to verify the calculations, it should be necessary to start a second independent implementation in another programming language. Therefore, we do not like to pretend that Figure 3.1 shows all of them. We conclude this section by showing that they are smallest possible.

Proposition 3.4. *The graphs in Figure 3.1 have smallest number of vertices among all uniquely 3-colourable triangle-free graphs with critical chromatic number 3.* ⋄

Proof. Since the critical chromatic number is 3, the number of vertices n of such graphs has to be divisible by 3. We left the case $n=3$ and $n=6$ to the reader and assume that there is such a graph G with $n=9$ vertices.

As mentioned above, each two colour classes induce a connected subgraph. Therefore let A and B be two colour classes of G, then $|A| = |B| = 3$ and $G[A \cup B]$ connected. If there was a vertex $v \in A$ with degree 3 in $G[A \cup B]$, then each neighbour $w \in C$ of v in the third colour class C would form a triangle with v and a suitable vertex from B. Hence $G[A \cup B]$ is either a path or a cycle on six vertices. In both cases, at least two vertices in A have degree 2 in $G[A \cup B]$. By a symmetric argument, there are two vertices in A having degree 2 in $G[A \cup C]$ and two vertices in B having degree 2 in $G[B \cup C]$. Thus, there is a vertex $v \in A$ with a neighbour $w \in B$ such that there is a common neighbour $u \in C$ of v and w; and we obtain a triangle on $\{u, v, w\}$, a contradiction. ■

Comparison with other results

In this section, we compare our new construction of uniquely colourable graphs with a straightforward construction, with a triangle-free construction by Nešetřil [Neš73], and a probabilistic proof by Bollobás and Sauer [BS76], which forces an arbitrary girth. To obtain, for an integer $k \geq 3$, a uniquely k-colourable K_k-free graphs with equal colour class sizes, none of the following constructions is suitable. To our best knowledge, there are no such other suitable constructions yet; thus, the construction ν and Corollary 3.3 seem to fill a gap.

Construction 1:

The complete graph K_k on k vertices is uniquely k-colourable. Given a uniquely k-colourable graph H, adding a new vertex and joining it to all vertices of H except those of one colour class, we obtain a new uniquely k-colourable graph. This is a way for increasing the critical chromatic number of uniquely k-colourable graphs by choosing repeatedly a smallest colour class.

However, if there is a clique in the original graph H containing a vertex from each non-maximal colour class, then we obtain a K_k after balancing all colour class sizes as described above. Hence, depending on our starting graph H, in some cases we cannot obtain a uniquely k-colourable K_k-free graphs with equal colour class sizes using Construction 1.

Construction 2 (Nešetřil [Neš73]):

To get a uniquely k-colourable graph, $k \geq 2$, choose $n > 16\,k \cdot (2(k-2))^{2k-1}$ and start with the uniquely 2-colourable path $P_n^0 = P_n$ on n vertices and colour classes A_1, A_2. The uniquely k-colourable graph $P_n^{(k-2)}$ is constructed iteratively. Assume that $P_n^{(j-1)}$, $j \geq 1$ with colour classes $A_1, \ldots, A_{(j+1)}$ is constructed and let $\mathcal{M}^j$ be the set of all independent sets M of $P_n^{(j-1)}$ with $|M| = j+2$ such that $M \cap A_i \neq \emptyset$, $1 \leq i \leq j+1$. Then $V(P_n^j) = V(P_n^{(j-1)}) \cup \mathcal{M}^j$ and $xy \in E(P_n^j)$ if $xy \in E(P_n^{(j-1)})$ or $x \in y \in \mathcal{M}^j$. The new colour class is $A_{(j+2)} = \mathcal{M}^j$.

By this construction, we obtain a triangle-free graph G that is uniquely k-colourable with a colouring $\mathcal{C} = \{A_1, \ldots, A_k\}$. It is $|A_1| = |A_2| = \Theta(n)$, $|A_3| = \Theta(n^3)$, $|A_4| = \Theta(n^8)$, ...

Thus, the size of the colour classes differs; moreover, the critical chromatic number tends to $k-1$ for $k \to \infty$.

Construction 3 (Bollobás, Sauer [BS76]):

Bollobás and Sauer used a probabilistic approach to show the existence of uniquely k-colourable graphs with an arbitrary minimum value g for the girth. To this extent, they started with k-partite graphs, each partition of size n and $m = \binom{k}{2} n^{1+\varepsilon}$ uniformly chosen edges with $0 < \varepsilon < \frac{1}{4g}$.

It is shown that many of these graphs contain only few cycles of length smaller than g and these cycles do not share a vertex. By removing a few edges to destroy these short cycles, most graphs are still uniquely k-colourable and we obtain the existence of a demanded graph.

Choosing $g = 4$ and analysing their arguments, there exists a uniquely k-colourable triangle-free graph on $\Theta(k^{129})$ vertices. However, this does not yield an explicit construction.

Acknowledgement

The author thanks Matthias Kriesell for his valuable suggestions.

published in: M. Kriesell and S. Mohr, Rooted complete minors in line graphs with a Kempe coloring, *Graphs and Combinatorics* 35.**2** (2019), pages 551–557.
Available at doi: 10.1007/s00373-019-02012-7, arXiv: 1804.06641.

Rooted Complete Minors in Line Graphs With a Kempe Coloring

4

Matthias Kriesell and Samuel Mohr[1]

Ilmenau University of Technology, Department of Mathematics, Ilmenau, Germany

It has been conjectured that if a finite graph has a vertex coloring such that the union of any two color classes induces a connected graph, then for every set T of vertices containing exactly one member from each color class there exists a complete minor such that T contains exactly one member from each branching set. Here we prove the statement for line graphs.

AMS classification: 05c15, 05c40.

Keywords: coloring, clique minor, Kempe coloring, line graph.

Hadwiger's Conjecture states that the order $h(G)$ of a largest clique minor in a graph G is at least its chromatic number $\chi(G)$ [Had43]. It is known to be true for graphs with $\chi(G) \leq 6$, where for $\chi(G) = 5$ or $\chi(G) = 6$, we have equivalence to the Four-Color-Theorem, respectively [RST93]. Instead of restricting the *number of color classes*, one could also uniformly bound the *order of the color classes*, but even when forbidding anticliques of order 3 (which bounds these orders by 2), the problem is wide open (cf. [Sey16]). In [Kri17], the first author suggested to bound the *number of colorings*, in particular to consider uniquely optimally colorable graphs; if G is uniquely k-colorable and $x_1, \dots, x_k$ have different colors, then it is easy to see that there exists a system of edge-disjoint x_i,x_j-paths ($i \neq j$ from $\{1, \dots, k\}$), a so-called *(weak) clique immersion* of order k at $x_1, \dots, x_k$, and the question suggests itself whether there exists a *clique minor* of the same order such that $x_1, \dots, x_k$ are in different bags. This has been answered affirmatively in [Kri17] if one forbids antitriangles

[1] Gefördert durch die Deutsche Forschungsgemeinschaft (DFG) – 327533333.

in G. The present paper gives an affirmative answer in the case that G is a line graph. It should be mentioned that HADWIGER's Conjecture is known to be true for line graphs in general by a result of REED and SEYMOUR [RS04], but it seems that their argument leaves no freedom for prescribing vertices in the clique minor at the expense of forcing any pair of color classes to be connected.

*

All *graphs* considered here are supposed to be finite, undirected, and loopless. They may contain parallel edges, graphs without these are called *simple*. For graph terminology not defined here, we refer to [Die17]. A *clique minor* of G is a set of connected, nonempty, pairwise disjoint, pairwise adjacent subsets of $V(G)$ (the *branching sets*), where a set $A \subseteq V(G)$ is *connected* if the subgraph $G[A]$ induced by A in G is connected, and disjoint $A, B \subseteq V(G)$ are *adjacent* if some vertex of A is adjacent to some vertex of B. An *anticlique* of G is a set of pairwise nonadjacent vertices. A *coloring* of a graph G is a partition $\mathfrak{C}$ into anticliques, the *color classes*, and we call it a KEMPE *coloring* if the union of any two of these induces a connected subgraph in G. Throughout, a (minimal) *transversal* of a set $\mathfrak{S}$ of pairwise disjoint sets is a set $T \subseteq \bigcup \mathfrak{S}$ such that $|T \cap A| = 1$ for all $A \in \mathfrak{S}$; in this case, we also say that $\mathfrak{S}$ is *traversed by* T. The *line graph* $L(H)$ of some graph H is the (simple) graph with vertex set $E(H)$ where two distinct vertices are adjacent if and only they are incident (as edges) in H.

The first author conjectured in [Kri17] that for every transversal T of every KEMPE coloring of a graph G there exists a complete minor in G traversed by T. Here we prove the conjecture for line graphs.

Theorem 4.1. *For every transversal of every* KEMPE *coloring of the line graph $L(H)$ of any graph H there exists a complete minor in $L(H)$ traversed by T.* ⋄

Of course this statement can be fomulated entirely without addressing to line graphs. Call a set F of edges of a graph H *connected* if any two of them are on a path of edges from F, and call two sets F, F' of edges *incident* if some edge of F is incident with some edge of F'. Theorem 4.1 translates as follows:

Theorem 4.2. *Let H be a graph and $\mathfrak{C}$ be a partition of $E(H)$ into (not necessarily maximum) matchings such that the union of any two of them is connected. Then for every transversal T of $\mathfrak{C}$ there exists a set of connected, pairwise disjoint, pairwise incident edge sets traversed by T.* ⋄

*

As our proof of Theorem 4.2 uses contraction at some places, it is reasonable to allow multiple edges (but no loops) in H. However, the precondition of Theorem 4.2 imposes a very special structure on H as soon as H contains a pair of parallel edges. We thus prefer to give a separate, simple proof for this situation instead of handling parallel edges in the proof of Theorem 4.2. Given a graph H, let us say that $F \subseteq E(H)$ *covers* $v \in V(H)$ if v is incident with at least one edge from F. By $E_H(v)$ we denote the set of all edges incident with $v \in V(H)$.

Lemma 4.3. *Let H be a graph with a pair of parallel edges and $\mathfrak{C}$ be a partition of $E(H)$ into (not necessarily maximum) matchings such that the union of any two of them is connected. Then for every transversal T of $\mathfrak{C}$, there exists a set of connected, pairwise disjoint, pairwise incident edge sets traversed by T.* ◇

Proof. Let e, f be parallel edges of H. They are in different matchings M_e, M_f of $\mathfrak{C}$, and they form a cycle of length 2, so that $M_e = \{e\}$ and $M_f = \{f\}$. Let $M \in \mathfrak{C} \setminus \{M_e, M_f\}$. As every edge in M is incident with some edge of M_e, every edge from M (and hence every edge from H) is incident with e, f, implying that M contains at most two edges. If $|M| = 1$, say, $M = \{g\}$, then, likewise, g is incident with every other edge of H and we apply induction to $H - g$, $\mathfrak{C} \setminus \{M\}$, and $T \setminus \{g\}$ as to find a set $\mathfrak{K}$ of connected, pairwise disjoint, pairwise incident edge sets traversed by T, and $\mathfrak{K} \cup \{\{g\}\}$ proves the statement for H (and $\mathfrak{C}, T$). — So we may assume that every M distinct from M_e, M_f consists of two edges. In particular, e, f is the only pair of parallel edges in H. Let x, y be the end vertices of e, f, and let $M_i = \{xa_i, yb_i\}$, $i \in \{1, \dots, \ell\}$ be the matchings from $\mathfrak{C} \setminus \{M_e, M_f\}$. The a_i are pairwise distinct, and so are the b_i. For $i \neq j$ from $\{1, \dots, \ell\}$, at least one of $a_i = b_j$ or $b_i = a_j$ holds since $M_i \cup M_j$ is connected. Since a_i, b_i have degree at most 2, for $i \in \{1, \dots, \ell\}$, there exists at most one $j \neq i$ with $a_i = b_j$ and at most one $j \neq i$ with $b_i = a_j$. It follows that $\ell \leq 3$. We may assume that the $\ell + 2$ transversal edges from T are not pairwise incident, as in this case $\{\{g\} : g \in T\}$ proves the statement. In particular, $\ell \geq 2$, and for $i \neq j$ only one of $a_i = b_j$ and $b_i = a_j$ holds, as otherwise $M_i \cup M_j$ induce a 4-cycle, $\ell = 2$, and the four transversal edges are pairwise incident. Hence, we may assume that if $\ell = 2$, then $b_1 = a_2$ and $T = \{e, f, xa_1, yb_2\}$ without loss of generality, and if $\ell = 3$, then $b_1 = a_2, b_2 = a_3, b_3 = a_1$ and $T = \{e, f, xa_1, yb_2, yb_3\}$ without loss of generality. In either case, $\{\{e\}, \{f\}, \{xa_1, xa_2, yb_1\}, \{yb_2\}, \dots, \{yb_\ell\}\}$ proves the statement. ■

If $\mathfrak{C}$ is the only coloring of order k in a graph G, then $\mathfrak{C}$ is a Kempe coloring, and there is no coloring with fewer than k colors (so k equals the chromatic number $\chi(G)$ of G). Even in this very special case, we do not know whether G has a complete minor of order k (disregarding transversals), that is, we do not know if Hadwiger's Conjecture [Had43] is true for uniquely k-colorable graphs. However, the situation for uniquely k-colorable line graphs is almost completely trivial, as for $k \geq 4$, the star $K_{1,k}$ is the only uniquely k-edge-colorable simple graph [Tho78] (and the non-simple ones are covered by Lemma 4.3).

At this point, one may wonder if the graphs considered in Theorem 4.2 are "rare". Let us show that this is not the case. A partition of the edge set of a graph into (perfect) matchings such that the union of any two of them induces a Hamiltonian cycle in G is called a *perfect 1-factorization*. Clearly, every graph with a perfect 1-factorization is k-regular and meets the assumptions of Theorem 4.2. There is an old conjecture by Kotzig stating that every complete graph of even order has a perfect 1-factorization [Kot63], indicating that it is difficult to determine whether a graph has a perfect 1-factorization. However, for our purposes it suffices to construct some variety of graphs which have one, as done in (A), (B) below.

(A) Given k, let $a = (a_1, \dots, a_k)$ be a sequence of pairwise distinct nonnegative integers and take m larger than all of these and relatively prime to any possible difference $a_i - a_j$, $i \neq j$; for example, just take a large prime number. Let $V = \mathbb{Z}_m \times \{0, 1\}$, set $M_i := \{(z, 0)(z + \overline{a_i}, 1) : z \in \mathbb{Z}_m\}$, where $\overline{a}$ denotes the residual class modulo m containing $a \in \mathbb{Z}$, and let $E := M_1 \cup \dots \cup M_k$. The resulting graph $(V, E) =: H(m, a) =: H$ is bipartite, k-regular, and $\{M_1, \dots, M_k\}$ is a partition of E into perfect matchings. The graph induced by the union of $M_i \cup M_j$ thus decomposes into cycles. Take a vertex $(z, 0)$ from such a cycle C; by following the M_i-edge, we reach $(z + \overline{a_i}, 1)$, and by following the M_j-edge from there we reach $(z + \overline{a_i} - \overline{a_j}, 0)$. So C contains all vertices of the form $(z + \overline{t \cdot (a_i - a_j)}, 0)$, $t \in \mathbb{Z}$, from $\mathbb{Z}_m \times \{0\}$, and since $a_i - a_j$ and m are relatively prime, it contains *all* vertices from $\mathbb{Z}_m \times \{0\}$. Consequently, C is a Hamiltonian cycle, so H admits a perfect 1-factorization of order k.

(B) For $i \in \{1, 2\}$, take any graph H_i with a perfect 1-factorization $\mathfrak{C}_i$ of order k. Take a vertex v_i in $V(H_i)$ and consider the edges $e_{i,1}, \ldots, e_{i,k}$ incident with v_i. Let $M_{i,j}$ be the member from $\mathfrak{C}_i$ containing $e_{i,j}$. Assuming that H_1, H_2 are disjoint, let H be obtained from the union of $H_1 - v_1$ and $H_2 - v_2$ by adding a new edge f_j from the end vertex of $e_{1,j}$ distinct from v_1 to the end vertex of $e_{2,j}$ distinct from v_2 for each $j \in \{1, \ldots, k\}$. Then $M_j := ((M_{1,j} \cup M_{2,j}) \setminus \{e_{1j}, e_{2,j}\}) \cup \{f_j\}$ for $j \in \{1, \ldots, k\}$ defines a perfect matching of H, and the union of any two of these induces a Hamiltonian cycleof H, so H has a perfect 1-factorization.

(C) If one deletes a vertex of any graph with a perfect 1-factorization, then the edge set of the resulting graph has an (obvious) partition into matchings such that the union of any two of them is a Hamiltonian path; so we get further graphs meeting the assumptions of Theorem 4.2 this way.

*

Back to Theorem 4.2, let us now consider the case that H is a complete graph. It turns out that for *any* set T of n edges (not necessarily being a transversal of some set of matchings as in Theorem 4.2) we can find connected, pairwise disjoint, pairwise incident edge sets traversed by T.

Lemma 4.4. *For every set T of n edges of the simple complete graph H on $n \geq 3$ vertices, there exists a set of connected, pairwise disjoint, pairwise incident edge sets traversed by T.* ⋄

Proof. For $n = 3$, the statement is obviously true. For $n > 3$, consider the subgraph $H[T] := (V(H), T)$ induced by T. It has average degree 2 and, therefore, a vertex v with at most two neighbors in $H[T]$.

If v has exactly one neighbor x in $H[T]$, then we apply induction to $H - v$ and find a set $\mathfrak{K}$ of $n-1$ connected, pairwise disjoint, pairwise incident edge sets traversed by $T \setminus \{vx\}$, and $\mathfrak{K} \cup \{E_H(v)\}$ proves the statement for H.

If v has exactly two neighbors x, y in $H[T]$, then we may assume that x is not incident with all the $n-2$ edges from $T \setminus \{vx, vy\}$, since otherwise these would form a spanning star in $H - v$ and one of the leaves of this star had degree 1 in $H[T]$ since $n > 3$ — a case which we have just considered. Therefore, there exists an edge xz in $E(H - v) \setminus T$. Induction applied to $H - v$ provides a set $\mathfrak{K}$ of $n-1$ connected, pairwise disjoint, pairwise incident edges traversed by $(T \setminus \{vx, vy\}) \cup \{xz\}$. Let F be the member of $\mathfrak{K}$ containing xz. The set $E_H(v) \setminus \{vx\}$ is incident to all of $\mathfrak{K}$ in H, as each of these cover at least two neighbors of v, so that $(\mathfrak{K} \setminus \{F\}) \cup \{F \cup \{vx\}, E_H(v) \setminus \{vx\}\}$ proves the statement for H.

If v has no neighbors in $H[T]$ and xy is any edge in T, then by induction there exists a set $\mathfrak{K}$ of $n-1$ connected, pairwise incident edges traversed by $T \setminus \{xy\}$ in $H - v$. If xy is not an edge of any member of $\mathfrak{K}$, then $\mathfrak{K} \cup \{E_H(v) \cup \{xy\}\}$ proves the statement for H. Otherwise, there is an $F \in \mathfrak{K}$ with $xy \in F$. Let $wz \neq xy$ be the edge from T contained in F, where we may assume that $w \notin \{x, y\}$. By symmetry, we may assume that w is in the component of $H[F] - xy$ containing x, so that $F' := (F \setminus \{xy\}) \cup \{vw, vy\}$ is connected and covers v and all vertices covered by F. The set $F'' := (E_H(v) \setminus \{vw, vy\}) \cup \{xy\}$ is connected and covers all vertices of H except for w, so that it is incident with F' and all sets from $\mathfrak{K}$ as each of these cover at least two neighbors of v. Consequently, $(\mathfrak{K} \setminus \{F\}) \cup \{F', F''\}$ proves the statement for H. ■

It remains open if Lemma 4.4 is best possible in the sense that we cannot prescribe a set T of more than n edges there. For $n = 3$ and $n = 4$, optimality is easy to check, for $n = 5$, one cannot prescribe six edges if they form a subgraph $K_{2,3}$. In general, for $n > 2$, one cannot prescribe $2n - 2$ edges if they form a graph $K_{2,n-2}$ plus two edges, one of them connecting the two vertices of degree $n - 2$ in $K_{2,n-2}$, but there should be better bounds.

*

Finishing preparation for the proof of our main result, let us recall and slightly extend Lemmas from [Kri17] and [Kri01].

Lemma 4.5 (Kriesell [Kri17]). *Suppose that $\mathfrak{C}$ is a Kempe coloring of order k of a graph G and let $S \subseteq V(G)$ be a separating set. Then **(i)** if $F \in \mathfrak{C}$ does not contain any vertex from S, then it contains a vertex from every component of $G - S$, and **(ii)** S contains vertices from at least $k-1$ members of $\mathfrak{C}$.* ◇

Proof. Let C, D be distinct components of $G - S$ and let $F \in \mathfrak{C}$ with $F \cap S = \emptyset$. Suppose, to the contrary, that $F \cap V(C) = \emptyset$. Take any vertex $x \in V(C)$ and the set A from $\mathfrak{C}$ containing x. Then x has no neighbors in F, so that $G[A \cup F]$ is not connected, contradiction. This proves **(i)**. For **(ii)**, suppose, to the contrary, that there exists $F \neq F'$ from $\mathfrak{C}$ with $F \cap S = \emptyset$ and $F' \cap S = \emptyset$. By **(i)**, there exist $x \in F \cap V(C)$ and $y \in F' \cap V(D)$, but no x,y-path in G avoiding S and, hence, no x,y-path in $G[F \cup F']$, contradiction. □

In particular, every graph with a Kempe coloring of order k must be $(k-1)$-connected. We repeat the following Lemma (and its proof) from [Kri01].

Lemma 4.6 (Kriesell [Kri01]). *Suppose that H is a graph such that $L(H)$ is k-connected. Then for all distinct vertices a, b of degree at least k, there exist k edge-disjoint a,b-paths in H.* ◇

Proof. If there were no such paths, then, by Menger's Theorem (cf. [Die17]), there exists an a,b-cut S in H with less than k edges. We may assume that S is a minimal a,b-cut, implying that $H - S$ has exactly two components C, D, where $a \in V(C)$ and $b \in V(D)$. Since both a, b have degree at least k, there exists an edge $e \in E_H(a) \setminus S$, that is, $e \in E(C)$, and at least one edge $f \in E(D)$. But then S separates e from f in H and, thus, e from f in $L(H)$, contradicting the assumption that $L(H)$ is k-connected. □

We are now ready to prove our main result. At some places, we will contract some subgraph X of H to a single vertex. In order to make object references easier, we choose a graph model where the edge set of the resulting graph actually *equals* $E(H) \setminus E(X)$, not just "corresponds to $E(H) \setminus E(X)$" in whatever way.

Proof of Theorem 4.2. We proceed by induction on $|E(H)|$. Let $\mathfrak{C}$ be a partition of $E(H)$ into matchings such that the union of any two of them is connected, set $k := |\mathfrak{C}|$, and let T be a transversal of $\mathfrak{C}$. We have to show that there exists a set of k connected, pairwise disjoint, pairwise incident edge sets traversed by T in H. Observe that the statement is easy to prove whenever $k \leq 2$. Hence, we may assume $k \geq 3$. In particular, $|E(H)| \geq 3$. By Lemma 4.3 we may assume that H is simple. Observe that $\mathfrak{C}$ is a Kempe coloring of $L(H)$.

As $\mathfrak{C}$ is a partition of $E(H)$ into k matchings, all vertices in H have degree at most k. Suppose first that there is a vertex v of degree k in H. Then $U := E_H(v)$ induces a clique of order k in $L(H)$. If there is a set of k disjoint U,T-paths in $L(H)$, then their vertex sets form a clique minor in $L(H)$ and, at the same time, a set of k connected, pairwise disjoint, pairwise incident edge sets in H, traversed by T. So we may assume that there are no k disjoint U,T-paths in $L(H)$. By Menger's Theorem (cf. [Die17]), there exists a vertex set S in $L(H)$ with $|S| < k$ separating U from T, and we may take a smallest such set, implying that $L(H) - S$ has only two components C, D (since for every vertex $v \in S$, $N_{L(H)}(v)$ contains a vertex from each component of $L(H) - S$ but is, at the same time, the union of at most two cliques of $L(H)$). We may assume that, say, C contains at least one vertex from U as $|U| > |S|$. But then C contains no vertex from T, and D contains at least one vertex from T and no vertex from U. Back in H, the set S is a minimal cut in H and $H - S$ has exactly two components, C', D', where $E(C') = V(C)$ and $E(D') = V(D)$. As $E(C')$ contains an edge from U, $v \in V(C')$ follows.

From Lemma 4.5 we know that S consists of $k-1$ objects, all coming from distinct members of $\mathfrak{C}$. Let F be the unique member from $\mathfrak{C}$ with $F \cap S = \emptyset$. Let H' be the graph obtained from H by contracting D' to a single vertex w. Since $F \in \mathfrak{C}$ contains an edge from $E(C')$ by Lemma 4.5 and all other classes contain an edge from S, we obtain a partition of $E(H')$ into matchings such that the union of any two of them is connected, by deleting all edges of $E(D')$ from their sets in $\mathfrak{C}$. Thus, we also have a Kempe coloring of $L(H')$, so that, by Lemma 4.5, $L(H')$ is $(k-1)$-connected. Since v has degree at least k and w has degree $k-1$ in H', there exist $k-1$ edge-disjoint v,w-paths in H' by Lemma 4.6. For $e \in S$, let P_e be the path among these containing e.

Now let H'' be the graph obtained from H by contracting C' to a single vertex (recall that $E(C') \neq \emptyset$). As above, by deleting all edges of $E(C')$ from their sets in $\mathfrak{C}$, we obtain a partition of $E(H'')$ into matchings such that the union of any two of them is connected. Moreover, T remains a transversal of the modified partition. Since $|E(H'')| < |E(H)|$, we may apply induction to H'' and find a set $\mathfrak{K}$ of connected, pairwise disjoint, pairwise incident edge sets traversed by T in H''. Setting $A' := A \cup \bigcup\{E(P_e) : e \in A,\, e \in S\}$ for $A \in \mathfrak{K}$, one readily checks that $\mathfrak{K}' := \{A' : A \in \mathfrak{K}\}$ proves the statement of the theorem for H.

Therefore, we may assume from now on that the maximum degree Δ of H is at most $k-1$. Let δ denote the minimum degree of H. Every edge xy is incident with at least one edge from each of the $k-1$ members of $\mathfrak{C}$ not containing xy, so that $d_H(x) + d_H(y) \geq k+1$. Consequently, $\delta \geq k+1-\Delta$ and $\Delta \geq (k+1)/2$. For distinct A, B from $\mathfrak{C}$, consider the subgraph $H(A,B)$ formed by all edges of $A \cup B$. It is either a path or a cycle, and we say that $H(A,B)$ *ends* in a vertex v if v has degree 1 in $H(A,B)$, or alternatively, if v is covered by exactly one edge of A, B. Now, if v has degree d in H, then it is covered by exactly d of the k matchings from $\mathfrak{C}$, so that exactly $d \cdot (k-d)$ of the subgraphs $H(A,B)$ end in v. Observe that $k - \Delta < \delta \leq d \leq \Delta$ and consider the quadratic function f defined by $f(d) := d \cdot (k-d) - \Delta \cdot (k-\Delta)$ with zeroes Δ and $k-\Delta$. We get $f(d) \geq 0$, that is, $d \cdot (k-d) \geq \Delta \cdot (k-\Delta)$, for $d \in (k-\Delta, \Delta]$ with equality only if $d = \Delta$. Consequently, in each vertex v, for at least $\Delta \cdot (k-\Delta) > 0$ pairs $A, B \in \mathfrak{C}$, the graph $H(A,B)$ ends in v. As there are only $\binom{k}{2}$ many subgraphs $H(A,B)$ and as each of them ends in two or zero vertices, we get

$|V(H)| \cdot \Delta \cdot (k - \Delta) \leq k \cdot (k - 1)$. Moreover, $|V(H)| \geq \Delta + 1$ since H is simple, so

$$(\Delta + 1) \cdot \Delta \cdot (k - \Delta) \leq k \cdot (k - 1). \tag{4.1}$$

Consider the cubic function g defined by $g(\Delta) := (\Delta + 1) \cdot \Delta \cdot (k - \Delta) - k \cdot (k - 1)$. It has zeros $k-1$ and $\pm\sqrt{k}$, so that it is positive for $\Delta \in (\sqrt{k}, k - 1)$, that is, $(\Delta + 1) \cdot \Delta \cdot (k - \Delta) > k \cdot (k - 1)$ for all $\Delta \in [(k + 1)/2, k - 1)$ as $k \geq 3$. Since $\Delta \in [(k + 1)/2, k - 1]$ and (4.1) holds, this necessarily implies $\Delta = k - 1$ and equality in (4.1). Backtracking through the arguments leading to (4.1) yields, subsequently: $|V(H)| = \Delta + 1$; in each vertex v, exactly $\Delta \cdot (k - \Delta)$ of the $H(A, B)$ end; and, finally, each vertex of H has degree $d = \Delta$.

It follows that H is the simple complete graph on $\Delta + 1 = k \geq 3$ vertices, and we obtain the statement of Theorem 4.2 for H from Lemma 4.4. ∎

submitted to: *Journal of Graph Theory*.
M. Kriesell and S. Mohr, Kempe chains and rooted minors.
Available at arXiv: 1911.09998.

Kempe Chains and Rooted Minors

5

Matthias Kriesell and Samuel Mohr[1]

Ilmenau University of Technology, Department of Mathematics, Ilmenau, Germany

A (minimal) *transversal* of a partition is a set which contains exactly one element from each member of the partition and nothing else. A *coloring* of a graph is a partition of its vertex set into anticliques, that is, sets of pairwise nonadjacent vertices. We study the following problem: Given a transversal T of a proper coloring $\mathfrak{C}$ of some graph G, is there a partition $\mathfrak{H}$ of a subset of $V(G)$ into connected sets such that T is a transversal of $\mathfrak{H}$ and such that two sets of $\mathfrak{H}$ are adjacent if their corresponding vertices from T are connected by a path in G using only two colors?

It has been conjectured by the first author that for any transversal T of a coloring $\mathfrak{C}$ of order k of some graph G such that any pair of color classes induces a connected graph, there exists such a partition $\mathfrak{H}$ with pairwise adjacent sets (which would prove Hadwiger's conjecture for the class of uniquely optimally colorable graphs); this is open for each $k \geq 5$, here we give a proof for the case that $k = 5$ and the subgraph induced by T is connected. Moreover, we show that for $k \geq 7$, it is not sufficient for the existence of $\mathfrak{H}$ as above just to force any two transversal vertices to be connected by a 2-colored path.

AMS classification: 05c40, 05c15.

Keywords: Kempe chain, rooted minor, Hadwiger's conjecture.

[1]Gefördert durch die Deutsche Forschungsgemeinschaft (DFG) – 327533333.

5.1 Introduction

All graphs in the present paper are supposed to be finite, undirected, and simple. For terminology not defined here we refer to contemporary text books such as [BM08] or [Die17]. By K_S the complete graph on a finite set S is denoted. A (minimal) *transversal* of a set $\mathfrak{C}$ of disjoint sets is a set T containing exactly one member of every $A \in \mathfrak{C}$ and nothing else; we also say that $\mathfrak{C}$ is *traversed* by T. A *coloring* of a graph G is a partition $\mathfrak{C}$ of its vertex set $V(G)$ into *anticliques*, that is, sets of pairwise nonadjacent vertices. The *chromatic number* $\chi(G)$ is the smallest order of a coloring of G. A *Kempe chain* is a connected component of $G[A \cup B]$ for some $A \neq B$ from $\mathfrak{C}$. For a transversal T of a coloring $\mathfrak{C}$ of G we define the graph $H(G, \mathfrak{C}, T)$ to be the graph on T where any two $s \neq t$ are adjacent if and only if they belong to the same Kempe chain in G. A graph H is a *minor* of a graph G if there exists a family $c = (V_t)_{t \in V(H)}$ of pairwise disjoint subsets of $V(G)$, called *bags*, such that V_t is nonempty and $G[V_t]$ is connected for all $t \in V(H)$ and there is an edge connecting V_t and V_s for all $st \in E(H)$. Any such c is called an *H-certificate* in G, and a *rooted H-certificate* if, moreover, $V(H) \subseteq V(G)$ and $t \in V_t$ for all $t \in V(H)$. If there exists a rooted H-certificate, then H is a *rooted minor* of G.

Let us say that a graph K *has property (*)* if for every transversal T of every coloring $\mathfrak{C}$ with $|\mathfrak{C}| = |V(K)|$ of every graph G such that K is isomorphic to a spanning subgraph H of $H(G, \mathfrak{C}, T)$, there exists a rooted H-certificate in G. It is obvious that property (*) holds for K_1 and transfers to isomorphic copies of K.

Theorem 5.1. *Property (*) inherits to subgraphs of K; furthermore, K has (*) if and only if every component of K has.* ⋄

Proof. First, assume K' has property (*) and K is a spanning subgraph of K'. Take an arbitrary graph G with a coloring $\mathfrak{C}$ with $|\mathfrak{C}| = |V(K)|$ and a transversal T of $\mathfrak{C}$ such that K is isomorphic to a spanning subgraph H of $H(G, \mathfrak{C}, T)$. For $e \in E(K') \setminus E(K)$ add a suitable edge between two transversal vertices to G (if not already present) to obtain a graph G'. Then K' is isomorphic to a spanning subgraph H' of $H(G', \mathfrak{C}, T)$. Since K' has property (*), there is a rooted H'-certificate c in G' and c is also a rooted H-certificate in G. Next, assume that K' has property (*) and let $K \neq K'$ be a component of K'. Take an arbitrary graph G with a coloring $\mathfrak{C}$ with $|\mathfrak{C}| = |V(K)|$ and a transversal T of $\mathfrak{C}$ such that K is isomorphic to a spanning subgraph H of $H(G, \mathfrak{C}, T)$. A graph G' can be obtained from G by the disjoint union with a complete graph K_S on vertex set $S := V(K) - V(K')$. Let $\mathfrak{C}' := \mathfrak{C} \cup \{\{s\} : s \in S\}$ and $T' := T \cup S$. Then K' is isomorphic to a spanning subgraph H' of $H(G', \mathfrak{C}', T')$. Since K' has property (*), there is a rooted H'-certificate $c' = (V_t)_{t \in V(K')}$ in G'. Then $c := (V_t)_{t \in V(K)}$ is a rooted H-certificate in G. If, conversely, every component of K has property (*) and there is an arbitrary graph G with a coloring $\mathfrak{C}$ with $|\mathfrak{C}| = |V(K)|$ and a transversal T of $\mathfrak{C}$ such that K is isomorphic to a spanning subgraph H of $H(G, \mathfrak{C}, T)$, then for each component $H_1, H_2, \ldots$ of H there are pairwise disjoint subgraphs $G_1, G_2, \ldots$ of G and $\mathfrak{C}_1, \mathfrak{C}_2, \ldots$ and $T_1, T_2, \ldots$ such that H_i is a subgraph of $H(G_i, \mathfrak{C}_i, T_i)$ for $i \in \{1, 2, \ldots\}$. Since property (*) holds for H_i, $i \in \{1, 2, \ldots\}$, there is a rooted H_i-certificate c_i in G_i. Then the union of $c_1, c_2, \ldots$ (considered as subsets of $V(H_i) \times \mathfrak{P}(V(G))$) is a rooted H-certificate in G, so that K has property (*). Finally, assume K' has property (*) and let K be a subgraph of K'. Then $K \cup (V(K') \setminus V(K), \emptyset)$ is a spanning subgraph of K' and has property (*), and so has K as one of its components. ■

A coloring $\mathfrak{C}$ is a *Kempe coloring* if any two vertices from distinct color classes belong to the same Kempe chain or, in other words, the union of any two color classes is connected. The following has been conjectured in [Kri17] by the first author.

Conjecture 10. *Let $\mathfrak{C}$ be a Kempe coloring of some graph G and let T be a transversal of $\mathfrak{C}$. Then there exists a set of connected, pairwise disjoint, pairwise adjacent subsets of $V(G)$ traversed by T.* ⋄

This would prove Hadwiger's Conjecture — that every graph with chromatic number k has a complete minor of order k [Had43] — for graphs with a Kempe coloring, in particular for uniquely k-colorable graphs. In the terminology defined above, the conjecture reads as follows: If $\mathfrak{C}$ is a Kempe coloring of G and T is a transversal of $\mathfrak{C}$ (so that $H := H(G, \mathfrak{C}, T)$ is the complete graph on T) then there exists a rooted H-certificate. This would follow if every complete graph — and hence *every* graph — K had property (*).

However, property (*) turns out to be too restrictive to be true: We will see that K_7 does not have property (*). This will not produce a counterexample to Conjecture 10 above; in fact, the coloring of our corresponding example is very far from being a Kempe coloring in the sense that only 8 of the $\binom{7}{2} = 21$ pairs of color classes induce a connected subgraph.

We also have a number of positive results. Graphs with at most four vertices do have property (*), so that the question for the largest b such that all graphs of order at most b have property (*) suggests itself (it must be one of $4, 5, 6$ by the results of the present paper). Moreover, graphs with at most one cycle have property (*). As a consequence, for example, we get (immediately from Lemma 5.5 below) that if $x_0, \dots, x_{\ell-1}$ belong to different color classes of some coloring $\mathfrak{C}$ of a graph G and x_i, x_{i+1} belong to the same Kempe chain for $i \in \{0, \dots, \ell-1\}$ (indices modulo ℓ), then there exists a cycle C in G and disjoint x_i,y_i-paths P_i with $V(C) \cap V(P_i) = \{y_i\}$ and $y_0, \dots, y_{\ell-1}$ occur in this order on C.

Apart from this, we determine a number of further 5-vertex graphs having property (*) and infer that if T is a *connected* transversal of a Kempe coloring of order 5 of some graph G, then there exists a rooted $H(G, \mathfrak{C}, T)$-certificate (where $H(G, \mathfrak{C}, T)$ is isomorphic to K_5).

5.2 Kempe Chains and Rooted K_7-minors

For a graph G we define the graph $Z(G)$ by $V(Z(G)) := V(G) \times \{1,2\}$ and $E(Z(G)) := \{(x,i)(y,j) : xy \in E(G) \wedge (i \neq 1 \vee j \neq 1)\}$. That is, $Z(G)$ is obtained from the lexicographic product of G with the graph $(\{1,2\}, \emptyset)$ by deleting all edges connecting vertices from $V(G) \times \{1\}$. For $s = (x,i) \in V(Z(G))$ let us define $\overline{s} := (x, 3-i)$. It follows that $\mathfrak{C} := \{\{(x,1),(x,2)\} : x \in V(G)\} = \{\{s, \overline{s}\} : s \in V(Z(G))\}$ is a coloring of $Z(G)$ — the *canonical coloring* — and that $T := V(G) \times \{1\}$ is a transversal of $\mathfrak{C}$. Observe that T induces an anticlique in $Z(G)$. As the union of two color classes $\{(x,1),(x,2)\}, \{(y,1),(y,2)\}$ induce **(i)** a path of length four between its transversal vertices if $xy \in E(G)$ and **(ii)** an edgeless graph if $xy \notin E(G)$, we see that $H := H(Z(G), \mathfrak{C}, T)$ is isomorphic to G (via $(x,1) \mapsto x$). Moreover, we find a copy of G induced in $Z(G)$ in a very natural way: $Z(G)[V(G) \times \{2\}]$ is isomorphic to G (via $(x,2) \mapsto x$). The question is if we find a rooted H-certificate in $Z(G)$; if not then G (!) fails to have property (*).

The bags of any *H*-certificate $c = (V_t)_{t \in T}$ in $Z(G)$ have average order at most 2. That is, as soon as there are bags of order at least 3, there must be bags of order 1; locally, the inverse implication is almost true, as follows:

Claim 1. *If $st \in E(H)$ is not on any triangle of H, then $|V_s| = 1$ implies $|V_t| \geq 3$.* ⋄

Proof. Suppose that $|V_s| = 1$, that is, $V_s = \{s\}$. Since s, t are not adjacent in $Z(G)$, $|V_t| \geq 2$. If $|V_t| = 2$, then $V_t = \{t, u\}$ for some $u \in V(Z(G))$, where t, u and s, u are adjacent in $Z(G)$ so that $u \in V(Z(G)) \setminus V(H)$, $\overline{u} \neq s$ and $t, \overline{u}$ must be adjacent in H. Since $s, t, \overline{u}$ do not form a triangle in H, $s, \overline{u}$ are nonadjacent in H so that s, u are nonadjacent in $Z(G)$; consequently, s has no neighbors in V_t, contradiction. This implies $|V_t| \geq 3$ as claimed. □

If all bags of c have order 2, then we look at the function $f \colon V(G) \to V(G)$ defined by $f(x) := y$ if $V_{(x,1)} = \{(x, 1), (y, 2)\}$. Since the bags are disjoint, f is an injection and, thus, a permutation of $V(G)$. Since the bags are connected, $xf(x) \in E(G)$, so that we may represent f as a partial orientation of G, where xy is oriented from x to y if $y = f(x)$ and from y to x if $x = f(y)$ (which may happen simultaneously). As c is a rooted *H*-certificate in $Z(G)$ we know that $xy \in E(G)$ implies that $V_{(x,1)}, V_{(y,1)}$ are adjacent, which is equivalent to saying that, in $Z(G)$, $f(x)$ is adjacent to one of $y, f(y)$ or $f(y)$ is adjacent to one of $x, f(x)$. Conversely, if f is a permutation of $V(G)$ such that **(i)** $xf(x) \in E(G)$ and **(ii)** $xy \in E(G)$ implies that $f(x)$ is adjacent to one of $y, f(y)$ or $f(y)$ is adjacent to one of $x, f(x)$, then $V_{(x,1)} := \{(x, 1), (f(x), 2)\}$ defines an *H*-certificate in $Z(G)$. Let us call a permutation of $V(G)$ with **(i)** and **(ii)** a *good* permutation throughout this section.

Claim 2. *If G has a good permutation, then every vertex of degree at least 3 in G is on a cycle of length at most 4 in G.* ⋄

Proof. Let f be a good permutation and suppose that w is a vertex of degree at least 3 in G and let x, y, z be three neighbors in G, where $f(w) = x$. We may assume that $f(y) \neq w$ (otherwise $f(z) \neq w$ and we swap the roles of y, z). If $u := f(y) \neq w$ is a neighbor of w then w, y, u form a triangle and we are done. Otherwise, $\{w, f(w) = x\}$, $\{y, f(y) = u\}$ are disjoint, and **(ii)** implies that, in G, u is a neighbor of w or x, or that x is a neighbor of y or u; in either case, w is on a cycle of length 3 or 4. This proves Claim 2. □

Let us specialize the considerations to the graph G obtained from a cycle G' of length 6 by adding another vertex x and two edges connecting x to two vertices a, b at distance 3 on G'. Assume, to the contrary, that $Z(G)$ has an *H*-certificate $(V_t)_{t \in T}$ with $T = V(G) \times \{1\}$. Let A be the set of vertices $t \in T$ with $|V_t| = 1$. By Claim 2, G cannot have a good permutation, so $|A| \geq 1$. A is an anticlique in H (and in $Z(G)$), so $|A| \leq 3$, and, by Claim 1, $|V_s| \geq 3$ for every vertex s in the neighborhood of A in H. For each case $|A| = 1, |A| = 2, |A| = 3$ one readily verifies $|N_H(A)| \geq |A| + 1$. It follows $q := \sum_{t \in T} |V_t| \geq 3 \cdot (|A| + 1) + 2 \cdot (7 - 2|A| - 1) + 1 \cdot |A| = 15$, contradicting $q \leq |V(Z(G))| = 14$. It follows that G does not have property (*). As property (*) inherits to spanning subgraphs we conclude that K_7 does not have it either. In fact, we could take $Z(G)$ with $\mathfrak{C}$ and T as above and just add all edges between transversal vertices $(x, 1), (y, 1)$ with $xy \notin E(G)$ as to obtain a graph G' without a rooted $H(G', \mathfrak{C}, T)$-certificate, where $H(G', \mathfrak{C}, T)$ is now the complete graph on the seven vertices from T. So we have proved:

Theorem 5.2. *K_7 does not have property (*).* ⋄

Let $d \geq 3$. We now specialize to a connected, d-regular, nonbipartite graph G of girth at least 5 and assume, to the contrary, that $Z(G)$ has an H-certificate $c = (V_t)_{t \in T}$. Let A, B, C be the set of vertices $x \in V(G)$ with $|V_{(x,1)}|$ being 1, 2, and at least 3, respectively. By Claim 2, there cannot be a good permutation, so that $|A| \geq 1$. The vertices from A and $A \times \{1\}$ induce an edgeless graph in G and $Z(G)$, respectively, and the neighbors of A in G are all from C by Claim 1. The number of edges between A and C in G is thus equal to $d|A|$ and at most $d|C|$ with equality only if every vertex from C has all its d neighbors in A. However, in the latter case, $G[A \cup C]$ is d-regular and bipartite and B is empty as G is connected, so that G is bipartite, contradiction. It follows $d|A| < d|C|$ and, consequently, $|Z(G)| \geq \sum_{t \in T} |V_t| \geq |A| + 2|B| + 3|C| > 2|A| + 2|B| + 2|C| = |Z(G)|$, contradiction. So we have proved

Theorem 5.3. *If G is connected, d-regular with $d \geq 3$, nonbipartite of girth at least 5, then it does not have property (*).* ⋄

The smallest graph meeting the assumptions of Theorem 5.3 is, incidentally, the Petersen graph.

5.3 Unicyclically Arranged Kempe Chains

We continue with a number of positive results. The main result of Fabila-Monroy and Wood in [FW13, Theorem 8] states that for any 3-connected graph G and distinct vertices $t_1, t_2, t_3, t_4 \in V(G)$ such that two vertex-disjoint t_i,t_j-path and t_k,t_ℓ-path exist for each choice of distinct $i, j, k, \ell \in \{1, 2, 3, 4\}$, then there exists a rooted H-certificate where H is the complete graph on $\{t_1, \ldots, t_4\}$. This generalizes to:

Theorem 5.4. *Every graph on at most four vertices has property (*).* ⋄

Proof. We prove that K_4 has property (*). As (*) inherits to subgraphs, this will complete the proof. Therefore, let G be a graph, $\mathfrak{C}$ be a coloring of G with $|\mathfrak{C}| = 4$, and let T be a transversal of $\mathfrak{C}$. For $x \in V(G)$, let A_x denote the member of $\mathfrak{C}$ containing x, and suppose that for all $x \neq y$ from T there exists an x,y-path P_{xy} in $G[A_x \cup A_y]$, that is, $H := H(G, \mathfrak{C}, T)$ is a complete graph on four vertices. Suppose that G was a minimal counterexample. Then G is connected and $E(G) = \bigcup_{x \neq y} E(P_{xy})$. By the previously stated result of Fabila-Monroy and Wood, G is not 3-connected.

We may assume that G has a separator S with $|S| \leq 2$. If $S \subseteq A_x$ for some $x \in T$, then $T \setminus \{x\}$ is contained in some component C of $G - S$. Let G' be the graph obtained from G by contracting $X := V(G) \setminus V(C)$ to a single vertex w. For $A \in \mathfrak{C}$ set $A' := (A \setminus X) \cup \{w\}$ if $A = A_x$ and $A' := A \setminus X$ otherwise, and for $z \in T$ set $z' := w$ if $z \in X$ and $z' := z$ otherwise. Then $\mathfrak{C}' := \{A' : A \in \mathfrak{C}\}$ is a coloring of G' and $T' := \{t' : t \in T\}$ is a transversal of $\mathfrak{C}'$. Moreover, $H' := H(G, \mathfrak{C}', T')$ is a complete graph on T'. By the choice of G, there exists a rooted H'-certificate in G', that can be extended to a rooted H-certificate of G by replacing its bag B containing w — if any — with $(B \setminus \{w\}) \cup X$, contradiction.

Otherwise, G is 2-connected and there exist $x \neq y$ from T such that each of $S \cap A_x$ and $S \cap A_y$ consists of a single vertex x_0, y_0, respectively. Again, $T \setminus \{x, y\}$ is contained in the same component C of $G - S$. Let G' be the graph obtained from G by deleting $X := V(G) \setminus (V(C) \cup S)$ and adding an edge x_0y_0 (if it does not already exist). For $A \in \mathfrak{C}$, let $A' := A \setminus X$, so that $\mathfrak{C}' := \{A' : A \in \mathfrak{C}\}$ is a coloring of G'. For $z \in T$, set $z' := z_0$ if $z \in \{x, y\} \cap X$ and $z' := z$ otherwise, so that $T' := \{z' : z \in T\}$ is a transversal of $\mathfrak{C}'$, and $H' := H(G', \mathfrak{C}', T)$ is a complete graph on T'. By the choice of G, G' admits a rooted H'-certificate c'. If X does not contain both x and y, say, $y \notin X$, then c' can be extended to a rooted H-certificate of G by replacing its bag B containing x_0 — if any — with $B \cup X$, contradiction. If, otherwise, X contains both x and y, then there are two vertex-disjoint paths P_1 and P_2 connecting x_0 and y_0 to $\{x, y\}$, respectively (by 2-connectivity of G and MENGER's Theorem). It is obvious, that $V(P_1), V(P_2) \subseteq S \cup X$. Let $X_0 := V(P_1)$ and let Y_0 be the vertex set of the component of $G[S \cup X] - X_0$ containing P_2, and therefore also y_0 and $\{x, y\} \setminus X_0$. Since G is 2-connected and all neighbors of $Y_0 \setminus \{y_0\} \neq \emptyset$ are in $X_0 \cup \{y_0\}$, X_0 and Y_0 are adjacent in G. Now, a rooted H-certificate can be obtained from c' by replacing the bags B_1 and B_2 containing x_0 and y_0 with $B_1 \cup X_0$ and $B_2 \cup Y_0$, respectively. Thus, all bags contain exactly one vertex from T and are pairwise adjacent, contradiction. ■

Here is an infinite class of connected graphs for which (*) is true.

Lemma 5.5. *Every cycle has property (*).* ⋄

Proof. Given ℓ, we have to prove for every graph G, every coloring $\mathfrak{C} = \{A_0, \dots, A_{\ell-1}\}$ and every choice $t_i \in A_i$ for $i \in \{0, \dots, \ell - 1\}$ such that there exists a t_i,t_{i+1}-path P_i in $G[A_i \cup A_{i+1}]$ for all $i \in \{0, \dots, \ell-1\}$, indices modulo ℓ, there exists a rooted $H := (\{t_0, \dots, t_{\ell-1}\}, \{t_0t_1, t_1t_2, \dots, t_{\ell-2}t_{\ell-1}, t_{\ell-1}t_0\})$-certificate. Suppose that G was a minimal counterexample. Then $E(G) = E(P_0) \cup \dots \cup E(P_{\ell-1})$. If $x \in A_i \setminus \{t_i\}$ had only two neighbors, say, y, z in A_j, where $j \in \{i+1, i-1\}$, then let G' be obtained from G by contracting yxz to a new vertex w, define $A'_i := A_i \setminus \{x\}$, $A'_j := (A_j \setminus \{y, z\}) \cup \{w\}$, $A'_k := A_k$ for $k \in \{0, \dots, \ell-1\} \setminus \{i, j\}$, $t'_k := w$ if $k = j \wedge t_k \in \{y, z\}$, and $t'_k := t_k$ otherwise. Letting $H' := (\{t'_0, \dots, t'_{\ell-1}\}, \{t'_0t'_1, t'_1t'_2, \dots, t'_{\ell-2}t'_{\ell-1}, t'_{\ell-1}t'_0\})$, we know that, by choice of G, there exists a rooted H'-certificate in G', from which we can construct a rooted H-certificate of G by replacing its bag B containing w — if any — with $(B \setminus \{w\}) \cup \{y, x, z\}$, contradiction. Hence we may assume that every vertex in $A_i \setminus \{t_i\}$ has degree 4, that is, it is on both P_i and P_{i-1}. In particular, all A_i have the same order $d \geq 1$. If $d = 1$, then $G = H$, so that G is not a counterexample, contradiction. Hence $d \geq 2$, and we consider $G' := G - \{t_0, \dots, t_{\ell-1}\}$, $A'_i := A_i \setminus \{t_i\}$ and let $t'_i \in A'_i$ be the neighbor of t_{i-1} on P_{i-1}. As t'_{i+1} is on $P_i - \{t_i, t_{i+1}\} = G[A'_i \cup A'_{i+1}]$, we know by choice of G that G' has a rooted H'-certificate (H' as above), from which we get a rooted H-certificate of G by extending the bag containing t'_i by the vertex t_{i-1}, contradiction. ■

Lemma 5.5 generalizes to unicylic graphs, as follows.

Theorem 5.6. *Every (connected) graph with at most one cycle has property (*).* ⋄

Proof. Let K be a connected graph with at most one cycle. Suppose that K has not property (*). Hence, we may assume by Lemma 5.5 that G is not a cycle and contains at least one edge. Therefore, K contains a vertex q of degree 1. Let r be the neighbor of q in K. By induction, we may assume that $K - q$ has property (*). Since K has not, there exists a graph G with a coloring $\mathfrak{C}$ and a transversal T of $\mathfrak{C}$ such that K is isomorphic to a spanning subgraph H of $H(G, \mathfrak{C}, T)$ but G has no rooted

H-certificate. Again we may take G minimal with respect to this property, implying that for all $A \neq B$ from $\mathfrak{C}$, $G[A \cup B]$ has a single nontrivial component which induces a path between the vertices $a \in A \cap T$, $b \in B \cap T$ if $ab \in E(H)$ and $E(G[A \cup B]) = \emptyset$ otherwise. Now let $Q \neq R$ be the members of $\mathfrak{C}$ with $q \in Q$, $r \in R$. If there existed a vertex $x \in Q \setminus \{q\}$ then x has degree 2, and we take its neighbors $y, z \in R$, contract yxz to a single vertex w. For $A \in \mathfrak{C}$ set $A' := (A \setminus \{y,z\}) \cup \{w\}$ if $A = R$, $A' := A \setminus \{x\}$ if $A = Q$, and $A' := A$ otherwise, and for $z \in T$ set $z' := w$ if $z \in \{y,z\}$ and $z' := z$ otherwise. Then $\mathfrak{C}' := \{A' : A \in \mathfrak{C}\}$ is a coloring of G' and $T' := \{t' : t \in T\}$ is a transversal of $\mathfrak{C}'$. By choice of G, G' has a rooted $H(G', \mathfrak{C}', T')$-certificate, and we get a rooted H-certificate of G by replacing its bag B containing w — if any — with $(B \setminus \{w\}) \cup \{y, x, z\}$, contradiction. Therefore, $Q = \{q\}$ so that, by induction, $G - q$ has a rooted $H(G - q, \mathfrak{C} \setminus \{Q\}, T \setminus \{q\})$-certificate, from which we get a rooted H-certificate by adding the bag $Q = \{q\}$. ■

5.4 The Graphs $Z(G)$ for $|V(G)| \leq 6$

One could ask if Theorem 5.2 extends to smaller complete graphs or, alternatively, if the bound ("four") in Theorem 5.4 can be increased. Both questions need new methods: In this section, we will see that the method used in Section 5.2 to identify graphs not satisfying (*) does not work out for graphs on less than seven vertices, whereas, in the next section, we will collect our knowledge on (*) for graphs on five vertices. We start with another positive result.

Lemma 5.7. *Let G be a graph, $\mathfrak{C}$ be a coloring of G, T be a transversal of $\mathfrak{C}$ and $H := H(G, \mathfrak{C}, T)$. Suppose that A is an anticlique in H and suppose that there is a matching M in H from $V(H) \setminus A$ into A (covering $V(H) \setminus A$, but not necessarily covering A). Suppose that for every edge $st \in M$ where $s \in V(H) \setminus A$ and $t \in A$, $G[P \cup Q] - t$ is connected, where P, Q denotes the color class from $\mathfrak{C}$ containing s, t, respectively. Then there is a rooted H-certificate in G.* ⋄

Proof. For $s \in A$ set $V_s := \{s\}$. For $s \in V(H) \setminus A$ let t be the vertex in A such that $st \in M$, let P, Q be the classes of $\mathfrak{C}$ containing s, t, respectively, and set $V_s := (P \cup Q) \setminus \{t\}$. One readily verifies that $(V_s)_{s \in V(H)}$ is a rooted H-certificate. □

Theorem 5.8. *Let G be any graph with at most 6 vertices. Consider $Z(G)$ defined as in Section 5.2 with its canonical coloring $\mathfrak{C} = \{\{x\} \times \{1,2\} : x \in V(G)\}$ and the transversal $T := V(G) \times \{1\}$. Then $Z(G)$ has a rooted $H(Z(G), \mathfrak{C}, T)$-certificate.* ⋄

Proof. Let G be a counterexample and set $H := H(Z(G), \mathfrak{C}, T)$. By the positive results of the Section 5.3, we may assume that $|V(H)| \in \{5, 6\}$, that H is connected, and that H contains more than one cycle. If H has a cutvertex s, then there is a component C of $H - s$ with one or two vertices; if $V(C) = \{t\}$, then we know that $Z(G) - \{s, \overline{s}, t, \overline{t}\}$ has a rooted $(H - \{s,t\})$-certificate, and we extend it to a rooted H-certificate of $Z(G)$ by setting $V_s := \{s, \overline{s}, \overline{t}\}$ and $V_t := \{t\}$; if, otherwise, $V(C) = \{t, u\}$, then $Z(G) - \{t, \overline{t}, u, \overline{u}\}$ has a rooted $(H - \{t,u\})$-certificate, and we extend it to a rooted H-certificate of $Z(G)$ by setting $V_t := \{t, \overline{u}\}$ and $V_u := \{u, \overline{t}\}$. Therefore, we may assume that H is 2-connected.

Let us call an anticlique *matchable* if there exists a matching M from $V(H) \setminus A$ into A. By Lemma 5.7, we may assume that H has no matchable anticlique. For $|V(H)| = 5$ it follows that there is no anticlique A of order larger than 2, as it would be matchable by 2-connectivity of H. Consequently, H has a spanning 5-cycle t_0, t_1, t_2, t_3, t_4, and setting $V_{t_i} := \{t_i, \overline{t_{i+1}}\}$ (indices mod 5) yields a family of pairwise adjacent cliques and, thus, a rooted H-certificate.

For $|V(H)| = 6$ it follows that there is not anticlique A of order larger than 3 (as, again, A would be matchable). If A was an anticlique of order 3, then there would be a matching M from A to $V(H) \setminus A$ by Hall's Theorem (see [Die17]), since every vertex in A has two neighbors in $V(H) \setminus A$ and $N_H(A) = V(H) \setminus A$ as there are no anticliques of order 4. M is a matching from $V(G) \setminus A$ into A, too, so A would be matchable, contradiction. It follows that G has no anticliques of order larger than 2.

Let us say that a matching $N = \{r_1s_1, r_2s_2, r_3s_3\}$ of H is *good* if every edge from $E(H) \setminus N$ is either on a triangle containing one edge from N or on a cycle of length 4 containing two edges from N. In this case, $V_{r_i} := \{r_i, \overline{s_i}\}$, $V_{s_i} := \{s_i, \overline{r_i}\}$ for $i \in \{1, 2, 3\}$ defines a rooted H-certificate.

If H has a spanning cycle $t_0t_1t_2t_3t_4t_5$, then setting $V_{t_i} := \{t_i, \overline{t_{i+1}}\}$ yields a family of cliques such that V_{t_i} is adjacent to $V_{t_{i+1}}$ and $V_{t_{i+2}}$ ((sub-)indices modulo 6). So we get a rooted H-certificate in the case that all three *long chords* t_0t_3, t_1t_4, t_2t_5 of the cycle are missing. If all long chords are present, then they form a good matching in H, and we are done, too.

Suppose that S is a smallest separator in H and that C, D are two components of $H - S$. Then $|S| \geq 2$, C, D are complete, and there are no further components of $H - S$ as H has no anticlique of order 3. If $|V(C)| = |V(D)| = |S| = 2$, then there is a spanning cycle $t_0t_1t_2t_3t_4t_5$ in G with $V(C) = \{t_0, t_1\}$, $S = \{t_2, t_5\}$, and $V(D) = \{t_3, t_4\}$; if $t_2t_5 \notin E(H)$, then all long chords are missing, and we are done, otherwise $\{t_0t_1, t_2t_5, t_3t_4\}$ is a good matching.

Suppose that $V(C) = \{s\}, S = \{t, u\}, V(D) = \{a, b, c\}$. We may assume that both t and u have more than one neighbor among a, b, c, for otherwise $N_H(\{s, t\})$ or $N_H(\{s, u\})$ would be a separator of order 2 as discussed in the previous paragraph. If $N_H(t) \setminus \{u\} = N_H(u) \setminus \{t\} = \{s, a, b\}$, then $\{st, ua, bc\}$ is a good matching, and we are done. Otherwise, $ta, tb, ub, uc \in E(H)$ without loss of generality, and if $tc \notin E(H)$ and $ua \notin E(H)$, then $stabcu$ is a cycle of length 6 without long chords, and we are done. So $tc \in E(H)$ without loss of generality. If $ua \in E(H)$, too, then su, ta, bc is a good matching, otherwise $V_s := \{s, \overline{u}\}$, $V_u := \{u, \overline{s}\}$, $V_t := \{t, \overline{c}\}$, $V_c := \{c, \overline{b}\}$, $V_b := \{b, \overline{t}\}$, $V_a := \{a\}$ defines a rooted H-certificate.

From now on we may assume that G is 3-connected. If H has a spanning wheel with center s and rim cycle $t_0, \ldots, t_4$, then $V_s := \{s\}$, $V_{t_i} := \{t_i, \overline{t_{i+1}}\}$ (indices mod 5) defines a rooted H-certificate. If H has a spanning prism with triangles $s_0s_1s_2$, $t_0t_1t_2$ and connecting edges s_it_i, then the connecting edges form a good matching.

If $|S| = 3$, then $V(C) = \{s\}$, $S = \{t, u, v\}$, $V(D) = \{a, b\}$, $ta, ua, ub, vb \in E(H)$ without loss of generality. If $tv \in E(H)$, then we have a spanning prism with triangles stv and uab. Otherwise, one of tu, uv is in $E(H)$ (as S is not an anticlique), say $tu \in E(H)$. If $uv \in E(H)$, then we have a spanning wheel with center u, otherwise $va \in E(H)$ and we have a spanning prism with triangles stu and vab.

Hence we may assume that G is 4-connected and, therefore, obtained from K_6 by deleting some edges of some perfect matching. Consequently, it has a spanning prism, and we are done. ∎

5.5 Connected Transversals of 5-colorings

By Theorem 5.4, all graphs with at most four vertices have property (*), whereas by Theorem 5.2, there exists a graph on seven vertices which has not. For graphs K on five vertices we do not have the full picture; since we may assume that such a K is connected and since a connected graph on five vertices and at most five edges contains at most one cycle, we know by Theorem 5.6 that all graphs on five vertices and at most five edges have property (*), too. This extends as follows:

Theorem 5.9. *Every graph on five vertices and at most six edges has property (*).* ⋄

Proof. Let K be a graph with $|V(K)| = 5$ and $|E(K)| \leq 6$. To verify (*) for K, we may assume that K is connected and $|E(K)| = 6$ by results and observations of the previous sections. If there is a vertex of degree 1 in K, by a similar argument as in the proof of Theorem 5.6 we can reduce the problem to a graph with four vertices which is already solved. Thus, up to isomorphism, there are only three remaining graphs to consider: The *hourglass* ⧖ obtained from the union of two disjoint triangles by identifying two nonadjacent vertices, the complete bipartite graph $K_{2,3}$ with color classes of order 2 and 3, respectively, and the graph C_5^+ obtained from a 5-cycle by adding a edge connecting some pair of nonadjacent vertices.

Assume, to the contrary, that K does not have property (*); then there exists a graph G with a coloring $\mathfrak{C}$ and a transversal T of $\mathfrak{C}$ such that K is isomorphic to a spanning subgraph H of $H(G, \mathfrak{C}, T)$ but G has no rooted H-certificate. Again we may take G minimal with respect to this property, implying that for all $A \neq B$ from $\mathfrak{C}$, $G[A \cup B]$ has a single nontrivial component which induces a path between the unique vertices $a \in A \cap T$, $b \in B \cap T$ if $ab \in E(H)$ and $E(G[A \cup B]) = \emptyset$ otherwise. In particular, $H(G, \mathfrak{C}, T)$ is isomorphic to K. As in the proof of Lemma 5.5, we may assume that all vertices in $V(G) \setminus T$ have degree at least 4.

In all three cases, let $T := \{t_1, \ldots, t_5\}$ and $\mathfrak{C} := \{A_1, \ldots, A_5\}$ such that $t_i \in A_i$ for all $i \in \{1, \ldots, 5\}$; the t_i will be specified differently in each case. In each case, we will find a rooted H-certificate c defined by its bags $B_i =: c(t_i)$

Case 1. K is isomorphic to the hourglass ⧖.

Let $t_1 \in T$ be the vertex of degree 4 in H, and let s_2, s_3 be two neighbors in G of t_1 in the color classes A_2 and A_3, respectively, such that $t_2t_3 \in E(H)$. Then there is a t_2,t_3-path P in $G[A_2 \cup A_3]$, and, because of the assumptions to G, $s_2, s_3 \in A_2 \cup A_3 = V(P)$. It is possible to partition $V(P)$ into two bags B_2 and B_3 such that each of them contains exactly one vertex from $\{s_2, s_3\}$ and one from $\{t_2, t_3\}$, and $G[B_2], G[B_3]$ are connected subgraphs. Repeating this step for the other two neighbors of t_1 in G, we obtain bags $B_2, \ldots, B_5$ forming a rooted H-certificate together with the fifth bag $B_1 := \{t_1\}$, contradiction.

Case 2. K is isomorphic to the graph $K_{2,3}$.

First, note that all Kempe chains in G have at least four vertices. (Otherwise remove one edge connecting two transversal vertices; the remaining graph is unicyclic and we are done by Theorem 5.6.) Additionally, assume $|V(G)|$ to be minimal.

Let t_1, t_2, t_3 be the vertices of T of degree 2 in H and let s_i (not necessarily distinct) be a neighbor of t_i for $i \in \{1, 2, 3\}$ such that $s_1, s_2, s_3 \in A_4$. The vertices s_i have at least two neighbors in a color class other than A_i. Assume first that two among s_1, s_2, s_3 have such neighbors in a common color class (this will always happens if s_1, s_2, s_3 are not pairwise distinct); say, without loss of generality, there are neighbors of s_1 and s_2 in A_3. We set $B_1 := \{t_1\}, B_2 := \{t_2\}, B_3 := \{t_3\}, B_4 := V(P_{34}) \setminus \{t_3\}, B_5 := (V(P_{15}) \setminus \{t_1\}) \cup (V(P_{25}) \setminus \{t_2\})$, where P_{ij} is the path from t_i to t_j in $G[A_i \cup A_j]$. Because each vertex in A_5 is a vertex of P_{15} or P_{25} — all these vertices have degree at least 3 — there are edges between B_5 and B_1, B_2, B_3. Since s_1 and s_2 have a neighbor in A_3, we conclude $\{s_1, s_2\} \subseteq V(P_{34})$, and $B_1, \dots, B_5$ form a rooted H-certificate of G, contradiction.

Thus, for $i \in \{1, 2, 3\}$, the vertices s_i are distinct and each has a neighbor in a color class $\tilde{A}_i \neq A_i$ such that $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$ are distinct. Then s_i is a vertex of the 2-colored path from t_4 to the transversal vertex of $\tilde{A}_i$. Let u_i be the neighbor of s_i in $\tilde{A}_i$ with shortest distance to t_4 on this 2-colored path. Moreover, u_1, u_2, u_3 are colored differently as $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$ are distinct. Since $u_i \notin \{t_1, t_2, t_3\}$, all s_i, t_i, u_i ($i \in \{1, 2, 3\}$) are distinct. Consider the graph $G' := G - \{s_1, t_1, s_2, t_2, s_3, t_3\}$ with the induced coloring $\mathfrak{C}' := \{A \cap V(G') : A \in \mathfrak{C}\}$, and $T' := \{u_1, u_2, u_3, t_4, t_5\}$. All vertices in $V(G) \setminus T$ have degree at least 4 in G, thus, u_i has neighbors in A_5 and is on the 2-colored path from the transversal vertex of $\tilde{A}_i$ to t_5. Because of the choice of u_1, u_2, u_3, there is a 2-colored path from u_i to t_4 for $i \in \{1, 2, 3\}$ in G'. Thus, $H(G', \mathfrak{C}', T')$ has a spanning subgraph H' isomorphic to K. Because of the minimality of G, there is a rooted H'-certificate in G'. Adding to its bag containing u_i the vertices s_i, t_i for $i \in \{1, 2, 3\}$, we obtain a rooted H-certificate of G, contradiction.

Case 3. K is isomorphic to the graph C_5^+.

Let $t_1 \in T$ be the vertex of degree 2 in H in the unique triangle of H, let $t_2, t_3 \in T$ be the two vertices of degree 3 in H, and let t_4, t_5 the remaining two transversal vertices such that $t_2 t_4 \in E(H)$. Choose an arbitrary partition of $A_1 \cup A_2 \cup A_3$ into B_1, B_2, B_3 such that $B_1 = \{t_1\}$ and $t_2 \in B_2, t_3 \in B_3$, $G[B_2], G[B_3]$ are connected subgraphs, and B_1, B_2, B_3 are bags of a rooted K_S-certificate with $S := \{t_1, t_2, t_3\}$. Note that t_1 has two neighbors on the t_2,t_3-path in $G[A_2 \cup A_3]$. If t_4 has a neighbor in B_2, then set $B_4 := \{t_4\}$ and $B_5 := (A_4 \cup A_5) \setminus \{t_4\}$ as to obtain a rooted H-certificate, contradiction. By symmetry, t_5 has no neighbor in B_3. But then, consider the t_4,t_2-path P in $G[A_2 \cup A_4]$. This one starts with t_4 followed by a vertex in B_3 and ends in $t_2 \in B_2$. Thus, there is a vertex $v \in A_4$ having neighbors in both B_2 and B_3. Since there is a t_5,t_3-path Q in $G[A_3 \cup A_5]$, disjoint from P, there is another vertex $w \in A_5$ having neighbors in both B_2 and B_3. Due to the assumptions to G, v and w have degree 4 and, therefore, they are vertices on the t_4,t_5-path S in $G[A_4 \cup A_5]$. Now take a partition of S into adjacent B_4 and B_5 such that $t_4 \in B_4, t_5 \in B_5$, $G[B_4], G[B_5]$ are connected subgraphs, and $|\{v, w\} \cap B_4| = |\{v, w\} \cap B_5| = 1$. Then the bags $B_2, \dots, B_5$ are pairwise adjacent, hence G has a rooted H-certificate, a contradiction. ∎

Corollary 5.10. *Let G be a graph with a* Kempe *coloring $\mathfrak{C}$ of order 5 and let T be a transversal of $\mathfrak{C}$ such that $G[T]$ is connected. Then there exists a rooted K_T-certificate in G.* ◇

Proof. Since G has a Kempe coloring, any pair of transversal vertices is connected by a 2-colored path. Hence, $H(G, \mathfrak{C}, T)$ is isomorphic to K_5. Let H be obtained from it by removing edges if they exist in $G[T]$, i.e. $V(H) = T$ and $E(H) = \{st : s, t \in T, s \neq t, st \notin E(G)\}$. Since $G[T]$ is connected, $|E(G[T])| \geq 4$ and $|E(H)| \leq 6$. Thus, H fulfills the conditions of Theorem 5.9 and has property (*). We find a rooted H-certificate c of G. It remains to show that c is a rooted $H(G, \mathfrak{C}, T)$-certificate of G. If for $s, t \in T$ the edge st is not in $E(G)$, then $st \in E(H)$ and $B_s = c(s), B_t = c(t)$ are adjacent. Otherwise, $st \in E(G)$, then B_s, B_t are connected by the edge st. ∎

5.6 Concluding Remarks

Assuming that K_5 has property (*), any graph G with a 5-coloring $\mathfrak{C}$ and a transversal T such that $H(G, \mathfrak{C}, T)$ is a complete graph on 5 vertices cannot be planar and even has a K_5-minor. We conclude by two remarks that such graphs are not planar and have a K_5-minor even if K_5 may not have property (*). The problem whether K_5 has property (*) remains open.

Remark 5.11. *Let G be a graph with a 5-coloring $\mathfrak{C}$ and T be a transversal of $\mathfrak{C}$. If for each distinct $s, t \in T$ there is a 2-colored path from s to t in G, then G is not planar.* ⋄

Proof. On the contrary assume that G is planar, and, again, we may assume G to be minimal, implying that for all $A \neq B$ from $\mathfrak{C}$, $G[A \cup B]$ has a single nontrivial component which induces a path between the unique vertices $a \in A \cap T, b \in B \cap T$. Consider a drawing of G into the plane. Then each of the ten 2-colored paths between the transversal vertices can be considered as a Jordan curve of a plane drawing of K_5 on T with crossings. Evoke Tutte-Hanani-Theorem [Tut70] which states that in any planar representation of a non-planar graph G there are two nonadjacent edges whose crossing number is odd. Since K_5 is non-planar, there must be two of the Jordan curves with different end vertices crossing and such a crossing is always a vertex of G. But then, these two Jordan curves share an end vertex in the same color as the crossing vertex, contradiction. ■

Remark 5.12. *Let G be a graph with a 5-coloring $\mathfrak{C}$ and T be a transversal of $\mathfrak{C}$. If for all $s \neq t$ from T there is a 2-colored path from s to t in G, then G has K_5 as minor.* ⋄

Proof. Assume that G does not contain K_5 as minor and let G be chosen as a minimal counterexample. If G was not 3-connected, then there would be a separator S with $|S| \leq 2$ and a component C of $G - S$ containing at least three vertices from T. Repeating the same arguments as in the proof of Theorem 5.4, we can reduce G to a smaller counterexample. Hence, we assume that G is 3-connected.

Denote by G^+ the graph obtained from G by repeatedly adding edges as long as the resulting graph does not contain a K_5-minor. Then, G^+ is a 3-connected maximal K_5-minor-free graph by construction. By a famous result of Wagner [Wag37], G^+ is a 3-clique-sum of maximal planar graphs or the 8-vertex Wagner graph. Since G is not planar by Remark 5.11 and not the 8-vertex Wagner graph (G has minimum degree at least 4), there is a clique S in G^+ with $|S| = 3$ that separates G^+. Since G is a spanning subgraph of G^+, S also separates G.

Let A_x denote the member of $\mathfrak{C}$ containing x with $x \in V(G)$. Then there are s ($s \geq 2$) distinct $x_1, x_2, \ldots, x_s \in T$ such that $A_{x_i} \cap S = \emptyset$ for $i \in \{1, \ldots, s\}$. Again as in the proof of Theorem 5.4, there is one component C of $G - S$ containing all x_i, $i \in \{1, \ldots, s\}$.

Let $X := V(G) \setminus (V(C) \cup S)$ and assume first that there is $y \in T$ such that $|A_y \cap S| \geq 2$. Let G' be obtained from G by contracting $Y := X \cup (S \cap A_y)$ to a single vertex w. For $A \in \mathfrak{C}$ set $A' := (A \setminus Y) \cup \{w\}$ if $A = A_y$ and $A' := A \setminus Y$ otherwise, so that $\mathfrak{C}' := \{A' : A \in \mathfrak{C}\}$ is a coloring of G'. For $z \in T$, set $z' := w$ if $z \in Y \cap A_y$, $z' := z_0$ with $z_0 \in S \setminus A_y$ uniquely determined if $z \in Y \setminus A_y$ and $z' := z$ otherwise, so that $T' := \{z' : z \in T\}$ is a transversal of $\mathfrak{C}'$, and $H' := H(G', \mathfrak{C}', T')$ is a complete graph on T'. By the choice of G, G' has a K_5-minor and so has G because G' is a minor of G, contradiction.

Thus, there are distinct $y_1, y_2, y_3 \in T$ such that $|A_{y_i} \cap S| = 1$ with $i \in \{1, 2, 3\}$. If $X = \{d\}$, then d is not in T and d cannot be a vertex of a 2-colored path of G, contradiction. Thus, X consists of at least two vertices with degree at least 3 in $G[X \cup S]$. If there was no cycle in $G[X \cup S]$, then $G[X \cup S]$

would be a tree. A leaf of this tree would be a vertex from S, contradiction because a tree with at least two vertices of degree at least three has at least four leaves.

Hence, there is a cycle D in $G[X \cup S]$ and by the 3-connectedness of G, there are three vertex-disjoint s_i,c_i-paths P_i, $i \in \{1, 2, 3\}$ in G such that $V(P_i) \cap S = \{s_i\}$ and $V(P_i) \cap V(D) = \{c_i\}$ (possibly $s_i = c_i$). It is easy to see that $V(P_i) \subseteq X \cup S$. Denote by D_i the subpath of D from c_i to c_{i+1} missing c_{i+2} (indices modulo 3). Let G' be obtained from $G[V(C) \cup D \cup \bigcup_{i=1}^{3} V(P_i)]$ by contracting $V(P_i) \cup (V(D_i) \setminus \{c_{i+1}\})$ to a single vertex w_i for $i \in \{1, 2, 3\}$. Observe that $w_1w_2w_3$ is a triangle in G'. For $A \in \mathfrak{C}$ set $A' := (A \cap V(C)) \cup \{w_i\}$ if $s_i \in A$ and $A' := A \cap V(C)$ otherwise, so that $\mathfrak{C}' := \{A' : A \in \mathfrak{C}\}$ is a coloring of G'. For $z \in T$, set $z' := w_i$ if $z \notin V(C)$ with $z \in A_{s_i}$ for suitable $i \in \{1, 2, 3\}$, and $z' := z$ otherwise, so that $T' := \{z' : z \in T\}$ is a transversal of $\mathfrak{C}'$. It is straightforward to check that $H' := H(G', \mathfrak{C}', T')$ is a complete graph on T'. By the choice of G, G' has a K_5-minor and so has G because G' is a minor of G, contradiction. ∎

submitted to: *Journal of Graph Theory*.
T. Böhme, J. Harant, M. Kriesell, S. Mohr, and J. M. Schmidt, Rooted Minors and Locally Spanning Subgraphs.
Available at arXiv: 2003.04011.

Rooted Minors and Locally Spanning Subgraphs[1]

6

Thomas Böhme, Jochen Harant, Matthias Kriesell, Samuel Mohr[2], and Jens M. Schmidt[3]

Ilmenau University of Technology, Department of Mathematics, Ilmenau, Germany

Given a graph G and $X \subseteq V(G)$, we say M is an *minor of G rooted at X*, if M is a minor of G such that each bag contains at most one vertex of X and X is a subset of the union of all bags. We consider the problem whether G has a highly connected minor rooted at X if $X \subseteq V(G)$ cannot be separated in G by removing a few vertices of G.

Our results constitute a general machinery for strengthening statements about k-connected graphs (for $1 \le k \le 4$) to locally spanning versions, i.e. subgraphs containing X, of graphs in which only a *vertex subset* X has high connectivity. As a first set of applications, we use this machinery to create locally spanning versions of six well-known results about degree-bounded trees, Hamiltonian paths and cycles, and subgraphs of planar graphs.

Keywords: Minor, rooted minor, connectedness, spanning subgraph.

AMS classification: 05c83, 05c40, 05c38.

[1]Partially supported by DAAD, Germany (as part of BMBF) and the Ministry of Education, Science, Research and Sport of the Slovak Republic within the project 57447800.

[2]Gefördert durch die Deutsche Forschungsgemeinschaft (DFG) – 327533333.

[3]This research is supported by the grant SCHM 3186/1-1 (270450205) from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)

6.1 Introduction and Main Result

In the present paper, we consider simple, finite, and undirected graphs; $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. For graph terminology not defined here, we refer to [Die17].

Let G be a graph and $\mathcal{M}$ be a family of pairwise disjoint subsets of $V(G)$ such that these sets — called *bags* — are non-empty and for each bag $A \subseteq V(G)$ the subgraph $G[A]$ induced by A in G is connected. Let the bags of $\mathcal{M}$ be represented by the vertex set $V(M)$ of a graph M, then we say $\mathcal{M} = (V_v)_{v \in V(M)}$ is an *M-certificate* and M is a *minor* of G if there is an edge of G connecting two bags V_u and V_v of $\mathcal{M}$ for every $uv \in E(M)$. As an equivalent definition (see [Die17]), a graph M is a minor of a graph G if it isomorphic to a graph that can be obtained from a subgraph of G by contracting edges.

Here, we want to keep a set $X \subseteq V(G)$ of *root vertices* alive in the minors. Therefore, we extend the concept of minors and introduce *rooted minors*. For adjacent vertices $x, y \in V(G)$, let G/xy denote the graph obtained from G by removing y and by adding a new edge xz for every z such that $yz \in E(G)$ and $xz \notin E(G)$. That is, the edge xy is *contracted* into the vertex x stated first (multiple edges do not occur); this is different from the standard notion of contraction, where a new artificial vertex v_{xy} is introduced as to replace both x and y. We call an edge xy of G *X-legal* if $y \notin X$. While this distinguishes xy from yx, both notions refer to the same undirected edge.

A graph M is a *minor of G rooted at X* or, shortly, an *X-minor of G* if it can be obtained from a subgraph of G containing $G[X]$ as a subgraph by a (possibly empty) sequence of contraction of X-legal edges. Lemma 6.1 shows that there is an equivalent definition of a minor of G rooted at X by using certificates:

Lemma 6.1. *Let G be a graph and $X \subseteq V(G)$. If M is a graph with $X \subseteq V(M)$ and there is an M-certificate $\mathcal{M} = (V_v)_{v \in V(M)}$ of G, then M is an X-minor of G if and only if $v \in V_v$ for all $v \in V(M)$ and $E(G[X]) \subseteq E(M)$.* ⋄

Proof of Lemma 6.1. Suppose M and $\mathcal{M}$ fulfil $v \in V_v$ for all $v \in V(M)$ and $xy \in E(M)$ for each $xy \in E(G[X])$. Then $G' = G[\bigcup_{v \in V(M)} V_v]$ is a subgraph of G. We obtain a subgraph G'' of G' by removing all edges between V_v and V_w for all distinct $v, w \in V(M)$ with $vw \notin E(M)$. Since $xy \in E(M)$ for $xy \in E(G[X])$, G'' contains $G[X]$ as a subgraph. Starting with G'' and repeatedly contracting X-legal edges vy with $v \in V(M)$ and $y \in V_v \setminus \{v\} \subseteq V(G) \setminus X$ as long as there is $v \in V(M)$ with $|V_v| \geq 2$, we obtain M. Hence, M is an X-minor of G.

Now, let M be an X-minor of G obtained from a subgraph G' of G by contracting edges. We partition $V(G')$ by defining V_v for every $v \in V(M)$. Let $V_v = \{v\}$ and iteratively add back all vertices $y \in V(G')$ to V_v if wy was contracted to $w \in V_v$. Then $\mathcal{M} = (V_v)_{v \in V(M)}$ is an M-certificate, $X \subseteq V(M)$, $v \in V_v$ for $v \in V(M)$ and $xy \in E(M)$ for $xy \in E(G[X])$ since $xy \in E(G')$. □

Note that an X-minor of G contains $G[X]$ as a subgraph and an $\emptyset$-minor of G is a minor of G in the usual sense whereas a minor of G is isomorphic to some $\emptyset$-minor of G. In this paper the set X is never empty.

If for an X-minor M of G there is an isomorphism φ from a subdivision of M into a subgraph of G such that all vertices of M are fixed by φ, then M is called a *topological X-minor* of G.

A set $S \subset V(G)$ is an *X-separator* of G if at least two components of $G - S$ obtained from G by removing S contain a vertex of X.

Let $\kappa_G(X)$ be the maximum integer less than or equal to $|X| - 1$ such that the cardinality of each X-separator $S \subset V(G)$ — if any exists — is at least $\kappa_G(X)$. It follows that $\kappa_G(X) = |X| - 1$ if $G[X]$ is complete; however, if X is a proper subset of $V(G)$, then the converse need not to be true. If $\kappa_G(V(G)) \geq k$ for a graph G, then we say that G is *k-connected*, and a $V(G)$-separator of G is a *separator* of G. This terminology corresponds to the commonly used definition of connectivity, e.g. in [Die17].

In the remainder of this section, we deal with the question whether, for a given graph G and $X \subseteq V(G)$, G has a highly connected X-minor or even a highly connected topological X-minor if $\kappa_G(X)$ is large. An answer is given by the forthcoming Theorem 6.2; we present examples showing that this theorem is best possible. The proof can be found in Section 6.2. As applications, we present, in the third Section 6.3, *local versions* of some theorems on the existence of special spanning subgraphs of graphs.

Theorem 6.2. *Let $k \in \{1, 2, 3, 4\}$, G be a graph, and $X \subseteq V(G)$ such that $\kappa_G(X) \geq k$. Then:*

(i) *G has a k-connected X-minor.*

(ii) *If $1 \leq k \leq 3$, then G has a k-connected topological X-minor.* ⋄

Observation 6.3. Theorem 6.2 (i) is best possible, because there are infinitely many (planar) graphs G with the property that G contains $X \subseteq V(G)$ such that $\kappa_G(X) = 6$ and G has no 5-connected X-minor. ⋄

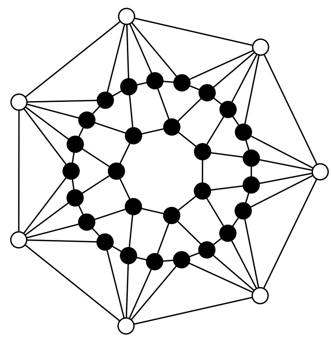


Figure 6.1: The graph G_7.

Proof. For an integer $t \geq 7$, the graph G_7 of Figure 6.1 can be readily generalized to a plane graph G_t containing a set X of t white vertices of degree 6 forming a t-gon of G_t and $4t$ black vertices of degree 4 such that $\kappa_{G_t}(X) = 6$. The assertion is proved, if there is no 5-connected X-minor M of G_t.

Assume that M exists and that M is obtained from a subgraph H of G_t by contractions of X-legal edges. If $|V(G) \setminus V(H)| = b$, then we can say that M is obtained from G_t by a number a of contractions of X-legal edges and by b removals of vertices not belonging to X. If an X-legal edge xy is contracted or a vertex $z \notin X$ is removed, then the degree of a vertex distinct from x, y or distinct from z, respectively, does not increase, respectively. Since G_t has $4t$ vertices of degree 4 and the minimum degree $\delta(M)$ of M is at least 5, each black vertex either must be removed or an incident edge must be contracted. Thus, it follows $2a + b \geq 4t$ implying $a + b \geq 2t$. Because $n = |V(M)| = |V(G_t)| - (a + b) = 5t - (a + b)$, we obtain $n \leq 3t$.

Note that M, as an X-minor of a planar graph, is planar. Since M is 5-connected, it has, up to the choice of the outer face, a unique embedding into the plane. It is clear (consider the drawing of G_7 in Figure 6.1) that the vertices of X remain boundary vertices of a t-gon α of such an embedding of M into the plane. For a vertex $x \in X$, let $N_M(x)$ be the set of neighbors of x in M, $|N_M(x)| \geq 5$. Furthermore, $|N^*(x)| \geq 3$ for $N^*(x) = N_M(x) \setminus X$ and $x \in X$, because otherwise the boundary cycle of α has a chord incident with x and the end vertices of this chord form a separator of M, contradicting the 3-connectedness, and therefore also the 5-connectedness of M. If $N^*(x_1) \cap N^*(x_2) \neq \emptyset$ for non-adjacent $x_1, x_2 \in X$, then $S = \{x_1, x_2, u\}$ with $u \in N^*(x_1) \cap N^*(x_2)$ is a separator of M, a contradiction. For the same reason $|N^*(x_1) \cap N^*(x_2)| \leq 1$ for adjacent $x_1, x_2 \in X$, and if $N^*(x_1) \cap N^*(x_2) = \{u\}$, then x_1, x_2, and u are the boundary vertices of a 3-gon of M. It follows

$$n = |V(M)| \geq |X| + |\bigcup_{x \in X} N^*(x)| \geq t + \sum_{x \in X} (|N^*(x)| - 1) \geq 3t.$$

All together, $n = 3t$, $V(M) = X \cup \bigcup_{x \in X} N^*(x)$, $|N^*(x)| = 3$ for $x \in X$, $|N^*(x_1) \cap N^*(x_2)| = 0$ for non-adjacent $x_1, x_2 \in X$, $|N^*(x_1) \cap N^*(x_2)| = 1$ for adjacent $x_1, x_2 \in X$, and if $N^*(x_1) \cap N^*(x_2) = \{u\}$ in this case, then x_1, x_2, and u are the boundary vertices of a 3-gon of M.

For $v \in \bigcup_{x \in X} N^*(x)$, it holds $|N_M(v) \cap X| \leq 2$, thus, $|N_M(v) \cap (V(M) \setminus X)| = |N_M(v) \cap (\bigcup_{x \in X} N^*(x))| \geq 3$ and it is checked readily that v has a neighbor $w \in N^*(x')$ such that $x \neq x'$ and $\{x, x', v, w\}$ is a separator of M, a contradiction to the 5-connectedness of M. □

Observation 6.4. Theorem 6.2 (ii) is best possible, because for an arbitrary integer ℓ, there is a (planar) graph G and $X \subseteq V(G)$ with $\kappa_G(X) \geq \ell$ such that every topological X-minor of G is not 4-connected. ⋄

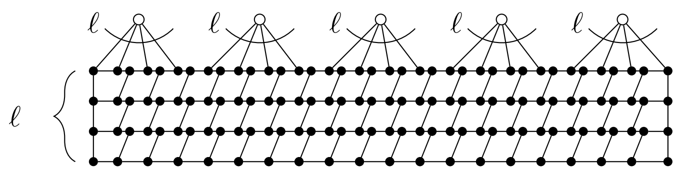


Figure 6.2: The graph F_ℓ.

Proof. For $\ell \geq 4$ consider the graph F_ℓ of Figure 6.2 and let X be the set of white vertices of F_ℓ. The vertices of X have degree $\ell \geq 4$ and all black vertices have degree at most 3 in F_ℓ. Moreover, it is easy to see that $\kappa_{F_\ell}(X) = \ell$. Suppose, to the contrary, that there is a 4-connected topological X-minor M of F_ℓ and an isomorphism φ from a subdivision of M into a subgraph H of F_ℓ. Then vertex $v \in V(M)$ is a vertex of H and has degree at least 4 in H and, therefore, also in F_ℓ, thus, $v \in X$. Since $X \subseteq V(M)$ it follows $X = V(M)$. The vertices of X are boundary vertices of a common face in F_ℓ, hence, also in M. Consequently, M is a simple outerplanar graph implying $\delta(M) = 2$, a contradicting $\delta(M) \geq 4$. □

This shows also, that there cannot be any integer ℓ such that $\kappa_G(X) \geq \ell$ implies the existence of a 4-connected topological X-minor. By the first example (Observation 6.3), it remains open whether an integer ℓ exists — it must be at least 7 — such that every graph G containing $X \subseteq V(G)$ with $\kappa_G(X) \geq \ell$ has a 5-connected X-minor. We conclude this section by showing:

Observation 6.5. There cannot be any integer ℓ such that $\kappa_G(X) \geq \ell$ implies the existence of a 6-connected X-minor. ◇

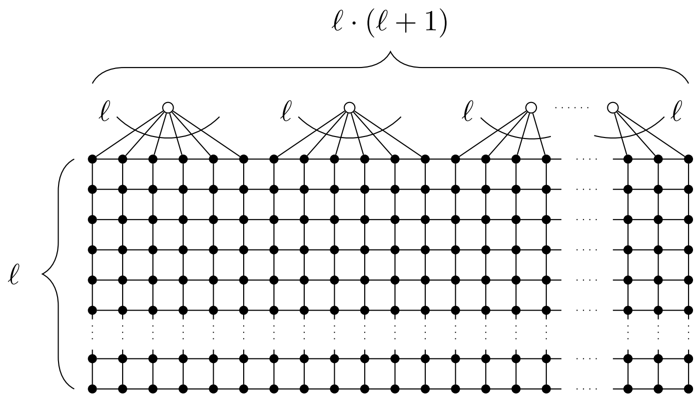


Figure 6.3: The graph H_ℓ.

Proof. Let $\ell \geq 6$ and consider the planar graph H_ℓ of Figure 6.3. It contains a set X of $\ell + 1$ white vertices of degree ℓ and further $\ell^2(\ell+1)$ black vertices. It is easy to see that $\kappa_{H_\ell}(X) = \ell$. An arbitrary X-minor M of H_ℓ is also planar and, since it contains X, it has at least $\ell + 1 \geq 7$ vertices. It is known that planar graphs are not 6-connected. □

6.2 Proof of Theorem 6.2

In the sequel, the following Lemma 6.6 — as a consequence of MENGER's Theorem [BGH01; Men27] — and Lemma 6.7 are used several times.

Lemma 6.6. *Let G be a graph, $X \subseteq V(G)$, $k \geq 1$, and $|X| \geq k+1$.*

Then $\kappa_G(X) \geq k$ if and only if for every $x, y \in X$ with $xy \notin E(G)$ there are k internally vertex disjoint paths connecting x and y. ◇

Let S be an X-separator of G, the union F of the vertex sets of at least one but not of all components of $G - S$ is called an *S-X-fragment*, if both F and $\overline{F} := V(G-S) \setminus F$ contain at least one vertex from X. In this case, $\overline{F}$ is an S-X-fragment, too. For a S-$V(G)$-fragment F, we again drop the $V(G)$ in the notion; thus, F is an *S-fragment* for a separator S of G. We say that some set $Y \subseteq V(G)$ is *X-free* if $Y \cap X = \emptyset$.

Lemma 6.7. *Let G be a graph, $S \subset V(G)$ be a separator of G, and F be an X-free S-fragment of G. Furthermore, let G' be the graph obtained from $G[\overline{F} \cup S]$ by adding all possible edges between vertices of S (if not already present).*
Then $\kappa_{G'}(X) \geq \kappa_G(X)$. ◇

Proof of Lemma 6.7. If $G'[X]$ is complete, then $\kappa_{G'}(X) = |X|-1 \geq \kappa_G(X)$, hence, Lemma 6.7 holds in this case.

Consider $x_1, x_2 \in X$ such that x_1 and x_2 are non-adjacent in G'. Since S forms a clique in G', we may assume that $x_2 \notin S$ (possibly $x_1 \in S$). According to Lemma 6.6, we have to show that there are at least $\kappa_G(X)$ internally vertex disjoint paths in G' connecting x_1 and x_2. Note that x_1 and x_2 are also non-adjacent in G and, again using Lemma 6.6, consider a set $\mathcal{P}$ of $\kappa_G(X)$ internally vertex disjoint paths of G connecting x_1 and x_2.

If some $P \in \mathcal{P}$ is not a path of G', then P contains at least one subpath Q on at least 3 vertices connecting two vertices $u, v \in S$ such that $V(Q) \cap V(G') = \{u, v\}$. We obtain a path connecting x_1 and x_2 from P by removing all inner vertices of Q and adding the edge uv. Note that $uv \in E(G')$ and repeating this procedure finally leads to a path P' of G'. If $P \in \mathcal{P}$ is a path of G', we put $P' = P$.

Since $V(P') \subseteq V(P)$ for all $P \in \mathcal{P}$, the set $\mathcal{P}' = \{P' : P \in \mathcal{P}\}$ is a set of $\kappa_G(X)$ internally vertex disjoint paths connecting x_1 and x_2. Since x_1 and x_2 have been chosen arbitrarily, Lemma 6.7 is proved. □

First we prove Theorem 6.2 (ii).

Proof of Theorem 6.2 (ii). Since X is connected in G, there is a component K of G containing all vertices from X. If K is k-connected, then K itself is a k-connected topological X-minor of G and (ii) is proved in this case.

Assume that (ii) is not true and let G be a smallest counterexample. Then G is connected and consider a smallest separator S of G, $|S| \leq k-1 \leq 2$. Since $\kappa_G(X) \geq k$, there is an X-free S-fragment F of G and $X \subseteq \overline{F} \cup S$.

Let G' be obtained from $G[\overline{F} \cup S]$ by adding all possible edges between vertices of S (if not already present), then, by Lemma 6.7, $\kappa_{G'}(X) \geq k$. Since G' has less vertices than G, G' contains a subgraph H' isomorphic to a subdivision of a k-connected X-minor M' of G'. Note that M' is also an X-minor of G, since we can contract F into one of the at most two vertices of S by performing only X-legal edge contractions.

If H' is also a subgraph of G, then this contradicts the choice of G. Thus, $k=3$, $\kappa_G(V(G)) = 2$, $S = \{u, v\}$ and $uv \in E(H') \setminus E(G)$. In this case, let H be obtained from H' by replacing uv with a path Q of G connecting u and v such that $V(Q) \cap \overline{F} = \emptyset$. Then H is a subgraph of G and also isomorphic to a subdivision of M', again a contradiction, and (ii) is proved. ■

Proof of Theorem 6.2 (i). Since a topological X-minor is an X-minor, Theorem 6.2 (ii) implies (i) if $k \in \{1, 2, 3\}$. It remains to consider the case $k=4$ and it suffices to show:

If $\kappa_G(X) \geq 4$, then there exists an X-legal edge xy such that $\kappa_{G/xy}(X) \geq 4$, unless G is 4-connected.

Claim 1. *If xy is an X-legal edge of G, then $\kappa_{G/xy}(X) \geq \kappa_G(X)$ or $\kappa_{G/xy}(X) = \kappa_G(X) - 1$ and the latter case holds if and only if there exists an X-separator of G of size $\kappa_G(X)$ containing x and y.* ⋄

Proof of Claim 1. We assume $\kappa_{G/xy}(X) < \kappa_G(X)$. Then $(G/xy)[X]$ is not complete, because otherwise $|X| - 1 = \kappa_{G/xy}(X) < \kappa_G(X)$, contradicting $\kappa_G(X) \leq |X| - 1$.

Let $x_1, x_2 \in X$ and $S \subset V(G/xy)$ be chosen such that $|S| = \kappa_{G/xy}(X)$ and S separates x_1 and x_2 in G/xy.

Then $x_1x_2 \notin E(G/xy)$ and it follows $x_1x_2 \notin E(G)$ because an edge in $E(G) \setminus E(G/xy)$ is incident with y. Since $\kappa_{G/xy}(X) < \kappa_G(X)$, $G - S$ contains a path P connecting x_1 and x_2. If $y \notin V(P)$, then P is also a path of $G/xy - S$, contradicting the choice of S. If $y \in V(P)$ and $x \notin S$, then $x \in V(G/xy)$, $N_G(y) \setminus \{x\} \subseteq N_{G/xy}(x)$ and, in both cases $x \in V(P)$ and $x \notin V(P)$, it is easy to see that $(G/xy) - S$ still contains a path connecting x_1 and x_2, again a contradiction.

All together, $x \in S$ and every path of $G - S$ connecting x_1 and x_2 contains y. It follows that $S \cup \{y\}$ separates x_1 and x_2 in G, hence, $\kappa_G(X) \leq |S \cup \{y\}| = \kappa_{G/xy}(X) + 1 \leq \kappa_G(X)$. □

So suppose that $\kappa_G(X) \geq 4$ and G is not 4-connected. Since $|V(G)| \geq |X| > 4$, there must exist a separator T with $|T| \in \{1, 2, 3\}$. Since at most one component of $G - T$ contains vertices from X, there exists an X-free T-fragment F. Let $x \in T$ and $y \in N_G(x) \cap F$, then xy is X-legal, and it turns out by Claim 1 that $\kappa_{G/xy}(X) \geq 4$ if $G[X]$ is complete or if $\kappa_G(X) \geq 5$. Assume that $|T| \in \{1, 2\}$. For all $u, v \in X$ with $uv \notin E(G)$, there are four internally vertex disjoint paths in G connecting u and v. If one of these paths contains y, then this path, say P, also contains x and there is a path in G/xy connecting u and v using only vertices from P; hence, $\kappa_{G/xy}(X) \geq 4$. Therefore, $\kappa_G(V(G)) = 3$ and $\kappa_G(X) = 4$.

Claim 2. *Let G be a graph, $X, X' \subseteq V(G)$, $G[X]$ and $G[X']$ not complete and S and S' be X-separators and X'-separators with $|S| = \kappa_G(X)$ and $|S'| = \kappa_G(X')$, respectively. For an S-X-fragment F and an S'-X'-fragment F', let $T(F, F') := (F \cap S') \cup (S' \cap S) \cup (S \cap F')$.*

Then

(i) *If $F \cap F' \neq \emptyset$, then $T(F, F')$ is a separator of G separating $F \cap F'$ from the remaining graph,*

(ii) $|T(F, F')| + |T(\overline{F}, \overline{F'})| = |S| + |S'|$. ⋄

Proof of Claim 2. Since S and S' are separators, $N_G(F \cap F') \subseteq S \cup F$ and $N_G(F \cap F') \subseteq S' \cup F'$. Hence, $N_G(F \cap F') \subseteq T(F, F')$. Obviously, $V(G) \neq T(F, F') \cup (F \cap F')$, thus $N_G(F \cap F')$ is a separator of G; and so is $T(F, F')$. This proves (i) and easy counting leads to (ii). □

Now, let us go back to the situation that there is a separator T of G with $|T| = \kappa_G(V(G)) = 3$.

Claim 3. *Let T be a separator of G with $|T| = 3$. Then*

(i) *T is an anticlique,*

(ii) *if F is an X-free T-fragment, $x \in T$, $y \in F \cap N_G(x)$, S is an X-separator with $x, y \in S$ of minimal size, and B is an S-X-fragment, then $|B \cap T| = 1$.* ⋄

Proof of Claim 3. Let F be an X-free T-fragment and $x \in T$. For $y \in F \cap N_G(x)$, the edge xy is X-legal. By Claim 1, there exists an X-separator S with $x, y \in S$. Let B be an S-X-fragment. If $T \cap B = \emptyset$, then $B \cap \overline{F}$ is not X-free and is separated by $T(B, \overline{F})$ from $\overline{B}$ (Claim 2 (i)). But $T(B, \overline{F}) = (B \cap T) \cup (T \cap S) \cup (S \cap \overline{F}) \subseteq S \setminus \{y\}$ has at most three vertices, a contraction to $\kappa_G(X) = 4$.

In the same vein, $T \cap \overline{B} \neq \emptyset$ and, because $|T| = 3$, it follows $|B \cap T| = |\overline{B} \cap T| = 1$ and the two vertices in $T \setminus \{x\}$ are non-adjacent. Since x has been chosen arbitrarily from T, T is an anticlique in G, and so is every separator T of G with $|T| = 3$. □

Claim 4. *Let T be a separator of G with $|T| = 3$ and F be an X-free T-fragment, then $|F| = 1$.* ⋄

Proof of Claim 4. Let $x \in T$, $y \in F \cap N_G(x)$, S be an X-separator with $x, y \in S$ of minimal size (by Claim 1), and B be an S-X-fragment. If $|F \cap S| \geq 2$, then $|T(B, \overline{F})| \leq 3$ and $|T(\overline{B}, \overline{F})| \leq 3$, and both $B \cap \overline{F}$ and $\overline{B} \cap \overline{F}$ are X-free, so that $X \subseteq T \cup (\overline{F} \cap S)$, contradicting $|X| \geq 5$. Hence $F \cap S = \{y\}$. Let x' be the unique vertex in $B \cap T$ by Claim 3 (ii).

It follows that $B \cap F = \emptyset$ for otherwise this set would be an $\{x, y, x'\}$-fragment as $T(B, F) = \{x, y, x'\}$ is a separator of G by Claim 2 (i); but $\{x, y, x'\}$ is not an anticlique since $xy \in E(G)$, which is a contradiction to Claim 3 (i). Likewise, $\overline{B} \cap F = \emptyset$, so that $F = \{y\}$, and again, this holds for every X-free T-fragment. □

Now, let $T = \{x, x', x''\}$ be a separator of G, $F = \{y\}$ be an X-free T-fragment (Claim 4), and S be an X-separator with $x, y \in S$ and $|S| = 4$. Then there is an S-X-fragment B and unique vertices x' and x'' in $B \cap T$ and $\overline{B} \cap T$, respectively (by Claim 3 (ii)). There exists an X-separator S' with $x', y \in S'$ by Claim 1 and we may take an S'-X-fragment B' such that $x \in B'$ and $x'' \in \overline{B'}$ (by Claim 3 (ii)). This situation is sketched in Figure 6.4.

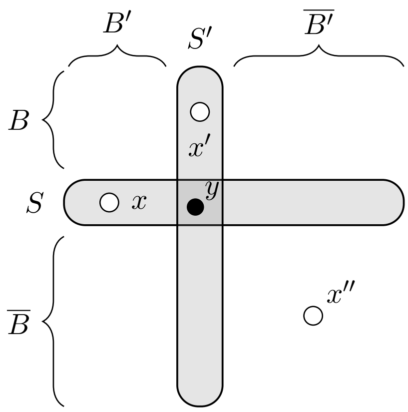


Figure 6.4

Claim 5. *The following holds:*

(i) *$B \cap \overline{B'}$ or $\overline{B} \cap B'$ is X-free,*

(ii) *$B \cap B'$ or $\overline{B} \cap \overline{B'}$ is X-free,*

(iii) *If $B \cap B' = \emptyset$, then $|T(B, B')| \geq 5$.* ⋄

Proof of Claim 5. To prove (i) assume that $B \cap \overline{B'}$ and $\overline{B} \cap B'$ are not X-free. Then, by Claim 2, $T(B, \overline{B'})$ and $T(\overline{B}, B')$ both are X-separators and since $|T(B, \overline{B'})| + |T(\overline{B}, B')| = |S| + |S'| = 8$, we have $|T(B, \overline{B'})| = |T(\overline{B}, B')| = 4$. But $T(B, \overline{B'}) \setminus \{y\}$ is also an X-separator because y has no neighbor in $B \cap \overline{B'}$, a contraction.

By the same arguments, $T(B, B') \setminus \{y\}$ is an X-separator of size 3 if $B \cap B'$ and $\overline{B} \cap \overline{B'}$ both are not X-free, and (ii) is shown.

To see (iii), assume that $B \cap B' = \emptyset$. Since x and x' must have neighbors in B and B', respectively, which can only be in $S' \setminus \{x'\}$ and $S \setminus \{x\}$, respectively, $T(B, B')$ has at least five vertices. □

Claim 6. *Let T be a separator of G with $|T| = 3$, then $T \cap X = \emptyset$.*
If additionally B and B' are an S-X-fragment and an S'-X-fragment, respectively, as defined before, then $\overline{B} \cap \overline{B'}$ is X-free. ⋄

Proof of Claim 6. Assume that $\overline{B} \cap \overline{B'}$ is not X-free. Thus, $B \cap B'$ is X-free by Claim 5 (ii). One checks that $|\overline{B} \cap S'| \geq |S \cap B'|$ and $|\overline{B'} \cap S| \geq |S' \cap B|$ (by Lemma 6.6, a vertex from X in $\overline{B} \cap \overline{B'}$ has at least four internally vertex disjoint paths to any vertex from X in B and B', respectively).

Furthermore, $T(\overline{B}, \overline{B'})$ is an X-separator by Claim 2 (i) and, therefore, $|T(\overline{B}, \overline{B'})| \geq 4$. Thus, $|T(B, B')| \leq 4$ (Claim 2 (ii)) and by Claim 5 (iii), $B \cap B' \neq \emptyset$, so that $\hat{T} = N_G(B \cap B') = T(B, B') \setminus \{y\}$ is a separator of size 3 in G. By Claim 4, $B \cap B'$ is a $\hat{T}$-fragment and its unique vertex b is adjacent to the three vertices in $\hat{T}$. Let v be the unique vertex from $T(B, B') \setminus \{x, x', y\}$. The situation is sketched in Figure 6.5.

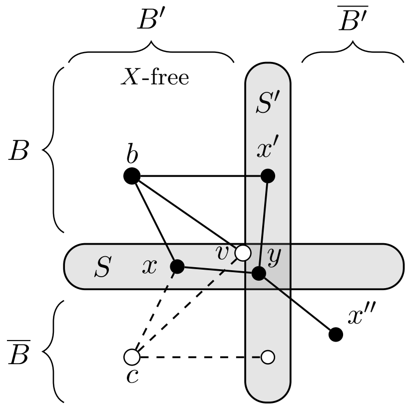


Figure 6.5

If $v \in S \cap B'$, then $|T(\overline{B}, B')| \geq 5$ (since $|\overline{B} \cap S'| \geq |S \cap B'|$), which implies that $B \cap \overline{B'}$ is empty. Thus, $B = \{b, x'\}$ and $x' \in X$, so that x' has degree at least 4 and must be adjacent to at least one of the two neighbors of b in S; this is not possible as $N_G(b) = \hat{T}$ is an anticlique (Claim 3). Analogously, the assertion $v \in S' \cap B$ is contradictory.

It follows that $v \in S \cap S'$. Thus, $|T(B, \overline{B'})| = |T(\overline{B}, B')| = 4$, where y has no neighbors in $B \cap \overline{B'}$ and $\overline{B} \cap B'$, so that the latter two sets are X-free. It follows that $X \cap B' = \{x\}$ and x has degree at least 4. Since x is non-adjacent to the two neighbors of b in S', it must have a neighbor in B' distinct from b, implying that $\overline{B} \cap B'$ is non-empty and, consequently, consists of a single vertex c. Since x is not adjacent to the two neighbors of c in S', the only neighbors of x are b, c, and y, a contraction.

Therefore, $\overline{B} \cap \overline{B'}$ is X-free and, in particular, $x'' \notin X$. By symmetry, $x, x' \notin X$, so that X is disjoint from T and, hence, from every separator T in G with $|T| = 3$. □

Let B, B' as before and note that $\overline{B} \cap \overline{B'} \neq \emptyset$ is X-free (by definition and by Claim 6). By symmetry we may assume that $\overline{B} \cap B'$ is X-free (see Claim 5 (i)), so that $\overline{B} \cap X \subseteq S' \cap \overline{B}$. This implies that $T(\overline{B}, \overline{B'})$ is not a separator in G of size 3 (since $T(\overline{B}, \overline{B'})$ is not X-free). Thus, $|T(\overline{B}, \overline{B'})| \geq 4$ and $|T(B, B')| \leq 4$ by Claim 2 (ii). By Claim 5 (iii), $B \cap B'$ is non-empty, and, as $N_G(B \cap B') = T(B, B') \setminus \{y\}$ is a separator of size 3, we get by Claim 4 that $B \cap B'$ consists of a single vertex b adjacent to all vertices in $T(B, B') \setminus \{y\}$, and, hence b is adjacent to all vertices in $B' \cap S$; among them, there is at least one vertex from $B' \cap X$ (since $\overline{B} \cap B'$ and $B \cap B'$ are X-free). This contradicts Claim 6 that $N_G(b)$ must be X-free; and Theorem 6.2 (i) is proved. ■

6.3 Locally Spanning Subgraphs

As examples of applications of Theorem 6.2, we show in this section how Theorem 6.2 can be used to ensure the existence of a subgraph H of a graph G such that H contains a specified $X \subseteq V(G)$, i.e. H is *X-spanning*, and H has certain properties if $\kappa_G(X)$ is large (Theorem 6.8 and Theorem 6.10). For a positive integer t, a *t-tree* is a tree with maximum degree at most t. Since an X-minor of a planar graph G is also planar, we first list four statements on the existence of subgraphs of a sufficiently highly connected planar graph G. In Statements 6.5 and 6.6, we consider non-planar graphs.

For 3-connected planar graphs, Barnette, Biedl, and Gao proved the following Statements 6.1 and 6.2, where Statement 6.1 is best possible since there are 3-connected planar graphs without a Hamiltonian path.

Statement 6.1 (D. Barnette [Bar66], T. Biedl [Bie14])**.** If G is a 3-connected planar graph and $uv \in E(G)$, then G has a spanning 3-tree, such that u and v are leaves of that tree. ⋄

Statement 6.2 (Z. Gao [Gao95])**.** A 3-connected planar graph G contains a 2-connected spanning subgraph of maximum degree at most 6. ⋄

In [Bar94], it is shown that the constant 6 in Statement 6.2 cannot be replaced with 5.

Tutte [Tut56] proved that every 4-connected planar graph has a Hamiltonian cycle, and Thomassen [Tho83] generalized this result by showing that every 4-connected planar graph has a Hamiltonian path connecting every given pair of vertices. Eventually, Sanders [San97] extended the results of Thomassen and of Tutte and proved the following statement.

Statement 6.3 (D. Sanders [San97])**.** Every 4-connected planar graph G has a Hamiltonian path between any two specified vertices x_1 and x_2 and containing any specified edge other than x_1x_2. ⋄

In [GH10], it is shown that Statement 6.3 is best possible in the sense that there are 4-connected maximal planar graphs with three edges of large distance apart such that any Hamiltonian cycle misses one of them.

Thomas and Yu proved Statement 6.4.

Statement 6.4 (R. Thomas, X. Yu [San97])**.** A graph obtained from a 4-connected planar graph G on at least five vertices by deleting two vertices is Hamiltonian. ⋄

Clearly, if three vertices of a 4-separator of a 4-connected planar graph are removed, then the resulting graph does not contain a Hamiltonian cycle, thus, Statement 6.4 is best possible.

For not necessarily planar graphs, Statements 6.5 and 6.6 hold.

Statement 6.5 (K. Ota, K. Ozeki [OO09])**.** Let $t \geq 4$ be an even integer and let G be a 3-connected graph. If G has no $K_{3,t}$-minor, then G has a spanning $(t-1)$-tree. ⋄

For a surface Σ, the EULER *characteristic* χ is defined by $\chi = 2 - 2g$ if Σ is an orientable surface of genus g, and by $\chi = 2 - g$ if Σ is a non-orientable surface of genus g. ELLINGHAM showed the following result.

Statement 6.6 (M. ELLINGHAM [Ell96], [OY11])**.** Let G be a 4-connected graph embedded on a surface of EULER characteristic $\chi < 0$. Then G has a spanning $\lceil \frac{10-\chi}{4} \rceil$-tree. ⋄

In the sequel, *X-spanning versions* of all six statements listed above are given. In Theorem 6.8, we present locally spanning subgraph versions of the Statements 6.1, 6.2, and 6.5, which are straight consequences of the statements and Theorem 6.2 (ii). The translation of the other three statements into local versions (see Theorem 6.10) needs more effort. Note that a minor of a graph G does not contain a graph U as a minor if already G does not contain U as a minor.

Theorem 6.8. (i) *If G is a planar graph, $X \subseteq V(G)$, and $\kappa_G(X) \geq 3$, then G contains an X-spanning 3-tree H_1. Moreover, if $uv \in E(G[X])$, then H_1 can be chosen such that u and v are leaves of H_1.*

(ii) *If G is a planar graph, $X \subseteq V(G)$, and $\kappa_G(X) \geq 3$, then G contains a 2-connected X-spanning subgraph H_2 of maximum degree at most 6.*

(iii) *If $t \geq 4$ is an even integer, $X \subseteq V(G)$ for a graph G, $\kappa_G(X) \geq 3$, and G has no $K_{3,t}$-minor, then G has an X-spanning $(t-1)$-tree.* ⋄

Proof of Theorem 6.8. Let G be a graph and $X \subseteq V(G)$ with $\kappa_G(X) \geq 3$ and properties requested as in Theorem 6.8. By Theorem 6.2 (ii), there is a topological X-minor M of G, which contains $G[X]$ as a subgraph, and let φ be an isomorphism from a certain subdivision of M into a subgraph of G such that all vertices of M are fixed by φ. Applying the suitable Statement 6.1, 6.2, or 6.5 on M, we obtain a spanning subgraph H of M containing all vertices from X. Using the isomorphism φ, a subdivision of H can be found in G which is X-spanning and has the properties in G that H has in M. ■

Given a graph G and $X \subseteq V(G)$, a subgraph H of G is an *X-spanning generalized cycle* of G if H is the edge disjoint union of a cycle C of G and $|X|$ pairwise vertex disjoint paths $P[x_i, y_i]$ of G connecting x_i and y_i (possibly $x_i = y_i$) such that $X \cap V(P[x_i, y_i]) = \{x_i\}$ and $V(C) \cap V(P[x_i, y_i]) = \{y_i\}$ for $i = 1, \ldots, |X|$. An *X-spanning generalized path* P of G is defined similarly if in the previous definition the *cycle* C is replaced with a *path* P of G. Note that an X-spanning path or an X-spanning cycle is also an X-spanning generalized path or an X-spanning generalized cycle, respectively, and we observationerve:

Observation 6.9. Let $X \subseteq V(G)$ for some graph G and M be an X-minor of G. If M has an X-spanning path or an X-spanning cycle as a subgraph, then G contains an X-spanning generalized path or an X-spanning generalized cycle, respectively. ⋄

Proof. Let P be an X-spanning path of M and $\mathcal{M} = (V_v)_{v \in V(M)}$ be an M-certificate. For each edge $uv \in E(P)$, there is an edge $e_{uv} \in E(G)$ between a vertex in V_u and a vertex in V_v. For each $v \in V(P)$ we define a set E_v of edges in V_v as follows: If v is an end vertex of P or if, for $uv, vw \in E(P)$ with

$u \neq w$, the end vertices of e_{uv} and e_{vw} in V_v coincide, then $E_v = \emptyset$. Otherwise, the end vertices of e_{uv} and e_{vw} in V_v can be connected by a path Q in $G[V_v]$, since $G[V_v]$ is connected, and we put $E_v = E(Q)$.

We obtain a path P' in G with $E(P') = \{e_{uv} : uv \in E(P)\} \cup \bigcup_{v \in V(P)} E_v$, which has non-empty intersection with V_v for all $v \in V(P)$. If $x \in X$ is not on P', then there is a path P_x in V_x connecting x to the subpath of P' in $G[V_x]$, i.e. $X \cap V(P_x) = \{x\}$ and $|V(P') \cap V(P_x)| = 1$. Eventually, P' together with all paths P_x for $x \in X \setminus V(P')$ forms an X-spanning generalized path of G.

Using the same arguments, the existence of a X-spanning generalized cycle of G can be proved if M contains an X-spanning cycle. □

Using Theorem 6.2 (i) and previous Observation 6.9, the Statements 6.3 and 6.4 can be immediately translated to locally spanning versions if the formulations "*X-spanning path*" and "*$X \setminus \{x_1, x_2\}$-spanning cycle*" in the forthcoming Theorem 6.10 (i) and (ii) are replaced by "*X-spanning generalized path*" and "*$X \setminus \{x_1, x_2\}$-spanning generalized cycle*", respectively. Theorem 6.10 (i) and (ii) do not follow directly from Theorem 6.2 since Theorem 6.2 (ii) is not true in case $k = 4$ (see Observation 6.4). We will use the theory of Tutte paths in 2-connected plane graphs (see [San97; Tho83; Tut56]) instead of Theorem 6.2 to prove the strong locally spanning versions, stated in Theorem 6.10 (i) and (ii), of Statements 6.3 and 6.4, respectively.

Furthermore, we show that Theorem 6.10 (iii) is a consequence of Statement 6.6 and Theorem 6.2 (i); thereby the upper bound on the maximum degree of the desired tree increases by "+1" compared with the one of Statement 6.6 (observe again that Theorem 6.2 (ii) does not hold in case $k = 4$).

Theorem 6.10. (i) *If G is a planar graph, $X \subseteq V(G)$, $\kappa_G(X) \geq 4$, $x_1, x_2 \in X$, $E' \subset E(G[X])$, $|E'| \leq 1$, and $x_1 x_2 \notin E'$, then G contains an X-spanning path P connecting x_1 and x_2 with $E' \subseteq E(P)$.*

(ii) *If G is a planar graph, $X \subseteq V(G)$, $\kappa_G(X) \geq 4$, and $x_1, x_2 \in X$, then $G - \{x_1, x_2\}$ contains an $X \setminus \{x_1, x_2\}$-spanning cycle.*

(iii) *Let G be a graph embedded on a surface of Euler characteristic $\chi < 0$, $X \subseteq V(G)$, $\kappa_G(X) \geq 4$. Then G has an X-spanning $(\lceil \frac{10-\chi}{4} \rceil + 1)$-tree.* ⋄

Proof of Theorem 6.10. In the following proof of Theorem 6.10, Observation 6.11 obtained from Lemma 6.7 is used several times.

Observation 6.11. Let G be a graph, $S \subset V(G)$ be a separator of G, and F be an X-free S-fragment of G. Furthermore, let G' be the graph obtained from $G[\overline{F} \cup S]$ by adding all possible edges between vertices of S (if not already present).

Then $\kappa_{G'}(X) \geq \kappa_G(X)$ and G' is planar if all following conditions hold: G is planar, $|S| \leq 3$, and S is a minimal separator. ⋄

Before we start to prove Theorem 6.10 (i), we introduce the concept of bridges and Tutte paths, on which the proofs of Statements 6.3 and 6.4 are principally based [Tut56]. Therefore, let G be a 2-connected graph embedded into the plane, H be a subgraph of G, $V(G)\setminus V(H) \neq \emptyset$, and K be a component of $G - V(H)$. If $N_G(K) \subseteq V(H)$ is the set of neighbors of K in $V(H)$, then the graph B with $V(B) = V(K) \cup N_G(K)$ and $E(B) = E(K) \cup \{uv \in E(G) : u \in V(K), v \in V(H)\}$ is a non-trivial *bridge* of H, where $N_G(K)$ and $V(K)$ are called the sets $T(B)$ of *touch vertices* and $I(B)$ of *inner vertices* of B, respectively. (A *trivial bridge* is an edge of $G - E(H)$ whose two end vertices are contained in H.) Since we are interested in bridges containing a vertex of X as an inner vertex, all references to bridges focus to non-trivial ones.

The *exterior cycle* of G is the cycle C_G bounding the infinite face of G. A path P of G on at least two vertices is a TUTTE *path* of G if each bridge of P has at most three touch vertices and each bridge containing an edge of C_G has exactly two touch vertices. (Note that a bridge of P cannot have less than two touch vertices since G is 2-connected.)

TUTTE [Tut56] proved that, for $x, y \in V(C_G)$ and $e \in E(C_G)$, G contains a TUTTE path from x to y containing e. THOMASSEN [Tho83] improved TUTTE's result by removing the restriction on the location of y, and, eventually, SANDERS ([San97]) established the following Lemma 6.12:

Lemma 6.12 (D. SANDERS [San97])**.** *If G is a 2-connected plane graph, $e \in E(C_G)$, and $x, y \in V(G)$, then G has a* TUTTE *path from x to y containing e.* ⋄

A bridge of a TUTTE path of a 3-connected planar graph G has exactly three touch vertices. Moreover, a 3-connected planar graph has a unique embedding in the plane up to the choice of the infinite face, thus, the following Observation 6.13 holds.

Observation 6.13. If G is 3-connected, then in Lemma 6.12 the condition $e \in E(C_G)$ can be replaced with $e \in E(G)$. ⋄

Thomas and Yu [San97] generalized the terms of TUTTE in the following sense. Let $E' \subseteq E(G)$ for a 2-connected graph G, then a path P of G on at least two vertices is an E'-*snake* of G if each bridge of P has at most three touch vertices and each bridge containing an edge of E' has exactly two touch vertices. Note that a TUTTE path in its original meaning is an $E(C_G)$-snake. A cycle C of G is an E'-*sling* of G if $C - e$ for some $e \in E(C)$ is an E'-snake. The following lemma generalizes TUTTE's result.

Lemma 6.14 (R. Thomas, X. Yu [San97])**.** *If G is a 2-connected plane graph with outer cycle C_G, another facial cycle D, and $e \in E(C_G)$, then G has an $(E(C_G) \cup E(D))$-sling C such that $e \in E(C)$ and no C-bridge contains edges of both C_G and D.* ⋄

If G is a plane graph, $X \subseteq V(G)$, $\kappa_G(X) \geq 4$, $E' \subseteq E(G)$, and Q is an E'-snake of G, then, by the forthcoming Lemma 6.15, all vertices of X either belong to a single Q-bridge or all belong to Q.

Lemma 6.15. *Let Q be an E'-snake of a 2-connected planar graph G, $E' \subseteq E(G)$, $X \subseteq V(G)$, and $\kappa_G(X) \geq 4$. If X is not a subset of $V(Q)$, then $X \subseteq V(B)$ for some bridge B of Q and, in this case, Q contains at most 3 vertices of X.* ⋄

Proof. Let $x \in X \setminus V(Q)$, then there is a bridge B of Q containing x as an inner vertex and B has at most three touch vertices on Q. Assume there is a vertex $y \in X \setminus V(B)$. Then the touch vertices $T(B)$ form an X-separator of G, contradicting $\kappa_G(X) \geq 4$. Hence, $X \subseteq V(B)$ and $|X \cap V(Q)| \leq |V(B) \cap V(Q)| = |T(B)| \leq 3$. □

Proof of Theorem 6.10 (i). Suppose, to the contrary, that Theorem 6.10 (i) does not hold and let G be a counterexample such that $|V(G)|$ is minimum.

If G is not 2-connected, then, because $\kappa_G(X) \geq 4$, $X \subseteq V(K)$ for a block K of G. Moreover, $E' \subset E(K)$ and, by Lemma 6.7, $\kappa_K(X) \geq \kappa_G(X) \geq 4$. Thus, K is a smaller counterexample than G, a contradiction.

Assume that G has a separator $S = \{u, v\} \subseteq V(G)$. Because $\kappa_G(X) \geq 4$, there is an S-fragment F, such that $X \subseteq F \cup S$. Let G_1 be obtained from $G[F \cup S]$ by adding the edge uv (if not already present). By Lemma 6.7 and Observation 6.11, it follows $X \subseteq V(G_1)$, $E' \subset E(G_1)$, and $\kappa_{G_1}(X) \geq 4$. Since $G - F$ contains S, there is a path Q of $G - F$ with ends u and v. The subgraph of G obtained from G_1 by replacing the path (u, uv, v) with Q shows that G_1 is a smaller counterexample than G, again a contradiction.

Hence, we may assume that G is 3-connected and consider two cases to complete the proof of Theorem 6.10 (i).

Case 1. $|E'| = 1$.

Let Q be a Tutte path of G connecting x_1 and x_2 such that $E' \subset E(Q)$ (Lemma 6.12 and Observation 6.13). If $X \subseteq V(Q)$, then Q is the desired path P, contradicting the choice of G. By Lemma 6.15, it follows $|V(Q) \cap X| \leq 3$ and there is a bridge B of Q such that $X \subseteq V(B)$, $I(B) \cap X \neq \emptyset$. Since E' consists of one edge e from $E(G[X])$ and $x_1x_2 \notin E'$, we may assume that $e = x_1u$ for some vertex $u \in X$. Hence, $T(B) = \{x_1, x_2, u\}$.

If $|V(Q)| \geq 4$ or Q has a second bridge distinct from B, then the graph G_1 obtained from $G[I(B) \cup T(B)]$ by adding all possible edges between vertices of $T(B)$ (if not already present — see Lemma 6.7 with $T(B)$ as separator), is a smaller counterexample than G, a contradiction.

Thus, $G = G_1$ if $x_1x_2 \in E(G)$ or G is obtained from G_1 by removing x_1x_2 otherwise. Moreover, $V(Q) = \{x_1, u, x_2\}$. Let $v \in I(B) \cap X$, w be an arbitrary neighbor of v distinct from x_1 (note that w exists because $\kappa_G(X) \geq 4$), and $G' = G - x_1$. Note that G' is 2-connected and assume that G' is embedded in the plane such that vw is an edge of the exterior cycle $C_{G'}$. Let R be a Tutte path of G' from u to x_2 through the edge vw. The path obtained from R by adding x_1 and $e = x_1u$ contains at least four vertices of X; hence, with Lemma 6.15, it contains X, a contradiction.

Case 2. $E' = \emptyset$.

Choose an arbitrary edge $e = uv$ of G such that $\{u, v\} \cap \{x_1, x_2\} = \emptyset$. To see that e exists, assume that each edge of G is incident with x_1 or with x_2. Then $G - \{x_1, x_2\}$ is edgeless, a contradiction to $\kappa_G(X) \geq 4$ and $|X| \geq 5$.

Now consider a Tutte path Q from x_1 to x_2 through e. Since $X \subseteq V(Q)$ contradicts the choice of G, there exists a bridge B of Q such that $X \subseteq V(B)$. It follows $X \cap I(B) \neq \emptyset$ and $x_1, x_2 \in T(B)$. Since $|V(Q)| \geq 4$, the graph obtained from $G[I(B) \cup T(B)]$ by adding all possible edges between vertices of $T(B)$ (if not already present), is a smaller counterexample than G, a contradiction. ■

Proof of Theorem 6.10 (ii). Suppose, to the contrary, that Theorem 6.10 (ii) does not hold and let G be a counterexample such that $|V(G)|$ is minimum.

If G is not 2-connected, then, as in the proof of Theorem 6.10 (i), there is a block K of G with $X \subseteq V(K)$ and $\kappa_K(X) \geq 4$. Thus, K is a smaller counterexample than G, a contradiction.

Assume that G is embedded in the plane such that x_1 is incident with the outer face and consider $G - \{x_1, x_2\}$. Since $|X| \geq 5$ (because $\kappa_G(X) \geq 4$) and $\kappa_{(G-\{x_1,x_2\})}(X \setminus \{x_1, x_2\}) \geq 2$, there is a block H containing $X \setminus \{x_1, x_2\}$.

Assume there is a component K of $G - (\{x_1, x_2\} \cup V(H))$ and let $N_G(K)$ be the neighbors of K in G. Because H, as a block of $G - \{x_1, x_2\}$, is a maximal 2-connected subgraph, it follows $|N_G(K) \cap V(H)| \leq 1$. Obviously, $N_G(K) \setminus V(H) \subseteq \{x_1, x_2\}$ and, therefore, $|N_G(K)| \leq 3$.

Consider the graph G_1 obtained from G by removing $V(K)$ and adding all edges between the vertices of $N_G(K)$ (if not already present). Then G_1 is planar since $|N_G(K)| \leq 3$ and, furthermore, $\kappa_{G_1}(X) \geq 4$ (see Lemma 6.7 and Observation 6.11). By the choice of G, there is a cycle C of G_1 containing all vertices of X except x_1 and x_2. Evidently, C misses all new edges between the vertices of $N_G(K)$, thus, C is also a cycle of G, a contraction. We conclude that $H = G - \{x_1, x_2\}$.

For $i = 1, 2$, there are (not necessarily distinct) faces α_i of H containing the vertex x_i in G and let C_i be the facial cycle of α_i in H. Because of the choice of the embedding of G, α_1 is the outer face of H, thus, $C_H = C_1$. We follow the proof in [San97].

Case 1. $C_1 = C_2$.

If $\alpha_1 \neq \alpha_2$, then $H = C_1$ and C_1 is the desired cycle. Otherwise, the vertices of $V(C_1)$ can be numbered with $v_1, v_2, \ldots, v_k$ according to their cyclic order in a such way that x_2 is not adjacent to vertices $v_2, v_3, \ldots, v_{\ell-1}$ and x_1 is not adjacent to vertices $v_{\ell+1}, v_{\ell+2}, \ldots, v_k$ for some integer ℓ with $3 \leq \ell \leq k-1$ (note that x_1 and x_2 have degree at least 4 in G). We apply Lemma 6.12 and consider a Tutte path Q of H from v_1 to v_2 containing $v_\ell v_{\ell+1}$ which can be joined by $v_1 v_2$ to a cycle. Since G is a counterexample, there is $x \in X \setminus (V(Q) \cup \{x_1, x_2\})$ and a bridge B of Q in H containing x as an inner vertex.

If $I(B) \cap V(C_1) = \emptyset$, then $N_G(x_1) \cap V(B) \subseteq T(B)$ and $T(B)$ separates x from x_1 in G, contracting $\kappa_G(X) \geq 4$. Otherwise, there is $v \in I(B) \cap V(C_1)$. Then the edge uv, where u is a neighbor of v at C_1, belongs to B. Especially, $u \in V(B)$ and B has exactly two attachment points s and t in $V(Q)$ and $s, t \in V(C_1)$. Thus, the subpath P of C_1 from s to t containing v is a path of B. If $(I(B) \cap V(C_1)) \setminus V(P) \neq \emptyset$, then there would be another subpath P' of C_1 connecting s and t with $V(P') \subseteq V(B)$, hence, $E(C_1) = E(P) \cup E(P')$, contradicting $v_\ell v_{\ell+1} \in E(Q)$.

Furthermore, $v_1, v_\ell \notin I(B)$ and $(I(B) \cap V(C_1)) \cap N_G(x_i) = \emptyset$ for one $i \in \{1, 2\}$. But then $N_G(x_i) \cap V(B) \subseteq T(B)$ and $T(B) \cup \{x_{3-i}\}$ separates x from x_i, contracting $\kappa_G(X) \geq 4$.

Case 2. $C_1 \neq C_2$.

By Lemma 6.14, there is an $(E(C_1) \cup E(C_2))$-sling C. Since G is a counterexample, there is $x \in X \setminus V(C)$ and a bridge B of C containing x as an inner vertex and, by Lemma 6.14, not simultaneously edges from both cycles C_1 and C_2. Hence, $I(B) \cap V(C_1) = \emptyset$ or $I(B) \cap V(C_2) = \emptyset$, and in both cases $T(B)$ separates x from x_1 or x_2 in G, contradicting $\kappa_G(X) \geq 4$. ∎

Proof of Theorem 6.10 (iii). Note that any minor of G is also embeddable on a surface of Euler characteristic χ. Using Theorem 6.2 (i) and Statement 6.6, let M be a 4-connected X-minor of G, $\mathcal{M} = (V_v)_{v \in V(M)}$ be an M-certificate of G, and T be a spanning tree of M of maximum degree at most $\lceil \frac{10-\chi}{4} \rceil$.

For each edge $e = uv \in E(T)$, let $e' \in E(G)$ be an arbitrary edge between a vertex in V_u and V_v. Furthermore, set $V_v' = V_v \cap (\bigcup_{e \in E(T)} V(e'))$ for $v \in V(T)$. Moreover, for $v \in V(T)$ and $w \in V_v'$, let $f(w) = |\{e \in E(T) : w \text{ is incident with } e'\}|$. Since $\sum_{w \in V_v'} f(w) = d_T(v)$, it follows $1 \leq f(w) \leq d_T(v) - |V_v'| + 1$ for all $w \in V_v'$. Since $G[V_v]$ is connected for $v \in V(T)$, the following Observation 6.16 can be seen readily by induction on $|V_v'|$.

Observation 6.16. For $v \in V(T)$, $G[V_v]$ contains a V_v'-spanning tree $T_{V_v'}$ such that, for all $w \in V(T_{V_v'})$, $d_{T_{V_v'}}(w) \leq |V_v'| - 1$ if $w \in V_v'$ and $d_{T_{V_v'}}(w) \leq |V_v'|$, otherwise. $\diamond$

Let T^* be the tree of G with

$$V(T^*) = \bigcup_{v \in V(T)} V(T_{V_v'}) \text{ and}$$
$$E(T^*) = (\bigcup_{v \in V(T)} E(T_{V_v'})) \ \cup \ \{e' : e \in E(T)\}.$$

Since $f(w) \leq d_T(v) - |V_v'| + 1$ and $d_{T_{V_v'}}(w) \leq |V_v'| - 1$, it follows $d_{T^*}(w) = f(w) + d_{T_{V_v'}}(w) \leq d_T(v)$ for $w \in V_v'$ and $v \in V(T)$. If $w \in (V_v \setminus V_v') \cap V(T^*)$ for some $v \in V(T)$, then $d_{T^*}(w) = d_{T_{V_v'}}(w) \leq |V_v'| \leq \sum_{u \in V_v'} f(u) = d_T(v)$. All together, the maximum degree of T^* is at most $\lceil \frac{10-\chi}{4} \rceil$.

Clearly, $X \subseteq \bigcup_{v \in V(T)} V_v$ and $|X \cap V_v| \leq 1$ for $v \in V(T)$. For every $v \in V(T)$ and $x \in X \cap (V_v \setminus V(T_{V_v'}))$, let P be a path of $G[V_v]$ connecting x with a vertex y of $T_{V_v'}$ such that $V(P) \cap V(T_{V_v'}) = \{y\}$ and add P to T^*. The resulting graph is the desired X-spanning $(\lceil \frac{10-\chi}{4} \rceil + 1)$-tree of G. $\blacksquare$

published in: I. Fabrici, J. Harant, S. Mohr, and J. M. Schmidt, Longer cycles in essentially 4-connected planar graphs, *Discussiones Mathematicae Graph Theory* 40.**1** (2020), pages 269–277.
Available at doi: 10.7151/dmgt.2133, arXiv: 1710.05619.

Longer Cycles in Essentially 4-Connected Planar Graphs[1]

7

Igor Fabrici[2a], Jochen Harant[b], Samuel Mohr[3b], and Jens M. Schmidt[4b]

[a]Pavol Jozef Šafárik University, Institute of Mathematics, Košice, Slovakia
[b]Ilmenau University of Technology, Department of Mathematics, Ilmenau, Germany

A planar 3-connected graph G is called *essentially 4-connected* if, for every 3-separator S, at least one of the two components of $G - S$ is an isolated vertex. Jackson and Wormald proved that the length $\mathrm{circ}(G)$ of a longest cycle of any essentially 4-connected planar graph G on n vertices is at least $\frac{2n+4}{5}$ and Fabrici, Harant and Jendroľ improved this result to $\mathrm{circ}(G) \geq \frac{1}{2}(n+4)$. In the present paper, we prove that an essentially 4-connected planar graph on n vertices contains a cycle of length at least $\frac{3}{5}(n+2)$ and that such a cycle can be found in time $\mathcal{O}(n^2)$.

Keywords: Essentially 4-connected planar graph, longest cycle, circumference, shortness coefficient.

AMS classification: 05c38, 05c10.

[1]Partially supported by DAAD, Germany (as part of BMBF) and by the Ministry of Education, Science, Research and Sport of the Slovak Republic within the project 57320575.

[2]Partially supported by Science and Technology Assistance Agency under the contract No. APVV-15-0116 and by the Slovak VEGA Grant 1/0368/16.

[3]Gefördert durch die Deutsche Forschungsgemeinschaft (DFG) – 327533333.

[4]Partially supported by the grant SCHM 3186/1-1 (270450205) from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation).

For a finite and simple graph G with vertex set $V(G)$ and edge set $E(G)$, let $N(x)$ and $d(x) = |N(x)|$ denote the neighborhood and the degree of any $x \in V(G)$ in G, respectively. The *circumference* $\text{circ}(G)$ of a graph G is the length of a longest cycle of G. A subset $S \subseteq V(G)$ is an *s-separator* of G if $|S| = s$ and $G - S$ is disconnected. From now on, let G be a 3-connected planar graph. For every 3-separator S of G, it is well-known that $G - S$ has exactly two components. We call S *trivial* if at least one component of $G - S$ is a single vertex. If every 3-separator S of G is trivial, we call the 3-connected planar graph G *essentially 4-connected*. In the present paper, we are interested in lower bounds on the circumference of essentially 4-connected planar graphs.

Jackson and Wormald [JW92] proved that $\text{circ}(G) \geq \frac{2n+4}{5}$ for every essentially 4-connected planar graph on n vertices and presented an infinite family of essentially 4-connected planar graphs G such that $\text{circ}(G) \leq c \cdot n$ for each real constant $c > \frac{2}{3}$. Moreover, there is a construction of infinitely many essentially 4-connected planar graphs with $\text{circ}(G) = \frac{2}{3}(n+4)$ (for example see [FHJ16]). It is open whether there exists an essentially 4-connected planar graph G on n vertices with $\text{circ}(G) < \frac{2}{3}(n + 4)$. Further results on the length of longest cycles in essentially 4-connected planar graphs can be found in [FHJ16; GM76; Zha87].

Fabrici, Harant and Jendrol' [FHJ16] extended the result of Jackson and Wormald by proving that $\text{circ}(G) \geq \frac{1}{2}(n + 4)$ for every essentially 4-connected planar graph G on n vertices.

Our result is presented in the following Theorem.

Theorem 7.1. *For any essentially 4-connected planar graph G on n vertices,* $\text{circ}(G) \geq \frac{3}{5}(n + 2)$. ⋄

We remark that the assertion of Theorem 7.1 can be improved to $\text{circ}(G) \geq \frac{3}{5}(n+4)$ if $n \geq 16$. This follows from using Lemma 5 in [FHJ16] and a more special version of the forthcoming inequality (7.1). We will also show how cycles of G of length at least $\frac{3}{5}(n + 2)$ can be found in quadratic time.

Let C be a plane cycle and let B be a set disjoint from $V(C)$. A plane graph H is called a *(B, C)-graph* if $B \cup V(C)$ is the vertex set of H, the cycle C is an induced subgraph of H, the subgraph of H induced by B is edgeless, and each vertex of B has degree 3 in H. The vertices in B are called *outer vertices* of C.

A face f of H is called *minor* (*major*) if it is incident with at most one (at least two) outer vertices. Note that f is incident with no outer vertex if and only if C is the facial cycle of f.

For every (B, C)-graph H, let $\mu(H)$ denote the number of minor faces of H. Then

$$\mu(H) \geq |V(H)| - |V(C)| + 2. \tag{7.1}$$

Proof of inequality (7.1). Let H be a smallest counterexample. Since $B = \emptyset$ implies $|V(H)| = |V(C)|$ and $\mu(H) = 2$, which satisfies the inequality (7.1), we may assume that B is non-empty. For each vertex $y \in B$, the three neighbors of y divide C into three internally disjoint paths $P_1(y)$, $P_2(y)$, and $P_3(y)$ with end vertices in $N(y)$. We may assume that $|V(P_1(y))| \leq |V(P_2(y))| \leq |V(P_3(y))|$ and define $\phi(y) = |V(P_1(y))| + |V(P_2(y))| - 1$ in this case.

Let $x \in B$ be chosen such that $\phi(x) = \min\{\phi(y) : y \in B\}$. Consider the two cycles A_1 and A_2 induced by $V(P_1(x)) \cup \{x\}$ and $V(P_2(x)) \cup \{x\}$, respectively. We claim that the interior of A_1 as well as the interior of A_2 is a face of H and, hence, both are minor faces. Suppose that there is a vertex z in the interior of A_i for $i \in \{1, 2\}$. Then $\phi(z) = |V(P_1(z))| + |V(P_2(z))| - 1 \leq \max\{\,|V(P_1(x))|, |V(P_2(x))|\,\} < |V(P_1(x))| + |V(P_2(x))| - 1 = \phi(x)$, which contradicts the choice of x.

Let $H' = H - x$. Note that H' is a $((B \setminus \{x\}), C)$-graph and has fewer vertices than H. Then $|V(H')| = |V(H)| - 1$, $\mu(H') \leq \mu(H) - 1$, and $\mu(H') \geq |V(H')| - |V(C)| + 2$, hence $\mu(H) \geq 1 + \mu(H') \geq 1 + |V(H')| - |V(C)| + 2 = |V(H)| - |V(C)| + 2$. ∎

Proof of Theorem 7.1. Let G be an essentially 4-connected plane graph on n vertices. If G has at most 10 vertices, then it is well known that G is Hamiltonian [Dil96]. In this case, we are done, since $n \geq \frac{3}{5}(n+2)$ for $n \geq 3$. Thus, we assume $n \geq 11$. A cycle C of G is called an *outer-independent-3-cycle* (OI3-cycle) if $V(G) \setminus V(C)$ is an independent set of vertices and $d(x) = 3$ for every $x \in V(G) \setminus V(C)$. An edge $a = xy \in E(C)$ of a cycle C is called an *extendable edge* of C if x and y have a common neighbor in $V(G) \setminus V(C)$.

In [FHJ16], it is shown that every essentially 4-connected planar graph G on $n \geq 11$ vertices contains an OI3-cycle. In this proof, let C be a longest OI3-cycle of G, let $c = |V(C)|$, and let H be the graph obtained from G by removing all *chords* of C, i.e. by removing all edges in $E(G) \setminus E(C)$ that connect vertices of C. Clearly, C does not contain an extendable edge. Obviously, H is a (B, C)-graph, with $B = V(H) \setminus V(C)$.

For the number μ of minor faces of H, we have by (7.1)

$$\mu \geq n - c + 2.$$

Moreover, we will show

$$6\,\mu \leq 4\,c \tag{7.2}$$

and then, Theorem 7.1 follows immediately.

Proof of inequality (7.2). An edge e of C is incident with exactly two faces f_1 and f_2 of H. In this case, we say f_1 is *opposite* to f_2 with respect to e. A face f of H is called *j-face* if it is incident with exactly j edges of C and the edges of C incident with f are called *C-edges* of f. Because C does not contain an extendable edge, we have $j \geq 2$ for every minor j-face of H.

We define a weight function w_0 on the set $F(H)$ of faces of H, by setting weight $w_0(f) = 6$ for every minor face f of H and weight $w_0(f) = 0$ for every major face f of H. Then $\sum_{f \in F(H)} w_0(f) = 6\,\mu$. Next, we redistribute the weights of faces of H by the rules **R1** and **R2**.

Rule R1. A minor 2-face f of H sends weight 1 through both C-edges to the opposite (possibly identical) faces.

Rule R2. A minor 3-face f of H with C-edges ux, xy, and yz sends weight 1 through its *middle C-edge xy* to the opposite face.

Let w_1 denote the new weight function; clearly, $\sum_{f \in F(H)} w_1(f) = 6\,\mu$ still holds.

For the proof of (7.2), we will show

$$w_1(f) \leq 2\,j \text{ for each } j\text{-face } f \text{ of } H. \tag{7.3}$$

To see that (7.2) is a consequence of (7.3), let each j-face f of H satisfying $j \geq 1$ send the weight $\frac{w_1(f)}{j}$ to each of its C-edges. Note that each 0-face f is major, thus $w_1(f) = 0$. Hence, the total weight of all minor and major faces is moved to the edges of C. Since every edge of C gets weight at most 4, we obtain $6\,\mu = \sum_{f \in F(H)} w_1(f) \leq 4\,c$, and (7.2) follows.

Proof of inequality (7.3). Next we distinguish several cases. In most of them, we construct a cycle $\tilde{C}$ that is obtained from C by replacing a subpath of C with another path. In every case, $\tilde{C}$ will be an OI3-cycle of G that is longer than C. This contradicts the choice of C and therefore shows that the considered case cannot occur. Note that all vertices of C in the following figures are different, because the length of the longest OI3-cycle C in a planar graph on $n \geq 11$ vertices is at least 8 [FHJ16, Lemma 4(ii)].

Case 1. *f is a major j-face.*

Because $w_0(f) = 0$ and f gets weight ≤ 1 through each of its C-edges, we have $w_1(f) \leq j$.

Case 2. *f is a minor 2-face (see Figure 7.1).*

We will show that f does not get any new weight by **R1** or by **R2**; this implies $w_1(f) = w_0(f) - (1+1) = 4$. Let xy and yz be the C-edges of f and a be the outer vertex incident with f (see Figure 7.1).

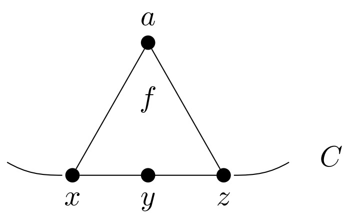


Figure 7.1

If f gets new weight by **R1** or by **R2** from a face f' opposite to f with respect to a C-edge of f, then f' is a minor 2-face or a minor 3-face of H. Without loss of generality, we may assume that f' is opposite to f with respect to the edge yz. Then yz is a common C-edge of f and f' and we distinguish the following subcases.

Case 2a. *f' is a 2-face and xy is a C-edge of f'.*

Then $\{x, z\}$ is the neighborhood of y in G, which contradicts the 3-connectedness of G.

Case 2b. *f' is a 2-face and xy is not a C-edge of f' (see Figure 7.2).*

Then a longer OI3-cycle $\tilde{C}$ is obtained from C by replacing the path (x, y, z, u) with the path (x, a, z, y, b, u), which gives a contradiction.

Case 2c. *f' is a 3-face.*

Since f' sends weight to f, then, by rule **R2**, a C-edge of f is the middle C-edge of f'. It follows that both C-edges of f are also C-edges of f' and the situation as shown in

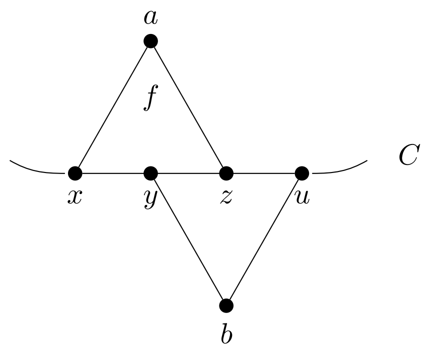


Figure 7.2

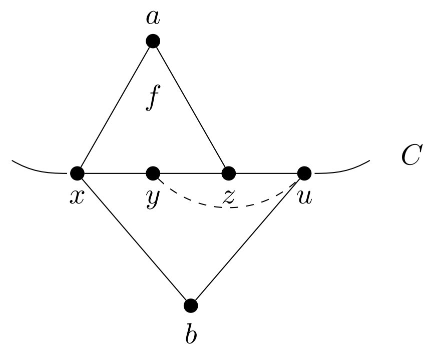


Figure 7.3

Figure 7.3 occurs. The edge yu exists in G, because otherwise $d(y) = 2$ and G would not be 3-connected. Then $\tilde{C}$ is obtained from C by replacing the path (x, y, z, u) with the path (x, a, z, y, u).

Case 3. *f is a minor 3-face (see Figure 7.4).*

Since f loses weight 1 by rule **R2** and possibly gets weight w by **R1** or by **R2**, we have $w_1(f') = 5 + w$.

If $w \leq 1$, then we are done.

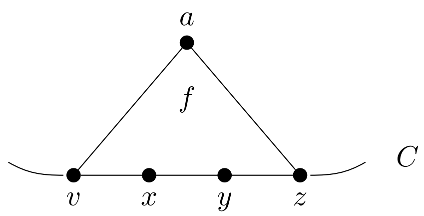


Figure 7.4

If $w \geq 2$, then f does not get any weight through the edge xy from the opposite face f'. Otherwise, if f' is a 2-face, then we have the situation as in Case 2c and if f' is a 3-face, then $w = 1$, with contradiction in both cases. Hence, f gets weight 1 through vx from the opposite face f_1 and weight 1 through yz from the opposite face f_2. Clearly, $f_1 \neq f_2$ and they are not simultaneously 3-faces.

Case 3a. *Both f_1 and f_2 are 2-faces.*

The situation is as illustrated in Figure 7.5 and $\tilde{C}$ is obtained from C by replacing the path (w, v, x, y, z, u) with the path $(w, b, x, v, a, z, y, c, u)$. Note that $b \neq c$, because $d(b) = d(c) = 3$.

Case 3b. *f_1 is a 2-face and f_2 is a 3-face.*

Then $e_2 = yz$ is the middle C-edge of f_2, as shown in Figure 7.6, and $\tilde{C}$ is obtained from C by replacing the path (w, v, x, y, z, u) with the path (w, v, a, z, y, x, c, u).

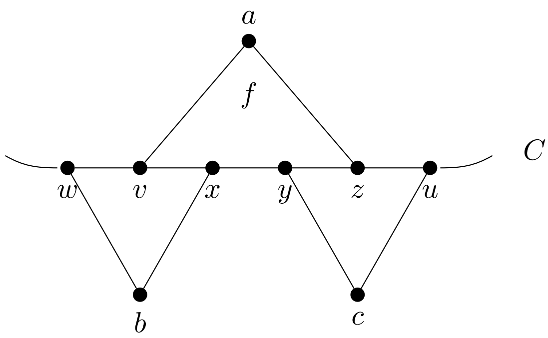


Figure 7.5

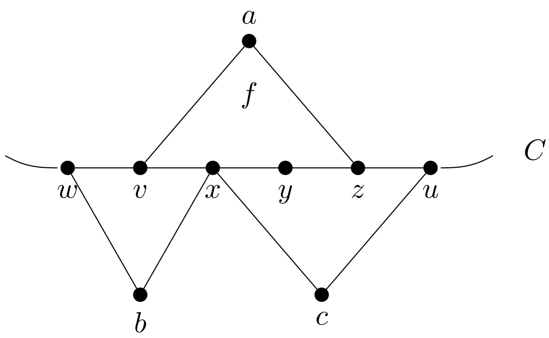


Figure 7.6

Case 4. *f is a minor 4-face (see Figure 7.7).*

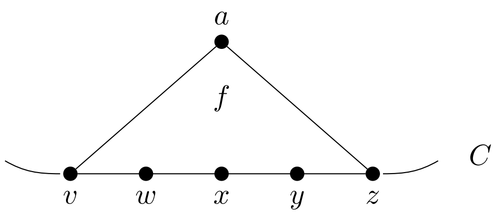


Figure 7.7

If $w_1(f) = w_0(f) + w = 6 + w$ and $w \leq 2$, then we are done.

If otherwise $w \geq 3$, there are at least three edges e_1, e_2, and e_3 among the four C-edges vw, wx, xy, and yz of f such that f gets weight from minor faces which are opposite to f with respect to e_1, e_2, and e_3, respectively.

Case 4a. *$w = 3$ and $\{e_1, e_2, e_3\} = \{vw, wx, xy\}$.*

Then no edge of $\{e_1, e_2, e_3\}$ is the middle C-edge of a minor 3-face and yz is not a C-edge of a minor 2-face. We have the situation of Figure 7.8 and one of the edges vx or xz exists in G, because otherwise x would have degree 2 in G.

Then $\tilde{C}$ is obtained again from C by replacing the path (v, w, x, y, z) with the path (v, x, w, c, y, z) or with the path (v, w, c, y, x, z), respectively.

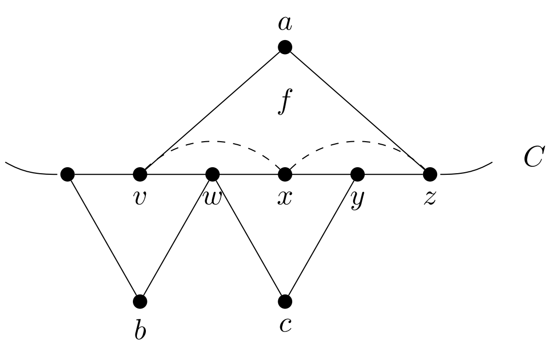


Figure 7.8

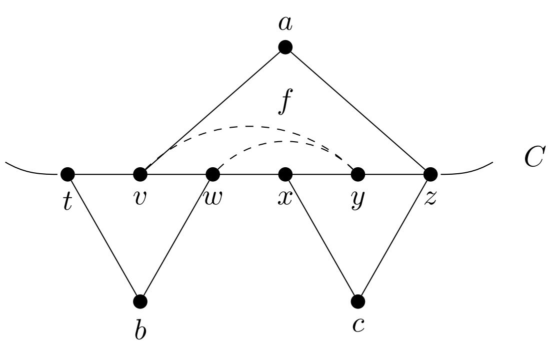


Figure 7.9

Case 4b. *$w = 3$, $\{e_1, e_2, e_3\} = \{vw, xy, yz\}$ and wx is not a C-edge of a minor 3-face.*

Then vw is not the middle C-edge of a minor 3-face opposite to f. We have the situation of Figure 7.9 and one of the edges vy or wy exists in G, because otherwise y would have degree 2 in G.

Note that $b \neq c$, because $d(b) = d(c) = 3$. Then $\tilde{C}$ is obtained from C by replacing the path (t, v, w, x, y, z) with the path (t, b, w, v, y, x, c, z) or with the path (t, v, w, y, x, c, z).

Case 4c. *$w = 3$, $\{e_1, e_2, e_3\} = \{vw, xy, yz\}$ and wx is a C-edge of a minor 3-face.*

Then vw is the middle C-edge of a minor 3-face opposite to f (see Figure 7.10).

Then at least one of the edges vy or wy exists, because otherwise y would have degree 2 in G, and $\tilde{C}$ is obtained from C by replacing the path (t, v, w, x, y, z) with the path (t, b, x, w, v, y, z) or with the path (t, v, w, y, x, c, z).

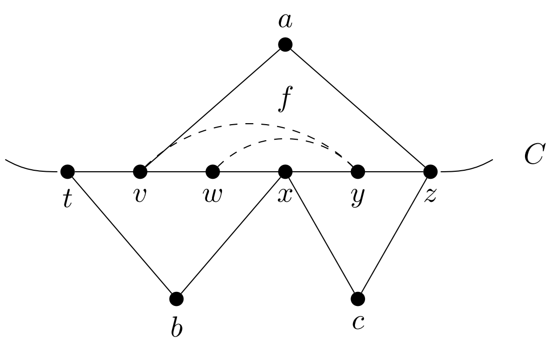


Figure 7.10

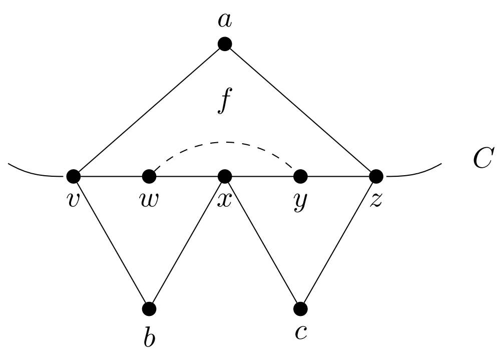


Figure 7.11

Case 4d. *$w = 4$.*

Then the edges vw, wx, xy, and yz are C-edges of minor 2-faces of H. Either a situation similar to Case 4a occurs, a contradiction, or the situation of Figure 7.11 follows.

Then the edge wy exists in G, because otherwise $d(w) = 2$ or $d(y) = 2$ in G, and $\tilde{C}$ is obtained from C by replacing the path (v, w, x, y, z) with the path (v, w, y, x, c, z).

Case 5. *f is a minor 5-face.*

Let $w_1(f) = w_0(f) + w = 6 + w$. If $w \leq 4$, then $w_1(f) \leq 10$ and we are done. If $w = 5$, then all five C-edges of f are also C-edges of minor 2-faces and we have the situation of Figure 7.12.

If the edge vx exists, then $\tilde{C}$ is obtained from C by replacing the path (s, v, w, x) with the path (s, b, w, v, x).

If vx does not exist, then, because $d(v) \geq 3$, y or z is a neighbor of v. If the edge vy exists, we get $d(x) = 2$, a contradiction. Hence, vz exists and, since $d(x) \geq 3$, xz exists as well. In this case, $\tilde{C}$ is obtained from C by replacing the path (w, x, y, z) with the path (w, c, y, x, z).

The remaining case completes the proof of (7.3) and therefore the proof of (7.2).

Case 6. *f is a minor j-face with $j \geq 6$.*

Then $w_1(f) = w_0(f) + w = 6 + w \leq 6 + j \leq 2j$. ∎

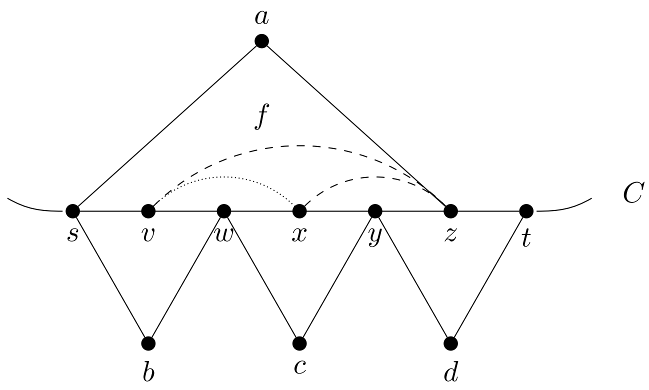


Figure 7.12

Algorithm.

We now show that a cycle of length at least $\frac{3}{5}(n+2)$ in any essentially 4-connected planar graph G on n vertices can be computed in time $\mathcal{O}(n^2)$. For $n \leq 10$, we may compute even a longest cycle in constant time, so assume $n \geq 11$. The existential proof of Theorem 7.1 proceeds by using a longest not extendable OI3-cycle of G. However, it is straightforward to observe that the proof is still valid when we replace this cycle by an OI3-cycle C that is not extendable and for which none of the local replacements described in the Cases 1–6 can be applied to increase its length (as argued, all these replacements preserve an OI3-cycle).

It suffices to describe how such a cycle C can be computed efficiently; the desired length of C is then implied by Theorem 7.1. In [FHJ16, Lemma 3], an OI3-cycle of G is obtained by constructing a special Tutte cycle with the aid of Sanders's result on Tutte paths [San97]. Using the recent result in [SS18], we can compute such Tutte paths and, by prescribing its end vertices accordingly, also the desired Tutte cycle in time $\mathcal{O}(n^2)$. This gives an OI3-cycle C_i of G.

If C_i is extendable, we compute an extendable edge of C_i and extend C_i to a longer cycle C_{i+1}; this takes time $\mathcal{O}(n)$ and preserves that C_{i+1} is an OI3-cycle. Otherwise, if there is no extendable edge of C_i (in this case, the length of C_i is at least 8 due to $n \geq 11$ and [FHJ16, Lemma 4(ii)]), we decide in time $\mathcal{O}(n)$ whether one of the local replacements of the Cases 1–6 can be applied to C_i. If so, we apply any such case and obtain the longer OI3-cycle C_{i+1} (which however may be extendable); since all replacements modify only subgraphs of constant size, this can be done in constant time. Iterating this implies a total running time of $\mathcal{O}(n^2)$, as the length of the cycle is increased at most $\mathcal{O}(n)$ times.

submitted to: *Journal of Graph Algorithms and Applications.*
I. Fabrici, J. Harant, S. Mohr, and J. M. Schmidt, Circumference of essentially 4-connected planar triangulations.

Circumference of Essentially 4-connected Planar Triangulations[1]

8

Igor Fabrici[2a], Jochen Harant[b], Samuel Mohr[3b], and Jens M. Schmidt[4b]

[a]Pavol Jozef Šafárik University, Institute of Mathematics, Košice, Slovakia
[b]Ilmenau University of Technology, Department of Mathematics, Ilmenau, Germany

A 3-connected graph G is essentially 4-connected if, for any 3-cut $S \subseteq V(G)$ of G, at most one component of $G - S$ contains at least two vertices. We prove that every essentially 4-connected maximal planar graph G on n vertices contains a cycle of length at least $\frac{2}{3}(n+4)$; moreover, this bound is sharp.

Keywords: Circumference, long cycle, triangulation, essentially 4-connected, planar graph.

AMS classification: 05c38, 05c10.

We consider finite, simple, and undirected graphs. The *circumference* $\mathrm{circ}(G)$ of a graph G is the length of a longest cycle of G. A cycle C of G is an *outer independent cycle* of G if the set $V(G) \setminus V(C)$ is independent. Note that an outer independent cycle is sometimes called a dominating cycle [Bro02], although this is in contrast to the more commonly used definition of a dominating subgraph H of G, where $V(H)$ dominates $V(G)$ in the usual sense. A set

[1]Partially supported by DAAD, Germany (as part of BMBF) and the Ministry of Education, Science, Research and Sport of the Slovak Republic within the project 57447800.

[2]Partially supported by the Slovak Research and Development Agency under contract No. APVV-15-0116.

[3]Gefördert durch die Deutsche Forschungsgemeinschaft (DFG) – 327533333.

[4]Partially supported by the grant SCHM 3186/1-1 (270450205) from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation).

$S \subseteq V(G)$ ($S \subseteq E(G)$) is a *k-cut* (a *k-edge-cut*) of G if $|S| = k$ and $G - S$ is disconnected. A 3-cut (a 3-edge-cut) S of a 3-connected (3-edge-connected) graph G is *trivial* if at most one component of $G - S$ contains at least two vertices and the graph G is *essentially 4-connected* (*essentially 4-edge-connected*) if every 3-cut (3-edge-cut) of G is trivial. A 3-edge-connected graph G is *cyclically 4-edge-connected* if for every 3-edge-cut S of G, at most one component of $G - S$ contains a cycle.

It is well-known that for (3-connected) cubic graphs different from the triangular prism $K_3 \times K_2$ (which is essentially 4-connected only) these three notions coincide (see e.g. [FJ89] and [VZ18]). Obviously, the line graph $H = L(G)$ of a 3-connected graph G is 4-connected if and only if G is essentially 4-edge-connected. These two observations are reasons for the quite great interest in studying all these three concepts of connectedness of graphs intensively.

Zhan [Zha86] proved that every 4-edge-connected graph has a Hamiltonian line graph. Broersma [Bro02] conjectured that even every essentially 4-edge-connected graph has a Hamiltonian and showed that this is equivalent to the conjecture of Thomassen [Tho86] stating that every 4-connected line graph is Hamiltonian (which is known to be equivalent to the conjecture by Matthews and Sumner [MS84] stating that every 4-connected claw-free graph is Hamiltonian, as shown by Ryjáček [Ryj97]). Among others, the subclass of essentially 4-edge-connected cubic graphs is interesting due to a conjecture of Fleischner and Jackson [FJ89] stating that every essentially 4-edge-connected cubic graph has an outer independent cycle which is equivalent to the previous three conjectures.

Regarding to the existence of long cycles in essentially 4-connected graphs we mention the following

Conjecture 11 (Bondy, see [Jac86]). *There exists a constant c, $0 < c < 1$, such that for every essentially 4-connected cubic graph on n vertices,* $\operatorname{circ}(G) \geq cn$. ⋄

Note that the conjecture of Fleischner and Jackson implies Conjecture 11 with $c = \frac{3}{4}$. Bondy's conjecture was later extended to all cyclically 4-edge-connected graphs (see [FJ89]). Máčajová and Mazák constructed essentially 4-connected cubic graphs on $n = 8m$ vertices with circumference $7m + 2$ [MM16]. We remark that the conjecture of Fleischner and Jackson and, therefore, also Bondy's conjecture with $c = \frac{3}{4}$ (this is the result of Grünbaum and Malkevitch [GM76]) are true for planar graphs, which can be seen easily by the forthcoming Lemma 8.1. Many results concerning the circumference of essentially 4-connected planar graphs G can be found in the literature.

For the class of essentially 4-connected cubic planar graphs, Tutte [Tut60] showed that it contains a non-Hamiltonian graph, Aldred, Bau, Holton, and McKay [Ald+00] found a smallest non-Hamiltonian graph on 42 vertices, and Van Cleemput and Zamfirescu [VZ18] constructed a non-Hamiltonian graph on n vertices for all even $n \geq 42$. As already mentioned, Grünbaum and Malkevitch [GM76] proved that $\operatorname{circ}(G) \geq \frac{3}{4}n$ for any essentially 4-connected cubic planar graph G on n vertices and Zhang [Zha87] (using the theory of Tutte paths) improved this lower bound on the circumference by 1. Recently, in [LS18], an infinite family of essentially 4-connected cubic planar graphs on n vertices with circumference $\frac{359}{366}n$ was constructed.

In [JW92], Jackson and Wormald extended the problem to find lower bounds on the circumference to the class of arbitrary essentially 4-connected planar graphs. Their result

$\mathrm{circ}(G) \geq \frac{2n+4}{5}$ was improved in [Fab+20c] to $\mathrm{circ}(G) \geq \frac{5}{8}(n+2)$ for every essentially 4-connected planar graph G on n vertices. On the other side, there are infinitely many essentially 4-connected maximal planar graphs G with $\mathrm{circ}(G) = \frac{2}{3}(n+4)$ [JW92]. To see this, let G' be a 4-connected maximal planar graph on $n' \geq 6$ vertices and let G be obtained from G' by inserting a new vertex into each face of G' and connecting it with all three boundary vertices of that face. Then G is an essentially 4-connected maximal planar graph on $n = 3n' - 4$ vertices and, since G' is HAMILTONian, it is easy to see that $\mathrm{circ}(G) = 2n' = \frac{2}{3}(n+4)$. It is still open whether there is an essentially 4-connected planar graph G that satisfies $\mathrm{circ}(G) < \frac{2}{3}(n+4)$. Indeed, we pose the following (to our knowledge so far unstated) Conjecture 12, which has been the driving force in that area for over a decade.

Conjecture 12. *For every essentially 4-connected planar graph on $n \geq 8$ vertices,* $\mathrm{circ}(G) \geq \frac{2}{3}(n+4)$. ⋄

By the forthcoming Theorem 8.2, Conjecture 12 is shown to be true for essentially 4-connected maximal planar graphs.

We remark that $G - S$ has exactly two components for every 3-connected planar graph G and every 3-cut S of G. Thus, in this case, G is essentially 4-connected if and only if S forms the neighborhood of a vertex of degree 3 of G for every 3-cut S of G. This property will be used frequently in the proof of Theorem 8.2.

A cycle C of G is a *good cycle* of G if C is outer independent and $\deg_G(x) = 3$ for all $x \in V(G) \setminus V(C)$. An edge xy of a good cycle C is *extendable* if x and y have a common neighbor $z \in V(G) \setminus V(C)$. In this case, the cycle C' of G, obtained from C by replacing the edge xy with the path (x, z, y) is again good (and longer than C). The forthcoming Lemma 8.1 is an essential tool in the proof of Theorem 8.2 (an implicit proof for cubic essentially 4-connected planar graphs can be found in [GM76], the general case is proved in [FHJ16]).

Lemma 8.1. *Every essentially 4-connected planar graph on $n \geq 11$ vertices contains a good cycle.* ⋄

Theorem 8.2. *For every essentially 4-connected maximal planar graph G on $n \geq 8$ vertices,*

$$\mathrm{circ}(G) \geq \frac{2}{3}(n+4). \qquad \diamond$$

Proof of Theorem 8.2. Suppose $n \geq 11$, as for $n \in \{8, 9, 10\}$, Theorem 8.2 follows from the fact that G is HAMILTONian [BJ70]. Using Lemma 8.1, let $C = [v_1, v_2, \ldots, v_k]$ (indices of vertices of C are taken modulo k in the whole paper) be a longest good cycle of length k of G (i.e. $\mathrm{circ}(G) \geq k$) and let $H = G[V(C)]$ be the graph obtained from G by removing all vertices of degree 3 which do not belong to C. Obviously, H is maximal planar and C is a HAMILTONian cycle of H. A face φ of H is an *empty face* of H if φ is also a face of G, otherwise φ is a *non-empty face* of H. Denote by $F_e(H)$ the set of empty faces of H and let $f_e(H) = |F_e(H)|$. Note that every face of G has at least two (of three) vertices on C. The three neighbors of a vertex of $V(G) \setminus V(C)$ induce a separating 3-cycle of G creating the boundary of a non-empty face of H, which has no edge in common with C because otherwise such an edge would be an extendable edge of C in G.

Let H_1 and H_2 be the spanning subgraphs of H consisting of the cycle C and of its chords lying in the interior and in the exterior of C, respectively. Note that $E(H_1) \cap E(H_2) = E(C)$ and H_1 and H_2 are maximal outerplanar graphs, both having k-gonal outer face and $k-2$ triangular faces. Let T_i be the weak dual of H_i, $i \in \{1,2\}$, which is the graph having all triangular faces of H_i as vertex set such that two vertices of T_i are adjacent if the triangular faces share an edge in H_i. Obviously, T_i is a tree of maximum degree at most three.

A face φ of H is a *j-face* if exactly j of its three incident edges belong to $E(C)$. Since $n \geq 11$, there is no 3-face in H and each face of H is a j-face with $j \in \{0,1,2\}$. Denote by $f_j(H_i)$ the number of empty j-faces of H_i. Since C does not contain any extendable edge, the following claim is obvious.

Claim 1. *Each face of H incident with an edge of any longest good cycle (in particular, each 1- or 2-face) is empty.* ⋄

An edge e of C incident with a j-face φ and an ℓ-face ψ, where $j, \ell \in \{1,2\}$, is a *(j,ℓ)-edge*. Let φ be a 2-face of H_i. The sequence $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$, $r \geq 2$, is the *φ-branch* if $\varphi_2, \dots, \varphi_{r-1}$ are 1-faces of H_i, φ_r is a 0-face of H_i, and φ_j, φ_{j+1} $(1 \leq j \leq r-1)$ are adjacent (i.e. B_φ is a minimal path in T_i with end vertices of degree 1 and 3). The *rim* $R(B_\varphi)$ of the φ-branch B_φ is the subgraph of C induced by all edges of C that are incident with an element of B_φ. Hence, it is easy to see:

Claim 2. *The rim of a φ-branch $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$ is a path of length r.* ⋄

Claim 3. *Let $\varphi = [v_1, v_2, v_3]$ be a 2-face of H_i, let $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$, $r \geq 2$, be the φ-branch of H_i, and let $v_0v_2 \in E(H_{3-i})$. If*

(i) *$R(B_\varphi) = (v_1, v_2, \dots, v_{r+1})$ is the rim of B_φ or*

(ii) *$R(B_\varphi) = (v_0, v_1, \dots, v_r)$ is the rim of B_φ and $v_{-1}v_2 \in E(H_{3-i})$, or*

(iii) *$R(B_\varphi) = (v_{3-r}, \dots, v_2, v_3)$ is the rim of B_φ and $v_{-1}v_2 \in E(H_{3-i})$,*

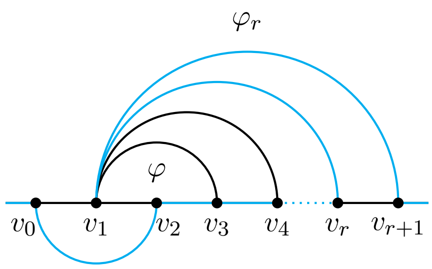
then φ_r is empty. ⋄

Proof. (i): The cycle C' obtained from C by replacing the path $(v_0, v_1, \dots, v_{r+1})$ with the path $(v_0, v_2, \dots, v_r, v_1, v_{r+1})$ (Figure 8.1a) is another longest good cycle of G and contains the edge v_1v_{r+1} incident with φ_r, thus φ_r is empty (by Claim 1).

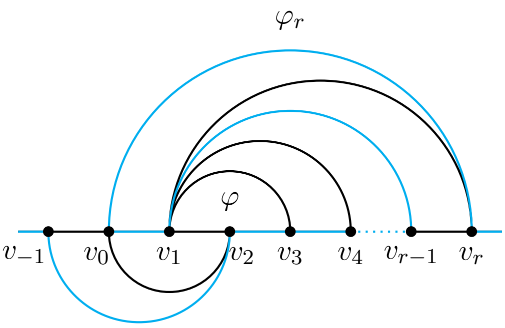
(ii): Let $\varphi_s = [v_0, v_1, v_s]$, for some s with $3 \leq s \leq r$, be a 1-face of H_i. The cycle C' obtained from C by replacing the path $(v_{-1}, v_0, \dots, v_r)$ by the path $(v_{-1}, v_2, \dots, v_{r-1}, v_1, v_0, v_r)$, for $s = r$ (Figure 8.1b), or by the path $(v_{-1}, v_2, v_1, v_3, \dots, v_{r-1}, v_0, v_r)$, for $s \leq r-1$ (Figure 8.1c), is a longest good cycle of G and contains the edge v_0v_r incident with φ_r, thus φ_r is empty (by Claim 1).

(iii): If $r \leq 3$, then φ_r is empty by (i) or (ii). If $r \geq 4$, then $v_0v_3, v_{-1}v_3 \in E(H_i)$, thus $\{v_{-1}, v_2, v_3\}$ is a non-trivial 3-cut, a contradiction. □

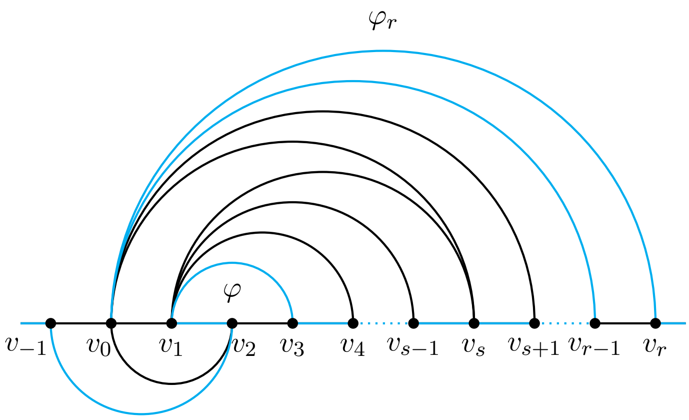
These tools will be used continuously in the following; we continue with the proof of Theorem 8.2. Hereby, we consider two cases. In the first case, both subgraphs H_1 and H_2 have some 0-faces. By using a customized discharging method, we distribute some weights from edges to faces to prove that sufficiently many faces are empty (each empty face will finally contain weight at most $\frac{2}{3}$). In the second case, there are only empty faces on one side of C, so that all vertices not in C are located on the other side of C. We have to prove that there are some additional empty faces on this side.



(a)



(b)



(c)

Figure 8.1: A longest good cycle (cyan) sharing an edge with φ_r.

CASE 1. Let H_1 and H_2 both contain at least two 0-faces or one non-empty 0-face.

For every edge e of C we define the weight $w_0(e) = 1$. Obviously, $\sum_{e \in E(C)} w_0(e) = |E(C)| = k$.

First redistribution of weights.

Each edge of C sends weight to both incident faces as follows

Rule R1. A (1,1)-edge sends $\frac{1}{2}$ to both incident 1-faces.

Rule R2. A (1,2)-edge sends $\frac{2}{3}$ to the incident 1-face and $\frac{1}{3}$ to the incident 2-face.

Rule R3. A (2,2)-edge sends $\frac{1}{2}$ to both incident 2-faces.

The edges of C completely redistribute their weights to incident 1- and 2-faces. For an empty face φ, let $w_1(\varphi)$ be the total weight obtained by φ (in first redistribution). Obviously, for an empty face φ, it is

$$w_1(\varphi) = \begin{cases} 1, & \text{if } \varphi \text{ is a 2-face incident with two (2,2)-edges,} \\ \frac{5}{6}, & \text{if } \varphi \text{ is a 2-face incident with a (1,2)-edge and a (2,2)-edge,} \\ \frac{2}{3}, & \text{if } \varphi \text{ is a 2-face incident with two (1,2)-edges,} \\ \frac{2}{3}, & \text{if } \varphi \text{ is a 1-face incident with a (1,2)-edge,} \\ \frac{1}{2}, & \text{if } \varphi \text{ is a 1-face incident with a (1,1)-edge,} \\ 0, & \text{if } \varphi \text{ is a 0-face.} \end{cases}$$

Moreover, $\sum_{\varphi \in F_{\mathrm{e}}(H)} w_1(\varphi) = |E(C)| = k$.

Second redistribution of weights.

The weight of 2-faces of H exceeding $\frac{2}{3}$ will be redistributed to 1-faces and empty 0-faces of H by the following rules. Let φ be a 2-face of H_i with $w_1(\varphi) > \frac{2}{3}$ (i.e. incident with at least one (2,2)-edge) and let $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$, $r \geq 2$, be the φ-branch. Moreover, let α be a 2-face of H_{3-i} adjacent to φ and let α_2 be the face of H_{3-i} adjacent to α.

Rule R4. φ sends $w_1(\varphi) - \frac{2}{3}$ to φ_r if φ_r is empty and $r \leq 3$.

Rule R5. φ sends $\frac{1}{6}$ to φ_j if φ_j $(2 \leq j \leq r-1)$ is a 1-face incident with a (1,1)-edge.

Rule R6. φ sends $\frac{1}{6}$ to φ_r if φ_r is empty and $r \geq 4$.

Rule R7. φ sends $\frac{1}{6}$ to α_2 if α is incident with a (1,2)-edge and α_2 is an empty 0-face.

Rule R8. φ sends $\frac{1}{6}$ to β_2, where β is a 2-face of H_{3-i} having exactly one common vertex with φ and incident with two (1,2)-edges and β_2 is an empty 0-face of H_{3-i} adjacent to β.

For an empty face φ, let $w_2(\varphi)$ be the total weight obtained by φ (after second redistribution). Obviously, $\sum_{\varphi \in F_{\mathrm{e}}(H)} w_2(\varphi) = |E(C)| = k$ (as non-empty faces do not obtain any weight). In the following, we will show that the weight $w_2(\varphi)$ of each (empty) face φ does not exceed $\frac{2}{3}$ which will mean $k = \sum_{\varphi \in F_{\mathrm{e}}(H)} w_2(\varphi) \leq \frac{2}{3} f_{\mathrm{e}}(H)$. The maximal planar graph G has exactly $2n-4$ faces. Each of $f_{\mathrm{e}}(H) \geq \frac{3}{2}k$ empty faces of H is a face of G as well, and each of $n-k$ (pairwise non-adjacent) vertices of G not belonging to C (whose removal has created a non-empty face of H) is incident with three ("private") faces of G. Hence $2n - 4 = |F(G)| = f_{\mathrm{e}}(H) + 3(n-k) \geq \frac{3}{2}k + 3n - 3k$ and finally $k \geq \frac{2}{3}(n+4)$ will follow.

Weight of a 2-face.

Let $\varphi = [v_1, v_2, v_3]$ be a 2-face of H_i and let $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$, $r \geq 2$, be the φ-branch. As already mentioned, $\frac{2}{3} \leq w_1(\varphi) \leq 1$. We check that the weight of φ exceeding $\frac{2}{3}$ will be shifted in the second redistribution.

1. Let φ be incident with two (2,2)-edges (note that $w_1(\varphi) = 1$). Denote $\alpha = [v_0, v_1, v_2]$ and $\beta = [v_2, v_3, v_4]$ the 2-faces of H_{3-i} adjacent to φ. Let α_2 and β_2 be the face of H_{3-i} adjacent to α and β, respectively. Each of the faces φ_2, α_2, and β_2 is either a 1-face or empty 0-face (by Claim 3(i)).

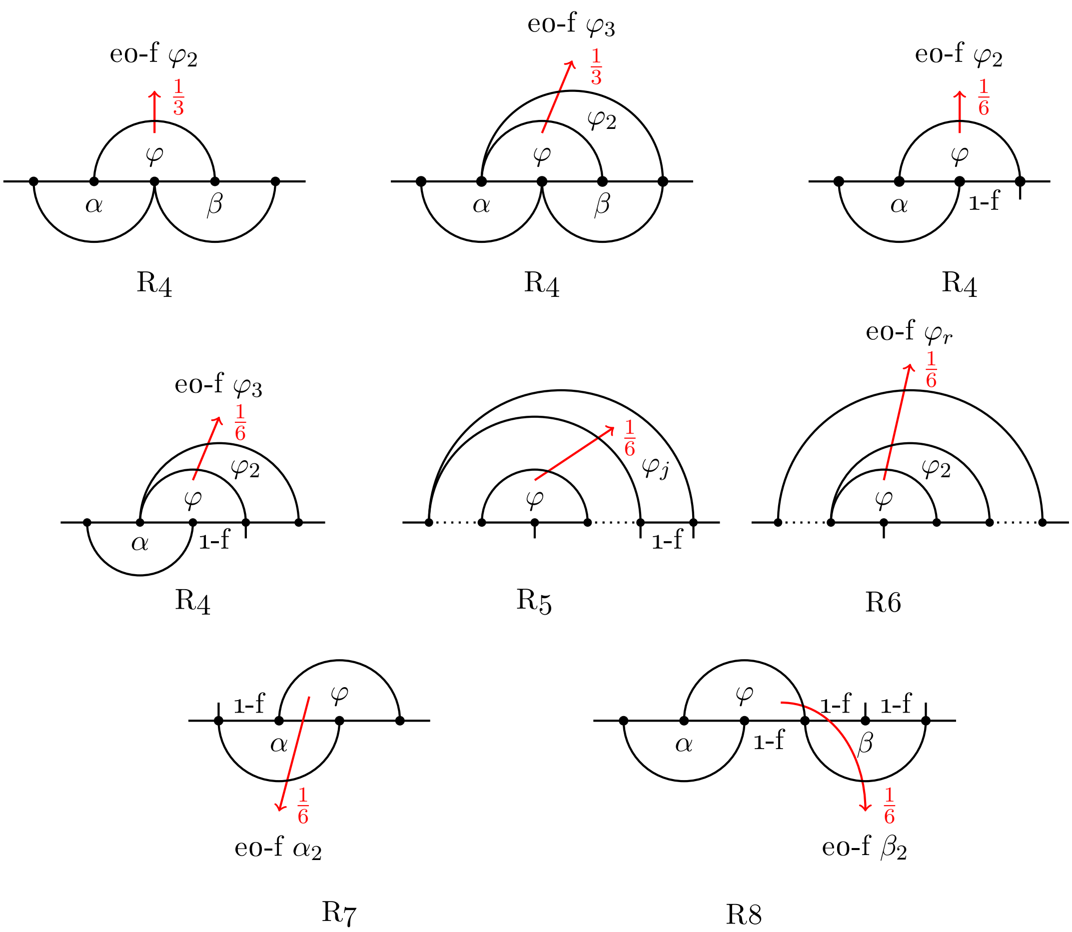


Figure 8.2: Redistribution rules R4–R8 (1-f is a 1-face and eo-f is an empty 0-face).

1.1. Let α_2 and β_2 be 0-faces (possibly $\alpha_2 = \beta_2$).

1.1.1. If edges v_0v_1 and v_3v_4 of C do not belong to the rim $R(B_\varphi)$ of B_φ, then $r = 2$, thus φ sends $\frac{1}{3}$ to empty 0-face φ_2 (by R4).

1.1.2. If v_0v_1 belongs to the rim $R(B_\varphi)$ and v_3v_4 does not belong to $R(B_\varphi)$, then $\varphi_2 = [v_0, v_1, v_3]$ is a 1-face and φ_r is empty (by Claim 3(i)). Thus φ sends weight $\geq \frac{1}{6}$ to φ_r (by R4 or R6) and $\frac{1}{6}$ to α_2 (by R7). (Similarly if v_0v_1 does not belong to $R(B_\varphi)$ and v_3v_4 belongs to $R(B_\varphi)$.)

1.1.3. If edges v_0v_1 and v_3v_4 belong to the rim $R(B_\varphi)$, then both are (1,2)-edges. Thus φ sends $\frac{1}{6}$ to α_2 and $\frac{1}{6}$ to β_2 (by R7).

1.2. Let $\alpha_2 = [v_{-1}, v_0, v_2]$ be a 1-face and β_2 be a 0-face. (Similarly if α_2 is a 0-face and β_2 is a 1-face.)

1.2.1. If v_3v_4 does not belong to the rim $R(B_\varphi)$, then $r \leq 3$ and φ_r is empty (by proof of Claim 3(iii)). Thus φ sends $\frac{1}{3}$ to φ_r (by R4).

1.2.2. If v_3v_4 belongs to the rim $R(B_\varphi)$ and v_0v_1 does not belong to $R(B_\varphi)$, then $\varphi_2 = [v_1, v_3, v_4]$ is a 1-face and φ_r is empty (by Claim 3(i)). Thus φ sends weight $\geq \frac{1}{6}$ to φ_r (by R4 or R6) and $\frac{1}{6}$ to β_2 (by R7).

1.2.3. Let edges v_3v_4 and v_0v_1 belong to the rim $R(B_\varphi)$, then both are (1,2)-edges. If v_0v_1 and v_3v_4 are incident with φ_2 and φ_3, then $\{v_0, v_2, v_4\}$ is a non-trivial 3-cut, a contradiction. If $\varphi_2 = [v_0, v_1, v_3]$ and $\varphi_3 = [v_{-1}, v_0, v_3]$, then $\{v_{-1}, v_2, v_3\}$ is a non-trivial 3-cut, a contradiction as well. Thus $\varphi_2 = [v_1, v_3, v_4]$ and $\varphi_3 = [v_1, v_4, v_5]$.

1.2.3.1. If $v_{-1}v_0$ does not belong to the rim $R(B_\varphi)$, then φ_r is empty (by Claim 3(ii)). Thus φ sends $\frac{1}{6}$ to φ_r (by R6) and $\frac{1}{6}$ to β_2 (by R7).

1.2.3.2. If $v_{-1}v_0$ belongs to the rim $R(B_\varphi)$, then $v_{-1}v_0$ is a (1,1)-edge. Thus φ sends $\frac{1}{6}$ to φ_j, a 1-face of B_φ incident with $v_{-1}v_0$ (by R5) and $\frac{1}{6}$ to β_2 (by R7).

1.3. Let $\alpha_2 = [v_{-1}, v_0, v_2]$ and $\beta_2 = [v_2, v_4, v_5]$ be 1-faces.

1.3.1. If v_3v_4 does not belong to the rim $R(B_\varphi)$, then $r \leq 3$ and φ_r is empty (by proof of Claim 3(iii)). Thus φ sends $\frac{1}{3}$ to φ_r (by R4). (Similarly if v_0v_1 does not belong to $R(B_\varphi)$.)

1.3.2. Let edges v_0v_1 and v_3v_4 belong to the rim $R(B_\varphi)$, then both are (1,2)-edges. If v_0v_1 and v_3v_4 are incident with φ_2 and φ_3, then $\{v_0, v_2, v_4\}$ is a non-trivial 3-cut, a contradiction. If $\varphi_2 = [v_0, v_1, v_3]$ and $\varphi_3 = [v_{-1}, v_0, v_3]$, then $\{v_{-1}, v_2, v_3\}$ is a non-trivial 3-cut, a contradiction as well. (Similarly if $\varphi_2 = [v_1, v_3, v_4]$ and $\varphi_3 = [v_1, v_4, v_5]$.)

2. Let φ be incident with (2,2)-edge v_1v_2 and (1,2)-edge v_2v_3 (note that $w_1(\varphi) = \frac{5}{6}$). Denote $\alpha = [v_0, v_1, v_2]$ the 2-face of H_{3-i} adjacent to φ and let α_2 be the face of H_{3-i} adjacent to α. Each of the faces φ_2 and α_2 is either a 1-face or empty 0-face (by Claim 3(i)).

2.1. Let α_2 be 0-face.

2.1.1. If v_0v_1 does not belong to the rim $R(B\varphi)$, then φ_r is empty (by Claim 3(i)). Thus φ sends $\frac{1}{6}$ to φ_r (by R4 or R6).

2.1.2. If v_0v_1 belongs to the rim $R(B\varphi)$, then v_0v_1 is a (1,2)-edge. Thus φ sends $\frac{1}{6}$ to α_2 (by R7).

2.2. Let α_2 be a 1-face incident with $v_{-1}v_0$ (i.e. $\alpha_2 = [v_{-1}, v_0, v_2]$).

2.2.1. If v_3v_4 does not belong to the rim $R(B_\varphi)$, then $r \leq 3$ and φ_r is empty (by proof of Claim 3(iii)). Thus φ sends $\frac{1}{6}$ to φ_r (by R4).

2.2.2. If v_3v_4 belongs to the rim $R(B_\varphi)$ and v_0v_1 does not belong to $R(B_\varphi)$, then $\varphi_2 = [v_1, v_3, v_4]$ is a 1-face and φ_r is empty (by Claim 3(i)). Thus φ sends $\frac{1}{6}$ to φ_r (by R4 or R6).

2.2.3. Let edges v_3v_4 and v_0v_1 belong to the rim $R(B_\varphi)$. If $v_{-1}v_0$ does not belong to $R(B_\varphi)$, then φ_r is empty (by Claim 3(ii)). Thus φ sends $\frac{1}{6}$ to φ_r (by R6). Otherwise $v_{-1}v_0$ belongs to $R(B_\varphi)$, thus it is a (1,1)-edge incident with a 1-face φ_j of B_φ. Hence φ sends $\frac{1}{6}$ to φ_j (by R5).

2.3. Let α_2 be a 1-face incident with v_2v_3 (i.e. $\alpha_2 = [v_0, v_2, v_3]$). Since $v_0v_3 \in E(H_{3-i})$, φ_2 cannot be the 1-face $[v_0, v_1, v_3]$ in H_i.

2.3.1. If v_3v_4 does not belong to the rim $R(B_\varphi)$, then $r = 2$, thus φ sends $\frac{1}{6}$ to φ_2 (by R4).

2.3.2. If v_3v_4 belongs to the rim $R(B_\varphi)$, then $r \geq 3$ and $\varphi_2 = [v_1, v_3, v_4]$.

2.3.2.1. If v_3v_4 is incident with a 1-face of H_{3-i} (i.e. v_3v_4 is a (1,1)-edge), then φ sends $\frac{1}{6}$ to φ_2 (by R5).

2.3.2.2. Let v_3v_4 be incident with a 2-face β of H_{3-i} (necessarily, $\beta = [v_3, v_4, v_5]$). If $r = 3$, then φ_3 is empty (by Claim 3(i)), thus φ sends $\frac{1}{6}$ to φ_3 (by R4). If $r = 4$, then $\varphi_3 = [v_1, v_4, v_5]$ (as $\{v_0, v_3, v_4\}$

is a non-trivial 3-cut if $\varphi_3 = [v_0, v_1, v_4]$) and φ_4 is empty (by Claim 3(i)), thus φ sends $\frac{1}{6}$ to φ_4 (by R6). Finally, let $r \geq 5$. Necessarily $\varphi_3 = [v_1, v_4, v_5]$ (as for $r = 4$) and $\varphi_4 = [v_1, v_5, v_6]$ (as $\{v_0, v_3, v_5\}$ is a non-trivial 3-cut if $\varphi_4 = [v_0, v_1, v_5]$) are 1-faces of B_φ. If v_5v_6 is a (1,1)-edge, then φ sends $\frac{1}{6}$ to φ_4 (by R5). Otherwise v_5v_6 is a (1,2)-edge, thus it does not belong to β-branch (in H_{3-i}) and therefore β_2 is a 0-face, which is, moreover, empty (as the cycle obtained from C by replacing the path $(v_0, \ldots, v_5)$ by the path $(v_0, v_2, v_1, v_4, v_3, v_5)$ is a longest good cycle of G and contains the edge v_3v_5 incident with β_2 (Claim 1)). Hence φ sends $\frac{1}{6}$ to β_2 (by R8).

Weight of a 1-face.

To estimate the weight of a 1-face, we use the following simple observation:

Claim 4. *Each 1-face of H belongs to at most one branch.* ⋄

Let ψ be a 1-face incident with an edge e of C. If e is a (1,2)-edge, then ψ obtains weight $\frac{2}{3}$ from e (by R2) only. Otherwise e is a (1,1)-edge, thus ψ obtains $\frac{1}{2}$ from e (by R1). Furthermore, in this case, ψ can get $\frac{1}{6}$ from a 2-face φ (by R5) if ψ belongs to the φ-branch. Hence $w_2(\psi) \leq \frac{2}{3}$.

Weight of an empty 0-face.

Each empty 0-face ω belongs to at most two branches (in Case 1). Let φ be a 2-face of H_i with the φ-branch $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ such that $\varphi_r = \omega$, and let e be the edge incident with φ_r and φ_{r-1} (where $\varphi_{r-1} = \varphi$ for $r = 2$).

If φ is adjacent to two 2-faces, then ω gets through e the weight $\frac{1}{3}$ (by R4) for $r \leq 3$ or the weight $\frac{1}{6}$ (by R6) for $r \geq 4$. If φ is adjacent to one 2-face, then ω gets through e the weight $\frac{1}{6}$ (by R4) and additionally $\frac{1}{6}$ (by R7) for $r = 2$ or the weight at most $\frac{1}{6}$ (by R4) for $r = 3$ or the weight $\frac{1}{6}$ (by R6) for $r \geq 4$. Finally, if φ is adjacent to no 2-face, then ω gets through e the weight $\frac{1}{6}$ (by R6) for $r \geq 4$ or the weight at most $2 \times \frac{1}{6}$ (by R8) for $r \leq 3$.

We showed that $w_2(\varphi) \leq \frac{2}{3}$ for each empty face φ and completed the Case 1.

Thus, we can assume that in H_i are only empty faces and among them, at most one face is a 0-face. To complete the proof, we have to show that there are some empty faces in H_{3-i} as well.

CASE 2. Let H_i contain no 0-face or exactly one 0-face which is additionally empty.

Obviously, if H_i contains no 0-face, then it contains two 2-faces α_1 and α_2 (since T_i is a path and 2-faces of H_i are leaves of T_i). Note that, (only) in this case, the branches in H_i are not defined.

Remember that $H = G[V(C)]$ has $k \geq 7$ vertices (as otherwise G with at most $k + 2 \leq 8$ vertices is Hamiltonian). If H_i contains exactly one 0-face, then it contains three 2-faces α_1, α_2 and α_3 (since T_i is a subdivision of $K_{1,3}$ and 2-faces of H_i are leaves of T_i). We assume that H_{3-i} contains at least two 0-faces as otherwise all but at most one faces of H_{3-i} are empty and G has $n \leq |V(H)|+1 = k+1$ vertices and Theorem 8.2 follows immediately (with $n \geq 11$).

Distribution of points.

To estimate the number of empty 0- and 1-faces in H_{3-i}, each 2-face α_j of H_i ($j \in \{1,2\}$ if H_i contains no 0-face and $j \in \{1,2,3\}$ if H_i contains one 0-face, respectively) will distribute 1 or 2 points to faces of H_{3-i}. Let α_j be adjacent to the faces φ and ψ of H_{3-i}.

Rule P1. If φ and ψ are 2-faces of H_{3-i} with branches $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ and $B_\psi = (\psi, \psi_2, \ldots, \psi_t)$, then φ_r and ψ_t will each receive 1 point (or 2 points if $\varphi_r = \psi_t$) from α_j.

Rule P2. If φ and ψ are 1-faces of H_{3-i}, then φ and ψ will each receive 1 point from α_j.

Rule P3. If φ is a 2-faces of H_{3-i} with φ-branch $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ and ψ is a 1-face of H_{3-i} not belonging to B_φ, then φ_r and ψ will each receive 1 point from α_j.

Rule P4. If φ is a 2-faces of H_{3-i} with φ-branch $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ and ψ is a 1-face of H_{3-i} belonging to B_φ, then only ψ will receive 1 point from α_j.

For a face φ of H_{3-i}, let $p(\varphi)$ be the total number of points carried by φ (in the distribution of points).

Claim 5. $f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq \sum\limits_{\varphi \in \mathcal{F}(H_{3-i})} p(\varphi)$. ⋄

Proof. We have to prove that each 1-face of H_{3-i} gets at most 1 point and that each 0-face of H_{3-i} gets points only if it is empty and it gets at most 2 points. Consequently, Claim 5 follows by simple counting.

Let β be a 1-face of H_{3-i}. Since β can only get points if it is adjacent to some α_j and there can only be one such face then $p(\beta) \leq 1$.

Let β be a 0-face of H_{3-i}. Since β can only get points if it belongs to a branch and it belongs to at most two branches (as there are at least two 0-faces in H_{3-i}), then $p(\beta) \leq 2$. Assume first that β gets a point by P1. Then there is α_j incident with two $(2,2)$-edges and adjacent 2-faces φ and ψ of H_{3-i}. Let $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ with $\varphi_r = \beta$ be the branch which ends in β. By Claim 3(i), $\varphi_r = \beta$ is an empty 0-face.

Thus, assume that β gets a point by P3. Then there is α_j incident with a $(1,2)$-edge with adjacent 1-face ψ in H_{3-i} and a $(2,2)$-edge with adjacent 2-face φ such that ψ does not belong to the branch $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ with $\varphi_r = \beta$. Since the common edge of α_j and ψ does not belong to the rim $R(B_\varphi)$, again by Claim 3(i), $\varphi_r = \beta$ is an empty 0-face. □

Claim 6. $f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq 4$. ⋄

Proof. If $\sum\limits_{\varphi \in \mathcal{F}(H_{3-i})} p(\varphi) \geq 4$, then $f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq 4$ (by Claim 5).
Assume $\sum\limits_{\varphi \in \mathcal{F}(H_{3-i})} p(\varphi) \leq 3$.

1. Let H_i contains exactly one 0-face. As there are three 2-faces $\alpha_1, \alpha_2, \alpha_3$ in H_i, then

$$\sum_{\varphi \in \mathcal{F}(H_{3-i})} p(\varphi) = 3.$$

Furthermore, only P4 was applied to each α_j ($j \in \{1, 2, 3\}$) hence there are three 1-faces with 1 point and they belong to three different branches.

Since $|V(H)| = k \geq 7$, there is $j \in \{1, 2, 3\}$ such that α_j is adjacent to a 1-face δ of H_i. Let φ be the adjacent 2-face of α_j in H_{3-i} and $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ be its branch.

1.1. If $r \geq 4$, then φ_2 and φ_3 are 1-faces of the same branch. Thus, at most one among φ_2 and φ_3 has a point and $f_1(H_{3-i}) \geq 4$.

1.2. If $r = 3$, then δ and φ are not adjacent (i.e. $\delta \neq \varphi_2$, since H has no multiple edges) and φ_3 is an empty 0-face (by Claim 3(ii)), hence $f_1(H_{3-i}) + f_0(H_{3-i}) \geq 4$.

2. Let H_i contains no 0-face. Since $\sum\limits_{\varphi \in \mathcal{F}(H_{3-i})} p(\varphi) \leq 3$, there is $j \in \{1, 2\}$ such that P4 was applied to α_j. Let δ be the 1-face of H_i adjacent to α_j (since $|V(H)| = k \geq 7$), let φ and ψ be the 2-face and 1-face of H_{3-i} adjacent with α_j, respectively, and let $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ be the branch of φ. We may assume $\alpha_j = [v_1, v_2, v_3]$ and $\varphi = [v_2, v_3, v_4]$.

2.1. Let $r \leq 4$.

2.1.1. If $\delta = [v_0, v_1, v_3]$, then v_0v_1 does not belong to the rim $R(B_\varphi)$ (otherwise $\varphi_2 = [v_1, v_2, v_4]$, $\varphi = [v_0, v_1, v_4]$ and v_0, v_3, v_4 is a non-trivial 3-cut, a contradiction) and φ_r is an empty 0-face (by Claim 3(ii)). By P1–4, there is a face in H_{3-i} other than ψ and φ_r with a point, thus $f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq 4$.

2.1.2. If $\delta = [v_1, v_3, v_4]$, then $\varphi_2 = [v_2, v_4, v_5]$ (since $v_1v_4 \in E(H_i)$), $\psi = \varphi_3 = [v_1, v_2, v_5]$, and $\{v_1, v_4, v_5\}$ is a non-trivial 3-cut, a contradiction.

2.2. Let $r = 5$. There are three 1-faces (in fact φ_2, φ_3, and φ_4) all belonging to the same branch B_φ. We may assume that P4 was applied to α_j and P2 was applied to α_{3-j}, and all three 1-faces are adjacent to α_1 or α_2 (since otherwise there is another 1-face or empty 0-face and Claim 6 follows).

2.2.1. If $\alpha_{3-j} = [v_{-1}, v_0, v_1]$, then rim $R(B_\varphi) = (v_{-1}, \ldots, v_4)$, thus $\varphi_2 = [v_1, v_2, v_4]$ and $\delta = [v_1, v_3, v_4]$, a contradiction to the simplicity of H.

2.2.2. If $\alpha_{3-j} = [v_4, v_5, v_6]$ and $\delta = [v_0, v_1, v_3]$, then rim $R(B_\varphi) = (v_1, \ldots, v_6)$ and φ_5 is an empty 0-face (by Claim 3(ii)), thus $f_1(H_{3-i}) + f_0(H_{3-i}) \geq 4$.

2.2.3. If $\alpha_{3-j} = [v_4, v_5, v_6]$ and $\delta = [v_1, v_3, v_4]$, then rim $R(B_\varphi) = (v_1, \ldots, v_6)$. Hence $v_1v_6 \in E(H_{3-i})$ and consequently $\{v_1, v_4, v_6\}$ is a non-trivial 3-cut, a contradiction.

2.3. If $r \geq 6$, then there are at least four 1-faces in B_φ, thus $f_1(H_{3-i}) \geq 4$. □

Remember that each j-face of H_{3-i} is incident with j ("private") edges of C, hence $2f_2(H_{3-i}) + f_1(H_{3-i}) = k$. As each of the $k-2$ triangular faces of H_i is empty, all non-empty faces of H belong to H_{3-i} and their number is $(k-2) - f_2(H_{3-i}) - f_1(H_{3-i}) - f_0(H_{3-i}) = (k-2) - \frac{1}{2}(k - f_1(H_{3-i})) - f_1(H_{3-i}) - f_0(H_{3-i}) = \frac{k}{2} - 2 - \frac{1}{2}(f_1(H_{3-i}) + 2f_0(H_{3-i})) \leq \frac{k}{2} - 4$ (by Claim 6). Finally, at most $\frac{k}{2} - 4$ vertices of G lie outside the cycle C (and exactly k vertices on C), hence $n \leq k + (\frac{k}{2} - 4)$ and $k \geq \frac{2}{3}(n + 4)$ follows, which completes the proof of Theorem 8.2. ∎

published in: I. Fabrici, J. Harant, T. Madaras, S. Mohr, R. Soták, and C. T. Zamfirescu, Long cycles and spanning subgraphs of locally maximal 1-planar graphs, *Journal of Graph Theory* 95.1 (2020), pages 125–137.
Available at DOI: 10.1002/jgt.22542, arXiv: 1912.08028.

Long Cycles and Spanning Subgraphs of Locally Maximal 1-planar Graphs[1][2]

9

Igor Fabrici[a], Jochen Harant[b], Tomáš Madaras[a], Samuel Mohr[3b], Roman Soták[a], and Carol T. Zamfirescu[4cd]

[a]Pavol Jozef Šafárik University, Institute of Mathematics, Košice, Slovakia
[b]Ilmenau University of Technology, Department of Mathematics, Ilmenau, Germany
[c]Ghent University, Department of Applied Mathematics, Computer Science and Statistics, Ghent, Belgium
[d]Babeş-Bolyai University, Department of Mathematics, Cluj-Napoca, Roumania

A graph is *1-planar* if it has a drawing in the plane such that each edge is crossed at most once by another edge. Moreover, if this drawing has the additional property that for each crossing of two edges the end vertices of these edges induce a complete subgraph, then the graph is *locally maximal 1-planar*. For a 3-connected locally maximal 1-planar graph G, we show the existence of a spanning 3-connected planar subgraph and prove that G is Hamiltonian if G has at most three 3-vertex-cuts, and that G is traceable if G has at most four 3-vertex-cuts. Moreover, infinitely many non-traceable 5-connected 1-planar graphs are presented.

Keywords: 1-planar graph, spanning subgraph, longest cycle, Hamiltonicity.

[1]Partially supported by DAAD, Germany (as part of BMBF) and the Ministry of Education, Science, Research and Sport of the Slovak Republic within the project 57447800.
[2]Partially supported by the Slovak Research and Development Agency under contract No. APVV-15-0116 and by the Science Grant Agency – project VEGA 1/0368/16.
[3]Gefördert durch die Deutsche Forschungsgemeinschaft (DFG) – 327533333.
[4]Supported by a Postdoctoral Fellowship of the Research Foundation Flanders (FWO).

9.1 Introduction and Results

We use standard terminology of graph theory and consider finite and simple graphs, where $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. These graphs are represented by *drawings* in the plane, such that vertices are distinct points and edges are arcs (non-self-intersecting continuous curves, i.e. open JORDAN curves) that join two points corresponding to their incident vertices. The arcs are supposed to contain no vertex points in their interior. Such a drawing of a graph G in the plane is denoted by $D(G)$. For more details on drawings of graphs in the plane, see [HMS12; PT97]. If two edges of $D(G)$ have an internal point in common, then these edges *cross* and we call the pair of these edges a *crossing*, and the aforementioned internal point their *crossing point*. It is easy to see that a drawing can be changed locally to a different drawing with fewer crossings if two edges with a shared end vertex cross or if two edges cross several times. Thus, in the sequel we will consider drawings with the property that if two edges cross, then they do so exactly once and their four end vertices are mutually distinct.

A graph G is *planar* if there exists a drawing $D(G)$ of G without crossings. There are several different approaches generalizing the concept of planarity. One of them is to allow a given constant number of crossings for each edge in a drawing $D(G)$. In particular, if there exists a drawing $D(G)$ of a graph G such that each edge of $D(G)$ is crossed at most once by another edge, then G is *1-planar*. In this case we call $D(G)$ a *1-planar drawing* of G. This class of graphs was introduced by RINGEL [Rin65] in connection with the simultaneous vertex-face coloring of plane graphs. Properties of 1-planar graphs are studied in [FM07; HMS12; KM13; Kor08; PT97].

PACH and TÓTH [PT97] proved that each 1-planar graph on n vertices, $n \geq 3$, has at most $4n - 8$ edges and this bound is attained for every $n \geq 12$. As a consequence, a 1-planar graph has a vertex of degree at most 7, hence, it is at most 7-connected. A 1-planar graph on n vertices is *optimal* if it has exactly $4n - 8$ edges. A graph G from a family $\mathcal{G}$ of graphs is *maximal* if $G + uv \notin \mathcal{G}$ for any two non-adjacent vertices $u, v \in V(G)$. It is remarkable that there exist maximal 1-planar graphs on n vertices that have significantly fewer than $4n - 8$ edges [Bra+12]. Thus, in contrast to the planar case, the properties “optimal” and “maximal” are not the same for 1-planar graphs. Obviously, an optimal 1-planar graph is also maximal 1-planar. It is clear that a maximal planar graph is not necessarily maximal 1-planar.

The length (number of vertices) of a longest cycle of a graph G (also called *circumference* of G) is denoted by $\mathrm{circ}(G)$. If $\mathrm{circ}(G) = n$ for a graph G on n vertices, then G is HAMILTONian and a longest cycle of G is a *HAMILTONian cycle*. In the same vein, a graph is *traceable* if it contains a path visiting every vertex of the graph.

In [HMS12], it is proved that an optimal 1-planar graph is HAMILTONian. This is in sharp contrast with the family of planar graphs since MOON and MOSER [MM63] constructed infinitely many maximal planar graphs G with $\mathrm{circ}(G) \leq 9|V(G)|^{\log_3 2}$ (in fact, MOON and MOSER even showed that every path in G is strictly shorter than the aforementioned length). It is known that a maximal planar graph on $n \geq 4$ vertices is 3-connected. In [HMS12], maximal 1-planar graphs with vertices of degree 2 are constructed and it remained open there whether every 3-connected maximal 1-planar graph is HAMILTONian. Moreover, the question arises whether such a construction as the one of Moon and Moser is also possible in the class of 3-connected maximal 1-planar graphs. An answer to both questions is given by Theorem 9.1 which has the consequence that there are positive constants c and $\alpha \leq \log_3 2 < 1$ such that infinitely many 3-connected maximal 1-planar graphs G with $\mathrm{circ}(G) \leq c \cdot |V(G)|^{\alpha}$ exist.

Theorem 9.1. *If H is a maximal planar graph on $n \geq 4$ vertices, then there is a 3-connected maximal 1-planar graph G on $7n - 12$ vertices such that $\mathrm{circ}(G) \leq 4 \cdot \mathrm{circ}(H)$.* ⋄

For an arbitrary 1-planar drawing $D(G)$ of a graph G, let $D^{\times}(G)$ be the plane graph obtained from $D(G)$ by turning all crossings into new 4-valent vertices. If uv and xy are two crossing edges of $D(G)$, then let c be the vertex of $D^{\times}(G)$ corresponding to the crossing point of uv and xy. Let α be the face of $D^{\times}(G)$ such that ucx is a subpath of the facial walk of α in $D^{\times}(G)$. If ux is an edge of G and ux is crossed by another edge in $D(G)$, then it is possible to redraw the edge ux in $D(G)$ such that ux lies in the region of $D(G)$ corresponding to the face α of $D^{\times}(G)$. It follows that ux is not crossed by another edge in $D(G)$ anymore. Thus, in the sequel we will consider 1-planar drawings $D(G)$ of a graph G with the property that if uv and xy are crossing edges of $D(G)$, then the edge xu (if it exists) is not crossed by another edge in $D(G)$.

Now we will consider much wider classes of 1-planar graphs than the class of maximal 1-planar graphs. Let K_4^- be the graph obtained from the complete graph K_4 on four vertices by removing one edge. Given a 1-planar drawing $D(G)$ of a graph G, we call a crossing of $D(G)$ *full* or *almost full* if the four end vertices of its edges induce a K_4 or a K_4^-, respectively.

If for a 1-planar graph G there exists a 1-planar drawing $D(G)$ such that all crossings of $D(G)$ are full or almost full, or all crossings of $D(G)$ are full, then, in the first case, we call G *weakly locally maximal 1-planar* and $D(G)$ a *weakly locally maximal 1-planar drawing* of G or, in the second case, G *locally maximal 1-planar* and $D(G)$ a *locally maximal 1-planar drawing* of G, respectively. Obviously, a planar graph is locally maximal 1-planar and it can easily be seen that a maximal 1-planar graph is also locally maximal 1-planar and that a locally maximal 1-planar graph is also weakly locally maximal 1-planar. For a positive integer $k \geq 2$, Figure 9.1 shows a graph on $4k$ vertices which is locally maximal 1-planar, obviously not maximal 1-planar, and also not planar since it contains a subdivision of K_5 with major (4-valent) vertices u_1, x_1, y_1, z_1, x_k.

Whitney [Whi31] showed that a 4-connected maximal planar graph is Hamiltonian. Later Tutte [Tut56] proved that an arbitrary 4-connected planar graph has a Hamiltonian cycle. We remark that non-Hamiltonian 4-connected 1-planar graphs are constructed in [HMS12]. In order to formulate the next result, we recall that for an infinite family $\mathcal{G}$ of graphs, its *shortness exponent* is defined as

$$\sigma(\mathcal{G}) = \liminf_{G \in \mathcal{G}} \frac{\log \mathrm{circ}(G)}{\log |V(G)|}.$$

See [GW73] for details concerning the theory of shortness exponents. We are able to prove the following theorem — however, it remains open whether a non-Hamiltonian 6-connected 1-planar graph exists.

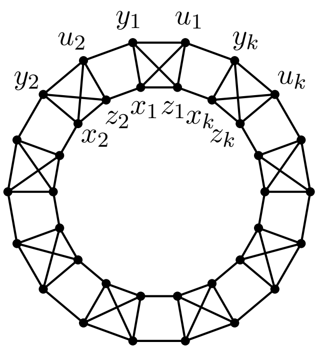


Figure 9.1

Theorem 9.2. *There are infinitely many non-traceable 5-connected weakly locally maximal 1-planar graphs. Moreover, for the class* Γ *of 5-connected 1-planar graphs we have* $\sigma(\Gamma) \leq \frac{\log 20}{\log 21}$. ⋄

We can infer from Theorem 9.2 that for an arbitrary $\varepsilon > 0$ there is a 5-connected 1-planar graph G such that $\mathrm{circ}(G) < \varepsilon \cdot |V(G)|$.

It is also not known whether every 7-connected 1-planar graph is HAMILTONian (see [HMS12]), so the intriguing question whether an analog of TUTTE's theorem holds for the family of 1-planar graphs remains open.

By Theorem 9.1, 3-connected maximal 1-planar graphs are far away from being HAMILTONian in general — nonetheless Theorem 9.3 and Theorem 9.4 both imply that a 4-connected locally maximal 1-planar graph is HAMILTONian, i.e. WHITNEY's theorem can be extended to the class of 4-connected locally maximal 1-planar graphs. For an overview of the minimum sufficient connectivity that leads to HAMILTONicity for the different kinds of 1-planar maximality discussed in this article, we refer the reader to Table 9.1 at the end of this paper (chapter).

As an extension of TUTTE's theorem, it is proved in [BZ19] that a 3-connected planar graph with at most three 3-cuts is HAMILTONian. (In this paper, all cuts are vertex-cuts.) By Theorem 9.2, this result cannot be extended to the class of 3-connected weakly locally maximal 1-planar graphs, however, Theorem 9.3 shows that the assertion is true for 3-connected locally maximal 1-planar graphs.

Theorem 9.3. *A 3-connected locally maximal 1-planar graph with at most three 3-cuts is HAMILTONian. Furthermore, every 3-connected locally maximal 1-planar graph with at most four 3-cuts is traceable.* ⋄

By Theorem 9.2, there are infinitely many non-HAMILTONian 5-connected weakly locally maximal 1-planar graphs. Theorem 9.4 shows that the situation changes if the number of almost full crossings in a weakly locally maximal 1-planar drawing of a graph is not too large, even if this graph is only 4-connected.

Theorem 9.4. *If a 4-connected graph has a weakly locally maximal 1-planar drawing with at most three almost full crossings, then it is HAMILTONian. Moreover, if a 4-connected* *graph has a weakly locally maximal 1-planar drawing with at most four almost full crossings, then it is traceable.* ⋄

Chen and Yu [CY02] showed that there is a constant c such that $\mathrm{circ}(G) \geq c \cdot |V(G)|^{\log_3 2}$ for an arbitrary 3-connected planar graph G. By Theorem 9.5, we show that the extension of the result of Chen and Yu (and of any other result concerning lower bounds on the length of a longest cycle of a 3-connected planar graph) to 3-connected locally maximal 1-planar graphs is possible. Moreover, by Theorem 9.5, every result on the existence of a certain subgraph of a 3-connected planar graph is also true for a 3-connected locally maximal 1-planar graph. Examples are the results of Barnette [Bar66] that a 3-connected planar graph has a spanning tree of maximum degree at most 3, and of Gao [Gao95] that a 3-connected planar graph has a spanning 2-connected subgraph of maximum degree at most 6.

Theorem 9.5. *Each 3-connected locally maximal 1-planar graph has a 3-connected planar spanning subgraph.* ⋄

In Figure 9.1, a 4-connected locally maximal 1-planar graph is presented. Because it is non-planar and 4-regular, it cannot contain a 4-connected planar spanning subgraph. Thus, Theorem 9.5 is best possible in this sense.

One obtains a weakly locally maximal 1-planar graph G (and one of its weakly locally maximal 1-planar drawings) if the edges $x_1y_1, \ldots, x_ky_k$ are removed from the graph of Figure 9.1. Assume this graph G contains a 3-connected planar spanning subgraph H. Since H has minimum degree at least three, all edges incident with a vertex from $\{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ belong to H. Thus, the graph H' obtained from H by removing the edges $u_1z_1, \ldots, u_kz_k$ is a subgraph of H. If $k \geq 3$, then it is easy to see that H' contains a subdivision of $K_{3,3}$ with major (3-valent) vertices x_1, y_1, x_k and u_1, z_1, x_2, a contradiction to the planarity of H. It follows that Theorem 9.5 is also best possible in the sense that “locally maximal 1-planar” cannot be replaced with “weakly locally maximal 1-planar”.

9.2 Proofs

Proof of

Theorem 9.1. *If H is a maximal planar graph on $n \geq 4$ vertices, then there is a 3-connected maximal 1-planar graph G on $7n - 12$ vertices such that $\mathrm{circ}(G) \leq 4 \cdot \mathrm{circ}(H)$.* ⋄

We construct G from H such that H is a subgraph of G. Therefore, the vertices of H are said to be *old* and these in $V(G) \setminus V(H)$ to be *new*.
It is well-known that a simple maximal planar graph on at least 4 vertices is 3-connected. Whitney [Whi32] (see also [Fle73]) proved, that a 3-connected planar graph has a unique (up to the choice of the outer face) planar drawing.
Let $D_0(H)$ be (in this sense) the unique planar drawing of H. Figure 9.2 shows a 1-planar embedding of K_6. A drawing $D_0(G)$ of G is obtained from $D_0(H)$ by inserting into each face of $D_0(H)$ with boundary vertices u, v, and w a triangle with three new vertices a, b, and c, and completed by further nine edges as shown in Figure 9.2.

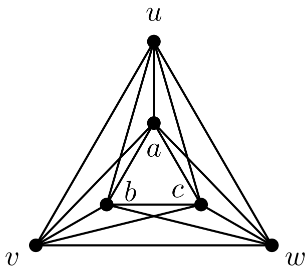


Figure 9.2

Obviously, $D_0(G)$ is a 1-planar drawing of G, hence, G is 1-planar. As H is maximal planar with $2n-4$ faces, G has $n+3(2n-4)=7n-12$ vertices.
Let C_G be a longest cycle of G. If C_G has at most eight vertices, then $\mathrm{circ}(G)=|V(C_G)| \leq 4 \cdot \mathrm{circ}(H)$ is true. If P is a subpath of C_G connecting two old vertices u and v such that $V(P) \setminus \{u,v\}$ does not contain an old vertex, then u and v are vertices of a face of H and $|V(P) \setminus \{u,v\}| \leq 3$ (see Figure 9.2). In case $V(P) \setminus \{u,v\} \neq \emptyset$, let Q be the u and v connecting path obtained from C_G by removing $V(P) \setminus \{u,v\}$. Note that Q contains at least three old vertices because $|V(C_G)| \geq 9$. We add the edge uv to Q and the resulting graph is again a cycle of G containing fewer new vertices than C_G. Repeating this step, we obtain a cycle C_H of H and it follows that $\mathrm{circ}(G)=|V(C_G)| \leq 4 \cdot |V(C_H)| \leq 4 \cdot \mathrm{circ}(H)$.

Now we show that G is 3-connected. Therefore, consider a minimal cut S of G. Since the neighborhood of each new vertex forms a complete graph, S does not contain new vertices, hence, $S \subset V(H)$. If S is also a cut of H, then $|S| \geq 3$ since a simple maximal planar graph is 3-connected. If $|S|<3$, then, since $H-S$ is still connected, $G-S$ has a component consisting of new vertices only. This is impossible since each new vertex has three old neighbors.

For the proof of Theorem 9.1, it remains to show that G is maximal 1-planar.

Let $D(G)$ be an arbitrary 1-planar drawing of G. Two subgraphs H_1 and H_2 of G are said to be k*-sharing* if H_1 and H_2 have at least k vertices in common. Moreover, H_1 and H_2 *share a crossing* in $D(G)$ if there are edges $e_1 \in E(H_1)$ and $e_2 \in E(H_2)$ that cross in $D(G)$. Bachmaier et al. [Bac+17] showed that if two subgraphs of G both isomorphic to K_5 share a crossing in $D(G)$, then they are 3-sharing. Using this result, we prove (i).

(i) *An edge of H is not crossed in $D(G)$.*

Let uv be an edge of H (see Figure 9.2) and $\{a,b,c\}$ and $\{a',b',c'\}$ be the disjoint sets of new vertices being inserted into the two faces of H both incident with uv, respectively. The subgraphs $G[\{u,v,a,b,c\}]$ and $G[\{u,v,a',b',c'\}]$ of G are both isomorphic to K_5. Assume there is an edge e of G that crosses uv in $D(G)$. Since each edge of G is an edge of a subgraph isomorphic to K_5, let $K(e)$ be a K_5-subgraph of G containing e.
By the aforementioned result in [Bac+17], it follows that $G[\{u,v,a,b,c\}]$ and $K(e)$ and also $G[\{u,v,a',b',c'\}]$ and $K(e)$ are 3-sharing. This implies $V(K(e)) \cap \{a,b,c\} \neq \emptyset$ and $V(K(e)) \cap \{a',b',c'\} \neq \emptyset$. Since there is no edge between the sets $\{a,b,c\}$ and $\{a',b',c'\}$, this contradicts the completeness of $K(e)$, and (i) is proved.

By (i), the restriction $D(H)$ of $D(G)$ to $V(H)$ is a planar embedding of H. The planar embedding of H is unique, thus, $D(H) = D_0(H)$. Consider a face F of $D_0(H)$ with boundary vertices u, v, and w. Since H has at least four vertices and again by (i), the three new vertices a, b, and c all adjacent to u, v, and w lie in the interior of the face F and, up to permutation of a, b, c, the situation of Figure 9.2 occurs. Thus, we may assume (ii).

(ii) *G has the unique 1-planar embedding $D_0(G)$.*

To show that G is maximal 1-planar, assume to the contrary that there are nonadjacent vertices x and y such that the graph $G + xy$ obtained from G by adding the edge xy is 1-planar. Therefore, let $D(G + xy)$ be a 1-planar drawing of $G + xy$. If xy is removed from $D(G+xy)$, then we obtain a 1-planar embedding of G and this embedding is $D_0(G)$ because of (ii). Thus, we may assume that in $D_0(G)$ the edge xy can be added in such a way that the resulting drawing is still 1-planar.

If x is new, then let $x = a$ (see Figure 9.2). Since $xy \notin E(G)$, y is not among the six vertices in Figure 9.2 and the edge xy has to cross at least one of the edges ub, uc, bw, cv in $D(G + xy)$, but each of them is already crossed in $D_0(G)$, a contradiction.

If x and y are old vertices, then, because H is maximal planar, $H+xy$ is not planar anymore. Thus, xy crosses an edge $e \in E(H)$ in $D(G + xy)$. Let $e = uv$ (see Figure 9.2), then xy crosses the edge av or bu in $D(G+xy)$. However, av and bu cross each other in $D_0(G)$, again a contradiction, and Theorem 9.1 is proved. ■

Proof of
Theorem 9.2. *There are infinitely many non-traceable 5-connected weakly locally maximal 1-planar graphs. Moreover, for the class Γ of 5-connected 1-planar graphs we have $\sigma(\Gamma) \leq \frac{\log 20}{\log 21}$.* ◇

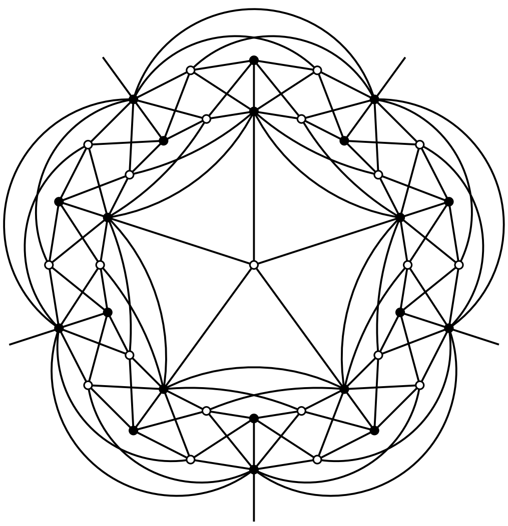


Figure 9.3: The structure H

Consider the structure H shown in Figure 9.3, add a new white vertex, and join the five half-edges of H to this new vertex. We obtain a weakly locally maximal 1-planar graph G. It is not difficult to see that G is 5-connected. Moreover, since G contains 20 black vertices and 22 white vertices, and the set of white vertices is independent, it follows that G is non-traceable. This construction can be generalized easily to obtain a weakly locally maximal 1-planar graph containing $4k$ black vertices and $4k+2$ independent white vertices, where k is an arbitrary integer at least five.

Now construct the 5-connected weakly locally maximal 1-planar graph G_0 from the graph H by removing the five half-edges. Starting with G_0, we construct an infinite sequence $\{G_i\}$ for $i \geq 0$ of 5-connected 1-planar graphs as follows. Let G_i be already constructed and G_{i+1} be obtained by replacing each white vertex v of G_i with a copy H_v of H (Figure 9.3) and connecting the five half-edges of H with the five neighbors of v in G_i. Let $M_i = \{H_v : v \text{ is white in } G_{i-1}\}$ and w_i be the number of white vertices of G_i. Then $|M_{i+1}| = w_i$, $w_0 = 21$, and $w_{i+1} = 21 \cdot w_i$, thus, $|V(G_i)| > w_i = 21^{i+1}$.

Let T_i be a longest closed trail of G_i visiting each black vertex of G_i at most once and let $t_i = |V(T_i)|$. Note that T_i visits a white vertex v of G_i at most twice, since v has degree 5 in G_i. Since a longest cycle of G_i is also a closed trail of G_i, it follows that $\operatorname{circ}(G_i) \leq t_i$ for all $i \geq 0$.

For $i \geq 1$, let $h_i = |\{H \in M_i : V(H) \cap V(T_i) \neq \emptyset\}|$ be the number of copies of H in G_i visited by T_i at least once; it is easy to see that $h_i \geq 2$. Moreover, since the 21 white vertices of $H \in M_i$ are independent and a half-edge of H is incident with a black vertex of H, it follows that $V(T_i) \cap V(H)$ contains at most 19 of the 21 white vertices of H. Thus, $|V(T_i) \cap V(H)| \leq 39$ and $t_i \leq b_i + 39h_i$, where b_i denotes the number of vertices of T_i not belonging to some $H \in M_i$. Let T be the closed trail of G_{i-1} obtained from T_i by shrinking all $H \in M_i$ to white vertices of G_{i-1} again.

Then $t_{i-1} \geq |V(T)| = b_i + h_i$ leads to $t_i - 20t_{i-1} \leq -19b_i + 19h_i$ and, because all $H \in M_i$ have distance at least 2 in G_i, $b_i \geq h_i$. Therefore, $t_i \leq 20t_{i-1}$, which gives $\operatorname{circ}(G_i) \leq t_i \leq 20^i t_0$. Finally, since

$$\frac{\log \operatorname{circ}(G_i)}{\log |V(G_i)|} \leq \frac{\log t_i}{\log |V(G_i)|} < \frac{\log 20 + \frac{1}{i}\log t_0}{\log 21 + \frac{1}{i}\log 21} \quad \text{for} \quad i \geq 1,$$

we have

$$\sigma(\Gamma) \leq \lim_{i \to \infty} \frac{\log \operatorname{circ}(G_i)}{\log |V(G_i)|} \leq \frac{\log 20}{\log 21}.$$

For the proofs of Theorem 9.3 and Theorem 9.4 we need the two forthcoming lemmas.

Lemma 9.6. *Let t be a non-negative integer and G be a non-planar 3-connected weakly locally maximal 1-planar graph that has a weakly locally maximal 1-planar drawing with t almost full crossings. Furthermore, among all weakly locally maximal 1-planar drawings of G with at most t almost full crossings let $D(G)$ be chosen with minimum number of crossings. Let G' with a drawing $D(G')$ be constructed by turning an arbitrary crossing X of $D(G)$ into a new 4-valent vertex v.*
Then

(i) *G' is weakly locally maximal 1-planar and $D(G')$ has one crossing less than $D(G)$.*

(ii) *If X is almost full, then G' has a weakly locally maximal 1-planar drawing with at most $t-1$ almost full crossings. Otherwise, G' has a weakly locally maximal 1-planar drawing with at most t almost full crossings.*

(iii) *G' is 3-connected.*

(iv) *Let S be a 3-cut of G'. If $S \subseteq V(G)$, then S is also a 3-cut of G. If $v \in S$, then X is almost full, the neighborhood of v in G' forms a path on vertices a, b, c, d that appear in this order, and there is $z \in V(G) \setminus N_{G'}(v)$ such that G has a 3-cut $S' = \{b, c, z\}$ which separates a and d.* $\diamond$

Proof of Lemma 9.6. Obviously, G' is weakly locally maximal 1-planar (remember that all considered 1-planar drawings $D(G)$ of a graph G have the property that if two edges uv and xy are crossing edges of $D(G)$, then the edge xu (if it exists) is not crossed by another edge in $D(G)$). Furthermore, $D(G')$ has one crossing less than $D(G)$, $D(G')$ has the desired number of almost full crossings, and (i) and (ii) immediately follow.

Assume that S is a minimal cut of G' and $v \in S$. Then there are $u, w \in N_{G'}(v)$ such that u and w belong to distinct components of $G' \setminus S$, thus, $G'[N_{G'}(v) \setminus S]$ has to be disconnected. Since the neighborhood $N_{G'}(v)$ of v in G' forms an induced cycle (if X is full) or an induced path (if X is almost full) on four vertices (note that G is simple), $S \cap N_{G'}(v) \neq \emptyset$.

If G' is not 3-connected, then G' has a minimal cut S such that $|S| \leq 2$ and $v \in S$. It follows that X is not full and the subgraph of G' spanned by $N_{G'}(v)$ is a path P with one of its inner vertices in S. But then both inner vertices of P form a 2-cut of G, in contradiction to the 3-connectedness of G and (iii) is proved.

Now, we prove (iv). First, let S be a 3-cut of G' with $S \subseteq V(G)$. Then there are components H_1 and H_2 of $G' \setminus S$ with $v \in V(H_1)$. Because $N_{G'}(v) \subseteq V(H_1) \cup S$ and $|S| = 3$, at most three of the four neighbors of v belong to S. Hence, S is a 3-cut of $G' \setminus \{v\}$ and also of G. Next, assume that G' contains a 3-cut S with $v \in S$. Let $e = xy$ and $e' = x'y'$ be two edges of the chosen crossing of $D(G)$, i.e. $N_{G'}(v) = \{x, y, x', y'\}$.

Case 1: *$G'[\{x, y, x', y'\}]$ is a cycle on 4 vertices.*
Because $G'[\{x, y, x', y'\} \setminus S]$ is disconnected, it follows that S contains two independent neighbors of v, say $S = \{v, x', y'\}$. Thus, $G' - S$ has two components each containing a vertex of $\{x, y\}$. If $G' - S$ has a further component H, then $V(H) \cap \{x, y, x', y'\} = \emptyset$ and $\{x', y'\}$ is a 2-cut of G', a contradiction. It is easy to see that there is either an open Jordan curve J of the plane connecting x' and y' such that $J \cap D(G) = \{x', y'\}$ or two edges, one from each component of $G' - S$, cross each other. The latter case cannot occur since the vertices of two crossing edges are connected in G'. Thus, if the edge $x'y'$ is replaced with J, then we get a drawing $D'(G)$ of G with fewer crossings than $D(G)$, a contradiction to the choice of $D(G)$. It follows that Case 1 does not occur.

Case 2: *$G'[\{x, y, x', y'\}]$ is a path on 4 vertices.*
Without loss of generality assume $yy' \notin E(G)$. Because $G'[\{x, y, x', y'\} \setminus S]$ is disconnected, it follows that S contains x or x'. Because of symmetry, let $x \in S$, i.e. $S = \{v, x, z\}$ with $z \in V(G)$.

Case 2.1: *$z \notin \{x, y, x', y'\}$.*
With a similar argument as in Case 1, $G' - S$ has exactly two components H_1 and H_2 with $x', y \in V(H_1)$ and $y' \in V(H_2)$. It is easy to see that $S' = \{x, x', z\}$ is a 3-cut of G. Moreover, $G - S'$ has two components $H_1 - x'$ and H_2 each containing one vertex from $\{y, y'\}$.

Case 2.2: *$z = y$.*
Then $S = \{v, x, y\}$ and we use the same arguments as in Case 1 for a contradiction.

Case 2.3: $z = x'$.
Then $S = \{v, x, x'\}$ and, since $G' - S$ is disconnected, $G - \{x, x'\}$ is disconnected, contradicting the 3-connectedness of G. ■

Lemma 9.7. *Let G be a 1-planar graph, $D(G)$ a 1-planar drawing of G, and $e = xy$ and $e' = x'y'$ two crossing edges of $D(G)$. Moreover, let G' be obtained from G by turning the crossing of e and e' into a new 4-valent vertex v.*

(i) *If this crossing is full and G' has a Hamiltonian cycle C' (Hamiltonian path P'), then G is Hamiltonian (traceable).*

(ii) *If this crossing is almost full with $xx' \notin E(G)$, and G' has a Hamiltonian cycle C' (Hamiltonian path P') not containing both vx and vx', then G is Hamiltonian (traceable).* ◇

Proof of Lemma 9.7. Let vu and vw be adjacent edges of C'. In either case we have $uw \in E(G)$, so replacing the subpath uvw of C' with the edge uw leads to a Hamiltonian cycle of G. The same arguments hold for P' if v is not an end vertex of P'. If it is, simply remove it and we obtain the desired Hamiltonian path in G. ■

Proof of
Theorem 9.3. *A 3-connected locally maximal 1-planar graph with at most three 3-cuts is Hamiltonian. Furthermore, every 3-connected locally maximal 1-planar graph with at most four 3-cuts is traceable.* ◇

Let G_1 be a 3-connected locally maximal 1-planar graph with at most three 3-cuts. We define a sequence of locally maximal 1-planar graphs $G_1, G_2, \ldots,$ where for all $i \geq 1$, G_{i+1} is the graph G' if G_i is the graph G according to Lemma 9.6 (with $t = 0$). By Lemma 9.6, there is an index k such that G_k is planar and 3-connected with at most three 3-cuts; no further 3-cut appears since all crossings of G_1 are full. A result in [BZ19] states that a 3-connected planar graph with at most three 3-cuts is Hamiltonian, thus, G_k is Hamiltonian. Applying assertion (i) of Lemma 9.7 repeatedly implies that G_1 is Hamiltonian.

In the same spirit, let now G_1 be a 3-connected locally maximal 1-planar graph with at most four 3-cuts. Define a sequence $G_1, G_2, \ldots$ as above. By Lemma 9.6, there is an index k such that G_k is planar and 3-connected with at most four 3-cuts. It was proved in [BZ19] that a 3-connected planar graph with at most four 3-cuts is traceable, thus, G_k is traceable. Again we apply assertion (i) of Lemma 9.7 repeatedly and obtain that G_1 is traceable. ■

Proof of
Theorem 9.4. *If a 4-connected graph has a weakly locally maximal 1-planar drawing with at most three almost full crossings, then it is Hamiltonian. Moreover, if a 4-connected graph has a weakly locally maximal 1-planar drawing with at most four almost full crossings, then it is traceable.* ◇

Let G_1 be a 4-connected 1-planar graph which has a weakly locally maximal 1-planar drawing with at most three almost full crossings. Among all weakly locally maximal 1-planar drawings of G_1 with at most three almost full crossings, let $D(G_1)$ be chosen with minimum number of crossings. If the number s of almost full crossings in $D(G_1)$ is zero, then G_1 is Hamiltonian by Theorem 9.3.

We assume $s \geq 1$, consider an almost full crossing X of $D(G_1)$ and apply Lemma 9.6 to this crossing with $G = G_1$ and $t = s$, and obtain $G_1' = G'$ with the new added vertex $v_1 = v$. Obviously, G_1' has a drawing with at most $s-1$ almost full crossings. Since G_1 is 4-connected, G_1' is 4-connected by Lemma 9.6. Let G_2 be obtained from G_1' by adding a vertex u_1, the edge u_1v_1 and the two edges connecting u_1 with both 2-valent vertices of the path $G_1'[N_{G_1'}(v_1)]$ (see Lemma 9.6 and Figure 9.4). Then, G_2 is 3-connected and $N_{G_2}(u_1)$ is the only 3-cut of G_2. Furthermore, G_2 has a weakly locally maximal 1-planar drawing with $s-1$ almost full crossings.

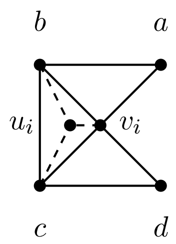


Figure 9.4

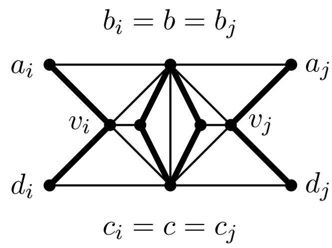


Figure 9.5

Note that a Hamiltonian cycle of G_2 (if it exists) leads to a Hamiltonian cycle of $G_1' = G_2 \setminus \{u_1\}$ containing at least one edge of $G_1'[N_{G_2}(u_1)]$.

If $s = 1$, then let $H = G_2$. Otherwise, we repeat this step $s-1$ times and obtain a graph $H = G_3$ or $H = G_4$. H is 3-connected locally maximal 1-planar. Assume there is a 3-cut S in G_{i+1} which is not a 3-cut of G_i for $i \in \{1, \ldots, s\}$. Then $S = N_{G_{i+1}}(u_i)$ or $S \neq N_{G_{i+1}}(u_i)$ and $v_i \in S$. In the second case, by Lemma 9.6, there is a 3-cut in G_i separating two vertices of $N_{G_{i+1}}(v_i) \setminus \{u_i\}$, a contradiction. Hence, H has exactly s 3-cuts, namely the neighborhoods of $u_1, \ldots, u_s$. Since $s \leq 3$, H is Hamiltonian by Theorem 9.3 and $G = H - \{u_1, \ldots, u_s\}$ is Hamiltonian because the neighborhoods of $u_1, \ldots, u_s$ are complete in H. By the previous remark, we may assume that G contains a Hamiltonian cycle C such that

$$E(G[N_H(u_i)]) \cap E(C) \neq \emptyset \text{ for } i = 1, \ldots, s. \qquad (*)$$

Consider an arbitrary vertex $v \in \{v_1, \ldots, v_s\}$ and let $\{a, b, c, d\}$ be the vertex set of the induced path on $N_G(v)$ in this order.

If $av, dv \in E(C)$, then $bc \in E(C)$ by property $(*)$. In this case let P_1 and P_2 be the subpaths of C obtained by removing v and the edge bc from C. If P_1 connects ab and P_2 connects cd, then the cycle obtained from P_1, P_2, ac, and bd is a Hamiltonian cycle of the graph G_v obtained from G by deleting v and adding the edges ac and bd. If P_1 connects ac and P_2 connects bd, then the cycle obtained from P_1, P_2, ab, and cd is a Hamiltonian cycle of the graph G_v. If not both edges av, dv belong to C, then G_v is Hamiltonian by assertion (ii) of Lemma 9.7.

Repeating this step s times, we get rid of $v_1, \ldots, v_s$, the resulting graph is G_1 and the existence of a Hamiltonian cycle of G_1 is shown. Note that if there exist distinct v_i and v_j sharing the same bc, then C misses at least one edge of $a_iv_i, d_iv_i, a_jv_j, d_jv_j$ (see Figure 9.5), since otherwise the edges bu_i, u_ic, cu_j, u_jb of C form a cycle.

The proof that any 4-connected 1-planar graph which has a weakly locally maximal 1-planar drawing with at most four almost full crossings is traceable uses very similar arguments and is therefore omitted. ■

Proof of
Theorem 9.5. *Each 3-connected locally maximal 1-planar graph has a 3-connected planar spanning subgraph.* ⋄

Among all locally maximal 1-planar drawings of G let $D(G)$ be chosen such that the number of crossings in $D(G)$ is minimal. If two edges of $D(G)$ cross each other, then remove an arbitrary one of them and let H be the resulting graph. Obviously, H is plane and a spanning subgraph of $D(G)$ (and of G).

It remains to show that H is 3-connected. Assume H is not 3-connected and, therefore, let $S \subset V(H)$ be a cut of H with $|S| \leq 2$ such that $H_1, \ldots, H_k$ $(k \geq 2)$ are the components of $H - S$. Since $D(G) - S$ is connected, there are at least $k-1$ *connecting* edges $xy \in E(D(G)) \setminus E(H)$ with $x \in V(H_i)$ and $y \in V(H_j)$ for suitable $i, j \in \{1, \ldots, k\}$ with $i \neq j$. These edges are crossed in $D(G)$ by edges from $E(H)$.

Let $x'y'$ be the edge crossing some connecting edge $xy \in E(D(G)) \setminus E(H)$ in $D(G)$.

Since $D(G)$ is locally maximal, $\{x, y, x', y'\}$ induces a complete subgraph of G, thus, both x' and y' are common neighbors of x and y.

It follows that $S = \{x', y'\}$. Hence xy is the only connecting edge (another connecting edge would also cross $x'y'$) and therefore $k = 2$. We argue as in Case 1 of the proof of Lemma 9.6: there is an open JORDAN curve J of the plane connecting x' and y' such that $J \cap D(G) = \{x', y'\}$ and, if the edge $x'y'$ is replaced with J, then we get a drawing $D'(G)$ with fewer crossings than $D(G)$, a contradiction. ■

9.3 Overview

We end this paper with a tabular overview of HAMILTONian properties of various families of graphs that we have discussed.

	Maximal planar	Planar	Optimal 1-planar	Maximal 1-planar	Locally maximal 1-planar	Weakly locally maximal 1-planar	1-planar
3	✗	✗	—	✗(D1)	✗	✗	✗
4	✓(A)	✓(B)	✓(C)	✓(D3)	✓(D3)	✗	✗(C)
5	✓	✓	✓	✓	✓	✗(D2)	✗(D2)
6	—	—	✓	✓	✓	?	?
7	—	—	✓	✓(C)	✓	?	?

A. WHITNEY [Whi32]
B. TUTTE [Tut56]
C. HUDÁK, MADARAS, SUZUKI [HMS12]
D1. This paper, Theorem 9.1
D2. This paper, Theorem 9.2
D3. This paper, Theorems 9.3 and 9.4

Table 9.1: Hamiltonicity of planar and 1-planar graphs, as well as some related families, listed by connectedness ranging from 3 to 7 (the maximum admissible value for 1-planar graphs). Green cells (marked ✓) indicate that every graph with the specified connectedness is HAMILTONian, red cells (marked ✗) signify that there exist such graphs which are not HAMILTONian, question marks designate open problems, and "—" stands for an impossible combination of properties.

Acknowledgement

We thank the referees, whose constructive suggestions helped to improve the quality of this article.

10 Outlook

This thesis presented an intermediate status of some topics in an extensive research field of structural graph theory. We focused on concepts like uniquely colourable graphs, graphs with a Kempe colouring, rooted minors with respect to Hadwiger's Conjecture, structure of local connectivity, and a new Tutte theory for graph classes apart from planar graphs. However, the research will not end here. The aim of this last section is to give a collection of related open problems that deserve further investigation.

Kempe colouring

Since all considerations meant to bound the number of colour classes or the size of the clique minors have not been leading to new proofs of the cases H(t) for $t \geq 7$ in Hadwiger's Conjecture, one part of the thesis dealt with graphs with few colourings. Based on uniquely colourable graphs, the presented Kempe colourings substantiated their utility with respect to Hadwiger's Conjecture.

By adapting Seymour's Conjecture (Conjecture 4) to graphs admitting a Kempe colouring, the following open problem naturally arises.

Problem 1. Let G be a graph with $\alpha(G) \leq 2$ and $\mathcal{C}$ be a Kempe colouring of size k. Does G contain a K_k-minor? ⋄

Even the weaker version asking for a K_ℓ-minor with $\ell \geq \frac{3}{4}k$ has remained still open:

Problem 2. Let G be a graph with $\alpha(G) \leq 2$ and $\mathcal{C}$ be a Kempe colouring of size k. Does G contain a K_ℓ-minor with $\ell \geq \frac{3}{4}k$? ⋄

Dropping the condition $\alpha(G) \leq 2$, the best-known result about Hadwiger's Conjecture has been published by M. Kriesell who showed that graphs with a Kempe colouring using up to ten colours fulfil the assertion of Hadwiger's Conjecture (Theorem 2.26). This was proved using a theorem of Song and Thomas (Theorem 2.7) about the maximum number of edges in K_p-minor-free graphs ($p \leq 9$); those investigations had been started by Mader (Theorem 2.6). It is an interesting task to continue research for $p \geq 10$ since new results may also lead to extensions of Kriesell's Theorem 2.26 and Hadwiger's Conjecture for uniquely colourable graphs.

Problem 3. Let $p \in \mathbb{N}$, $p \geq 10$. How can K_p-minor-free graphs on n vertices and at least $(p-2)\,n - \binom{p-1}{2} + 1$ edges be characterised? $\diamond$

Rooted Minors

Most of the graphs considered in this thesis contain a clique immersion, i.e. for vertices $x_1, x_2, \ldots, x_k$ with pairwise different colours, there exists a system of edge-disjoint x_i,x_j-paths. It is for this reason that we investigated minors rooted by the transversal of a KEMPE colouring. We were hoping to solve the following very ambitious Problem 4; however, in this thesis we have only managed to obtain some results indicating an affirmative answer and refuted a slight generalisation.

Problem 4. Let G be a graph with a KEMPE colouring $\mathcal{C}$ of size k. Does G contain a K_k-minor rooted at T for every transversal T of $\mathcal{C}$? $\diamond$

The answer to Problem 4 for $k \leq 4$ is inferred from Theorem 2.33 published by FABILA-MONROY and WOOD. This result about rooted K_4-minors has a wide range of applications. We would be pleased with knowing a generalisation:

Problem 5.

(i) What is the full characterisation of all graphs and all tuples of vertices $\{a, b, c, d, e\}$ of G such that there is no K_5-minor rooted at $\{a, b, c, d, e\}$?

(ii) What are sufficient conditions for a graph G and a fixed vertex set $X \subseteq V(G)$ of k vertices in order to force a K_k-minor rooted at these vertices? $\diamond$

Related to the above problem is the following, again asking for a K_k-minor rooted at k vertices with some additional assumptions to these vertices. First, these vertices are supposed to be a transversal T of a (not necessarily optimal) colouring. Furthermore, it is demanded that each pair of distinct vertices $x, y \in T$ is contained in a connected component of $G[A \cup B]$, where $A, B \in \mathcal{C}$ with $x \in A, y \in B$.

Problem 6. Which is the largest integer k such that the following is true for all $k' \leq k$: Let G be a graph and $\mathcal{C}$ be a k'-colouring. Assume that for a transversal T of $\mathcal{C}$ each pair of distinct vertices in T belongs to a common KEMPE chain. Then G has a $K_{k'}$-minor rooted at T. $\diamond$

We have already seen in Chapter 5 that Problem 6 can only hold for $k < 7$.

Cycle Lengths

Another major part of this thesis investigated the capabilities of TUTTE paths. Those had initially been used to answer the long-standing question whether 4-connected planar graphs are HAMILTONian. Over time, they have evolved as a strong tool for many questions about cycle lengths, e.g. determining the spectrum of graphs. I like to restate the following wide open problem about the spectrum of planar graphs (MALKEVITCH's Conjecture 8).

Problem 7. Does each 4-connected planar graph G have cycles of length k for every $k \in \{3, 5, 6, \ldots, |V(G)|\}$, i.e. does G contain cycles of all possible length except 4? $\diamond$

At the end of Chapter 2, we presented a new idea on how to implement a Tutte theory to 1-planar graphs. To conclude this thesis I would like to phrase this issue in the last Problem 8 and encourage further research on the new theory.

Problem 8. Is every 6-connected (not maximal) 1-planar graph Hamiltonian? $\diamond$

Bibliography

All cited publication are included in Primary Sources. Publications printed in this thesis can be found thereafter in Publications and Preprints.

Primary Sources

[AG18] B. Albar and D. Gonçalves, On triangles in K_r-minor free graphs, *Journal of Graph Theory* 88.**1** (2018), pages 154–173.

[AH89] K. I. Appel and W. Haken, *Every planar map is four colorable*, volume 98, American Mathematical Society, 1989.

[Ald+00] R. E. Aldred, S. Bau, D. A. Holton, and B. D. McKay, Nonhamiltonian 3-connected cubic planar graphs, *SIAM Journal on Discrete Mathematics* 13.**1** (2000), pages 25–32.

[AMS01] S. Akbari, V. S. Mirrokni, and B. S. Sadjad, K_r-free uniquely vertex colorable graphs with minimum possible edges, *Journal of Combinatorial Theory, Series B* 82.**2** (2001), pages 316–318.

[Bac+17] C. Bachmaier, F. J. Brandenburg, K. Hanauer, D. Neuwirth, and J. Reislhuber, NIC-planar graphs, *Discrete Applied Mathematics* 232 (2017), pages 23–40.

[Bar66] D. Barnette, Trees in polyhedral graphs, *Canadian Journal of Mathematics* 18 (1966), pages 731–736.

[Bar94] D. W. Barnette, 2-connected spanning subgraphs of planar 3-connected graphs, *Journal of Combinatorial Theory, Series B* 61.**2** (1994), pages 210–216.

[BCE80] B. Bollobás, P. A. Catlin, and P. Erdős, Hadwiger's conjecture is true for almost every graph., *European Journal of Combinatorics* 1.**3** (1980), pages 195–199.

[BGH01] T. Böhme, F. Göring, and J. Harant, Menger's Theorem, *Journal of Graph Theory* 37.**1** (2001), pages 35–36.

[Bie14] T. Biedl, Trees and co-trees with bounded degrees in planar 3-connected graphs, *Scandinavian Workshop on Algorithm Theory* (2014), pages 62–73.

[BJ70] D. Barnette and E. Jucovič, Hamiltonian circuits on 3-polytopes, *Journal of Combinatorial Theory* 9.**1** (1970), pages 54–59.

[BJW11] J. Barát, G. Joret, and D. R. Wood, Disproof of the list Hadwiger conjecture, *Electronic Journal of Combinatorics* (2011), P232.

[Bla07] J. Blasiak, A special case of Hadwiger's conjecture, *Journal of Combinatorial Theory, Series B* 97.**6** (2007), pages 1056–1073.

[BM08] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer Graduate Texts in Mathematics, Springer-Verlag, 2008, isbn: 978-1-84628-969-9.

[Bon71] J. A. Bondy, Pancyclic graphs I, *Journal of Combinatorial Theory, Series B* 11.**1** (1971), pages 80–84.

[Bra+12] F. J. Brandenburg, D. Eppstein, A. Gleissner, M. T. Goodrich, K. Hanauer, and J. Reislhuber, On the density of maximal 1-planar graphs, *International Symposium on Graph Drawing* (2012), pages 327–338.

[Bro02] H. J. Broersma, On some intriguing problems in hamiltonian graph theory—a survey, *Discrete mathematics* 251.**1-3** (2002), pages 47–69.

[Bro17] C. Brosse, *Hadwiger's conjecture in a particular case*, technical report, Ilmenau University of Technology, 2017.

[BS76] B. Bollobás and N. Sauer, Uniquely colourable graphs with large girth, *Canadian Journal of Mathematics* 28.**6** (1976), pages 1340–1344.

[BZ19] G. Brinkmann and C. Zamfirescu, Polyhedra with few 3-cuts are hamiltonian, *Electronic Journal of Combinatorics* 26.**1** (2019), P1.39.

[Cat79] P. A. Catlin, Hajós' graph-coloring conjecture: Variations and counterexamples, *Journal of Combinatorial Theory, Series B* 26 (1979), pages 268–274.

[Cay79] A. Cayley, On the colouring of maps, *Proceedings of the Royal Geographical Society and Monthly Record of Geography* 1.**4** (1879), pages 259–261.

[CFY04] G. Chen, G. Fan, and X. Yu, Cycles in 4-connected planar graphs, *European Journal of Combinatorics* 25.**6** (2004), pages 763–780.

[CG69] G. Chartrand and D. P. Geller, On uniquely colorable planar graphs, *Journal of Combinatorial Theory* 6.**3** (1969), pages 271–278.

[CHW09] Q. Cui, Y. Hu, and J. Wang, Long cycles in 4-connected planar graphs, *Discrete mathematics* 309.**5** (2009), pages 1051–1059.

[CY02] G. Chen and X. Yu, Long cycles in 3-connected graphs, *Journal of Combinatorial Theory, Series B* 86.**1** (2002), pages 80–99.

[Die17] R. Diestel, *Graph Theory: 5th edition*, Springer Graduate Texts in Mathematics, Springer-Verlag, 2017, ISBN: 978-3-96134-005-7.

[Dil96] M. B. Dillencourt, Polyhedra of small order and their hamiltonian properties, *journal of combinatorial theory, Series B* 66.**1** (1996), pages 87–122.

[Dir52] G. A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, *Journal of the London Mathematical Society* 1.**1** (1952), pages 85–92.

[EHK98] T. Emden-Weinert, S. Hougardy, and B. Kreuter, Uniquely colourable graphs and the hardness of colouring graphs of large girth, *Combinatorics, Probability and Computing* 7.**4** (1998), pages 375–386.

[Ell96] M. N. Ellingham, Spanning paths, cycles, trees and walks for graphs on surfaces, *Congressus Numerantium* (1996), pages 55–90.

[Eul41] L. Euler, Solutio problematis ad geometriam situs pertinentis, *Commentarii academiae scientiarum Petropolitanae* (1741), pages 128–140.

[Fab+20c] I. Fabrici, J. Harant, S. Mohr, and J. M. Schmidt, On the circumference of essentially 4-connected planar graphs, *Journal of Graph Algorithms and Applications* 24.**1** (2020), pages 21–46, DOI: 10.7155/jgaa.00516, arXiv: 1806.09413.

[FHJ16] I. Fabrici, J. Harant, and S. Jendroľ, On longest cycles in essentially 4-connected planar graphs, *Discussiones Mathematicae Graph Theory* 36.**3** (2016), pages 565–575.

[FJ89] H. Fleischner and B. Jackson, A note concerning some conjectures on cyclically 4-edge-connected 3-regular graphs, *Graph Theory in Memory of GA Dirac, Annals of Discrete Mathematics* 41 (1989), pages 171–178.

[FKO08] N. Fountoulakis, D. Kühn, and D. Osthus, The order of the largest complete minor in a random graph, *Random Structures & Algorithms* 33.**2** (2008), pages 127–141.

[Fle73] H. Fleischner, The uniquely embeddable planar graphs, *Discrete Mathematics* 4.**4** (1973), pages 347–358.

[FM07] I. Fabrici and T. Madaras, The structure of 1-planar graphs, *Discrete Mathematics* 307.**7-8** (2007), pages 854–865.

[Fra12] A. Fradkin, Clique minors in claw-free graphs, *Journal of Combinatorial Theory, Series B* 102.**1** (2012), pages 71–85.

[FW13] R. Fabila-Monroy and D. R. Wood, Rooted K_4-minors, *Electronic Journal of Combinatorics* 20.**2** (2013), P64.

[Gao95] Z. Gao, 2-connected coverings of bounded degree in 3-connected graphs, *Journal of Graph Theory* 20.**3** (1995), pages 327–338.

[GH10] F. Göring and J. Harant, Hamiltonian cycles through prescribed edges of 4-connected maximal planar graphs, *Discrete Mathematics* 310.**9** (2010), pages 1491–1494.

[GM75] G. R. Grimmett and C. J. McDiarmid, On colouring random graphs, *Mathematical Proceedings of the Cambridge Philosophical Society* 77.**2** (1975), pages 313–324.

[GM76] B. Grünbaum and J. Malkevitch, Pairs of edge-disjoint Hamiltonian circuits, *aequationes mathematicae* 14.**1-2** (1976), pages 191–196.

[GW73] B. Grünbaum and H. Walther, Shortness exponents of families of graphs, *Journal of Combinatorial Theory, Series A* 14.**3** (1973), pages 364–385.

[Had43] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich* 88.**2** (1943), pages 133–142.

[Hea98] P. J. Heawood, On the four-colour map theorem, *The Quarterly Journal of Pure and Applied Mathematics* 29 (1898), pages 270–285.

[Hee69] H. Heesch, *Untersuchungen zum Vierfarbenproblem*, volume 810, Bibliographisches Institut, 1969.

[HHR69] F. Harary, S. T. Hedetniemi, and R. W. Robinson, Uniquely colorable graphs, *Journal of Combinatorial Theory* 6.**3** (1969), pages 264–270.

[HMS12] D. Hudák, T. Madaras, and Y. Suzuki, On properties of maximal 1-planar graphs, *Discussiones Mathematicae Graph Theory* 32.**4** (2012), pages 737–747.

[Jac86] B. Jackson, Longest cycles in 3-connected cubic graphs, *Journal of Combinatorial Theory, Series B* 41.**1** (1986), pages 17–26.

[Jør94] L. K. Jørgensen, Contractions to K_8, *Journal of Graph Theory* 18.**5** (1994), pages 431–448.

[JW90] B. Jackson and N. C. Wormald, k-walks of graphs., *Australasian Journal of Combinatorics* 2 (1990), pages 135–146.

[JW92] B. Jackson and N. C. Wormald, Longest cycles in 3-connected planar graphs, *Journal of Combinatorial Theory, Series B* 54.**2** (1992), pages 291–321.

[Kem79] A. B. Kempe, On the geographical problem of the four colours, *American journal of mathematics* 2.**3** (1879), pages 193–200.

[KM13] V. P. Korzhik and B. Mohar, Minimal obstructions for 1-immersions and hardness of 1-planarity testing, *Journal of Graph Theory* 72.**1** (2013), pages 30–71.

[KO03] D. Kühn and D. Osthus, Minors in graphs of large girth, *Random Structures & Algorithms* 22.**2** (2003), pages 213–225.

[Kom00] J. Komlós, Tiling Turán theorems, *Combinatorica* 20.**2** (2000), pages 203–218.

[Kor08] V. P. Korzhik, Minimal non-1-planar graphs, *Discrete Mathematics* 308.**7** (2008), pages 1319–1327.

[Kos84] A. V. Kostochka, Lower bound of the Hadwiger number of graphs by their average degree, *Combinatorica* 4.**4** (1984), pages 307–316.

[Kot63] A. Kotzig, Hamilton graphs and Hamilton circuits, *Theory of Graphs and its Applications, Proceedings of the Symposium of Smolenice* (1963), pages 63–82.

[Kri01] M. Kriesell, Induced paths in 5-connected graphs, *Journal of Graph Theory* 36.**1** (2001), pages 52–58.

[Kri17] M. Kriesell, Unique colorability and clique minors, *Journal of Graph Theory* 85.**1** (2017), pages 207–216.

[Kri20a] M. Kriesell, A note on uniquely 10-colorable graphs, *arXiv preprint* (2020), arXiv: 2002.08074.

[Kri20b] M. Kriesell, Maximal ambiguously k-colorable graphs, *Journal of Combinatorial Theory, Series B* 140 (2020), pages 248–262.

[KT05] K. Kawarabayashi and B. Toft, Any 7-chromatic graphs has K_7 or $K_{4,4}$ as a minor, *Combinatorica* 25.**3** (2005), pages 327–353.

[Lo19] O.-H. S. Lo, Find subtrees of specified weight and cycles of specified length in linear time, *arXiv preprint* (2019), arXiv: 1902.06484.

[Lov72] L. Lovász, A characterization of perfect graphs, *Journal of Combinatorial Theory, Series B* 13.**2** (1972), pages 95–98.

[LS18] O.-H. S. Lo and J. M. Schmidt, Longest cycles in cyclically 4-edge-connected cubic planar graphs., *Australasian Journal of Combinatorics* 72 (2018), pages 155–162.

[Mad68] W. Mader, Homomorphiesätze für Graphen, *Mathematische Annalen* 178.**2** (1968), pages 154–168.

[Mal88] J. Malkevitch, Polytopal graphs, *Selected Topics in Graph Theory* 3.**1699188** (1988), page 1211.

[Men27] K. Menger, Zur allgemeinen Kurventheorie, *Fundamenta Mathematicae* 10.**1** (1927), pages 96–115.

[MM16] E. Máčajová and J. Mazák, Cubic graphs with large circumference deficit, *Journal of Graph Theory* 82.**4** (2016), pages 433–440.

[MM63] J. Moon and L. Moser, Simple paths on polyhedra, *Pacific Journal of Mathematics* 13.**2** (1963), pages 629–631.

[Moh19] S. Mohr, On the circumference of 3-connected maximal 1-planar graphs (2019), Bordeaux Graph Workshop, pages 222–223, url: http://bgw.labri.fr/2019/booklet.pdf.

[MS17] T. Madaras and R. Soták, *personal communication with the graph theory group form Pavol Jozef Šafárik University during a research visit in Košice*, Nov. 7, 2017.

[MS84] M. M. Matthews and D. P. Sumner, Hamiltonian results in $K_{1,3}$-free graphs, *Journal of Graph Theory* 8.**1** (1984), pages 139–146.

[Neš73] J. Nešetřil, On uniquely colorable graphs without short cycles, *Časopis pro pěstování matematiky* 98.**2** (1973), pages 122–125.

[OO09] K. Ota and K. Ozeki, Spanning trees in 3-connected $K_{3,t}$-minor-free graphs, *Electronic Notes in Discrete Mathematics* 34 (2009), pages 145–149.

[OY11] K. Ozeki and T. Yamashita, Spanning trees: A survey, *Graphs and Combinatorics* 27.**1** (2011), pages 1–26.

[Pet98] J. Petersen, Sur le théoreme de Tait, *L'intermédiaire des Mathématiciens* 5 (1898), pages 225–227.

[Plu75] M. D. Plummer, Problem in infinite and finite sets, *Colloquia Mathematica Societatis Janos Bolyai* 10 (1975), pages 1549–1550.

[PT97] J. Pach and G. Tóth, Graphs drawn with few crossings per edge, *Combinatorica* 17.**3** (1997), pages 427–439.

[Rin65] G. Ringel, Ein Sechsfarbenproblem auf der Kugel, *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 29.**1-2** (1965), pages 107–117.

[Rob+96] N. Robertson, D. Sanders, P. Seymour, and R. Thomas, A new proof of the four-colour theorem, *Electronic Research Announcements of the American Mathematical Society* 2.**1** (1996), pages 17–25.

[RP77] G. Ravindra and K. Parthasarathy, Perfect product graphs, *Discrete Mathematics* 20 (1977), pages 177–186.

[RS04] B. Reed and P. Seymour, Hadwiger's conjecture for line graphs, *European Journal of Combinatorics* 25.**6** (2004), pages 873–876.

[RS83] N. Robertson and P. D. Seymour, Graph minors. I. Excluding a forest, *Journal of Combinatorial Theory, Series B* 35.**1** (1983), pages 39–61.

[RS90] N. Robertson and P. D. Seymour, Graph minors. IX. Disjoint crossed paths, *Journal of Combinatorial Theory, Series B* 49.**1** (1990), pages 40–77.

[RST93] N. Robertson, P. Seymour, and R. Thomas, Hadwiger's conjecture for K_6-free graphs, *Combinatorica* 13.**3** (1993), pages 279–361.

[Ryj97] Z. Ryjáček, On a closure concept in claw-free graphs, *Journal of Combinatorial Theory, Series B* 70.**2** (1997), pages 217–224.

[San97] D. P. Sanders, On paths in planar graphs, *Journal of Graph Theory* 24.**4** (1997), pages 341–345.

[Sey16] P. Seymour, Hadwiger's conjecture, *Open problems in mathematics*, Springer, 2016, pages 417–437.

[SS18] A. Schmid and J. M. Schmidt, Computing tutte paths, *45th International Colloquium on Automata, Languages, and Programming* 107 (2018), 98:1–98:14.

[ST06] Z.-X. Song and R. Thomas, The extremal function for K_9-minors, *Journal of Combinatorial Theory, Series B* 96.**2** (2006), pages 240–252.

[Tai80] P. G. Tait, Remarks on the colouring of maps, *Proceedings of the Royal Society of Edinburgh* 10.**729** (1880), pages 501–503.

[Tho78] A. G. Thomason, Hamiltonian cycles and uniquely edge colourable graphs, *Annals of Discrete Mathematics* 3 (1978), pages 259–268.

[Tho83] C. Thomassen, A theorem on paths in planar graphs, *Journal of Graph Theory* 7.**2** (1983), pages 169–176.

[Tho84] A. Thomason, An extremal function for contractions of graphs, *Mathematical Proceedings of the Cambridge Philosophical Society* 95.**2** (1984), pages 261–265.

[Tho86] C. Thomassen, Reflections on graph theory, *Journal of Graph Theory* 10.**3** (1986), pages 309–324.

[Tof96] B. Toft, A survey of Hadwiger's conjecture, *Congressus Numerantium* (1996), pages 249–283.

[Tut46] W. T. Tutte, On hamiltonian circuits, *Journal of the London Mathematical Society* 1.**2** (1946), pages 98–101.

[Tut56] W. T. Tutte, A theorem on planar graphs, *Transactions of the American Mathematical Society* 82.**1** (1956), pages 99–116.

[Tut60] W. T. Tutte, A non-Hamiltonian planar graph, *Acta Mathematica Hungarica* 11.**3-4** (1960), pages 371–375.

[Tut61] W. T. Tutte, A theory of 3-connected graphs, *Indagationes Mathematicae* 23.**441-455** (1961).

[Tut70] W. T. Tutte, Toward a theory of crossing numbers, *Journal of Combinatorial Theory* 8.**1** (1970), pages 45–53.

[Tut77] W. T. Tutte, Bridges and Hamiltonian circuits in planar graphs, *Aequationes mathematicae* 15.**1** (1977), pages 1–33.

[TY94] R. Thomas and X. Yu, 4-connected projective-planar graphs are Hamiltonian, *Journal of Combinatorial Theory, Series B* 62.**1** (1994), pages 114–132.

[TYZ05] R. Thomas, X. Yu, and W. Zang, Hamilton paths in toroidal graphs, *Journal of Combinatorial Theory, Series B* 94.**2** (2005), pages 214–236.

[Voi93] M. Voigt, List colourings of planar graphs, *Discrete Mathematics* 120.**1-3** (1993), pages 215–219.

[VZ18] N. Van Cleemput and C. T. Zamfirescu, Regular non-hamiltonian polyhedral graphs, *Applied Mathematics and Computation* 338 (2018), pages 192–206.

[Wag37] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Mathematische Annalen* 114.**1** (1937), pages 570–590.

[Wag60] K. Wagner, Bemerkungen zu Hadwigers Vermutung, *Mathematische Annalen* 141.**5** (1960), pages 433–451.

[Whi31] H. Whitney, A theorem on graphs, *Annals of Mathematics* (1931), pages 378–390.

[Whi32] H. Whitney, Congruent graphs and the connectivity of graphs, *American Journal of Mathematics* 54.**1** (1932), pages 150–168.

[Wil13] R. Wilson, *Four Colors Suffice: How the Map Problem Was Solved-Revised Color Edition*, volume 30, Princeton university press, 2013.

[Wol05] P. Wollan, Extremal functions for graph linkages and rooted minors, PhD thesis, Georgia Institute of Technology, 2005.

[Wol08] P. Wollan, Extremal functions for rooted minors, *Journal of Graph Theory* 58.**2** (2008), pages 159–178.

[Woo19] D. Wood, *personal communication at the CanaDAM conference in Vancouver*, May 30, 2019.

[Xu90] X. Shaoji, The size of uniquely colorable graphs, *Journal of Combinatorial Theory, Series B* 50.**2** (1990), pages 319–320.

[Zha86] S. M. Zhan, Hamiltonian connectedness of line graphs, *Ars Combinatoria* 22 (1986), pages 89–95.

[Zha87] C.-Q. Zhang, Longest cycles and their chords, *Journal of graph theory* 11.**4** (1987), pages 521–529.

[Zyp12] D. van der Zypen, Hadwiger's conjecture for graphs with infinite chromatic number, *arXiv preprint* (2012), arXiv: 1212.3093.

Publications and Preprints

In connection with the work on this thesis, the following publications were published in international scientific journals. DOI and arXiv links are provided if available. They are sorted in ascending chronological order.

[KM19] M. Kriesell and S. Mohr, Rooted complete minors in line graphs with a Kempe coloring, *Graphs and Combinatorics* 35.**2** (2019), pages 551–557, doi: 10.1007/s00373-019-02012-7, arXiv: 1804.06641.

[Fab+20a] I. Fabrici, J. Harant, T. Madaras, S. Mohr, R. Soták, and C. T. Zamfirescu, Long cycles and spanning subgraphs of locally maximal 1-planar graphs, *Journal of Graph Theory* 95.**1** (2020), pages 125–137, doi: 10.1002/jgt.22542, arXiv: 1912.08028.

[Fab+20b] I. Fabrici, J. Harant, S. Mohr, and J. M. Schmidt, Longer cycles in essentially 4-connected planar graphs, *Discussiones Mathematicae Graph Theory* 40.**1** (2020), pages 269–277, doi: 10.7151/dmgt.2133, arXiv: 1710.05619.

[Böh+] T. Böhme, J. Harant, M. Kriesell, S. Mohr, and J. M. Schmidt, Rooted Minors and Locally Spanning Subgraphs, submitted, arXiv: 2003.04011.

[Fab+] I. Fabrici, J. Harant, S. Mohr, and J. M. Schmidt, Circumference of essentially 4-connected planar triangulations, submitted.

[KM] M. Kriesell and S. Mohr, Kempe chains and rooted minors, submitted, arXiv: 1911.09998.

[Moh] S. Mohr, A construction of uniquely colourable graphs, submitted, arXiv: 2001.08801.

Index

Symbols

$G(n, p)$, 13–14
  sparse, 14
X-free, 72, 74–76
X-minor, *see* rooted minor rooted at X
X-spanning, *see* locally spanning subgraph
$\kappa_G(X)$, 28
φ-branch, 95
k-connected, 6, 15, 70
  cyclically edge, 7, 93
  edge, 7
  essentially, *see* essentially 4-connected
  minor, 70–72
k-critical, 6, 15
k-degenerate, 11, 39
  colouring, 12
k-sharing, 108
  K_5, 108
k-subset, 5
1-planar, 37–42, 103–115
  α-band, 41
  associated graph, 41, 105
  boundary stripe, 41
  drawing, 104
  face, 41
  facial stripe, 41
  facial vertices, 41
  half-edges, 41
  locally maximal, 105
  maximal, 37, 104, 109
  number of edges, 104
  optimal, 104
  outer stripe, 41
  outer stripe vertices, 41
  parallel edges, 41
  weakly locally maximal, 105, 109

A

adjacent, 5
Akbari, S., 44
Albar, B., 11
Aldred, R., 93
algorithmic, 3, 32, 91
  linear time, 91
  quadratic time, 3, 32, 91
  Tutte path, *see* Tutte path computation
anticlique, 6, 48
  matchable, 62
apex, 6, 20, 24
  vertex, 4, 6, 10
Appel, K., 2

B

Bachmaier, C., 108
bag, *see* branch set
Barnette, D., 28, 77, 107
Bau, S., 93
Biedl, T., 77
Bollobás, B., 1, 13, 44, 47, 48
Bondy, J., 35, 93
  Bondy's conjecture, 93
boundary cycle, *see* facial cycle
branch
  φ-branch, *see* φ-branch
branch set, 8, 18, 19, 25, 59–60
bridge, 31, 34, 39, 79, 81–82
  F-bridge, 39
  inner vertex, 34, 80

- non-trivial, 80
- trivial, 31, 80

Broersma, H., 93
Brosse, C., 22

C

Catlin, P., 1, 4, 13
Cayley, A., 1
certificate, 8, 17, 22, 57–67
- rooted, 23–25, *see also* rooted minor

charging-discharging, 34, 86–87, 95–97, 101–102
Chen, G., 33, 35, 37, 38, 107
chromatic number, 6
- critical, *see* critical chromatic number
- random graphs, 13

circumference, 32–34, 37, 104
- 1-planar (4- and 5-connected), 38
- 3-connected maximal 1-planar, 37–38, 105
- 3-connected planar, 32–33
- essentially 4-connected cubic, 93
- essentially 4-connected maximal planar, 34, 94–102
- essentially 4-connected planar, 33–34, 85–90, 93–94

claw-free, 11, 93
clique, 6
- number, 6, 46
- random graphs, 13
- sum, 7, 10, 12, 66

clique minor, *see* K_k-minor
colouring, *see* k-critical, 6
- colour classes, 6, 49
- Kempe, *see* Kempe colouring
- optimal, 6
- proper, 6
- vertex-face, 37, 104

complete graph, 5, 26, 28, 48, 52, 58
- drawing of K_6, 107

complete join, *see* graph join
component, 6, *see also* fragment
computer aided, 2, 46
connected, 6
- k-connected, *see* k-connected
- 4-connected, 71
- 4-connected planar, 3, 30–32, 34–36, 77
- 5-connected, 70
- 6-connected, 72
- 7-connected, 39, 104
- essentially, *see* essentially 4-connected
- Hamilton, *see* Hamilton-connected
- maximal 1-planar, 104

contraction, 8, 35
- X-legal, 69
- into a vertex, 69

critical chromatic number, 44, 46, 48
crossing, 37
- almost full, 105
- full, 105
- point, 104
- remove an edge, 38, 114
- to new vertex, 110

Cui, Q., 35
cut, *see* separator
cycle, 6, 48, 59, 61, 65
- all lengths, *see* pancyclic
- chord, 6, 34, 63, 71, 86, 95
- dominating, *see also* outer-independent-cycle
- extendable, 86, 91, 94
- exterior, 7, 40
- facial, *see* facial cycle
- generalised, 78
- interior, 7, 31, 40
- long cycle, *see* circumference
- odd, 22
- OI3-cycle, *see* outer-independent-3-cycle

D

degeneracy, *see* k-degenerate
degree, 5
- average, 12, 13
- degenerate, *see* k-degenerate

Descartes, R.
- Cartesian product, *see* graphs

Deutsche Forschungsgemeinschaft (DFG), vi, 4
De Morgan, A., 1

Dirac, G., 12
drawing, *see* graph drawing

E
edge, 5
  C-edge, 86
  j,ℓ-edge, 95
  connected edge set, 50–55
  contraction, *see* contraction
  cover vertices, 50
  extendable, 86
edges
  multi, 50–51
Ellingham, M., 78
Erdős, P., 1, 13
essentially 4-connected, 32–34, 85, 93
  4-edge-connected, 35, *see also* k-connected, cyclically edge connected, 93
  circumference, *see* circumference
Euclid
  Euclidean plane, 7, 40
Euler, L., 1
  Euler characteristic, 78, 79, 82
  Euler's formula, 7, 33, 36
exterior, *see* cycle exterior

F
Fabila-Monroy, R., 23, 24, 26, 27, 60, 117
Fabrici, I., 33, 85
face, 7, 32, 34, 36
  j-face, 86, 95
  empty face, 94
  facial cycle, 7, 71, 80, 82
  facial walk, 40
  major, 85–91
  minor, 85–91
  opposite, 86
  outer, 7, 31, 95
factor
  1-factorisation, 51–52
Fan, G., 35
Five-Colour-Theorem, 2
Fleischner, H., 93
forest, 11
Fountoulakis, N., 14
Four-Colour-Theorem, 1–2, 10, 49
Fradkin, A., 11
fragment, 72, 74–76
  connectedness, 72

G
Gao, Z., 77, 107
generalised cycle, 58
genus, *see also* Euler characteristic, 36
  projective plane, 37
German Research Foundation, *see* Deutsche Forschungsgemeinschaft
girth, 14, 35, 48, 60
Gonçalves, D., 11
Grünbaum, B., 93
graph, 5
  K_k-free, 16, 46
  K_k-free, 16
  antitriangle-free, 18–20, 49
  Cartesian product, 22
  complement, 5
  cubic, 93
  drawing, 7, 37, 104
    unique planar embedding, 107
  embedding, *see* genus
  few colourings, 4, 15, 21
  from matrix, 21–22
  Hamiltonian, *see* Hamiltonian graph
  isomorphism, 5
  join, 5, 16, 44
  lexicographic product, 22, 58
  multipartite, 12, 48
  perfect, *see* perfect
  planar, *see* planar
  random, *see* $G(n, p)$
  regular, 51, 60
  rooted, 23
  triangle-free, 46–48
  union, 5
  uniquely colourable, *see* unique colouring
  with 2 colourings, 21
Grimmett, G., 13
Guthrie, F., 1

H

Hadwiger's Conjecture, 1, 3, 4, 6, 9–19, 22, 49, 58, 116
  C_4-free, 14
  $G(n,p)$, 14
  claw-free graph, 11
  girth, 14
  graph product, 22
  introduction, 3–4
  Kempe coloured graph, 58
  line graph, 11, 50
  matrix graph, 22
  number of edges, 11–13, 21, 117
  proofs of special cases, 9–11
  uniquely colourable, 17, 18, 51
Hadwiger, H., 1, 3, 9
Hajós, G., 4
Haken, W., 2
Hall, P.
  Hall's Theorem, 63
Hamilton, W., 1, 3, 32
  Hamilton-connected, 7, 32
  Hamiltonian cycle, 2, 7, 36–38, 51, 52, 77, 94, 104, 105, 112, 113
  Hamiltonian path, 52, 77, 112
Hamiltonicity, 7, 37
  of 1-planar graphs, 39, 105, 106, 115
  of 3-connected locally maximal 1-planar graphs with few 3-cuts, 106, 112
  of 3-connected maximal 1-planar graphs, 106
  of 3-connected planar graphs with few 3-cuts, 106, 112
  of 3-connected planar graphs with few vertices, 86, 94
  of 4-connected claw-free graphs, 93
  of 4-connected line graphs, 93
  of 4-connected locally maximal 1-planar graphs, 106
  of 4-connected maximal 1-planar graphs, 38
  of 4-connected maximal planar graphs, 30
  of 4-connected planar graphs, 30–32, 77
  of 4-connected weakly locally maximal 1-planar graphs, 106, 112
  of 5-connected 1-planar graphs, 38
  of 6-connected 1-planar graphs, 39, 118
  of cubic 3-connected planar graphs, 2
  of essentially 4-connected cubic planar graphs, 93
  of line graphs of 4-edge-connected graphs, 93
  of line graphs of essentially 4-edge-connected graphs, 93
  of maximal 1-planar graphs, 37, 104
  of of 4-connected planar graphs, 117
  of optimal 1-planar graphs, 104
Hanani, H., 66
  Tutte-Hanani-Theorem, 66
Harant, J., 33, 85
Harary, F., 16, 44
Heawood, P., 2
Hedetniemi, S., 16, 44
Heesch, H., 2
Holton, D., 93
hourglass, 64
Hu, Y., 35
Hudák, D., 115

I

immersion minor, 22, 49, 117
incident, 5
inclusion, 5
independent, *see* anticlique, 86
  independence number, 6
  random graphs, 13
interior, *see* cycle interior
isomorph, *see* graph isomorphism

J

Jackson, B., 33, 85, 93
Jendroľ, S., 33, 85
Jordan, C.
  Jordan curve, 37, 66, 104, 111, 114
  Jordan curve theorem, 7
journal publications, 4
Jørgensen, L., 12

K

Kawarabayashi, Ken-ichi., 11
Kempe, A., 2
  Kempe chain, 25–27, 57, 58, 64, 117
  Kempe change, 15, 18, 44
  Kempe colouring, 18–26, 50, 51, 53, 54, 58, 65, 116, 117
    properties, 53
Komlós, J., 44
Kostochka, A., 13
Kotzig, A., 51
  Kotzig's conjecture, 51
Kriesell, M., 14, 18–21, 23, 24, 48, 53, 116
  Kriesell's conjecture, 24–26, 50, 58, 65
    line graph, 50–55
Kühn, D., 14

L

line graph, 11, 25, 35, 93
  uniquely colourable, 51
  with Kempe colouring, 51–52
linkage, 8, 24, 27, 60
Lo, S., 36
locally spanning subgraph, 28, 77–83
  cycle, 79
  path, 79
  tree, 78
Lovász, L., 6

M

Madaras, T., 35, 115
Mader, W., 12, 116
  Mader's theorem, 12
Malkevitch, J., 35, 93, 117
matching, 20, 50–55, 62–63
  perfect, 20, 51–52
matrix, 21
Matthews, M., 93
maximal, 5, 37
Mazák, J., 93
McDiarmid, C., 13
McKay, B., 93
Menger, K., 6, 7
  Menger's Theorem, 7, 17, 53, 54, 61, 72
minor, 8
  K_1, 9
  K_2, 9
  K_3, 9, 24
  K_4, 9
  K_5, 10
  K_6, 10–11
  K_7, 11
  K_8, 11, 12
  K_9, 12, 21
  K_k, 12, 13, 17–20, 25, 49–55
  K_{10}, 21
  $K_{3,5}$, 11
  $K_{3,t}$, 77
  $K_{4,4}$, 11
  $\emptyset$-minor, 69
  number of edges, 12–13
  rooted, *see* rooted minor
  topological, *see* topological minor
Mirrokni, V., 44
Moon, J., 33, 37, 38, 104, 105
Moser, L., 33, 37, 38, 104, 105
Máčajová, E., 93

N

natural numbers, 5
neighbourhood, 5
Nešetřil, J., 44, 47, 48

O

OI3-cycle, *see* outer-independent-3-cycle
order, *see* size
Osthus, D., 14
Ota, K., 77
outer-independent
  cycle, 92
  outer-independent-3-cycle, 86–91, 94–97
outerplanar, 71
  maximal, 95
Ozeki, K., 77

P

Pach, J., 104
pancyclic, *see also* spectrum, 35, 36
  4-cycle, 35
paper, *see* journal publications

Parthasarathy, K., 22
partition, 5
path, 6, 48
  x,y-path, 6, 49, 54, 60–62, 72
  2-coloured, 20, 22, 24, 27, 49, 57, 65, 66
  generalised, 78
  system, *see also* linkage
perfect (graph), 6, 22
  weak perfect graph theorem, 6, 22
permutation, 59
Petersen, J., 2
  Petersen graph, 60
planar, 7, 37, 66, 71–72, 79–82, 84–91
  4-connected, *see* connected
  dual graph, 7
  generalisation, 37, 104
  maximal planar, 7, 30–32, 94–102
  subgraphs, 77
  weak dual, 36, 95
Plummer, M., 34
power set, 5
probabilistic proof, 48
property (*), 57–66
  K_4, 60
  K_5, 65
  K_7, 60
  cycle, 61
  five vertex graph, 64
  subgraph, 57
  unicyclic, 61
publications, *see* journal publications

Q

quadratic time, *see* algorithmic

R

random graph, *see* $G(n, p)$
Ravindra, G., 22
Reed, B., 11, 50
regular, 35
rim, 95
Ringel, G., 37, 104
Robertson, N., 2, 4, 11, 23
Robinson, R., 16, 44
root vertex, 69
rooted minor, 22–30, 50–55, 57–65, 69
  $G[X]$ as subgraph, 69
  π-rooted, 23, 26
  π-rooted relaxed, 27
  bipartite, 23
  property (*), *see* property (*)
  rooted K_4, 24–26, 60
  rooted K_5, 25, 26, 117
  rooted at X, 23–27, 69–72, 117
  topological, *see* topological minor rooted at X
rules, *see* charging-discharging rules, 96–97
Ryjáček, Z., 93

S

Sadjad, B., 44
Sanders, D., 2, 32, 77, 80, 91
Sauer, N., 44, 47, 48
Schmid, A., 32
Schmidt, J., 32
separation, 6
separator, 6, 28, 72, 106
  X-separator, 28, 70, 72–76
  s-separator, 85
  maximal 1-planar, 114
  trivial, 33, 85, 93
Seymour, P., 2–4, 9, 11, 13, 18, 23, 50
  Seymour's Conjecture, 14, 18–19, 49, 116
sharing, *see* k-sharing
shortness exponent, 105
size, 5
sling, *see also* Tutte cycle, 80
snake, *see also* Tutte path, 80
Song, Z., 12, 21, 116
Soták, R., 35
spectrum, 34–36, 117
  middle, 35
structure of the thesis, 4
subdivision, 8
subgraph, 5
  2-connected, 77
  dominating, 92
  induced, 5
  locally, *see* locally spanning subgraph

spanning subgraph, 38, 77, 107, 114
weighted subgraph, 36
Sumner, D., 93
Suzuki, Y., 115

T
Tait, P., 2
Tait's conjecture, 2
Thomas, R., 2, 4, 11–13, 21, 34, 37, 77, 116
Thomason, A., 13
Thomassen, C., 32, 34, 42, 77, 80, 93
Toft, B., 9, 11
topological minor, 8
π-rooted, 27
rooted at X, 69–72
touch vertex, *see* vertex of attachment
traceable, *see also* Hamiltonian path, 104
trail, 110
transversal, 6, 18, 23, 25, 26, 50–55, 57–67
of branch sets, *see* rooted minor
tree, 23, 36, 95
t-tree, 77
rooted, 23
spanning tree, 77, 78
triangulation, *see* planar, maximal
trivial separator, *see* separator
Turán, P.
Turán number, 44
Tutte, W., 2, 29–33, 37, 39, 40, 66, 77, 80, 93, 105, 115, 116, 118
Tutte cycle, 3, 30–42, 91
Tutte path, 2–3, 30–42, 79–82, *see also* outer-independent-3-cycle, 91, 93, 117
Tutte path computation, 3, 32, 91
Tutte path in non-planar graphs, 36–42, 106
Tutte's theorem, 33, 37, 38, 42, 106
Tóth, G., 104

U
unavoidable sets, 2
unicyclic, 61
unique colouring, *see also* Kempe colouring, 15–18, 24, 43–49, 51
K_k, 16, 48
K_k as subgraph, 16, 44
K_k-free, 46–48
3-colourable, 46–47
connected, 15
construction, 16, 45–48
Kempe property, 15
minimum degree, 15
number of edges, 44
properties, 15–16, 44
same size, 16–17, 24, 44–47

V
Van Cleemput, N., 93
vertex, 5
of attachment, 10, 31, 80

W
Wagner, K., 3, 4, 10, 66
planarity criteria, *see* Wagner's theorem
Wagner graph, 10, 66
Wagner's theorem, 3, 10, 66
Wang, J., 35
weights, *see* charging-discharging function, 36, 86, 96–97
Whitney, H., 30, 31, 105–107, 115
Wollan, P., 23
Wood, D., 23–27, 60, 117
Wormald, N., 33, 85, 93

X
Xu S., 44

Y
Yu, X., 33–35, 37, 38, 77, 107

Z
Zamfirescu, C., 93
Zhan, S., 93
Zhang, C., 93