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Spin-base-invariant Formulation of Hilbert-Palatini Gravity

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"And death shall have no dominion.
Dead man naked they shall be one
With the man in the wind and the west moon;
When their bones are picked clean and the clean bones gone,
They shall have stars at elbow and foot;
Though they go mad they shall be sane,
Though they sink through the sea they shall rise again;
Though lovers be lost love shall not;
And death shall have no dominion."

– Dylan Thomas

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ABSTRACT

In this thesis we derive a novel formulation of General Relativity that uses curved (spacetime-dependent) Dirac matrices and generalized spin connections as degrees of freedom and preserves full spin-base invariance - a symmetry of all fermionic matter sectors in nature. The curved Dirac matrices are thereby defined by virtue of a generalized Clifford algebra, which is postulated to hold locally. These Dirac matrices are the natural variables of the so called spin-base formalism, which allows to define spinors in curved spacetimes without introducing a coframe, i.e. a vielbein field is not required. In the generalization of the spin connection, we find an unconstrained degree of freedom, which uncovers a new symmetry of the novel formulation aside from the spin-base symmetry explicitly implemented. We investigate terms quadratic in the curvature, which give rise to a kinetic term for this unconstrained degree of freedom promoting it to a dynamical constituent of the theory.

In dieser Arbeit leiten wir eine neue Formulierung der Allgemeinen Relativitätstheorie ab, welche koordinatenabhängige Dirac Matrizen und einen verallgemeinerten Spin-Zusammenhang als Freiheitsgrade verwendet und die volle Spin-Basis-Invarianz - eine Symmetrie aller fermionischen Materie in der Natur - bewahrt. Die koordinatenabhängigen Dirac Matrizen werden dabei durch eine verallgemeinerte Clifford Algebra definiert, welche lokal postuliert wird. Diese Dirac Matrizen sind die natürlichen Variablen des so genannten Spin-Basen Formalismus, der es erlaubt, Spinore in gekrümmten Raumzeiten zu definieren, ohne ein spezielles Koordinatensystem einzuführen zu müssen, d.h. ein Vielbein-Feld ist nicht erforderlich. In der Verallgemeinerung des Spin-Zusammenhangs finden wir einen uneingeschränkten Freiheitsgrad, der neben der explizit implementierten Spin-Basis-Symmetrie eine weitere Symmetrie der neuen Formulierung aufdeckt. Weiter untersuchen wir in der Krümmung quadratische Terme, aus denen ein kinetischer Term für diesen unbeschränkten Freiheitsgrad hervorgeht, der ihn zu einem dynamischen Bestandteil der Theorie macht.

Contents

Notations

- Complex conjugation: $\overline{a+ib} = a - ib \quad (a, b \in \mathbb{R})$.
- Greek indices $(\alpha, \beta, ...)$ denote curved spacetime indices.
- Capital latin indices $(A, B, ...)$ denote flat vielbein indices.
- Small latin indices $(a, b, ...)$ denote spinor indices.
- Flat Minkowski spacetime is given by the metric η_{IJ} with signature (1,-1,-1,-1).
- The covariant derivative ∇_μ acts on curved and flat indices
  with the connections $\Gamma^\alpha{}_{\beta\gamma}$ and $\omega_\alpha{}^I{}_J$.
- The covariant derivative $\mathcal{D}_\mu$ acts also on spinor indices
  with the connection $\Gamma_\mu{}^i{}_j =: \Gamma_\mu$
- Connections not corresponding to the Levi-Civita connection
  shall be denoted as $\widetilde{\Gamma}^\alpha{}_{\beta\gamma}$, $\widetilde{\omega}_\alpha{}^I{}_J$, $\widetilde{\Gamma}_\mu$.
- Complete index antisymmetrization without prefactors is marked by $[\alpha_1 \dots \alpha_n]$.
- Complete index symmetrization without prefactors is marked by $(\alpha_1 \dots \alpha_n)$.
- We use the abbreviation: $\gamma_{\alpha_1} \dots \gamma_{\alpha_n} =: \gamma_{\alpha_1 \dots \alpha_n}$.

1. Introduction

One of the biggest challenges of modern theoretical physics is the formulation of a quantum theory for gravity. The standard procedure to quantize the underlying classical theory from Einstein via canonical quantization or path integral methods have failed, since the theory can be shown to be non renormalizable [GSS85], [GS86], [van92]. There are many other candidates for a quantum theory of gravity like string theory, loop quantum gravity or asymptotic safety scenarios just to name a few. All of these approaches are based on different features of general relativity and quantum theory to remain unchanged and others to be modified in a quantum theory for gravity [Ish93], [Sor97]. Since it is assumed that quantum gravity effects will only play a relevant role near the Planck scale, it is almost impossible to access this regime with today's experiments due to the huge energy that would be required. Therefore, there has not been much guidance through experiments that could indicate how to formulate or what to include into a quantum theory of gravity. Thus we rely on theoretical assumptions and instinct on the way to quantum gravity until experiments help to exclude unsuitable candidates.

Since fermions are present in our universe, for example as part of the standard model, we would wish for quantum gravity to be compatible with spinors, the mathematical construct describing fermions. Spinors were first introduced to differential geometry by Cartan [Car13] in 1913. Fifteen years later Dirac used Dirac spinors to describe the quantum nature of electrons [Dir28] and established the deep connection of spinors to fundamental physics. Ever since, spinors have been essential to many theories and concepts that form the basis of modern physics today. But formulating Dirac spinors in curved spacetimes brought up many difficulties due to the definition of Dirac spinors via the covering of the Lorentz group. In a curved space one would naively define spinors via the covering of general coordinate transformations, which does not exist. This problem was solved by using vielbeins, introduced to general relativity by Weyl [Wey29] in 1929. Fock and Ivanenko used vielbeins to give a consistent description of spinors by finding a coordinate independent formulation of the Dirac equation [FI29], [Foc29]. Though there are a few aesthetic subtleties coming with this formalism [Wel01], [GL14], this is still the standard approach to describe fermions in curved spacetimes. Schrödinger pioneered in developing a different approach without the explicit usage of vielbeins [Sch32] in 1932, but couldn't give a consistent picture thereof as he didn't include a generalization of the so called spin metric. Soon after Schrödinger's publication, this remaining obstacle was resolved by Bargmann [Bar32] to result in the first consistent description of the so called spinbase formalism. This new spinbase formalism was only occasionally picked up again, e.g. [FSY99], [Wel01], [CDN+13]. A compact description with extension to spin torsion and arbitrary integer dimensions is given in the publications by Gies and Lippoldt [GL14], [Lip15] and in the Ph.D thesis of Lippoldt [Lip16]. The advantages of the spinbase formalism became clear in the explicit construction of a global realization of the Clifford algebra on a 2-sphere [GL15], which is not possible within the vielbein formalism. Gies and Lippoldt thus suggested to include spinbase invariance as a possible feature of gravity or a quantum theory for gravity.

Following this idea, we will establish a classical theory of gravity which is manifestly invariant under spinbase and coordinate transformations in this thesis. The action will be written in terms of coordinate dependent Dirac matrices, which yield the metric by virtue of a generalized Clifford algebra for curved spacetimes. In a further step, we will include arbitrary spin connections in spirit of the publication of Palatini [Pal19]. This generalization can be included without any extra terms in the action. We will prove the validity of these theories on a classical level by deriving the Einstein equations form the equations of motion. In the perspective of quantum field theory, we will extend the theory and consider terms quadratic in the spin curvature, which corresponds to gravity theories quadratic in the Riemann tensor. Doing so, will uncover a new symmetry contained in the theory aside from the spinbase symmetry explicitly implemented.

We try to structure the thesis in coherent way in order to explain and state the formalisms underlying our considerations. Therefore, we will start in chapter 2 by recapitulating basic constructs from differential geometry. Here we will concentrate on the construction of tangent spaces at manifolds and the bases which can be chosen. This builds a bridge for the introduction to the vielbein formalism in section 2.2, that will be used to explain the description of spinors in curved spaces in chapter 3. In chapter 4 we will extend the vielbein construction of spinors and state the spinbase formalism. For later use and to draw analogies, we will introduce Palatini's idea [Pal19] in metric formulation in section 5.1 and in vielbein formulation in section 5.2. Chapter 6 is dedicated to the derivation of the spinbase invariant action for gravity, where we also will derive the equation of motion and show that it is equivalent to Einstein's equations. In chapter 7 we will apply Palatini's idea to this new spinbase invariant action and we will also find Einstein's equations as the equation of motion. For the last chapter 8 we will consider terms quadratic in the spin curvature and summarize our results in chapter 9.

2. Mathematical Foundations

As an introduction to the spinbase formalism, some general remarks and concepts should be mentioned that are essential for a basic understanding of the underlying mathematical constructs. Using differential geometry as an approach seems natural since the spinbase formalism is primarily used to describe spinors in curved spacetime. Thus we want to explain the basic terms of differential geometry and introduce the idea of vielbeins. Then, these vielbeins can be used to describe spinors in curved spacetimes, which will be done in chapter 3. Extending the vielbein formalism for spinors in chapter 4, will then lead to a coherent description of the spinbase formalism. There will be no explicit proofs given in this chapter, as they can be looked up in many text books about differential geometry and General Relativity (e.g. [Lee09], [Wal84], [Nak03], [Car19]).

2.1. Differential Geometry

If spacetime is understood as a four dimensional *topological manifold* $\mathcal{M}$ carrying a smooth *atlas* $\mathcal{A}$, we can construct the so called *tangent space* $T_p\mathcal{M}$ at each point p of the manifold $\mathcal{M}$. The tangent space $T_p\mathcal{M}$ is defined as the vector spaces of all directional derivatives $\vartheta_{\gamma,p}$ of functions $f \in C^\infty(\mathcal{M})$ along arbitrary smooth curves γ at a point p of the manifold $\mathcal{M}$

$$T_p\mathcal{M} = \left\{\vartheta_{\gamma,p} \mid \gamma \text{ smooth curve through } p\right\} . \tag{2.1}$$

A smooth curve γ through $p \in \mathcal{M}$ is thereby defined as a map $\gamma : \mathbb{R} \to \mathcal{M}$ with $\gamma(\lambda_0) = p$ for some $\lambda_0 \in \mathbb{R}$, which is called smooth if it is smooth in the chart representation of $\mathcal{M}$. The directional derivative w.r.t γ at point p is then understood as the linear map (indicated by ~)

$$\begin{aligned} \vartheta_{\gamma,p} : C^\infty(\mathcal{M}) &\xrightarrow{\sim} \mathbb{R} \\ f &\mapsto \vartheta_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0) \end{aligned} \tag{2.2}$$

where the prime indicates the derivative w.r.t. the curve parameter λ. Similarly we can construct the *dual tangent space* $T_p^*\mathcal{M}$ (also called co-tangent space) as the vector space of all linear maps v from $T_p\mathcal{M}$ into the real numbers

$$T_p^*\mathcal{M} := \left\{v : T_p\mathcal{M} \xrightarrow{\sim} \mathbb{R}\right\} . \tag{2.3}$$

It is convenient to introduce the *gradient* $(\mathrm{d}f)_p \in T_p^*\mathcal{M}$ of a function $f \in C^\infty(\mathcal{M})$ at a point $p \in \mathcal{M}$ as a vector of the dual tangent space by defining

$$\begin{aligned} (\mathrm{d}f)_p : T_p\mathcal{M} &\xrightarrow{\sim} \mathbb{R} \\ V &\mapsto (\mathrm{d}f)_p(V) := V(f) . \end{aligned} \tag{2.4}$$

The charts (U,x) with the bijective chart map $x: U \to \mathbb{R}^{\dim\mathcal{M}}$, chart map components $x^\mu : U \to \mathbb{R}$ and the corresponding chart domain $U \subseteq \mathcal{M}$ induce a basis to the tangent space T_pU. This can be shown since for every vector $V \in T_pU$ exists a curve γ such that $V = \vartheta_{\gamma,p}$ with $\gamma(\lambda_0) = p \in U$. Applied to a function $f \in C^\infty(U)$ this gives

$$V(f) = \vartheta_{\gamma,p}(f) = (f \circ \gamma)'(\lambda_0)\ .$$

Inserting the identity $\mathrm{id}_U = x^{-1} \circ x$ between f and γ yields

$$V(f) = (\underbrace{(f \circ x^{-1})}_{\mathbb{R}^{\dim\mathcal{M}} \to \mathbb{R}} \circ \underbrace{(x \circ \gamma)}_{\mathbb{R} \to \mathbb{R}^{\dim\mathcal{M}}})'(\lambda_0)\ ,$$

where we can use the multi dimensional chain rule in $\mathbb{R}^{\dim\mathcal{M}}$ with the Einstein sum convention implicitly being understood

$$V(f) = \partial_\mu (f \circ x^{-1})(x(p))\ (x^\mu \circ \gamma)'(\lambda_0)\ .$$

If we define the symbols

$$\left(\frac{\partial}{\partial x^\mu}\right)_p (f) := \partial_\mu (f \circ x^{-1})(x(p)) \tag{2.5}$$

$$V^\mu_{(x)} := (x^\mu \circ \gamma)'(\lambda_0)\ , \tag{2.6}$$

we can finally write for a vector $V \in T_pU$ in the chart (U,x)

$$V(f) = V^\mu_{(x)} \left(\frac{\partial}{\partial x^\mu}\right)_p (f)\ , \tag{2.7}$$

where the set $\mathfrak{B}$,

$$\mathfrak{B} := \left\{ \left(\frac{\partial}{\partial x^\mu}\right)_p \in T_pU \ \middle|\ 0 \leq \mu \leq \dim\mathcal{M} - 1 \right\}\ , \tag{2.8}$$

forms a basis for the tangent space T_pU. Similarly the gradients of the chart map components x^μ induce a basis in the dual tangent space $T^*_p\mathcal{M}$, which furthermore have the property

$$(\mathrm{d}x^\mu)_p \left[\left(\frac{\partial}{\partial x^\nu}\right)_p \right] = \partial_\nu (x^\mu \circ x^{-1})(x(p)) = \partial_\nu(\mathrm{proj}_\mu)(x(p)) = \delta^\mu_\nu\ , \tag{2.9}$$

which also makes $\{(\mathrm{d}x^\mu)_p\}$ the *dual basis* w.r.t. to the basis $\mathfrak{B}$ in $T_p\mathcal{M}$. The vectors and dual vectors are independent concepts with regard to the chart induced basis and do not change with a general change of basis. To facilitate this independence, the components of vectors or dual vectors have to transform accordingly with a change of basis. A special change of basis is given if we change the chart (or change of coordinates) $(U,x) \to (W,y)$ in an overlap region $U \cap W \subseteq \mathcal{M}$. The transformation behavior for the vector and dual vector components is then given by

$$V^\mu_{(y)} = \left(\frac{\partial}{\partial x^\nu}\right)_p (y^\mu)\, V^\nu_{(x)}\ , \tag{2.10}$$

$$v_{(y)\mu} = \left(\frac{\partial}{\partial y^\mu}\right)_p (x^\nu)\, v_{(x)\nu}\ . \tag{2.11}$$

In the following the subscript $_{(x)}$ etc. will be suppressed. The concept of tangent and dual tangent vectors can be further generalized to so called (p,q)-*tensors* T over the vector space $T_p\mathcal{M}$, which are defined as multi linear maps to $\mathbb{R}$

$$T : \underbrace{T_p^*\mathcal{M} \otimes \cdots \otimes T_p^*\mathcal{M}}_{p \text{ copies}} \otimes \underbrace{T_p\mathcal{M} \otimes \cdots \otimes T_p\mathcal{M}}_{q \text{ copies}} \xrightarrow{\sim} \mathbb{R}\,, \tag{2.12}$$

that again are equipped with a vector space structure over $\mathbb{R}$. In a chart (U,x) the tensor component carries one upstair index for each copy of T_pU and one downstair index for each copy of T_p^*U and each index transforms analogously to the transformations in (2.10) and (2.11) if the charts are changed (coordinates transformed).

One can proceed further and equip spacetime with a special (0,2)-tensor g which satisfies the following two conditions

$$\begin{aligned} &\text{(i)} && \text{symmetric:} && g(X,Y) = g(Y,X) \quad \forall X,Y \in T_p\mathcal{M} \\ &\text{(ii)} && \text{non degeneracy:} && T_p^*\mathcal{M} \ni \wp(X) := g(X,\cdot) \neq 0 \quad \forall X \in T_p\mathcal{M}\,. \end{aligned} \tag{2.13}$$

A tensor which satisfies these conditions is called a *metric* on $\mathcal{M}$. The *"inverse" metric* g^{-1} is a (2,0)-tensor which satisfies analogous conditions to (2.13) and is defined w.r.t to a metric g by

$$g^{-1}(v,w) = v\big(\underbrace{\wp^{-1}(w)}_{\in T_p\mathcal{M}}\big)\,, \qquad v,w \in T_p^*\mathcal{M}\,. \tag{2.14}$$

In a chart (U,x) this can be written in a more known form namely

$$\left.\begin{aligned} g(X,Y) &= g_{\mu\nu}X^\mu Y^\nu \\ \big(\wp(X)\big)_\nu &= \big(g(X,\cdot)\big)_\nu = g_{\mu\nu}X^\mu =: X_\nu \end{aligned}\right\} \quad \forall\ X,Y \in T_pU \tag{2.15}$$

$$\left.\begin{aligned} g^{-1}(v,w) &= \big(g^{-1}\big)^{\mu\nu}\, v_\mu w_\nu \\ \big(\wp^{-1}(v)\big)^\nu &= \big(g^{-1}\big)^{\mu\nu}\, v_\mu =: v^\nu \end{aligned}\right\} \quad \forall\ v,w \in T_p^*U\,. \tag{2.16}$$

From this follows the relation

$$\begin{aligned} X^\mu = \big(\wp^{-1}\left(\wp(X)\right)\big)^\mu = \big(g^{-1}\big)^{\mu\nu} g_{\nu\sigma}X^\sigma \quad \forall X \in T_pU \\ \Rightarrow \big(g^{-1}\big)^{\mu\nu} g_{\nu\sigma} = \delta^\nu_\sigma\,, \end{aligned} \tag{2.17}$$

that justifies the term "inverse" metric and we will simply write $g^{\mu\nu} := \big(g^{-1}\big)^{\mu\nu}$. Also we can find a basis $\{\hat{e}_I\}$ in $T_p\mathcal{M}$ with which the values of $g(\hat{e}_I,\hat{e}_J) =: \eta_{IJ}$ are either 1,-1, or 0 and vanish for $I \neq J$. The numbers η_{IJ} are then called the *signature* of g and can be ordered in the way that $\eta_{IJ} = \mathrm{diag}(-1,\ldots,-1,1\ldots,1,0,\ldots,0)$. In the case of $\eta_{IJ} = \mathrm{diag}(1,-1,\ldots,-1)$ we will call g a Lorentzian metric.

So far all the concepts introduced were defined only locally at a point $p \in \mathcal{M}$ or in a chart (U,x) with the implicit usage of the atlas $\mathcal{A}$ covering the whole manifold and being well defined in the overlaps of the charts contained in $\mathcal{A}$. The notion of vectors, dual (co-) vectors and tensors can be pushed further to the whole manifold by using the construction of the *tangent bundle* TM, which is defined as the disjoint union of all tangent spaces associated to $\mathcal{M}$

$$TM := \coprod_{p \in \mathcal{M}} T_p\mathcal{M} \,. \tag{2.18}$$

The tangent bundle can be equipped with a manifold structure inherited from $\mathcal{M}$. This way we can define the bundle $(TM, \pi, \mathcal{M})$ with the *projection map* $\pi : TM \to \mathcal{M}$ as a continuous surjective map. Simply by projecting down any given vector from the bundle to its base point $\pi(V) := q \in \mathcal{M}$ for $V \in T_q\mathcal{M} \subseteq TM$ we get a map satisfying these constraints. "Putting" a vector at each point of the manifold now corresponds to a map $\sigma : \mathcal{M} \to TM$ called *section*, which satisfies $\mathrm{id}_{\mathcal{M}} = \sigma \circ \pi$. This condition ensures that a section σ only "attaches" vectors from $T_p\mathcal{M}$ to a point p of the manifold $\mathcal{M}$. The section σ is then also called a *vector field* on $\mathcal{M}$. Graphically the situation can be illustrated in the following way

$$\begin{tikzcd} TM \arrow[d, shift left, "\pi"] \\ \mathcal{M} \arrow[u, shift left, "\sigma"] \end{tikzcd}$$

In the following, this is implicitly understood when speaking about vectors on the whole manifold. Thus, we can suppress the subscript $_p$ in the definition of basis vectors etc. Dual vectors can then be defined by using the dual tangent bundle $T^*\mathcal{M}$ and tensor can be constructed with tensor products of $T\mathcal{M}$ and $T^*\mathcal{M}$.

2.2. Vielbein Formalism

For the *vielbein formalism* we make use of the choice of basis in $T_p\mathcal{M}$. Before, a chart (U, x) induced the basis in every tangent space, but we are not constrained to that choice of basis. If $\mathcal{M}$ is equipped with a metric g we might choose a basis in which the metric coefficients take an easy from. In chapter 2, we used a special basis to explain the signature of the metric. This basis shall now be used as well in the vielbein formalism. So we define the *vielbein basis* vectors $\hat{e}_I$ as

$$g(\hat{e}_I, \hat{e}_J) = \eta_{IJ} = \begin{cases} 1, -1 \text{ or } 0 & \text{for } I = J \\ 0 & \text{for } I \neq J \,. \end{cases} \tag{2.19}$$

For a four dimensional Lorentzian spacetime, η_{IJ} is equal to the Minkowski metric, which is also the case we want to focus on. To reinterpret our findings, we can say that at each point of $\mathcal{M}$ a flat Minkowskian tangent space is constructed. As $\left\{\frac{\partial}{\partial x^\mu}\right\}$ and $\{\hat{e}_I\}$ both form a basis, we can use both of them to span any vector in the tangent space. Making use of this we can write for the basis vectors themselves

$$\frac{\partial}{\partial x^\mu} = e_\mu{}^I \hat{e}_I \,, \tag{2.20}$$

$$\hat{e}_I = e^\nu{}_I \frac{\partial}{\partial x^\nu} \,, \tag{2.21}$$

where the coefficients $e_\mu{}^I$ and $e^\nu{}_I$ are very different objects at this point They can be related to another by reinserting the expressions (2.20) and (2.21) into each other

$$\frac{\partial}{\partial x^\mu} = e_\mu{}^I \hat{e}_I = e_\mu{}^I e^\nu{}_I \frac{\partial}{\partial x^\nu} \quad \Rightarrow \quad e_\mu{}^I e^\nu{}_I = \delta^\nu_\mu \tag{2.22}$$

$$\hat{e}_I = e^\nu{}_I \frac{\partial}{\partial x^\nu} = e^\nu{}_I e_\nu{}^J \hat{e}_J \quad \Rightarrow \quad e^\nu{}_I e_\nu{}^J = \delta^J_I \,. \tag{2.23}$$

Similarly we can find for the dual tangent space the dual vielbein basis $\{\hat{E}^I\}$ by demanding

$$\hat{E}^I(\hat{e}_J) \stackrel{!}{=} \delta^I_J \,, \tag{2.24}$$

which also implies

$$\mathrm{d}x^\mu = \tilde{e}^\mu{}_I \hat{E}^I \tag{2.25}$$

$$\hat{E}^I = \tilde{e}_\nu{}^I \mathrm{d}x^\nu \,. \tag{2.26}$$

With the constraint (2.24) we can connect the coefficients for vectors and dual vectors by

$$\delta^I_J \stackrel{!}{=} \hat{E}^I(\hat{e}_J) = \tilde{e}_\nu{}^I e^\mu{}_J \mathrm{d}x^\nu \left(\frac{\partial}{\partial x^\mu}\right) = \tilde{e}_\nu{}^I e^\nu{}_J \quad \Rightarrow \quad \tilde{e}_\nu{}^I = e_\nu{}^I \tag{2.27}$$

$$\delta^\mu_\nu \stackrel{!}{=} \mathrm{d}x^\mu \left(\frac{\partial}{\partial x^\nu}\right) = \tilde{e}^\mu{}_I e_\nu{}^J \hat{E}^I(\hat{e}_J) = \tilde{e}^\mu{}_I e_\nu{}^I \quad \Rightarrow \quad \tilde{e}^\mu{}_I = e^\mu{}_I \,. \tag{2.28}$$

Thus the coefficients $e_\mu{}^I$ and $e^\mu{}_I$ are the only relevant degrees of freedom and can be used further to transfer vector, dual vector or tensor components from the coordinate to the vielbein basis and back

$$V^I = e_\mu{}^I V^\mu \quad , \quad V_I = e^\mu{}_I V_\mu \,, \tag{2.29}$$

$$V^\mu = e^\mu{}_I V^I \quad , \quad V_\mu = e_\mu{}^I V_I \,. \tag{2.30}$$

This is especially true in the case of the metric tensor g for which we can write

$$g_{\mu\nu} e^\mu{}_I e^\nu{}_J = \eta_{IJ} \quad , \quad g_{\mu\nu} = e_\mu{}^I e_\nu{}^J \eta_{IJ} \,. \tag{2.31}$$

Together with the transformations (2.29) and (2.30) between bases, this implies we can raise and lower indices of both types with either $g_{\mu\nu}$, $g^{\mu\nu}$, η_{IJ} and η^{IJ}.

There is still some freedom left in the choice of a vielbein basis, namely any basis satisfying the definition (2.19) is well suited. This means any transformation leaving the coefficients η_{IJ} invariant is legitimate

$$\hat{e}_I \to \hat{e}_{I'} = (\Lambda)_{I'}{}^I(x)\hat{e}_I \quad , \quad \hat{E}^I \to \hat{E}^{I'} = (\Lambda)_I{}^{I'} \hat{E}^I \,, \tag{2.32}$$

$$\eta_{I'J'} = (\Lambda)_{I'}{}^I (\Lambda)_{J'}{}^J \eta_{IJ} \,. \tag{2.33}$$

In our case of a Lorentzian metric, this corresponds to the *(local) Lorentz transformations*, which can act differently at any point of the manifold $\mathcal{M}$ in general. As a result, it is possible to change charts (coordinates) and to do local Lorentz transformations in the tangent spaces independently of each other. This is not possible in the pure coordinate induced bases formalism, due to the link between the basis vectors and the charts used to cover the chosen patch of the manifold. So we get a general transformation behavior for a general tensor carrying mixed indices

$$T^{\mu' I'}{}_{\nu' J'} = \frac{\partial x^{\mu'}}{\partial x^\mu} (\Lambda(x))_I{}^{I'} \frac{\partial x^\nu}{\partial x^{\nu'}} (\Lambda(x))_{J'}{}^J \, T^{\mu I}{}_{\nu J} \,, \tag{2.34}$$

where $\Lambda(x)$ stresses the locality of the Lorentz transformations.

The locality of the transformations makes it necessary to introduce two connections in order to be able to covariantly differentiate tensors. We introduce the *event connection* $\widetilde{\Gamma}^{\alpha}{}_{\beta\gamma}$ to compensate partial derivatives of coordinate transformations and the *vielbein connection* $\widetilde{\omega}_{\alpha}{}^{I}{}_{J}$ to compensate partial derivatives of local Lorentz transformations. The covariant derivative $\widetilde{\nabla}$ induced from these connections acts differently on tensor components in different bases

$$\widetilde{\nabla}_\alpha T^\mu{}_\nu = \partial_\alpha T^\mu{}_\nu + \widetilde{\Gamma}^\mu{}_{\alpha\sigma} T^\sigma{}_\nu - \widetilde{\Gamma}^\sigma{}_{\alpha\nu} T^\mu{}_\sigma \,, \tag{2.35}$$

$$\widetilde{\nabla}_\alpha T^I{}_J = \partial_\alpha T^I{}_J + \widetilde{\omega}_\alpha{}^I{}_K T^K{}_J - \widetilde{\omega}_\alpha{}^K{}_J T^I{}_K \,. \tag{2.36}$$

In order to identify $\widetilde{\nabla}_\alpha T^\mu{}_\nu$ and $\widetilde{\nabla}_\alpha T^I{}_J$ as tensor components, they have to transform accordingly to (2.34) under general transformations. To achieve this, the coordinate dependent connections have to transform inhomogenously under the associated transformations

$$\widetilde{\Gamma}^{\alpha'}{}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\gamma}{\partial x^{\gamma'}}\widetilde{\Gamma}^\alpha{}_{\beta\gamma} - \frac{\partial^2 x^{\alpha'}}{\partial x^\beta x^\gamma}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\gamma}{\partial x^{\gamma'}} \tag{2.37}$$

$$\widetilde{\omega}_{\alpha'}{}^{I'}{}_{J'} = \Lambda_I{}^{I'}\Lambda_{J'}{}^J\widetilde{\omega}_\alpha{}^I{}_J - \Lambda_{J'}{}^K\partial_\alpha\left(\Lambda_K{}^{I'}\right) \,. \tag{2.38}$$

This ensures a homogeneous transformation behavior for the tensor components $\widetilde{\nabla}_\alpha T^\mu{}_\nu$ and $\widetilde{\nabla}_\alpha T^I{}_J$. Further both connections can be related to each other, because the tensor $\widetilde{\nabla}V$ has to be independent of its basis representation. In the coordinate basis we get

$$\begin{aligned}\widetilde{\nabla}V &= \left(\widetilde{\nabla}_\alpha V^\beta\right)\,\mathrm{d}x^\alpha\otimes\frac{\partial}{\partial x^\beta}\\ &= \left(\partial_\alpha V^\beta + \widetilde{\Gamma}^\beta{}_{\alpha\sigma}V^\sigma\right)\,\mathrm{d}x^\alpha\otimes\frac{\partial}{\partial x^\beta}\,,\end{aligned} \tag{2.39}$$

on the other hand, we can also use the vielbein basis and transform back to the coordinate basis

$$\begin{aligned}\widetilde{\nabla}V &= \left(\widetilde{\nabla}_\alpha V^I\right)\,\mathrm{d}x^\alpha\otimes\hat{e}_I\\ &= \Big(\partial_\alpha\underbrace{(e_\sigma{}^I V^\sigma)}_{=V^I} + \widetilde{\omega}_\alpha{}^I{}_K\underbrace{(e_\sigma{}^K V^\sigma)}_{=V^K}\Big)\,\mathrm{d}x^\alpha\otimes\left(e^\beta{}_I\frac{\partial}{\partial x^\beta}\right)\\ &= e^\beta{}_I\left(e_\sigma{}^I\partial_\alpha V^\sigma + V^\sigma\partial_\alpha e_\sigma{}^I + \widetilde{\omega}_\alpha{}^I{}_K e_\sigma{}^K V^\sigma\right)\,\mathrm{d}x^\alpha\otimes\frac{\partial}{\partial x^\beta}\\ &= \left(\partial_\alpha V^\beta + e^\beta{}_I V^\sigma\partial_\alpha e_\sigma{}^I + \widetilde{\omega}_\alpha{}^I{}_K e_\sigma{}^K e^\beta{}_I V^\sigma\right)\,\mathrm{d}x^\alpha\otimes\frac{\partial}{\partial x^\beta}\,.\end{aligned} \tag{2.40}$$

Comparing this result to (2.39) gives the relation between the event connection and the vielbein connection, the so called *vielbein postulate*, which is true for all connections

$$\begin{aligned}\widetilde{\Gamma}^\beta{}_{\alpha\sigma} &= e^\beta{}_I\partial_\alpha e_\sigma{}^I + \widetilde{\omega}_\alpha{}^I{}_K e_\sigma{}^K e^\beta{}_I\\ \Leftrightarrow 0 &= \partial_\alpha e_\sigma{}^J + \widetilde{\omega}_\alpha{}^J{}_K e_\sigma{}^K - \widetilde{\Gamma}^\beta{}_{\alpha\sigma}e_\beta{}^J = \widetilde{\nabla}_\alpha e_\sigma{}^J\,.\end{aligned} \tag{2.41}$$

Demanding metric compatibility and zero torsion

$$\widetilde{\nabla}_\alpha g_{\beta\gamma} \stackrel{!}{=} 0 \tag{2.42}$$

$$T^\alpha{}_{\beta\gamma} = \widetilde{\Gamma}^\alpha{}_{\beta\gamma} - \widetilde{\Gamma}^\alpha{}_{\gamma\beta} \stackrel{!}{=} 0 \tag{2.43}$$

defines unique connections $\Gamma^\alpha{}_{\beta\gamma}$ and $\omega_\alpha{}^I{}_J$, called the *Levi-Civita connections*, which only depend on $g_{\mu\nu}$ and $e_\mu{}^I$ respectively [1]

$$\Gamma^\gamma{}_{\alpha\beta} = \Gamma^\gamma{}_{\beta\alpha} = \frac{1}{2} g^{\gamma\delta} \left(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}\right) , \tag{2.44}$$

$$\omega_\alpha{}^{IJ} = -\omega_\alpha{}^{JI} = \frac{1}{2} e_\alpha{}^K \left(e^\rho{}_K e^{\sigma I} \partial_{[\rho} e_{\sigma]}{}^J + e^{\rho J} e^\sigma{}_K \partial_{[\rho} e_{\sigma]}{}^I - e^{\rho I} e^{\sigma J} \partial_{[\rho} e_{\sigma]k} \right) . \tag{2.45}$$

These Levi-Civita connections and all objects referring to it shall be denoted without tilde in comparison to a general connection. Furthermore, the covariant derivative ∇ making use of the Levi-Civita connection allows for writing total derivative terms in volume integrals as we can show

$$\nabla_\alpha T^\alpha = \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g}\, T^\alpha\right) \quad \text{and} \tag{2.46}$$

$$\nabla_{\alpha_1} T^{[\alpha_1 \dots \alpha_n]} = \frac{1}{\sqrt{-g}} \partial_{\alpha_1} \left(\sqrt{-g}\, T^{[\alpha_1 \dots \alpha_n]}\right) , \tag{2.47}$$

where $[\alpha_1 \dots \alpha_n]$ indicates complete index antisymmetrization.[2] The difference between a general event connection $\widetilde{\Gamma}^\alpha{}_{\beta\gamma}$ and the Levi-Civita event connection $\Gamma^\alpha{}_{\beta\gamma}$ is called the *contorsion tensor* $C^\alpha{}_{\beta\gamma}$

$$C^\alpha{}_{\beta\gamma} := \widetilde{\Gamma}^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\beta\gamma} , \tag{2.48}$$

which transforms as a proper tensor. This allows to write for the covariant derivative $\widetilde{\nabla}_\alpha$

$$\widetilde{\nabla}_\alpha = \partial_\alpha + \Gamma^{\cdot}{}_{\alpha\,\cdot} + C^{\cdot}{}_{\alpha\,\cdot} \,. \tag{2.49}$$

With the general covariant derivative we can define two *curvature tensors* $\widetilde{R}_{\alpha\beta\gamma}{}^\delta$ and $\widetilde{F}_{\alpha\beta I}{}^J$

$$\begin{aligned}[]
[\widetilde{\nabla}_\alpha, \widetilde{\nabla}_\beta] T_\gamma &= \widetilde{R}_{\alpha\beta\gamma}{}^\delta T_\delta \\
&= \left(\partial_\beta \widetilde{\Gamma}^\delta{}_{\alpha\gamma} - \partial_\alpha \widetilde{\Gamma}^\delta{}_{\beta\gamma} + \widetilde{\Gamma}^\delta{}_{\beta\rho} \widetilde{\Gamma}^\rho{}_{\alpha\gamma} - \widetilde{\Gamma}^\delta{}_{\alpha\rho} \widetilde{\Gamma}^\rho{}_{\beta\gamma}\right) T_\delta
\end{aligned} \tag{2.50}$$

$$\begin{aligned}[]
[\widetilde{\nabla}_\alpha, \widetilde{\nabla}_\beta] T_I &= \widetilde{F}_{\alpha\beta I}{}^J T_J \\
&= \left(\partial_\beta \widetilde{\omega}^J{}_{\alpha I} - \partial_\alpha \widetilde{\omega}^J{}_{\beta I} + \widetilde{\omega}^J{}_{\beta K} \widetilde{\omega}^K{}_{\alpha I} - \widetilde{\omega}^J{}_{\alpha K} \widetilde{\omega}^K{}_{\beta I}\right) T_J ,
\end{aligned} \tag{2.51}$$

which are related to each other by

$$\widetilde{R}_{\alpha\beta\gamma\delta} = \widetilde{F}_{\alpha\beta IJ}\, e_\gamma{}^I e_\delta{}^J . \tag{2.52}$$

The vielbein formalism has a few more interesting features. One example is the application to spinors in curved spaces discussed in the next chapter 3. A second one is the vielbein formulation for General Relativity introduced at the beginning of section 5.1.

[1] For a derivation see e.g. appendix B in [Pel94].
[2] In this thesis antisymmetrization is used without prefactors see Notations.

3. Spinors in Curved Spaces

Ever since spinors have been introduced to physics by Dirac in 1928 [Dir28], they have been essential to many theories and concepts that form the basis of modern physics today. However, Dirac spinors originally were defined on a flat Minkowski spacetime which made it difficult to extend the concept to curved spacetimes, because there is no covering group for the group of general coordinate transformations. The vielbein formalism gave first guidance, when it was introduced to General Relativity by Weyl in 1929 [Wey29]. Fock and Ivanenko picked up vielbeins to present a way to formulate the Dirac equation on curved spaces [FI29], [Foc29]. This approach via vielbeins shall be discussed here as it will be the starting point for the spinbase formalism following in chapter 4.

Vielbeins can be used to introduce Dirac spinors into Lorentzian curved spacetimes. The *Dirac matrices* γ_I in flat Minkowski space are well known from the flat *Clifford algebra*

$$\left\{\gamma_I,\gamma_J\right\} = 2\eta_{IJ}\mathbb{1}_{4\times 4}\ . \tag{3.1}$$

With the vielbeins, we can lift the flat Dirac matrices to coordinate dependent (curved) Dirac matrices, carrying a proper coordinate basis index μ, with the definition

$$\gamma_\mu(x) = e_\mu{}^I(x)\gamma_I\ . \tag{3.2}$$

The curved Dirac matrices will then satisfy a general Clifford algebra for the curved spacetime

$$\left\{\gamma_\mu(x),\gamma_\nu(x)\right\} = 2g_{\mu\nu}(x)\mathbb{1}_{4\times 4}\ . \tag{3.3}$$

Thus we can equip every tangent space with flat Dirac matrices and introduce a Dirac structure. This also ensures that we have well-defined Graßmann-valued *spinors* ψ and *dual spinors* $\bar\psi$ in the tangent spaces. The dual spinors $\bar\psi$ are thereby constructed by using a *spin metric* h for *Dirac conjugation* indicated with a bar

$$\bar\psi = \psi^\dagger h \quad , \quad |\det h| = 1\ , \tag{3.4}$$

which is equal to γ_0 in the Dirac representation for flat Minkowski spacetime, but can be different in other representations or coordinate systems [Lip16]. The determinant of the spin metric is further set to 1 in order to introduce no scale between the spinor ψ and its dual spinor $\bar\psi$. As the vielbeins are fixed up to local Lorentz transformations, we have to identify the transformation behavior of spinors under local Lorentz transformations and coordinate transformations, whereas spinors transform (trivially) as scalars under coordinate transformations. A Lorentz transformation in flat space can be related to transformations in spinor space by virtue of the identity

$$\Lambda_I{}^J\gamma_J = \mathcal{T}^{-1}\gamma_I\mathcal{T}\ , \tag{3.5}$$

where $\Lambda \in \mathrm{SO}^+(1,3)$ and $\mathcal{T} = D_{\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)}(\Lambda)$ being the associated spin transformation, in the Dirac representation $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$, of the Lorentz transformation Λ. This does not mean that the Dirac matrices themself change under Lorentz transformations. Rather, the Dirac matrices can be understood as the tool to translate Lorentz transformations into spin transformations. Demanding that objects like $\bar{\psi}\gamma_I\psi$ transform as usual tangent vectors with index I under Lorentz transformations, implies that spinors transform like

$$\psi \to \mathcal{T}\psi \quad , \quad \bar{\psi} \to \bar{\psi}\mathcal{T}^{-1} \, , \tag{3.6}$$

under the associated spin transformations. But applying local Lorentz transformations to a curved Dirac matrix corresponds to transforming the vielbein only, so we get

$$\gamma_\mu \to \gamma'_\mu = e'_\mu{}^I \gamma_I = e_\mu{}^J \Lambda_J{}^I \gamma_I = \mathcal{T}^{-1}\gamma_\mu \mathcal{T} \, . \tag{3.7}$$

This is analogous to the behavior in flat space (3.5), which suggests again that spinors ψ and dual spinors $\bar{\psi}$ in curved space transform according to (3.6). As these transformations are local, we require a *spin connection* $\widetilde{\Gamma}_\mu{}^i{}_j$ in order to properly transport spinors on the manifold, where the indices i, j denote components in spinor space. This way we get a covariant derivative $\widetilde{\mathcal{D}}$ also respecting spinor indices. We demand the usual properties for derivatives also for $\widetilde{\mathcal{D}}$ namely

linearity: $$\widetilde{\mathcal{D}}_\mu(\lambda\psi_1 + \psi_2) = \lambda\, \widetilde{\mathcal{D}}_\mu\psi_1 + \widetilde{\mathcal{D}}_\mu\psi_2 \, , \quad \lambda \in \mathbb{C} \, , \tag{3.8}$$

product rule: $$\widetilde{\mathcal{D}}_\mu(\psi\bar{\psi}) = (\widetilde{\mathcal{D}}_\mu\psi)\bar{\psi} + \psi(\widetilde{\mathcal{D}}_\mu\bar{\psi}) \, , \tag{3.9}$$

spin metric compatability: $$\widetilde{\mathcal{D}}_\mu\bar{\psi} = \overline{\widetilde{\mathcal{D}}_\mu\psi} \, . \tag{3.10}$$

Furthermore we demand that $\widetilde{\mathcal{D}}$ reduces to the usual covariant derivative $\widetilde{\nabla}$ if applied to tensors, i.e.

$$\widetilde{\mathcal{D}}_\mu\left(\bar{\psi}\psi\right) = \partial_\mu\left(\bar{\psi}\psi\right) \quad , \quad \widetilde{\mathcal{D}}_\mu\left(\bar{\psi}\gamma_\nu\psi\right) = \widetilde{\nabla}_\mu\left(\bar{\psi}\gamma_\nu\psi\right) \, . \tag{3.11}$$

Using the expressions in (3.11) and the properties (3.8)-(3.10) yields for $\widetilde{\mathcal{D}}$ acting on spinors ψ, dual spinors $\bar{\psi}$ and Dirac matrices γ_ν

$$\widetilde{\mathcal{D}}_\mu\psi^i = \partial_\mu\psi^i + \widetilde{\Gamma}_\mu{}^i{}_j\psi^j = \partial_\mu\psi + \widetilde{\Gamma}_\mu\psi \, , \tag{3.12}$$

$$\widetilde{\mathcal{D}}_\mu\bar{\psi}_i = \partial_\mu\bar{\psi}_i - \widetilde{\Gamma}_\mu{}^j{}_i\bar{\psi}_j = \partial_\mu\bar{\psi} - \bar{\psi}\widetilde{\Gamma}_\mu \, , \tag{3.13}$$

$$\widetilde{\mathcal{D}}_\mu\gamma_\nu = \widetilde{\nabla}_\mu\gamma_\nu + \left[\widetilde{\Gamma}_\mu, \gamma_\nu\right] \, . \tag{3.14}$$

The Levi-Civita spin connection Γ_μ, defined through $\mathcal{D}_\mu\gamma_\nu \stackrel{!}{=} 0$ and using the Levi-Civita event and vielbein connection, is then given by the solution to the equation

$$\begin{aligned} 0 &= \mathcal{D}_\mu\gamma_\nu = e_\nu{}^I\left(\nabla_\mu\gamma_I + \left[\Gamma_\mu, \gamma_I\right]\right) \\ \Leftrightarrow 0 &= \omega_\mu{}^J{}_I\gamma_J + \left[\gamma_I, \Gamma_\mu\right] \, . \end{aligned} \tag{3.15}$$

The connection Γ_μ can be spanned using the elements of the flat Clifford basis (see appendix A)

$$\Gamma_\mu = p_\mu\gamma_* + v_\mu{}^I\gamma_I + a_\mu{}^I\gamma_*\gamma_I + t_\mu{}^{IJ}\left[\gamma_I, \gamma_J\right] \, , \tag{3.16}$$

where the scalar part, coincides with the trace of Γ_μ, which can be neglected as it corresponds to an external field [Pol10], [GL14]. This is will be further discussed in section 4. Using this decomposition in equation (3.15) yields

$$p_\mu = 0 \, , \quad a_\mu{}^I = 0 \, , \quad v_\mu{}^I = 0 \, , \quad t_\mu{}^{IJ} = \frac{1}{8}\omega_\mu{}^{IJ} \, , \tag{3.17}$$

and thus we get for the spin connection Γ_μ

$$\Gamma_\mu = \frac{1}{8}\omega_\mu{}^I{}_J \left[\gamma_I, \gamma_J\right] \ . \tag{3.18}$$

Merging these results enables us to construct a kinetic term for spinors using a covariant derivative in a curved spacetime

$$\bar{\psi}\not{\mathcal{D}}\psi = \bar{\psi}\gamma^\mu \mathcal{D}_\mu \psi = \bar{\psi}\gamma^\mu \left(\partial_\mu + \Gamma_\mu\right)\psi \ . \tag{3.19}$$

Finally we want to mention that the matrix γ_*, defined for even dimensions only, does not change from the flat description compared to the curved description which can be seen in 3+1 dimensions by

$$\begin{aligned}
\gamma_* &:= -\frac{i}{4!}\varepsilon^{\mu\nu\rho\lambda}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda && (3.20)\\
&= -\frac{i}{4!}\underbrace{\frac{\epsilon^{\mu\nu\rho\lambda}}{e} e_\alpha{}^I e_\beta{}^J e_\gamma{}^K e_\delta{}^L}_{\text{with (5.33): } = \epsilon^{IJKL} = \varepsilon^{IJKL}} \gamma_I\gamma_J\gamma_K\gamma_L \\
&= -i\gamma_{(f)0}\gamma_{(f)1}\gamma_{(f)2}\gamma_{(f)3} \ , \ , && (3.21)
\end{aligned}$$

where we have used that $\det\eta = -1$ in Minkowski spacetime, and the subscript (f) is used in the last line to explicitly indicate the use of flat Dirac matrices.

4. Spinbase Formalism

The *spinbase formalism* extends the description of spinors using vielbeins to a more general formalism [Wel01], [GL14]. As we are only interested in spacetimes with 3+1 dimensions, we will reduce the complexity and focus on 3+1 dimensional spacetimes. This also fixes the dimensions for Dirac spinors to 4. General results in other dimensions can be found in [Lip15], [Lip16].

To introduce the spinbase formalism we postulate the curved Clifford algebra to hold locally

$$\{\gamma_\mu(x), \gamma_\nu(x)\} = 2 g_{\mu\nu}(x) \mathbb{1}_{4\times 4}\,, \tag{4.1}$$

and do not give an explicit construction for the curved Dirac matrices like (3.2) a priori. Different irreducible representations for the Clifford algebra are connected to each other via similarity transformations [Pau36], [Cor89]. Thus a transformation

$$\gamma^\mu \to \mathcal{S}\gamma^\mu\mathcal{S}^{-1}, \tag{4.2}$$

with $\mathcal{S} \in \mathrm{SL}(4,\mathbb{C})$ leaves the curved Clifford algebra (4.1) invariant. We demonstrated in chapter 3 how to transfer Lorentz transformations to spin transformations for spinors by means of the relation (3.5). This, however, does not allow to transfer general coordinate transformations to spin transformations as, for example, scaling of one coordinate is not covered by this (see [Lip16], p. 18, 19 and [GL15]). To derive a suitable formalism, we investigate non trivial coordinate transformations of the metric g and use the Clifford algebra (4.1) to relate to transformations of spinors. A coordinate transformation changes the metric to

$$g_{\mu\nu} \to \frac{\partial x^\sigma}{\partial x^\mu}\frac{\partial x^\lambda}{\partial x^\nu} g_{\sigma\lambda}\,. \tag{4.3}$$

The Clifford algebra (4.1) indicates that the Dirac matrices also have to transform nontrivially

$$2g_{\mu\nu}\mathbb{1}_{4\times 4} = \{\gamma_\mu(x), \gamma_\nu(x)\} \quad \to \quad 2\frac{\partial x^\sigma}{\partial x^\mu}\frac{\partial x^\lambda}{\partial x^\nu} g_{\sigma\lambda}\mathbb{1}_{4\times 4} = \left\{\frac{\partial x^\sigma}{\partial x^\mu}\gamma_\sigma(x),\ \frac{\partial x^\lambda}{\partial x^\nu}\gamma_\lambda(x)\right\}\,. \tag{4.4}$$

Also incorporating similarity transformations (4.2) gives the most general transformation law for Dirac matrices under general coordinate transformations

$$\gamma_\mu \to \frac{\partial x^\sigma}{\partial x^\mu}\mathcal{S}\gamma_\sigma(x)\mathcal{S}^{-1}\,. \tag{4.5}$$

Hence a coordinate transformation of the Dirac matrices is composed out of the transformation of the spacetime vector part and a similarity transformation in spinor space. Introducing Graßmann-valued spinors ψ and dual spinors $\bar{\psi}$, where the dual spinors are again related to spinors with a spin metric h [1] (*Dirac conjugation*)

$$\bar{\psi} = \psi^\dagger h\,, \tag{4.6}$$

[1] A complete description of the spin metric is given in appendix F of [Lip16]. We only need the spin metric to consistently define Dirac conjugated spinors $\bar{\psi}$.

enables a different point of view on these similarity transformations. Since objects like $\bar{\psi}\gamma_\nu\psi$ and $\bar{\psi}\psi$ shall transform trivially under transformations with SL(4,ℂ), the transformation laws need to satisfy

$$\gamma^\mu \to \mathcal{S}\gamma^\mu\mathcal{S}^{-1}, \quad \psi \to \mathcal{S}\psi, \quad \bar{\psi} \to \bar{\psi}\mathcal{S}^{-1}, \quad h \to (\mathcal{S}^\dagger)^{-1}h\mathcal{S}^{-1}\,. \tag{4.7}$$

This can be understood as bases transformations in spinor space, so we name these transformations *spinbase transformations* instead of similarity transformations. The choice of SL(4,ℂ) as the transformation group is proven in appendix B of [Lip16] as the minimal transformation group, which contains all similarity transformations and which does not miss any representation of the Clifford algebra. Furthermore, spinbase transformations can also be local, since we may choose a different representation of the Dirac matrices from point to point of the spacetime manifold. This in turn makes it necessary to introduce a spin connection $\widetilde{\Gamma}_\mu{}^i{}_j$ for spinbase transformations, where the indices i and j again label spinor components. The existence of the spin connection is further guaranteed by the Weldon theorem [Wel01], [Lip15]

$$\delta\gamma_\mu = \frac{1}{2}(\delta g_{\mu\nu})\gamma^\nu + [\delta M, \gamma_\mu]\,, \quad \delta M \in \mathrm{SL}(4,\mathbb{C})\,, \quad \mathrm{Tr}\,(\delta M) = 0\,. \tag{4.8}$$

The Weldon theorem considers arbitrary variations of the Dirac matrices $\delta\gamma_\mu$ compatible with the Clifford algebra (4.1) and connects them to the corresponding variation of the metric $\delta g_{\mu\nu}$ and an arbitrary infinitesimal spinbase transformation δM. Thus we can always construct a connection respecting the Clifford algebra. A proof of the theorem is given in appendix A of [Lip16] or [Wel01]. The covariant derivative $\widetilde{\mathcal{D}}$ induced from this connection is required to satisfy the usual properties for derivatives and to reduce to the usual covariant derivative $\widetilde{\nabla}$ for objects like $\bar{\psi}\gamma_\nu\psi$ and $\bar{\psi}\psi$

linearity: $$\widetilde{\mathcal{D}}_\mu(\lambda\psi_1 + \psi_2) = \lambda\,\widetilde{\mathcal{D}}_\mu\psi_1 + \widetilde{\mathcal{D}}_\mu\psi_2\,, \quad \lambda \in \mathbb{C}\,, \tag{4.9}$$

product rule: $$\widetilde{\mathcal{D}}_\mu(\psi\bar{\psi}) = (\widetilde{\mathcal{D}}_\mu\psi)\bar{\psi} + \psi(\widetilde{\mathcal{D}}_\mu\bar{\psi})\,, \tag{4.10}$$

spin metric compatability: $$\widetilde{\mathcal{D}}_\mu\bar{\psi} = \overline{\widetilde{\mathcal{D}}_\mu\psi}\,, \tag{4.11}$$

covariance: $$\widetilde{\mathcal{D}}_\mu\psi \to \frac{\partial x^\sigma}{\partial x^\mu}\mathcal{S}\left(\widetilde{\mathcal{D}}_\sigma\psi\right)\,, \tag{4.12}$$

reduction: $$\widetilde{\mathcal{D}}_\mu(\bar{\psi}\gamma_\nu\psi) = \widetilde{\nabla}_\mu(\bar{\psi}\gamma_\nu\psi)\,, \quad \widetilde{\mathcal{D}}_\mu(\bar{\psi}\psi) = \partial_\mu(\bar{\psi}\psi)\,. \tag{4.13}$$

Using (4.9), (4.10) and (4.13) we find analogously to (3.12)

$$\widetilde{\mathcal{D}}_\mu\psi^i = \partial_\mu\psi^i + \widetilde{\Gamma}_\mu{}^i{}_j\psi^j = \partial_\mu\psi + \widetilde{\Gamma}_\mu\psi\,, \tag{4.14}$$

$$\widetilde{\mathcal{D}}_\mu\bar{\psi}_i = \partial_\mu\bar{\psi}_i - \widetilde{\Gamma}_\mu{}^j{}_i\bar{\psi}_j = \partial_\mu\bar{\psi} - \bar{\psi}\widetilde{\Gamma}_\mu\,, \tag{4.15}$$

$$\widetilde{\mathcal{D}}_\mu\gamma_\nu = \widetilde{\nabla}_\mu\gamma_\nu + [\widetilde{\Gamma}_\mu, \gamma_\nu]\,. \tag{4.16}$$

Considering (4.11) and the definition for Dirac conjugation of matrices

$$\bar{M} = h^{-1}M^\dagger h\,, \quad M \in \mathrm{SL}(4,\mathbb{C})\,, \tag{4.17}$$

gives the condition for *spin metric compatibility*

$$h^{-1}\partial_\mu h = \widetilde{\Gamma}_\mu + \bar{\widetilde{\Gamma}}_\mu\,. \tag{4.18}$$

Finally, from (4.12) in combination with (4.13) we can deduce the transformation behavior for the spin connection

$$\widetilde{\Gamma}_\mu \to \mathcal{S}\widetilde{\Gamma}_\mu \mathcal{S}^{-1} - (\partial_\mu \mathcal{S})\,\mathcal{S}^{-1}\,, \quad \widetilde{\Gamma}_\mu \to \frac{\partial x^\sigma}{\partial x^\mu}\widetilde{\Gamma}_\sigma\,, \tag{4.19}$$

with $\mathcal{S} \in \mathrm{SL}(4,\mathbb{C})$, which also shows that $\widetilde{\Gamma}_\mu$ transforms as a connection for spinbase transformations and as a dual vector for coordinate transformations. Again we define the Levi-Civita spin connection Γ_μ by using the Levi-Civita event connection $\Gamma^\sigma{}_{\mu\nu}$ and demanding

$$\mathcal{D}_\mu \gamma_\nu \stackrel{!}{=} 0 = \underbrace{\partial_\mu \gamma_\nu - \Gamma^\sigma{}_{\mu\nu}\gamma_\sigma}_{=\nabla_\mu \gamma_\nu} + [\Gamma_\mu, \gamma_\nu]\ . \tag{4.20}$$

The spin connection can be decomposed using the Clifford algebra basis

$$\Gamma_\mu = p_\mu \gamma_* + v_\mu{}^\sigma \gamma_\sigma + a_\mu{}^\sigma \gamma_* \gamma_\sigma + t_\mu{}^{\sigma\rho}[\gamma_\sigma, \gamma_\rho]\,, \tag{4.21}$$

where we can neglect the scalar part as the generators of $\mathrm{SL}(4,\mathbb{C})$ are traceless matrices $\mathrm{Mat}(4,\mathbb{C})$ and thus we have $\mathrm{Tr}\,\widetilde{\Gamma}_\mu = 0$. The trace can be included if we extend the transformation group to $\mathcal{G} \otimes \mathrm{SL}(4,\mathbb{C})$, with $\mathcal{G}$ being a compact Lie group. In that case we have to include a connection for $\mathcal{G}$ also and would get

$$\widetilde{\Gamma}_{(\mathcal{G}\otimes \mathrm{SL}(4,\mathbb{C}))\ \mu} = i\, s_{(\mathcal{G})\mu} \otimes \mathbb{1}_{(\mathrm{SL}(4,\mathbb{C}))} + \mathbb{1}_{(\mathcal{G})} \otimes \widetilde{\Gamma}_{(\mathrm{SL}(4,\mathbb{C}))}\ . \tag{4.22}$$

The field s_μ would then correspond to an external gauge field [GL14], [Lip16] and transform trivially under spinbase transformations. Using the defining equation (4.20) and using the decomposition (4.21) yields for the components of Γ_μ (see appendix C.1)

$$p_\mu = \frac{1}{32}\,\mathrm{Tr}(\gamma_* \gamma^\lambda \partial_\mu \gamma_\lambda)\,, \tag{4.23}$$

$$v_{\mu\sigma} = \frac{1}{48}\,\mathrm{Tr}\Big([\gamma_\sigma, \gamma^\lambda]\partial_\mu \gamma_\lambda\Big)\,, \tag{4.24}$$

$$a_{\mu\sigma} = -\frac{1}{8}\,\mathrm{Tr}(\gamma_* \partial_\mu \gamma_\sigma)\,, \tag{4.25}$$

$$t_{\mu\sigma\rho} = \frac{1}{32}\,\mathrm{Tr}(\gamma_\rho \nabla_\mu \gamma_\sigma) = \frac{1}{32}\,\mathrm{Tr}(\gamma_\rho \partial_\mu \gamma_\sigma) - \frac{1}{8}\Gamma_{\rho\mu\sigma} \equiv -t_{\mu\rho\sigma}\ . \tag{4.26}$$

The antisymmetry in (4.26) follows from the construction in (4.21) and also from the metric compatibility of the covariant derivative ∇_μ

$$\nabla_\mu g_{\alpha\beta} = 0 = \frac{1}{4}\nabla_\mu \mathrm{Tr}(\gamma_\alpha \gamma_\beta) \Leftrightarrow 0 = \mathrm{Tr}(\nabla_\mu(\gamma_\alpha)\gamma_\beta) + \mathrm{Tr}(\nabla_\mu(\gamma_\beta)\gamma_\alpha)\,,$$

where the cyclicity of the trace has been used. The coefficients (4.23)-(4.26) are written in a slightly different way from the ones stated in [Wel01] or [GL14]. The difference is due to the partial derivatives acting on Dirac matrices with lower spacetime index only. One can raise and lower indices even though there is a partial derivative present. This only introduces a minus sign in front of

$$\mathrm{Tr}(\gamma_\rho \partial_\mu \gamma_\sigma) g^{\rho\nu} = -\mathrm{Tr}(\gamma_\sigma \partial_\mu \gamma^\nu)$$

in equation (4.26). Here we also want to mention a small typo in [GL14] in the coefficient $a_\mu{}^\alpha$, which has to be multiplied with -1, as the axial vektor in [Wel01] is defined as $a_\mu{}^\alpha \gamma_\alpha \gamma_*$ in contrast to the definition in [GL14], where it is defined as $a_\mu{}^\alpha \gamma_* \gamma_\alpha$. All this is further explained in more detail in appendix B.1 and C.1.

We will call the difference of a general spin connection $\widetilde{\Gamma}_\mu$ to the Levi-Civita spin connection Γ_μ the *spin torsion* $\Delta\Gamma_\mu$

$$\Delta\Gamma_\mu := \widetilde{\Gamma}_\mu - \Gamma_\mu\,, \quad \mathrm{Tr}(\Delta\Gamma_\mu) = 0\,, \tag{4.27}$$

where we can set the trace to zero as it again coincides to an external gauge field (see (4.22)). The spin torsion will then transform homogeneously under spinbase transformations and as a dual vector under coordinate transformations

$$\Delta\Gamma_\mu \to \mathcal{S}\Delta\Gamma_\mu\mathcal{S}^{-1}\,, \quad \Delta\Gamma_\mu \to \frac{\partial x^\mu}{\partial x^\nu}\Delta\Gamma_\nu \tag{4.28}$$

Using spin torsion and (2.48) we can rewrite the covariant derivative $\widetilde{\mathcal{D}}_\mu$ as

$$\widetilde{\mathcal{D}}_\mu = \underbrace{\partial_\mu + \Gamma^{\cdot}{}_{\mu\,\cdot} + C^{\cdot}{}_{\alpha\,\cdot}}_{=\widetilde{\nabla}_\mu} + \underbrace{\Gamma_\mu + \Delta\Gamma_\mu}_{=\widetilde{\Gamma}_\mu}\,. \tag{4.29}$$

Having a covariant derivative $\widetilde{\mathcal{D}}_\mu$ allows to construct a spinor valued *spin curvature tensor* $\widetilde{\Phi}_{\mu\nu}{}^i{}_j$

$$[\widetilde{\mathcal{D}}_\mu, \widetilde{\mathcal{D}}_\nu]\psi = \widetilde{\Phi}_{\mu\nu}\psi = \left(\partial_\mu\widetilde{\Gamma}_\nu - \partial_\nu\widetilde{\Gamma}_\mu + [\widetilde{\Gamma}_\mu, \widetilde{\Gamma}_\nu]\right)\psi\,, \tag{4.30}$$

where we suppressed the spinor indices. The spin curvature built from the Levi-Civita spin connection Γ_μ can be further related to the Riemann tensor in metric or vielbein formulation by the identities (see appendix C.2)

$$\Phi_{\alpha\beta} = \frac{1}{8}R_{\alpha\beta}{}^{\sigma\tau}\left[\gamma_\sigma, \gamma_\tau\right] = \frac{1}{4}R_{\alpha\beta}{}^{\sigma\tau}\gamma_\sigma\gamma_\tau \tag{4.31}$$

$$= \frac{1}{8}F_{\alpha\beta}{}^{IJ}\left[\gamma_I, \gamma_J\right] = \frac{1}{4}F_{\alpha\beta}{}^{IJ}\gamma_I\gamma_J\,. \tag{4.32}$$

In a last step, we want to constrain the spin connection by imposing reasonable requirements for a dynamical theory of spinors. We want the different constituents in the action to be real, so we demand

$$\bar{\psi}\psi = \psi^\dagger h\psi \stackrel{!}{=} (\bar{\psi}\psi)^* = \psi^\top h^*\psi^* = \left(\psi^\top h^*\psi^*\right)^\top = -\psi^\dagger h^\dagger\psi\,, \tag{4.33}$$

where the minus sign in the last step comes from a commutation of the Graßmann-valued spinors. From this we conclude that the spin metric is antihermitean

$$h^\dagger = -h\,. \tag{4.34}$$

The same is required for the kinetic term

$$\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\;\bar{\psi}\widetilde{\mathcal{D}}\!\!\!/\,\psi = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\;\left(\bar{\psi}\widetilde{\mathcal{D}}\!\!\!/\,\psi\right)^*\,. \tag{4.35}$$

To evalute (4.35) we have to use some identities

$$\left(\bar{\chi}M\psi\right)^* = \chi^\dagger h^* M^*\psi^* = \left(\chi^\top h^* M^*\psi^*\right)^\top = \psi^\dagger M^\dagger(-h^\dagger)\chi = \bar{\psi}\bar{M}\chi \tag{4.36}$$

where the minus sign is due to the commutation of the spinors and

$$
\begin{aligned}
0 &= \int\limits_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\, \nabla_\mu \left(\bar{\psi}\bar{\gamma}^\mu\psi\right) \\
\Leftrightarrow \int\limits_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\, \partial_\mu(\bar{\psi})\bar{\gamma}^\mu\psi &= -\int\limits_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\, \left(\bar{\psi}\bar{\gamma}^\mu\partial_\mu\psi + \bar{\psi}\left(\nabla_\mu\bar{\gamma}^\mu\right)\psi\right) .^{2}
\end{aligned}
\tag{4.37}
$$

Also we need the derivative $\nabla_\mu\bar{\gamma}_\nu$

$$
\begin{aligned}
\nabla_\mu\bar{\gamma}_\nu &= h^{-1}\left(\nabla_\mu\gamma_\nu^\dagger\right)h + \left(\partial_\mu h^{-1}\right)\gamma_\nu^\dagger h + h^{-1}\gamma_\nu^\dagger\left(\partial_\mu h\right) \\
&= h^{-1}\left(\nabla_\mu\gamma_\nu\right)^\dagger h + \left[\bar{\gamma}_\nu, h^{-1}\left(\partial_\mu h\right)\right] ,
\end{aligned}
$$

where we have used

$$
\partial_\mu(h^{-1}h) = 0 = \left(\partial_\mu h^{-1}\right)h + h^{-1}\left(\partial_\mu h\right) ,
$$

in the second term and property (4.12) of the covariant derivative in the first term. With (4.18) and (4.20) we get

$$
\begin{aligned}
\nabla_\mu\bar{\gamma}_\nu &= -\left[\bar{\Gamma}_\mu, \bar{\gamma}_\nu\right] + \left[\bar{\gamma}_\nu, \widetilde{\Gamma}_\mu + \bar{\widetilde{\Gamma}}_\mu\right] \\
&= \left[\bar{\gamma}_\nu, \widetilde{\Gamma}_\mu - \Delta\bar{\Gamma}_\mu\right]
\end{aligned} .
\tag{4.38}
$$

Now coming back to evaluate the second requirement (4.35). In a first step, we get using (4.36) and (4.11)

$$
\int\limits_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\, \left(\bar{\psi}\widetilde{\mathcal{D}}\psi\right)^* = \int\limits_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\, \left(\left(\widetilde{\mathcal{D}}_\mu\bar{\psi}\right)\bar{\gamma}^\mu\psi\right) = \int\limits_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\, \left(\left(\partial_\mu\bar{\psi}\right)\bar{\gamma}^\mu\psi - \bar{\psi}\widetilde{\Gamma}_\mu\bar{\gamma}^\mu\psi\right) .
$$

Replacing $\left(\partial_\mu\bar{\psi}\right)\bar{\gamma}^\mu\psi$ with (4.37) yields

$$
\int\limits_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\, \left(\bar{\psi}\widetilde{\mathcal{D}}\psi\right)^* = -\int\limits_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\, \left(\bar{\psi}\bar{\gamma}^\mu\partial_\mu\psi + \bar{\psi}\left(\nabla_\mu\bar{\gamma}^\mu\right)\psi + \bar{\psi}\widetilde{\Gamma}_\mu\bar{\gamma}^\mu\psi\right) .
$$

Now using (4.38) for the second term results in

$$
\int\limits_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\, \left(\bar{\psi}\widetilde{\mathcal{D}}\psi\right) = \int\limits_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\, \left(\bar{\psi}\widetilde{\mathcal{D}}\psi\right)^* = \int\limits_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\, \left(-\bar{\psi}\bar{\gamma}^\mu\widetilde{\mathcal{D}}_\mu\psi + \bar{\psi}\left[\bar{\gamma}^\mu, \Delta\bar{\Gamma}_\mu\right]\psi\right) ,
\tag{4.39}
$$

which has to be satisfied for arbitrary spinors. So we conclude

$$
\gamma_\mu = -\bar{\gamma}_\mu ,
\tag{4.40}
$$

$$
\left[\gamma^\mu, \Delta\Gamma_\mu\right] = 0 .
\tag{4.41}
$$

The spin torsion can be spanned again by the Clifford decomposition

$$
\Delta\Gamma_\mu = p_{(\Delta\Gamma)\mu}\gamma_* + v_{(\Delta\Gamma)\mu}{}^{\sigma}\gamma_\sigma + a_{(\Delta\Gamma)\mu}{}^{\sigma}\,\gamma_*\gamma_\sigma + t_{(\Delta\Gamma)\mu}{}^{\sigma\tau}\left[\gamma_\sigma, \gamma_\tau\right] .
\tag{4.42}
$$

[2] We consider only manifolds $\mathcal{M}$ with no boundary, so that we can neglect surface terms.

The constraint (4.41) then implies for the decomposition coefficients

$$p_{(\Delta\Gamma)\mu} = 0\,, \tag{4.43}$$

$$v_{(\Delta\Gamma)[\mu\sigma]} = 0\,, \tag{4.44}$$

$$a_{(\Delta\Gamma)\mu}{}^{\mu} = 0\,, \tag{4.45}$$

$$t_{(\Delta\Gamma)\mu}{}^{\sigma\mu} = 0\,. \tag{4.46}$$

One further constraint can be deduced for the possible trace of the spin torsion (see appendix F, (F.28) in [Lip16]), if we consider the spin metric compatibility condition (4.18). It follows that the a priori complex trace has to be purly imaginary

$$\mathrm{Re}(s_\mu) = 0\,, \tag{4.47}$$

which we already have taken into account for the inclusion of the external gauge field (4.22).
We want to emphasize here that the constraints (4.40) (4.41) and (4.47) arise only if we couple the connection to spinors. If we want to write down a theory without spinors, but still containing the connection, we can lift these constraints.
The standard vielbein formalism for spinors from section 3 can be related to the spinbase formalism, if there is no spacetime torsion present. This can be seen from (4.29)

$$\widetilde{\mathcal{D}}_\mu\gamma_\nu = C^{\sigma}{}_{\mu\nu}\gamma_\sigma + \left[\Delta\Gamma_\mu, \gamma_\nu\right] \overset{(4.41)}{=} C^{\sigma}{}_{\mu\nu}\gamma_\sigma\,. \tag{4.48}$$

Thus, generally the Dirac matrices are not covariantly constant, in contrast to the standard vielbein formalism where we have (3.15). But in absence of spacetime torsion we can always choose a spinbase transformation to transform the Dirac matrices such that they are spanned as in (3.2) [Wel01], [GL14].

5. The Palatini Formulation for Gravity

Einstein's theory for gravity is described by a dynamical metric obeying the Einstein equations [Ein15] (here stated in vacuum)

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\, R = 0\,. \tag{5.1}$$

The *Ricci tensor* $R_{\alpha\beta}$ and the *Ricci scalar* R in the Einstein equations are defined via the curvature tensor (2.50) obtained from the Levi-Civita event connection (2.44) and are given by

$$\Gamma^{\gamma}{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}\right)\,, \tag{5.2}$$

$$R_{\alpha\beta\gamma}{}^{\delta} = \partial_\beta \Gamma^{\delta}{}_{\alpha\gamma} - \partial_\alpha \Gamma^{\delta}{}_{\beta\gamma} + \Gamma^{\delta}{}_{\beta\rho}\Gamma^{\rho}{}_{\alpha\gamma} - \Gamma^{\delta}{}_{\alpha\rho}\Gamma^{\rho}{}_{\beta\gamma}\,, \tag{5.3}$$

$$R_{\alpha\beta} = R_{\alpha\delta\beta}{}^{\delta}\,, \tag{5.4}$$

$$R = g^{\alpha\beta} R_{\alpha\beta}\,. \tag{5.5}$$

This makes the Einstein equations second order partial differential equations for the metric $g_{\alpha\beta}$. Test bodies are said to move on geodesics $x^\alpha(\lambda)$ given by the solution to the geodesic equation

$$\frac{d^2 x^\alpha}{d^2\lambda} + \Gamma^{\alpha}{}_{\beta\gamma} \frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0\,, \tag{5.6}$$

with the curve parameter λ. The Einstein equations can be derived from the *Einstein-Hilbert action* S_{EH} [Hil15] via the variational principal (see e.g. [Wal84], [Car19])

$$S_{EH}[g] = \int_{\mathcal{M}} \mathrm{d}^4x\, \sqrt{-\det g}\; R\,. \tag{5.7}$$

This is a theory in which the metric is the only dynamical degree of freedom. As introduced in chapter 2, the event connection $\widetilde{\Gamma}^{\alpha}{}_{\beta\gamma}$ and the metric $g_{\alpha\beta}$ are both independent concepts from differential geometry and therefore do not need to be linked as in Einstein's theory of gravity. There are many other theories that correspond to Einstein's theory on the classical level [Rom93], [Pel94]. A natural generalization of Einstein's theory is to detach the connection from the metric and consider both as independent degrees of freedom of the theory. This idea has its origin in the work of Palatini [Pal19] and is explained in terms of metric and vielbein formulation in the following. In chapter 7 this procedure will be applied to a gravity theory in the spinbase formalism derived in chapter 6.

5.1. Metric Palatini Formulation

The metric Palatini formulation [Pal19] of gravity considers the metric $g_{\alpha\beta}$ and event connection $\Gamma^{\alpha}{}_{\beta\gamma}$ as two independent degrees of freedom. The starting point is the usual Einstein Hilbert action

$$S_{EH}[g] = \int_{\mathcal{M}} \mathrm{d}^4x\, \sqrt{-\det g}\; R\,. \tag{5.8}$$

To promote the event connection to be an independent degree of freedom we study arbitrary event connections $\widetilde{\Gamma}^\alpha{}_{\beta\gamma}$. Following [DP12] and [Bla19], the general curvature $\widetilde{R}_{\alpha\beta\gamma}{}^\delta$, Ricci tensor $\widetilde{R}_{\alpha\beta}$ and Ricci scalar $\widetilde{R}$ following from this new event connection are then given analogously to (5.3), (5.4) and (5.5)

$$\widetilde{R}_{\alpha\beta\gamma}{}^\delta = \partial_\beta \widetilde{\Gamma}^\delta{}_{\alpha\gamma} - \partial_\alpha \widetilde{\Gamma}^\delta{}_{\beta\gamma} + \widetilde{\Gamma}^\delta{}_{\beta\rho}\widetilde{\Gamma}^\rho{}_{\alpha\gamma} - \widetilde{\Gamma}^\delta{}_{\alpha\rho}\widetilde{\Gamma}^\rho{}_{\beta\gamma} , \tag{5.9}$$

$$\widetilde{R}_{\alpha\beta} = \widetilde{R}_{\alpha\delta\beta}{}^\delta , \tag{5.10}$$

$$\widetilde{R} = g^{\alpha\beta}\widetilde{R}_{\alpha\beta} , \tag{5.11}$$

and one gains a new *Einstein-Palatini action* depending now on $g_{\alpha\beta}$ and $\widetilde{\Gamma}^\alpha{}_{\beta\gamma}$

$$S_{EP}[g,\widetilde{\Gamma}] = \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\, \widetilde{R} . \tag{5.12}$$

To obtain the equations of motion for $g_{\alpha\beta}$ and $\widetilde{\Gamma}^\alpha{}_{\beta\gamma}$ we have to consider the variations $\delta g_{\alpha\beta}$ and $\delta\widetilde{\Gamma}^\alpha{}_{\beta\gamma}$. The calculation of these variations turns out to be less complicated if the event connection $\widetilde{\Gamma}^\alpha{}_{\beta\gamma}$ is decomposed into the Levi-Civita connection $\Gamma^\alpha{}_{\beta\gamma}$ and a general deviation therefrom in form of the contorsion tensor $C^\alpha{}_{\beta\gamma}$

$$\widetilde{\Gamma}^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} + C^\alpha{}_{\beta\gamma} , \tag{5.13}$$

and hence we can conclude for the variation $\delta\widetilde{\Gamma}^\alpha{}_{\beta\gamma}$

$$\delta\widetilde{\Gamma}^\alpha{}_{\beta\gamma} = \delta C^\alpha{}_{\beta\gamma} . \tag{5.14}$$

Using (5.13) we can rewrite the curvature tensor $\widetilde{R}_{\alpha\beta\gamma}{}^\delta$ into the following form

$$\widetilde{R}_{\alpha\beta\gamma}{}^\delta = R_{\alpha\beta\gamma}{}^\delta + \nabla_\beta C^\delta{}_{\alpha\gamma} - \nabla_\alpha C^\delta{}_{\beta\gamma} + C^\delta{}_{\beta\rho}C^\rho{}_{\alpha\gamma} - C^\delta{}_{\alpha\rho}C^\rho{}_{\beta\gamma} , \tag{5.15}$$

where ∇_α stands for the covariant derivative associated to the Levi-Civita connection $\Gamma^\alpha{}_{\beta\gamma}$ which is connected to the general covariant derivative $\widetilde{\nabla}_\alpha$ by

$$\begin{aligned} \widetilde{\nabla}_\alpha T^\beta = \partial_\alpha T^\beta + \widetilde{\Gamma}^\beta{}_{\alpha\gamma}T^\gamma = \partial_\alpha T^\beta + \Gamma^\beta{}_{\alpha\gamma}T^\gamma + C^\beta{}_{\alpha\gamma}T^\gamma \\ \Leftrightarrow \widetilde{\nabla}_\alpha T^\beta = \nabla_\alpha T^\beta + C^\beta{}_{\alpha\gamma}T^\gamma , \end{aligned} \tag{5.16}$$

with the usual extension to higher rank tensors including lower indices etc. The second and third term in (5.15) will not contribute to the equations of motions as they correspond to surface terms in the action (5.12), due to the explicit properties of the Levi-Civita covariant derivative, namely

$$\nabla_\alpha T^\alpha = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\, T^\alpha\right) \quad \text{and} \tag{5.17}$$

$$\nabla_{\alpha_1} T^{[\alpha_1\ldots\alpha_n]} = \frac{1}{\sqrt{-g}}\partial_{\alpha_1}\left(\sqrt{-g}\, T^{[\alpha_1\ldots\alpha_n]}\right) , \tag{5.18}$$

which do not hold for arbitrary covariant derivatives $\widetilde{\nabla}_\alpha$. The action (5.12) can now be rewritten as

$$\begin{aligned} S_{EP}[g,C] &= \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left(R + \nabla_\alpha C^{\alpha}{}_{\beta}{}^{\beta} - \nabla_\alpha C^{\beta}{}_{\beta}{}^{\alpha} + C^{\alpha}{}_{\alpha\rho}C^{\rho}{}_{\beta}{}^{\beta} - C^{\alpha}{}_{\beta\rho}C^{\rho}{}_{\alpha}{}^{\beta}\right) \\ &= S_{EH}[g] + \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left(C^{\alpha}{}_{\alpha\rho}C^{\rho}{}_{\beta}{}^{\beta} - C^{\alpha}{}_{\beta\rho}C^{\rho}{}_{\alpha}{}^{\beta}\right) + \text{surface terms}\,. \end{aligned} \tag{5.19}$$

Requiring the action to be stationary under the variations $\delta g_{\alpha\beta}$ and $\delta C^{\alpha}{}_{\beta\gamma}$ and neglecting surface terms, we get the equations of motion for $g_{\alpha\beta}$ and $C^{\alpha}{}_{\beta\gamma}$

$$0 \overset{!}{=} \frac{\delta S_{\mathrm{EP}}}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\left(C^{\alpha}{}_{\alpha\rho}C^{\rho}{}_{\beta}{}^{\beta} - C^{\alpha}{}_{\beta\rho}C^{\rho}{}_{\alpha}{}^{\beta}\right), \tag{5.20}$$

$$0 \overset{!}{=} \frac{\delta S_{\mathrm{EP}}}{\delta C^{\sigma}{}_{\mu\nu}} = \sqrt{-g}\left(\delta^{\mu}_{\sigma}C^{\nu}{}_{\beta}{}^{\beta} + C^{\alpha}{}_{\alpha\sigma}g^{\mu\nu} - C^{\nu}{}_{\sigma}{}^{\mu} - C^{\mu\nu}{}_{\sigma}\right), \tag{5.21}$$

where the first two parts of (5.20) arises from the usual variation of the Einstein-Hilbert action (see e.g. [Car19] or [Bla19]). Contracting (5.21) with $g_{\mu\nu}$ or $g_{\mu}{}^{\sigma}$ we arrive at the two equations

$$C_{\sigma\alpha}{}^{\alpha} + 4C^{\alpha}{}_{\alpha\sigma} - C^{\alpha}{}_{\sigma\alpha} - C^{\alpha}{}_{\alpha\sigma} = C_{\sigma\alpha}{}^{\alpha} + 3C^{\alpha}{}_{\alpha\sigma} - C^{\alpha}{}_{\sigma\alpha} = 0\,, \tag{5.22}$$

$$4C^{\nu}{}_{\alpha}{}^{\alpha} + C^{\alpha}{}_{\alpha}{}^{\nu} - C^{\nu}{}_{\alpha}{}^{\alpha} - C^{\alpha\nu}{}_{\alpha} = 3C^{\nu}{}_{\alpha}{}^{\alpha} + C^{\alpha}{}_{\alpha}{}^{\nu} - C^{\alpha\nu}{}_{\alpha} = 0\,. \tag{5.23}$$

Properly rearranging the indices and taking the difference between (5.22) and (5.23) one can show

$$C_{\sigma\alpha}{}^{\alpha} = C^{\alpha}{}_{\alpha\sigma}\,, \tag{5.24}$$

which brings (5.21) into the form

$$0 = g_{\mu\nu}C_{\sigma\alpha}{}^{\alpha} + g_{\mu\sigma}C_{\nu\alpha}{}^{\alpha} - C_{\nu\sigma\mu} - C_{\mu\nu\sigma}\,. \tag{5.25}$$

Adding and subtracting permutations of (5.25) in the following way gives

$$(5.25)_{\mu\nu\sigma} - (5.25)_{\sigma\mu\nu} + (5.25)_{\nu\sigma\mu} \Leftrightarrow C_{\nu\sigma\mu} = g_{\mu\nu}C_{\sigma\alpha}{}^{\alpha}\,, \tag{5.26}$$

and yields for the connection $\widetilde{\Gamma}^{\alpha}{}_{\beta\gamma}$

$$\widetilde{\Gamma}^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} + \delta^{\alpha}_{\gamma}A_{\beta}\,, \tag{5.27}$$

with an arbitrary covector $A_\beta := C_{\sigma\beta}{}^{\alpha}$. Investigating metric compatibility (5.28) and the torsion tensor $T^{\alpha}{}_{\beta\gamma}$ for the general covariant derivative $\widetilde{\nabla}_\alpha$

$$\widetilde{\nabla}_\alpha g_{\beta\gamma} = \underbrace{\nabla_\alpha g_{\beta\gamma}}_{=0} - C^{\rho}{}_{\alpha\beta}g_{\rho\gamma} - C^{\rho}{}_{\alpha\gamma}g_{\beta\rho} = -2g_{\beta\gamma}A_\alpha\,, \tag{5.28}$$

$$T^{\alpha}{}_{\beta\gamma} = \widetilde{\Gamma}^{\alpha}{}_{\beta\gamma} - \widetilde{\Gamma}^{\alpha}{}_{\gamma\beta} = \delta^{\alpha}_{\gamma}A_\beta - \delta^{\alpha}_{\beta}A_\gamma \quad \Rightarrow \quad T^{\alpha}{}_{\beta\alpha} = 3A_\beta\,, \tag{5.29}$$

shows that demanding either metric compatibility (5.28) or the vanishing of the trace of the torsion tensor in (5.29) results in $A_\beta = 0$. Consequently, we obtain the Levi-Civita connection from the connection equation of motion.

This contradicts the *Palatini principle*, which states that either metric compatibility or vanishing of the entire torsion tensor must be required in advance to yield the Levi-Civita connection [Pal19], [Bla19]. In principle it should be sufficient to demand tracelessness for $T^{\alpha}{}_{\beta\gamma}$. But we can even drop all a priori assumptions [DP12], since the metric equation of motion reduces to the Einstein equations

$$0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,, \tag{5.30}$$

if the general solution for the connection (5.27) is being used. This is because the second part of (5.20) yields

$$C^{\alpha}{}_{\alpha\rho} C^{\rho}{}_{\beta}{}^{\beta} - C^{\alpha}{}_{\beta\rho} C^{\rho}{}_{\alpha}{}^{\beta} = A^{\alpha} A_{\alpha} - A^{\beta} A_{\beta} = 0 \,. \tag{5.31}$$

This way, we get a valid theory for gravity that is equal to Einstein's theory, if the connection is not further coupled to matter degrees of freedom etc.[1]

5.2. Vielbein Palatini Formulation

To also apply the idea from Palatini in the vielbein frame work, we have to us an equivalent action for gravity expressed by vielbeins only. We start again with the Einstein-Hilbert action S_{EH}

$$S_{EH}[g] = \int_{\mathcal{M}} \mathrm{d}^4x \, \sqrt{-\det g} \; g^{\alpha\beta} R_{\sigma\alpha}{}^{\sigma}{}_{\beta} \,. \tag{5.32}$$

We begin rewriting the action with the determinant of the metric $\det g$, which can be computed using a general formula for determinants

$$\begin{aligned} \epsilon_{j_1 \dots j_n} \det(A) &= \epsilon^{i_1 \dots i_n} \, A_{i_1 j_1} \dots A_{i_n j_n} \\ \Leftrightarrow \det(A) &= \frac{\epsilon^{i_1 \dots i_n} \, \epsilon^{j_1 \dots j_n}}{n!} \, A_{i_1 j_1} \dots A_{i_n j_n} \,, \end{aligned} \tag{5.33}$$

with ϵ being the *Levi-Civita symbol*. One has to distinguish here between the *Levi-Civita symbol* ϵ and the *Levi-Civita tensor* ε, where the later has to be used for contracting indices, in order to preserve the tensorial character of contracted quantities

$$\begin{aligned} \varepsilon^{\mu_1 \dots \mu_{\dim \mathcal{M}}} &= \frac{1}{\sqrt{-\det g}} \, \epsilon^{\mu_1 \dots \mu_{\dim \mathcal{M}}} \\ \varepsilon_{\mu_1 \dots \mu_{\dim \mathcal{M}}} &= \sqrt{-\det g} \, \epsilon_{\mu_1 \dots \mu_{\dim \mathcal{M}}} \,. \end{aligned} \tag{5.34}$$

Then, the determinant of the metric can be related to the determinant of the vielbein by using the defining equation (2.19)

$$\begin{aligned} \det g &= \underbrace{\det \eta}_{=-1} (\det e)^2 \\ &= -(\det e)^2 \end{aligned} \tag{5.35}$$

Using the relation between the curvature tensors in the metric and vielbein formalism (2.52) yields for the Einstein Hilbert action (5.32)

$$S_{VEH}[e] = \int_{\mathcal{M}} \mathrm{d}^4x \; e e^{\alpha}{}_{I} e^{\beta J} F_{\alpha\beta}{}^{I}{}_{J} \,, \tag{5.36}$$

[1] See [Bla19], [Pel94] and [ART89] for a further discussion.

where omitting the tilde means that the Levi-Civita connection is being used.

Now we can apply the idea from Palatini to the vielbein formulation of gravity. Again, we state the formulas for curvature and vielbein connection

$$F_{\alpha\beta}{}^{I}{}_{J} = \partial_\alpha \omega_\beta{}^{I}{}_{J} - \partial_\beta \omega_\alpha{}^{I}{}_{J} + \omega_\alpha{}^{I}{}_{K}\omega_\beta{}^{K}{}_{J} - \omega_\beta{}^{I}{}_{K}\omega_\alpha{}^{K}{}_{J}\,, \tag{5.37}$$

$$\omega_\alpha{}^{I}{}_{J} = \frac{1}{2} e_\alpha{}^{K}\left(e^\rho{}_K e^{\sigma I}\partial_{[\rho}e_{\sigma]J} + e^\rho{}_J e^\sigma{}_K \partial_{[\rho}e_{\sigma]}{}^{I} - e^{\rho I} e^\sigma{}_J \partial_{[\rho}e_{\sigma]k}\right)\,. \tag{5.38}$$

The action (5.36) is an equivalent theory for gravity and yields the Einstein equations after variation w.r.t. $\delta e_\alpha{}^I$. Using an arbitrary connection $\widetilde{\omega}_\alpha{}^I{}_J$, one obtains a general curvature $\widetilde{F}_{\alpha\beta}{}^I{}_J$ analogous to (5.37) in the form of

$$\widetilde{F}_{\alpha\beta}{}^{I}{}_{J} = \partial_\alpha \widetilde{\omega}_\beta{}^{I}{}_{J} - \partial_\beta \widetilde{\omega}_\alpha{}^{I}{}_{J} + \widetilde{\omega}_\alpha{}^{I}{}_{K}\widetilde{\omega}_\beta{}^{K}{}_{J} - \widetilde{\omega}_\beta{}^{I}{}_{K}\widetilde{\omega}_\alpha{}^{K}{}_{J}\,. \tag{5.39}$$

With the decomposition

$$\widetilde{\omega}_\alpha{}^I{}_J = \omega_\alpha{}^I{}_J + D_\alpha{}^I{}_J\,, \tag{5.40}$$

where $D_\alpha{}^I{}_J$ is an a priori arbitrary mixed tensor, we can rewrite the general curvature $\widetilde{F}_{\alpha\beta}{}^I{}_J$ into

$$\widetilde{F}_{\alpha\beta}{}^I{}_J = F_{\alpha\beta}{}^I{}_J + \nabla_{[\alpha}D_{\beta]}{}^I{}_J + D_{[\alpha}{}^I{}_{|K|}D_{\beta]}{}^K{}_J\,. \tag{5.41}$$

Here we introduced ∇_α as the covariant derivative utalizing the Levi-Civita connection with an symmetric extension to spacetime indices. Remember that any covariant derivative annihilates the vielbein (*vielbein postulate* (2.41)). The extension can be done because in (5.41) only the anti symmetric part regarding the indices α and β is considered and hence any contribution from a symmetric event connection drops out of equation (5.41). Furthermore the extension is done explicitly with the Levi-Civita event connection (5.2) to use again its properties stated in (5.17) and (5.18).

Here we want to comment on the literature being used as references. The extension to spacetime indices is a crucial point and not mentioned in [Pel94]. Thus it is not clear how to arrive at equation (2.12) of [Pel94], unless the extension is used and $\widetilde{\nabla}_\alpha e_\beta{}^I = 0$ [2] is agreed on in general. From our point of view this is of course true (see (2.41)). This in turn creates problems in the equations of motion for the connection (2.14) of [Pel94], as it is trivially satisfied and hence does not allow to determine the connection.

A different subtlety can be found in [Rom93]. There the convention uses a connection equal to the connection in this thesis apart from a minus sign. When computing the equations of motion for the connection, the identity

$$\delta\widetilde{F}_{\alpha\beta}{}^{IJ} = \widetilde{\nabla}_{[\alpha}\delta\widetilde{\omega}_{\beta]}{}^{IJ}\,,$$ [3]

is used. But this is only true if written as

$$\delta\widetilde{F}_{\alpha\beta}{}^{I}{}_{J} = \widetilde{\nabla}_{[\alpha}\delta\widetilde{\omega}_{\beta]}{}^{I}{}_{J}\,,$$

which can easily be checked. Raising the index J to get the first equation is not allowed as the derivative $\widetilde{\nabla}_\alpha$ does not annihilate the Minkowski metric η_{IJ}. Also the same problems with the covariant derivative acting on the vielbein mentioned above are contained in (6.10) of [Rom93]

[2] Here we use our convention for the covariant derivatives. In the convention of [Pel94] this equation has to be written as $\mathcal{D}_\alpha e_\beta{}^I = 0$.

[3] In the convention of [Rom93] this equation reads $\delta\,{}^4F_{ab}{}^{IJ} = {}^4\mathcal{D}_{[a}\delta\,{}^4A_{b]}{}^{IJ}$.

etc. We suggest to follow [DP12] and use the decomposition (5.40) as this prevents one from running into these subtleties.
Now following again the same line of reasoning as in section 5.1, we obtain a new action depending on the vielbein and the deviation from the Levi-Civita vielbein connection $D_\alpha{}^I{}_J$

$$S_{VEP}[e,D] = \int_{\mathcal{M}} \mathrm{d}^4x\; e e^\alpha{}_I e^{\beta J} \widetilde{F}_{\alpha\beta}{}^I{}_J\,, \tag{5.42}$$

which can also be written as

$$S_{VEP}[e,D] = S_{VEH}[e] + \int_{\mathcal{M}} \mathrm{d}^4x\; e \underbrace{\nabla_{[\alpha}\left(D_{\beta]}{}^I{}_J e^\alpha{}_I e^{\beta J}\right)}_{\to\text{ surface term}} + e e^\alpha{}_I e^{\beta J} D_{[\alpha}{}^I{}_{|K|} D_{\beta]}{}^K{}_J\,. \tag{5.43}$$

Neglecting the surface term, the equations of motion for this action are

$$0 \stackrel{!}{=} \frac{\delta S_{VEP}}{\delta e_\mu{}^L} = \frac{\delta S_{VEH}}{\delta e_\mu{}^L} + \frac{\delta\left(e e^\alpha{}_I e^{\beta J}\right)}{\delta e_\mu{}^L} D_{[\alpha}{}^I{}_K D_{\beta]}{}^K{}_J\,, \tag{5.44}$$

$$0 \stackrel{!}{=} \frac{\delta S_{VEP}}{\delta D_\mu{}^L{}_M} = e\left(D_\beta{}^M{}_J e^\mu{}_L e^{\beta J} + D_\alpha{}^I{}_L e^\alpha{}_I e^{\mu M} - D_\alpha{}^M{}_J e^\alpha{}_L e^{\mu J} - D_\beta{}^I{}_L e^\mu{}_I e^{\beta M}\right)\,. \tag{5.45}$$

The first variation in (5.44) is equivalent to the Einstein equations in vielbein formalism (see e.g.[Pel94]) and gives

$$\frac{\delta S_{VEH}}{\delta e_\mu{}^L} = -2e\, e^\mu{}_I e^\alpha{}_L e^{\mu J} F_{\alpha\beta}{}^I{}_J + e\, e^\mu{}_L e^\alpha{}_I e^{\beta J} F_{\alpha\beta}{}^I{}_J\,. \tag{5.46}$$

Equation (5.45) can be contracted with $e_\sigma{}^L e^\nu{}_M$ which yields, with the tensor $E^\beta{}_{\alpha\gamma}$ introduced for reasons of convenience and defined by

$$E^\beta{}_{\alpha\gamma} \coloneqq D_\alpha{}^I{}_J e^\beta{}_I e_\gamma{}^J\,, \tag{5.47}$$

$$0 = e\left(\delta^\mu_\sigma E^\nu{}_\beta{}^\beta + E^\alpha{}_{\alpha\sigma} g^{\mu\nu} - E^\nu{}_\sigma{}^\mu - E^{\mu\nu}{}_\sigma\right)\,, \tag{5.48}$$

where the first two indices of $E^\beta{}_{\alpha\gamma}$ have been swapped to achieve the same canonical form for (5.48) as in (5.21). This again implies the same solution for $E^\beta{}_{\alpha\gamma}$ as for $C^\alpha{}_{\beta\gamma}$ in (5.27)

$$E^\beta{}_{\alpha\gamma} = \delta^\beta_\gamma A_\alpha\,, \tag{5.49}$$

$$\Leftrightarrow D_\alpha{}^I{}_J = \delta^I_J A_\alpha\,, \tag{5.50}$$

with an arbitrary covector A_α. As before $A_\alpha = 0$ corresponds to metric compatibility as well as the vanishing of the trace of the torsion tensor $T^I{}_{\alpha\beta}$.

The torsion tensor $T^I{}_{\alpha\beta}$ and metric compatibility (5.51) in vielbein terms are given by

$$\widetilde{\nabla}_\alpha g_{\beta\gamma} = e_\beta{}^I e_\gamma{}^J \widetilde{\nabla}_\alpha \eta_{IJ} = -e_\beta{}^I e_\gamma{}^J \Big(\underbrace{\omega_{\alpha JI} + \omega_{\alpha IJ}}_{=0} + D_{\alpha JI} + D_{\alpha IJ} \Big) = -2 g_{\beta\gamma} A_\alpha \,, \tag{5.51}$$

$$T^I{}_{\alpha\beta} = \Big(\underbrace{\partial_{[\alpha} e^I_{\beta]} + \omega_{[\alpha}{}^{IJ} e_{\beta]J}}_{=0} + D_{[\alpha}{}^{IJ} e_{\beta]J} \Big) = A_\alpha e_\beta{}^I - A_\beta e_\alpha{}^I \quad \Rightarrow \quad T^I{}_{\alpha\beta} e^\beta{}_I = 3 A_\alpha \,, \tag{5.52}$$

which proves the previous statement. Similar to the discussion in 5.1, the general solution (5.50) will suffice to give the Einstein equations as

$$D_{[\alpha}{}^I{}_{|K|} D_{\beta]}{}^K{}_J = \delta^I_J A_{[\alpha} A_{\beta]} = 0 \,. \tag{5.53}$$

This reduces (5.44) to (5.46). With the relation between $F_{\alpha\beta}{}^I{}_J$ and the usual Riemann tensor (5.3)

$$F_{\alpha\beta}{}^I{}_J = R_{\alpha\beta}{}^\gamma{}_\delta e_\gamma{}^I e^\delta{}_J \,, \tag{5.54}$$

the remaining equations of motion equal the Einstein equations in (5.30).

6. Derivation of a Spinbase Invariant Form of the Einstein-Hilbert Action

As mentioned at the end of chapter 4, the vielbein formalism is included in the spinbase formalism if there is no torsion present (see (4.48)). Since we could construct a theory for gravity by means of the vielbeins only in section 5.2, it would be logical to ask weather there is a different theory for gravity using the spinbase formalism. The vielbein would be replaced by the Dirac matrices which are connected to the metric by virtue of the Clifford algebra for a curved space time (4.1). Thus we get an equation of motion for the Dirac matrices which will yield the metric for the underlying spacetime. The equation of motion can further be simplified to yield the Einstein equations.

6.1. Derivation

We try to reformulate the classical Einstein-Hilbert action in the vielbein formalism (5.36)

$$S_{VEH,\Lambda}[e] = \int_{\mathcal{M}} \mathrm{d}^4x \, \left[e e^{\mu}{}_{I} e^{\nu J} F_{\mu\nu}{}^{I}{}_{J} - 2e\Lambda \right] \tag{6.1}$$

now including the cosmological constant Λ[1], into a manifestly spinbase invariant form. We take this approach since we should be able to come back to this action once we have found a spinbase invariant action. Rearranging the action gives

$$S_{VEH,\Lambda}[e] = \int_{\mathcal{M}} \mathrm{d}^4x \, \left[\frac{e}{2} \left(e^{[\mu}{}_{I} e^{\nu]J} + e^{(\mu}{}_{I} e^{\nu)J} \right) F_{\mu\nu}{}^{I}{}_{J} - 2e\Lambda \right] , \tag{6.2}$$

where we can make use of the antisymmetry in the first two indices of $F_{\mu\nu}{}^{I}{}_{J}$ to cancel the symmetric part $e^{(\mu}{}_{I} e^{\nu)}{}_{J}$. In a next step we use the two identities

$$e = \frac{1}{4!} \epsilon^{\mu\nu\rho\lambda} \epsilon_{IJKL} e_{\mu}{}^{I} e_{\nu}{}^{J} e_{\rho}{}^{K} e_{\lambda}{}^{L} , \tag{6.3}$$

$$e \, e^{[\mu}{}_{I} e^{\nu]}{}_{J} = \frac{1}{2} \epsilon^{\mu\nu\rho\lambda} \epsilon_{IJKL} e_{\rho}{}^{K} e_{\lambda}{}^{L} . \tag{6.4}$$

A proof for (6.4) can be found in appendix C.3 and (6.3) follows from the general formula for determinants (5.33).

[1] Since we do not couple the theory to matter, it is not necessary to include the appropriate constant factor $\frac{1}{2\kappa} = \frac{c^4}{16\pi G}$. In the presence of matter we would need to write $S = \frac{c^4}{16\pi G} S_{VEH,\Lambda} + S_{\mathrm{matter}}$.

To eliminate the vielbein in (6.3) and (6.4) we use the identity (6.5) for the Levi-Civita symbol ϵ_{IJKL} and the definition for the curved Dirac matrices in vielbein formulation (3.2)

$$\epsilon_{IJKL} = -\frac{i}{4}\mathrm{Tr}[\gamma_*\gamma_I\gamma_J\gamma_K\gamma_L]\,, \tag{6.5}$$

$$\gamma_\mu = e_\mu{}^I\gamma_I\,. \tag{6.6}$$

Merging the results yields

$$e = -\frac{i}{96}\epsilon^{\mu\nu\rho\lambda}\mathrm{Tr}[\gamma_*\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda]\,, \tag{6.7}$$

$$e\, e^{[\mu}{}_I e^{\nu]}{}_J = -\frac{i}{8}\epsilon^{\mu\nu\rho\lambda}\mathrm{Tr}[\gamma_*\gamma_I\gamma_J\gamma_\rho\gamma_\lambda]\,. \tag{6.8}$$

Hence the action can be written as

$$S_{VEH,\Lambda}[e] = \int\limits_{\mathcal{M}} \mathrm{d}^4x \left[-i\epsilon^{\mu\nu\rho\lambda}\left(\frac{F_{\mu\nu}{}^{IJ}}{16}\mathrm{Tr}[\gamma_*\gamma_I\gamma_J\gamma_\rho\gamma_\lambda] - \frac{\Lambda}{48}\mathrm{Tr}[\gamma_*\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda]\right)\right]\,. \tag{6.9}$$

Here we emphasize the Levi-Civita symbol $\epsilon^{\mu\nu\rho\lambda}$ contained in the action. Expanding with $\sqrt{-\det g}$ would reinstall the usual integration measure and the Levi-Civita tensor $\varepsilon^{\mu\nu\rho\lambda}$ in front. In a last step we have to replace the curvature $F_{\mu\nu}{}^{IJ}$ with the relation to the spin curvature $\Phi_{\mu\nu}$ (4.32)

$$\Phi_{\mu\nu} = \frac{1}{4}F_{\mu\nu}{}^{IJ}[\gamma_I,\gamma_J]\,. \tag{6.10}$$

Pulling $F_{\mu\nu}{}^{IJ}$ inside the trace and using the curvature relation yields for the action

$$\begin{aligned} S_{SEH,\Lambda}[\gamma] &= \int\limits_{\mathcal{M}} \mathrm{d}^4x \left[\frac{-i\epsilon^{\mu\nu\rho\lambda}}{4}\left(\mathrm{Tr}[\gamma_*\Phi_{\mu\nu}\gamma_\rho\gamma_\lambda] - \frac{\Lambda}{12}\mathrm{Tr}[\gamma_*\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda]\right)\right] \\ &= \int\limits_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-\det g}\left[\frac{-i\varepsilon^{\mu\nu\rho\lambda}}{4}\left(\mathrm{Tr}\left[\gamma_*\Phi_{\mu\nu}\gamma_\rho\gamma_\lambda\right] - \frac{\Lambda}{12}\mathrm{Tr}[\gamma_*\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda]\right)\right]\,. \end{aligned} \tag{6.11}$$

This action is now spinbase invariant. Interestingly it also does not make use of any inverse Dirac matrices and is further fully contracted by using the Levi-Civita symbol or tensor if expanded with $\sqrt{-\det g}$. The determinant $\det g$ in γ_* for example can be computed with

$$g_{\mu\nu} = \frac{1}{4}\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\right)\,, \tag{6.12}$$

$$\det g = \frac{1}{4!}\epsilon^{\alpha\beta\delta\psi}\epsilon^{\mu\nu\rho\lambda}g_{\alpha\mu}\,g_{\beta\nu}\,g_{\delta\rho}\,g_{\psi\lambda}\,. \tag{6.13}$$

Thus the action is fully determined by the Dirac matrices obeying the Clifford algebra (4.1).

At this point we found no way of rewriting the action in pure metric formulation similar to the form of (6.11). But we can start with the analog of the action depending only on the metric

$$S_{EH,\Lambda}[g] = \int\limits_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-\det g}\,\left[g^{\mu\rho}g^{\nu\sigma}R_{\mu\nu\rho\sigma} - 2\Lambda\right]\,, \tag{6.14}$$

and use a rewritten form of relation (4.31)

$$R_{\mu\nu\rho\sigma} = -\frac{1}{8}\operatorname{Tr}\left(\Phi_{\mu\nu}\left[\gamma_\rho,\gamma_\sigma\right]\right) . \tag{6.15}$$

Hence we get for

$$g^{\mu\rho}g^{\nu\sigma}R_{\mu\nu\rho\sigma} = -\frac{1}{8}\operatorname{Tr}\left(\Phi_{\mu\nu}\left[\gamma^\mu,\gamma^\nu\right]\right) = \frac{1}{4}\operatorname{Tr}\left(\gamma^\nu\Phi_{\nu\mu}\gamma^\mu\right) , \tag{6.16}$$

where we have used the antisymmetry of $\Phi_{\mu\nu} = -\Phi_{\nu\mu}$. This leads to the action

$$\widetilde{S}_{SEH,\Lambda}[\gamma] = \int_{\mathcal{M}} \mathrm{d}^4x\, \sqrt{-\det g}\, \left[\frac{1}{4}\operatorname{Tr}\left(\gamma^\nu\Phi_{\nu\mu}\gamma^\mu\right) - 2\Lambda\right] , \tag{6.17}$$

where $\det g$ can be computed with (6.12) and (6.13). To get a similar form as in (6.11) one would need a similar identity as in (6.4) which is, to our knowledge, not possible using the metric instead of vielbeins. So this would again be a spinbase invariant action, but which contains inverse Dirac matrices. The equation of motion for this action yields the Einstein equations and thus validates the theory. A computation of the equation of motion for (6.17) can be found in [Lip12].

6.2. Equations of Motion

Since we found a new action (6.11) for gravity, we should check whether or not we get the Einstein equations if we require the action to be stationary under variations $\delta\gamma_\psi$. To compute the variation w.r.t. γ_ψ we need the variations from appendix C.4

$$\delta \det g = (-\det g)\operatorname{Tr}\left(\mathfrak{K}^\psi(\delta\gamma_\psi)\right) , \quad \mathfrak{K}^\psi := \frac{1}{4\cdot 3}\varepsilon^{\psi\beta\sigma\tau}\varepsilon^{\mu\nu\rho\lambda}g_{\beta\nu}g_{\sigma\rho}g_{\tau\lambda}\,\gamma_\mu \tag{6.18}$$

$$\begin{aligned}\delta\gamma_* = \frac{1}{2}\operatorname{Tr}\left[\mathfrak{K}^\psi(\delta\gamma_\psi)\right]\gamma_* - \frac{i}{4!}\varepsilon^{\alpha\beta\sigma\tau}\Big(&-(\delta\gamma_\tau)\gamma_{\beta\sigma\alpha} - \gamma_\alpha(\delta\gamma_\tau)\gamma_{\sigma\beta}\\ &-\gamma_{\alpha\beta}(\delta\gamma_\tau)\gamma_\sigma + \gamma_{\alpha\beta\sigma}(\delta\gamma_\tau)\Big) ,\end{aligned} \tag{6.19}$$

$$\frac{1}{4}\varepsilon^{\mu\nu\rho\lambda}\delta\left(R_{\mu\nu}{}^{\delta\eta}\right)\operatorname{Tr}\left[\gamma_{*\delta\eta\rho\lambda}\right] = -2iR^{\mu\psi}(\gamma_\mu)^j{}_i(\delta\gamma_\psi)^i{}_j . \tag{6.20}$$

With these identities the variation of the action (6.11) yields (see appendix C.4)

$$\delta S_{SEH,\Lambda}[\gamma] = -\frac{i}{4}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-\det g}\left[\mathfrak{A}^\psi - 2iR^{\mu\psi}\gamma_\mu + \mathfrak{B}^\psi\right]^j{}_i(\delta\gamma_\psi)^i{}_j , \tag{6.21}$$

$$\mathfrak{A}^\psi = \frac{1}{4}\varepsilon^{\mu\nu\rho\lambda}R_{\mu\nu}{}^{\delta\eta}\left[\delta^\psi_\delta\left\{\gamma_\eta,\gamma_{\rho\lambda}\right\}\gamma_* + \delta^\psi_\lambda\gamma_*\left\{\gamma_\rho,\gamma_{\delta\eta}\right\} + 2i\varepsilon_{\delta\eta\rho\lambda}\mathfrak{K}^\psi + \mathfrak{T}^\psi{}_{\delta\eta\rho\lambda}\right] , \tag{6.22}$$

$$\mathfrak{B}^\psi = -\frac{\Lambda}{12}\varepsilon^{\mu\nu\rho\lambda}\left[4\delta^\psi_\mu\gamma_{\nu\rho\lambda*} + 2i\varepsilon_{\mu\nu\rho\lambda}\mathfrak{K}^\psi + \mathfrak{T}^\psi{}_{\mu\nu\rho\lambda}\right] , \tag{6.23}$$

$$\mathfrak{T}^\psi{}_{\mu\nu\rho\lambda} = -\frac{i}{4!}\varepsilon^{\alpha\beta\sigma\psi}\left(-\gamma_{\beta\sigma\alpha\mu\nu\rho\lambda} - \gamma_{\sigma\beta\mu\nu\rho\lambda\alpha} - \gamma_{\sigma\mu\nu\rho\lambda\alpha\beta} + \gamma_{\mu\nu\rho\lambda\alpha\beta\sigma}\right) . \tag{6.24}$$

Here we used the abbreviation for products of Dirac matrices

$$\gamma_{\alpha_1}\cdots\gamma_{\alpha_n} =: \gamma_{\alpha_1\ldots\alpha_n} .$$

The equation of motion for γ_ψ can be read of

$$0 = \mathfrak{A}^\psi - 2iR^{\mu\psi}\gamma_\mu + \mathfrak{C}^\psi \,. \tag{6.25}$$

Multiplying (6.25) by $\frac{1}{8i}\gamma_\chi$ and taking the trace gives the vacuum Einstein equations

$$R^\psi{}_\chi - \frac{1}{2}g^\psi{}_\chi R + \Lambda g^\psi{}_\chi = 0 \,.$$

This is most conveniently checked by using MATHEMATICA and FEYNCALC. Thus we obtained a valid theory for gravity which yields the Einstein equations after variation.

7. Palatini Principle in the Spinbase Framework

With the action derived in chapter 6 we found a manifestly spinbase invariant action which only depends on the Dirac matrices γ_α. The spin curvature contained in the action makes explicit usage of the Levi-Civita spin connection (4.21), (4.23) - (4.26) fully determined by the Dirac matrices and its partial derivatives. In the spirit of chapter 5, we want to investigate arbitrary spin connections $\widetilde{\Gamma}^{\alpha}{}_{\beta\gamma}$ by using the idea of Palatini.

Starting from the action (6.11)

$$S_{SEH,\Lambda}[\gamma] = \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-\det g} \left[\frac{-i\varepsilon^{\mu\nu\rho\lambda}}{4} \left(\mathrm{Tr}[\gamma_* \Phi_{\mu\nu}\gamma_\rho\gamma_\lambda] - \frac{\Lambda}{12} \mathrm{Tr}[\gamma_*\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda] \right) \right] \tag{7.1}$$

we can apply again Palatini's idea and promote the spin connection Γ_μ to an independent degree of freedom by allowing arbitary deviations (spin torsion see (4.27)) $\Delta\Gamma_\mu$ from the Levi-Civita spin connection Γ_μ

$$\widetilde{\Gamma}_\mu = \Gamma_\mu + \Delta\Gamma_\mu \,. \tag{7.2}$$

Hence the new connection induces a new general spin curvature $\widetilde{\Phi}_{\mu\nu}$ given analogously to (4.30)

$$\widetilde{\Phi}_{\mu\nu} = \partial_\mu \widetilde{\Gamma}_\nu - \partial_\nu \widetilde{\Gamma}_\mu + [\widetilde{\Gamma}_\mu, \widetilde{\Gamma}_\nu] \,, \tag{7.3}$$

which also implies a new action

$$S_{SEP,\Lambda}[\gamma, \widetilde{\Gamma}] = \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-\det g} \left[\frac{-i\varepsilon^{\mu\nu\rho\lambda}}{4} \left(\mathrm{Tr}[\gamma_* \widetilde{\Phi}_{\mu\nu}\gamma_\rho\gamma_\lambda] - \frac{\Lambda}{12} \mathrm{Tr}[\gamma_*\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda] \right) \right] . \tag{7.4}$$

Analogously to the approaches in section 5 we can rewrite the spin curvature $\widetilde{\Phi}_{\mu\nu}$

$$\widetilde{\Phi}_{\mu\nu} = \Phi_{\mu\nu} + \mathcal{D}_{[\mu}\Delta\Gamma_{\nu]} + \left[\Delta\Gamma_\mu, \Delta\Gamma_\nu\right] \,, \tag{7.5}$$

using the Levi-Civita covariant derivative $\mathcal{D}_\mu$ which annihilates the Dirac matrices and which can be used to arrange for surface terms in an action. Inserting the rewritten form (7.5) of $\widetilde{\Phi}_{\mu\nu}$ into the action S_{SEP} yields

$$\begin{aligned} S_{SEP,\Lambda}[\gamma, \Delta\Gamma] = S_{SEH,\Lambda} - \frac{i}{4} \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-\det g} \Bigg[& \underbrace{\mathcal{D}_\mu \mathrm{Tr}\left(\varepsilon^{\mu\nu\rho\lambda}\gamma_*\Delta\Gamma_\nu\gamma_\rho\gamma_\lambda\right)}_{\to \text{ surface term}} \\ & + \mathrm{Tr}\left(\varepsilon^{\mu\nu\rho\lambda}\gamma_* \left[\Delta\Gamma_\mu, \Delta\Gamma_\nu\right]\gamma_\rho\gamma_\lambda\right) \Bigg] . \end{aligned} \tag{7.6}$$

The second part corresponds to a surface term and can be neglected. This is due to the Dirac structure being eliminated by the trace and the derivative $\mathcal{D}_\mu$ then corresponding to ∇_μ.

Thus the equation of motion for $\Delta\Gamma_\mu$ follows from the last term and reads (see C.5)

$$0 \stackrel{!}{=} \frac{\delta S_{\mathrm{SEP}}}{\delta \left(\Delta\Gamma_\mu\right)^j{}_i} = \left(\Delta\Gamma_\nu G^{\mu\nu}\right)^i{}_j - \left(G^{\mu\nu}\Delta\Gamma_\nu\right)^i{}_j \ , \tag{7.7}$$

where $G^{\alpha\mu}$ is defined for convenience as

$$G^{\mu\nu} = -G^{\nu\mu} := \epsilon^{\mu\nu\rho\lambda}\gamma_\rho\gamma_\lambda\gamma_* \ . \tag{7.8}$$

To solve this equation it is useful to span $\Delta\Gamma_\nu$ in the Clifford basis

$$\Delta\Gamma_\nu = p_{(\Delta\Gamma)\nu}\gamma_* + v_{(\Delta\Gamma)\nu}{}^\sigma\gamma_\sigma + a_{(\Delta\Gamma)\nu}{}^\sigma\,\gamma_*\gamma_\sigma + t_{(\Delta\Gamma)\nu}{}^{\sigma\tau}[\gamma_\sigma,\gamma_\tau] \ . \tag{7.9}$$

Here we explicitly lift the constrains (4.43) - (4.46) for the spin torsion, which for example allows for the pseudoscalar part $p_{(\Delta\Gamma)\mu}$, because we did not couple the spin connection to spinors. Projecting out the different parts from (7.7) yields

$$(s) \qquad 0 = 0 \ , \tag{7.10}$$

$$(p) \qquad 0 = 0 \ , \tag{7.11}$$

$$(v) \qquad 0 = v_{(\Delta\Gamma)\sigma}{}^\sigma\delta^\mu_\nu - v_{(\Delta\Gamma)\nu}{}^\mu \ , \tag{7.12}$$

$$(a) \qquad 0 = a_{(\Delta\Gamma)\nu}{}^\mu - a_{(\Delta\Gamma)\sigma}{}^\sigma\delta^\mu_\nu \ , \tag{7.13}$$

$$(t) \qquad 0 = \delta^\rho_\nu t_{(\Delta\Gamma)}{}^{\sigma\mu}{}_\sigma - \delta^\mu_\nu t_{(\Delta\Gamma)}{}^{\sigma\rho}{}_\sigma - t_{(\Delta\Gamma)}{}^{[\rho\mu]}{}_\nu \ . \tag{7.14}$$

This indicates that $p_{(\Delta\Gamma)\alpha}$ is not constrained by the equations of motion. Contracting the indices μ and ν in (7.12), (7.13) and (7.14) yields

$$0 = v_{(\Delta\Gamma)\sigma}{}^\sigma \ , \tag{7.15}$$

$$0 = a_{(\Delta\Gamma)\sigma}{}^\sigma \ , \tag{7.16}$$

$$0 = t_{(\Delta\Gamma)}{}^{\sigma\rho}{}_\sigma - 4t_{(\Delta\Gamma)}{}^{\sigma\rho}{}_\sigma + \underbrace{t_{(\Delta\Gamma)}{}^{\rho\sigma}{}_\sigma}_{=0} - t_{(\Delta\Gamma)}{}^{\sigma\rho}{}_\sigma = -4t_{(\Delta\Gamma)}{}^{\sigma\rho}{}_\sigma \ , \tag{7.17}$$

which then results in

$$0 = v_{(\Delta\Gamma)\nu}{}^\mu \ , \tag{7.18}$$

$$0 = a_{(\Delta\Gamma)\nu}{}^\mu \ , \tag{7.19}$$

$$0 = t_{(\Delta\Gamma)}{}^{[\rho\mu]}{}_\nu \ . \tag{7.20}$$

Equation (7.20) implies that $t_{(\Delta\Gamma)}$ must be symmetric in the first two indices, $t_{(\Delta\Gamma)\rho\mu\nu} = t_{(\Delta\Gamma)\mu\rho\nu}$. As $t_{(\Delta\Gamma)\rho\mu\nu}$ is antisymmetric in the last two indices by definition, this allows us to write

$$t_{(\Delta\Gamma)\rho\mu\nu} = -t_{(\Delta\Gamma)\rho\nu\mu} = -t_{(\Delta\Gamma)\nu\rho\mu} = t_{(\Delta\Gamma)\nu\mu\rho} = t_{(\Delta\Gamma)\mu\nu\rho} = -t_{(\Delta\Gamma)\mu\rho\nu} = -t_{(\Delta\Gamma)\rho\mu\nu} \ . \tag{7.21}$$

Hence we conclude

$$0 = t_{(\Delta\Gamma)\rho\mu\nu} \ . \tag{7.22}$$

This constrains the spin torsion $\Delta\Gamma_\nu$ to

$$\Delta\Gamma_\nu = p_{(\Delta\Gamma)\nu}\gamma_* \;, \tag{7.23}$$

as the pseudo scalar part could not be constrained by the connection equation of motion. The equation of motion for the Dirac matrices will then correspond to (6.25) from $\delta S_{SEH,\Lambda}$ plus extra terms from the last part in (7.6). The additional terms will always include $\left[\Delta\Gamma_\mu, \Delta\Gamma_\nu\right]$. But inserting the solution (7.23) yields

$$\left[\Delta\Gamma_\mu, \Delta\Gamma_\nu\right] = p_{(\Delta\Gamma)\mu} p_{(\Delta\Gamma)\nu}\left[\gamma_*, \gamma_*\right] = 0 \;.$$

Thus very similar to the results in chapter 5 we get an unrestricted vector degree of freedom, which does not contribute to the equation of motion, which yields the Einstein equations.

8. Higher Curvature Terms

In chapter 7 we found that the theory described by the action

$$S_{SEP,\Lambda}[\gamma,\widetilde{\Gamma}] = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-\det g}\left[\frac{-i\varepsilon^{\mu\nu\rho\lambda}}{4}\left(\mathrm{Tr}[\gamma_*\widetilde{\Phi}_{\mu\nu}\gamma_\rho\gamma_\lambda] - \frac{\Lambda}{12}\mathrm{Tr}[\gamma_*\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda]\right)\right] \tag{8.1}$$

has an arbitrary spin connection $\widetilde{\Gamma}_\mu$ as a new degree of freedom. The connection equation of motion (7.7) restricted the spin connection $\widetilde{\Gamma}_\mu$ to

$$\widetilde{\Gamma}_\mu = \Gamma_\mu + p_\mu\gamma_* \; . \tag{8.2}$$

Here Γ_μ is again fully determined by the Dirac matrices and their partial derivatives (4.21), (4.23) - (4.26). Since the covector p_μ remains unaffected by the connection equation of motion and does not alter the equation of motion for the Dirac matrices, it is not necessary to artificially restrict p_μ by additional terms in the action. However, from the perspective of quantum field theory, the natural question arises whether the apperently redundant degrees of freedom in p_μ could develop their own dynamics on a higher curvature level. Thus we investigate terms quadratic in the spin curvature $\widetilde{\Phi}_{\mu\nu}$ which conserve parity and don't introduce any inverse metrics as additional constraints. This means, we neglect terms with an in-total odd number of Levi-Civita tensors and γ_* matrices due to parity and neglect terms with objects carrying upstairs indices except for the Levi-Civita tensor to not introduce inverse metrics. Also, we consider only the covector p_μ as the remaining freedom for the spin torsion and, for reasons of generality, include a scalar vector s_μ degree of freedom which is known to carry an abelian gauge field [GL14]. Thus we write

$$\Delta\Gamma_\mu = s_\mu \mathbb{1}_{4\times 4} + p_\mu\gamma_* = s_\mu + p_\mu\gamma_* \; , \tag{8.3}$$

as the remaining freedom for the spin torsion. This allows us to focus on the unconstrained covector p_μ and keeps the calculations manageable.

To simplify the calculations we first reexamine (7.5)

$$\widetilde{\Phi}_{\mu\nu} = \Phi_{\mu\nu} + \mathcal{D}_{[\mu}\Delta\Gamma_{\nu]} + \left[\Delta\Gamma_\mu, \Delta\Gamma_\nu\right] \; .$$

Inserting (8.3) yields for the spin curvature $\widetilde{\Phi}_{\mu\nu}$

$$\widetilde{\Phi}_{\mu\nu} = \Phi_{\mu\nu} + \mathcal{D}_{[\mu}s_{\nu]} + \mathcal{D}_{[\mu}p_{\nu]}\gamma_* \; ,$$

because the commutator vanishes and the derivative $\mathcal{D}_\mu$ annihilates the Dirac matrices. If we define the field strength tensors

$$S_{\mu\nu} := \partial_{[\mu}s_{\nu]} = \nabla_{[\mu}s_{\nu]} = \mathcal{D}_{[\mu}s_{\nu]} \; , \tag{8.4}$$

$$P_{\mu\nu} := \partial_{[\mu}p_{\nu]} = \nabla_{[\mu}p_{\nu]} = \mathcal{D}_{[\mu}p_{\nu]} \; , \tag{8.5}$$

and use the relation to the Riemann tensor (4.31), we can write for $\widetilde{\Phi}_{\mu\nu}$

$$\widetilde{\Phi}_{\mu\nu} = S_{\mu\nu} + P_{\mu\nu}\gamma_* + \frac{1}{8}R_{\mu\nu}{}^{\rho\lambda}\left[\gamma_\rho, \gamma_\lambda\right] . \tag{8.6}$$

Keeping the restrictions from above in mind, we investigated the following additional terms

$$\varepsilon^{\mu\nu\rho\lambda}\,\mathrm{Tr}\left(\gamma_*\widetilde{\Phi}_{\mu\nu}\widetilde{\Phi}_{\rho\lambda}\right) = 8\varepsilon^{\mu\nu\rho\lambda}S_{\mu\nu}P_{\rho\lambda} + \frac{i}{4}\varepsilon^{\mu\nu\rho\lambda}\varepsilon^{\alpha\beta\sigma\tau}R_{\mu\nu\alpha\beta}R_{\rho\lambda\sigma\tau} , \tag{8.7}$$

$$\varepsilon^{\mu\nu\rho\lambda}\varepsilon^{\alpha\beta\sigma\tau}\,\mathrm{Tr}\left(\widetilde{\Phi}_{\mu\nu}\gamma_\rho\gamma_\lambda\widetilde{\Phi}_{\alpha\beta}\gamma_\sigma\gamma_\tau\right) = 32P_{\mu\nu}P^{\mu\nu} + 32S_{\mu\nu}S^{\mu\nu} + 32i\varepsilon^{\mu\nu\rho\lambda}P_{\mu\nu}S_{\rho\lambda} - 4R^2 , \tag{8.8}$$

$$\varepsilon^{\mu\nu\rho\lambda}\varepsilon^{\alpha\beta\sigma\tau}\,\mathrm{Tr}\left(\widetilde{\Phi}_{\mu\nu}\gamma_\rho\gamma_\sigma\widetilde{\Phi}_{\alpha\beta}\gamma_\lambda\gamma_\tau\right) = -32i\varepsilon^{\mu\nu\rho\lambda}P_{\mu\nu}S_{\rho\lambda} + 8R_{\mu\nu}R^{\mu\nu} - 4R^2 , \tag{8.9}$$

$$\begin{aligned}\varepsilon^{\mu\nu\rho\lambda}\varepsilon^{\alpha\beta\sigma\tau}\,\mathrm{Tr}\left(\widetilde{\Phi}_{\mu\alpha}\gamma_\rho\gamma_\lambda\widetilde{\Phi}_{\nu\beta}\gamma_\sigma\gamma_\tau\right) &= 16P_{\mu\nu}P^{\mu\nu} + 16S_{\mu\nu}S^{\mu\nu} - 32i\varepsilon^{\mu\nu\rho\lambda}P_{\mu\nu}S_{\rho\lambda} \\ &\quad - 4R^2 + 12R_{\mu\nu}R^{\mu\nu} - 2R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} ,\end{aligned} \tag{8.10}$$

$$\varepsilon^{\mu\nu\rho\lambda}\varepsilon^{\alpha\beta\sigma\tau}\,\mathrm{Tr}\left(\widetilde{\Phi}_{\mu\alpha}\gamma_\rho\gamma_\sigma\widetilde{\Phi}_{\nu\beta}\gamma_\lambda\gamma_\tau\right) = 32i\varepsilon^{\mu\nu\rho\lambda}P_{\mu\nu}S_{\rho\lambda} - 2R^{\mu\nu}R_{\mu\nu} - 2R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda} . \tag{8.11}$$

Other terms quadratic in $\widetilde{\Phi}_{\mu\nu}$ can be constructed from these terms. This is analogous to R^2-gravity theory where we have in principal three building blocks $R_{\mu\nu}R^{\mu\nu}$, R^2 and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ to build the Lagrangian, minus the topological invariant Gauss-Bonnet term [BOS92]. The Gauss-Bonnet term is given by

$$\frac{1}{4}\varepsilon^{\mu\nu\rho\lambda}\varepsilon^{\alpha\beta\sigma\tau}R_{\mu\nu\alpha\beta}R_{\rho\lambda\sigma\tau} = R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

and can be shown to be a surface term in $d = 4$ spacetime dimensions. This allows to cancel one of the three terms quadratic in the Riemann tensor. Thus we have two remaining terms quadratic in R, plus three for $P_{\mu\nu}P^{\mu\nu}$, $S_{\mu\nu}S^{\mu\nu}$ and $i\varepsilon^{\mu\nu\rho\lambda}S_{\mu\nu}P_{\rho\lambda}$, so in total five terms. This means that the five stated terms should suffice to write down all possible Lagrangians. Furthermore, we want to restrict the terms (8.7) - (8.11) to be real, in order to appropriately include them into a Lagrangian. Because we have three types of terms containing field strength tensors $P_{\mu\nu}P^{\mu\nu}$, $S_{\mu\nu}S^{\mu\nu}$ and $i\varepsilon^{\mu\nu\rho\lambda}S_{\mu\nu}P_{\rho\lambda}$, we have to choose p_μ real and s_μ imaginary or p_μ imaginary and s_μ real to restrict all three terms to be real simultaneously. Here we want to remind again, that if we couple spinors to the connection, we would have to implement the constraints (4.40) (4.41) and (4.47). This would constrain the spin torsion to

$$\Delta\Gamma_\mu = is_\mu , \tag{8.12}$$

with s_μ being real, in contrast to (8.3).
Looking at (8.7) - (8.11) we see that we find terms to promote p_μ to a dynamical field in the theory, at least to this order in the spin curvature. The way p_μ is made dynamical, strongly reminds us of the kinetic term for gauge fields from quantum field theory. Thus the new ingredient seems to correspond to a new abelian symmetry of the theory, similar to the trace part contained in s_μ. Indeed we can show for the theories on the Einstein-Hilbert and -Palatini level (6.11) and (7.4), that the transformations

$$\Gamma_\mu \;\to\; \Gamma_\mu + p_\mu\gamma_* , \quad \widetilde{\Gamma}_\mu \;\to\; \widetilde{\Gamma}_\mu + p_\mu\gamma_* \tag{8.13}$$

are local symmetries of the Lagrangians. This can easily be seen in the rewritten Lagrangian (7.6), where the additional part $\Delta\Gamma_\mu = p_\mu\gamma_*$ drops out, due to the commutator. Hence, we can allow for $\mathbb{C}^d$-shifts in p_μ

$$p_\mu \;\to\; p_\mu + p'_\mu(x) , \tag{8.14}$$

because these also leave the Lagrangain invariant on this level, due to the same argument as above. For theories quadratic in the spin curvature, we would require shifts in p_μ to leave the field strength tensor $P_{\mu\nu}$ invariant. Thus we would restrict to ($\mathbb{C}$-)shifts with gradients of appropriately differentiable complex functions, $p'_\mu(x) \stackrel{!}{=} \partial_\mu p'(x)$. These would leave the field strength tensor invariant, because we have

$$P'_{\mu\nu} = \partial_{[\mu}\left(p_{\nu]} + \partial_{\nu]} p'(x)\right) = \partial_{[\mu} p_{\nu]} + \underbrace{\partial_{[\mu}\partial_{\nu]} p'(x)}_{=0} = P_{\mu\nu} \, . \tag{8.15}$$

As already indicated, these shifts can be local so that we can understand these shifts as gauge transformations. But the underlying symmetry group has to be different from the spinbase transformations, because the term containing p_μ transforms homogeneously under spinbase transformations, due to the transformation behavior of the spin torsion (4.28). Hence it can't be an element of the Lie algebra $\mathrm{sl}(4,\mathbb{C})$ and must correspond to a different gauge group.
This is almost analogous to the conclusions drawn in [DP12] about the remaining degrees of freedom in the Palatini formalism, discussed in chapter 5. The difference to our result is that we considered arbitrary spin connections by including spin torsion, not torsion in the event connection as in [DP12]. In [DP12], the new gauge group is identified with the R^d *gauge symmetry of projective transformations*, originally discussed in [JS98]. Since we uncovered a similar behavior, but for an inclusion of spin torsion, we don't want to identify the new group with the R^d gauge symmetry or for example the $\mathrm{U}(1)$ group a priori. To specify the origin of this symmetry may be a starting point for a future investigation.

9. Conclusion

In this thesis we have revisited two formalisms for describing spinors in curved spacetimes. The first one being the description using the construction of vielbeins in the tangent spaces of the spacetime manifold. Historically, this corresponds to the first consistent description from Fock and Ivanenko in 1929 [FI29], [Foc29]. The spinbase formalism, proposed secondly, is based on the pioneering publication from Schrödinger in 1932 [Sch32] and completed by Bargman shortly after [Bar32]. It was shown that the vielbein description can be reinstalled within the spinbase formalism as a special choice of spin basis [Wel01], [GL14]. Further in 2015, Gies and Lippoldt depicted a superiority of the spinbase formalism by constructing a global realization of the Clifford algebra on a 2-sphere, which is not possible within the vielbein formalism [GL15]. The publications from Weldon [Wel01] and Gies and Lippoldt [GL14] also demonstrated a relation between the spin curvature and the Riemann tensor. This hence illustrates an underlying connection to general relativity.

Following the suggestion of Gies and Lippoldt to include spinbase invariance as a new feature for a theory for gravity [GL14], we presented such a candidate theory. The derived theory is manifestly invariant under spinbase and coordinate transformations by construction and is only written in terms of coordinate dependent Dirac matrices. By virtue of a generalized Clifford algebra we can reinstall the metric degree of freedom in the theory. We further computed the equations of motion and showed that these yield the Einstein equations on a classical level. As a generalization, we followed the publication from Palatini [Pal19] and Dadhich and Pons [DP12] and allowed for an a priori unspecified spin connection. This new spin connection can be understood as the inclusion of spin torsion alongside the corresponding Levi-Civita spin connection. Also, the concept of spin torsion can be introduced in parallel to the usual torsion in the event connection. We showed that the connection equation of motion constrained the spin torsion to be equivalent to the Levi-Civita spin connection, except for the pseudo scalar part in the Clifford decomposition. This pseudo scalar part remained unconstrained, but also does not influence the dynamics of the Dirac matrices or the rest of the spin connection. Due to this behavior, we conjectured that the pseudo scalar part might develop a kinetic term, if we were to consider terms quadratic in the spin curvature. We showed indeed, that the pseudo scalar will obtain own dynamics in this case. Because of the form of the kinetic term, it seems appropriate to think of the pseudo scalar as a novel abelian gauge symmetry uncovered at this level in the spin curvature. This is based on the fact, that the pseudo scalar transforms homogeneously under spinbase transformations and hence should be associated to a different gauge symmetry group then $\mathrm{SL}(4,\mathbb{C})$.

The striking similarity to the R^d gauge symmetry [JS98] within the Palatini formulation for torsion in the event connection [DP12], might be a good starting point to investigate the origin and nature of this symmetry. Also an approach to quantize the new theory with path integral or functional renormalization methods could be the basis of future considerations. Since the action does not require an a prior inverse metric and is also fully contracted with the Levi-Civita tensor, calculations build on tensor networks or lattice gauge theory techniques may give further guidance in this regard.

A. Clifford Decomposition

The elements of the Clifford algebras in flat (f) or curved (c) spacetime

$$\{\gamma_I, \gamma_J\} = 2\eta_{IJ}\mathbb{1}_{4\times 4} , \quad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}(x)\mathbb{1}_{4\times 4} , \tag{A.1}$$

are capable of spanning any 4×4 complex matrix. The basis elements $\mathfrak{B}$ of this span are

$$\mathfrak{B}_{(f)} = \{\mathbb{1}_{4\times 4} , \gamma_* , \gamma_I , \gamma_*\gamma_I , [\gamma_I, \gamma_J]\} , \quad \mathfrak{B}_{(c)} = \{\mathbb{1}_{4\times 4} , \gamma_* , \gamma_\mu , \gamma_*\gamma_\mu , [\gamma_\mu, \gamma_\nu]\} . \tag{A.2}$$

The following properties hold for both bases $\mathfrak{B}_{(f)}$ or $\mathfrak{B}_{(c)}$ by replacing $g_{\mu\nu}$ with η_{IJ}, so stick to $\mathfrak{B}_{(c)}$ and drop the label (c). We introduce the scalar product $\langle \cdot, \cdot \rangle$

$$\langle A, B\rangle := \frac{1}{4}\mathrm{Tr}(AB) , \quad A,\ B \in \mathfrak{B} , \tag{A.3}$$

for which elements of $\mathfrak{B}$ are orthogonal but not normalized, $\langle A, B\rangle \propto \delta_{A,B}$. With this scalar product we can compute the proportionality factors

$$\langle \mathbb{1}_{4\times 4}, \mathbb{1}_{4\times 4}\rangle = 1 , \tag{A.4}$$

$$\langle \gamma_*, \gamma_*\rangle = 1 , \tag{A.5}$$

$$\langle \gamma_\mu, \gamma_\nu\rangle = g_{\mu\nu} , \tag{A.6}$$

$$\langle \gamma_*\gamma_\mu, \gamma_*\gamma_\nu\rangle = -g_{\mu\nu} , \tag{A.7}$$

$$\langle [\gamma_\mu, \gamma_\nu] , [\gamma_\rho, \gamma_\lambda]\rangle = -4\left(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}\right) . \tag{A.8}$$

Any 4×4 complex matrix M has 16 complex degrees of freedom which can be encoded into the form

$$M = s\,\mathbb{1}_{4\times 4} + p\,\gamma_* + v^\alpha\,\gamma_\alpha + a^\alpha\,\gamma_*\,\gamma_\alpha + t^{\alpha\beta}\,[\gamma_\alpha, \gamma_\beta] , \tag{A.9}$$

since we have 1 complex degree of freedom in s and p, 4 in v^α and a^α and 6 in $t^{\alpha\beta} = -t^{\beta\alpha}$. Further we call s the *scalar part*, p the *pseudo scalar part*, v^α the *vector part*, a^α the *axial vector part* and $t^{\alpha\beta}$ the *tensor part*. The trace of M is related to the scalar part

$$\mathrm{Tr}(M) = 4s . \tag{A.10}$$

We can use the scalar product $\langle\cdot,\cdot\rangle$ also to project whole equations onto the different parts, e.g. for

$$P = M + N\,,$$

we get

$$
\begin{array}{cccc}
(s) & \langle P, \mathbb{1}_{4\times4}\rangle = \langle M+N, \mathbb{1}_{4\times4}\rangle & \Leftrightarrow & s_P = s_M + s_N\,, \\
(p) & \left\langle P, \gamma_*\right\rangle = \left\langle M+N, \gamma_*\right\rangle & \Leftrightarrow & p_P = p_M + p_N\,, \\
(v) & \left\langle P, \gamma_\mu\right\rangle = \left\langle M+N, \gamma_\mu\right\rangle & \Leftrightarrow & v_{P\mu} = v_{M\mu} + v_{N\mu}\,, \\
(a) & \left\langle P, \gamma_*\gamma_\mu\right\rangle = \left\langle M+N, \gamma_*\gamma_\mu\right\rangle & \Leftrightarrow & -a_{P\mu} = -a_{M\mu} - a_{N\mu}\,, \\
(t) & \left\langle P, \left[\gamma_\mu, \gamma_\nu\right]\right\rangle = \left\langle M+N, \left[\gamma_\mu, \gamma_\nu\right]\right\rangle & \Leftrightarrow & -8t_{P\mu\nu} = -8t_{M\mu\nu} - 8t_{N\mu\nu}\,.
\end{array}
$$

This is an effective method used throughout thesis to determine components of matrices from defining equations etc.

B. Clifford Algebra Constraints

B.1. Constraints on the Partial Derivatives $\partial_\mu \gamma_\nu$

The Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \mathbb{1}_{4\times 4} \, , \tag{B.1}$$

can be used to constrain the different parts from the Clifford decomposition of the partial derivatives acting on the Dirac matrices, $\partial_\alpha \gamma_\mu$. Taking the derivative of the Clifford algebra we find a relation that $\partial_\alpha \gamma_\mu$ has to satisfy

$$\{\partial_\alpha \gamma_\mu, \gamma_\nu\} + \{\gamma_\mu, \partial_\alpha \gamma_\nu\} = 2 \partial_\alpha g_{\mu\nu} \mathbb{1}_{4\times 4} \, . \tag{B.2}$$

With the relation

$$\partial_\alpha \left(\delta^\mu_\nu\right) = \partial_\alpha \left(g^{\mu\rho} g_{\rho\nu}\right) = 0 \;\Leftrightarrow\; \partial_\alpha \left(g^{\mu\rho}\right) g_{\rho\nu} = -g^{\mu\rho} \partial_\alpha \left(g_{\rho\nu}\right) \, , \tag{B.3}$$

we can show, as a corollary, that the index placement with regard to μ and ν is not relevant

$$\{\partial_\alpha \gamma_\mu, \gamma_\nu\} + \{\gamma_\mu, \partial_\alpha \gamma_\nu\} = 2 \partial_\alpha g_{\mu\nu} \mathbb{1}_{4\times 4} \tag{B.4}$$

$$\Leftrightarrow \quad \{g^{\mu\rho} \partial_\alpha \left(\gamma_\mu\right), \gamma_\nu\} + \{\gamma^\rho, \partial_\alpha \gamma_\nu\} = 2 \partial_\alpha \left(g_{\mu\nu}\right) g^{\mu\rho} \mathbb{1}_{4\times 4}$$

$$\Leftrightarrow \quad \{\partial_\alpha \left(\gamma^\rho\right) - \partial_\alpha \left(g^{\mu\rho}\right) \gamma_\mu, \gamma_\nu\} + \{\gamma^\rho, \partial_\alpha \gamma_\nu\} = -2 \partial_\alpha \left(g^{\mu\alpha}\right) g_{\mu\nu} \mathbb{1}_{4\times 4}$$

$$\Leftrightarrow \quad \{\partial_\alpha \left(\gamma^\rho\right), \gamma_\nu\} + \{\gamma^\rho, \partial_\alpha \gamma_\nu\} - \partial_\alpha \left(g^{\mu\rho}\right) \{\gamma_\mu, \gamma_\nu\} = -2 \partial_\alpha \left(g^{\mu\alpha}\right) g_{\mu\nu} \mathbb{1}_{4\times 4}$$

$$\Leftrightarrow \quad \{\partial_\alpha \gamma^\rho, \gamma_\nu\} + \{\gamma^\rho, \partial_\alpha \gamma_\nu\} = 0 \tag{B.5}$$

$$\Leftrightarrow \quad \left\{\partial_\alpha \gamma^\rho, \gamma^\lambda\right\} + \left\{\gamma^\rho, g^{\nu\lambda} \partial_\alpha \left(\gamma_\nu\right)\right\} = 0$$

$$\Leftrightarrow \quad \left\{\partial_\alpha \gamma^\rho, \gamma^\lambda\right\} + \left\{\gamma^\rho, \partial_\alpha \left(\gamma^\lambda\right) - \gamma_\nu \partial_\alpha \left(g^{\nu\lambda}\right)\right\} = 0$$

$$\Leftrightarrow \quad \left\{\partial_\alpha \gamma^\rho, \gamma^\lambda\right\} + \left\{\gamma^\rho, \partial_\alpha \left(\gamma^\lambda\right)\right\} = \partial_\alpha g^{\nu\lambda} \{\gamma^\rho, \gamma_\nu\}$$

$$\Leftrightarrow \quad \left\{\partial_\alpha \gamma^\rho, \gamma^\lambda\right\} + \left\{\gamma^\rho, \partial_\alpha \left(\gamma^\lambda\right)\right\} = 2 \partial_\alpha g^{\rho\lambda} \, \mathbb{1}_{4\times 4} \, . \tag{B.6}$$

If we span $\partial_\alpha \gamma_\mu$ by the Clifford basis

$$\partial_\alpha \gamma_\mu = s_{\alpha\mu} + p_{\alpha\mu} \gamma_* + v_{\alpha\mu}{}^\rho \gamma_\rho + a_{\alpha\mu}{}^\rho \gamma_* \gamma_\rho + t_{\alpha\mu}{}^{\rho\lambda} \left[\gamma_\rho, \gamma_\lambda\right] \, , \tag{B.7}$$

and insert this representaton into (B.4), we get for the different Clifford parts of the whole equation

$$(s) \qquad v_{\alpha(\mu\nu)} = \partial_\alpha g_{\mu\nu} \, , \tag{B.8}$$

$$(p) \qquad 0 = 0\,, \tag{B.9}$$

$$(v) \qquad s_{\alpha\mu} = 0\,, \tag{B.10}$$

$$(a) \qquad \varepsilon^{\rho\mu\sigma\lambda} t_{\alpha\mu\rho\lambda} = 0\,, \tag{B.11}$$

$$(t) \qquad a_\alpha{}^{\rho\lambda} \varepsilon_{\rho\lambda\mu\nu} = 0\,. \tag{B.12}$$

Thus we obtain the constraints

$$v_{\alpha(\mu\nu)} = \partial_\alpha g_{\mu\nu}\,, \tag{B.13}$$

$$s_{\alpha\mu} = 0\,, \tag{B.14}$$

$$t_{\alpha[\mu\rho\lambda]} = 0\,, \tag{B.15}$$

$$a_\alpha{}^{[\rho\lambda]} = 0\,. \tag{B.16}$$

We can link $\partial_\alpha \gamma_\mu$ to $\partial_\alpha \gamma^\mu$ simply by considering

$$\partial_\alpha \gamma^\mu = \partial_\alpha \left(\gamma_\nu g^{\mu\nu}\right) = \partial_\alpha \left(\gamma_\nu\right) g^{\mu\nu} + \gamma_\nu \partial_\alpha \left(g^{\mu\nu}\right)\,. \tag{B.17}$$

Multiplying the relation (B.2) with $g^{\nu\lambda}$ yields

$$\partial_\alpha \left(g^{\mu\lambda}\right) = -g^{\mu\rho} g^{\nu\lambda} \partial_\alpha \left(g_{\rho\nu}\right)\,, \tag{B.18}$$

so that we can write for (B.17)

$$\partial_\alpha \gamma^\mu = \partial_\alpha \left(\gamma_\nu\right) g^{\mu\nu} - \gamma_\nu g^{\mu\rho} g^{\nu\lambda} \partial_\alpha \left(g_{\rho\lambda}\right)\,. \tag{B.19}$$

If we use the span (B.7) and the constraints (B.13) and (B.14) we get

$$\partial_\alpha \gamma^\mu = p_\alpha{}^\mu \gamma_* - v_\alpha{}^{\rho\mu} \gamma_\rho + a_\alpha{}^{\mu\rho} \gamma_* \gamma_\rho + t_\alpha{}^{\mu\rho\lambda} \left[\gamma_\rho, \gamma_\lambda\right]\,, \tag{B.20}$$

$$\partial_\alpha \gamma_\mu = p_{\alpha\mu} \gamma_* + v_{\alpha\mu}{}^\rho \gamma_\rho + a_{\alpha\mu}{}^\rho \gamma_* \gamma_\rho + t_{\alpha\mu}{}^{\rho\lambda} \left[\gamma_\rho, \gamma_\lambda\right]\,, \tag{B.21}$$

with the same components for both derivatives of the Dirac matrices. We emphasize that one has to change the sign and the index placement in the vector part, if one wishes to change from one representation to the other.

B.2. Constraints on γ_μ

In appendix A we mentioned that any set of matrices satisfying *a* Clifford algebra is well suited to span any 4×4 complex matrices. Thus we could try to span a curved Dirac matrix $\gamma_\mu(x)$ with the set of flat Dirac matrices γ_I satisfying

$$\left\{\gamma_I, \gamma_J\right\} = 2\eta_{IJ} \mathbb{1}_{4\times 4}\,. \tag{B.22}$$

The vielbein formalism establishes this idea implicitly, because there the curved Dirac matrices are spanned solely by the vector part

$$\gamma_\mu = e_\mu{}^I \gamma_I\,.$$

As the curved Dirac matrices have to fulfill a generalized Clifford algebra, we can derive constraints, which have to be satisfied by the parts of the Clifford decomposition.

We start with the ansatz

$$\gamma_\mu := s_\mu \mathbb{1}_{4\times 4} + p_\mu \gamma_* + v_\mu{}^I \gamma_I + a_\mu{}^I \gamma_* \gamma_I + t_\mu{}^{IJ} [\gamma_I, \gamma_J] \; . \tag{B.23}$$

Inserting (B.23) into the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \mathbb{1}_{4\times 4} \; , \tag{B.24}$$

we can use the projection onto the Clifford basis, which yield

$$(s) \qquad g_{\mu\nu} = s_\mu s_\nu + p_\mu p_\nu + v_\mu{}^I v_{\nu J} - a_\mu{}^I a_{\nu J} - 8 t_\mu{}^{IJ} t_{\nu IJ} \; , \tag{B.25}$$

$$(p) \qquad 0 = 4i\varepsilon^{IJKL} t_{\mu IK} t_{\nu JL} + p_{(\mu} s_{\nu)} \; , \tag{B.26}$$

$$(v) \qquad 0 = s_{(\nu} p_{\mu)} - 2i a_{(\nu|I|} t_{\mu) JK} \varepsilon^{IJKM} \; , \tag{B.27}$$

$$(a) \qquad 0 = -s_{(\mu} a_{\nu)}{}^M + 2i v_{(\mu|I|} t_{\nu) JK} \, \varepsilon^{IJKM} \; , \tag{B.28}$$

$$(t) \qquad 0 = -i a_{(\nu|I|} v_{\mu) J} \, \varepsilon^{IJMN} - 2i p_{(\nu} t_{\mu) IJ} \, \varepsilon^{IJMN} - 4 s_{(\nu} t_{\mu)}{}^{MN} \; . \tag{B.29}$$

The constraints (B.25) - (B.29) can be written in a better way if we would use a basis which is orthonormal. If we chose the basis

$$\mathfrak{B} = \left\{ \mathbb{1}_{4\times 4}, \, \gamma_*, \, \gamma_0, \, i\gamma_i, -i\gamma_*\gamma_0, \, -\gamma_*\gamma_i, \, -\frac{1}{2}[\gamma_0, \gamma_i], \, \frac{i}{2}[\gamma_i, \gamma_j] \right\} \; , \tag{B.30}$$

and the scalar product (A.3), we can show that this is in deed a orthonormal basis. The small latin indices now label spacial components, instead of spinor components as in the rest of the thesis. We will label the A-th element from $\mathfrak{B}$ as $\mathfrak{g}_A$ and implicitly sum over repeated indices. Thus we could use the ansatz for γ_μ

$$\gamma_\mu = E_\mu{}^A \mathfrak{g}_A \; . \tag{B.31}$$

Inserting the ansatz (B.31) in to the curved Clifford algebra (B.24), we can write

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \mathbb{1}_{4\times 4} \;\Leftrightarrow\; E_\mu{}^A E_\nu{}^B \{\mathfrak{g}_A, \mathfrak{g}_B\} = 2 g_{\mu\nu} \, \mathfrak{g}_1 \; . \tag{B.32}$$

Applying the scalar product with an arbitrary element $\mathfrak{g}_C$ from $\mathfrak{B}$, we get 16 equations

$$E_\mu{}^A E_\nu{}^B \langle \{\mathfrak{g}_A, \mathfrak{g}_B\}, \mathfrak{g}_C \rangle = 2 g_{\mu\nu} \langle \mathfrak{g}_1, \mathfrak{g}_C \rangle \tag{B.33}$$

$$\Leftrightarrow \qquad E_\mu{}^A E_\nu{}^B F_{ABC} = g_{\mu\nu} \delta_{1,C} \; , \tag{B.34}$$

where we defined

$$F_{ABC} = \frac{1}{2} \langle \{\mathfrak{g}_A, \mathfrak{g}_B\}, \mathfrak{g}_C \rangle = \frac{1}{8} \mathrm{Tr}\left[\{\mathfrak{g}_A, \mathfrak{g}_B\} \mathfrak{g}_C \right] \; . \tag{B.35}$$

The object F_{ABC} is totally symmetric in the indices $A, \, B, \, C$ and further reduces to $\delta_{A,B}$ if $C = 1$. Thus we have two qualitatively different types of equations for $C = 1$ and $C \neq 1$

$$E_\mu{}^A E_{\nu A} = g_{\mu\nu} \; , \tag{B.36}$$

$$E_\mu{}^A E_\nu{}^B F_{ABC} = 0 \; , \quad C = 2, 3, \ldots, 16 \; . \tag{B.37}$$

Then, the equation (B.25) corresponds to (B.36) and equations (B.26) - (B.29) correspond to (B.37). In this form it is easier to compute constraints in different dimensions then $d + 1 = 4$.

C. Calculations and Proofs

C.1. Derivation of the Levi-Civita Spin Connection

The Levi-Civita spin connection in the spinbase framework is implicitly defined by (4.20)

$$0 = \underbrace{\partial_\mu \gamma_\nu - \Gamma^\sigma{}_{\mu\nu}\gamma_\sigma}_{=\nabla_\mu\gamma_\nu} + \left[\Gamma_\mu, \gamma_\nu\right] \,. \tag{C.1}$$

With the span for Γ_μ and for $\partial_\mu \gamma_\nu$

$$\Gamma_\mu = p_\mu \gamma_* + v_\mu{}^\sigma \gamma_\sigma + a_\mu{}^\sigma \gamma_* \gamma_\sigma + t_\mu{}^{\sigma\rho}[\gamma_\sigma, \gamma_\rho] \,, \tag{C.2}$$

$$\partial_\mu \gamma_\nu = \tilde{p}_{\mu\nu}\gamma_* + \tilde{v}_{\mu\nu}{}^\sigma \gamma_\sigma + \tilde{a}_{\mu\nu}{}^\sigma \gamma_* \gamma_\sigma + \tilde{t}_{\mu\nu}{}^{\sigma\rho}[\gamma_\sigma, \gamma_\rho] \,, \tag{C.3}$$

we can project out the Clifford parts of (C.1)

$$(s) \qquad 0 = 0 \,, \tag{C.4}$$

$$(p) \qquad 0 = 2a_{\mu\nu} + \tilde{p}_{\mu\nu} \,, \tag{C.5}$$

$$(v) \qquad 0 = -8t_{\mu\nu\rho} + \tilde{v}_{\mu\nu\rho} - \Gamma_{\rho\mu\nu} \,, \tag{C.6}$$

$$(a) \qquad 0 = -2p_\mu g_{\nu\rho} - \tilde{a}_{\mu\nu\rho} \,, \tag{C.7}$$

$$(t) \qquad 0 = g_{\nu\rho} v_{\mu\lambda} - g_{\nu\lambda} v_{\mu\rho} - 2\tilde{t}_{\mu\nu\rho\lambda} \,. \tag{C.8}$$

From these equations we conclude the following

$$(p) \qquad a_{\mu\nu} = -\frac{1}{2}\tilde{p}_{\mu\nu} = -\frac{1}{8}\mathrm{Tr}\left(\gamma_* \partial_\mu \gamma_\nu\right) \,, \tag{C.9}$$

$$(v) \qquad t_{\mu\nu\rho} = \frac{1}{8}\tilde{v}_{\mu\nu\rho} - \frac{1}{8}\Gamma_{\rho\mu\nu} = \frac{1}{32}\mathrm{Tr}\left(\gamma_\rho \partial_\mu \gamma_\nu\right) - \frac{1}{8}\Gamma_{\rho\mu\nu} \,, \tag{C.10}$$

$$(a) \qquad p_\mu = -\frac{1}{8} g^{\nu\rho}\tilde{a}_{\mu\nu\rho} = \frac{1}{32} g^{\nu\rho}\ \mathrm{Tr}\left[\gamma_* \gamma_\rho \partial_\mu \gamma_\nu\right] = \frac{1}{32}\ \mathrm{Tr}\left[\gamma_* \gamma^\nu \partial_\mu \gamma_\nu\right] \,, \tag{C.11}$$

$$\begin{aligned}(t) \qquad v_{\mu\lambda} &= \frac{2}{3} g^{\nu\rho}\tilde{t}_{\mu\nu\rho\lambda} = \frac{1}{3} g^{\nu\rho}\left(g_{\rho\alpha}g_{\lambda\beta} - g_{\rho\beta}g_{\lambda\alpha}\right)\tilde{t}_{\mu\nu}{}^{\alpha\beta} \\ &= -\frac{1}{48}\mathrm{Tr}\left(\left[\gamma_\rho, \gamma_\lambda\right]\partial_\mu \gamma_\nu\right) g^{\nu\rho} = \frac{1}{48}\mathrm{Tr}\left(\left[\gamma_\lambda, \gamma^\nu\right]\partial_\mu \gamma_\nu\right) \,.\end{aligned} \tag{C.12}$$

If we want to raise the indices on the components to get the same canonical form as in [Wel01] we can do so for p, v and a due to the relations between (B.20) and (B.21). For the tensor part we can just raise the index ρ and expand with -1 in the first term

$$t_{\mu\nu}{}^\rho = -\frac{1}{8}\left(-g_{\nu\sigma}\tilde{v}_\mu{}^{\sigma\rho}\right) - \frac{1}{8}\Gamma^\rho{}_{\mu\nu} = -\frac{1}{32}\mathrm{Tr}\left(\gamma_\nu \partial_\mu \gamma^\rho\right) - \frac{1}{8}\Gamma^\rho{}_{\mu\nu} \,, \tag{C.13}$$

to acquire the from stated in [GL14]. We easily raise the index ν also to get the form stated in [Wel01], where the author has used $\Gamma^{\rho\nu}{}_\mu = \Gamma^\rho{}_{\alpha\mu} g^{\alpha\nu} = \frac{1}{4}\Gamma^\rho{}_{\alpha\mu}\mathrm{Tr}\left(\gamma^\alpha \gamma^\nu\right)$.

C.2. Curvature Relation

Using the property of the Levi-Civita covariant derivative in the spinbase framework we can consider

$$\mathcal{D}_\alpha \gamma_\mu = 0 \quad \Rightarrow \quad \mathcal{D}_{[\alpha}\mathcal{D}_{\beta]}\gamma_\mu = 0\,.$$

This will allow to relate $R_{\mu\nu}^{\alpha\beta}$ and ${F_{\mu\nu}}^{IJ}$ with the spin curvature $\Phi_{\mu\nu}$ built from the Levi-Civita spin connection Γ_μ. Applying twice the rule (4.16) for differentiating objects with one upstairs and one downstairs spinor index yields

$$\begin{aligned}0 = \mathcal{D}_{[\alpha}\mathcal{D}_{\beta]}\gamma_\mu &= \mathcal{D}_{[\alpha}\left(\nabla_{\beta]}\gamma_\mu + [\Gamma_{\beta]},\gamma_\mu]\right)\\ &= \nabla_{[\alpha}\left(\nabla_{\beta]}\gamma_\mu + [\Gamma_{\beta]},\gamma_\mu]\right) + \left[\Gamma_{[\alpha},\left(\nabla_{\beta]}\gamma_\mu + [\Gamma_{\beta]},\gamma_\mu]\right)\right]\,.\end{aligned}$$

Expanding the last line gives

$$0 = \nabla_{[\alpha}\nabla_{\beta]}\gamma_\mu + [\underbrace{\nabla_{[\alpha}\Gamma_{\beta]}}_{=\partial_{[\alpha}\Gamma_{\beta]}},\gamma_\mu] + \underbrace{[\Gamma_{[\beta},\nabla_{\alpha]}\gamma_\mu] + [\Gamma_{[\alpha},\nabla_{\beta]}\gamma_\mu]}_{=0} + [\Gamma_{[\alpha},\Gamma_{\beta]}\gamma_\mu] - [\Gamma_{[\alpha},\gamma_{|\mu|}\Gamma_{\beta]}]\,.$$

Investigating the last two terms we can show

$$\begin{aligned}[\Gamma_{[\alpha},\Gamma_{\beta]}\gamma_\mu] - [\Gamma_{[\alpha},\gamma_{|\mu|}\Gamma_{\beta]}] &= \Gamma_{[\alpha}\Gamma_{\beta]}\gamma_\mu \underbrace{-\Gamma_{[\beta}\gamma_{|\mu|}\Gamma_{\alpha]} - \Gamma_{[\alpha}\gamma_{|\mu|}\Gamma_{\beta]}}_{=0} + \gamma_\mu\Gamma_{[\beta}\Gamma_{\alpha]}\\ &= \left[[\Gamma_\alpha,\Gamma_\beta],\gamma_\mu\right]\,,\end{aligned}$$

which yields in total

$$\begin{aligned}0 &= \nabla_{[\alpha}\nabla_{\beta]}\gamma_\mu + \left[\left(\partial_{[\alpha}\Gamma_{\beta]} + [\Gamma_\alpha,\Gamma_\beta]\right),\gamma_\mu\right]\\ &= \nabla_{[\alpha}\nabla_{\beta]}\gamma_\mu + [\Phi_{\alpha\beta},\gamma_\mu]\,.\end{aligned} \tag{C.14}$$

This allows to relate the spin curvature to the Riemann tensor ${R_{\alpha\beta\mu}}^\delta$ or to the curvature ${F_{\alpha\beta}}^{IJ}$ if the Dirac matrices are spanned with the vielbein $\gamma_\mu = {e_\mu}^I\gamma_I$

$$0 = {R_{\alpha\beta\mu}}^\delta\gamma_\delta + [\Phi_{\alpha\beta},\gamma_\mu]\,, \tag{C.15}$$

$$0 = {e_\mu}^I\left({F_{\alpha\beta I}}^J\gamma_J + [\Phi_{\alpha\beta},\gamma_I]\right)\,. \tag{C.16}$$

Spanning $\Phi_{\alpha\beta}$ with the curved (c) and flat (f) Dirac matrices for (C.15) and (C.16) respectively

$$\Phi_{\alpha\beta} = p_{(c)\alpha\beta}\gamma_* + {v_{(c)\alpha\beta}}^\sigma\gamma_\sigma + {a_{(c)\alpha\beta}}^\sigma\gamma_*\gamma_\sigma + {t_{(c)\alpha\beta}}^{\sigma\tau}[\gamma_\sigma,\gamma_\tau]\,, \tag{C.17}$$

$$\Phi_{\alpha\beta} = p_{(f)\alpha\beta}\gamma_* + {v_{(f)\alpha\beta}}^I\gamma_I + {a_{(f)\alpha\beta}}^I\gamma_*\gamma_I + {t_{(f)\alpha\beta}}^{IJ}[\gamma_I,\gamma_J]\,, \tag{C.18}$$

yields

$$p_{(c)\alpha\beta} = 0\,,\quad {v_{(c)\alpha\beta}}^\sigma = 0\,,\quad {a_{(c)\alpha\beta}}^\sigma = 0\,,\quad {t_{(c)\alpha\beta}}^{\sigma\tau} = \frac{1}{8}{R_{\alpha\beta}}^{\sigma\tau}\,, \tag{C.19}$$

$$p_{(f)\alpha\beta} = 0\,,\quad {v_{(f)\alpha\beta}}^I = 0\,,\quad {a_{(f)\alpha\beta}}^I = 0\quad {t_{(f)\alpha\beta}}^{IJ} = \frac{1}{8}{F_{\alpha\beta}}^{IJ}\,. \tag{C.20}$$

Thus we can write for the spin curvature

$$\Phi_{\alpha\beta} = \frac{1}{8}{R_{\alpha\beta}}^{\sigma\tau}[\gamma_\sigma,\gamma_\tau] = \frac{1}{4}{R_{\alpha\beta}}^{\sigma\tau}\gamma_\sigma\gamma_\tau \tag{C.21}$$

$$= \frac{1}{8}{F_{\alpha\beta}}^{IJ}[\gamma_I,\gamma_J] = \frac{1}{4}{F_{\alpha\beta}}^{IJ}\gamma_I\gamma_J\,. \tag{C.22}$$

C.3. Identity (6.4)

For proving identity (6.4) we start with the formula for the determinant (5.33)

$$e = \frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\,\epsilon_{IJKL}e_\alpha{}^I e_\beta{}^J e_\gamma{}^K e_\delta{}^L \tag{C.23}$$

which is equivalent to

$$\epsilon^{IJKL}e = \epsilon^{\alpha\beta\gamma\delta}e_\alpha{}^I e_\beta{}^J e_\gamma{}^K e_\delta{}^L \,. \tag{C.24}$$

Multiplying with $e^\mu{}_I e^\nu{}_J$ and contracting with ϵ_{KLMN} we get

$$\epsilon^{IJKL}\epsilon_{KLMN}e^\mu{}_I e^\nu{}_J e = \epsilon^{\alpha\beta\gamma\delta}\epsilon_{KLMN}e^\mu{}_I e^\nu{}_J e_\alpha{}^I e_\beta{}^J e_\gamma{}^K e_\delta{}^L \,. \tag{C.25}$$

With the definition of the vielbein in (2.19) it follows that

$$e^\mu{}_I e_\alpha{}^I = e^{\mu J}e_\alpha{}^I\eta_{IJ} = g^\mu{}_\nu = \delta^\mu_\nu \,.$$

From the definition for the Levi-Civita symbol we can show that

$$\epsilon^{IJKL}\epsilon_{KLMN} = \epsilon^{KLIJ}\epsilon_{KLMN} = 2\left(\delta^I_M\delta^J_N - \delta^I_N\delta^J_M\right) \,.$$

Using these two equations results in

$$\begin{aligned} 2ee^\mu{}_I e^\nu{}_J\left(\delta^I_M\delta^J_N - \delta^I_N\delta^J_M\right) &= \epsilon^{\mu\nu\gamma\delta}\epsilon_{KLMN}e_\gamma{}^K e_\delta{}^L \\ \Leftrightarrow e\, e^{[\mu}_M e^{\nu]}_N &= \frac{1}{2}\epsilon^{\mu\nu\gamma\delta}\epsilon_{MNKL}e_\gamma{}^K e_\delta{}^L \,. \end{aligned} \tag{C.26}$$

C.4. Variations for Section 6.2

To compute the equation of motion of (6.11) we introduce the abrevation

$$\gamma_{\alpha_1}\cdots\gamma_{\alpha_n} =: \gamma_{\alpha_1\ldots\alpha_n} \,,$$

and use the relation (4.31)

$$\Phi_{\mu\nu} = \frac{1}{4}R_{\mu\nu}{}^{\delta\eta}\left[\gamma_\delta,\gamma_\eta\right] \,.$$

In a first step we get for the variation $\delta S_{SEH,\Lambda}$

$$\begin{aligned} \delta S_{SEH,\Lambda}[\gamma] &= -\frac{i}{4}\int_{\mathcal{M}} \mathrm{d}^4x\,\epsilon^{\mu\nu\rho\lambda}\,\mathrm{Tr}\left[\frac{1}{4}R_{\mu\nu}{}^{\delta\eta}\delta(\gamma_{*\delta\eta\rho\lambda}) + \frac{1}{4}\delta\left(R_{\mu\nu}{}^{\delta\eta}\right)\gamma_{*\delta\eta\rho\lambda} - \frac{\Lambda}{12}\delta(\gamma_{*\mu\nu\rho\lambda})\right] \\ &= -\frac{i}{4}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-\det g}\,\varepsilon^{\mu\nu\rho\lambda}\,\mathrm{Tr}\left[\frac{1}{4}R_{\mu\nu}{}^{\delta\eta}\delta(\gamma_{*\delta\eta\rho\lambda}) + \frac{1}{4}\delta\left(R_{\mu\nu}{}^{\delta\eta}\right)\gamma_{*\delta\eta\rho\lambda} - \frac{\Lambda}{12}\delta(\gamma_{*\mu\nu\rho\lambda})\right] \,. \end{aligned} \tag{C.27}$$

We also need

$$\delta g_{\alpha\beta} = \frac{1}{4}\mathrm{Tr}\left((\delta\gamma_\alpha)\gamma_\beta + \gamma_\alpha(\delta\gamma_\beta)\right) \,, \tag{C.28}$$

from which we can compute

$$
\begin{aligned}
\delta \det g &= \frac{1}{4!}\epsilon^{\alpha\beta\sigma\tau}\epsilon^{\mu\nu\rho\lambda}\delta(g_{\alpha\mu}g_{\beta\nu}g_{\sigma\rho}g_{\tau\lambda}) \\
&= \frac{1}{4!}\epsilon^{\alpha\beta\sigma\tau}\epsilon^{\mu\nu\rho\lambda}g_{\beta\nu}g_{\sigma\rho}g_{\tau\lambda}\, 4\,\delta(g_{\alpha\mu}) \\
&= \frac{-\det g}{4!}\frac{\epsilon^{\alpha\beta\sigma\tau}}{\sqrt{-\det g}}\frac{\epsilon^{\mu\nu\rho\lambda}}{\sqrt{-\det g}}g_{\beta\nu}g_{\sigma\rho}g_{\tau\lambda}\,\mathrm{Tr}\left[(\delta\gamma_\alpha)\gamma_\mu+\gamma_\alpha(\delta\gamma_\mu)\right] \\
&= \frac{-\det g}{4!}\underbrace{\varepsilon^{\alpha\beta\sigma\tau}\varepsilon^{\mu\nu\rho\lambda}g_{\beta\nu}g_{\sigma\rho}g_{\tau\lambda}}_{\to\text{ symmetric in }\alpha\text{ and }\mu}\mathrm{Tr}\left[(\delta\gamma_\mu)\gamma_\alpha+\gamma_\alpha(\delta\gamma_\mu)\right] \\
&= (-\det g)\,\mathrm{Tr}\Big[\underbrace{\frac{1}{4\cdot 3}\varepsilon^{\alpha\beta\sigma\tau}\varepsilon^{\mu\nu\rho\lambda}g_{\beta\nu}g_{\sigma\rho}g_{\tau\lambda}\,\gamma_\alpha}_{:=\mathfrak{K}^\mu}(\delta\gamma_\mu)\Big] \\
&= (-\det g)\,\mathrm{Tr}\left[\mathfrak{K}^\mu(\delta\gamma_\mu)\right]\,.
\end{aligned}
\tag{C.29}
$$

This allows us to compute

$$
\begin{aligned}
\delta\gamma_* &= -\frac{i}{4!}\delta\left(\varepsilon^{\alpha\beta\sigma\tau}\gamma_{\alpha\beta\sigma\tau}\right) \\
&= -\frac{i}{4!}\delta\left(\frac{1}{\sqrt{-\det g}}\right)\epsilon^{\alpha\beta\sigma\tau}\gamma_{\alpha\beta\sigma\tau}-\frac{i}{4!}\varepsilon^{\alpha\beta\sigma\tau}\delta\left(\gamma_{\alpha\beta\sigma\tau}\right) \\
&= -\frac{i}{4!}\cdot\left(\frac{1}{2}\cdot\frac{-\det g}{-\det g}\mathrm{Tr}\left[\mathfrak{K}^\psi(\delta\gamma_\psi)\right]\right)\frac{\epsilon^{\alpha\beta\sigma\tau}}{\sqrt{-\det g}}\gamma_{\alpha\beta\sigma\tau}-\frac{i}{4!}\varepsilon^{\alpha\beta\sigma\tau}\delta\left(\gamma_{\alpha\beta\sigma\tau}\right) \\
&= \frac{1}{2}\mathrm{Tr}\left[\mathfrak{K}^\psi(\delta\gamma_\psi)\right]\gamma_*-\frac{i}{4!}\varepsilon^{\alpha\beta\sigma\tau}\left((\delta\gamma_\alpha)\gamma_{\beta\sigma\tau}+\gamma_\alpha(\delta\gamma_\beta)\gamma_{\sigma\tau}+\gamma_{\alpha\beta}(\delta\gamma_\sigma)\gamma_\tau+\gamma_{\alpha\beta\sigma}(\delta\gamma_\tau)\right) \\
&= \frac{1}{2}\mathrm{Tr}\left[\mathfrak{K}^\psi(\delta\gamma_\psi)\right]\gamma_*-\frac{i}{4!}\varepsilon^{\alpha\beta\sigma\tau}\left(-(\delta\gamma_\tau)\gamma_{\beta\sigma\alpha}-\gamma_\alpha(\delta\gamma_\tau)\gamma_{\sigma\beta}-\gamma_{\alpha\beta}(\delta\gamma_\tau)\gamma_\sigma+\gamma_{\alpha\beta\sigma}(\delta\gamma_\tau)\right)\,.
\end{aligned}
\tag{C.30}
$$

Thus we can compute the variations

$$
\begin{aligned}
-\frac{\Lambda}{12}\varepsilon^{\mu\nu\rho\lambda}\mathrm{Tr}\left[\delta(\gamma_{*\mu\nu\rho\lambda})\right] = -\frac{\Lambda}{12}\varepsilon^{\mu\nu\rho\lambda}\mathrm{Tr}\Big[&(\delta\gamma_*)\gamma_{\mu\nu\rho\lambda}+\gamma_*(\delta\gamma_\mu)\gamma_{\nu\rho\lambda}+\gamma_{*\mu}(\delta\gamma_\nu)\gamma_{\rho\lambda} \\
&+\gamma_{*\mu\nu}(\delta\gamma_\rho)\gamma_\lambda+\gamma_{*\mu\nu\rho}(\delta\gamma_\lambda)\Big]\,.
\end{aligned}
\tag{C.31}
$$

The last four terms can be simplified to

$$
\begin{aligned}
-\frac{\Lambda}{12}\varepsilon^{\mu\nu\rho\lambda}\mathrm{Tr}\Big[&\gamma_*(\delta\gamma_\mu)\gamma_{\nu\rho\lambda}+\gamma_{*\mu}(\delta\gamma_\nu)\gamma_{\rho\lambda} \\
&+\gamma_{*\mu\nu}(\delta\gamma_\rho)\gamma_\lambda+\gamma_{*\mu\nu\rho}(\delta\gamma_\lambda)\Big] = -\frac{\Lambda}{3}\varepsilon^{\mu\nu\rho\lambda}(\gamma_{\nu\rho\lambda*})^j{}_i(\delta\gamma_\mu)^i{}_j\,,
\end{aligned}
\tag{C.32}
$$

and the first term in the trace can be shown to be

$$
-\frac{\Lambda}{12}\varepsilon^{\mu\nu\rho\lambda}\mathrm{Tr}\left[(\delta\gamma_*)\gamma_{\mu\nu\rho\lambda}\right] = -\frac{\Lambda}{12}\varepsilon^{\mu\nu\rho\lambda}\Big[\frac{1}{2}\mathrm{Tr}\left(\gamma_{*\mu\nu\rho\lambda}\right)(\mathfrak{K}^\psi)^j{}_i-\frac{i}{4!}\varepsilon^{\alpha\beta\sigma\psi}\Big(-\gamma_{\beta\sigma\alpha\mu\nu\rho\lambda}
$$

$$
\left. - \gamma_{\sigma\beta\mu\nu\rho\lambda\alpha} - \gamma_{\sigma\mu\nu\rho\lambda\alpha\beta} + \gamma_{\mu\nu\rho\lambda\alpha\beta\sigma}\Big)^{j}{}_{i}\right] (\delta\gamma_{\psi})^{i}{}_{j}
$$

$$
= -\frac{\Lambda}{12}\varepsilon^{\mu\nu\rho\lambda}\left[2i\varepsilon_{\mu\nu\rho\lambda}(\mathfrak{K}^{\psi})^{j}{}_{i} + \left(\mathfrak{T}^{\psi}{}_{\mu\nu\rho\lambda}\right)^{j}{}_{i}\right](\delta\gamma_{\psi})^{i}{}_{j}\,, \tag{C.33}
$$

with $\mathfrak{T}^{\psi}{}_{\mu\nu\rho\lambda}$ defined as

$$
\mathfrak{T}^{\psi}{}_{\mu\nu\rho\lambda} := -\frac{i}{4!}\varepsilon^{\alpha\beta\sigma\psi}\Big(-\gamma_{\beta\sigma\alpha\mu\nu\rho\lambda} - \gamma_{\sigma\beta\mu\nu\rho\lambda\alpha} - \gamma_{\sigma\mu\nu\rho\lambda\alpha\beta} + \gamma_{\mu\nu\rho\lambda\alpha\beta\sigma}\Big)\,. \tag{C.34}
$$

Thus we get for the last term in (C.27)

$$
\begin{aligned}
-\frac{\Lambda}{12}\varepsilon^{\mu\nu\rho\lambda}\delta(\gamma_{*\mu\nu\rho\lambda}) &= -\frac{\Lambda}{12}\varepsilon^{\mu\nu\rho\lambda}\left[4\delta^{\psi}_{\mu}(\gamma_{\nu\rho\lambda*})^{j}{}_{i} + 2i\varepsilon_{\mu\nu\rho\lambda}(\mathfrak{K}^{\psi})^{j}{}_{i} + \left(\mathfrak{T}^{\psi}{}_{\mu\nu\rho\lambda}\right)^{j}{}_{i}\right](\delta\gamma_{\psi})^{i}{}_{j} \\
&=: \left(\mathfrak{B}^{\psi}\right)^{j}{}_{i}(\delta\gamma_{\psi})^{i}{}_{j}\,.
\end{aligned} \tag{C.35}
$$

Now we turn to the first term in (C.27)

$$
\begin{aligned}
\frac{1}{4}\varepsilon^{\mu\nu\rho\lambda}R_{\mu\nu}{}^{\delta\eta}\,\mathrm{Tr}\left[\delta(\gamma_{*\delta\eta\rho\lambda})\right] = \frac{1}{4}R_{\mu\nu}{}^{\delta\eta}\varepsilon^{\mu\nu\rho\lambda}\,\mathrm{Tr}\Big[&(\delta\gamma_{*})\gamma_{\delta\eta\rho\lambda} + \gamma_{*}(\delta\gamma_{\delta})\gamma_{\eta\rho\lambda} \\
&+ \gamma_{*\delta}(\delta\gamma_{\eta})\gamma_{\rho\lambda} + \gamma_{*\delta\eta}(\delta\gamma_{\rho})\gamma_{\lambda} + \gamma_{*\delta\eta\rho}(\delta\gamma_{\lambda})\Big]\,.
\end{aligned} \tag{C.36}
$$

the last four terms of the trace can again be simplified to

$$
\begin{aligned}
&\frac{1}{4}\varepsilon^{\mu\nu\rho\lambda}R_{\mu\nu}{}^{\delta\eta}\,\mathrm{Tr}\left[\gamma_{*}(\delta\gamma_{\delta})\gamma_{\eta\rho\lambda} + \gamma_{*\delta}(\delta\gamma_{\eta})\gamma_{\rho\lambda} + \gamma_{*\delta\eta}(\delta\gamma_{\rho})\gamma_{\lambda} + \gamma_{*\delta\eta\rho}(\delta\gamma_{\lambda})\right] \\
&\qquad = \frac{1}{4}\varepsilon^{\mu\nu\rho\lambda}R_{\mu\nu}{}^{\delta\eta}\left[\delta^{\psi}_{\delta}\left\{\gamma_{\eta},\gamma_{\rho\lambda}\right\}\gamma_{*} + \delta^{\psi}_{\lambda}\gamma_{*}\left\{\gamma_{\rho},\gamma_{\delta\eta}\right\}\right]^{j}{}_{i}(\delta\gamma_{\psi})^{i}{}_{j}\,,
\end{aligned} \tag{C.37}
$$

and the first term of the trace is similar to (C.33) instead of the indices μ and ν being replaced by δ and η. Hence we get

$$
\frac{1}{4}\varepsilon^{\mu\nu\rho\lambda}R_{\mu\nu}{}^{\delta\eta}\,\mathrm{Tr}\left[(\delta\gamma_{*})\gamma_{\delta\eta\rho\lambda}\right] = \frac{1}{4}\varepsilon^{\mu\nu\rho\lambda}R_{\mu\nu}{}^{\delta\eta}\left[2i\varepsilon_{\delta\eta\rho\lambda}(\mathfrak{K}^{\psi})^{j}{}_{i} + \left(\mathfrak{T}^{\psi}{}_{\delta\eta\rho\lambda}\right)^{j}{}_{i}\right](\delta\gamma_{\psi})^{i}{}_{j}\,. \tag{C.38}
$$

Combining the results allows to write for the first part of (C.27)

$$
\begin{aligned}
\frac{1}{4}\varepsilon^{\mu\nu\rho\lambda}R_{\mu\nu}{}^{\delta\eta}\,\mathrm{Tr}\left[\delta(\gamma_{*\delta\eta\rho\lambda})\right] = \frac{1}{4}\varepsilon^{\mu\nu\rho\lambda}R_{\mu\nu}{}^{\delta\eta}\Big[&\delta^{\psi}_{\delta}\left\{\gamma_{\eta},\gamma_{\rho\lambda}\right\}\gamma_{*} + \delta^{\psi}_{\lambda}\gamma_{*}\left\{\gamma_{\rho},\gamma_{\delta\eta}\right\} \\
&+ 2i\varepsilon_{\delta\eta\rho\lambda}\mathfrak{K}^{\psi} + \mathfrak{T}^{\psi}{}_{\delta\eta\rho\lambda}\Big]^{j}{}_{i}(\delta\gamma_{\psi})^{i}{}_{j} \\
=: \left(\mathfrak{A}^{\psi}\right)^{j}{}_{i}(\delta\gamma_{\psi})^{i}{}_{j}
\end{aligned} \tag{C.39}
$$

In a final step we need to compute the second term from (C.27), where we make use of the relation

$$
\varepsilon^{\mu\nu\rho\lambda}\,\mathrm{Tr}\left[\gamma_{*\delta\eta\rho\lambda}\right] = 4i\varepsilon^{\mu\nu\rho\lambda}\varepsilon_{\delta\eta\rho\lambda} = 8i\left(\delta^{\mu}_{\delta}\delta^{\nu}_{\eta} - \delta^{\nu}_{\delta}\delta^{\mu}_{\eta}\right)
$$

to write

$$\frac{1}{4}\varepsilon^{\mu\nu\rho\lambda}\delta\left(R_{\mu\nu}{}^{\delta\eta}\right)\operatorname{Tr}\left[\gamma_{*\delta\eta\rho\lambda}\right] = 4i\delta\left(R_{\mu\nu}{}^{\mu\nu}\right) = 4i\left[\delta(g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}\delta\left(R_{\mu\nu}\right)\right] . \tag{C.40}$$

If integrated, the term $g^{\mu\nu}\delta\left(R_{\mu\nu}\right)$ corresponds to a surface term and can be neglected. For this to be true we need the exact index placement, $g^{\mu\nu}\delta\left(R_{\mu\nu}\right)$. The standard derivation for this surface term can be found in [Wal84], [Car19] or [Bla19]. To proceed we need to rewrite $\delta g^{\mu\nu}$ to

$$\begin{aligned}\delta g^{\mu\nu} &= \delta(g_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}) \\ &= (\delta g_{\alpha\beta})g^{\mu\alpha}g^{\nu\beta} + g_{\alpha\beta}\delta(g^{\mu\alpha})g^{\nu\beta} + g_{\alpha\beta}g^{\mu\alpha}\delta(g^{\nu\beta}) \\ &= (\delta g_{\alpha\beta})g^{\mu\alpha}g^{\nu\beta} - \delta(g_{\alpha\beta})g^{\mu\alpha}g^{\nu\beta} - \delta(g_{\alpha\beta})g^{\mu\alpha}g^{\nu\beta} \\ &= -(\delta g_{\alpha\beta})g^{\mu\alpha}g^{\nu\beta} ,\end{aligned} \tag{C.41}$$

where we have used

$$\delta(\delta^{\mu}_{\nu}) = \delta(g^{\mu\rho}g_{\nu\rho}) = 0 \;\Leftrightarrow\; \delta(g^{\mu\rho})g_{\nu\rho} = -g^{\mu\rho}\delta(g_{\nu\rho}) .$$

Using (C.28) and (C.41) we can further reduce (C.40) to

$$\frac{1}{4}\varepsilon^{\mu\nu\rho\lambda}\delta\left(R_{\mu\nu}{}^{\delta\eta}\right)\operatorname{Tr}\left[\gamma_{*\delta\eta\rho\lambda}\right] = -iR^{\mu\nu}\operatorname{Tr}\left[(\delta\gamma_\mu)\gamma_\nu + \gamma_\mu(\delta\gamma_\nu)\right] = -2iR^{\mu\psi}(\gamma_\mu)^j{}_i(\delta\gamma_\psi)^i{}_j . \tag{C.42}$$

Merging all the results yields for (C.27)

$$\delta S_{\text{SEH},\Lambda}[\gamma] = -\frac{i}{4}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-\det g}\left[\mathfrak{A}^\psi - 2iR^{\mu\psi}\gamma_\mu + \mathfrak{B}^\psi\right]^j{}_i(\delta\gamma_\psi)^i{}_j , \tag{C.43}$$

$$\mathfrak{A}^\psi = \frac{1}{4}\varepsilon^{\mu\nu\rho\lambda}R_{\mu\nu}{}^{\delta\eta}\left[\delta^\psi_\delta\left\{\gamma_\eta,\gamma_{\rho\lambda}\right\}\gamma_* + \delta^\psi_\lambda\gamma_*\left\{\gamma_\rho,\gamma_{\delta\eta}\right\} + 2i\varepsilon_{\delta\eta\rho\lambda}\mathfrak{K}^\psi + \mathfrak{T}^\psi{}_{\delta\eta\lambda}\right] , \tag{C.44}$$

$$\mathfrak{B}^\psi = -\frac{\Lambda}{12}\varepsilon^{\mu\nu\rho\lambda}\left[4\delta^\psi_\mu\gamma_{\nu\rho\lambda*} + 2i\varepsilon_{\mu\nu\rho\lambda}\mathfrak{K}^\psi + \mathfrak{T}^\psi{}_{\mu\rho\lambda}\right] , \tag{C.45}$$

$$\mathfrak{T}^\psi{}_{\mu\rho\lambda} = -\frac{i}{4!}\varepsilon^{\alpha\beta\sigma\psi}\left(-\gamma_{\beta\sigma\alpha\mu\nu\rho\lambda} - \gamma_{\sigma\beta\mu\nu\rho\lambda\alpha} - \gamma_{\sigma\mu\nu\rho\lambda\alpha\beta} + \gamma_{\mu\nu\rho\lambda\alpha\beta\sigma}\right) . \tag{C.46}$$

We require the action to be stationary under the variations and hence obtain the equation of motion

$$\mathfrak{A}^\psi - 2iR^{\mu\psi}\gamma_\mu + \mathfrak{B}^\psi = 0 . \tag{C.47}$$

Multiplying by $\frac{1}{8i}\gamma_\chi$ and taking the trace yields Einstein's equations

$$R^\psi{}_\chi - \frac{1}{2}g^\psi{}_\chi R + \Lambda g^\psi{}_\chi = 0 .$$

C.5. Variations for Chapter 7

The connection equation of motion (7.7) in the spinbase Palatini frame work is computed from

$$\begin{aligned}S_{\text{SEP},\Lambda}[\gamma,\Delta\Gamma] = S_{\text{SEH},\Lambda} - \frac{i}{4}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-\det g}\Big[&\underbrace{\mathcal{D}_\mu\operatorname{Tr}\left(\varepsilon^{\mu\nu\rho\lambda}\gamma_*\Delta\Gamma_\nu\gamma_\rho\gamma_\lambda\right)}_{\text{surface term}} \\ &+ \operatorname{Tr}\left(\varepsilon^{\mu\nu\rho\lambda}\gamma_*\left[\Delta\Gamma_\mu,\Delta\Gamma_\nu\right]\gamma_\rho\gamma_\lambda\right)\Big] ,\end{aligned} \tag{C.48}$$

by considering only variations in $\Delta\Gamma_\mu$. Thus only the last term will be non trivial. Therefore we get

$$\delta S_{SEP,\Lambda}[\gamma,\Delta\Gamma] = -\frac{i}{4}\int\limits_{\mathcal{M}} \mathrm{d}^4x\sqrt{-\det g}\left[\operatorname{Tr}\left(\varepsilon^{\mu\nu\rho\lambda}\gamma_*\left(\delta\left[\Delta\Gamma_\mu,\Delta\Gamma_\nu\right]\right)\gamma_\rho\gamma_\lambda\right)\right]. \tag{C.49}$$

We also find

$$\begin{aligned}\varepsilon^{\mu\nu\rho\lambda}\left(\delta\left[\Delta\Gamma_\mu,\Delta\Gamma_\nu\right]\right) &= \varepsilon^{\mu\nu\rho\lambda}\left(\left[\delta\Delta\Gamma_\mu,\Delta\Gamma_\nu\right]+\left[\Delta\Gamma_\mu,\delta\Delta\Gamma_\nu\right]\right)\\ &= \varepsilon^{\mu\nu\rho\lambda}\left(\left[\delta\Delta\Gamma_\mu,\Delta\Gamma_\nu\right]-\left[\Delta\Gamma_\nu,\delta\Delta\Gamma_\mu\right]\right)\\ &= 2\varepsilon^{\mu\nu\rho\lambda}\left[\delta\Delta\Gamma_\mu,\Delta\Gamma_\nu\right]\,.\end{aligned} \tag{C.50}$$

Applying this to (C.49) we obtain

$$\begin{aligned}\operatorname{Tr}\left(\varepsilon^{\mu\nu\rho\lambda}\gamma_*\left(\delta\left[\Delta\Gamma_\mu,\Delta\Gamma_\nu\right]\right)\gamma_\rho\gamma_\lambda\right) &= 2\operatorname{Tr}\left(\varepsilon^{\mu\nu\rho\lambda}\gamma_*\left[\delta\Delta\Gamma_\mu,\Delta\Gamma_\nu\right]\gamma_\rho\gamma_\lambda\right)\\ &= 2\varepsilon^{\mu\nu\rho\lambda}\operatorname{Tr}\left(\gamma_*(\delta\Delta\Gamma_\mu)\Delta\Gamma_\nu\gamma_\rho\gamma_\lambda-\gamma_*\Delta\Gamma_\nu(\delta\Delta\Gamma_\mu)\gamma_\rho\gamma_\lambda\right)\\ &= 2\varepsilon^{\mu\nu\rho\lambda}\left(\Delta\Gamma_\nu\gamma_\rho\gamma_\lambda\gamma_*-\gamma_\rho\gamma_\lambda\gamma_*\Delta\Gamma_\nu\right)^i{}_j\,(\delta\Delta\Gamma_\mu)^j{}_i\,.\end{aligned} \tag{C.51}$$

Defining

$$G^{\mu\nu} := \varepsilon^{\mu\nu\rho\lambda}\gamma_\rho\gamma_\lambda\gamma_*\,, \tag{C.52}$$

the total equation reads

$$\delta S_{SEP,\Lambda}[\gamma,\Delta\Gamma] = -\frac{i}{2}\int\limits_{\mathcal{M}} \mathrm{d}^4x\sqrt{-\det g}\left[\Delta\Gamma_\nu G^{\mu\nu}-G^{\mu\nu}\Delta\Gamma_\nu\right]^i{}_j\,(\delta\Delta\Gamma_\mu)^j{}_i\,. \tag{C.53}$$

From this we can read off the connection equation of motion

$$0 = \Delta\Gamma_\nu G^{\mu\nu}-G^{\mu\nu}\Delta\Gamma_\nu\,. \tag{C.54}$$

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