Asymptotically Free Gauged-Yukawa Systems

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Gutachter

1. Prof. Dr. Holger Gies (Friedrich-Schiller-Universität Jena)
2. Priv-Doz. Dr. Michael Scherer (Universität zu Köln)
3. Prof. Dr. Jan Martin Pawlowski (Ruprecht-Karls-Universität Heidelberg)

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A Maria Cristina

Abstract

Recent studies have provided evidence for the existence of asymptotically free trajectories in non-Abelian Higgs models without asymptotic symmetry in the high-energy limit. These solutions are not evident within standard perturbation theory.

This discovery served as our main motivation to investigate another class of quantum field theories which already exhibits a regime of asymptotic freedom in all their marginal couplings in standard perturbation theory: namely the *gauged-Yukawa models*. We start with a minimalistic toy-model containing only a Yukawa and a QCD-like gauge sector. We then extend the analysis to a larger class of models which include the whole non-Abelian sector of the Standard Model.

In both models we discover the existence of further and novel asymptotically free trajectories by exploiting generalized boundary conditions. We construct such trajectories as quasi-fixed points for the Higgs scalar potential, whose couplings approach the noninteracting Gaußian fixed point with specific scalings with respect to the asymptotically free gauge couplings. We corroborate our findings in an effective-field-theory approach, and subsequently we obtain a comprehensive picture using the functional renormalization group. The latter method allows us to study the stability of the scalar potential for large field amplitudes.

In contrast to standard perturbation theory, these new solutions become visible beyond the deep-Euclidean-regime, because of the important role of mass-threshold effects. Since one-loop universality is no longer guaranteed once threshold corrections are included, we investigate whether the existence of these ultraviolet complete trajectories is universal, i.e., a scheme-independent feature. We consider a wide class of regularization schemes that account for threshold behavior persisting in the infinite-energy limit, firstly focusing on the conventional $\overline{\text{MS}}$ scheme and subsequently on mass-dependent schemes based on general momentum-space infrared regularizations.

We argue that the existence of these asymptotically free solutions is a scheme-independent phenomenon. A change of scheme induces a map of the theory's coupling space onto itself, which in the present case also translates into a reparametrization of the space of asymptotically free solutions.

Zusammenfassung

Aktuelle Untersuchungen haben Hinweise auf die Existenz asymptotisch freier Trajektorien ohne asymptotische Symmetrie im Hochenergielimes in nicht-abelschen Higgsmodellen geliefert. Diese Lösungen können nicht mittels Störungstheorie untersucht werden.

Motiviert von diesen Ergebnissen untersuchen wir eine weitere Klasse von Quantenfeldtheorien mit asymptotischer Freiheit in allen Kopplungen: geeichte Yukawa-Modelle. Wir beginnen die Untersuchung mit einem minimalen Modell, welches nur einen an die Starke Wechselwirkung erinnernden Eichsektor sowie einen Yukawa Sektor enthält. Danach verallgemeinern wir zu einer breiteren Klasse an Modellen, die den gesamten nichtabelschen Eichsektor des Standardmodells enthalten.

In beiden Fällen finden wir durch Ausnutzung verallgemeinerter Randbedingungen weitere, neue asymptotisch freie Trajektorien. Wir konstruieren solche Trajektorien als quasi-Fixpunkte des skalaren Higgspotentials, dessen Kopplungen sich mit bestimmten Potenzgesetzen in der Eichkopplung dem nichtwechselwirkenden Gaußschen Fixpunkt nähern. Wir untermauern die Ergebnisse durch Berechnungen in effektiven Feldtheorien, und erhalten dann, mit Hilfe der Funktionalen Renormierungsgruppe, ein verständliches Bild. Mit den letzteren Methoden lassen sich auch die Stabilitätseigenschaften des skalaren Potentials für große Feldamplituden untersuchen.

Im Gegensatz zu der normalen Störungstheorie lassen sich diese Lösungen aufgrund von wichtigen Schwelleneffekten auch außerhalb der tiefeuklidschen Region sehen. Da die Universalität von Ein-Loop-Berechnungen unter Berücksichtigung dieser Schwelleneffekte nicht mehr gegeben ist, untersuchen wir ob es sich bei diesen Trajektorien um universelle, also schemenunabhängige, Eigenschaften handelt. Wir untersuchen eine große Klasse an Regularisierungmethoden, die Schwelleneffekte auch im Hochenergielimes berücksichtigen, angefangen mit der $\overline{\text{MS}}$-Methode, und untersuchen im Folgenden dann massenabhängige Renormoierungsmethoden, die auf allgemeinen Impulsraumregulatoren basieren.

Wir behaupten dass die Existenz dieser asymptotisch freien Lösungen universell ist. Wechselt man die Methode, so ergibt sich eine Abbildung von dem Raum aller Kopplungen der Theorie in sich selbst, was also einer Reparametrisierung des Raumes der asymptotisch freien Lösungen entspricht.

Contents

1. Introduction

The Standard Model (SM) theory of particle physics represents an enormous success in capturing the present day phenomenology of elementary particles in Nature and their interaction forces, up to the present accessible energy scales at the Large Hadron Collider (LHC) (currently the highest collision energy is roughly 13 TeV). Only a few experimental data appear to show some anomalies in comparison with the prediction of the SM [1–3]. The origin and motivations which led to the final formulation of the SM stem from the theoretical necessity to move beyond earlier phenomenological models which provided, even though very accurately, explanations of the weak and strong interactions among hadrons. One such phenomenological model was the popular "Vector minus Axial" (V-A) theory, a generalization of the Fermi theory used to describe the weak interaction. However, the V-A theory is perturbatively non-renormalizable due to the fact that the Fermi coupling constant G_{F} has a canonical dimension of inverse mass. The very same issue appears also in any attempt to construct a quantum field theory description of the Einstein-Hilbert gravity theory (above its critical dimension $d_{\mathrm{c}} = 2$).

Quantum Electrodynamics (QED) had a great success in describing the electromagnetic force as an interaction among fermionic fields (whose quantum excitations are the electron/positron particles) mediated by the exchange of a vector boson field (whose quantum excitation is the photon). In the language of gauge field theories, this means that a gauge covariant derivative

$$D_\mu = \partial_\mu - \mathrm{i}gV_\mu, \tag{1.1}$$

acting on the fermionic field, has to be introduced. Guided by this idea, the four-fermion interaction in the V-A theory can be thought of as a Quantum Field Theory (QFT) where the interaction, instead of being point-like, is again mediated by the exchange of a vector boson. The latter has to be massive, with a mass proportional to $G_{\mathrm{F}}^{-1/2}$, setting a finite length for the interaction scale. However, an explicit mass term for vector fields, such as

$$m_V^2 V^\mu V_\mu, \tag{1.2}$$

would violate the gauge symmetry and such a theory can not be perturbatively renormalizable.

Regardless of this issue, in 1961, Glashow [4] realized that the weak and electromagnetic forces can be described by a gauge theory based on the gauge group symmetry $\mathrm{SU}(2)_\mathrm{W} \times \mathrm{U}(1)_\mathrm{Y}$, provided the existence of a massive, charged W_μ field and a massive neutral Z_μ field (in addition to the massless photon field). Later on, at the end of the sixties, Weinberg and Salam [5, 6] circumvented the problem of massive vector bosons by invoking a complex scalar field doublet which has the purpose to spontaneously break the gauge symmetry into

$$\mathrm{SU}(2)_\mathrm{W} \times \mathrm{U}(1)_\mathrm{Y} \longrightarrow \mathrm{U}(1)_\mathrm{Q}, \tag{1.3}$$

such that the only symmetry left is the global abelian electromagnetic symmetry associated to the electric charge Q. The scalar potential acquires a "Mexican hat"-like shape due to the presence of a nonzero scalar vacuum expectation value (*vev*), and the would-be Goldstone modes are "eaten-up" by the vector fields which become massive. [1] This mechanism is general for gauge field theories and was introduced in 1964 by Brout and Englert [10], Higgs [11–13], and independently shortly after by Kibble, Hagen and Guralnik [14].

Finally, in 1971 and 1972, the Glashow-Weinberg-Salam (GWS) theory of the electroweak (EW) interaction could then be proven to be perturbatively renormalizable by 't Hooft [15] and others [16–19], using generalized Ward-Takahashi identities for gauge bosons, the so-called Slavnov-Taylor identities [20, 21]. It therefore became clear and widely accepted that, for a Yang-Mills theory to have massive vector fields, it must have a Higgs mechanism. It did not take long before the experiments could provide proof of the theory. Neutral currents were indeed discovered at CERN one year later in the "Gargamelle" collaboration experiment at CERN [22–24]; and eventually, 10 years later, resonances corresponding to the massive vector bosons $W_\mu^\pm$ and Z_0 were observed by the Rubbia's UA1 group at CERN [25].

In the early sixties, due to the plethora of particles produced in hadronic scatterings, it was still not clear how to include the strong interaction in a perturbative QFT description. Experiments in the high energy domain, such as deep inelastic scattering of electrons and protons, showed that the constituents of the hadrons behave, at short distances, as they were free. This peculiar asymptotic behavior, called the "Bjorken scaling" [26], was not understood within the framework of renormalizable QFTs until 1973, when Politzer [27], Gross and Wilczek [28], and 't Hooft (unpublished) showed that non-Abelian gauge theories are asymptotically free (AF). In the same year, it was also demonstrated by Gross and Coleman [29, 30] that asymptotic freedom can not occur in theories with fermions and scalars without the inclusion of a non-Abelian gauge group. This goes by the name of *Coleman-Gross theorem.* The final missing tail towards a complete and uniform description of the strong interaction, the Quantum Chromodynamics (QCD) theory, was found in the same years when it became clear that hadrons can be understood as made of quarks, transforming

[1] The non-Abelian SU(2) gauge group represents a very special case since the presence of a condensate for the complex scalar doublet breaks the whole group such that, no massless Goldstone bosons are present in the theory. Quite in general, the number of "unbroken" generators of a gauge group corresponds to the number of massless Goldstone bosons. This goes under the name of *Goldstone theorem* [7–9].

according to the fundamental representation of the SU(3) "color" group. This internal degree of freedom was introduced first by Greenberg, Han and Nambu [31, 32], and proposed as a description of the quark structure by Gell-Mann, Fritzsch and Leutwyler [33, 34]. Still, there was a deep puzzle of the QCD theory which was unclear, and this was the fact that quarks were never seen as free particles in the experiments at CERN. The explanation of this behavior was found in the so-called *quark confinement*, which got substantiated by lattice simulations of QCD carried out by Creutz et al. [35]. Still, an analytical description of this phase transition is left unproven.

The program of building the SM was complete. It is based on a QFT description of elementary particles with a local gauge symmetry

$$\mathcal{G}_{\mathrm{SM}} = \mathrm{SU}(3)_{\mathrm{c}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}, \tag{1.4}$$

where three of the four fundamental forces in Nature (excluding gravity), i.e., the electromagnetic, weak and strong interactions, are described in a unified framework: the forces are mediated via exchanges of vector gauge boson particles. The beauty of this model stems from its astonishing capability to accurately describe essentially all of the present day experimental particle physics data and, moreover, from its success in predicting new phenomena. The discovery of the Higgs boson by the ATLAS and CMS collaborations [36–38] was the last missing piece to further consolidate the SM.

Despite the beauty of this model, the SM possesses its own flaws and predicts its own failure. As we said, gravity is not included in this unified description of the fundamental forces. Indeed it is still unclear how to consistently include the gravitational force into a QFT framework, without spoiling the renormalizability. [2] Therefore, the SM sets intrinsically the cutoff of its maximal extension to the Planck scale $M_{\mathrm{Pl}} \sim 10^{19}$ GeV; above that, quantum gravitational fluctuations should be accounted for. Aside from gravity, the SM presents several other limitations. Some of these are related to the scalar sector, in particular they are the (maybe) meta-stability of the Higgs vacuum [47–66], the naturalness problem, the triviality problem [67–72] and the huge negative contribution to the cosmological constant [73] compared with the observations [74]. Other issues are, for example, related to the Yukawa sector, for example, the unjustified hierarchy structure of the Yukawa couplings, reflected also by the hierarchy of the lepton and quark masses. Therefore, the SM is often considered as an effective field theory description of the fundamental forces, extremely successful only up to the energy scales reachable nowadays in collider experiments. Any other attempt to

[2] Several methods have been proposed over the years in order to consistently quantize gravity. We mention for example string theory [39, 40], loop quantum gravity [41–43] or *asymptotic safety* [44–46]. The latter was first suggested by Weinberg in 1976, by conjecturing that the renormalizability problem of quantum gravity can be solved by the presence of an interacting ultraviolet fixed point, keeping under controlled the ultraviolet behavior of the theory.

push the validity of the SM above these scales, or construct beyond Standard Model (BSM) theories to account for new phenomena, are "speculations" which have to be confirmed experimentally, or at least justified on the basis of theoretical arguments.

Among all these issues, we choose to focus on the *triviality problem*, intimately related to the presence of Landau pole singularities in the perturbative renormalization group (RG) flow of the coupling constants. The presence of such divergences is a severe consistency problem. On one hand, it invalidates standard perturbation theory as a perturbation about small couplings and, on the other hand, it invalidates the actual QFT as a fundamental theory. In fact, in order to be fundamental, a theory needs to be valid at arbitrary energy scales, without the need to introduce an ultraviolet (UV) cutoff scale of maximal extension of the theory. Insisting on UV completeness by enforcing this cutoff to be sent to infinity typically requires to send the renormalized coupling to zero, such that the theory becomes trivial, i.e., non interacting, and thus unphysical.

The SM of particle physics is trivial in its hypercharge Abelian group $\mathrm{U}(1)_{\mathrm{Y}}$ [75, 76], which feeds back into the scalar sector. The latter feature is even "worse" for two reasons: it can generically occur also in the absence of the Abelian group, and it threatens the Brout-Englert-Higgs mechanism, which is crucial in order to have massive gauge bosons as well as massive chiral fermions while preserving the gauge symmetry. This is one of the reasons why, in this thesis, we will focus only on the triviality of the scalar sector of the SM. Therefore, we will consider non-Abelian Yang-Mills theories coupled to some quark-like fermionic degrees of freedom, the latter interacting with some Higgs-like scalar fields via the Yukawa channel. Generally speaking, these theories are called in the literature *gauged-Yukawa models*.

Gauged-Yukawa systems provide interesting routes to construct theories which are UV complete, in the sense that no singularities appear in the RG flow at all. In fact, already standard perturbation theory can reveal the existence of gauged-Yukawa models which are *asymptotically free* (AF), whose couplings constants approach the noninteracting Gaußian fixed point (GFP) in the limit of infinite energy scales [29, 30, 77–85]. Recently, studies in the direction of a classification of these AF UV complete gauged-Yukawa models have been performed [86–88], together with some construction of phenomenologically acceptable models [89–92]. Moreover, gauged-Yukawa models also offer the possibility of constructing *asymptotically safe* (AS) theories [93–96]. The discovery of such AS gauged-Yukawa theories, perturbatively under control, was a break-through and attracted a lot of attention in the recent years, since it provided a novel route to construct UV complete BSM theories [94, 97–106]. In general, the existence of such UV complete theories and the fate of their asymptotic behavior at large RG scales depends typically on their matter field content and corresponding representations with respect to the gauge group. These constructions usually require additional matter fields (for example vector-like fermions or scalars in higher dimensional representations) and/or additional gauge sectors with respect to the SM case. However, a unique route to an unequivocal model appears not obvious.

The aim of this work is to bring new perspectives into this pool of studies regarding gauged-Yukawa models. In trying to do so, we first restrict our target to a minimal number of matter degrees of freedom, without advocating the presence of BSM fields. In this respect, for example, we do not rely our analysis on the presence of an infinite number of fermionic flavors such as required from the Veneziano-Witten limit or $1/N_{\mathrm{f}}$ expansions [107–109]. Secondly, we want to depart from a standard perturbative analysis and also properly take into account the possibility of *mass-threshold effects*. In other words, we want to relax the standard simplification of dealing with beta functions deduced in the deep Euclidean regime (DER), where all mass scales are regarded as negligible with respect to the loop momenta of the quantum fluctuations. In case these mass scales are generated through a spontaneous-symmetry-breaking (SSB) regime of the scalar potential, the DER assumption can be summarized as *asymptotic symmetry* [110]. [3] However, these assumptions may hide possible UV completions of a given QFT. In fact, recent studies [111, 112] on theories where a scalar field is coupled to a Yang-Mills theory (we will refer to that as non-Abelian Higgs models) have shown that, the occurrence of mass-threshold effects also in the UV limit of large RG scales opens the possibility for new total AF trajectories. This result has also been astonishing as it was obtained in a class of models which does not exhibit asymptotic freedom within standard perturbation theory.

Therefore, also in our analysis of gauged-Yukawa models, we want to systematically account for mass-threshold effects, and check whether their inclusion is of crucial importance to enrich the possibilities for UV completion. In order to do so, we will consider nonperturbative techniques mainly focusing on the *functional renormalization group* (FRG) method, which is intrinsically a mass-dependent regularization scheme. As a drawback, the occurrence of threshold phenomena in the RG flow of the coupling constants entails the loss of universality for the one-loop beta functions. The latter, in fact, become dependent on the renormalization scheme adopted, as a manifestation of the details of the physical decoupling of massive modes.

Universality in physics characterizes the fact that long-range (i.e. low energy) effective properties of a system can be largely independent of the microscopic details of the theory. In statistical physics, a notable example is the Ising model which has the same critical behavior as a ϕ^4 scalar field theory. In particle physics universality is often quantified in terms of observables which must be independent of the choice of the regularization and renormalization scheme. On a technical level, the latter property becomes manifest on the level of the RG flow since the perturbative one-loop beta functions for couplings with vanishing mass dimension are scheme-independent (in a mass-independent scheme, also the

[3] This is the usual approach adopted, for example, while computing the stability bound for the Higgs mass. The instability scale of the scalar potential is defined as the scale where the quartic self-interaction becomes negative. Since this scale is much higher than the EW scale, it is reasonable to assume that the momenta of the fluctuations are dominant. Therefore one often approximates the potential as $U_{\mathrm{eff}}(\phi) \sim \lambda_{\mathrm{eff}}\phi^4$.

two-loop coefficients are universal).

These observations naturally raise the question of the *scheme dependence* of the results obtained for the non-Abelian Higgs models in [111, 112], as well as of the novel total AF UV completions of the gauge-Yukawa models which will be presented soon. In the second part of this thesis we thus mainly focus on the universality of our statements.

This thesis is structured in the following way. We start in Chap. 2 by presenting a detailed overview of the triviality problem, providing a few didactical examples to end up with a presentation of the SM issues. After that, we introduce our technical machinery based on the nonperturbative FRG method, and, in the last section, we present two examples of AF theories which serve as a motivation behind this work. In Chap. 3 we start our analysis and we present the procedure for constructing total AF trajectories for gauge-Yukawa models by

- including mass-threshold effects, thus going beyond standard perturbation theory which usually assumes the DER;
- retaining only a "minimal" amount of degrees of freedom as a subsector of the SM.

To start with, we consider only the color $\mathrm{SU}(3)_\mathrm{c}$ gauge group. In the following Chap. 4 we generalize the previous toy model by adding the weak $\mathrm{SU}(2)_\mathrm{L}$ group, thus considering the whole non-Abelian gauge sector of the SM. While studying this generalization we will also address the problem of the scheme dependence of our results, by extending our analysis to any generic FRG scheme as well as also to the more widely used minimal subtraction scheme $\overline{\mathrm{MS}}$. At this point the universality test of our statements can be considered concluded and we finally present our conclusions in Chap. 5. We leave the details of the most tedious calculations, as well as further numerical and analytic analyses which corroborate our results, to the Appendices.

The compilation of this thesis is solely due to the author. However, parts of this work have been developed in collaboration with members of the Theoretical Physical Institute in Jena. The novel asymptotically free solutions in $\mathbb{Z}_2$-Yukawa-QCD models, described in Chap. 3, have been discovered in collaboration with H. Gies, R. Sondenheimer and L. Zambelli and published in [113]. The study of the scheme dependence of asymptotically free solutions in $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{c}$ *models, described in Chap. 4, has been elaborated together with H. Gies, R. Sondenheimer and L. Zambelli and published in [114].*

2. Theoretical Foundations

As anticipated in the Introduction, we dedicate this chapter to presenting the theoretical concepts which are on the basis of this work. First we want to introduce the triviality problem which plagues several QFTs, in particular the SM, and possible solutions thereof. See for example [85] for an extensive review. In this thesis we will focus in particular on the AF scenario where a theory approaches the GFP in the UV limit. In the last section we summarize the results of recent studies [111, 112], since they serve as a motivation to enlarge the discussion of asymptotic freedom to different and also more generic non-Abelian gauge field theories, including fermionic matter fields. We refer to these models as *gauged-Yukawa models*.

2.1. The Triviality Problem

The concept of *triviality* has a very long history and its origin can be traced back in the '50s when Landau, Pomeranchuk and collaborators [115–124], while studying the asymptotic behavior for large transferred momentum of the photon propagator, discovered some inconsistencies in the full theory. For a deeper understanding of the nature of such inconsistencies, it is useful to study the problem by means of the RG in its historical approach. The basic prescription, as described in any introductory textbook of QFT, consists mainly of two steps. The first one is meant to regularize the theory by introducing appropriate counter terms in the bare action in order to remove the divergences of the loop integrals, the latter representing the quantum fluctuations of the system. A theory is called *perturbatively renormalizable* if a finite number of counter terms needs to be added in such a way that all quantum loops, to any order in perturbation theory, stay finite. The second step corresponds to the renormalization prescription: given a certain momentum scale μ_0 accessible from the experiments, a renormalized coupling constant λ_0 is defined by the measurement of that coupling constant at the momentum scale μ_0. The choice of the renormalization point μ_0 is completely arbitrary, thus the theory itself must be equally valid for any rescaling of μ_0 as

$$\mu_0 \to \mu_0 \mathrm{e}^t \equiv \mu(t), \tag{2.1}$$

provided that the physical couplings are appropriately redefined too. In other words, if $\mathscr{O}(\lambda_0, \mu_0)$ is any physical observable measured at some scale μ_0 for some coupling λ_0, then

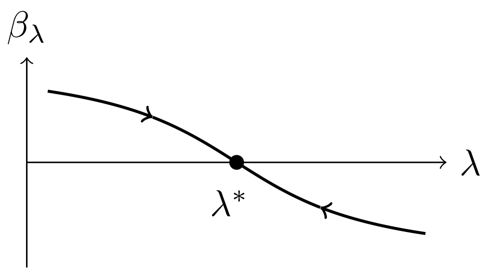


Fig. 2.1.: Example of an attractive ultraviolet fixed point. The arrows are pointing in the direction of increasing energy scales.

the previous statement can be expressed by demanding that

$$\mathscr{O}(\lambda_0, \mu_0) = \mathscr{O}\left(\lambda(t), \mu(t)\right), \tag{2.2}$$

where the dependence of $\lambda(t)$ upon t is represented by the so-called *beta function*:

$$\beta(\lambda) = \frac{\partial}{\partial t}\lambda(t) = \mu\frac{\partial}{\partial\mu}\lambda(\mu). \tag{2.3}$$

The renormalized coupling λ is thus called *running coupling constant* due to the fact that the renormalization group criteria induces a momentum-scale dependence of the coupling constant. In particular the value $\lambda(t)$ is equal to the value of the coupling evaluated at the momentum scale $\mu(t)$.

The famous “Feldman-Landau ghost” appears whenever the running coupling diverges at some finite momentum scale. The necessary condition to avoid this scenario is thus that the renormalized coupling stays finite as long as the momentum scale is finite. To understand this criterion, let us consider the solution of the former equation

$$t = \int_{\lambda_0}^{\lambda(t)} \frac{\mathrm{d}\lambda}{\beta(\lambda)}. \tag{2.4}$$

The superior extrema $\lambda(t)$ must be bounded for all positive finite t. This can be achieved if $\beta(\lambda)$ is a continuous function and obeys one of the following conditions:

(a) $\lambda(t)$ approaches a FP λ^*, i.e., $\beta(\lambda^*) = 0$, as $t \to \infty$ such that $\beta(\lambda)/(\lambda^* - \lambda)$ is bounded as $\lambda \to \lambda^*$.

(b) $\lambda(t)$ increases without bound as $t \to \infty$ but $\beta(\lambda)/\lambda$ is bounded in the same limit.

In the first case, let us suppose that the beta function behaves as in Fig. 2.1 and consider $\lambda < \lambda^*$. Because of the assumption (a), $(\lambda^* - \lambda)/\beta(\lambda) \geq c$, where c is some positive constant, thus Eq. (2.4) can be rewritten as

$$t = \int_{\lambda_0}^{\lambda} \frac{\mathrm{d}\lambda}{\lambda^* - \lambda}\frac{\lambda^* - \lambda}{\beta(\lambda)} \geq -\frac{1}{c}\ln(\lambda^* - \lambda) + \text{const.}, \tag{2.5}$$

which tells us that $t \to \infty$ while λ approaches λ^* from below. The same considerations hold for $\lambda > \lambda^*$. On the other hand, the second case can be understood by introducing the inverse coupling $\varpi = \lambda^{-\epsilon}$ (with ϵ positive and arbitrarily small). Its beta function then reads

$$\frac{\partial}{\partial t}\varpi = -\epsilon\varpi\frac{\beta(\lambda)}{\lambda}, \tag{2.6}$$

and vanishes at $\varpi = 0$ because of the hypothesis (b). The running coupling $\lambda(t)$ diverges but the related one $\varpi(t)$ approaches a trivial FP in the UV limit. Both conditions (a) and (b) can then be summarized by the following statement:
A theory, to be nontrivial at any finite momentum scale, should possess necessarily an ultraviolet fixed point.
Depending on the value of λ^*, we distinguish two different scenarios:

- *asymptotic freedom* in case λ^* is the Gaußian fixed point, i.e., $\lambda^* = 0$,
- *asymptotic safety* in case λ^* is an interacting fixed point, i.e., $\lambda^* \neq 0$.

Let us now give some didactic examples of triviality.

Triviality in Quantum Electrodynamics

The beta function of the fine structure constant α in QED can be found in any introductory textbook of QFT. The agreement with the experimental results at a low energy regime is astonishing and represents one of the great successes of perturbation theory. Recent experimental results can be found in [125, 126]. The leading one-loop contribution in α is [75, 127]

$$\beta(\alpha) = \frac{2}{3\pi}\alpha^2 + \dots \qquad \Longrightarrow \qquad \alpha(t) = \frac{\alpha_0}{1 - \frac{2\alpha_0}{3\pi}t}, \tag{2.7}$$

where α_0 fixes the renormalized coupling at the renormalization scale μ_0^2. From the latter equation it is evident that a pole in the running coupling appears at the finite energy scale $\mu = \mu_0 e^{3\pi/(2\alpha_0)}$. The presence of this “Landau ghost” persists in the perturbative domain also by adding higher loop corrections. One might therefore conclude that QED is “trivial” in the sense that the theory can be considered consistent only for $\alpha_0 = 0$.

The physical explanation for the increase of the QED coupling constant with the energy scale can be understood qualitatively by considering the polarization of the vacuum: quantum effects constantly induce creation and annihilation of electron-positron pairs in the vacuum. These virtual particles eventually form a cloud around any electric charge placed in the vacuum such that the effective charge seen by a probe is screened at large distances. However, a disaster occurs when this effective electric charge becomes infinite at a finite, even though large momentum scale, i.e., at small distances.

Triviality in Scalar Field Theories

Another example for such a mechanism is provided by a pure scalar field theory in $d = 4$ dimensions with a pure $\lambda\phi^4/4!$ self-interaction term. In fact, for a one-component real scalar field, the one-loop beta function is positive [128–131]

$$\beta(\lambda) = \frac{3}{16\pi^2}\lambda^2 + \dots \qquad \Longrightarrow \qquad \lambda(t) = \frac{\lambda_0}{1 - \frac{3\lambda_0}{16\pi^2}t}, \tag{2.8}$$

such that the same conclusions as for the QED case can be drawn. The situation is not altered even when N copies of the same scalar field are considered. In this case the beta function reads

$$\beta(\lambda) = (N+8)\lambda^2/(48\pi^2) + \dots, \tag{2.9}$$

with a larger positive one-loop coefficient since more scalar degrees of freedom can run inside the loop.

One might doubt about the true existence of the "Landau ghost" as the lowest order approximation for $\beta(\lambda)$ in perturbation theory is clearly valid only in the regime of small coupling constant, i.e., $\lambda \ll 1$. However a counter example which invalidates this critic is given by the large N limit of an $\mathrm{O}(N)$ model. For an interaction term like $\lambda(\phi^a\phi^a)^2/(4!N)$, the beta function for λ, at the leading order in N^{-1}, is exactly the same as in Eq. (2.8). Moreover, this result is obtained by resumming an infinite series of loop diagrams to all orders in λ, thus being valid even outside the regime $\lambda \ll 1$. Strong evidence for triviality in $d = 4$ has been collected also by lattice simulation [132–138] as well as by FRG studies [139]. We can therefore conclude that a pure scalar field theory in $d = 4$ dimensions, even though it is a perturbatively renormalizable model, cannot be a fundamental QFT valid up to arbitrary scales unless it is trivial, i.e., non interacting.

2.2. Standard Model as an Effective Field Theory

As we anticipated in the Introduction, the SM of particle physics is composed of two subsectors, one describing the electroweak interaction and the other one describing the strong interaction. The former one is a unified description of the electromagnetic and weak forces in terms of a local gauge symmetry based on the $\mathrm{SU}(2)_\mathrm{W} \times \mathrm{U}(1)_\mathrm{Y}$ group [4–6]. In order to account for the massive gauge bosons $W^\pm_\mu$ and Z^0_μ, which are the mediators of the weak force, the latter gauge group is spontaneously broken by the Brout-Englert-Higgs-Kibble mechanism [11–14]. The second subsector is represented by QCD, which also unifies the strong interaction as a gauge theory where the force is mediated by gauge bosons (the gluons). The gauge symmetry of QCD is represented by the internal "color" $\mathrm{SU}(3)_\mathrm{c}$ group. Therefore, in its final formulation, the SM of particle physics is described by a QFT based on a Yang-Mills

Vector Fields		Fermionic Fields		Scalar Fields	
		Quarks			
G^I_μ	$(\mathbf{8},\mathbf{1},0)$	$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(\mathbf{3},\mathbf{2},1/3)$	$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$(\mathbf{1},\mathbf{2},1)$
W^i_μ	$(\mathbf{1},\mathbf{3},0)$	u_R	$(\mathbf{3},\mathbf{1},4/3)$		
B_μ	$(\mathbf{1},\mathbf{1},0)$	d_R	$(\mathbf{3},\mathbf{1},-2/3)$		
		$\ell_L = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}$	$(\mathbf{1},\mathbf{2},-1)$		
		e_R	$(\mathbf{1},\mathbf{1},-2)$		
		Leptons			

Fig. 2.2.: The Standard Model field content with the respective quantum numbers for the gauge group components $(\mathrm{SU}(3)_c, \mathrm{SU}(2)_L, \mathrm{U}(1)_Y)$. For simplicity we have reported only one fermionic generation instead of three, corresponding to the up/down quarks and the electron/electron-neutrino leptons. The other two generations exhibit the same quantum numbers. Among the fermionic fields we have quarks, the only ones charged under $\mathrm{SU}(3)_c$, and leptons.

theory structured around the local gauge symmetry of Eq. (1.4).

In order to construct a Lagrangian density invariant under the local gauge group $\mathcal{G}_{\mathrm{SM}}$, it is necessary to account for several vector bosons associated to each gauge group. These fields of spin (or helicity in the case of massless representations) 1 have different quantum numbers with respect to $\mathcal{G}_{\mathrm{SM}}$, as shown in Fig. 2.2. The gluons G^I_μ, mediators of the strong interaction, form an octet with respect to $\mathrm{SU}(3)_c$, belonging therefore to the adjoint representation of the "colored" group. The same holds for the W^a_μ bosons, mediator of the weak force, which form a triplet being in the adjoint representation of $\mathrm{SU}(2)_L$. [4] Last but not least, is the vector boson B_μ, mediator of the "electromagnetic" force, which is a singlet with respect to the non-Abelian gauge groups.

In addition to the vector bosons, the particle content of the SM is composed of fermionic fields of spin 1/2 with different representations of $\mathcal{G}_{\mathrm{SM}}$. The fermions are divided into three generations and for simplicity, in Fig. 2.2 (middle column), we have represented only the first one. Each generation is further divided into quarks and leptons. The quarks are charged with respect to the strong interaction, while the leptons are sensible only to the electroweak force. Notice that only the left-handed components of these fermionic fields are structured as doublets under $\mathrm{SU}(2)_L$, whereas the right-handed components are singlets with respect to the electroweak group. This feature is ultimately related to the chiral symmetry of the SM and, as a consequence, to the fact that Dirac-like mass terms are forbidden since the latter

[4] For a general non-Abelian SU(N)-group, the adjoint representation has dimension $N^2 - 1$, while the smallest nontrivial representation is the fundamental one with dimension N.

would involve scalar combinations of left- and right-handed spinors. [5]

The SM field content also includes a scalar field of spin zero in the fundamental representation of $SU(2)_L$, in order to provide for the Higgs mechanism. Namely, the electroweak gauge group breaks down into the electromagnetic $U(1)_Q$ group, provided that the scalar field acquires a *vev*. This mechanism is responsible for the mass generation of the W^i_μ gauge bosons without spoiling the unitarity of the theory; as well as for the mass generation of leptons and quarks coupled with the scalar condensate via Yukawa channels. In the SSB regime, B_μ and the third component W^3_μ can be rotated into the photon field A_μ (massless) and the neutral vector boson Z^0_μ (massive) through a rotation matrix whose angle is called the Weinberg or weak-mixing angle. A complex linear combination of the first and second components W^1_μ and W^2_μ defines instead the charged vector bosons $W^\pm_\mu$.

The vector fields can be thought of as the mediators for the interactions since they enter in the appropriate definition of the covariant derivatives, acting on the matter fields charged under the $\mathcal{G}_{\mathrm{SM}}$ gauge group. In fact we have (in Euclidean spacetime)

$$D_\mu = \partial_\mu + \bar{g}_s \sum_{I=1}^{8} G^I_\mu T^I + \bar{g} \sum_{a=1}^{3} W^i_\mu t^i + \bar{g}_Y \frac{Y}{2} B_\mu, \tag{2.10}$$

where $\bar{g}_s$, $\bar{g}$ and $\bar{g}_Y$ are the (bare) couplings for each of the three interactions, and the matrices T^I, t^i are the generators for a gauge transformation under $SU(3)_c$ or $SU(2)_L$ respectively. [6] Given the general definition of covariant derivative in the latter Eq. (2.10), we can construct gauge-invariant kinetic terms for all matter fields. For the scalar field it would be $\mathcal{L}^{\phi}_{\mathrm{kin}} = (D_\mu\phi)^\dagger(D_\mu\phi)$ and for a left-handed quark it would be $\mathcal{L}^{q_L}_{\mathrm{kin}} = \bar{q}_L i\gamma^\mu D_\mu q_L$ (a similar expression holds true also for the leptons and the corresponding right-handed fermions).

The Higgs mechanism is guaranteed by the presence of Yukawa-type interaction terms between the scalar field ϕ and the quarks and leptons. As an example, let us consider only one fermionic flavor as reported in Fig. 2.2. For quarks, gauge-invariant Yukawa terms can be constructed with both up and down quarks; while for leptons, due to the absence of right-handed neutrinos, [7] only the Yukawa interaction with the electron field can be constructed.

[5] A Dirac fermion ψ_D is indeed made of a left- and a right-handed Weyl spinor such that $\psi_D = (\psi_L, \psi_R)^T$. For such a field it is possible to construct a Lorentz invariant mass term which is $\bar{\psi}_D\psi_D = \psi^\dagger_R\psi_L + \psi^\dagger_L\psi_R$, involving therefore a mixing between different chiralities.

[6] Of course Eq. (2.10) simplifies in case the matter field, the covariant derivative is acting on, is charged only under a subgroup of $\mathcal{G}_{\mathrm{SM}}$. For example the scalar field ϕ does not have a direct interaction term with the gluons G^I_μ, since it is uncharged with respect to $SU(3)_c$. Quite in general, the appropriate definition of a covariant derivative D_μ is such as that when acting on a matter field f, it should possess the same transformation rule as f itself. In other words, if $f' = Uf$ under a gauge transformation, then $(D_\mu f)' = U(D_\mu f)$. This is essential in order to be able to construct gauge-invariant kinetic terms. For example, if f belongs to the fundamental representation of $SU(2)_L$, then its finite gauge transformation is $f' = U_\alpha f$, where U_α is the local unitary matrix obtained from the exponential map $U_\alpha = \exp\left(i\bar{g}\alpha_i(x)t^i\right)$. The spacetime dependent functions $\alpha_i(x)$ are the local gauge functions and the infinitesimal generators of the corresponding Lie group are $t^i = \sigma^i/2$, where σ^i are the Pauli matrices.

[7] This is also called "minimal" SM which is justified only in case the neutrino masses are neglected. However, the observed neutrino oscillations among different flavors provide an evidence for their non zero (yet very

Therefore we can write (in Euclidean spacetime)

$$\mathcal{L}^{\mathrm{u,d}}_{\mathrm{Yukawa}} = \mathrm{i}h_{\mathrm{d}}\,(\bar{q}_{\mathrm{L}}\phi)d_{\mathrm{R}} + \mathrm{i}h_{\mathrm{u}}\,(\bar{q}_{\mathrm{L}}\phi^c)u_{\mathrm{R}} + \mathrm{h.c.}, \qquad \mathcal{L}^{\mathrm{e}}_{\mathrm{Yukawa}} = \mathrm{i}h_{\mathrm{e}}\,(\bar{\ell}_{\mathrm{L}}\phi)e_{\mathrm{R}} + \mathrm{h.c.}, \tag{2.11}$$

where $\phi^c = \mathrm{i}\sigma_2\phi^*$ is the charged-conjugated scalar field and h_{u}, h_{d} and h_{e} are the Yukawa couplings for the up quark, down quark and electron respectively. The inclusion of all quark flavors implicates a richer structure in the Yukawa sector, since interactions among different flavors are possible. In this most general case, the couplings h_{u} and h_{d} become matrices with flavor indices. As a consequence, also the quark-mass matrix becomes off-diagonal, which nevertheless can be diagonalized by a bi-unitary transformation. In the corresponding mass eigenstates (or physical bases) the kinetic terms, mass terms and Yukawa matrices are all diagonal in flavor space; only a nontrivial mixing matrix remains in the interactions terms between quarks and charged electroweak vector bosons $W^{\pm}_{\mu}$, the Cabibbo-Kobayashi-Maskawa matrix.

In case the scalar field acquires a nontrivial *vev*, the Yukawa terms in the previous equation induce masses to quarks and leptons as a result of their interaction with the condensate. Suppose for example to choose the parameterization $\phi = (0, v + H)/\sqrt{2}$ where H represents the Higgs excitation about the nontrivial minimum. Then the mass terms can be read off from Eq. (2.11)

$$\mathrm{i}\frac{h_{\mathrm{d}}v}{\sqrt{2}}\left(\bar{d}_{\mathrm{L}}d_{\mathrm{R}} + \bar{d}_{\mathrm{R}}d_{\mathrm{L}}\right) + \mathrm{i}\frac{h_{\mathrm{u}}v}{\sqrt{2}}\left(\bar{u}_{\mathrm{L}}u_{\mathrm{R}} + \bar{u}_{\mathrm{R}}u_{\mathrm{L}}\right) + \mathrm{i}\frac{h_{\mathrm{e}}v}{\sqrt{2}}\left(\bar{e}_{\mathrm{L}}e_{\mathrm{R}} + \bar{e}_{\mathrm{R}}e_{\mathrm{L}}\right), \tag{2.12}$$

which are exactly the Dirac mass terms for the up/down quarks and the electron.

Perturbative RG Equations

After this brief overview on the main structure of the SM, we want to present now the RG running of its coupling constants in order to understand whether the SM can be considered as a UV complete QFT or not. In order to simplify the discussion, we reduce the number of couplings to the three gauge couplings, the top-Yukawa and the quartic scalar self-interaction couplings. This simplification is justified from the fact that the values for the other Yukawas are negligible compared to the large value of the top-Yukawa coupling. Moreover, the qualitative picture important for our discussion does not change. Therefore let us consider the one-loop beta functions computed, for example, in the $\overline{\mathrm{MS}}$ scheme in dimensional regularization

$$\partial_t g_{\mathrm{s}}^2 = -\frac{7}{8\pi^2}g_{\mathrm{s}}^4, \qquad \partial_t g^2 = -\frac{19}{48\pi^2}g^4, \qquad \partial_t g_{\mathrm{Y}}^2 = \frac{41}{48\pi^2}g_{\mathrm{Y}}^4, \tag{2.13}$$

small) masses. The inclusion of right-handed neutrinos too, would provide a "minimal-effort" extension of the SM without compromising its basic structure.

$$\partial_t h_{\rm t}^2 = \frac{h_{\rm t}^2}{16\pi^2}\left(9h_{\rm t}^2 - 16g_{\rm s}^2 - \frac{9}{2}g^2 - \frac{17}{6}g_{\rm Y}^2\right), \tag{2.14}$$

$$\partial_t \lambda = \frac{1}{16\pi^2}\left(12\lambda^2 + 12\lambda h_{\rm t}^2 - 12h_{\rm t}^4 - 3\lambda\left(3g^2 + g_{\rm Y}^2\right) + \frac{9}{4}g^4 + \frac{3}{4}g_{\rm Y}^4 + \frac{3}{2}g^2 g_{\rm Y}^2\right), \tag{2.15}$$

where the normalization of the quartic self-interacting scalar coupling is $\lambda\phi^4/8$. At the current state of the art, the beta functions for the top-Yukawa and scalar self interaction are computed up to three-loop [140–144], and the beta functions for the gauge couplings at four-loop order [145–147]. [8]

In Fig. 2.3 (left panel) we show the RG running for the gauge couplings, the top/bottom-Yukawa couplings and the scalar quartic coupling λ, as reported in [153]. [9] As a first observation, we can notice that all couplings remain perturbatively small up to the Planck scale $M_{\rm Pl} \simeq 10^{19}$ GeV. Moreover, the running of λ shows a very important behavior: it decreases with the energy and crosses the point $\lambda = 0$ at a scale of about 10^{10} GeV. Subsequently it runs very slowly as if it were in a "walking-regime", eventually becoming positive again above $M_{\rm Pl}$. The fact that the scalar coupling λ becomes negative is often consider as a manifestation of a phase transition for the scalar potential from a stable regime (where the potential is bounded and possesses only one minimum at the electroweak scale) to a metastable state (where the potential acquires a second deeper minimum at higher energy) or even to a unstable regime (where the potential is unbounded).

The vacuum stability problem, namely the fate of the scalar potential at high energy scales, is strongly determined by the physical mass parameters for the Higgs and top-quark particles. As depicted in the middle and right panels of Fig. 2.3, it is clear that these masses are bounded within certain ranges in order for the scalar potential to be stable (up to a given UV cutoff scale). In fact, for a given cutoff scale and fixed value for the top mass, the Higgs mass has to exceed a lower bound in order to avoid a metastability [64, 66, 153–156], and

[8] Notice that here and in the following of this thesis we do not consider the fact that the perturbative beta functions of the SM (or extensions thereof) have a natural hierarchy due to the Weyl consistency conditions [148–152]

$$\frac{\partial \beta_i}{\partial g_j} = \frac{\partial \beta_j}{\partial g_i}.$$

A consistent solution of these equations requires that any gauge coupling should have the highest order in the loop expansion, while the Yukawa couplings and the quartic scalar interactions should be considered at one and two orders less respectively. So far it is not known how to implement these Weyl consistency conditions also in nonperturbative approaches like, for example, in FRG.

[9] In computing the RG flow of the SM parameters, the authors in [153] used the three-loop RG equations together with two-loop matching conditions for the initial values at the top pole mass. With respect to our notation, the couplings shown in Fig. 2.3 are normalized as $g_1 = g_{\rm Y}\sqrt{5/3}\sqrt{2}$, $g_2 = g\sqrt{2}$, $g_3 = g_{\rm s}\sqrt{2}$, $y_{t/b} = h_{\rm t/b}\sqrt{2}$. The extra prefactor of $\sqrt{5/3}$ in front of $g_{\rm Y}$ comes from the standard normalization used in SU(5) GUT theories. In fact, from Fig. 2.3, it is possible to see a partial unification of the gauge couplings g_1, g_2 and g_3 at a scale of about 10^{16} GeV. This "unification" is within the 10% of accuracy and may be viewed as an indication for an underlying GUT theory. Studies in this direction have been conducted, indeed, by considering scenarios where the SM gauge group $\mathcal{G}_{\rm SM}$ comes from the breaking of an high-energy SU(5) gauge theory.

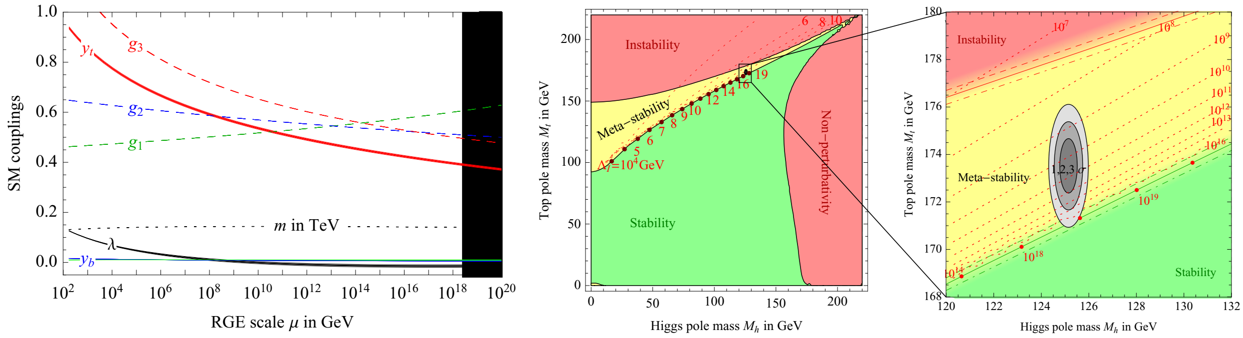


Fig. 2.3.: From [153]. *Left Panel:* RG running of the SM parameters up to the Planck scale $M_{\rm Pl} \simeq 10^{19}$ GeV. Within our notation $g_1 = g_{\rm Y}\sqrt{10/3}$, $g_2 = g\sqrt{2}$, $g_3 = g_{\rm s}\sqrt{2}$ and $y_{\rm t} = h_{\rm t}\sqrt{2}$ (also the bottom-Yukawa coupling $y_{\rm b}$ is represented). The numerical results have been obtained using three-loop RG equations and two-loop matching conditions for the initial conditions at the top pole mass. *Middle and Right Panels:* phase diagram of the SM in terms of the physical mass parameters for the Higgs and top-quark masses. The dashed red lines corresponds to the instability scale where the conventional perturbative running coupling λ turns negative.

has to be smaller than an upper bound in order not to encounter a Landau pole singularity. These bounds are usually computed, in a standard perturbative approach, by considering an RG-improved effective potential of the form $U_{\rm eff} \sim \lambda_{\rm eff}(\mu = \phi)\phi^4$, where the effective quartic self-interaction is turned to negative values due to the fermionic fluctuations. Currently, according to the present experimental results [155], the physical Higgs mass seems to violate the lower bound within 2.8 σ-accuracy [153].

However, the assumptions of standard perturbation theory can be too restrictive, since a full comprehensive study of the Higgs mass bounds should usually involve nonperturbative methods. In fact, lattice simulations have shown that the presence of an instability scale is questionable and it can disappears once the underlying microscopic theory is well-defined [47, 157–163]. In particular, the lower bound arises from the mere criterion of having a physical meaningful bare action defined on the lattice, and no reference to a low-energy stability issue has to be made. Recently, also the functional methods have reinforced the lattice results by showing that, the Higgs-mass bounds arise from consistency conditions imposed on the bare action S_Λ, defined at a finite UV cutoff scale Λ [164–172]. For example, it has been shown that the presence of higher-dimensional operators in the bare scalar potential can modify the RG running close to Λ, leading to a relaxation of the conventional lower bound. In general, determining the consistency bounds for the Higgs mass still remains an open question, and very much depends on the consistency assumptions chosen for the microscopic bare action.

The flow reported in Fig. 2.3 does not show the RG running of the SM couplings beyond $M_{\rm Pl}$. This is usually motivated by the belief that, above that scale, some gravitational effects are important and should be included in order to eventually describe all fundamental forces in Nature. Even though it may be an incomplete viewpoint, we restrict our interest only on the matter sector and ignore any gravitational force. Demanding perturbativity up to arbitrary scales, as observed in [153], the hypercharge coupling g_Y hits a Landau pole at about 10^{42} GeV. Since the pure gauge contributions to the beta function $\partial_t \lambda$ are positive, cf. Eq. (2.15), it is inevitable that also the quartic scalar coupling presents a divergence too. It may look like that the Landau pole in the scalar sector is triggered by the singularity in the gauge sector. However, even in the absence of the abelian $U(1)_Y$ gauge sector, the scalar sector of the SM features a Landau pole due to the positive contribution of the pure scalar loop. In the language adopted in Sec. 2.1, we can conclude that the SM represents another example of a trivial QFT. The occurrence of Landau poles at finite energy scales signals the failure of perturbation theory and endorses a scale of maximal extension of the theory itself. Therefore the SM cannot be trusted as a fundamental QFT description of Nature, but only as an effective field theory description of it valid only at electroweak scales.

While presenting the RG behavior of the SM, we focused mainly on two issues, namely the near-criticality of the scalar potential below M_{Pl} and the presence of Landau poles above M_{Pl}. For the purposes of this thesis, these two problems are of our main concern, even though they are not the only issues. [10] In the following of this thesis, we will try to convince the reader that these issues may be overcome once some of the implicit assumptions of standard perturbation theory are relaxed. In particular, we will present routes for solving the triviality problem of the Higgs sector of the SM, and also the metastability of the Higgs potential as a byproduct of the latter. While presenting these solutions, we restrict ourselves to the matter-field content of the SM, or subsector of it; thus we do not advocate the presence of BSM matter. Moreover, we also neglect any gravitational contributions.

2.3. Functional Renormalization Group

In the previous section we have presented the triviality problem for some QFTs, and in particular for the SM of particle physics. Since the occurrence of a Landau pole is intrinsically related to the failure of perturbation theory, a comprehensive study of this issue, and how it could be cured, should naturally require some nonperturbative tools. In this respect, the major technique which we are going to apply in the next chapters will be the functional renormalization group (FRG). Therefore let us briefly present here this machinery which combines functional methods with the modern Wilsonian interpretation of the RG.

[10] Another issue is the hierarchy problem, also named as naturalness problem. It corresponds to an enormous fine-tuning of initial conditions for the Higgs mass parameter m_Λ^2, in order to have a large separation between the electroweak scale (or Fermi scale) and the cutoff scale Λ. For example, if the cutoff scale of the SM is chosen to be that of a GUT theory, the value m_{GUT}^2 need to be fine-tuned within a precision of $\Lambda_{EW}^2/\Lambda_{GUT}^2 \sim 10^{-28}$ from its critical value. This naturalness problem stems from the fact that the Higgs mass renormalizes quadratically with respect to the cutoff scale. Moreover, the SM also has a non-explained hierarchy in the values of the Yukawa couplings, which reflects the same hierarchy upon the fermionic masses. The values for these parameters are fixed only by the experiments and there is not a clear reasoning for their structure.

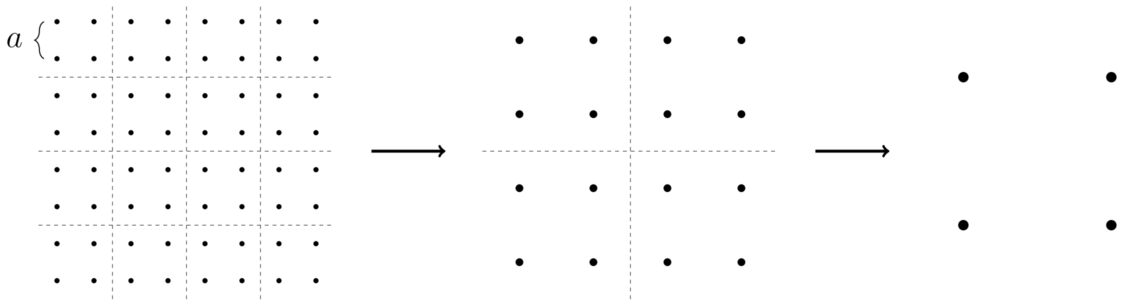


Fig. 2.4.: Diagrammatic interpretation of the Kadanoff's idea of averaging over blocks of spins.

2.3.1. Underlying Philosophy

The major intuition of the Wilsonian approach to the renormalization group [173, 174] is based on the hypothesis that, for a system with a large or even an infinite number of degrees of freedom (like in field theories), the dynamics of the system can be described by fewer degrees of freedom at the expense of a change in the interactions. As a consequence the original theory, defined usually at a certain UV cutoff scale Λ and expressed in terms of an action S_Λ, is mapped onto a sequence of theories, each with fewer degrees of freedom than its predecessor, and each characterized by a new set of coupling constants. The sequence of such sets of coupling constants defines a flow as the number of degrees of freedom is steadily reduced. In fact, the action for the theory changes its functional form along the flow, and in general an infinite set of couplings is generated.

This change of the action is described by *exact flow equations*, whose first formulations have been provided by Wegner and Houghton [175], Wilson [174, 176] as well as Polchinski [177]. In this thesis we use a more modern formulation based on the Effective Average Action (EAA) Γ_k formulated by Wetterich [178–181]. The basic idea behind all these formulations is that the quantum fluctuations associated to a momentum $q \leq \Lambda$ are integrated out not all at once, but momentum shell by momentum shell [174, 182]. [11] It is thus necessary to introduce, by hand, an intermediate momentum scale k, such that only the "high frequency" momenta $k \lesssim q \leq \Lambda$ (also called "fast" modes) are integrated out. As a consequence we obtain an *effective action* S_k which describes effectively the dynamics of the system at the energy scale k.

The above procedure of progressively integrating out degrees of freedom has a qualitative interpretation if we think about the Kadanoff's idea [183] to describe the Ising model at a critical point where the correlation length diverges. Consider a system of spins localized on a two dimensional square lattice with lattice spacing a as in Fig. 2.4. A small but finite lattice spacing morally corresponds to a finite but large cut-off Λ for a microscopic field theory.

[11] This momentum integration is performed with the introduction of an IR cutoff in the formulation for the EA, or the introduction of a UV cutoff in the earlier formulations by Wegner, Houghton, Wilson and Polchinski.

Near a critical point, the correlation length is very large and spins which are close to each other are strongly correlated. As a consequence, if we divide the plane into blocks of four spins, then the spins in each block act much like a single spin, and the original lattice can be replaced by an effective lattice where the interactions are between blocks of spins rather than between the spins themselves. This process of averaging can be iterated and it is also named "coarse-graining" procedure. Let us remark an important point of this discussion; a continuum limit of a lattice theory must occur at a phase transition of (at least) second order, where a dimensionless correlation length diverges. In other words such continuum limit exists if and only if a FP of the coarse-graining transformation exists.

Going back to the language of field theory, lowering the k scale for the effective action S_k corresponds morally to averaging over more and more spins, and the limit $k \to 0$ corresponds to the case where all quantum fluctuations are taken into account. The microscopic action S_Λ defined at some cutoff scale Λ is totally arbitrary, and can even contain higher dimensional operators which are not renormalizable from the naive power-counting analysis. After the integration of the higher-momentum fluctuations has been performed; at a momentum scale much lower than Λ, the perturbatively nonrenormalizable operators of the original action can be accounted for by a redefinition of the renormalized coupling constants at the lower scale. Specifically, the nonrenormalizable interactions correspond to irrelevant directions in the flow space of coupling constants. On the other hand, it is well known that the number of renormalizable couplings in a field theory equals the number of relevant directions (or directions of instability) at a fixed point [173, 184]. The reason is that for a UV complete field theory to exist it must possess a FP such that the limit $\Lambda \to \infty$ can be taken; while this limit is taken it is necessary to adjust as many renormalized parameters as there are relevant directions in order to approach the critical surface the FP lays on. Therefore the critical surface in a theory space has as many degrees of instability as there are renormalized coupling constants.

For the case of the Gaußian fixed point, this is a rephrasing of *Weinberg's theorem* which says that a Feynman graph gives rise to a convergent integral if it is convergent by power counting in all its sectors. If the number of different type of Feynman graphs which need to be regularized is finite then the theory is renormalizable [185]. In our functional language this corresponds exactly to the requirement that only a finite number of relevant directions exist.

Linearized Flow

The difference between relevant and irrelevant directions can be intuitively understood from Fig. 2.5, where we have represented, for simplicity, a two dimensional critical surface embedded in a three dimensional coupling space. The magenta point corresponds to a FP on the critical surface while the three highlighted lines represent three eigendirections of the flow. Arrows are pointing towards higher energies such that the relevant directions, which identify

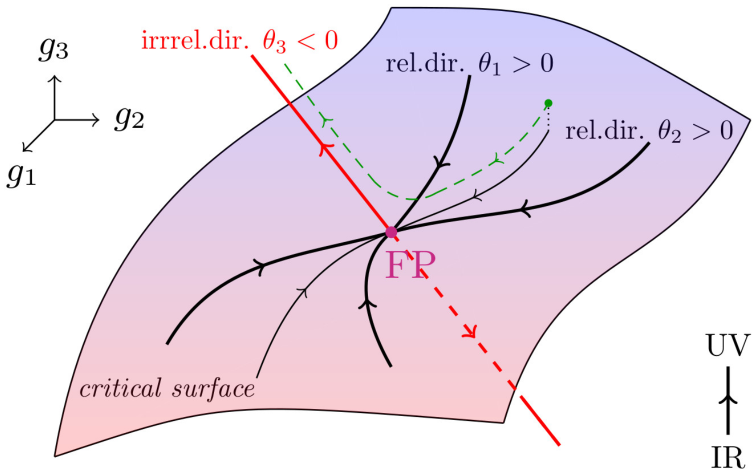


Fig. 2.5.: Example of a two dimensional critical surface embedded in a three dimensional coupling space parameters. Arrows are pointing towards the UV. Thick black lines represent relevant directions (UV attractive) while the red line represents an irrelevant direction (UV repulsive). The green dashed line exemplifies the RG behavior of a trajectory which starts very close to the critical surface in the IR and escapes from the FP while approaching the UV limit.

the critical surface (black solid lines), are attractive in the UV limit. The opposite situation occurs for the irrelevant directions which are repulsive in the UV (red solid line).

As a consequence, only those trajectories belonging to the critical surface can approach the FP in the UV limit, thus corresponding to well defined UV complete field theories. Consider for example an initial condition for S_Λ very close to the critical surface but not belonging to that. Then the trajectory (green dashed line) seems to reach the FP, however, at a certain finite energy scale it will escape from the FP since perturbations along the irrelevant direction become unstable at large energy. Of course the picture will be reversed if we consider the flow towards the IR.

Mathematically speaking, we can describe this mechanism by linearizing the flow around the FP. Suppose we have a set of couplings $\{g_i\}$ and corresponding beta functions $\{\beta_i\}$. For small perturbations about the FP $\{g_i^*\}$, we can Taylor expand the flow at the first order as

$$\beta_i = \beta_i\Big|_{g^*} + \frac{\partial \beta_i}{\partial g_j}\Big|_{g^*} (g_j - g_j^*) + \dots , \tag{2.16}$$

where the collection of the first derivatives $\partial\beta_i/\partial g_j \equiv M_{ij}$ is also called *stability matrix*. The latter can be diagonalized by means of an invertible matrix S such that $S^{-1}MS = D$, where D is a diagonal matrix whose elements are the eigenvalues of M. One typically works with the scaling exponents $\theta_i = -\mathrm{eig}\, S = -D_{ii}$. In fact, by changing the set of variables from

$\{g_i\}$ to $\{z_i\}$ with $z_i = (S^{-1})_{ij}(g_j - g_j^*)$ we obtain

$$\frac{\mathrm{d}z_i}{\mathrm{d}t} = -\theta_i z_i \quad \Longrightarrow \quad z_i(t) \sim \mathrm{e}^{-\theta_i t} \sim \left(\frac{k}{k_0}\right)^{-\theta_i}. \tag{2.17}$$

From the latter expression we can give a rigorous definition of (ir)relevant directions depending on the sign of the related scaling exponents.

- *Negative* scaling exponents $\theta_i < 0$ are associated to *irrelevant* directions since a perturbation z_i along them increases for increasing energies. These trajectories are repelled from the FP for $t \to \infty$, such that they are UV unstable/repulsive and IR stable.

- *Positive* scaling exponents $\theta_i > 0$ are associated to *relevant* directions since a perturbation z_i along them decreases for increasing energies. These trajectories are attracted towards the FP for $t \to \infty$, such that they are UV stable/attractive and IR unstable/repulsive. For example, at the free Gaußian FP, the relevant directions correspond to couplings with positive mass dimension.

- *Null* scaling exponents $\theta_i = 0$ are associated to *marginal* directions and the behavior of a perturbation along them is determined beyond the linear order of Eq. (2.16). The calculation of the second derivatives $\partial^2\beta_i/(\partial g_j \partial g_k)$ is required. Marginal directions are further classified into marginally irrelevant or marginally relevant, according to the fact that perturbations along them respectively increase or decrease for increasing energies.

2.3.2. Exact RG Flow Equation

As previously anticipated, we are interested in the Wetterich formulation of the RG flow, which consists in providing a flow equation for the *Effective Average Action* (EAA) Γ_k [178, 186–191]. The latter functional smoothly interpolates between the microscopic action defined at the cutoff scale Λ, i.e., $\lim_{k\to\Lambda}\Gamma_k = S_\Lambda$, and the full EA which is obtained when all quantum fluctuations for all momenta are integrated out, i.e., $\lim_{k\to 0}\Gamma_k = \Gamma$.

Let us sketch briefly the main ingredients necessary in order to derive the desired flow equation. In a functional approach, the coarse-graining procedure described before can be realized at the level of the partition function $\mathcal{Z}$, by adding to the action S an IR regulated functional ΔS_k, which is dependent on the characteristic averaging scale k. We thus obtain an effective partition function $\mathcal{Z}_k$ [12], dependent on k and on the source J coupled to the

[12] Let us remind the reader that, in a QFT, the partition function $\mathcal{Z}[J] = \int\!\mathcal{D}\varphi\ \mathrm{e}^{-S[\varphi]+J\cdot\varphi}$ incorporates all the correlation functions, which can be obtained by functional differentiation with respect to the source J. Namely $\langle 0|\,\varphi_1 \cdots \varphi_n\,|0\rangle = \mathcal{Z}[0]^{-1}\,\delta^n\mathcal{Z}[J]/(\delta J_1 \cdots \delta J_n)|_{J=0}$. Correlations are all we need in order to compute physical observables.

microscopic quantum field φ, which reads (in Euclidean spacetime)

$$\mathcal{Z}_k[J] = \int_\Lambda \mathcal{D}\varphi \; \mathrm{e}^{-S[\varphi]-\Delta S_k[\varphi]+J\cdot\varphi} \equiv \mathrm{e}^{\mathcal{W}_k[J]}. \tag{2.18}$$

The UV cutoff scale Λ has been explicitly introduced in the functional measure. For simplicity we consider only one scalar field φ, but a straightforward generalization can be done also for generic matter fields. Notice also that the coupling between the source J and the scalar field φ involves an integration over the spacetime, i.e., $J \cdot \varphi = \int \mathrm{d}^d x \, J(x)\varphi(x)$.

From the partition function in the latter equation, we can define the EAA Γ_k as the generating functional for the one-particle irreducible (1PI) correlation functions, by a modified Legendre transformation involving the regulator functional ΔS_k

$$\Gamma_k[\Phi] = \sup_J \left\{ J \cdot \Phi - \mathcal{W}_k[J] \right\} - \Delta S_k[\Phi]. \tag{2.19}$$

At the supremum, the source is a functional of the *classical field* Φ, i.e., $J = J_{\text{sup}} = J[\Phi]$. The classical field is defined as the J-dependent expectation value of the quantum field φ such that

$$\Phi(x) = \frac{\delta \mathcal{W}_k[J]}{\delta J(x)} = \langle \varphi(x) \rangle_J \,. \tag{2.20}$$

We can therefore interpret $\Phi(x)$ as a macroscopic field whose dynamics is governed by Γ_k.

The insertion $\Delta S_k[\varphi]$ in Eq. (2.18) is chosen to be quadratic in the quantum field φ,

$$\Delta S_k[\varphi] = \frac{1}{2} \int \mathrm{d}^d x \, \varphi(x) R_k(x,y) \varphi(y) = \frac{1}{2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \, \varphi(-p) R_k(p^2) \varphi(p), \tag{2.21}$$

such that it can be viewed as a momentum-dependent mass term for φ. This is precisely the reason why the FRG approach, differently from the $\overline{\text{MS}}$ scheme based on dimensional regularization, represents a mass-dependent scheme. In order to correctly implement the averaging procedure of the "fast" modes, and the correct limits for Γ_k, the regulator function R_k should satisfy the following relations:

$$\lim_{p^2/k^2 \to 0} R_k(p^2) > 0, \qquad \lim_{k^2/p^2 \to 0} R_k(p^2) = 0, \qquad \lim_{k^2 \to \Lambda \to \infty} R_k(p^2) \to \infty. \tag{2.22}$$

The first relation represents the IR regularization since the "slow" modes, i.e., small p^2, are screened in a mass-like fashion. If you consider for example the piece-wise linear regulator [192, 193], $R_k(p^2) = (k^2 - p^2)\theta(k^2 - p^2)$, then $R_k(p^2) \sim k^2$ for $p^2 \ll k^2$ such that these modes appear like having a mass $m^2 \sim k^2$. The second condition guarantees the correct limit $\Gamma_{k \to 0} = \Gamma$, which means that for $k = 0$ all the quantum fluctuations are integrated out and the full EA is restored. The third condition instead ensures that the EAA reduces, in the deep UV limit, to the bare action evaluated on the classical field configuration.

Derivation

Now we are in the position to derive the exact RG flow equation for Γ_k [178–181]. From the definition of the k-dependent partition function $\mathcal{Z}_k$ and EAA Γ_k, we have that

$$e^{-\Gamma_k[\Phi]} = \int \mathcal{D}\chi \; e^{-S[\Phi+\chi]-\Delta S_k[\Phi+\chi]+\Delta S_k[\Phi]+J\cdot\chi}, \tag{2.23}$$

where we have changed the integration variable from φ to the fluctuation χ around the classical field configuration defined as $\chi = \varphi - \Phi$. Since the regulator functional is quadratic in the field we can write

$$\Delta S_k[\Phi+\chi] = \Delta S_k[\Phi] + \Delta S_k[\chi] + \chi \cdot \frac{\delta \Delta S_k}{\delta \Phi}. \tag{2.24}$$

Moreover, from Eq. (2.19) we obtain an expression for the source J as a functional derivative

$$J[\Phi(x)] = \frac{\delta}{\delta\Phi(x)} \left(\Gamma_k[\Phi] + \Delta S_k[\Phi]\right), \tag{2.25}$$

such that, by substituting the latter two expressions into Eq. (2.23), we obtain an integral-differential equation for the EAA

$$e^{-\Gamma_k[\Phi]} = \int \mathcal{D}\chi \; e^{-S[\Phi+\chi]-\Delta S_k[\chi]+\chi\cdot\frac{\delta\Gamma_k}{\delta\Phi}}. \tag{2.26}$$

Applying the RG-time derivative $\partial_t = k\partial/\partial k$ (with $t = \log(k/\Lambda)$) to the latter expression we obtain

$$e^{-\Gamma_k[\Phi]} \left(-\partial_t \Gamma_k[\Phi]\right) = \int \mathcal{D}\chi \; e^{(\cdots)} \left\{ -\partial_t \Delta S_k[\chi] + \chi \cdot \partial_t \frac{\delta \Gamma_k[\Phi]}{\delta\Phi} \right\} \tag{2.27}$$

$$= -\frac{1}{2} \int d^d x \int d^d y \;\; (\partial_t R_k(x,y)) \int \mathcal{D}\chi \; e^{(\cdots)} \chi(x)\chi(y) + \int d^d x \left(\partial_t \frac{\delta\Gamma_k[\Phi]}{\delta\Phi(x)} \right) \int \mathcal{D}\chi \; e^{(\cdots)} \chi(x), \tag{2.28}$$

such that

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \int d^d x \int d^d y \;\; [\partial_t R_k(x,y)] \frac{\int \mathcal{D}\chi \; e^{(\cdots)} \chi(x)\chi(y)}{\int \mathcal{D}\chi \; e^{(\cdots)}} + \int d^d x \left(\partial_t \frac{\delta\Gamma_k[\Phi]}{\delta\Phi(x)} \right) \frac{\int \mathcal{D}\chi \; e^{(\cdots)} \chi(x)}{\int \mathcal{D}\chi \; e^{(\cdots)}}, \tag{2.29}$$

where $(\cdots)$ stands for the exponent on the right hand side of Eq. (2.26). We are left with two functional integrals to compute, which are nothing more than the expectation values $\langle \chi(x)\chi(y) \rangle$ and $\langle \chi(x) \rangle$, if we interpret the quantity $\mathcal{P}_\chi \equiv e^{(\cdots)} / \int \mathcal{D}\chi e^{(\cdots)}$ as densityprobability functional for the fluctuation field χ.

By construction $\langle\chi(x)\rangle = \int \mathcal{D}\chi \mathcal{P}_\chi \chi(x) = 0$ since the field χ is the fluctuation around the average field configuration Φ. Indeed, from the very definition of the latter in Eq. (2.20), we have

$$\Phi(x) = \frac{1}{\mathcal{Z}_k[J]} \int \mathcal{D}\varphi \; \varphi(x) \; \mathrm{e}^{-S[\varphi]-\Delta S_k[\varphi]+J[\Phi]\cdot\varphi} \equiv \int \mathcal{D}\varphi \; \mathcal{P}_\varphi \, \varphi(x), \tag{2.30}$$

and, by changing the integration variable from φ to $\chi = \varphi - \Phi$ on the right hand side, we obtain that $\Phi(x) = \Phi(x) + \int \mathcal{D}\chi \mathcal{P}_\chi \chi(x)$ (notice that Eqs. (2.25) and (2.24) have been used). By taking the functional derivative of Eq. (2.30) with respect to $\Phi(y)$, we obtain

$$\begin{aligned}
\delta(x-y) &= -\frac{1}{\mathcal{Z}_k[J]} \int \mathrm{d}^d z \frac{\delta \mathcal{Z}_k[J]}{\delta J(z)} \frac{\delta J(z)}{\delta \Phi(y)} \int \mathcal{D}\varphi \; \mathcal{P}_\varphi \, \varphi(x) + \int \mathcal{D}\varphi \; \mathcal{P}_\varphi \, \varphi(x) \int \mathrm{d}^d z \frac{\delta J(z)}{\delta \Phi(y)} \varphi(z) \\
&= \int \mathrm{d}^d z \left(\Gamma_k^{(2)}[\Phi] + R_k\right)(y,z) \left[-\Phi(z)\Phi(x) + \int \mathcal{D}\varphi \; \mathcal{P}_\varphi \, \varphi(x)\varphi(z)\right] \\
&= \int \mathrm{d}^d z \left(\Gamma_k^{(2)}[\Phi] + R_k\right)(y,z) \int \mathcal{D}\chi \; \mathcal{P}_\chi \, \chi(x)\chi(z),
\end{aligned} \tag{2.31}$$

where again a change of the integration variable from φ to χ has been taken. From an operator point of view, the latter identity reads

$$\langle\chi(x)\chi(y)\rangle = G_k^{(2)}(x,y) = \left(\Gamma_k^{(2)}[\Phi] + R_k\right)^{-1}(x,y), \tag{2.32}$$

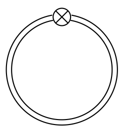
where $G_k^{(2)}(x,y)$ represents the modified (due to the regulator insertion) two-point correlation function, i.e., the propagator of the scalar field φ. We can conclude that Eq. (2.29) reduces to

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \int \mathrm{d}^d x \int \mathrm{d}^d y \; (\partial_t R_k)(x,y) \, G_k^{(2)}(y,x) = \frac{1}{2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \; [\partial_t R_k(p^2)] \, G_k^{(2)}(p^2), \tag{2.33}$$

where the integrals represent the fact that a trace over the spacetime indices, or over the momentum index, has to be taken.

The derivation can be generalized in a straightforward manner for generic matter field content, giving as a result the Wetterich equation (known also as Exact RG Equation or FRG Equation) [178], which in the operator language reads

$$\begin{aligned}
\partial_t \Gamma_k &= \frac{\hbar}{2} \mathrm{STr} \left[(\partial_t R_k) \left(\Gamma_k^{(2)} + R_k\right)^{-1}\right] = \frac{1}{2}
\end{aligned}$$



$$= \frac{\hbar}{2} \mathrm{STr} \, \tilde{\partial}_t \log\left(\Gamma_k^{(2)} + R_k\right), \tag{2.34}$$

where $\tilde{\partial}_t$ acts only on the regulator. In the most general case $\Gamma_k^{(2)}$ is the Hessian matrix

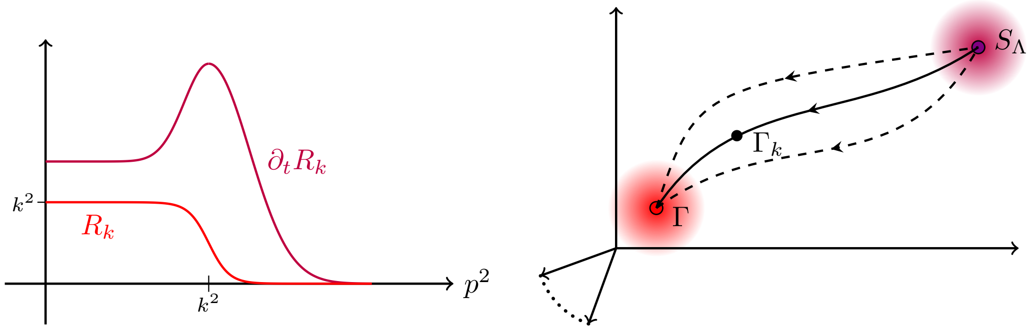


Fig. 2.6.: *Left Panel*: example of a regulator function. The peak in the derivative $\partial_t R_k$ implements the Wilsonian idea of integrating out fluctuation of momentum shell around $p^2 \sim k^2$. *Right Panel*: intuitive representation of RG trajectories in the theory space, connecting a UV microscopic action S_Λ to the IR Effective Action Γ. At an intermediate energy scale, the dynamics is governed by the Effective Average Action Γ_k. The dashed trajectories exemplify the regulator dependence.

with all second functional derivatives with respect to all fields. R_k and $\Gamma_k^{(2)}$ are square $n \times n$ matrices with n the number of independent fields in the theory. The supertrace has to be performed in case of Graßmann-valued fields, which implies a minus sign for each fermionic or ghost loop integrals. On a technical level, the latter equation represents a functional differential equation for Γ_k. Moreover it is exact since it involves the exact full regularized propagator $G_k^{(2)} = \Gamma_k^{(2)} + R_k$ and it has a one-loop structure which can be diagrammatically represented by the Feynman diagram in Eq. (2.34) (the cross represents the insertion $\partial_t R_k$ and the double line the full dressed propagator). The regulator function R_k at the denominator encodes – by construction – the IR regularization, whereas the RG-time derivative at the numerator acts as a UV regularization since it goes rapidly to zero for $p^2 \gg k^2$. In addition the Wilsonian idea of integrating out momentum shell by momentum shell is guaranteed by the fact that the predominant support of $\partial_t R_k$ lies on a smeared momentum shell around $p^2 \sim k^2$ (in this sense we can say that the Wetterich equation is "local in momentum space"). [13] These features are qualitatively depicted on the left panel of Fig. 2.6. Suppose now we have a solution of Eq. (2.34), then this solution corresponds to a specific trajectory in the theory space spanned by an infinite dimensional basis compatible with the underlying symmetries. This trajectory – by construction – has to connect the microscopic bare action S_Λ to the full Effective Action Γ in the IR limit. The precise path of this trajectory is not unique but it depends on the choice of the regulator R_k; a different choice for the latter leads to a different trajectory connecting S_Λ to Γ. This is precisely the reason for the non-universality of the beta functionals obtained within the FRG scheme. The regulator dependence is pictorially represented on the right panel of Fig. 2.6.

[13] In Wilson's discussion of the RG, the basic object is the partition function $\mathcal{Z} = \int \mathcal{D}\phi\, \mathrm{e}^{-S_\Lambda(\phi)}$, where Λ is a UV cutoff and S_Λ an action valid at the scale Λ. Wilson's RG describes how the latter changes when Λ is lowered infinitesimally, in such a way that $\mathcal{Z}$ remains the same. The definition of the EAA is an implementation of the same idea for the 1PI-generating functional.

Truncation and Optimization

Given the microscopic bare action S_Λ, the RG flow usually generates, due to quantum fluctuations, an infinite amount of operators compatible with the symmetries of the system under investigation. As a matter of fact, the exact RG flow equation (2.34) can be solved only if an infinite amount of operators are taken into account, which is certainly beyond any realistic expertise. Therefore some nonperturbative approximation schemes should be devised if we want to make use of Eq. (2.34). A first example is given by the so-called *vertex expansion*, i.e., a series of $\Gamma_k[\Phi]$ in powers of the classical field configuration Φ

$$\Gamma_k[\Phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_n \Gamma_k^{(n)}(x_1, \ldots, x_n) \Phi(x_1) \cdots \Phi(x_n). \tag{2.35}$$

Upon substituting the latter expansion in Eq. (2.34), we obtain an infinite tower of flow equations for all n-point vertex functions $\Gamma_k^{(n)}$. The corresponding system of differential equations can be solved by truncating the series at some order $n = n_{\max}$.

In this thesis we will use, instead, a second type of approximation called *derivative expansion* [14], which consists of expanding the EAA into operators of increasing powers of derivatives, but no approximation is made in the field dependence. As an example, for a real scalar field theory, this implies

$$\Gamma_k[\Phi] = \int \mathrm{d}^d x \left[U_k(\Phi) + \frac{1}{2} Z_k(\Phi)(\partial_\mu \Phi)^2 + (\partial_\mu \Phi)^4 \, Y_{1,k}(\Phi) + (\partial_\mu \partial^\mu \Phi)^2 \, Y_{2,k}(\Phi) + \ldots \right],$$

where the coefficient functions $U_k(\Phi)$, $Z_k(\Phi)$ and $Y_{j,k}(\Phi)$ are arbitrary functionals of Φ. (In the latter equation we have not written all 3 four-derivative terms). The lowest order of the derivative expansion, which is also called *local potential approximation* (LPA), only includes the effective scalar potential $U_k(\Phi)$ and a bare kinetic term, i.e., $Z_k = 1$. The first correction includes a wave function renormalization $Z_k(\Phi)$ as a generic function of the field Φ, the second correction involves invariants with four derivatives, and so on for the higher orders. Within the LPA, the kinetic term keeps its bare (unrenormalized) form and the anomalous dimension of the scalar field is therefore zero. The latter, can nevertheless be accounted for by considering an intermediate approximation between the LPA and the first correction. In consists of including a k-dependent field renormalization factor Z_k, constant with respect to the scalar field. This improvement of the LPA is also called in the literature LPA'.

Generally speaking, any choice of truncation has to be *systematic* and *consistent*. Once a certain order of expansion is chosen then all possible operators admissible up to that order must be retained in the flow. Differently from standard perturbation theory, the

[14] The derivative expansion has been used quite extensively for the RG analysis of the Higgs-potential flow [111, 112, 164–171, 194–198]. Moreover, it has been proven useful and accurate also for Yukawa theories in many contexts [199–204]. Combinations of the derivative expansion and the vertex expansion are also possible.

approximation schemes adopted within the FRG framework, most often, do *not* rely on some small-parameter expansion. For this reason, and due to the rather different nature of the systematic errors in the FRG approaches, the common lore of attributing error bars or addressing the convergence of truncations can be a challenge. One way of trying to do so is to investigate the influence of higher-order operators in the expansion of Γ_k. Whereas such computations can become particularly time-consuming, another more direct check can be performed at the level of regulator dependencies of the results. These spurious scheme-dependent differences from the universal physical results can be used as indicators for the influence of higher-order terms.

Moreover, the freedom of choosing R_k arbitrary, besides the conditions in Eq. (2.22), can become fruitful to optimize the flow. In this sense, a flow is optimized when the RG flow is the "shortest" trajectory in the theory space, i.e, the results for physical observables lie as close as possible to the true results. In this thesis, especially in Chap. 3, we will make use of the piece-wise linear regulator which is optimized at next-to-leading order in the derivative expansion [192, 193]. [15] In Chap. 4, while discussing the universality of our results, we will not be tied to this specific choice of regulator, but we will explore all possible regulators compatible with the FRG framework.

2.3.3. Perturbative Expansion

Perturbation theory can be re-derived from the exact RG flow equation as a perturbative expansion in powers of $\hbar$ [208, 209]. In fact let us consider such an expansion

$$\Gamma_k = S_\mathrm{B} + \sum_{L\geq 1} \hbar^L \Gamma_{L,k}\,, \tag{2.36}$$

where the functional S_B plays the role of the bare action and L is the loop order of the expansion. By substituting the latter expansion into the exact RG flow equation (2.34), we obtain

$$\partial_t S_\mathrm{B} + \sum_{L\geq 1} \hbar^L \partial_t \Gamma_{L,k} = \frac{\hbar}{2}\tilde{\partial}_t \log\left(S_\mathrm{B}^{(2)} + R_k + \sum_{L\geq 1} \hbar^L \Gamma^{(2)}_{L,k} \right), \tag{2.37}$$

such that, by expanding the logarithm in powers of $\hbar$, we reduce the Wetterich equation into a tower of flow equations, one for each loop order L. The first two loop orders are

$$\partial_t S_\mathrm{B} = 0, \tag{2.38}$$

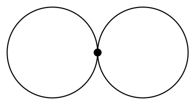
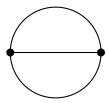
$$\partial_t \Gamma_{1,k} = \frac{1}{2}\,\mathrm{STr}\,\tilde{\partial}_t \log\big(R_k + S_\mathrm{B}^{(2)}\big) = \frac{1}{2}\,\mathrm{STr}\,(G_{\mathrm{B},k}\partial_t R_k), \tag{2.39}$$

[15] For more details about the optimization procedure we refer the reader to [205–207], and for a full functional approach to optimization we refer to [187].

$$\partial_t \Gamma_{2,k} = \frac{1}{2}\,\mathrm{STr}\,\tilde{\partial}_t (G_{\mathrm{B},k}\Gamma^{(2)}_{1,k}), \tag{2.40}$$

where the modified bare propagator $G_{\mathrm{B},k} = (S^{(2)}_{\mathrm{B}} + R_k)^{-1}$ differs from that one in Eq. (2.32), since the EAA is replaced by the bare action. Each right hand sides of the latter system of equations can be brought into the form of a total t-derivative. However, the commutation of the derivative ∂_t with the (super)trace STr usually spoils the UV finiteness of the result. Therefore, a regularization of the trace operator has to be introduced in order for the commutation $\mathrm{STr}\,\partial_t = \partial_t\,\mathrm{STr}_{\mathrm{reg}}$ to be valid. Once such a regularization is provided, the left-hand sides of the latter tower of equations can be integrated. For example, the one and two-loop terms become

$$\Gamma_{1,k} = \frac{1}{2}\mathrm{STr}_{\mathrm{reg}} \log\bigl(S^{(2)}_{\mathrm{B}} + R_k\bigr), \tag{2.41}$$

$$\Gamma_{2,k} = \frac{1}{8}$$  $$- \frac{1}{12}$$  $,\quad (2.42)$

where the lines represent $G_{\mathrm{B},k}$ and the interaction vertices highlighted by black dots are $S^{(4)}_{\mathrm{B}}$ and $S^{(3)}_{\mathrm{B}}$. The functional traces can be regularized for example by means of dimensional regularization, i.e., performing an analytic continuation of the dimension d around the upper critical dimension d_c of the theory, such that, the highest divergences appear as poles of the form $1/\epsilon^L$ with $\epsilon = d - d_c$. Subsequently, one should also perform a renormalization procedure introducing the expansion

$$S_{\mathrm{B}} = S_{\mathrm{R}} + \sum_{L\geq 1} \hbar^L \delta S_L, \tag{2.43}$$

where S_{R} is the tree-level renormalized action and δS_L are the counterterms which need to be added in order to cancel the divergences of $\Gamma_{L,k}$, order by order in $\hbar$. For example, in the minimal-subtraction (MS) scheme, the finite parts of the loop integrals are set to zero such that $\delta S_L = -\Gamma^{\mathrm{div}}_{L,k}$.

A crucial property is that the leading divergence of the Lth loop does not depend on the FRG scale k and $\Gamma^{\mathrm{div}}_{L,k} = \Gamma^{\mathrm{div}}_{L,k=0}$. This fact ensures the existence of an explicit and perturbative map between the FRG method and standard perturbation theory based on $\overline{\mathrm{MS}}$. This map is such that a renormalized coupling within the FRG scheme can be expanded in terms of the renormalized coupling within the $\overline{\mathrm{MS}}$ scheme; at the leading order they coincide while higher terms will encode the mass-dependence of the FRG scheme [210, 211]. In particular, this mass-dependence disappears once the renormalized mass(es) of the theory are set to zero, i.e., the DER limit is taken. In other words

$$\lambda_{\mathrm{FRG}} = \lambda_{\overline{\mathrm{MS}}} + \mathcal{F}(m^2_{\mathrm{R}})\lambda^2_{\overline{\mathrm{MS}}} + \dots, \qquad \mathcal{F}(m^2_{\mathrm{R}}) \xrightarrow[m^2_{\mathrm{R}}\to 0]{} 0, \qquad \beta_{\mathrm{FRG}} \xrightarrow[m^2_{\mathrm{R}}\to 0]{} \beta_{\overline{\mathrm{MS}}}. \tag{2.44}$$

2.4. Asymptotic Freedom

We have already observed at the beginning of this chapter that the triviality problem, intimately related to the existence of a finite cutoff scale of maximal UV extension of a theory, can be circumvented when the theory itself becomes AF or AS in the UV limit. In other words, a QFT becomes UV complete whenever there are either Gaußian or interacting FPs in the theory space, corresponding to the AF and AS scenario respectively.

Asymptotically safe field theories are particularly attractive for studying consistent extensions of nonrenormalizable theories like quantum gravity in $d = 4$ dimension. The first suggestions that an AS scenario for quantum gravity might exist can be traced back to the early work by Weinberg [44], and it has become nowadays a very promising and successful field of research [46, 212–219]. For reviews on this topic see for example [220, 221]. The asymptotic safety program has been successfully applied not only to gravity but also to a number of different models involving four-fermion interactions [222–226], simple Yukawa models [227, 228], nonlinear sigma models in $d > 2$ [229] and extra-dimensional gauge theories [230]. A break-through in the asymptotic safety program, and aside from the functional RG methods, came recently with the work of Litim and Sannino [93], who discovered the existence of interacting fixed points for gauged-Yukawa models which are perturbatively under control. General conditions for the existence of such FPs have been discussed in [96], and applications to BSM physics have also been addressed [94, 97–101, 103, 105, 106].

However, in this thesis we are interested in those UV complete QFTs which feature *asymptotic freedom*. The interesting property of these theories is that they can be studied already within standard perturbation theory, as the coupling constants approach naturally a noninteracting GFP. Yet, in the following chapters, we will try to convince the reader that, given a certain set of fields and symmetries, standard perturbation theory is not able to give a definite answer about the whole spectrum of possible AF trajectories existing in a theory space.

Which type of QFTs are AF in the UV limit? The answer to this question can be traced back in the *Coleman-Gross theorem* [231, 232] which states that

- *no renormalizable field theory (in $d = 4$ dimensions and assuming unitarity and Lorentz symmetry) without non-Abelian gauge fields can be asymptotically free.*

It is therefore essential that some gauge fields associated to a non-Abelian local gauge symmetry exist. The first discoveries go back in the early 70's with the study of non-Abelian gauge theories by Gross and Politzer [27, 28], soon extended also to scalars in order to include the possibility of a Higgs mechanism [29], or scalars coupled to fermions via Yukawa interaction [77, 79].

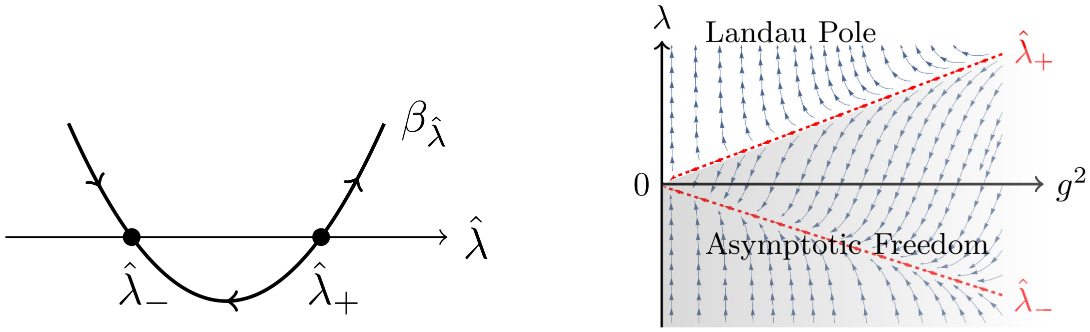


Fig. 2.7.: *Left Panel:* example of a beta function for the rescaled coupling λ/g^2. The arrows are pointing toward the ultraviolet direction such that the fixed points $\hat\lambda_-$ and $\hat\lambda_+$ correspond to ultraviolet attractive and repulsive fixed points, respectively.
Right Panel: example of RG flow in the plane (λ, g^2) in case $\hat\lambda_+ > 0$ and $\hat\lambda_- < 0$. The region free from Landau poles $\lambda \leq \hat\lambda_+ g^2$ is highlighted in gray.

2.4.1. Within Perturbation Theory

In order to give an example to the reader about how asymptotic freedom is detectable in standard perturbation theory, let us present the mechanism in the case of an $\mathrm{O}(N)$ gauge theory interacting with an N-component scalar fundamental representation, the latter self-interacting via a quartic potential $\lambda(\phi^a\phi^a)^2/8$ [77]. The renormalization group equations for λ and the gauge coupling g are:

$$\partial_t\lambda = \frac{1}{16\pi^2}\left[(N+8)\lambda^2 - 3(N-1)\lambda g^2 + \frac{3}{4}(N-1)g^4\right], \tag{2.45}$$

$$\partial_t g^2 = -2b_g g^4, \tag{2.46}$$

where b_g is a constant depending on the representation of the gauge group, such that g^2 is AF for any positive b_g. Introducing a rescaled coupling $\hat\lambda$ defined as

$$\hat\lambda = \frac{\lambda}{g^2}, \tag{2.47}$$

then its beta function becomes a parabola with two roots. Indeed

$$\partial_t\hat\lambda \;\propto\; (\hat\lambda - \hat\lambda_-)(\hat\lambda - \hat\lambda_+), \tag{2.48}$$

where the values $\hat\lambda_\pm$ are real as long as there are enough gauge fields to render the scalar coupling AF, namely $3(N-1)(2N-11) > 0$.

The beta function for the rescaled coupling $\hat\lambda$ is qualitatively depicted on the left panel of Fig. 2.7, for some real values of $\hat\lambda_\pm$. The arrows are pointing in the UV direction such that $\hat\lambda_-$ represents a UV attractive FP while the $\hat\lambda_+$ root is a UV repulsive FP. The latter distinction implies that in the two-dimensional plane (λ, g) there can be three different types of RG trajectories depending of the initial conditions (λ_0, g_0) at t_0 (as represented on the right panel of Fig. 2.7):

- for $\lambda_0/g_0^2 > \hat{\lambda}_+$ the quartic scalar coupling will hit a Landau pole while running towards the UV;
- for $\lambda_0/g_0^2 < \hat{\lambda}_+$ the quartic scalar coupling becomes AF and the way how it approaches the GFP is governed by the root $\hat{\lambda}_-$, i.e., $\lambda \underset{g^2\to 0}{\sim} \hat{\lambda}_- g^2$;
- for $\lambda_0/g_0^2 = \hat{\lambda}_+$ the corresponding trajectory becomes a separatrix between the two regions just mentioned, and λ remains proportional to g^2 along the whole flow towards the GFP, i.e., $\lambda = \hat{\lambda}_+ g^2$. Since this special trajectory is UV unstable, the initial conditions at low energy need to be "fine tuned" in order to approach $\hat{\lambda}_+$ in the UV. For this reason, this scenario is in general not an attractive perspective. However, if $\hat{\lambda}_+$ is the only positive root of $\partial_t \hat{\lambda} = 0$, then the UV completion of the theory would be determined by consistency conditions, which is, on the contrary, a very promising possibility. In fact, in this case there would be an enhancement of the predictive power of the model at low energy since the special trajectory $\hat{\lambda}_-$ is IR attractive.

Therefore, for a non-Abelian O(N) gauge theory, an upper bound exists on the renormalized parameters λ and g^2 in order to feature total asymptotic freedom in all its couplings, namely $\hat{\lambda} \leq \hat{\lambda}_+$. In case the scalar potential is in the SSB regime, this bound translates naively into an upper bound for the mass-ratio between the renormalized mass of the Higgs excitation m_{H} and the renormalized mass of the gauge bosons m_{W} [233, 234]:

$$\frac{m_{\text{H}}^2}{m_{\text{W}}^2} \leq \hat{\lambda}_+. \tag{2.49}$$

This analysis can also be extended including fermionic degrees of freedom, as for example described by Cheng et. al. in [77]. We will discuss this case in more detail in Sec. 3.1. For the moment, we hope to have made clear that non-Abelian gauge field theories can rescue a pure self-interacting scalar theory from triviality, and that the requirement of having a total AF model imposes some restriction on the coupling constants and particle masses which can ultimately lead to experimental predictions in the IR.

In the rest of this thesis, we will call any fixed point for the beta function of any gauge-rescaled coupling (such as the FPs $\hat{\lambda}_\pm$) as *quasi-fixed-point* (QFP). This name is justified by the fact that a QFP corresponds to a FP even though the gauge coupling assumes a finite value. In fact, QFPs are more a description of how the GFP is approached in the UV limit than a standard FP in the traditional RG language. This concept has been used in the literature under different names, such as eigenvalue conditions [79, 85] or fixed flows [86]. Therefore, a defining requirement for a viable QFP solution to represent an AF trajectory is that, any gauge-rescaled coupling should approach some finite constant value in the limit where the gauge coupling becomes AF, namely in the limit where the RG time t approaches infinity.

2.4.2. Beyond Perturbation Theory

Standard perturbation theory seems perfectly suited for studying totally AF QFTs, as the running coupling constants become arbitrary small in the UV limit. However, it makes some implicit assumptions which may, in principle (and actually they do), reduce the number of viable AF trajectories. One of these assumptions is that at large Euclidean momentum scale, all masses or mass-like parameters become negligible with respect to the momenta of the quantum fluctuations, i.e., the system is assumed to be in the *Deep Euclidean Regime* (DER) while approaching the UV limit. In addition, perturbation theory does not consider the RG running of irrelevant couplings since they are non-renormalizable from a perturbative point of view. These couplings are, for example, associated to higher dimensional operators such as $\phi^6, \phi^8, \dots$ in a scalar theory, and have canonical dimension of inverse mass in $d = 4$ dimensions.

The confirmation that standard perturbation theory is not able to detect some of the AF trajectories has been recently observed for non-Abelian Higgs models [111, 112]. Since those results have been one of the main motivations for our studies, we want to briefly introduce here some of the key ingredients which are of fundamental importance for this work. Let us consider therefore a non-Abelian Higgs model with an SU(N) gauge sector coupled to a charged (complex) scalar field ϕ^a in the fundamental representation. The one-loop beta functions for the gauge coupling and quartic scalar coupling normalized as $\lambda(\phi^{a\dagger}\phi^a)^2/2$ are [29]

$$\partial_t g^2 = -b_g g^4, \qquad \partial_t \lambda = A\lambda^2 - B\lambda g^2 + C g^4, \tag{2.50}$$

where b_g, A, B and C are group theoretical factors depending on N and are positive for any integer $N \geq 1$. If we change the set of couplings into $(g^2, \hat{\lambda})$ as in Eq. (2.47), the condition which has to be satisfied in order for the parabola $\partial_t \hat{\lambda}$ to have two real roots is that $\Delta = (B + b_g)^2 - 4AC > 0$. However, it turns out that $\Delta < 0$ for all $N \geq 2$ and the scalar sector is inevitably trivial.

Suppose now to leave one of the implicit assumptions of perturbation theory, for example by including the higher dimensional operator $\lambda_3(\phi^\dagger\phi)^3/3!$. This interaction term will contribute to the running of λ with a negative term which lowers the parabola and can produce roots. Indeed

$$\partial_t \lambda = A\lambda^2 - B\lambda g^2 + C g^4 - D\lambda_3, \tag{2.51}$$

such that for finite values of g the coefficient C changes into $C' = C - D\lambda_3/g^4$. By keeping the ratio λ_3/g^4 fixed, and varying its value, we can tune the sign of the discriminant $\Delta' = (B + b_g)^2 - 4AC'$ and obtain AF trajectories. It might look like that these trajectories can be obtained by naively following the spirit of effective field-theory (EFT) approaches and adding higher dimensional couplings. However let us point out that no running of the coupling λ_3 has been taken into account. If we had tried to look for scaling solutions for the system $\partial_t\lambda = 0$ and $\partial_t\lambda_3 = 0$ with respect to the rescaled couplings $(\hat{\lambda}, \lambda_3/g^4)$, the triviality of the scalar sector would not have been solved.

This simple example tells us already an important concept: in order to detect these new AF trajectories it is fundamental to have an approximation scheme allowing for the presence of *free parameters*, which can be encoded, for example, in the UV behavior of higher dimensional operators. These parameters play the role of *boundary conditions* selecting different theories; the approach is therefore completely different from a perturbative framework where the parameters of a theory are fixed within the theory while boundary conditions are merely implicitly assumed. The fact that the RG behavior of a model depends on its boundary conditions is well known in statistical field theory. For instance, the scalar potential at the interacting Wilson-Fisher FP describes the correct behavior (i.e., critical exponents) of the Ising model only for suitable boundary conditions [235–238]. In the following of this thesis we will also explain the importance of relaxing the implicit assumption of being in the DER. Allowing for the presence of mass-threshold effects over arbitrary energy scales is another fundamental aspect to be taken into account, in order to reveal new UV complete QFTs.

In the light of these considerations, we are going to present in the next chapters two different gauged-Yukawa toy models which include part of the SM matter field content. Therefore, not all the fundamental degrees of freedom described by the SM are considered. For example, we will not address the triviality problem of the abelian $\mathrm{U}(1)_\mathrm{Y}$ gauge sector. In this sense our studies are "minimalistic" and do not advocate the presence of exotic matter fields beyond the SM. We will give instead a fresh look at AF gauged-Yukawa models providing a perspective beyond standard perturbation theory, which could lead to a new path towards a full UV completion of the SM.

The core of our analysis will rely on nonpertubative techniques necessary for taking into account threshold effects (we will mainly use the FRG approach). Of course these techniques carry their own limitations: the main one of our concern is the scheme dependence, an intrinsic feature in any mass-dependent regularization scheme. In the light of this awareness, in the last Chap. 4, we will reinforce the robustness of our results by checking for their scheme (in)dependence. In this respect, the FRG methods provide us with a powerful tool since, as we will explain, they allow us to address generic IR regularization schemes.

3. $\mathbb{Z}_2$-Yukawa-QCD Models

Let us start by describing our first toy-model, which includes a real scalar field H (interpreted as the Higgs field), N_c copies of a Dirac field ψ (interpreted as the "colored" top quark) and some gauge fields G^I_μ (interpreted as the gluons). The corresponding bare action, in Euclidean spacetime, reads

$$S = \int \mathrm{d}^4x \left[\frac{1}{4}\left(G^I_{\mu\nu}\right)^2 + \frac{1}{2}\left(\partial_\mu H\right)^2 + \frac{\bar{m}}{2}H^2 + \frac{\bar{\lambda}}{8}H^4 + \bar{\psi}^A \mathrm{i}\not{D}^{AB}\psi^B + \frac{\mathrm{i}\bar{h}}{\sqrt{2}}H\bar{\psi}^A\psi^A\right]. \qquad (3.1)$$

The gauge sector is a non-Abelian Yang-Mills theory for the field strength tensor $G^I_{\mu\nu} = \partial_\mu G^I_\nu - \partial_\nu G^I_\mu - \bar{g}_\mathrm{s} f^{IJK} G^J_\mu G^K_\nu$. The fermion field ψ belongs to the fundamental representation of $\mathrm{SU}(N_\mathrm{c})$ and is coupled to the gauge bosons through the covariant derivative in the color space

$$D^{AB}_\mu = \delta^{AB}\partial_\mu + \mathrm{i}\bar{g}_\mathrm{s} G^I_\mu T^{AB}_I, \qquad (3.2)$$

where T^{AB}_I are the generators of the $\mathrm{su}(N_\mathrm{c})$ Lie algebra satisfying the anti-commutation relation $[T_I, T_J]^{AB} = \mathrm{i}f_{IJK}T^{AB}_K$. Moreover, there is a Yukawa-type interaction term which couples ψ to the real scalar field $H(x)$. Due to the absence of a Dirac mass term, the model exhibits a discrete chiral symmetry which mimics the electroweak sector of the SM

$$\psi'(x) = e^{\mathrm{i}\pi\gamma_5/2}\psi(x), \qquad \bar{\psi}'(x) = \bar{\psi}(x)e^{\mathrm{i}\pi\gamma_5/2}, \qquad H'(x) = -H(x). \qquad (3.3)$$

Of course, we need to add a gauge fixing condition to the action and the related Faddeev-Popov action. We choose

$$S_\mathrm{GF} = \int \mathrm{d}^4x\, \frac{1}{2\zeta_\mathrm{s}}(\partial_\mu G^I_\mu)^2, \qquad S_\mathrm{gh} = \int \mathrm{d}^4x\, \bar{u}^I \partial_\mu D^{IJ}_\mu u^J, \qquad (3.4)$$

where ζ_s is an arbitrary gauge-fixing parameter, and the covariant derivative in the adjoint representation is $D^{IJ}_\mu = \delta^{IJ}\partial_\mu + \bar{g}_\mathrm{s} f^{IJK} G^K_\mu$. The Faddeev-Popov determinant is complemented by two independent set of ghost fields u^I and $\bar{u}^I$. From now on we will associate the indices $A, B, C, \cdots$ starting at the beginning of the alphabet to the fundamental representations of $\mathrm{SU}(N_\mathrm{c})$, namely $A = \{1, \ldots, N_\mathrm{c}\}$; and the indices $I, J, K, \cdots$ starting in the middle of the alphabet to the adjoint representations of $\mathrm{SU}(N_\mathrm{c})$, namely $I = \{1, \ldots, N_\mathrm{c}^2 - 1\}$.

In the subsequent of the present chapter we explore the UV behavior of this model. In the first place, we review the well-known existence of AF trajectories discovered long time ago by Cheng, Eichten, and Li [77]. In the second place, we will improve the analysis by means of the FRG approach, which gives us access to several approximation schemes that allow to address the global properties of the Higgs potential as well as to go beyond some implicitly made assumptions of standard perturbation theory.

3.1. Perturbative Analysis

Let us then start by analyzing the present model, at one-loop level, retaining only perturbatively renormalizable couplings, and assuming to be in the DER such that the mass parameter $\bar{m}$ can be set to zero [28, 77]. We are then left with a set of three couplings: the Higgs quartic self-interaction $\bar{\lambda}$, the top-Yukawa coupling $\bar{h}$ and the strong gauge coupling $\bar{g}_{\mathrm{s}}$. Moreover, it is customary to switch to the dimensionless couplings λ, h and g_{s} in order to study the scaling properties of the model in the vicinity of a FP. Their definitions in terms of the bare couplings and wave function renormalizations are postponed in Sec. 3.2 for the moment.

Gauge Sector

Since the scalar field H is not charged under the "colored" gauge group, the anomalous dimension of the gluons η_{G} receives contributions, of opposite sign, only from the gauge and fermion loops. Therefore, the one-loop beta function for the gauge coupling reads

$$\partial_t g_{\mathrm{s}}^2 = \eta_{\mathrm{G}} g_{\mathrm{s}}^2, \qquad \eta_{\mathrm{G}} = -\frac{g_{\mathrm{s}}^2}{3(4\pi)^2}\left(22N_{\mathrm{c}} - d_\gamma^{\mathrm{c}} N_{\mathrm{f}}^{\mathrm{c}}\right), \tag{3.5}$$

where d_γ^{c} denotes the dimension of the Dirac algebra ($d_\gamma^{\mathrm{c}} = 4$ in $d = 4$ dimensions), and $N_{\mathrm{f}}^{\mathrm{c}}$ is the number of quark flavors. The latter equation slightly generalizes the model in Eq. (3.1) since now we allow for in total $N_{\mathrm{f}}^{\mathrm{c}}$ quark Dirac fermions in the fundamental $\mathrm{SU}(N_{\mathrm{c}})$ representation. This would correspond to a model which includes additional $\mathrm{SU}(N_{\mathrm{c}})$-invariant kinetic terms for the other $N_{\mathrm{f}}^{\mathrm{c}} - 1$ quark flavors but their Yukawa couplings to the scalar field are set to zero. That choice for the action is motivated by the fact that the top-Yukawa coupling plays a dominant role in the RG running of the Higgs potential and all other Yukawa couplings are negligibly small. Therefore, we consider the correct running of the gauge sector, with all the quark flavors taken into account, while we retain only the top-quark contribution into the flow equation of the Higgs sector.

In the specific case of the SM field content, with $N_{\mathrm{c}} = 3$ and $N_{\mathrm{f}}^{\mathrm{c}} = 6$, the one-loop beta function for g_{s} is negative

$$\partial_t g_{\mathrm{s}}^2 = -\frac{7}{8\pi^2} g_{\mathrm{s}}^2, \tag{3.6}$$

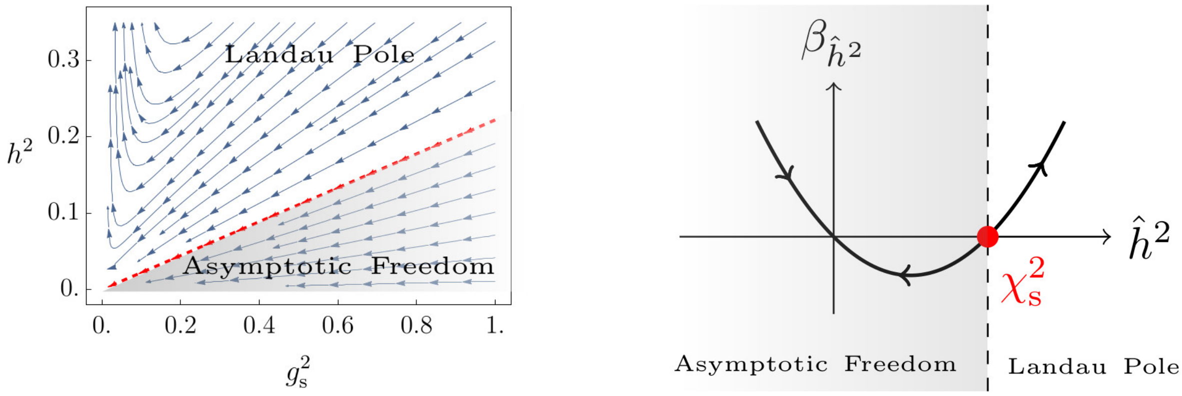


Fig. 3.1.: *Left Panel*: one-loop perturbative RG flow of the $\mathbb{Z}_2$-Yukawa-QCD model projected on the plane of the top-Yukawa coupling h^2 and the strong gauge coupling g_s^2. The color and flavor numbers are fixed to that of the SM, i.e., $N_c = 3$ and $N_f^c = 6$. *Right panel*: beta function for the gauge-rescaled Yukawa coupling $\hat{h}^2$ defined in Eq. (3.8). The UV repulsive QFP is highlighted by a red point corresponding to the solution of Eq. (3.11).

such that the strong gauge coupling is AF in the UV limit.

Yukawa Sector

Concerning the Yukawa sector, the one-loop beta function for h^2 is

$$\partial_t h^2 = \frac{h^2}{16\pi^2}\left[(3+2N_c)h^2 - 6\frac{N_c^2-1}{N_c}g_s^2\right]. \tag{3.7}$$

The latter equation together with Eq. (3.5) entails that AF trajectories can exist in the (g_s^2, h^2) plane, as it is visible in the left panel of Fig. 3.1, where the RG flow is represented with arrows pointing towards the UV. There is indeed a region, below the dashed red line, where both the top-Yukawa coupling as well as the strong gauge coupling are AF in the UV limit. Above this region, on the contrary, the coupling h^2 hits a Landau pole at a fine RG energy scale. The dashed red line can thus be viewed as an upper bound for the AF region and, in fact, represents a special trajectory, along which h^2 exhibits an asymptotic scaling proportional to g_s^2. [16] This behavior is best characterized in terms of the rescaled coupling

$$\hat{h}^2 = \frac{h^2}{g_s^2}, \tag{3.8}$$

and its beta function

$$\partial_t \hat{h}^2 = \frac{3+2N_c}{16\pi^2} g_s^2 \hat{h}^2 \left(\hat{h}^2 - \chi_s^2\right), \qquad \chi_s^2 = \frac{1}{3+2N_c}\left[\frac{4}{3}\left(N_f^c - N_c\right) - \frac{6}{N_c}\right]. \tag{3.9}$$

[16] Below this special trajectory where $h^2 \propto g_s^2$, the top-Yukawa coupling approaches the GFP faster than g_s in the UV limit. These further AF solutions have already been discussed in [77] as well as in later analyses [79, 86]; for completeness, we review them in App. A.1.

The latter equation has a nontrivial QFP solution at $\hat{h}^2 = \chi_\mathrm{s}^2$, such that when the ratio $h_0^2/g_{\mathrm{s}0}^2$ equals χ_s^2 at some initialization RG-time t_0, the flow is "frozen" in the sense that $h^2(t)$ runs proportional to $g_\mathrm{s}^2(t)$ while approaching the GFP. The QFP χ_s is UV unstable and can be viewed as the upper bound for the ratio h^2/g_s^2 in order to have total asymptotic freedom in the plane (h^2, g_s^2). This behavior is depicted qualitatively on the right panel of Fig. 3.1, where it should be clear that the condition $\hat{h}^2 \leq \chi_\mathrm{s}^2$ has to hold in order not to fall into a Landau pole.

Let us stress here that Eqs. (3.5) and (3.7) can (but not necessarily does) imply the existence of AF trajectories. Such trajectories are present only if a suitable matter content is provided, implying a range of possible values for the parameter N_f^c at fixed N_c. The upper bound for such window is given by the requirement that g_s^2 stays AF, i.e., $\eta_\mathrm{G} < 0$; while the lower bound is obtained from Eq. (3.9) by demanding the positivity of the QFP χ_s^2 in order to preserve unitarity. Thus, we obtain

$$N_\mathrm{c} + \frac{9}{2N_\mathrm{c}} < N_\mathrm{f}^\mathrm{c} < \frac{11}{2} N_\mathrm{c}. \tag{3.10}$$

The SM case with $N_\mathrm{c} = 3$ and $N_\mathrm{f}^\mathrm{c} = 6$ falls inside this window, resulting in a QFP at

$$\chi_\mathrm{s}^2 = \frac{2}{9}. \tag{3.11}$$

Throughout the main text of this chapter, we will concentrate on the implications of the RG flow for this particular ratio, represented by the special red trajectory (or point) in Fig. 3.1.

Scalar Sector

In order to investigate the implications for the Higgs sector, we consider the beta function for the renormalized quartic coupling at one-loop level which is

$$\partial_t \lambda = \frac{9}{16\pi^2}\lambda^2 - \frac{N_\mathrm{c}}{4\pi^2} h^4 + 2\eta_\mathrm{H}\lambda, \qquad \eta_\mathrm{H} = \frac{N_\mathrm{c}}{8\pi^2} h^2, \tag{3.12}$$

where η_H is the anomalous dimension of the Higgs scalar field.

As before, we can classify AF trajectories for λ by a similar QFP condition for a suitable ratio among λ and the AF strong gauge coupling. Instead of choosing the particular rescaling of Eq. (2.47), we want to consider a generalization,

$$\hat{\lambda}_2 = \frac{\lambda}{g_\mathrm{s}^{4P}}, \qquad P > 0, \tag{3.13}$$

where the power P has to be determined by the self-consistent condition that $0 < \hat{\lambda}_2 < \infty$. The flow equation for this rescaled quartic Higgs coupling will then receive contributions from the running coupling g_s^2. Moreover, it is useful to introduce two rescaled anomalous dimensions, by factoring out $g_{\rm s}$

$$\hat{\eta}_{\rm G} \equiv \frac{\eta_{\rm G}}{g_{\rm s}^2} = -\frac{1}{3(4\pi)^2}\left(22N_{\rm c} - d_\gamma^{\rm c} N_{\rm f}^{\rm c}\right), \qquad \hat{\eta}_{\rm H} \equiv \frac{\eta_{\rm H}}{g_{\rm s}^2} = \frac{N_{\rm c}}{8\pi^2}\chi_{\rm s}. \tag{3.14}$$

Therefore we can express the beta function for $\hat{\lambda}_2$ as

$$\partial_t \hat{\lambda}_2 = \frac{9}{16\pi^2} g_{\rm s}^{4P}\hat{\lambda}_2^2 - \frac{N_{\rm c}}{4\pi^2} g_{\rm s}^4 \chi_{\rm s}^4 + \left(2\hat{\eta}_{\rm H} - 2P\hat{\eta}_{\rm G}\right) g_{\rm s}^2 \hat{\lambda}_2, \tag{3.15}$$

and it turns out that the only possible QFP occurs at $P = 1/2$. This particular value for the scaling power P is not surprising. Indeed, the beta function in Eq. (3.12), as a function of λ, is a parabola with two roots that are proportional to h^2 and thus, to $g_{\rm s}^2$ since we are on the trajectory $\hat{h}^2 = \chi_s^2$. For the SM matter-field content these roots are

$$\hat{\lambda}_2^{\pm} = \frac{1}{27}\left(-25 \pm \sqrt{673}\right), \qquad P = \frac{1}{2}. \tag{3.16}$$

The UV properties of the QFPs $\hat{\lambda}_2^{\pm}$ go along with the discussion presented in Sec. 2.4.1 for a gauged O(N) scalar model. Here we just highlight the fact that, since the root $\hat{\lambda}_2^{-}$ is negative, the requirement of a stable and UV complete theory enforces the condition $\hat{\lambda}_2 = \hat{\lambda}_2^{+}$. This special trajectory is UV repulsive but IR attractive, hence the low-energy behavior is governed by the QFP itself, enhancing the predictive power of the model. From now on, we refer to this solution as the *Cheng–Eichten–Li* (CEL) solution, since it was first described in [77].

Whereas classically $g_{\rm s}$, h and λ are independent renormalized couplings to be determined by experiment, the special AF trajectory represented by $\hat{h} = \chi_{\rm s}$ and $\hat{\lambda}_2 = \hat{\lambda}_2^{+}$ locks the running of h and λ to that of $g_{\rm s}$. Physically, this implies that the mass of the top quark as well as that of the Higgs boson will be determined in terms of the initial conditions for the SU($N_{\rm c}$) gauge sector at a given energy scale, say the Fermi scale.

Let us finally emphasize that all throughout this chapter we will focus mainly on the asymptotic UV running for the top-Yukawa coupling described by the QFP in Eq. (3.9) (or Eq. (3.11) whenever we want to restrict to the SM case). Regarding the scalar sector, instead, we will investigate whether the CEL solution is the only viable AF solution with the same field content and symmetries of the action in Eq. (3.1). To address the possibility for new solutions, we take inspiration from the discussion about the limitations of standard perturbation theory, presented at the end of the previous chapter. Therefore, we now turn to the computation of the beta function(al)s for the $\mathbb{Z}_2$-Yukawa-QCD model within the FRG method, in order to take into account mass threshold effects.

3.2. Nonperturbative RG Flow Equations

In order to make use of the Wetterich equation (2.34), we truncate the EAA at the next-to-leading order in the derivative expansion, such as

$$\Gamma_k = \int d^dx \Bigg[\frac{Z_\mathrm{G}}{4} \left(G^I_{\mu\nu}\right)^2 + \frac{Z_\mathrm{H}}{2} \left(\partial_\mu H\right)^2 + U\left(H^2/2\right) + Z_\mathrm{F} \bar{\psi}^A \mathrm{i} \not{D}^{AB} \psi^B + \frac{\mathrm{i}\bar{h}}{\sqrt{2}} H \bar{\psi}^A \psi^A \\ + \frac{Z_\mathrm{G}}{2\zeta_\mathrm{s}} (\partial_\mu G^I_\mu)^2 + Z_u \bar{u}^I \partial_\mu D^{IJ}_\mu u^J \Bigg], \tag{3.17}$$

where the effective average potential U is an *arbitrary* function of the discrete $\mathbb{Z}_2$-invariant field $H^2/2$. Also the wave function renormalizations Z_Φ (with $\Phi = \{\mathrm{G}, \mathrm{H}, \mathrm{F}, u\}$) are functions of the corresponding field Φ. Since we want to investigate the UV behavior of our model and in particular its scaling properties, it is useful to introduce dimensionless renormalized fields in order to fix the usual RG invariance of field rescalings

$$\rho = Z_\mathrm{H} k^{2-d} \frac{H^2}{2}, \qquad \Psi = Z_\mathrm{F} k^{\frac{1-d}{2}} \psi. \tag{3.18}$$

In a similar manner, we introduce dimensionless renormalized couplings which read

$$h^2 = \bar{h}^2 \frac{k^{d-4}}{Z_\mathrm{H} Z_\mathrm{F}^2}, \qquad g_\mathrm{s}^2 = \bar{g}_\mathrm{s}^2 \frac{k^{d-4}}{Z_\mathrm{G}}. \tag{3.19}$$

Inserting our truncation for Γ_k into the Wetterich equation we can extract the flow equations for various operators by appropriate projection rules. All the details on these computations can be found in App. C. For instance we can derive the RG flow for the dimensionless renormalized potential defined as

$$u(\rho) = k^{-d} U(Z_\mathrm{H}^{-1} k^{d-2} \rho), \tag{3.20}$$

as well as the beta function for the top-Yukawa coupling. Similarly, we can obtain the anomalous dimensions of the fields from the following definitions

$$\eta_\mathrm{H} = -\partial_t \log Z_\mathrm{H}, \qquad \eta_\psi = -\partial_t \log Z_\mathrm{F}, \tag{3.21}$$

encoding the running of the wave function renormalizations.

Matter Content

Let us start by inspecting the functional flow equation for the full dimensionless renormalized potential, given by

$$\partial_t u(\rho) = -du + (d-2+\eta_\mathrm{H})\rho u' + 2v_d \left[l_0^{(\mathrm{H})d}(\omega_\mathrm{H}, \eta_\mathrm{H}) - N_\mathrm{c} d_\gamma^\mathrm{c}\, l_0^{(\mathrm{F})d}(\omega_\mathrm{F}, \eta_\psi) \right], \tag{3.22}$$

$\partial_t U_k$:

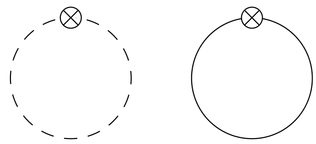


Fig. 3.2.: Diagrammatic representation for the two contributions to the RG flow equation of the scalar potential. Dashed and solid lines denote scalar and fermionic propagators respectively. All the internal propagators are considered as fully dressed, due to the regulator insertion $\partial_t R_k$ represented by a circle with a cross.

where $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma\left(\frac{d}{2}\right)$ and ω_{H} as well as ω_{F} are defined as

$$\omega_{\mathrm{H}} = u'(\rho) + 2\rho u''(\rho), \qquad \omega_{\mathrm{F}} = h^2\rho. \tag{3.23}$$

In comparison to the simple Higgs-Yukawa model [197], the beta function in Eq. (3.22) differs by an N_{c} prefactor in front of the fermionic loop, since in this model we have N_{c} copies of the top quark. Moreover, we have ignored the contributions to $\partial_t u(\rho)$ coming from the pure gluon and ghost loops. In fact, because these loops are field-independent, they contribute only to the running of a cosmological constant which is irrelevant for the purpose of our investigations.

The threshold functions $l_0^{(\mathrm{H})d}$ and $l_0^{(\mathrm{F})d}$ correspond to the contribution from a pure scalar and fermionic loop integral respectively. They can be diagrammatically represented as in Fig. 3.2, where the circles with a cross emphasize the fact that the propagators connect to a regulator insertion $\partial_t R_k$. The regulator dependence is precisely the reason for the nonuniversal behavior of the loop threshold functions, which describe the decoupling of massive modes. Indeed, for any choice of the regulator compatible with the physical conditions in Eq. (2.22), the loop integrals approach finite values for zero arguments while they decrease to zero for very large arguments. The general definitions for these threshold functions, as well as explicit representations for a convenient piece-wise linear regulator [192, 193], to be used in the following of this chapter, are listed in App. B.

In the following sections we will see more directly how Eq. (3.22) can be exploited in order to derive the beta functions for scalar self-couplings up to an arbitrary order. Additionally, the key feature of having the beta function for the full effective potential is that we can keep track of all the important scales, the RG scale as well as the field amplitudes which are not any more demanded to be small. Thus, it allows to study global properties of the Higgs potential which we will discuss with regard to AF trajectories in the following.

Concerning the Yukawa sector, some comments are in order. Typically we can obtain the beta function for the top-Yukawa coupling by projecting Eq. (2.34) on the three-point function $\Gamma^{(3)}_{H\bar{\psi}\psi}$, as one would do for a model with a field-independent Yukawa coupling [164, 194, 227, 228]. This projection would give Eq. (C.32), as reviewed in details in App. C.4. However, if we want to enlarge the truncation of Γ_k by including higher-order Yukawa interactions, we could promote the top-Yukawa coupling to be an arbitrary functional $\bar{h}(H^2/2)$ of the field invariant $H^2/2$, in such a way that the discrete chiral symmetry in Eq. (3.3) is still preserved [168, 196, 199, 239, 240]. In this case, the beta function for the Yukawa functional can be obtained by following two equivalent projection rules. One way is to project onto the field-dependent two-point function $\Gamma^{(2)}_{\bar{\psi}\psi}(H)$ and factoring out the scalar field, such that

$$\partial_t[H\bar{h}(H)] = H\beta_{\bar{h}}(H) \quad \Longrightarrow \quad \partial_t\bar{h}(H) = \beta_{\bar{h}}(H), \tag{3.24}$$

whose particular expression is given in App. C.4.1, cf. Eq. (C.42). Notice that the RG-time derivative is taken at constant field configuration. On the other hand, one can also project onto the operator $H\bar{\psi}\psi$, by taking an additional derivative with respect to the scalar field. In this second case we would have instead

$$\partial_t\bar{h}(H) = \beta_{\bar{h}}(H) + H\beta'_{\bar{h}}(H) - H\partial_t\bar{h}'(H). \tag{3.25}$$

Obviously the latter two formulations coincide as long as no further approximation on the Yukawa potential $\bar{h}(H)$ is taken. In fact, $\partial_t\bar{h} = \beta_{\bar{h}}$ solves consistently Eq. (3.25) and the flow equation for the Yukawa potential is unique. The differences between these two formulations arise whenever some finite-order truncation for $\bar{h}(H)$ is considered, for example by doing a polynomial expansion in H and truncating the series to a finite number of higher-dimensional Yukawa couplings. It is also a key aspect that only in the SSB regime these discrepancies become manifest. Moreover, it has recently been observed that the projection onto $\Gamma^{(2)}_{\bar{\psi}\psi}$ shows better convergence than the projection onto $\Gamma^{(3)}_{H\bar{\psi}\psi}$, when one analyzes the RG flow of the Yukawa couplings towards the IR [169, 239]. It can even happen that convergence is lost when using Eq. (3.25). In addition, there are also physical reasons to prefer the projection rule given by Eq. (3.24). In fact, it has been observed that the nontrivial UV FP discovered for simple Yukawa model in [227] as well as Higgs-top-bottom chiral Yukawa model in [165] are artifacts of the second projection scheme of Eq. (3.25), leaving the trivial Gaußian FP as the only physical acceptable FP solution for those models.

For the above mentioned reasons, we will make use of the first projection rule given by Eq. (3.24) for the rest of this thesis, and we specialize to our case where the Yukawa potential $h(\rho)$ is just a constant dependent on the RG scale k. In the case of the present $\mathbb{Z}_2$-Yukawa-QCD model, the RG flow equation for h^2 reads

$$\begin{aligned}\partial_t h^2 = {} & (d-4+\eta_{\mathrm{H}}+2\eta_\psi)h^2 + 4v_d h^4\, l^{(\mathrm{FH})d}_{11}(\omega_{\mathrm{F}},\omega_{\mathrm{H}};\eta_\psi,\eta_{\mathrm{H}}) \\ & - 8v_d\frac{N_{\mathrm{c}}^2-1}{2N_{\mathrm{c}}}(d+\zeta_{\mathrm{s}}-1)\, h^2 g_{\mathrm{s}}^2 l^{(\mathrm{FG})d}_{11}(\omega_{\mathrm{F}},0;\eta_\psi,\eta_{\mathrm{G}})\big|_{\rho=\kappa},\end{aligned} \tag{3.26}$$

where the general definitions and explicit expressions for the threshold functions $l^{(\mathrm{FH})d}_{n_1n_2}$ and $l^{(\mathrm{FG})d}_{n_1n_2}$ can be found in App. B. The diagrammatic picture of the loop contributions for the top-Yukawa coupling is given in the upper line of Fig. 3.3, from which it is clear that the only

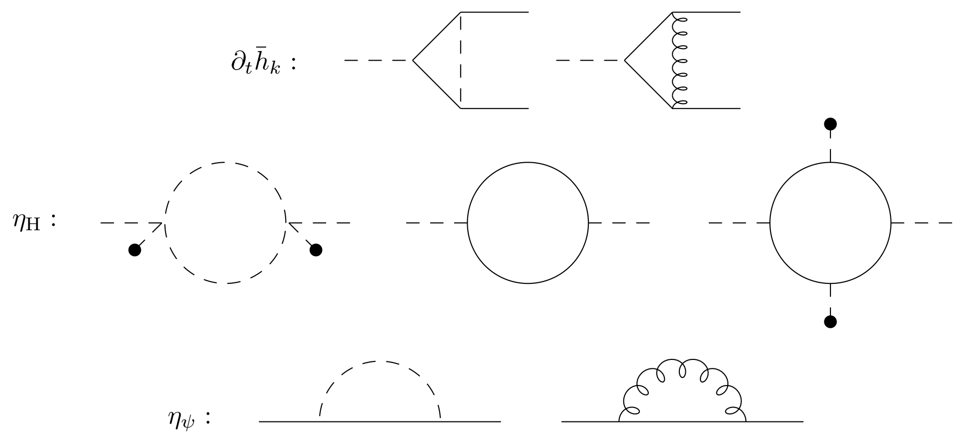


Fig. 3.3.: Diagrammatic representation for the RG flow equations of the top-Yukawa coupling as well as the wave functions renormalization of the scalar and fermionic fields. Curly lines denote the propagation of gluon fields. Moreover, for convenience, we have decided to drop the regulator insertion out of any internal lines. Thick black dots represent couplings to the condensate, i.e., the nontrivial ground state for the scalar potential.

two contributions are given by a one-loop diagram where an internal Higgs scalar or gluon is exchanged among two Dirac fields. The first contribution is represented by the threshold function $l_{11}^{(\mathrm{FH})d}$, while the second contribution by $l_{11}^{(\mathrm{FG})d}$. Indeed the two fundamental vertices of the present theory, involving Dirac fermions, are the Yukawa-type vertex $\sim H\bar{\psi}\psi$, and the vertex with an exchanged gluon $\sim \bar{\psi}G\psi$ arising from the covariant derivative.

Finally, the scalar and spinor anomalous dimensions read respectively

$$\begin{aligned}\eta_{\mathrm{H}} = &\frac{8v_d}{d}\rho\,(3u'' + 2\rho u''')^2\, m_2^{(\mathrm{H})d}(\omega_{\mathrm{H}},\eta_{\mathrm{H}}) \\ &+ \frac{4v_d N_{\mathrm{c}} d_\gamma^{\mathrm{c}}}{d} h^2 \left[m_4^{(\mathrm{F})d}(\omega_{\mathrm{F}},\eta_\psi) - \rho h^2\, m_2^{(\mathrm{F})d}(\omega_{\mathrm{F}},\eta_\psi)\right]\Big|_{\rho=\kappa}, \end{aligned} \tag{3.27}$$

and

$$\begin{aligned}\eta_\psi = &\frac{4v_d}{d} h^2 m_{12}^{(\mathrm{FH})d}(\omega_{\mathrm{F}},\omega_{\mathrm{H}};\eta_\psi,\eta_{\mathrm{H}}) + \frac{8v_d}{d}\frac{N_{\mathrm{c}}^2-1}{2N_{\mathrm{c}}} g_{\mathrm{s}}^2 \Big[(d-\zeta_{\mathrm{s}}-1) m_{12}^{(\mathrm{FG})d}(\omega_{\mathrm{F}},0;\eta_\psi,\eta_{\mathrm{G}}) \\ &-(d-1)(1-\zeta_{\mathrm{s}})\widetilde{m}_{11}^{(\mathrm{FG})d}(\omega_{\mathrm{F}},0;\eta_\psi,\eta_{\mathrm{G}})\Big]\Big|_{\rho=\kappa}, \end{aligned} \tag{3.28}$$

with further threshold functions $m_{\cdots}^{\cdots}$ and $\tilde{m}_{\cdots}^{\cdots}$ which are also presented in detail in App. B. The Feynman graph representation of the loop contributions to the anomalous dimensions η_{H} and η_ψ is given in the second and third lines of Fig. 3.3 respectively. Let us observe that there are also two loop contributions to the running of Z_{H} proportional to the nontrivial minimum κ of the scalar potential. These are the scalar loop function $m_2^{(\mathrm{H})d}$ and the fermionic loop $m_2^{(\mathrm{F})d}$. The condensate of the scalar field is depicted as a thick black dot. In the SSB regime, indeed, there are more vertices allowed: in particular vertices among two Dirac or three scalar Higgs fields coupled with the *vev*. These kind of contributions are, in fact, not taken into account in a conventional perturbative computation, which relies generally on the assumption of the DER regime. To the spinor anomalous dimension η_ψ there are, instead, mass threshold contributions due to the exchange of a Higgs scalar or gluon fluctuations.

The arguments ω_{F} and ω_{H} in Eqs. (3.26–3.28) are evaluated at the minimum κ of the scalar potential $u(\rho)$, which means that $\kappa = 0$ in the SYM regime whereas $u'(\kappa) = 0$ with $\kappa \neq 0$ in the SSB regime. In the following of this chapter, for any practical computation, we will take the Landau gauge limit $\zeta_{\mathrm{s}} \to 0$. This choice is also of particular interest since it is a FP of the RG flow of the gauge-fixing parameter itself [241, 242]. It is interesting to notice that, within this limit and in the DER, the mass threshold effects due to the gluon exchange in η_ψ cancel each other. Moreover we will drop the index d from the threshold functions, as we work in $d = 4$ from now on. We also fix the dimension of the Dirac algebra to its value $d_\gamma^{\mathrm{c}} = 4$.

Gauge Sector

To complete the set of flow equations within the truncation encoded in Eq. (3.17), we should also include the running of the strong gauge coupling. We might be tempted to treat the RG flow for g_{s}^2 on a perturbative level like in Eq. (3.5). This approximation would be justified only if the UV behavior of our $\mathbb{Z}_2$-Yukawa-QCD model is compatible with the assumption that all the masses are negligible with respect the RG scale, which is the case if the scalar potential is in the SYM regime. However, threshold effects might play a role if the scalar potential develops a nontrivial minimum κ. In this case, indeed, the fermions acquire masses due to the nonvanishing Higgs expectation value and these masses can be neglected or not according to the asymptotic behavior of κ in the UV limit.

Let us have a look on how this discussion plays a role in our case. The most convenient way to compute the one-loop beta function for the gauge coupling in non-Abelian Yang-Mills theories proceeds in the background-field method [188, 194]. At one-loop level, the gluon wave function renormalization reads

$$\eta_{\mathrm{G}} = -\frac{g_{\mathrm{s}}^2}{3(4\pi)^2}\left[22N_{\mathrm{c}} - d_\gamma^{\mathrm{c}} \sum_{j=1}^{N_{\mathrm{f}}^{\mathrm{c}}} L_{\mathrm{G}}(\mu_{\mathrm{Q}_j}^2)\right], \tag{3.29}$$

where j is a multi-index labeling the quark flavors and the threshold function L_{G} guarantees the decoupling of the massive quarks in the SSB regime. This threshold function takes the particular form

$$L_{\mathrm{G}}(\mu_{\mathrm{Q}_j}^2) = \frac{1}{1+\mu_{\mathrm{Q}_j}^2}, \qquad \mu_{\mathrm{Q}_j}^2 = h_{\mathrm{Q}_j}^2 \kappa, \tag{3.30}$$

where $\mu_{\mathrm{Q}_j}^2$ are the renormalized dimensionless quark masses proportional to the *vev* κ of the scalar potential. Moreover the threshold $L_{\mathrm{G}}(\mu_{\mathrm{Q}_j}^2)$ is normalized such that at zero argument it is equal to 1, implying that we obtain again the standard result in the DER, i.e., Eq. (3.5).

As a last comment, we want to remark that in the following we will also discuss some RG trajectories where κ diverges in the UV limit faster than h^{-2}, entailing a complete decoupling of the quarks from the dynamics. In this scenario, in fact, the assumption of the DER would then fail and the gluon anomalous dimension would read

$$\eta_{\rm G} = -\frac{11 N_{\rm c}}{24\pi^2} g_{\rm s}^2. \tag{3.31}$$

Limiting Case: Deep Euclidean Regime

In Sec. 3.1 we have presented the universal one-loop beta functions for the top-Yukawa coupling as well as the quartic Higgs self-interaction. Being universal, these one-loop coefficients must be independent from the renormalization scheme used to regularized the theory, thus their contribution can be extracted also from the functional RG improved beta functions in Eqs. (3.26–3.28). For this purpose, one has to set all the anomalous dimensions occurring in the threshold functions to zero, but keep the η's entering the dimensional scaling of the renormalized couplings. The latter in fact contribute to the perturbative one-loop flow equation via one-particle reducible graphs. Furthermore, one has to take the limit toward the DER, by setting the scalar mass parameters as well as the scalar vacuum expectation value to zero, in order to neglect threshold effects. In other words we assume that the scalar potential is a purely self-interacting potential $u \sim \lambda\rho^2/2$. With these substitutions, the anomalous dimensions reduce to

$$\eta_{\rm H} = \frac{N_{\rm c}}{8\pi^2} h^2, \qquad \eta_\psi = \frac{1}{32\pi^2} h^2, \tag{3.32}$$

and the flow equation for the top-Yukawa coupling boils down to

$$\partial_t h^2 = (\eta_{\rm H} + 2\eta_\psi) h^2 + \frac{h^4}{8\pi^2} - \frac{3}{8\pi^2} \frac{N_{\rm c}^2 - 1}{N_{\rm c}} h^2 g_{\rm s}^2, \tag{3.33}$$

which coincides precisely with Eq. (3.7), if we substitute Eq. (3.32). Moreover, also the one-loop beta function for λ in the DER can be obtained from its FRG-improved flow equation. In fact, by assuming a purely self-interacting potential $u = \lambda\rho^2/2$, Eq. (3.22) reduces exactly to Eq. (3.12).

3.3. Effective Field Theory Analysis for the Scalar Potential

As we emphasized in the previous section, the threshold functions in the FRG flow equations for $u(\rho)$, h^2 and the anomalous dimensions take properly into account the decoupling of massive modes along the entire RG flow. In this section we want to systematically include these threshold effects by means of an *effective-field-theory* (EFT) analysis, and in doing so, we investigate whether the results in a more general mass-dependent scheme are sensitive to the assumption of working in the DER as a special case. It will turn out, in fact, that the restriction to the DER is severe and legitimate only for the CEL solution. Therefore, we now generalize the discussion outlined in Sec. 3.1, by including a mass term for the scalar Higgs field as well as higher-dimensional scalar interaction terms, the latter being perturbatively nonrenormalizable. This approach slightly differs from the standard EFT paradigm, which consists of adding, in the classical action, *all* operators allowed from the symmetries of the model, up to some given dimensionality. In fact, we want to make the following further choices on the higher-dimensional operators we add:

1. consider only momentum-independent scalar self-interactions,
2. consider the scale dependence of one coupling as a free parameter in order to play the role of a boundary condition.

Even though the focus on point-like scalar self-interactions can be relaxed, including in the effective Lagrangian *all* interactions up to some given dimensionality, the second ingredient is of crucial importance in order to reveal new AF solutions, as was already anticipated in Sec. 2.4.2.

In order to set the stage for this approximation, we start from a systematic polynomial expansion of $u(\rho)$ around the flowing minimum κ, which can be either at vanishing field amplitude (SYM regime) or at some nontrivial value (SSB regime). Assuming that the system is in the SSB regime, the potential is parameterized as

$$u(\rho) = \sum_{n=2}^{N_\mathrm{p}} \frac{\lambda_n}{n!}(\rho - \kappa)^n, \tag{3.34}$$

where each coefficient λ_n corresponds to the point-like scalar self-interaction coupling between $2n$ Higgs scalar fields and the corresponding beta function is

$$\partial_t \lambda_n = \frac{\mathrm{d}^n}{\mathrm{d}\rho^n}\bigg|_{\rho=\kappa} \partial_t u(\rho) + \frac{\mathrm{d}^{n+1} u(\rho)}{\mathrm{d}\rho^{n+1}}\bigg|_{\rho=\kappa} \partial_t \kappa. \tag{3.35}$$

Together with the latter equation we should also consider the running of the nontrivial minimum

$$\partial_t \kappa = -\frac{\partial_t u'(\rho)}{u''(\rho)}\bigg|_{\rho=\kappa}, \tag{3.36}$$

obtained from the definition of the minimum itself $u'(\kappa) = 0$. Generically, we expect all couplings to be generated by fluctuations, i.e., $N_\mathrm{p} \to \infty$, whereas truncating the sum at some finite N_p corresponds to a polynomial approximation of the potential.

3.3.1. Deep Euclidean Regime

As a preliminary investigation, we want to restrict the EFT-like analysis to the DER. To implement this regime we just need to take the limit $\kappa \to 0$, since it is the only mass parameter in the classical action. By substituting the polynomial ansatz for the scalar potential (in the SYM regime) into Eq. (3.22), and further setting the anomalous dimensions inside the threshold functions to zero, we recover the set of one-loop beta functions $\partial_t \lambda_n$ for $n = 2, \ldots, N_\mathrm{p}$ in the DER. Since we are interested in constructing AF trajectories, we allow for any arbitrary scaling of the quartic coupling λ_2 with respect to the AF strong gauge coupling, and introduce the finite ratio $\hat{\lambda}_2$ defined in Eq. (3.13) for $\lambda = \lambda_2$. In the same way we define also rescaling for the higher-order couplings and corresponding beta functions

$$\hat{\lambda}_n = \frac{\lambda_n}{g_\mathrm{s}^{2P_n}}, \qquad \partial_t \hat{\lambda}_n = \frac{\partial_t \lambda_n}{g_\mathrm{s}^{2P_n}} - P_n \hat{\lambda}_n \eta_\mathrm{G}. \tag{3.37}$$

For example, the beta function for the quartic coupling reads [17]

$$\partial_t \hat{\lambda}_2 = \frac{9\hat{\lambda}_2^2 g_\mathrm{s}^{4P}}{16\pi^2} + g_\mathrm{s}^2 \hat{\lambda}_2 \left(\frac{1}{6\pi^2} + \frac{7P}{4\pi^2} \right) - \frac{5\hat{\lambda}_3 g_\mathrm{s}^{2P_3 - 4P}}{32\pi^2} - \frac{g_\mathrm{s}^{4-4P}}{27\pi^2}, \tag{3.38}$$

which depends on the powers P, P_3 and the rescaled coupling $\hat{\lambda}_3$. From the latter equation we can observe that one possible combination for the asymptotic powers are $P = 1/2$ and $P_3 > 4P$, in a such a way that the contribution from the coupling $\hat{\lambda}_3$ is subleading. This turns out to be in fact the only viable solution, which corresponds precisely to the CEL solution given in Eq. (3.16). Let us therefore choose this particular value of P and look for the corresponding values of P_n and $\hat{\lambda}_n$ for $n \geq 3$. The beta function for $\hat{\lambda}_3$ becomes

$$\partial_t \hat{\lambda}_3 = 2\hat{\lambda}_3 + \left(\frac{2}{81\pi^2} - \frac{81}{16\pi^2} \hat{\lambda}_2^3 \right) g_\mathrm{s}^{2(3-P_3)} + \mathcal{O}(g_\mathrm{s}^2), \tag{3.39}$$

where subleading contributions proportional to g_s^2 have been omitted. Because of the definition of QFP, $\hat{\lambda}_3$ has to approach a finite and nonzero value in the $g_\mathrm{s}^2 \to 0$ limit, thus the only possible solution occurs at $P_3 = 3$. In the very same way it is possible to fix the scaling for the rest of the couplings, and to conclude that the only solution corresponds to

$$P = \frac{1}{2}, \qquad P_{n\geq 3} = n. \tag{3.40}$$

This choice for the scaling powers P_n, together with the polynomial truncation in Eq. (3.34) for $\kappa = 0$, provides a closed system of N_p equations in N_p variables, when one looks at the

[17] Let us remind the reader once more that, all throughout this chapter, we focus our attention on the special trajectory where the top-Yukawa coupling is proportional to the strong gauge coupling. Therefore, any occurrence of h^2 in the beta function for $\hat{\lambda}_n$ can be substituted by $h^2 = \chi_\mathrm{s}^2 g_\mathrm{s}^2$, where the QFP value χ_s^2 is given by Eq. (3.11).

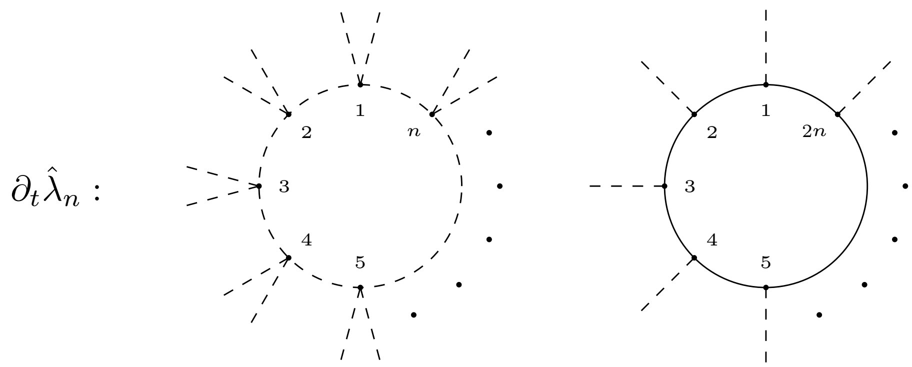


Fig. 3.4.: Diagrammatic representation for the two contributions to the RG flow equation (3.42) of $\hat{\lambda}_n$ in the DER and in the UV limit, where the rescaled Higgs quartic coupling $\hat{\lambda}_2$ plays a dominant role.

QFP condition. To give an example, the first four beta functions are shown here:

$$
\begin{aligned}
\partial_t \hat{\lambda}_2 &= g_s^2 \left(\frac{9}{16\pi^2} \hat{\lambda}_2^2 + \frac{25}{24\pi^2} \hat{\lambda}_2 - \frac{1}{27\pi^2} \right) + \mathcal{O}(g_s^4), \\
\partial_t \hat{\lambda}_3 &= 2\hat{\lambda}_3 - \frac{81}{16\pi^2} \hat{\lambda}_2^3 + \frac{2}{81\pi^2} + \mathcal{O}(g_s^2), \\
\partial_t \hat{\lambda}_4 &= 4\hat{\lambda}_4 + \frac{243}{4\pi^2} \hat{\lambda}_2^4 - \frac{16}{729\pi^2} + \mathcal{O}(g_s^2), \\
\partial_t \hat{\lambda}_5 &= 6\hat{\lambda}_5 - \frac{3645}{4\pi^2} \hat{\lambda}_2^5 + \frac{160}{6561\pi^2} + \mathcal{O}(g_s^2).
\end{aligned} \tag{3.41}
$$

By neglecting the subleading contributions in g_s, we have that the QFP solution for the scalar quartic coupling is $\hat{\lambda}_2 = \hat{\lambda}_2^{\pm}$ as in Eq. (3.16), and all the other higher-order couplings are functions of $\hat{\lambda}_2$ only. For the positive root $\hat{\lambda}_2^+$ the sign of $\hat{\lambda}_n$ with $n > 2$ is alternating, whereas for the negative root $\hat{\lambda}_2^-$ all the higher order couplings stay negative. Furthermore, by solving numerically the system of QFP equations at the next-to-leading order in g_s^2, it is possible to see that only the positive root for $\hat{\lambda}_2$ leads to a fully real solution for all $3 \leq n \leq N_p$.

In order to investigate the stability of the scalar potential for the solution $\hat{\lambda}_2 = \hat{\lambda}_2^+$, we resum the series in Eq. (3.34) at the leading order in g_s^2. The structure of the higher order beta functions is

$$
\partial_t \hat{\lambda}_{n\geq 3} = 2(n-2)\hat{\lambda}_n + \frac{(-1)^n n!}{32\pi^2} \left[(3\hat{\lambda}_2)^n - 12 \left(\frac{2}{9} \right)^n \right], \tag{3.42}
$$

where $2(2-n)$ is the canonical dimension of $\hat{\lambda}_n$, the second term (being proportional to $\hat{\lambda}_2^n$) corresponds to a scalar loop contribution with n quartic vertices and the third term (being proportional to the QFP value χ_s^{2n}) corresponds to a fermion loop with $2n$ top-Yukawa vertices. These two contributions are diagrammatically represented in Fig. 3.4.

Effective Potential in the ϕ^4-approximation

Equation (3.42) entails that among all possible scalar self-interactions, the $\hat{\lambda}_2$ coupling plays a dominant role in the UV limit. We can call indeed this regime ϕ^4-*dominance regime* and it can be studied within the effective potential paradigm. In other words, we can specify a pure ϕ^4-interaction, i.e., $u(\rho) = \lambda_2\rho^2/2$, in the bosonic threshold function $l_0^{(\mathrm{H})d}$ appearing on the right-hand side of Eq. (3.22), such that

$$\omega_{\mathrm{H}} = 3\lambda_2\rho. \tag{3.43}$$

At this point, we should anticipate here a technical aspect which will turn out to be particularly useful from Sec. 3.4 on: all the rescalings for $\lambda_{n\geq 3}$ can be embedded in a suitable redefinition of ρ. This can be achieved by defining a new field variable

$$z = g_{\mathrm{s}}^2\rho. \tag{3.44}$$

In fact, the polynomial expansion in Eq. (3.34) for $\kappa = 0$ can be written as

$$u(z) = \frac{1}{2}\hat{\lambda}_2\frac{z^2}{g_{\mathrm{s}}^2} + \sum_{n=3}^{N_{\mathrm{p}}}\frac{1}{n!}\hat{\lambda}_n z^n\Big|_{z=g_{\mathrm{s}}^2\rho}. \tag{3.45}$$

With the above field redefinition, the RG flow equation for $u(z)$ is different from the one for $u(\rho)$, since typically the t-scale derivative is taken at constant field argument. Indeed the two RG flows are related to each other as follows

$$\partial_t u(z)\big|_{z=\mathrm{const}} = \partial_t u(\rho)\big|_{\rho=\mathrm{const}} - \eta_{\mathrm{G}} z u'(z), \tag{3.46}$$

with an additional contribution which only modifies the scaling part in the flow equation. This suggests us to define also a total anomalous dimension for the z field which is

$$\eta_z = \eta_{\mathrm{H}} - \eta_{\mathrm{G}}. \tag{3.47}$$

We can now project the RG flow for $u(z)$ onto the ansatz in Eq. (3.45), and restrict the analysis to the ϕ^4-dominance regime with $\omega_{\mathrm{H}} = 3\hat{\lambda}_2 z$. The solution to the QFP system of equations $\partial_t\hat{\lambda}_n = 0$ for $2 \leq n \leq N_{\mathrm{p}}$ reads

$$\hat{\lambda}_2 = \hat{\lambda}_2^{\pm}, \qquad \hat{\lambda}_{n\geq 3} = \frac{(-1)^n n!}{32\pi^2}\,\frac{(3\hat{\lambda}_2)^n - 12\left(\frac{2}{9}\right)^n}{4 - n(2+\eta_z)}. \tag{3.48}$$

The simple structure of the above solution, being direct consequence of the assumption made for ω_{H}, allows us to take the $N_{\mathrm{p}} \to \infty$ limit. The resummation of the series has, indeed, an analytic expression in terms of the Gauß hypergeometric function ${}_2F_1(a, b, c; z)$, which is given by

$$u(z) = \left[\frac{1}{2}\hat{\lambda}_2 \frac{z^2}{g_s^2} + \frac{1}{32\pi^2} \frac{(3\hat{\lambda}_2 z)^3}{2+3\eta_z} \, {}_2F_1\left(1, \frac{2+3\eta_z}{2+\eta_z}, \frac{4+4\eta_z}{2+\eta_z}; -3\hat{\lambda}_2 z\right) \right.$$
$$\left. - \frac{3}{8\pi^2} \frac{\left(\frac{2}{9}z\right)^3}{2+3\eta_z} \, {}_2F_1\left(1, \frac{2+3\eta_z}{2+\eta_z}, \frac{4+4\eta_z}{2+\eta_z}; -\frac{2}{9}z\right) \right]_{z=g_s^2\rho} . \tag{3.49}$$

The effective potential $u(z)$ has the following property:

$$\lim_{z\to 0} u''(z) = \frac{\hat{\lambda}_2}{g_s^2}, \tag{3.50}$$

clearly reasonable from the chosen polynomial ansatz in Eq. (3.45).

Stability of the Effective Potential

Now we are ready to discuss the stability property for the effective potential $u(z)$ given in Eq. (3.49). In other words, we ask ourselves whether the scalar potential remains bounded from below even if the UV limit, where the theory approaches the GFP, is taken. Since the solution has been constructed by a resummation of a local expansion for small field amplitudes, it might depart from the actual FP potential at large values of ρ, due to nonanalytic terms which cannot be represented by simple integer powers of ρ. It is also true that we are interested in studying the stability issue of the scalar potential mainly in the asymptotic region where the two limits

$$\rho \to \infty, \qquad\qquad g_s^2 \to 0, \tag{3.51}$$

are taken simultaneously. However, since the effective potential is a function of both variables ρ and g_s^2, there might be several such asymptotic regions, corresponding to different ways of taking this combined limit. To classify these possible regions, we address the dependence of the loop integrals on g_s and ρ. By substituting the asymptotic UV scaling for λ_2 and h^2, the threshold functions $l_0^{(\mathrm{H})d}$ and $l_0^{(\mathrm{F})d}$ in Eq. (3.22) become functions of $\omega_\mathrm{H} = 3\hat{\lambda}_2 z$ and $\omega_\mathrm{F} = 2z/9$ respectively. Thus, the variable entering the loop integrals is z, as defined in Eq. (3.44). Therefore, it is natural to define an outer region where $z \gg 1$ and an inner region where $z \ll 1$. In App. D we will address in more detail these two cases and show that, in both asymptotic regions, the combined limit presented in Eq. (3.51) exist and is given by

$$u(\rho) \underset{\rho\to\infty}{\sim} \frac{1}{2}\, \hat{\lambda}_2 \, g_s^2 \rho^2 > 0. \tag{3.52}$$

The effective potential in Eq. (3.49) is thus stable and remains stable for any arbitrarily small value of g_s^2. This proves for the first time that the CEL solution corresponds to a bounded potential in the DER.

3.3.2. Inclusion of Threshold Effects

In this section we finally relax the restriction to be in the DER, and we account for the running of the scalar mass term. In other words, we include the possibility for a nontrivial minimum, by choosing a polynomial expansion of the scalar potential around $\rho = \kappa \neq 0$, as in Eq. (3.34). By projecting the left-hand side of the Eq. (3.22) onto this ansatz, we can derive the flow equations for the rescaled couplings $\hat{\lambda}_n$ as defined in Eqs. (3.13) and (3.37). Similarly, also the coupling κ may scale asymptotically as a certain power of g_s^2. We define therefore

$$\hat{\kappa} = g_\mathrm{s}^{2Q} \kappa, \tag{3.53}$$

where the real power Q is a priori arbitrary.

In order to construct polynomial solutions of the QFP equations for the rescaled couplings $\hat{\lambda}_n$ and $\hat{\kappa}$, we set up the following recursive problem: we solve the equation $\partial_t \hat{\kappa} = 0$ for $\hat{\lambda}_2$, and $\partial_t \hat{\lambda}_n = 0$ for $\hat{\lambda}_{n+1}$. Upon truncating the series of equations at some N_p, this can be achieved only if one more coupling $\hat{\lambda}_{N_\mathrm{p}+1}$ is retained. In the spirit of what has been explained at the beginning of Sec. 3.3, we treat this extra coupling $\hat{\lambda}_{N_\mathrm{p}+1}$ as a free parameter. In addition, the anomalous dimensions inside the threshold functions in Eqs. (3.22) and (3.27) are set to zero in order to boil down to the one-loop beta functions for the scalar couplings. Moreover, we restrict our analytical analysis to the leading order in g_s^2 and we truncate the scalar potential up to $N_\mathrm{p} = 2$.

$\mathbf{P \in (0, 1/2)}$

Because of the qualitative similarity between the flow equations of the present model and those analyzed in Refs. [111, 112] (indeed the leading order contributions in g_s only come from the scalar loop), we can expect that $\hat{\kappa} = \kappa$ for P being equal or smaller then 1/2. Thus, we make the ansatz $Q = 0$, which nevertheless turns out to be the correct solution. The leading orders in g_s^2 in the flow equations of the rescaled couplings are

$$\partial_t \hat{\lambda}_2 = -\frac{\hat{\lambda}_3\, g_\mathrm{s}^{2P_3-4P}}{16\pi^2} + \frac{9\hat{\lambda}_2^2\, g_\mathrm{s}^{4P}}{16\pi^2} + \frac{\kappa\hat{\lambda}_3^2\, g_\mathrm{s}^{4P_3-8P}}{16\pi^2\hat{\lambda}_2}, \tag{3.54}$$

$$\partial_t \kappa = -2\kappa + \frac{3}{32\pi^2} + \frac{\kappa\hat{\lambda}_3\, g_\mathrm{s}^{2P_3-4P}}{16\pi^2\hat{\lambda}_2}. \tag{3.55}$$

One solution corresponds to the case where the contribution coming from $\hat{\lambda}_3$ is subleading in Eq. (3.55), i.e., $P_3 > 2P$, and it reads

$$\kappa = \frac{3}{64\pi^2}, \qquad \hat{\lambda}_2^2 = \frac{\hat{\lambda}_3}{9}, \qquad\qquad Q = 0, \qquad P_3 = 4P, \tag{3.56}$$

where $\hat{\lambda}_3$ must be positive, but is otherwise arbitrary. By contrast, a second solution is found to correspond to the case where the $\hat{\lambda}_3$ term contributes to the flow equation for κ in the UV limit, i.e., $P_3 = 2P$. In this other case we have

$$\kappa = \frac{5}{64\pi^2}, \qquad \hat{\lambda}_2 = \frac{5\hat{\lambda}_3}{64\pi^2}, \qquad\qquad Q = 0, \qquad P_3 = 2P, \tag{3.57}$$

where again the rescaled cubic scalar coupling remains a free parameter.

We have therefore found two roots for the QFP system of equations, each of them corresponds to a *one-parameter family* of solutions. [18]

P = 1/2

For this value of P, the analysis is less straightforward, due to the interplay between the scalar and fermionic loop. The leading g_s^2 contributions to the RG flow of $\hat{\lambda}_2$ and κ are

$$\partial_t \hat{\lambda}_2 = g_s^2 \left(\frac{9\hat{\lambda}_2^2}{16\pi^2} + \frac{25\hat{\lambda}_2}{24\pi^2} - \frac{1}{27\pi^2} \right) + \frac{\kappa \hat{\lambda}_3^2 \, g_s^{4P_3-4}}{16\pi^2 \hat{\lambda}_2} - g_s^{2P_3-2} \frac{\hat{\lambda}_3}{4\pi^2} \left(\frac{1}{4} + \frac{1}{3\hat{\lambda}_2} \right), \tag{3.58}$$

$$\partial_t \kappa = -2\kappa + \frac{3}{32\pi^2} - \frac{1}{12\pi^2 \hat{\lambda}_2} + \frac{\kappa \hat{\lambda}_3 \, g_s^{2P_3-2}}{16\pi^2 \hat{\lambda}_2}. \tag{3.59}$$

If $P_3 > 2$ the contributions due to $\hat{\lambda}_3$ are negligible in the UV limit and we recover the CEL solution of Eq. (3.16). Moreover a positive (negative) solution for $\hat{\lambda}_2$ leads to a negative (positive) solution for κ, suggesting that the stable CEL potential possesses only the trivial minimum $\kappa = 0$. For $1 < P_3 < 2$, the contribution coming from $\hat{\lambda}_3$ plays the dominant role in the RG flow of $\hat{\lambda}_2$ but is subleading for κ. The solution of the corresponding QFP equations is $\kappa = 5/(64\pi^2)$ and $\hat{\lambda}_2 = -4/3$, which has to be rejected since it does not fulfill the requirement that κ should be a minimum of the potential.

Therefore, the only two new solutions which may occur correspond to $P_3 = 1$ and $P_3 = 2$. In the first case, the QFP solution to the previous system of equations is

$$\kappa = \frac{5}{64\pi^2}, \qquad \hat{\lambda}_2 = \frac{5\hat{\lambda}_3}{64\pi^2} - \frac{4}{3}, \qquad\qquad Q = 0, \qquad P_3 = 1. \tag{3.60}$$

[18] Let us notice that the first class of solutions given by Eq. (3.56) was already discovered in [111, 112], whereas the second one corresponding to Eq. (3.57) is new. This latter solution was not observed in [111, 112] because of simplifying approximations in the analysis of the RG equations. In fact, only linear insertions of the coupling λ_3 into the beta function of lower-dimensional couplings were considered.

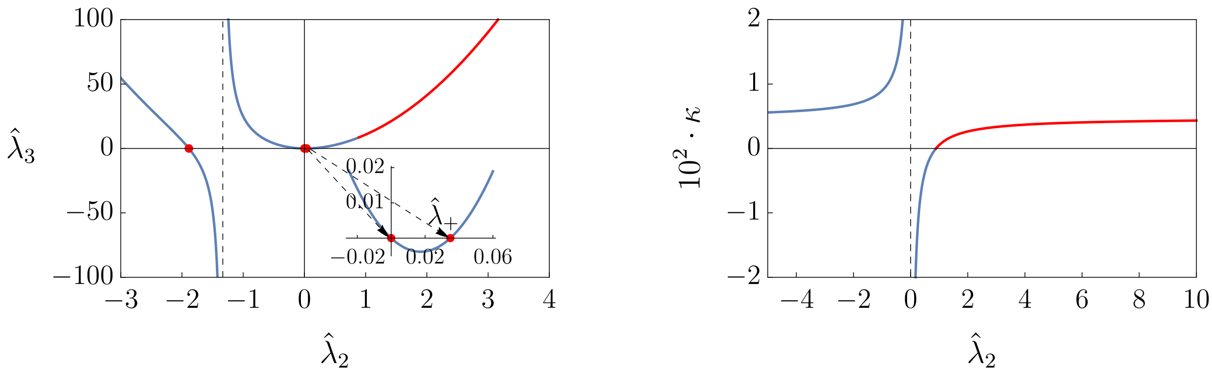


Fig. 3.5.: Effective field theory analysis including thresholds for the rescaling powers $P = 1/2$ and $P_3 = 2$, cf. Eq. (3.61). We show the QFP solutions for $\hat\lambda_3$ (left) and κ (right) as a function of $\hat\lambda_2$ and at the leading order in the $g_s^2 \to 0$ limit. The magnified inset near the origin highlights the CEL solution $\hat\lambda_2 = \hat\lambda_2^+$ as in Eq. (3.16). The right panel shows that there are solutions with a positive nontrivial minimum and scalar quartic coupling only for $\hat\lambda_2 > 4$.

In the second case where $P_3 = 2$, instead, we have a solution for κ and $\hat\lambda_3$ depending on $\hat\lambda_2$ via the expressions

$$\kappa = \frac{1}{192\pi^2}\frac{9\hat\lambda_2 - 8}{\hat\lambda_2}, \qquad \hat\lambda_3 = \frac{\hat\lambda_2}{9}\,\frac{243\hat\lambda_2^2 + 450\hat\lambda_2 - 16}{3\hat\lambda_2 + 4}, \qquad Q = 0, \quad P_3 = 2. \tag{3.61}$$

The second solution for $P_3 = 2$ is shown in Fig. 3.5. The three red dots in the left panel highlight the three roots corresponding to $\hat\lambda_3 = 0$. For one of these roots we find $\hat\lambda_2 = 0$, which must be discarded as the QFP value for κ becomes singular. The other two roots are exactly the $\hat\lambda_2^\pm$ of Eq. (3.16). Moreover, it is clear from Eq. (3.61) that the condition $\hat\lambda_2 > 8/9$ has to hold in order to obtain a positive nontrivial minimum and, at the same time, a positive quadratic scalar coupling. This can also be seen from the right panel of Fig. 3.5, and this *one-parameter family* of solution is highlighted by a red line.

For completeness, we refer the reader to App. E for the discussion relating the values $P > 1/2$. We do not present that analysis here since no physical solutions, with the desired properties $\hat\lambda_2 > 0$ and $\kappa > 0$, can be found within this range of the scaling power P.

3.4. Full Effective Potential in the ϕ^4-dominance Approximation

So far, we have projected the RG flow of the potential onto a polynomial basis and studied the running of the various coefficients. Here we want to investigate the *full* functional RG flow for an *arbitrary* scalar potential which can also include nonpolynomial structures [243, 244]. These structures, which would be hidden in any polynomial truncation, can be taken into account by performing a one-loop computation where the thresholds $l_0^{(\mathrm{H})}$ and $l_0^{(\mathrm{F})}$ in

Eq. (3.22) are considered as field-dependent functions. Moreover, in order to be able to perform calculations, we choose a convenient piece-wise linear regulator [192, 193] as the regularization scheme for the massive modes. However, we will see in the following Chap. 4 that the restriction of choosing a particular IR regulator can be relaxed and even a general analysis, for arbitrary IR regularization schemes, can be performed.

We already know from the previous sections that AF trajectories, can be constructed by searching for QFPs for the g_s^2-rescaled couplings. To implement this condition in a functional set-up we define, in a similar way to what we have done in Sec. 3.3.1, cf. Eqs. (3.44) and (3.46), a new field variable and its potential

$$x = g_s^{2P} \rho, \qquad\qquad f(x) = u(\rho). \tag{3.62}$$

We also denote the minimum $g_s^{2P}\kappa$ by x_0 and the couplings by ξ_n, such that

$$f'(x_0) = 0, \qquad\qquad \frac{d^n f(x)}{dx^n}\Big|_{x=x_0} = \xi_n. \tag{3.63}$$

The arbitrary rescaling power P is chosen to be that of Eq. (3.13), in order to have $\xi_2 = \hat{\lambda}_2$, because we specifically look for QFPs where ξ_2 approaches a finite value in the UV limit. [19]

As a first-level approximation, we consider here an intermediate step between a polynomial truncation and a fully functional approach, which is based on the expectation that the marginal quartic scalar coupling plays a dominant role in the UV limit. This is precisely what we have called ϕ^4-*dominance approximation* in Sec. 3.3.1. More precisely, we substitute Eq. (3.43) on the right-hand side of Eq. (3.22), but we consider the scalar potential as an unknown arbitrary function in the scaling term and on the left-hand side of the flow equation itself. This leads to the following beta function for the rescaled effective potential

$$\partial_t f(x) = -4f(x) + d_{x,\mathrm{s}} x f'(x) + \frac{1}{32\pi^2}\left(\frac{1}{1+3\xi_2 g_s^{2P} x} - \frac{12}{1+\frac{2}{9} g_s^{2-2P} x}\right), \tag{3.64}$$

$$d_{x,\mathrm{s}} = 2 + \eta_{\mathrm{H}} - P\eta_{\mathrm{G}} \equiv 2 + \eta_{x,\mathrm{s}}, \tag{3.65}$$

where the anomalous dimensions $\eta_{x,\mathrm{s}}$ of the rescaled invariant field x includes also a contribution from the running of the strong gauge coupling. The anomalous dimensions η_{H} and η_{G} are also evaluated by neglecting the possible appearance of higher-dimensional couplings, as well as contributions from the scalar mass term. Therefore, in the following, we will make use of Eqs. (3.5) and (3.12) which rely on the DER. Moreover, let us remind the reader again that, regarding the top-Yukawa coupling, we are focusing all throughout this chapter

[19] It might happen however that, at a QFP, the nontrivial minimum x_0 and the couplings $\xi_{n\geq 3}$ differ from their actual scaling solutions $\hat{\kappa}$ and $\hat{\lambda}_{n\geq 3}$ respectively. Consequently, also the QFP solution $f(x)$ will differ from the actual scaling potential. Therefore, the rescaling of Eq. (3.62), together with the condition $\xi_2 = \hat{\lambda}_2$, is expected to be useful as long as the quartic scalar coupling is the dominant term in the approach of the scalar potential towards a total AF trajectory.

on the AF trajectory along which $h^2 = \chi_s^2 g_s^2$ with the corresponding QFP value χ_s^2 given by Eq. (3.11), which also rely on the DER. [20]

3.4.1. Scaling Solution for the Scalar Potential

Thanks to the simple approximation just described, the equation $\partial_t f(x) = 0$ reduces to a first-order linear ordinary differential equation (ODE) which can be solved analytically for generic P and its QFP solution reads

$$f(x) = C_f\, x^{\frac{4}{2+\eta_{x,s}}} + \frac{1}{128\pi^2}\, {}_2F_1\left(1, -\frac{4}{2+\eta_{x,s}}, \frac{-2+\eta_{x,s}}{2+\eta_{x,s}}; -3\xi_2 g_s^{2P} x\right) - \frac{3}{32\pi^2}\, {}_2F_1\left(1, -\frac{4}{2+\eta_{x,s}}, \frac{-2+\eta_{x,s}}{2+\eta_{x,s}}; -\frac{2}{9} g_s^{2-2P} x\right). \tag{3.66}$$

This expression consists of a sum of the homogeneous solution, proportional to the integration constant C_f, and of a particular solution, given by the hypergeometric functions ${}_2F_1(a, b, c; x)$, obtained by integrating the non-homogeneous part.

As a first analysis, we want to understand the asymptotic properties of the full g_s^2-dependent solution $f(x)$. Specifically, we want to identify those parameter ranges for C_f and ξ_2 for which the potential is stable, namely bounded from below. To this end, we focus on the asymptotic behavior of $f(x)$ at $x \to \infty$ and, in particular, we are interested in the UV regime where $g_s^2 \to 0$. Since the QFP potential for given C_f (which might also depend on g_s^2) is a function of the two variables x and g_s^2, we have to take the limit process with care in order to investigate the asymptotic behavior of $f(x)$ in the far UV regime. In order to address these informations, we analyze the flow for fixed arguments,

$$z_{\rm H} = 3\xi_2 g_s^{2P} x, \qquad\qquad z_{\rm F} = \frac{2}{9} g_s^{2(1-P)} x, \tag{3.67}$$

of the hypergeometric functions. For small enough g_s^2 and $P \gtrless 1/2$, we have $z_{\rm F} \gtrless z_{\rm H}$. Thus, one can divide the interval $x \in [0, \infty)$ into three distinct domains. Suppose $P < 1/2$, then we define the g_s^2-dependent boundary $x_1(g_s^2)$ of an inner interval $x \in [0, x_1)$ by requiring $z_{\rm H} = 1$ and the boundary $x_2(g_s^2)$ of an outer interval (x_2, ∞) by $z_{\rm F} = 1$, for fixed P and ξ_2. Whereas, for $P > 1/2$, the requirement $z_{\rm H} = 1$ and $z_{\rm F} = 1$ will define x_2 and x_1 respectively. In addition, in the case of our interest with $P < 1$, the two boundaries x_1 and x_2 grow towards larger values and always fulfill $x_2 > x_1$ when we send $g_s^2 \to 0$. For a graphical

[20] At this point the reader might be puzzled since we have stressed since the beginning that it is one of our main goals to go beyond the DER assumption of standard perturbation theory. In fact, we allow for a generic solution of Eq. (3.64) which can also develop a nontrivial minimum x_0 with a nontrivial scaling dependence on g_s. Yet, in order to keep our analysis as simple as possible, we assume that the anomalous dimensions and the QFP for the top-Yukawa coupling can be approximated with their DER expressions. Of course, the consistency of these assumptions has to be verified once a scaling solution for $f(x)$ is provided. In fact, in Sec. 4.4.1 and App. G.1 we will discuss more clearly how this consistency check is not fulfilled for the scaling powers $P < 1/4$, but fulfilled for $1/4 \leq P \leq 1/2$.

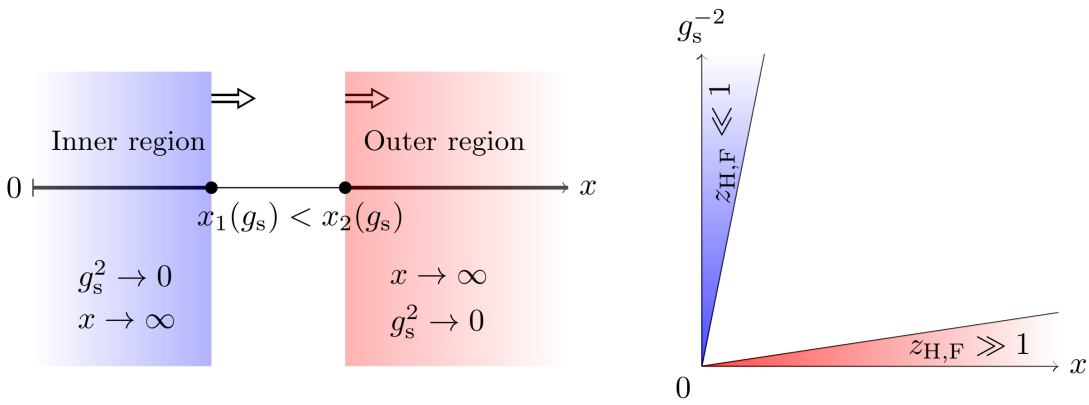


Fig. 3.6.: Diagrammatic representation for the two possible ways of performing the limit $g_s^2 \to 0$ and $x \to \infty$. *Left Panel:* The boundary points x_1 and x_2 (for the inner and outer region respectively) diverge while $g_s^2 \to 0$, but always satisfying $x_1 < x_2$ for $P < 1$. In the inner region, the asymptotic behavior of $f(x)$ is captured by taking first $g_s^2 \to 0$ and then $x \to \infty$. In the outer region this order is reverse. *Right Panel:* Schematic example for the two asymptotic regions in the (x, g_s^{-2}) plane. The limit $z_{H,F} \gg 1$ describes properly the outer region since the asymptotic region in the plane is reached by taking first the limit $x \to \infty$ and then $g_s^2 \to 0$. The limit $z_{H,F} \ll 1$ describes instead the inner region.

interpretation see the left panel of Fig. 3.6.

Approximating the hypergeometric functions for small but fixed arguments $z_{H,F} \ll 1$, we obtain a valid approximation of the potential in the first inner interval as this also implies $x \ll x_1$. Thus, we are able to reliably check the asymptotic behavior in this region by first performing the limit $g_s^2 \to 0$ and afterwards $x \to \infty$. On the contrary, in case the hypergeometric functions shall be expanded for large arguments $z_{H,F} \gg 1$, we have to perform first the limit $x \to \infty$ before sending $g_s^2 \to 0$ such that one stays in the outer interval. On the right panel of Fig. 3.6 there is a schematic representation of the asymptotic areas in the (x, g_s^{-2}) plane which are accessible by considering the expansions $z_{H,F} \ll 1$ or $z_{H,F} \gg 1$.

Large-field Behavior

For finite values of g_s^2, we can investigate the asymptotic behavior in the interval $x \in (x_2, \infty)$ by expanding the QFP potential in Eq. (3.66) around $x = \infty$. The analytic expansion yields

$$f(x) = C_{f,\infty}\, x^{\frac{4}{2+\eta_{x,s}}} + \mathcal{O}(x^{-1}), \tag{3.68}$$

where the asymptotic coefficient in front of the scaling term is a function $C_{f,\infty}(C_f, \xi_2, g_s^2, P)$ depending on the different parameters characterizing the RG trajectory. The full expression is given in App. D.2, cf. Eq. (D.11). We investigate its g_s^2-dependence in the far UV by an expansion at vanishing g_s. This yields a scaling $C_{f,\infty} \sim g_s^{4P-2}$ for $P \in (0, 1/2)$ and $C_{f,\infty} \sim g_s^{2-4P}$ for $P \in (1/2, 1)$ at fixed C_f. Let us call $\hat{C}_{f,\infty}$ the corresponding finite ratio.

For the sake of clarity, it is therefore useful to define a new variable

$$\hat{C}_f = \begin{cases} g_\mathrm{s}^{2-4P} C_f & \text{if} \quad P \in (0, 1/2), \\ C_f & \text{if} \quad P = 1/2, \\ g_\mathrm{s}^{4P-2} C_f & \text{if} \quad P \in (1/2, 1), \end{cases} \tag{3.69}$$

such that the asymptotic coefficient, at the leading order in g_s^2, is

$$\hat{C}_{f,\infty} = \begin{cases} \hat{C}_f - \dfrac{9\xi_2^2}{64\pi^2 \hat{\eta}_{x,\mathrm{s}}} & \text{if} \quad P \in (0, 1/2), \\ C_f - \dfrac{243\,\xi_2^2 - 16}{1728\pi^2\,\hat{\eta}_{x,\mathrm{s}}} & \text{if} \quad P = 1/2, \\ \hat{C}_f + \dfrac{1}{108\pi^2 \hat{\eta}_{x,\mathrm{s}}} & \text{if} \quad P \in (1/2, 1), \end{cases} \tag{3.70}$$

where $\hat{\eta}_{x,\mathrm{s}} = \eta_{x,\mathrm{s}}/g_\mathrm{s}^2$ is evaluated according to Eqs. (3.5), (3.12) and (3.65). The locus of points that satisfies the condition $\hat{C}_{f,\infty} = 0$ for $P \leq 1/2$ are plotted in Fig. 3.7 by black solid lines. They characterize the transition from the region in the $(\hat{C}_f, \xi_2)$ plane where the potential is bounded from below (right side of the black line) to the region where the potential is unbounded (left side).

Small-field Behavior and the CEL Solution

Next, we study the properties of the solution $f(x)$ for small arguments $x \ll 1$. This is relevant to address both the $x \to 0$ limit at fixed g_s^2, and also to inspect the large field asymptotics for $P < 1$ in the limit where $g_\mathrm{s}^2 \to 0$ and $x \to \infty$ at $z_{\mathrm{H,F}} \ll 1$. For this purpose, we start from the expansion of the QFP potential $f(x)$ for small x, which can be found in App. D.2, cf. Eq. (D.14). The first derivative at the origin is

$$f'(0) = \frac{8 g_\mathrm{s}^{2-2P} - 9\xi_2 g_\mathrm{s}^{2P}}{96\pi^2 (2 - \eta_{x,\mathrm{s}})}, \tag{3.71}$$

and by keeping the leading order in the gauge coupling we have

$$f'(0) = \begin{cases} -\dfrac{3\,\xi_2}{64\pi^2} g_\mathrm{s}^{2P} & \text{if} \quad P \in (0, 1/2), \\ \dfrac{8 - 9\xi_2}{192\pi^2}\, g_\mathrm{s} & \text{if} \quad P = 1/2, \\ \dfrac{1}{24\pi^2}\, g_\mathrm{s}^{2-2P} & \text{if} \quad P \in (1/2, 1). \end{cases} \tag{3.72}$$

Thus, we observe that $f'(0)$ is negative for $P < 1/2$ and $\xi_2 > 0$ while it is always positive for $P > 1/2$.

For $P = 1/2$, the first derivative at the origin changes sign at $\xi_2 = 8/9$. In this case, we

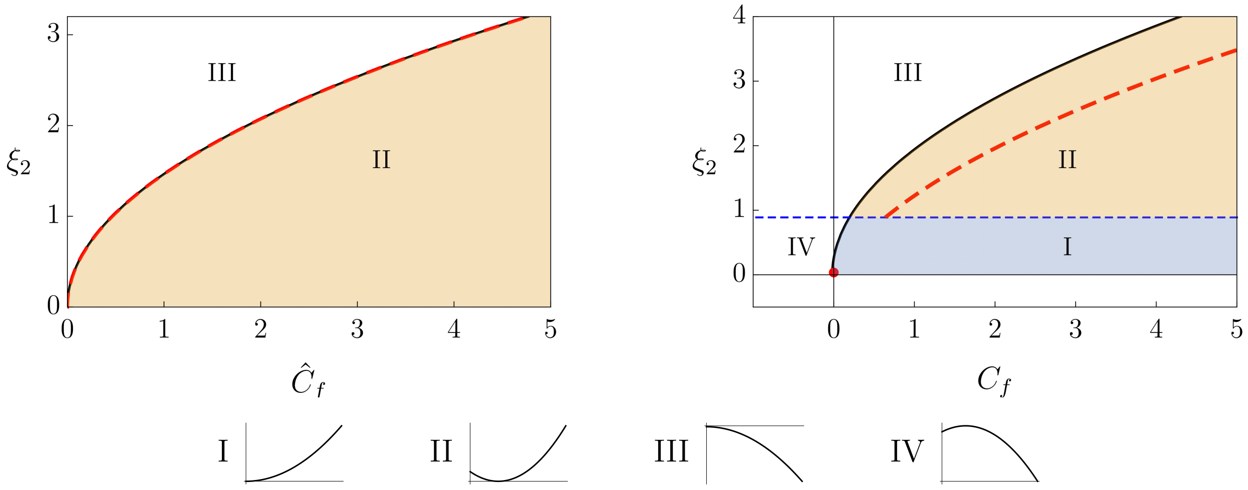


Fig. 3.7.: Stability properties of the effective potential $f(x)$, see Eq. (3.66), for $P \in (0, 1/2)$ (upper left) and $P = 1/2$ (upper right). The two black lines separate the left-hand side regions where the potential is unbounded from below from the right-hand side regions where it is bounded, and in the $g_s^2 \to 0$ limit. Their equations are obtained imposing the condition $\hat{C}_{f,\infty} = 0$ in Eq. (3.70). Sketches of the potential shapes in the different regions are given in the lower panels. Upper right: the blue dashed line $\xi_2 = 8/9$ identifies the locus of points where $f'(0) = 0$. For $\xi_2 < 8/9$, the potential has an unstable minimum in region IV and it is monotonically increasing to $+\infty$ in region I. For $\xi_2 > 8/9$, $f(x)$ has a stable minimum in region II and it is monotonically decreasing to $-\infty$ in region III. The red point at $\hat{C}_f = 0$ and $\xi_2 = \hat{\lambda}_2^+$ corresponds to the CEL solution, see Eq. (3.16), being regular at $x = 0$. The red dashed line in region II shows the *one-parameter family* of new solutions satisfying the consistency condition $f''(x_0) = \xi_2$, as expressed in Eq. (3.80). Upper left: only the regions of type II and III are present and the *one-parameter family* of QFP solutions (red dashed line) lies exactly at the boundary between the stable and unstable regions.

find that the two lines $C_{f,\infty} = 0$ and $\xi_2 = 8/9$ divide the (C_f, ξ_2) plane in four regions with different qualitative behavior for $f(x)$. This is represented in the right panel of Fig. 3.7, where the solid black line corresponds to $C_{f,\infty} = 0$ and the dashed blue line to $\xi_2 = 8/9$. In region II the QFP potential is bounded from below and has a nontrivial stable minimum. In region IV the potential has a nontrivial maximum but is unbounded from below. Instead in regions I and III the function $f(x)$ is monotonically increasing towards $+\infty$ and decreasing to $-\infty$ respectively. For $P < 1/2$, there are only the regions of type II and III.

In the region I, the potential is bounded from below and has a minimum located at the origin, such that the consistency condition to be imposed is $f''(0) = \xi_2$. At the origin the Gauß hypergeometric functions ${}_2F_1$ are analytic in x, whereas the scaling term represents a source of nonanalyticity. In fact, the second derivative at $x = 0$ is not defined as long as $\eta_{x,\mathrm{s}} > 0$, which is generically the case for a potential in the SYM regime and for the present $\mathbb{Z}_2$-Yukawa-QCD model. As a consequence, the consistency condition can be imposed only if the log-type singularity in the second derivative at the origin is removed by requiring $C_f = 0$. With this choice, we obtain the following equation for the finite rescaled quartic scalar coupling ξ_2

$$\xi_2 = \frac{16\, g_s^{4-4P} - 243\, \xi_2^2\, g_s^{4P}}{864\pi^2\, \eta_{x,s}} \qquad \text{if} \qquad C_f = 0, \tag{3.73}$$

whose only solution is the CEL potential described in Sec. 3.1, namely $\xi_2 = \hat{\lambda}_2^{\pm}$ and $P = 1/2$. The positive root $\hat{\lambda}_2^{+}$ is highlighted by a red dot in the right panel of Fig. 3.7.

Having constructed a full effective potential for the CEL solution, we can ask whether this is stable for large field amplitudes and how it is related to the functional $u(z)$ of Eq. (3.49). As shown in App. D.3, we have

$$\lim_{\eta_{x,s}\to\eta_z} f(x) = u(z)\Big|_{z=g_s x} + \frac{8-9\xi_2}{96\pi^2(2-\eta_{x,s})} g_s x. \tag{3.74}$$

Therefore the full solution $f(x)$ includes all the information about $u(\rho)$, plus a linear term that was discarded in Sec. 3.3.1 by definition of the DER. Furthermore, Eqs. (3.68) and (3.70) apply to all values of C_f, thus by choosing $P = 1/2$ and $C_f = 0$ in those equations, and specifying the value $\xi_2 = \hat{\lambda}_2^{+}$, we deduce that the asymptotic behavior for the CEL potential is

$$f(x) \underset{x\to\infty}{\sim} \frac{\hat{\lambda}_2^{+}}{2} x^2. \tag{3.75}$$

We conclude that the CEL solution is stable for arbitrary small values of the strong gauge coupling, even in the UV limit where $g_s^2 \to 0$.

3.4.2. New Solutions with a Nontrivial Minimum

Let us now consider the region II where the potential has a nontrivial minimum x_0. This feature implies an important consequence; as it can be seen from Eq. (3.66), the QFP potential behaves as a nonrational power of x at the origin. In fact its second order derivative is not defined at $x = 0$ as long as $\eta_{x,s} > 0$. However, this problem might be avoided if there is at least one nontrivial minimum x_0, in the spirit of the Coleman-Weinberg mechanism [243], in such a way that we can define the self-interacting scalar couplings as derivatives of the potential at x_0. This is precisely our case, thus we impose the consistency condition $f''(x_0) = \xi_2$.

To simplify the discussion we adopt the same small-field expansion discussed above, which corresponds to neglecting subleading powers of x_0, for small values of the vacuum expectation value. The defining condition for the minimum, $f'(x_0) = 0$, provides the expression for C_f

$$C_f = x_0^{\frac{-2+\eta_{x,s}}{2+\eta_{x,s}}} \frac{2+\eta_{x,s}}{2-\eta_{x,s}} \frac{9\xi_2 g_s^{2P} - 8 g_s^{2-2P}}{384\pi^2}, \tag{3.76}$$

while the consistency condition

$$f''(x_0) = \frac{81\xi_2 g_s^{2P} - 72 g_s^{2-2P}}{864\pi^2(2+\eta_{x,s})x_0} = \xi_2, \tag{3.77}$$

provides the expression for the nontrivial minimum

$$x_0 = \frac{9\xi_2 g_s^{2P} - 8 g_s^{2-2P}}{96\pi^2\xi_2(2+\eta_{x,s})}. \tag{3.78}$$

Different powers of P lead to different leading behaviors in g_s^2 for the latter expression. These can be summarized in the following way

$$x_0 = \begin{cases} \frac{3}{64\pi^2} g_s^{2P} & \text{if} \quad P \in (0, 1/2), \\ \frac{9\xi_2 - 8}{192\pi^2\xi_2} g_s & \text{if} \quad P = 1/2, \\ -\frac{1}{24\pi^2\xi_2} g_s^{2-2P} & \text{if} \quad P \in (1/2, 1). \end{cases} \tag{3.79}$$

These results are in agreement with the EFT analysis including thresholds presented in Sec. 3.3.2. In fact Eqs. (3.56) and (3.61) are identical to those in Eq. (3.79), recalling the relation $x_0 = g_s^{2P}\kappa$. Also the (unphysical) solution for $P \in (1/2, 1)$ is in agreement with the EFT-like analysis (see App. E, in particular Eq. (E.3)).

Moreover, we can substitute the expression for the minimum $x_0(\xi_2, g_s^2)$ inside the parameterization for C_f in Eq. (3.76) for $P = 1/2$. Considering the leading order in g_s^2, we find

$$C_f = \frac{243\,\xi_2^2 - 16}{1728\pi^2\,\hat{\eta}_{x,s}} + \frac{\xi_2}{2} \qquad \text{if} \qquad P = \frac{1}{2}, \tag{3.80}$$

which describes a *one-parameter family* of QFP solutions satisfying the consistency condition at the nontrivial minimum, i.e., $f''(x_0) = \xi_2$. These solutions are represented in the right panel of Fig. 3.7 as a red dashed line laying in region II. The asymptotic behavior for these solutions is obtained by plugging Eq. (3.80) into Eq. (3.70). It turns out that these solutions obey the same asymptotic behavior as for the CEL solution, which is given by a quadratic function in x:

$$f(x) \underset{x\to\infty}{\sim} \frac{\xi_2}{2} x^2 \qquad \text{if} \qquad P = \frac{1}{2}. \tag{3.81}$$

Also for $P < 1/2$ it is possible to find a parameterization $C_f(\xi_2)$ for the QFP solutions with a nontrivial minimum satisfying the consistency condition in x_0. Its leading order contribution in g_s^2 reads

$$\hat{C}_f = \frac{9\,\xi_2^2}{64\pi^2\,\hat{\eta}_{x,s}} \qquad \text{if} \qquad P \in (0, 1/2), \tag{3.82}$$

and coincides exactly with the solution to the condition $\hat{C}_{f,\infty} = 0$. Therefore, we find the asymptotic behavior

$$f(x) \underset{x\to\infty}{\sim} 0 \qquad \text{if} \qquad P \in (0, 1/2), \tag{3.83}$$

which corresponds to an asymptotically flat QFP potential.

Along these two families of QFP solutions for $P \leq 1/2$, it is interesting to evaluate the rescaled cubic coupling at x_0. It is given by the third derivative of the homogenous scaling part with respect to x which reads

$$f'''(x_0) = -C_f \frac{8\eta_{x,s}(2-\eta_{x,s})}{(2+\eta_{x,s})^3} x_0^{-\frac{2+3\eta_{x,s}}{2+\eta_{x,s}}}. \tag{3.84}$$

By inserting $x_0(\xi_2, g_s^2)$ and $C_f(\xi_2)$, the leading contribution in g_s^2 is given by

$$\xi_3 = \begin{cases} -\,6\xi_2^2 g_s^{2P} & \text{if} \quad P \in (0, 1/2), \\ -\,2\xi_2 \dfrac{243\xi_2^2 + 864\pi^2\hat{\eta}_{x,s}\xi_2 - 16}{9(9\xi_2 - 8)}\, g_s & \text{if} \quad P = 1/2. \end{cases} \tag{3.85}$$

From the definitions in Eqs. (3.37) and (3.63), we deduce that the transformation between the rescaled cubic coupling for $f(x)$ and the finite ratio $\hat{\lambda}_3$ is

$$\xi_3 = \hat{\lambda}_3 g_s^{2P_3 - 6P}. \tag{3.86}$$

The scaling powers in Eq. (3.85) yield that $P_3 = 2$ for $P = 1/2$ and $P_3 = 4P$ for $P \in (0, 1/2)$, which is again in agreement with the EFT analysis including thresholds described in Sec. 3.3.2. However, the expression for the finite ratio $\hat{\lambda}_3$ is different. This is not surprising since here we are approximating the threshold functions by assuming a ϕ^4-dominance regime in the UV. On the contrary, in the previous EFT approach, we took into account the full threshold effects, even though we were considering a polynomial truncation for the scalar potential.

Summary

Finally, let us summarize the results presented so far for the effective potential $f(x)$, at the *quasi-fixed-point*, and for general $P < 1$. Starting from a "bare" potential where the quartic self-interaction is the only coupling, we obtain a quantum one-loop effective potential $f(x)$ which is of the same type (and satisfying the consistency property $f''(0) = \xi_2$) only for the particular choice of parameters $P = 1/2$, $C_f = 0$, and $\xi_2 = \hat{\lambda}_2^{\pm}$. This solution corresponds to the CEL potential. We argued that it is stable with a well defined asymptotic behavior in the combined limit $x \to \infty$ and $g_s^2 \to 0$.

In addition, for $P \leq 1/2$, we discovered in the $(\hat{C}_f, \xi_2)$ plane the existence of a *one-parameter family* of novel solutions. Despite the presence of a log-type singularity at the origin, these solutions possess a nontrivial minimum x_0 satisfying the condition $f''(x_0) = \xi_2$. For $P = 1/2$ these new solutions are stable and present the same quadratic asymptotic behavior as for the CEL solution. For $P < 1/2$, the QFP potential becomes asymptotically flat in the combined limit $x \to \infty$ and $g_s^2 \to 0$, due to the vanishing of the asymptotic coefficient $\hat{C}_{f,\infty}$.

Last but not least, we want to highlight again the importance of consistently check the validity of the scaling solutions, accordingly to the further assumption that we made: namely the approximation of the anomalous dimensions and the RG running of h^2 with their expressions in the DER limit. The consistency check of these assumptions will be provided in the following chapter, while discussing the scheme (in)dependence of our results for an even larger class of models. In fact, in Sec. 4.4.1 and App. G.1, we will clarify that the scaling solutions for $P < 1/4$ have to be rejected since, for these values of P, QFP solutions for h^2 are not viable.

4. $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{c}$-Gauged-Higgs-Yukawa Models

The aim of the present chapter is to generalize the results of Chap. 3 to a larger family of gauged-Yukawa theories which include the whole non-Abelian sector of the SM. In addition to the strong gauge group $\mathrm{SU}(N_\mathrm{c})$, discussed in the previous chapter, we now add the $\mathrm{SU}(N_\mathrm{L})$ gauge group as part of the electroweak interaction coupled to scalars and chiral fermions.

We have seen, in the previous chapter, the importance of including mass-threshold effects, over the whole RG running of couplings, in order to reveal novel AF trajectories. As a drawback, the inclusion of threshold phenomena introduces naturally a scheme dependence in all the beta functions. Therefore, in this chapter we want not only to enlarge the complexity of our toy-models, but also investigate the scheme (in)dependence of our results.

Let us start first by introducing the model. As we have already mentioned, it consists of a Yang-Mills theory based on the local $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{c}$ subgroup of $\mathcal{G}_\mathrm{SM}$. In order to be as much general as possible, we keep the group dimensions N_L and N_c generic and we specify them to be $N_\mathrm{L} = 2$ and $N_\mathrm{c} = 3$ whenever concrete calculations are concerned. Therefore, we introduce a complex scalar field ϕ and a fermionic field ψ_L belonging to the fundamental representation of $\mathrm{SU}(N_\mathrm{L})$. The field ψ_L is composed of N_L left-handed Weyl fermions. Correspondingly we introduce also right-handed Weyl components, which transform trivially with respect to $\mathrm{SU}(N_\mathrm{L})$ (in order to exhibit the same chiral symmetry of the SM). We will refer to the scalar and fermionic components as ϕ^a and ψ_L^a respectively, where $a \in \{1, \dots, N_\mathrm{L}\}$. For $N_\mathrm{L} = 2$, we identify these components as the top t_L and bottom b_L quarks or their corresponding counter-parts for a different flavor. In addition, each left- and right-handed Weyl components are also charged under the $\mathrm{SU}(N_\mathrm{c})$ gauge group, as was also assumed in the previous toy-model discussed in Chap. 3. Therefore the bare action for this model, in Euclidean spacetime, reads

$$S = \int d^4x \bigg[\frac{1}{4}\left(F^i_{\mu\nu}\right)^2 + \frac{1}{4}\left(G^I_{\mu\nu}\right)^2 + (D_\mu\phi)^{\dagger a}(D_\mu\phi)^a + \bar{m}^2\tilde{\rho} + \frac{\bar{\lambda}}{2}\tilde{\rho}^2 + \bar{\psi}^A_\mathrm{R}\mathrm{i}\not{D}^{AB}\psi^B_\mathrm{R} + \bar{\psi}^{aA}_\mathrm{L}\mathrm{i}\not{D}^{abAB}\psi^{bB}_\mathrm{L} + \mathrm{i}\bar{h}(\bar{\psi}^{aA}_\mathrm{L}\phi^a\psi^A_\mathrm{R} + \bar{\psi}^A_\mathrm{R}\phi^{\dagger a}\psi^{aA}_\mathrm{L}) \bigg], \tag{4.1}$$

where the scalar field amplitude $\tilde{\rho}$ is the $\mathrm{SU}(N_\mathrm{L})$ invariant $\tilde{\rho} = \phi^{\dagger a}\phi^a$. In comparison with the previous simpler $\mathbb{Z}_2$-Yukawa-QCD model, here we have in addition a pure Yang-Mills term for the vector bosons W_μ^i. The field strength tensor for the latter field is $F_{\mu\nu}^i = \partial_\mu W_\nu^i - \partial_\nu W_\mu^i - \bar{g} f^{ijk} W_\mu^j W_\nu^k$, where f^{ijk} are the structure constants for the $\mathrm{su}(N_{\mathrm{L}})$ Lie algebra. The interaction terms with the gauge bosons are generated from the covariant derivatives. Right-handed fermions interact only with gluons through the covariant derivative in color-space D_μ^{AB}, already defined in Eq. (3.2). In addition here we have the covariant derivative acting on ϕ, namely $D_\mu^{ab} = \delta^{ab}\partial_\mu + \mathrm{i}\bar{g}W_\mu^i t_i^{ab}$ where $\bar{g}$ is the bare weak gauge coupling of $\mathrm{SU}(N_{\mathrm{L}})$. Left-handed fermions are instead charged under both gauge group therefore the covariant derivative acting on them is more involved since $D_\mu^{abAB} = \delta^{AB}\left(\delta^{ab}\partial_\mu + \mathrm{i}\bar{g}W_\mu^i t_i^{ab}\right) + \mathrm{i}\bar{g}_{\mathrm{s}} G_\mu^I T_I^{AB} \delta^{ab}$.

Concerning the Yukawa sector, we consider only one Yukawa coupling $\bar{h}$, which plays the role of the top-Yukawa coupling, since it is quantitatively the most relevant one for the running of the Higgs potential. This concretely means that only the bottom right-handed Weyl spinor, with its "colored" structure, has been considered in Eq. (4.1). However, as already mentioned in Sec. 3.1, whenever feasible and relevant, we will specialize to the whole SM matter content including its flavor and generation substructure. While doing this generalization, additional kinetic terms in Eq. (4.1) need to be added.

4.1. Perturbative Analysis

Following the same structure as in the previous simpler $\mathbb{Z}_2$-Yukawa-QCD model, we start our analysis from standard perturbation theory at one-loop level and in the DER. Therefore, we present now the RG beta functions for the renormalized dimensionless couplings g^2, g_{s}^2, h^2 and λ (their general definitions with respect to the bare couplings will be given in Sec. 4.2).

Gauge Sector

Concerning the strong gauge coupling, its RG equation is the same as for the previous model, cf., Eq. (3.5). In fact, the number of fermionic degrees of freedom which can run in a spinor loop contributing to the anomalous dimension η_{G} is still the same; both left- and right-handed Weyl spinors are charged under the color gauge group. In addition, for the present model we should consider also the one-loop diagrams contributing to the anomalous dimension η_{W} for the gauge boson fields W_μ^i. The RG beta function for g^2 is [29]

$$\partial_t g^2 = \eta_{\mathrm{W}} g^2, \qquad \eta_{\mathrm{W}} = -\frac{g^2}{3(16\pi)^2}(22N_{\mathrm{L}} - d_\gamma^{\mathrm{L}} N_{\mathrm{f}}^{\mathrm{L}} - N_{\mathrm{sc}}), \tag{4.2}$$

where also a contribution due to the scalar loop is present. N_{sc} represents in fact the number of complex scalar N_{L}-tuples. The fermionic contribution to the anomalous dimension η_{W} is different from the $\mathrm{SU}(N_{\mathrm{c}})$ case: since only interaction vertices between W_μ^i and ψ_{L} are allowed, this contribution counts precisely the number of possible left-handed Weyl spinors which can run in the fermion loop. N_f^L is indeed the number of spinor N_L-tuples and $d_\gamma^\mathrm{L} = 2$ denotes the dimension of the corresponding Clifford-algebra representation.

In the SM, we have 3 doublets for the leptons and 3×3 doublets for the quarks, therefore $N_\mathrm{f}^\mathrm{L} = 12$. Moreover there is only one scalar doublet $N_\mathrm{sc} = 1$. To summarize we have for the SM that

$$N_\mathrm{L} = 2, \qquad N_\mathrm{c} = 3, \qquad N_\mathrm{f}^\mathrm{L} = 12, \qquad N_\mathrm{f}^\mathrm{c} = 6, \qquad N_\mathrm{sc} = 1, \tag{4.3}$$

and for these values, both one-loop beta functions in Eqs. (3.5) and (4.2) are negative such that g^2 and g_s^2 approach the AF Gaußian FP in the UV limit.

Yukawa Sector

For the top-Yukawa coupling, the standard one-loop RG flow equation in the DER reads

$$\partial_t h^2 = (\eta_\mathrm{H} + \eta_\mathrm{L} + \eta_\mathrm{R}) h^2 - \frac{3}{8\pi^2} \frac{N_\mathrm{c}^2 - 1}{N_\mathrm{c}} h^2 g_\mathrm{s}^2, \tag{4.4}$$

where the anomalous dimensions for the scalar, the left- and right-handed Weyl spinors are

$$\eta_\mathrm{H} = \frac{N_\mathrm{c}}{8\pi^2} h^2 - \frac{3}{16\pi^2} \frac{N_\mathrm{L}^2 - 1}{N_\mathrm{L}} g^2, \qquad \eta_\mathrm{L} = \frac{1}{16\pi^2} h^2, \qquad \eta_\mathrm{R} = \frac{N_\mathrm{L}}{16\pi^2} h^2. \tag{4.5}$$

Let us notice that, by setting $g^2 = 0$ and $N_\mathrm{L} = 2$ in the previous equation, we recover exactly the beta function of the $\mathbb{Z}_2$-Yukawa-QCD model, cf. Eq. (3.7). Therefore, if we project the system on the (h^2, g_s^2) plane, we would find the same behavior and conclusions as for the $\mathbb{Z}_2$-Yukawa-QCD model discussed in Sec. 3.1. A different situation occurs if we instead, project the RG flow of the top-Yukawa onto the (h^2, g^2) plane, which corresponds to take the $g_\mathrm{s}^2 \to 0$ limit. Setting $N_\mathrm{c} = 3$ for illustration, we can search for AF trajectories along which h^2 becomes proportional to g^2. Namely, we can search for QFPs for a new rescaled coupling $\check{h}^2 = h^2/g^2$, whose corresponding RG flow equation reads

$$\partial_t \check{h}^2 = \frac{N_\mathrm{L} + 7}{16\pi^2} g^2 \check{h}^2 \left(\check{h}^2 - \chi_g^2 \right), \qquad \chi_g^2 = \frac{2}{3(N_\mathrm{L} + 7)} \left(N_\mathrm{f}^\mathrm{L} + \frac{1}{2} - \frac{13}{2} N_\mathrm{L} - \frac{9}{2N_\mathrm{L}} \right). \tag{4.6}$$

The matter content is constrained, in order for AF trajectories to be present, by the following condition

$$\frac{1}{2} \left(13 N_\mathrm{L} + \frac{9}{N_\mathrm{L}} - 1 \right) < N_\mathrm{f}^\mathrm{L} < 11 N_\mathrm{L} - \frac{1}{2}. \tag{4.7}$$

However, the lower bound is not fulfilled for the SM, resulting in a negative value for $\chi_g^2 = -11/54$. Therefore nontrivial solutions for the QFP equation $\partial_t \check{h}^2 = 0$ do not exist in the

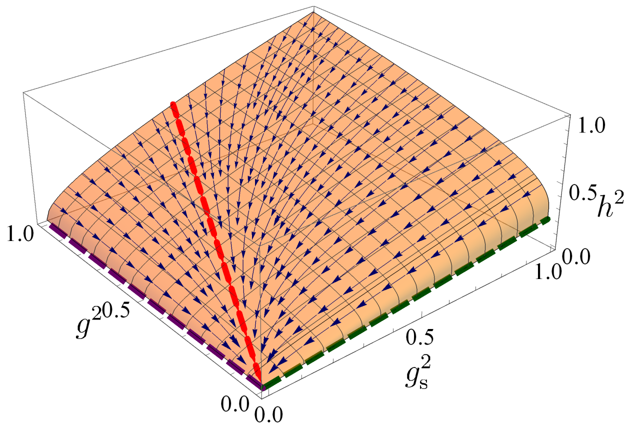


Fig. 4.1.: The upper critical surface $h^2 = \Omega(g_\mathrm{s}^2, g^2)$ of total asymptotic freedom for the perturbatively renormalizable model in the DER and for the SM set of parameters summarized in Eq. (4.3). The special trajectory in Eq. (4.14), along which h^2 is proportional both to g^2 and g_s^2, is highlighted by a red line. It is a UV attractive (repulsive) trajectory along the directions tangent (orthogonal) to Ω. The intersection of Ω with the $g^2 = 0$ plane is highlighted by a green line with a slope given by Eq. (3.11); the intersection of Ω with the $g_\mathrm{s}^2 = 0$ plane is shown as a purple line satisfying $h^2 = g_\mathrm{s}^2 = 0$. The arrows of the RG stream flow on the top of the critical surface are pointing towards the UV.

positive physical quadrant of the (h^2, g^2) plane, and the only possible solution is the trivial one $\breve{h}_*^2 = 0$. In a pictorial representation of the flow similar to that in Fig. 3.1 (left panel), one would find that the red AF trajectory is squeezed to the $h^2 = 0$ axis.

If we consider the three-dimensional $(h^2, g_\mathrm{s}^2, g^2)$ space, the beta function $\partial_t h^2$ in Eq. (4.4) can be analytically integrated together with equation $\partial_t g_\mathrm{s}^2$ (cf. (3.5)) and equation $\partial_t g^2$ (cf. (4.2)). As explained in App. A.2, the matter content parameters N_f^L and N_f^c must fulfill the following necessary but not sufficient condition in order to feature total asymptotic freedom,

$$\chi^2 = \frac{9(N_\mathrm{c}^2 - 1)}{N_\mathrm{c}(11N_\mathrm{c} - 2N_\mathrm{f}^\mathrm{c})} + \frac{9(N_\mathrm{L}^2 - 1)}{N_\mathrm{L}(22N_\mathrm{L} - 2N_\mathrm{f}^\mathrm{L} - 1)} - 1 > 0, \tag{4.8}$$

which generalizes the lower bound $\chi_\mathrm{s}^2 > 0$ for the $\mathbb{Z}_2$-Yukawa-QCD model as well as the condition $\chi_g^2 > 0$ in Eq. (4.6). In the SM case, the latter inequality is satisfied since $\chi^2 = 227/266$. Moreover, if the condition in Eq. (4.8) holds, we can identify a critical surface Ω, parameterized by a function $h^2 = \Omega(g_\mathrm{s}^2, g^2)$, which represents the upper bound for total asymptotic freedom. In other words, for any initial condition such that $h_0^2 \le \Omega(g_{\mathrm{s}0}^2, g_0^2)$ the top-Yukawa coupling becomes AF and approaches the GFP in the UV limit. As a matter of fact, this surface is UV repulsive along its normal directions, while all the trajectories on the surface itself are attracted towards a special one where the top-Yukawa coupling and both gauge couplings become proportional to each other (see App. A.2 for details). The RG flow on the critical surface $\Omega(g_\mathrm{s}^2, g^2)$ is depicted in Fig. 4.1, where the lines are pointing in the UV direction. The special trajectory on the surface itself is highlighted by a red line. In addition, also the intersections of Ω with the planes $g^2 = 0$ and $g_s^2 = 0$ are represented in the figure; they correspond to the QFP solutions in Eq. (3.11) (green line) and $\breve{h}_*^2 = 0$ (purple line) respectively.

In order to find the corresponding equation for the special UV attractive trajectory on Ω, let us use the QFP criteria and consider the ratio among the two gauge couplings

$$\hat{g}^2 = \frac{g^2}{g_s^2}, \tag{4.9}$$

together with its RG flow

$$\partial_t \hat{g}^2 = \frac{g^2}{48\pi^2}\left[22N_c - 4N_f^c - (22N_L - 2N_f^L - 1)\hat{g}^2\right]. \tag{4.10}$$

The latter possesses a QFP solution for $g^2 \neq 0$ at

$$\hat{g}_*^2 = \frac{2(11N_c - 2N_f^c)}{22N_L - 2N_f^L - 1}. \tag{4.11}$$

The solution $g^2 = \hat{g}_*^2 g_s^2$ identifies a plane in the three dimensional space of parameters (g^2, g_s^2, h^2), whose intersection with Ω is a trajectory along which $h^2 \propto g_s^2 \propto g^2$. Assuming $g^2 = \hat{g}_*^2 g_s^2$, we perform the same rescaling as in Eq. (3.8), arriving at the beta function for $\hat{h}^2$ which reads

$$\partial_t \hat{h} = \hat{h}^2 g_s^2 \frac{2N_c + N_L + 1}{16\pi^2}\left[\hat{h}^2 - \frac{2(11N_c - 2N_f^c)\chi^2}{3(2N_c + N_L + 1)}\right], \tag{4.12}$$

whose nontrivial QFP solution is

$$\hat{h}_*^2 = \frac{2(11N_c - 2N_f^c)}{3(2N_c + N_L + 1)}\chi^2. \tag{4.13}$$

For the SM set of parameters we have

$$\hat{g}_*^2 = \frac{42}{19}, \qquad \hat{h}_*^2 = \frac{227}{171}. \tag{4.14}$$

Scalar Sector

Now we investigate the scalar sector and thus include also the running of the quartic scalar coupling λ. Its one-loop perturbative beta function in the DER is

$$\partial_t \lambda = 2\eta_H \lambda + \frac{3(N_L - 1)(N_L^2 + 2N_L - 2)}{32\pi^2 N_L^2} g^4 + \frac{N_L + 4}{8\pi^2}\lambda^2 - \frac{N_c}{4\pi^2} h^4, \tag{4.15}$$

where η_{H} is given by Eq. (4.5). Since we are interested in the special trajectory described by Eqs. (4.11) and (4.13), along which the top-Yukawa and the gauge couplings are proportional among each other, we can express h^2 and g^2 as a function of g_{s}^2. Thus the beta function $\partial_t\lambda$ turns out to be just a function of λ and g_{s}^2, and any AF solution must correspond to a particular power scaling P of the quartic coupling with respect to the strong gauge coupling. The beta function for $\hat{\lambda}_2$, as defined in Eq. (3.13), becomes

$$\partial_t\hat{\lambda}_2 = 2\hat{\eta}_{\mathrm{H}}\hat{\lambda}_2 g_{\mathrm{s}}^2 + \frac{3(N_{\mathrm{L}}-1)(N_{\mathrm{L}}^2+2N_{\mathrm{L}}-2)}{32\pi^2 N_{\mathrm{L}}^2}\hat{g}_*^4 g_{\mathrm{s}}^{4-4P} + \frac{N_{\mathrm{L}}+4}{8\pi^2}\hat{\lambda}_2^2 g_{\mathrm{s}}^{4P} - \frac{N_{\mathrm{c}}}{4\pi^2}\hat{h}_*^4 g_{\mathrm{s}}^{4-4P} + 2P\hat{\lambda}_2\hat{\eta}_{\mathrm{G}} g_{\mathrm{s}}^2, \tag{4.16}$$

where $\hat{\eta}_{\mathrm{G}}$ is the same as in Eq. (3.14) while the rescaled scalar anomalous dimension

$$\hat{\eta}_{\mathrm{H}} = \frac{\eta_{\mathrm{H}}}{g_{\mathrm{s}}^2} = \frac{N_{\mathrm{c}}}{8\pi^2}\hat{h}_*^2 - \frac{3(N_{\mathrm{L}}^2-1)}{16\pi^2 N_{\mathrm{L}}}\hat{g}_*^2, \tag{4.17}$$

is different from Eq. (3.14) due to the presence of a gauge bosons contribution. A nontrivial finite QFP solution for $\hat{\lambda}_2$ is present only for $P = 1/2$. By choosing the SM set of parameters, we find the QFP roots

$$\hat{\lambda}_2^{\pm} = \frac{1}{342}\left(-143 \pm \sqrt{119402}\right), \qquad P = \frac{1}{2}, \tag{4.18}$$

whose stability properties are the same as for the CEL solution described in Sec. 3.1. In particular, the positive root $\hat{\lambda}_2 = \hat{\lambda}_2^+$ corresponds to the only trajectory along which the theory is UV complete with a perturbatively stable potential. Moreover it corresponds to a UV repulsive, i.e., IR attractive, trajectory.

Comparing our toy-model flow to that of the SM, see Fig. 2.3 (left panel), current data suggests that the SM flow is governed by its vicinity to the analogue of the critical surface Ω, with the gauge couplings, the top-Yukawa coupling h and the scalar coupling λ all exhibiting a flow towards smaller values above the Fermi scale. As the strong coupling g_{s} is larger than the weak coupling g ($g_3 > g_2$ in Fig. 2.3), the gauge sector has not yet reached its QFP value of Eq. (4.11). Also, the top-Yukawa coupling is below its QFP value in Eq. (4.13). The scalar coupling appears to be near critical [153, 245–249], with $\hat{\lambda}_2$ being slightly below (the analogue of) $\hat{\lambda}_2^+$, such that λ appears to approach zero or potentially drop below zero towards higher scales. Of course, the contribution of the hypercharge U(1) group that would dominate the flow far above the Planck scale is ignored in the present discussion.

4.2. Nonpertubative RG Flow Equations

Our next step is to present the nonperturbative FRG flow equations for the present model. In order to do so, we start from a truncation of the EAA based on the leading-order derivative expansion, such that Γ_k reads in generic d dimensions

$$\Gamma_k = \int \mathrm{d}^d x \Bigg[\frac{Z_{\mathrm{W}}}{4} (F_i^{\mu\nu})^2 + \frac{Z_{\mathrm{G}}}{4} (G_I^{\mu\nu})^2 + Z_{\mathrm{H}} (D_\mu \phi)^{\dagger a} (D^\mu \phi)^a + U(\phi^\dagger \phi) + Z_{\mathrm{L}} \bar{\psi}_{\mathrm{L}}^{aA} \mathrm{i} \not{D}^{abAB} \psi_{\mathrm{L}}^{bB} \\ + Z_{\mathrm{R}} \bar{\psi}_{\mathrm{R}}^{A} \mathrm{i} \not{D}^{AB} \psi_{\mathrm{R}}^{B} + \mathrm{i} \bar{h} (\bar{\psi}_{\mathrm{L}}^{aA} \phi^a \psi_{\mathrm{R}}^{A} + \bar{\psi}_{\mathrm{R}}^{A} \phi^{\dagger a} \psi_{\mathrm{L}}^{aA}) + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{gh}} \Bigg]. \tag{4.19}$$

All couplings $\bar{g}$, $\bar{g}_{\mathrm{s}}$, $\bar{h}$, wave function renormalizations $Z_{\mathrm{W,G,H,L,R}}$, and the effective potential U are k dependent. For simplicity, we refer to the $N_{\mathrm{L}} = 2$ case for the remainder of this section, but we will consider N_{c} and the spacetime dimension d as arbitrary parameters.

In order to take into account also the threshold effects coming from the SSB regime, we decompose the scalar field into the bare nontrivial minimum $\bar{v}$ plus fluctuations around it. Without loss of generality we choose the radial mode in the first real component such that [112, 194]:

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{v} \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} H + \mathrm{i}\theta_3 \\ \theta_2 + \mathrm{i}\theta_1 \end{pmatrix} \equiv \frac{\bar{v}}{\sqrt{2}} \hat{n} + \Delta\phi, \tag{4.20}$$

where the radial fluctuation $H(x)$ corresponds to the Higgs excitation and the Goldstone fields form a triplet. The second equality in the latter equation allows us to generalize the discussion in the case we want to keep arbitrary the unitary radial direction $\hat{n}$, in the fundamental space of the $\mathrm{SU}(N_{\mathrm{L}})$ gauge group.

We also need to fix both gauge groups through a Lagrangian

$$\mathcal{L}_{\mathrm{GF}} = \frac{Z_{\mathrm{W}}}{2\zeta} (\mathsf{F}^i)^2 + \frac{Z_{\mathrm{G}}}{2\zeta_{\mathrm{s}}} (\mathsf{F}_{\mathrm{s}}^I)^2, \tag{4.21}$$

where the gauge-fixing condition F^i, for the weak gauge group, is usually chosen in such a way that no mixing terms between the Goldstone modes $\theta_{1,2,3}$ and the vector bosons W_i^μ appear in the propagators. This is obtained for the following choice of F^i

$$\mathsf{F}^i = \partial_\mu W_\mu^i - \mathrm{i}\zeta \frac{Z_{\mathrm{H}}}{Z_{\mathrm{W}}} \frac{\bar{g}\bar{v}}{\sqrt{2}} \left(\hat{n}^\dagger t^i \Delta\phi - \Delta\phi^\dagger t^i \hat{n} \right) = \partial_\mu W_\mu^i + \zeta \frac{Z_{\mathrm{H}}}{Z_{\mathrm{W}}} \frac{\bar{g}\bar{v}}{2} \left(\delta^{i1}\theta_1 - \delta^{i2}\theta_2 + \delta^{i3}\theta_3 \right). \tag{4.22}$$

Notice that, within this choice, only the Goldstone modes are involved in the gauge-fixing condition but not the Higgs field. For the strong gauge group, we choose the standard Lorentz gauge-fixing condition as we did for the previous model, thus we have $\mathsf{F}_{\mathrm{s}}^I = \partial_\mu G_\mu^I$.

The ghost Lagrangian should encode the determinants of the Faddeev-Popov operators

$$\begin{aligned} \mathsf{M}^{ij} = \frac{\delta \mathsf{F}^i}{\delta \alpha_j} &= -\partial_\mu D^{ij}_\mu + \zeta \frac{Z_\mathrm{H}}{Z_\mathrm{W}} \frac{\bar{g}^2 \bar{v}}{\sqrt{2}} \left[\hat{n}^\dagger t^i t^j \left(\frac{\bar{v}}{\sqrt{2}} \hat{n} + \Delta\phi \right) + \left(\frac{\bar{v}}{\sqrt{2}} \hat{n}^\dagger + \Delta\phi^\dagger \right) t^j t^i \hat{n} \right], \\ &= -\partial_\mu D^{ij}_\mu + \zeta \frac{Z_\mathrm{H}}{Z_\mathrm{W}} \frac{\bar{g}^2 \bar{v}}{4} \left[\delta^{ij} (\bar{v} + H) - \epsilon^{ij1}\theta_1 + \epsilon^{ij2}\theta_2 - \epsilon^{ij3}\theta_3 \right], \end{aligned} \tag{4.23}$$

$$\mathsf{M}^{IJ}_\mathrm{s} = \frac{\delta \mathsf{F}^I_\mathrm{s}}{\delta \alpha_{\mathrm{s}J}} = -\partial_\mu D^{IJ}_\mu = -\Box \delta^{IJ} - \bar{g}_\mathrm{s} f^{IJK} \partial_\mu G^K_\mu, \tag{4.24}$$

where α_j and $\alpha_{\mathrm{s}J}$ are the local gauge parameters parameterizing any infinitesimal gauge transformation under $\mathrm{SU}(N_\mathrm{L})$ and $\mathrm{SU}(N_\mathrm{c})$ respectively (ϵ^{ijk} is the Levi-Civita total anti-symmetric tensor). At the functional level, the determinants for M and M_s are obtained by introducing ghost Graßman fields c_i, $\bar{c}_i$, b_I, $\bar{b}_I$ and defining the Lagrangian density

$$\mathcal{L}_\mathrm{gh} = -\bar{c}_i \mathsf{M}^{ij} c_j - \bar{b}_I \mathsf{M}^{IJ}_\mathrm{s} b_J. \tag{4.25}$$

In the following of this chapter, we will mostly quote the results obtained in the Landau gauge limit, where the gauge-fixing parameters are set to zero $\{\zeta, \zeta_\mathrm{s}\} \to 0$.

Since we are ultimately interested in the scaling solutions of this model, we introduce the dimensionless renormalized $\mathrm{U}(N_\mathrm{L})$-invariant scalar field amplitude

$$\rho = Z_\mathrm{H} \frac{\phi^{\dagger a} \phi^a}{k^{d-2}}, \tag{4.26}$$

and the dimensionless renormalized couplings

$$h^2 = \frac{\bar{h}^2 k^{d-4}}{Z_\mathrm{H} Z_\mathrm{L} Z_\mathrm{R}}, \qquad g^2 = \frac{\bar{g}^2 k^{d-4}}{Z_\mathrm{W}}, \qquad g^2_\mathrm{s} = \frac{\bar{g}^2_\mathrm{s} k^{d-4}}{Z_\mathrm{G}}. \tag{4.27}$$

Scalar Potential

Given the truncated EAA in Eq. (4.19), we can derive from the Wetterich equation (cf. Eq. (2.34)) the exact RG flow equations for the different operators included in Γ_k, by applying appropriate projection rules. [21] For example, the beta function for the dimensionless potential $u(\rho)$, defined in Eq. (3.20), reads [113, 194, 250]

$$\begin{aligned} \partial_t u = -du + (d - 2 + \eta_\mathrm{H})\rho u' + 2v_d \Big\{ & l_0^{(\mathrm{H})d}(\omega_\mathrm{H}, \eta_\mathrm{H}) + 3 l_0^{(\theta)d}(\omega_\theta, \eta_\mathrm{H}) \\ & + 3(d-1) l_0^{(\mathrm{W})d}(\omega_\mathrm{W}, \eta_\mathrm{W}) - 4N_\mathrm{c} l_0^{(\mathrm{F})d}(\omega_\mathrm{F}, \eta_\psi) \Big\}, \end{aligned} \tag{4.28}$$

[21] Details on this procedure are presented in App. C for the $\mathbb{Z}_2$-Yukawa-QCD model. Generalization to the present more involved $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{c}$ model is nevertheless straightforward. Let us just notice her that the choice of the Landau gauge is a convenient one, since the matrix $\Gamma^{(2)}_k + R_k$ can be easily inverted in order to obtain the exact full propagator.

where the arguments $\omega_{\rm H,F}$ are the same as in Eq. (3.23), while $\omega_{\theta,\rm W}$ are defined by

$$\omega_\theta = u'(\rho), \qquad\qquad \omega_{\rm W} = \frac{g^2\rho}{2}. \tag{4.29}$$

The anomalous dimensions are, as usual, given by $\eta_\Phi = -\partial_t \log Z_\Phi$, where $\Phi = \{\rm H, G, W, L, R\}$. From Eq. (4.28), we can extract also the flow equation for the nontrivial minimum κ

$$\kappa = \frac{Z_{\rm H}\bar{v}^2}{2k^{d-2}}, \qquad\qquad \partial_t\kappa = -\frac{\partial_t u'(\rho)}{u''(\rho)}\bigg|_{\rho=\kappa}, \tag{4.30}$$

as well as the RG flow equations for the scalar self-couplings to any order by polynomial expansion. For the general definition of the threshold functions we refer the reader to App. B.

Let us emphasize a few aspects of the flow equation (4.28). Regarding the scalar loops, there are two threshold contributions: one is associated to the radial or Higgs excitation labeled by H and the other one refers to the longitudinal Goldstone modes labeled by θ. [22] In fact the latter occurs with a prefactor $2N_{\rm L} - 1 = 3$ which counts the number of Goldstone modes. Regarding the fermionic contribution, we have the same as for the $\mathbb{Z}_2$-Yukawa-QCD model, since we consider only one right-handed Weyl spinor. As a consequence, only one component of the $N_{\rm L}$-plet $\psi_{\rm L}$ acquires a Dirac mass via the Yukawa interaction and is referred to as the top-quark. [23] In addition to the $\mathbb{Z}_2$-Yukawa-QCD model, there is also a contribution coming from the gauge boson fields W^i_μ. [24] Further contributions from gluon and ghost loops can be ignored since they are field independent. All loop contributions in Eq. (4.28) can be diagrammatically represented as in Fig. 4.2; here dashed, solid and wavy lines represent the scalar, top-fermion and W propagators respectively.

Top-Yukawa Coupling

In order to obtain the beta function for the top-Yukawa coupling, we proceed as discussed in Sec. 3.2 and project $\partial_t\Gamma_k$ onto a generalized field-dependent Yukawa potential $h(\rho)$. Subsequently, we specify this functional to the constant configuration $h(\rho) = h$. We obtain

[22] The eigenvalues of the symmetric scalar sector of $\Gamma^{(2)}_k$ are $U'(\phi^\dagger\phi)$, with a multiplicity of $2N_{\rm L}-1$, and $U'(\phi^\dagger\phi) + 2\phi^\dagger\phi U''(\phi^\dagger\phi)$, with a degeneracy of one. These eigenvalues define exactly the arguments ω_θ and $\omega_{\rm H}$ as given in Eqs. (4.29) and (3.23).

[23] In case kinetic terms for the other $N_{\rm L}-1$ bottom-like Weyl-spinors were included (keeping their Yukawas to zero), their loop contributions would be field-independent since $\omega_{\rm F} = 0$. Therefore these contributions can be neglected since they would influence an unimportant cosmological constant.

[24] Within the present choice of Landau gauge, only the transverse gauge bosons contribute to the flow $\partial_t u$ and the factor in front of this contribution counts exactly the dimension of the adjoint representation ($N_{\rm L}^2 - 1 = 3$) times the trace over the transverse propagator ($\Pi^{\rm T}_{\mu\mu} = d-1$). The longitudinal gauge bosons do not contribute to the scalar potential flow since the corresponding threshold loop would be $l_0^{({\rm W})d}(0, \eta_{\rm W})$, thus field independent.

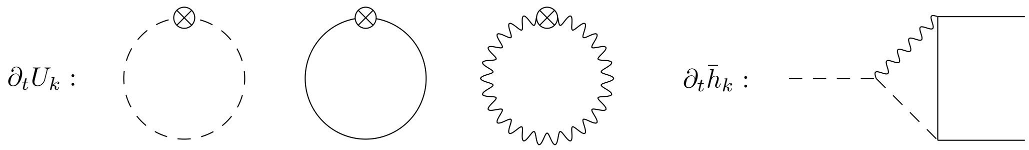


Fig. 4.2.: *Left Panel:* Diagrammatic representation of the three contributions to the RG flow equation of the scalar potential. Dashed, solid and wavy lines denote the propagation of scalar, fermionic and W-gauge-boson fields respectively. All the internal propagators, even when it is not represented, are considered as fully dressed, due to the regulator insertion $\partial_t R_k$ identified by a circle with a cross. *Right Panel:* 1PI contribution to the beta function for the top-Yukawa coupling which is suppressed in the Landau gauge since it involves a longitudinal W gauge boson.

therefore in the Landau gauge [113]

$$\partial_t h^2 = (d-4+\eta_\mathrm{H}+2\eta_\psi)h^2 + 4v_d h^4 \left\{ l_{11}^{(\mathrm{FH})d}(\omega_\mathrm{F},\omega_\mathrm{H};\eta_\psi,\eta_\mathrm{H}) - l_{11}^{(\mathrm{F}\theta)d}(\omega_\mathrm{F},\omega_\theta;\eta_\psi,\eta_\mathrm{H}) \right\}$$
$$- 8v_d \frac{N_\mathrm{c}^2-1}{2N_\mathrm{c}}(d-1)h^2 g_\mathrm{s}^2 l_{11}^{(\mathrm{FG})d}(\omega_\mathrm{F},0;\eta_\psi,\eta_\mathrm{G})\Big|_{\rho=\kappa}, \tag{4.31}$$

where the spinor anomalous dimension η_ψ is defined as the average of the left- and right-handed Weyl spinor anomalous dimensions,

$$2\eta_\psi = \eta_\mathrm{L} + \eta_\mathrm{R}. \tag{4.32}$$

In equation (4.31) there are no 1PI contributions from the W fields. The only contribution from these fields could come from a triangular one-loop diagram as depicted in Fig. 4.2 (right panel). However, this loop involves a Goldstone boson and in particular a longitudinal W gauge boson, which is removed from the theory in the Landau gauge [194, 251]. The same conclusion also holds for the unitary gauge in the SSB regime due to the decoupling of the involved Goldstone modes, which are gapped.

Anomalous Dimensions

In order to derive the flows for the wave functions renormalization Z_H, Z_L and Z_R we project the Wetterich equation onto the kinetic terms for the radial Higgs excitation, the left-handed top quark and the right-handed top-quark respectively. [25] The final results, again in the Landau gauge, read:

$$\eta_\mathrm{H} = \frac{8v_d}{d}\Big\{ \rho(3u''+2\rho u''')^2 m_2^{(\mathrm{H})d}(\omega_\mathrm{H},\eta_\mathrm{H}) + 3\rho(u'')^2 m_2^{(\theta)d}(\omega_\theta,\eta_\mathrm{H})$$

[25] Different projections, leading to different expressions for the anomalous dimensions, are also possible [194]. This ambiguity stems from the splitting of the scalar field ϕ into its *vev* plus fluctuations around it, cf., Eq. (4.20). For example, one could evaluate the anomalous dimension for the scalar field by projecting the Wetterich equation onto the kinetic operator of the Goldstone modes. In a similar manner, a projection on the kinetic term for the left-handed top-quark or the left-handed bottom-quark lead to different beta functions.

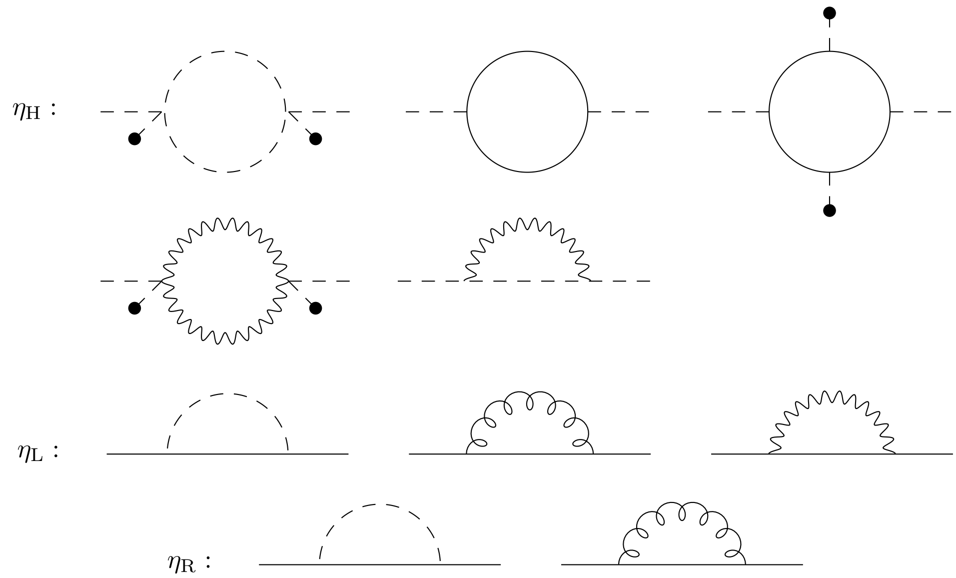


Fig. 4.3.: Diagrammatic representation for the RG flow equations of the scalar wave function renormalization Z_H and the left and right top-quark wave functions renormalization Z_L and Z_R. Thick black dots represent couplings to the condensate, i.e., the nontrivial ground state for the scalar potential. For convenience, we have decided to drop the regulator insertion in any internal lines.

$$
\begin{aligned}
&+ 2N_c h^2 \big[m_4^{(F)d}(\omega_F,\eta_\psi) - \rho h^2 m_2^{(F)d}(\omega_F,\eta_\psi)\big] - \frac{3(d-1)g^2}{2} l_{11}^{(\theta W)d}(\omega_\theta,\omega_W;\eta_H,\eta_W) \\
&+ \frac{3(d-1)\omega_W^2}{\rho}\big[2\,\widetilde{m}_2^{(W)d}(\omega_W,\eta_W) + m_2^{(W)d}(\omega_W,\eta_W)\big]\Big\}\Big|_{\rho=\kappa}, \qquad (4.33)
\end{aligned}
$$

$$
\begin{aligned}
\eta_R = \frac{4v_d}{d}\Big\{ & h^2\big[m_{12}^{(LH)d}(\omega_F,\omega_H;\eta_L,\eta_H) + m_{12}^{(L\theta)d}(\omega_F,\omega_\theta;\eta_L,\eta_H) + 2m_{12}^{(L\theta)d}(0,\omega_\theta;\eta_L,\eta_H)\big] \\
&+ \frac{N_c^2-1}{2N_c}2g_s^2(d-1)\big[m_{12}^{(RG)d}(\omega_F,0;\eta_\psi,\eta_G) - \widetilde{m}_{11}^{(RG)d}(\omega_F,0;\eta_\psi,\eta_G)\big]\Big\}\Big|_{\rho=\kappa}, \qquad (4.34)
\end{aligned}
$$

$$
\begin{aligned}
\eta_L = \frac{4v_d}{d}h^2 & \Big\{m_{12}^{(RH)d}(\omega_F,\omega_H;\eta_R,\eta_H) + m_{12}^{(R\theta)d}(\omega_F,\omega_\theta;\eta_R,\eta_H)\Big\} \\
&+ \frac{(d-1)g^2}{2}\big[m_{12}^{(LW)d}(\omega_F,\omega_W;\eta_L,\eta_W) - \widetilde{m}_{11}^{(LW)d}(\omega_F,\omega_W;\eta_L,\eta_W) \\
&+ 2m_{12}^{(LW)d}(0,\omega_W;\eta_L,\eta_W) - 2\widetilde{m}_{11}^{(LW)d}(0,\omega_W;\eta_L,\eta_W)\big] \\
&+ \frac{N_c^2-1}{2N_c}2g_s^2(d-1)\big[m_{12}^{(LG)d}(\omega_F,0;\eta_\psi,\eta_G) - \widetilde{m}_{11}^{(LG)d}(\omega_F,0;\eta_\psi,\eta_G)\big]\Big\}\Big|_{\rho=\kappa}, \qquad (4.35)
\end{aligned}
$$

where, as usual, different labels in the threshold functions, defined in App. B, identify different propagators in the corresponding one-loop integrals.

A diagrammatic representation of the latter equations is given in Fig. 4.3. As a first observation, we want to emphasize the fact that the SU(N_c) gauge group does not interfere with the electroweak gauge group. For example, there are no fundamental vertices among fermions and gluons which involve a change in the chirality of the Weyl spinors. As a consequence, the gluons contributions to the right- and left-handed top-quark are exactly of the same form as for the $\mathbb{Z}_2$-Yukawa-QCD model, see Sec. 3.2. Notice that, due to the trivial gauge transformation of ψ_{R} with respect to the $\mathrm{SU}(2)_{\mathrm{L}}$ gauge group, there are no additional contributions in η_{R} coming from an exchange of a W gauge boson. The opposite situation occurs for η_{L}. For both spinor anomalous dimensions, there are threshold contributions due to an exchange of a scalar field which can be either the Higgs or the Goldstone excitation.

Let us observe that, the fundamental two fermions-one scalar vertex involves a change in the chirality of the fermion. Therefore, to the anomalous dimension η_{L} (η_{R}), there is a contribution from a scalar-fermion loop where a right-handed (left-handed) Weyl spinor is exchanged. [26]

Regarding the Higgs anomalous dimension, there are pure scalar loop contributions where either the Higgs or the Goldstone excitations are exchanged, as well as pure W gauge boson loop. The fermion loop contribution to η_{H} involves only the top-quark since we are projecting on the kinetic term of the radial scalar fluctuation. The threshold function $l_{11}^{(\theta\mathrm{W})}$ corresponds to a loop integral where a Goldstone mode and a W gauge boson are exchanged. Indeed, the theory allows for a two scalars-one gauge boson elementary vertex where the scalars can be either two Goldstone fields or one Higgs and one Goldstone field. The latter vertex is the one of our interest due to the projection on the radial $H(x)$ field.

Limiting Case: Deep Euclidean Regime

In the previous chapter, in particular in Sec. 3.2, we have observed that the universal one-loop beta functions can be straightforwardly obtained from the FRG results by going into the DER and setting to zero all η's inside the threshold functions. As a matter of fact, within these limits, $\partial_t h^2$ reduces to Eq. (4.4) and $\eta_{\mathrm{H,L,R}}$ take the same form as in Eq. (4.5) for $N_{\mathrm{L}} = 2$. Once we have the beta functions in the DER for the general $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{c}}$ model, we can wonder how we can obtain the universal one-loop beta function for the two limiting models: the $\mathbb{Z}_2$-Yukawa-QCD and the non-Abelian Higgs models. The latter limit can be straightforwardly recovered by setting $h^2 \to 0$ and $g_{\mathrm{s}}^2 \to 0$, which is valid also at the level of the FRG equations.

However, the limit to the $\mathbb{Z}_2$-Yukawa-QCD model is more subtle: if we consider the top-Yukawa coupling, Eq. (4.4) reduces to Eq. (3.33) by taking the limits $g^2 \to 0$ and $N_{\mathrm{L}} \to 2$; whereas, if we consider the quartic scalar self-interaction λ, Eq. (4.15) would reduce to Eq. (3.12) by taking the limits $g^2 \to 0$ and $N_{\mathrm{L}} \to 1/2$. This seeming contradiction can be resolved by taking the unitary-gauge limit $\zeta \to \infty$ in the FRG equations before approaching the DER. This choice for the gauge-fixing parameter has the advantage that projects onto the physical degrees of freedom whenever the theory has some hidden gauge symmetries. In

[26] The mass-threshold contributions to η_{L} only involve an exchange of a right-handed top quark, since no right-handed bottom quarks are present in the present model. On the other hand, the mass-threshold contributions to η_{R} can either involve an exchange of a left-handed top quark or a left-handed bottom quark. In the latter case the fermionic argument is $\omega_{\mathrm{F}} = 0$ due to the massless nature of the bottom-quark for our toy-model.

fact, the Goldstone modes would acquire infinite mass, being proportional to ζ, and decouple from the theory. As a consequence, the $\mathbb{Z}_2$-Yukawa-QCD model would simply correspond to the limit $g^2 \to 0$ of the present general $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{c}$ model, and the DER limit can now be safely taken. The importance of taking the unitary gauge before going into the DER becomes manifest in the FRG equations. In fact, taking the reverse order of these two limits would lead to ill-defined quantities such as a divergent η_H [251].

Gauge Sector

The functional treatment of the RG flow also goes along with generalizations of the beta functions for the gauge couplings g^2 and g_s^2. In fact a nontrivial minimum κ for $u(\rho)$ gives rise to mass-threshold corrections to the anomalous dimensions η_G and η_W. So far we have considered Eqs. (4.2) and (3.5) which represent these flows in the DER. A functional generalization of the latter equations beyond the DER can be computed in the background-field formalism [194]. For the $\mathrm{SU}(2)_\mathrm{L}$ gauge group and for N_f^L left-handed fermion doublets, the anomalous dimension for the W_μ^i gauge bosons is [27]

$$\eta_\mathrm{W} = -\frac{g^2}{48\pi^2}\left[44 L_\mathrm{W}(\mu_\mathrm{W}^2) - d_\gamma^\mathrm{L} \sum_{j=1}^{N_\mathrm{f}^\mathrm{L}} L_\mathrm{F}(\mu_{\mathrm{t}_j}^2, \mu_{\mathrm{b}_j}^2) - L_\mathrm{H}(\mu_\mathrm{H}^2)\right], \tag{4.36}$$

where the renormalized dimensionless mass parameters are proportional to κ.

$$\mu_\mathrm{W}^2 = \frac{g^2\kappa}{2}, \qquad \mu_{\mathrm{t}_j/\mathrm{b}_j}^2 = h_{\mathrm{t}_j/\mathrm{b}_j}^2 \kappa, \qquad \mu_\mathrm{H}^2 = 2\lambda_2\kappa. \tag{4.37}$$

The threshold functions $L_\mathrm{W,F,H}$ are normalized in such a way that η_W reduces to Eq. (4.2) in the DER, namely $L_\mathrm{W,F,H}(0) = 1$. Ignoring higher-loop resummations and taking the Landau gauge limit $\zeta \to 0$, the threshold functions read

$$L_\mathrm{W}(\mu_\mathrm{W}^2) = \frac{1}{44}\left(21 + \frac{21}{1+\mu_\mathrm{W}^2} + 2\right), \qquad L_\mathrm{F}(\mu_{\mathrm{t}_j}^2, \mu_{\mathrm{b}_j}^2) = \frac{1}{2}\left(\frac{1}{1+\mu_{\mathrm{t}_j}^2} + \frac{1}{1+\mu_{\mathrm{b}_j}^2}\right), \tag{4.38}$$

$$L_\mathrm{H}(\mu_\mathrm{H}^2) = \frac{1}{2}\left(1 + \frac{1}{1+\mu_\mathrm{H}^2}\right). \tag{4.39}$$

For the $\mathrm{SU}(3)_\mathrm{c}$ gauge group, the gluons wave function renormalization is

$$\eta_\mathrm{G} = -\frac{g_\mathrm{s}^2}{48\pi^2}\left[22N_\mathrm{c} - d_\gamma^\mathrm{c} \sum_{j=1}^{N_\mathrm{f}^\mathrm{c}} L_\mathrm{F}'(\mu_{\mathrm{Q}_j}^2)\right], \qquad L_\mathrm{F}'(\mu_{\mathrm{Q}_j}^2) = \frac{1}{1+\mu_{\mathrm{Q}_j}^2}, \tag{4.40}$$

where $\mu_{\mathrm{Q}_j}^2 = h_{\mathrm{Q}_j}^2\kappa$ is the mass for the j-th quark where j has to be understood as a multiindex, labeling the position within the left-handed doublet as well as possible generation copies.

[27] Here, we also allow for a bottom-type Yukawa coupling $h_{\mathrm{b}_j}^2$ in addition to the top-type Yukawas $h_{\mathrm{t}_j}^2$ associated to the j-th generation in order to model the decoupling of all quark mass thresholds.

Let us briefly comment on the case where the rescaling power P of λ is $P > 1$. In this case, as it will be shown exhaustively in App. G.2, the W gauge bosons and fermionic fluctuations decouple from the dynamics since their masses diverge in the UV limit, in striking contradiction with the assumption of the DER. We will refer to this scenario as the *decoupling regime*. As a consequence, all loop contributions from these massive modes drop out of the gauge coupling flows. Also the neutrinos might or might not contribute to the running of g^2 depending on their nature. They either decouple as well if they are Dirac neutrinos with a mass term generated trough the coupling with the scalar condensate. Or as Majorana neutrinos, they could essentially behave as nearly massless particles in the DER and thus would not decouple from η_{W}. Counting the massless neutrinos by n_ν, we obtain

$$\eta_{\mathrm{G}} = -\frac{11N_{\mathrm{c}}}{24\pi^2} g_{\mathrm{s}}^2, \qquad \eta_{\mathrm{W}} = -\frac{g^2}{48\pi^2}\left(23 - d_\gamma^{\mathrm{L}}\frac{n_\nu}{2} - 1\right). \tag{4.41}$$

In this case, the ratio of the two gauge couplings, defined in Eq. (4.9), takes the QFP value

$$\hat{g}_*^2 = \frac{66}{19}. \tag{4.42}$$

On the other hand, if we treat the neutrinos as Dirac particles, their contribution decouples from η_{W} and the latter QFP value changes into

$$\hat{g}_*^2 = 3. \tag{4.43}$$

4.3. Regularization in the $\overline{\mathrm{MS}}$ Scheme

In the present section we want to temporarily abandon the FRG method, based on a mass-dependent IR regularization scheme, and we evaluate the loop integrals in dimensional regularization [252–254]). In particular we use here the $\overline{\mathrm{MS}}$ prescription, which represents a mass-independent regularization scheme. Within this scheme, the beta functions equal the residue of the $(d-4)^{-1}$ poles of the loop integrals. This prescription can be extended in a functional form, providing functional perturbative beta functions [255, 256]. Even though the $\overline{\mathrm{MS}}$ prescription corresponds to a mass-independent scheme, it can be understood within the functional language of FRG: it corresponds indeed to a very special choice of regulator such that the loop threshold functions $l_0^{(\dots)d}$ become in the $\overline{\mathrm{MS}}$ scheme, in $d = 4$ dimension,

$$l_0^{(\overline{\mathrm{MS}})}(\omega) = \frac{\omega^2}{2}. \tag{4.44}$$

Given the latter equation, we can immediately read out the RG flow of $u(\rho)$ in $\overline{\text{MS}}$, for the general $\text{SU}(2)_\text{L} \times \text{SU}(3)_\text{c}$ model, from Eq. (4.28) and is

$$\partial_t u = -4u + (2+\eta_\text{H})\rho u' + \frac{1}{32\pi^2}(\omega_\text{H}^2 + 3\omega_\theta^2 + 9\omega_\text{W}^2 - 12\omega_\text{F}^2), \tag{4.45}$$

where the arguments for the different threshold contributions have been already defined in Eqs. (3.23) and (4.29). It is also straightforward to write down the corresponding beta functionals for the two limiting models: for the $\mathbb{Z}_2$-Yukawa-QCD model we have

$$\partial_t u = -4u + (2+\eta_\text{H})\rho u' + \frac{1}{32\pi^2}(\omega_\text{H}^2 - 12\omega_\text{F}^2), \tag{4.46}$$

and for the non-Abelian Higgs model the beta functional is

$$\partial_t u = -\, 4u + (2+\eta_\text{H})\rho u' + \frac{1}{32\pi^2}(\omega_\text{H}^2 + 3\omega_\theta^2 + 9\omega_\text{W}^2). \tag{4.47}$$

Let us notice that the expressions for the anomalous dimensions as well as the beta function for the top-Yukawa coupling are given, in the $\overline{\text{MS}}$ scheme, by their definitions in the DER.

4.3.1. Full Effective Potential in the ϕ^4-dominance Approximation

We start the analysis of the beta functions in Eqs. (4.45-4.47), by adopting the same approximation that we used in Sec. 3.4. Namely, we assume that the scalar fluctuations are dominated, in the UV limit, by the marginal quartic self-interaction coupling. This assumption, whose consistency has to be checked once we have the QFP solution, allows us to write the arguments ω_H and ω_θ in the simpler form

$$\omega_\text{H} = 3\lambda_2\rho, \qquad\qquad \omega_\theta = \lambda_2\rho. \tag{4.48}$$

Yet, we retain the full ρ-dependence in the scaling terms and on the left-hand sides of the RG flows. Subsequently, in order to implement the QFP condition at the functional level, we introduce a rescaled field variable x together with its potential $f(x)$ according to Eq. (3.62) (for the non-Abelian Higgs model we simply have to replace the strong gauge coupling with the weak gauge coupling). According to this rescaling, the beta function for $f(x)$ becomes

$$\partial_t f(x) = \partial_t u(\rho) - P\,\eta_{\text{G/W}}\, x f'(x), \tag{4.49}$$

depending on whether we use g_s^2 or g^2 to rescale the field amplitude ρ.

$\text{SU}(2)_\text{L} \times \text{SU}(3)_\text{c}$ **Model**

Regarding the general model under investigation, we specifically consider the trajectory described by Eq. (4.14). This is exactly the special red trajectory in Fig. 4.1, along which

h^2 and g^2 become proportional to g_{s}^2. This yields a beta function for the rescaled scalar potential $f(x)$ that depends only on the AF strong gauge coupling g_{s}^2:

$$\partial_t f = -4f + d_{x,\mathrm{s}} x f' + \frac{3\,x^2}{128\pi^2}\left[16\xi_2^2 g_{\mathrm{s}}^{4P} - \left(16\hat{h}_*^4 - 3\hat{g}_*^4\right) g_{\mathrm{s}}^{4-4P}\right], \tag{4.50}$$

where the scaling dimension $d_{x,\mathrm{s}}$ includes also a contribution from the running of g_{s}, cf., Eq. (3.65). The anomalous dimensions $\eta_{\mathrm{H,G}}$ are given by Eqs. (4.5) and (3.5). The QFP values for the ratios $\hat{g}_*^2$ and $\hat{h}_*^2$ are listed in Eq. (4.14). The QFP equation, which is obtained by the requirement that the left-hand side of Eq. (4.50) is vanishing, is solved by

$$f(x) = C_f x^{4/d_{x,\mathrm{s}}} - \frac{3\,x^2}{256\pi^2\eta_{x,\mathrm{s}}}\left[16\xi_2^2 g_{\mathrm{s}}^{4P} - \left(16\hat{h}_*^4 - 3\hat{g}_*^4\right) g_{\mathrm{s}}^{4-4P}\right], \tag{4.51}$$

where C_f is a free integration constant, parameterizing the solution for the associated homogeneous equation. Setting $C_f = 0$ and requiring the consistency condition $f''(0) = \xi_2$ singles out the same solution with $P = 1/2$ and $\xi_2 = \hat{\lambda}_2^+$ as described in Sec. 4.1, cf. Eq. (4.18).

For any nonvanishing $C_f \neq 0$, the QFP potential behaves as a nonrational power of x at the origin, thus, its second order derivative at $x = 0$ is singular for any $\eta_{x,\mathrm{s}} > 0$. If the system is in the SYM regime, the anomalous dimension for the rescaled field is indeed positive for the $\mathbb{Z}_2$-Yukawa-QCD model, for all values of P. Hence, the singularity would affect large classes of correlation functions expanded about the symmetric ground state, such that we consider these solutions as unphysical. On the other hand, for the general model and for the non-Abelian Higgs model, η_x can be negative for small enough values of P, because of the negative gauge-loop contribution entering in η_{H}.

The singular behavior at the origin can be avoided if there is at least one nontrivial minimum $x_0 > 0$ for $f(x)$, satisfying the consistency condition $f''(x_0) = \xi_2$. In fact, the system of two equations that arises by setting $n = 2$ in Eq. (3.63) can be solved for C_f and ξ_2 as functions of x_0. The additional requirement that $0 < \xi_2 < \infty$ in the $g_{\mathrm{s}}^2 \to 0$ limit can be fulfilled only when $P = 1$. We obtain therefore the following expressions for C_f and ξ_2, at the leading order in g_{s}^2,

$$C_f = -\frac{3(16\hat{h}_*^4 - 3\hat{g}_*^4)}{256\pi^2}\left(\frac{1}{\eta_{x,\mathrm{s}}} + \frac{1 + 2\log x_0}{2}\right), \tag{4.52}$$

$$\xi_2 = \frac{3(16\hat{h}_*^4 - 3\hat{g}_*^4)}{128\pi^2} > 0, \qquad P = 1. \tag{4.53}$$

If x_0 attains a finite value in the $g_{\mathrm{s}}^2 \to 0$ limit, this corresponds to a potential that has a finite minimum as well as finite derivatives at this minimum, which are given by

$$\xi_n = (-1)^{n+1}\frac{3(16\hat{h}_*^4 - 3\hat{g}_*^4)}{128\pi^2}\frac{(n-3)!}{x_0^{n-2}}, \qquad n \geq 3. \tag{4.54}$$

Therefore, we can construct a family of solutions parameterized by x_0, with the desired property that the rescaled quartic coupling at x_0 is finite in the UV limit. These solutions represent a *two-parameter family* of solutions, as the latter equation is compatible with an arbitrary asymptotic scale dependence of x_0 in the form

$$x_0 = g_\mathrm{s}^{2(P-Q)}\hat{\kappa} = g_\mathrm{s}^{2(1-Q)}\hat{\kappa}, \tag{4.55}$$

where $\hat{\kappa}$ and Q remain two undetermined parameters (see the rescaling expressions in Eqs. (3.53) and (3.62)).

In order to address the global stability of the scaling solution $f(x)$ in the UV limit, we have to study its asymptotic behavior for large amplitudes x and in the limit $g_\mathrm{s}^2 \to 0$. Following the same discussion as for Sec. 3.4.1, we can perform this combined limit by considering the product $g_\mathrm{s}^{2P}x$ either small or large. The former asymptotic region is addressed by taking first the $g_\mathrm{s}^2 \to 0$ limit and then the $x \to \infty$ limit, where we find the following asymptotic behavior

$$f(x) \underset{x\to\infty}{\sim} x^2 \frac{3(16\hat{h}_*^4 - 3\hat{g}_*^4)}{128\pi^2}\frac{1}{4}\left[-1 + 2\log\left(\frac{x}{x_0}\right)\right]. \tag{4.56}$$

In the latter asymptotic region the order of the two limits is exchanged and the asymptotic behavior is

$$f(x) \underset{x\to\infty}{\sim} \frac{3(16\hat{h}_*^4 - 3\hat{g}_*^4)}{128\pi^2}\frac{x^2}{2\eta_{x,\mathrm{s}}} > 0. \tag{4.57}$$

For both cases, we find a stable potential, providing evidence for global stability.

$\mathbb{Z}_2$-Yukawa-QCD Model

Regarding this limiting model, the beta function for $f(x)$ is obtained by substituting Eq. (4.48) into Eq. (4.46). We thus have

$$\partial_t f = -4f + d_{x,\mathrm{s}} x f' + \frac{3}{32\pi^2}\left(3\xi_2^2 g_\mathrm{s}^{4P} - 4\chi_\mathrm{s}^4 g_\mathrm{s}^{4-4P}\right)x^2, \tag{4.58}$$

where the QFP value χ_s^2 in given in Eq. (3.11). The scaling solution of the latter equation is

$$f(x) = C_f x^{4/d_{x,\mathrm{s}}} - \frac{3}{64\pi^2\eta_{x,\mathrm{s}}}\left(3\xi_2^2 g_\mathrm{s}^{4P} - 4\chi_\mathrm{s}^4 g_\mathrm{s}^{4-4P}\right)x^2, \tag{4.59}$$

where C_f is again a free integration constant, parameterizing the general solution for the associated homogeneous equation. Setting $C_f = 0$ and requiring the consistency condition $f''(0) = \xi_2$ singles out the CEL solution with $P = 1/2$ and $\xi_2 = \hat{\lambda}_2^{\pm}$ as in Eq. (3.16).

However, the potential has a log-type singularity in its second derivative at the origin for any $C_f \neq 0$, since $\eta_{x,\mathrm{s}} > 0$ in this model. This problem can be avoided if $f(x)$ admits a nontrivial minimum x_0 satisfying the consistency conditions $f'(x_0) = 0$ and $f''(x_0) = \xi_2$. It becomes then possible to solve for C_f and ξ_2 at the leading order in g_s^2

$$C_f = -\frac{3\chi_\mathrm{s}^4}{16\pi^2}\left(\frac{1}{\eta_{x,\mathrm{s}}} + \frac{1 + 2\log x_0}{2}\right), \quad \xi_2 = \frac{3\chi_\mathrm{s}^4}{8\pi^2} > 0, \qquad P = 1, \tag{4.60}$$

where $P = 1$ is again the only value leading to a finite ξ_2 . Moreover, the higher-dimensional couplings at the nontrivial minimum are

$$\xi_{n\geq 3} = (-1)^{n+1}\frac{3\chi_\mathrm{s}^4}{8\pi^2}\frac{(n-3)!}{x_0^{n-2}}. \tag{4.61}$$

In addition, we can also analyze $f(x)$ in the two asymptotic regions. The large-field expansion at finite values of g_s^2 provides the asymptotic behavior

$$f(x) \underset{x\to\infty}{\sim} \frac{3\chi_\mathrm{s}^4}{16\pi^2\eta_{x,\mathrm{s}}}x^2 > 0, \tag{4.62}$$

whereas in the opposite asymptotic region where the $g_\mathrm{s}^2 \to 0$ limit is taken before addressing the large-field expansion provides the following asymptotic behavior

$$f(x) \underset{x\to\infty}{\sim} x^2\frac{3\chi_\mathrm{s}^4}{32\pi^2}\left[-1 + 2\log\left(\frac{x}{x_0}\right)\right]. \tag{4.63}$$

Despite the different limits, we can nevertheless deduce that the potential appears stable while approaching the UV limit.

We conclude that also for the $\mathbb{Z}_2$-Yukawa-QCD model, within the $\overline{\mathrm{MS}}$ scheme, we find a *two-parameter family* of admissible QFP potential. These two parameters are encoded in the dependence of x_0 through $\hat{\kappa}$ and Q as in Eq. (4.55).

Non-Abelian Higgs Model

For this second limiting case of the general SU(2)$_L$ × SU(3)$_c$ model, the RG flow equation for $f(x)$ is obtained by substituting Eq. (4.48) into Eq. (4.47). We thus have

$$\partial_t f = -4f + d_x x f' + \frac{3}{128\pi^2}\left(16\xi_2^2 g^{4P} + 3g^{4-4P}\right)x^2, \tag{4.64}$$

where the quantum dimension d_x includes a contribution from the anomalous dimension of the W gauge boson fields

$$d_x = 2 + \eta_\mathrm{H} - P\eta_\mathrm{W} \equiv 2 + \eta_x. \tag{4.65}$$

In fact the scalar field is rescaled according to a certain power of the weak gauge coupling $x = g^{2P}\rho$. The solution for the QFP equation $\partial_t f = 0$ is

$$f(x) = C_f x^{4/d_x} - \frac{3\left(16\xi_2^2 g^{4P} + 3g^{4-4P}\right)}{256\pi^2\eta_x} x^2, \tag{4.66}$$

which again features a log-type singularity for $f''(0)$, as long as the integration constant $C_f \neq 0$ and $\eta_x > 0$.

In contrast to the general case or the $\mathbb{Z}_2$-Yukawa-QCD model, there is no real solution compatible with the consistency condition $f''(0) = \xi_2$ for $C_f = 0$. This reflects the conventional conclusion of triviality as evidenced by the presence of Landau-pole singularities in perturbation theory, see the discussion in Sec. 2.1.

A different situation occurs for $C_f \neq 0$. In this case indeed, the second derivative at the nontrivial minimum is finite only for $P = 1$ and takes the g^2-leading order value

$$\xi_2 = -\frac{9}{128\pi^2}, \qquad\qquad P = 1. \tag{4.67}$$

However, the negativity of the QFP value for the quartic scalar interaction contradicts one of our selection criteria, therefore also this second solution has to be rejected.

4.3.2. Effective Field Theory Analysis for the Scalar Potential

As a second approximation which we want to apply in order to study the beta functional $\partial_t u(\rho)$ in the $\overline{\text{MS}}$ scheme, we present now an EFT analysis following the strategy already explained in Sec. 3.3. Therefore, let us consider a polynomial expansion for $u(\rho)$ in the SSB regime as given in Eq. (3.34), and let us truncate the series up to $N_\text{p} = 2$ while retaining λ_3 as a free parameter, the latter playing the role of a boundary condition in the theory space.

Within the $\overline{\text{MS}}$ scheme, it is known that higher-dimensional scalar operators do not influence the running of the lower dimensional ones, once the DER is taken. This is because the $1/(d-4)$ divergences introduced by the higher-dimensional couplings are always multiplied by a certain power of the masses. As a consequence, in our SSB parametrization, we expect that the contributions to the running of λ_2 coming from $\lambda_{n\geq 3}$ are always multiplied by certain powers of κ.

For reasons of simplicity, we present the EFT study only for the two limiting models, the $\mathbb{Z}_2$-Yukawa-QCD model and the non-Abelian Higgs model. For both cases we consider the SM matter-field content, see Eq. (4.3). We also want to anticipate that, for the non-Abelian Higgs model, the EFT approach is capable to unveil solutions which were not accessible within the ϕ^4-dominance approximation, and we will soon explain the reasons for this disagreement.

$\mathbb{Z}_2$-Yukawa-QCD Model

Substituting the polynomial expansion of $u(\rho)$ inside Eq. (4.46), we obtain the beta functions

$$\partial_t \kappa = \left(-2 - \frac{3h^2}{8\pi^2} - \frac{3\lambda_2}{8\pi^2} + \frac{3h^4}{4\pi^2 \lambda_2} - \frac{\kappa \lambda_3}{4\pi^2} \right) \kappa, \tag{4.68}$$

$$\partial_t \lambda_2 = \frac{9\lambda_2^2}{16\pi^2} + \frac{3h^2 \lambda_2}{4\pi^2} - \frac{3h^4}{4\pi^2} + \kappa \lambda_3 \left(\frac{\lambda_2}{\pi^2} + \frac{3h^4}{4\pi^2 \lambda_2} \right), \tag{4.69}$$

where it is immediate to notice that, the contribution to the running of the quartic coupling, coming from the higher-dimension operator λ_3, vanishes in the DER. In order to unveil AF scaling solutions for the latter system of beta functions, we look for QFP solutions for the rescaled couplings $\hat{\lambda}_n$ and $\hat{\kappa}$, as defined through Eqs. (3.37) and (3.53), where $P = P_2/2$ and Q are a priori arbitrary. The dependence on the top-Yukawa coupling can be absorbed by considering the special trajectory along which $h^2 = \chi_\mathrm{s}^2 g_\mathrm{s}^2$, where χ_s^2 is given in Eq. (3.11). Also for the running of the top-Yukawa coupling, the same consideration as for η_H holds, such that the expression for $\partial_t h^2$ equals its form in the DER limit. Consequently, the RG flow equations for $\hat{\kappa}$ and $\hat{\lambda}_2$ become

$$\partial_t \hat{\kappa} = \left(-2 - \frac{g_\mathrm{s}^2}{12\pi^2} - \frac{3 g_\mathrm{s}^{4P} \hat{\lambda}_2}{8\pi^2} + \frac{g_\mathrm{s}^{4(1-P)}}{27\pi^2 \hat{\lambda}_2} - Q \frac{7 g_\mathrm{s}^2}{8\pi^2} - \frac{g_\mathrm{s}^{-2Q} \hat{\kappa} \lambda_3}{4\pi^2} \right) \hat{\kappa}, \tag{4.70}$$

$$\partial_t \hat{\lambda}_2 = \frac{9 g_\mathrm{s}^{4P} \hat{\lambda}_2^2}{16\pi^2} + \frac{g_\mathrm{s}^2 \hat{\lambda}_2}{6\pi^2} - \frac{g_\mathrm{s}^{4(1-P)}}{27\pi^2} + P \frac{7 g_\mathrm{s}^2 \hat{\lambda}_2}{4\pi^2} + \left(\frac{\hat{\lambda}_2}{\pi^2} + \frac{g_\mathrm{s}^{4(1-2P)}}{27\pi^2 \hat{\lambda}_2} \right) g_\mathrm{s}^{-2Q} \hat{\kappa} \lambda_3. \tag{4.71}$$

where the two terms proportional to the rescaled powers P and Q are the contributions coming from the running of the strong gauge coupling, whose RG flow, in the $\overline{\mathrm{MS}}$ scheme, equals the beta function in the DER, namely Eq. (3.6). It is straightforward to rediscover the CEL solution, for $P = 1/2$, $\hat{\kappa} = 0$, and $\lambda_3 = 0$ [77, 113]. In this case, we find indeed the two QFP solutions given by Eq. (3.16).

Compared to the standard one-loop flow in the DER, which is contained in Eq. (4.69) in the limit $\kappa \to 0$, it appears that nonvanishing values of $\kappa\lambda_3$ can considerably influence the flow of the quartic coupling. In fact, let us generalize our discussion by considering the case $P > 1/2$ and let us focus on the Eq. (4.71) at the leading order in g_s^2. Solving the QFP equation $\partial_t \hat{\lambda}_2 = 0$ for λ_3, we obtain the scaling relation

$$\lambda_3 = g_\mathrm{s}^{2(Q+2P)} \frac{\hat{\lambda}_2}{\hat{\kappa}}. \tag{4.72}$$

Inserting the latter into Eqs. (4.70) and (4.71), and again keeping only the leading g_s^2 terms, we obtain the QFP solution

$$\hat{\lambda}_2 = \frac{1}{54\pi^2}, \qquad \hat{\lambda}_3 = \frac{1}{54\pi^2 \hat{\kappa}}, \qquad\qquad P = 1, \qquad P_3 = Q + 2, \tag{4.73}$$

where $\hat{\kappa} > 0$ is retained as free, and $Q > -2$ since we request that $\lambda_3 \to 0$ in the UV limit. Thus, there is a two-parameter family of AF solutions labeled by $\hat{\kappa}$ and Q.

After having worked out the problem at order $N_\mathrm{p} = 2$, one might easily increase N_p and check the stability of the known solution. It turns out that any $\hat{\lambda}_n$ is a function of $\hat{\kappa}$. For instance, at $N_\mathrm{p} = 3$, we find again the same solution as in Eq. (4.73), complemented by

$$\hat{\lambda}_4 = \frac{\lambda_4}{g_\mathrm{s}^{4(Q+1)}} = -\frac{1}{54\pi^2\hat{\kappa}^2}. \tag{4.74}$$

Increasing further the value of N_p, we find that all couplings $\hat{\lambda}_{n\geq 3}$ have alternating signs. Therefore, in order to have a definite answer regarding the stability of the scalar potential, we should treat all scalar couplings at once. In fact, within the functional approach described in Sec. 4.3.1, we were able to conclude that the full scalar potential $u(\rho)$ is overall stable for arbitrarily small g_s^2 values. This functional ϕ^4-dominance approximation allowed us also to obtain an expression for all the higher-dimensional couplings, given by the Eq. (4.61), which agrees with the present EFT analysis. In fact, by inserting the scaling relation of x_0 as a function of $\hat{\kappa}$ and Q, we obtain

$$\hat{\lambda}_{n\geq 3} = \frac{\lambda_n}{g_\mathrm{s}^{2(n-2)Q+4}}, \tag{4.75}$$

which reproduces the expressions in Eqs. (4.73) and (4.74).

For completeness, let us mention that, by following the same procedure, also for the $P < 1/2$ case no acceptable solutions have been found.

Non-Abelian Higgs Model

For the second limiting case, the non-Abelian Higgs model, the functional RG flow equation (4.47) provides for the beta functions of κ and λ_2, which are

$$\partial_t\kappa = \left[-2 + \frac{9g^2}{32\pi^2} - \frac{3\lambda_2}{8\pi^2} - \frac{9g^4}{64\pi^2\lambda_2} - \frac{\kappa\lambda_3}{4\pi^2}\right]\kappa, \tag{4.76}$$

$$\partial_t\lambda_2 = \frac{3\lambda_2^2}{4\pi^2} - \frac{9\lambda_2 g^2}{16\pi^2} + \frac{9g^4}{64\pi^2} + \kappa\lambda_3\left(\frac{\lambda_2}{\pi^2} - \frac{9g^4}{64\pi^2\lambda_2}\right). \tag{4.77}$$

In order to look for AF trajectories, we proceed by rescaling the couplings with certain powers of the weak gauge coupling

$$\breve{\lambda}_2 = \frac{\lambda_2}{g^{4P}}, \qquad \breve{\lambda}_{n>2} = \frac{\lambda_n}{g^{2P_n}}, \qquad \breve{\kappa} = g^{2Q}\kappa, \tag{4.78}$$

and search for corresponding QFP solutions of the beta functions

$$\partial_t \breve{\kappa} = \left(-2 - \frac{9g^{4-4P}}{64\pi^2 \breve{\lambda}_2} - \frac{\breve{\kappa}\breve{\lambda}_3 g^{2(P_3-Q)}}{4\pi^2} + \frac{9g^2}{32\pi^2} - \frac{43Qg^2}{48\pi^2} - \frac{3\breve{\lambda}_2 g^{4P}}{8\pi^2} \right) \breve{\kappa}, \tag{4.79}$$

$$\partial_t \breve{\lambda}_2 = \frac{9g^{4(1-P)}}{64\pi^2} - \frac{9\breve{\lambda}_2 g^2}{16\pi^2} + \frac{43P\breve{\lambda}_2 g^2}{24\pi^2} + \frac{3\breve{\lambda}_2^2 g^{4P}}{4\pi^2} + \frac{\breve{\lambda}_3 \breve{\kappa}}{64\pi^2 \breve{\lambda}_2} \left(64\breve{\lambda}_2^2 - 9g^{4(1-2P)} \right) g^{2(P_3-Q)}. \tag{4.80}$$

Apart from the present use of the $\overline{\mathrm{MS}}$ scheme, these equations generalize the ones discussed in [112] by an independent Q rescaling of the minimum κ.

Because of the definition of QFP, $\breve{\kappa}$ should approach a finite value in the UV limit. Therefore, it is possible to recognize from Eq. (4.79) that only three combinations for the rescaling powers are allowed: $P = 1$, or $P_3 = Q$ or $P_3 = Q + 2 - 2P$. In the first case where $P = 1$, P_3 is fixed to be $P_3 = Q + 2$ in order for the beta function $\partial_t \breve{\lambda}_2$ to have a finite $g^2 \to 0$ limit. We obtain therefore the solution

$$\breve{\lambda}_2 = -\frac{9}{128\pi^2}, \qquad \breve{\kappa} = -\frac{9}{128\pi^2 \breve{\lambda}_3}, \qquad P = 1, \qquad P_3 = Q + 2, \tag{4.81}$$

where $\breve{\lambda}_3 < 0$ and $Q \geq -2$ remain two free parameters. However, this solution has to be rejected since it is not compatible with the positivity of the scalar quartic self-interaction. Instead, for the second possibility with $P_3 = Q$, the system admits a suitable solution corresponding to

$$\breve{\lambda}_2 = \pm\frac{3}{8}, \qquad \breve{\kappa} = -\frac{8\pi^2}{\breve{\lambda}_3}, \qquad P = \frac{1}{2}, \qquad P_3 = Q, \tag{4.82}$$

where the requirement $\hat{\kappa} > 0$ implies $\breve{\lambda}_3 < 0$. For completeness let us stress that the third possibility with $P_3 = Q + 2 - 2P$ does not lead to any real solution since the QFP equation $\partial_t \breve{\lambda}_2 = 0$ admits only complex roots, at the leading order in g^2.

Also for this model, we can enlarge the polynomial expansion to higher orders. As a representative case, let us consider $N_\mathrm{p} = 4$. The solution $P = 1$ still survives and leads to

$$\breve{\lambda}_4 = \frac{9}{128\pi^2 \breve{\kappa}^2}, \quad \breve{\lambda}_5 = -\frac{9}{64\pi^2 \breve{\kappa}^3}, \qquad P_4 = 2Q + 2, \quad P_5 = 3Q + 2, \tag{4.83}$$

where all couplings $\breve{\lambda}_{n\geq 2}$ have alternating signs for $\breve{\kappa} > 0$. The solution for $P = 1/2$ acquires a different QFP value for $\breve{\lambda}_2$, since its beta function receives leading-order contributions both from $\breve{\lambda}_3$ as well as from $\breve{\lambda}_4$. In fact we have

$$\breve{\lambda}_2 = \pm\frac{\sqrt{3}}{2}, \quad \breve{\lambda}_4 = \frac{26\pi^2}{\breve{\kappa}^2}, \quad \breve{\lambda}_5 = -\frac{187\pi^2}{2\breve{\kappa}^3}, \qquad P = \frac{1}{2}, \quad P_4 = 2Q, \quad P_5 = 3Q, \tag{4.84}$$

while $\breve{\lambda}_3$ and P_3 are still given by Eq. (4.82).

Like the previous model, the presence of solutions with alternating sign for λ_n, imposes that, in order to discuss the stability of the potential, we need to account for the full functional structure of $u(\rho)$. In the previous Sec. 4.3.1, we indeed used the ϕ^4-dominance approximation. However, within that approach, we were able to reveal only the unphysical $P = 1$ solution. On the contrary, the present EFT analysis can reveal an acceptable solution associated to the value $P = 1/2$. The fact that the $P = 1/2$ solution is not accessible within the ϕ^4-dominance approximation may stem from the fact that the higher-order couplings λ_3 and λ_4 are crucial for finding the QFP value in Eq. (4.84). Therefore, we expect that neglecting the presence of these interactions within the scalar threshold loops results in the impossibility to detect this additional AF solution. This fact clearly illustrates that these two different approximation schemes come with their own advantages and limitations such that they should be thought of as complementary tools.

To conclude this first part dedicated to the $\overline{\text{MS}}$ scheme, we want to emphasize that, for the general $SU(2)_L \times SU(3)_c$ model as well as for the limiting $\mathbb{Z}_2$-Yukawa-QCD model, we have revealed the existence of novel AF trajectories by means of the ϕ^4-dominance approximation. Also the non-Abelian Higgs model possesses an AF solution, detectable from an EFT approach. All these trajectories have been obtained as QFP solutions of the scalar potential, provided the presence of threshold effects induced by a nontrivial vacuum expectation value over an arbitrarily large RG scale.

These results enforce even more the findings presented in the previous Chap. 3, where we restricted ourselves to the FRG scheme. We concluded Chap. 3 by saying that, for $P < 1/2$ there is a *one-parameter family* of stable QFP solutions. Those solutions were parameterized by the finite rescaled quartic scalar coupling ξ_2. Here instead, the two parameters labeling the QFP solutions are $\hat{\kappa}$ and Q. We have therefore obtained a first glance on how different regularization schemes, the $\overline{\text{MS}}$ and the FRG schemes, induce a map in the parameters space: the set (P, ξ_2) is mapped into $(Q, \hat{\kappa})$ and vice versa.

4.4. Full Effective Potential in the Weak-coupling Expansion for a General Scheme

As anticipated at the beginning of this chapter, we now want to substantiate all these analyses by addressing the important question of the scheme (in)dependence of our results. In order to do so, we present now a weak-coupling expansion of the full beta functional $\partial_t f(x)$. In other words, this functional approximation consists of an expansion of $\partial_t f(x)$ in powers of the gauge couplings, while retaining x, $f(x)$ and its derivatives as finite quantities. Within this approximation, we do not restrict anymore the scalar fluctuations to be dominated by the quartic self-interaction coupling in the far UV, as we did in Sec. 4.3.1, and in the previous chapter in Sec. 3.4. Technically, we retain the full functional expression for the threshold functions appearing in Eq. (4.28), while expanding them in powers of g_s^2 (or g^2). It will be clear soon that this approximation allows us to account for the general scheme dependence of the corresponding solutions. In fact, we can address the one-loop flow equation of $f(x)$ in any arbitrary regularization and renormalization scheme.

Let us then start by first giving the RG flow equations of $f(x)$ for the different models under investigation. For the general $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{c}$ model, and by choosing the SM matter field content (cf. Eq. (4.3)), the beta functional for f reads

$$\partial_t f = -4f + d_{x,\mathrm{s}} x f' + \frac{1}{16\pi^2}\left[l_0^{(\mathrm{H})}(\omega_{\mathrm{H}f}) + 3 l_0^{(\theta)}(\omega_{\theta f}) + 9 l_0^{(\mathrm{W})}(z_\mathrm{W}) - 12 l_0^{(\mathrm{F})}(z_\mathrm{F})\right], \qquad (4.85)$$

which is derived straightforwardly from Eqs. (4.28) and (3.62). The full quantum dimension of x is given by Eq. (3.65). The arguments of the threshold functions can be obtained from Eqs. (3.23) and (4.29), after having rescaled the field amplitude ρ. We thus have

$$\omega_{\mathrm{H}f} = g_\mathrm{s}^{2P}\left(f' + 2xf''\right), \qquad \omega_{\theta f} = g_\mathrm{s}^{2P} f', \qquad z_\mathrm{F} = \hat{h}_*^2 g_\mathrm{s}^{2-2P} x, \qquad z_\mathrm{W} = \hat{g}_*^2 g_\mathrm{s}^{2-2P}\frac{x}{2}, \qquad (4.86)$$

where, as usual, the special trajectories along which $h^2 \propto g_\mathrm{s}^2$ and $g^2 \propto g_\mathrm{s}^2$ is taken into account, cf. Eq. (4.14). In the limiting case of the $\mathbb{Z}_2$-Yukawa-QCD model, the degrees of freedom associated to the W gauge bosons and the Goldstone modes are not present therefore, the beta function of $f(x)$ boils down to

$$\partial_t f = -4f + d_{x,\mathrm{s}} x f' + \frac{1}{16\pi^2}\left[l_0^{(\mathrm{H})}(\omega_{\mathrm{H}f}) - 12 l_0^{(\mathrm{F})}(z_\mathrm{F})\right], \qquad (4.87)$$

where z_F depends, in this case, on the QFP value χ_s^2 given in Eq. (3.11). For the non-Abelian Higgs model, suppressing the fermionic contribution, we obtain

$$\partial_t f = -4f + d_x x f' + \frac{1}{16\pi^2}\left[l_0^{(\mathrm{H})}(\omega_{\mathrm{H}f}) + 3 l_0^{(\theta)}(\omega_{\theta f}) + 9 l_0^{(\mathrm{W})}(z_\mathrm{W})\right], \qquad (4.88)$$

where the quantum dimension d_x is given by Eq. (4.65). Let us point out that, even though we have used the same notation for the threshold functions as for the FRG case, here we interpret $l_0^{(\dots)}$ as threshold functions in a *generic* regularization and renormalization scheme. As an example, the $\overline{\mathrm{MS}}$ scheme discussed in Sec. 4.3, without RG improvement, i.e., suppressing the anomalous dimensions in the threshold functions, would correspond to Eq. (4.44).

Given the general beta functionals for the rescaled scalar potential $f(x)$, the weak-coupling approximation provides us with an expansion as

$$\partial_t f = \left[\beta_f\right]_0 + \delta\beta_f, \qquad (4.89)$$

where $[\beta_f]_0$ is the leading order (LO) contribution corresponding to $g_s = 0$, and $\delta\beta_f$ is the next-to-leading-order (NLO) term in g_s. According to the rescaling in Eq. (3.62), it should be clear that quantum fluctuations can contribute to the LO term only for $P = 1$. For example, we would have for the $\mathbb{Z}_2$-Yukawa-QCD model

$$[\beta_f]_0 = -4f + 2xf' - \frac{3}{4\pi^2} l_0^{(\mathrm{F})}\left(\chi_s^2 x\right), \qquad P = 1, \tag{4.90}$$

and for the non-Abelian Higgs model

$$[\beta_f]_0 = -4f + 2xf' + \frac{9}{16\pi^2} l_0^{(\mathrm{W})}\left(\frac{x}{2}\right), \qquad P = 1, \tag{4.91}$$

such that a particular choice for the regularization scheme is required in order to address explicit properties of their QFP solutions. We will discuss this aspect below. On the contrary, for $P < 1$ the LO contribution is trivial since $[\beta_f]_0 = -4f + 2xf'$, and corresponds to the classical scaling of a scalar field where no quantum corrections are retained.

Aiming at the NLO contribution $\delta\beta_f$, the values of $P \neq 1$ are simpler to address in a generic scheme, since the vertices of the theory, by assumption, scale like positive powers of the gauge couplings. In fact, $\delta\beta_f$ is obtained by Taylor expanding the threshold functions to the first order in the gauge couplings. This gives rise to several coefficients which account for all the scheme dependence of the beta functions, namely

$$\mathcal{A}_\Phi = -\frac{1}{16\pi^2}\left[\partial_z l_0^{(\Phi)}(z)\right]_{z=0}, \qquad \Phi \in \{\mathrm{H}, \theta, \mathrm{F}, \mathrm{W}\}, \tag{4.92}$$

with Φ labeling the fluctuation modes. Within an FRG scheme these coefficients can also be written as [28]

$$\mathcal{A}_\Phi = \frac{1}{2k^2}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\tilde{\partial}_t P_\Phi(p^2)}{\left[P_\Phi(p^2)\right]^2}, \tag{4.93}$$

such that, for any admissible regulator functions providing an IR regularization, they are always positive $\mathcal{A}_\Phi > 0$. In fact, the requirement by which the shape functions provide a physical coarse-graining is that they have to monotonically increase with respect to k, at a given p^2, see Eq. (2.22). For example the piecewise linear regulator [192, 193], discussed in App. B, leads to $\mathcal{A}_\Phi = 1/(32\pi^2)$.

[28] Here, the operator $\tilde{\partial}_t$ denotes an RG-time differentiation acting only on the regulator. Its precise definition can be found in App. B, cf. Eq. (B.21). Moreover P_Φ is the inverse regularized propagator of each field in momentum space. In the FRG formalism P_Φ depends on the regularization kernel R_k entering in the Wetterich equation (2.34). For more details and explicit expressions see App. B.

Consistency Check

In the following part of this chapter, we will address all cases for the rescaling power $P \leq 1$, and for all the three models of our interest; the general $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{c}$ model as well as the two limiting cases, the $\mathbb{Z}_2$-Yukawa-QCD and the non-Abelian Higgs models. We do not present here the weak-coupling analysis for $P > 1$, since it does not provide any AF scaling solution for the gauged-Yukawa models. We refer the reader to App. G.2, for a discussion of this latter case.

Before presenting the weak-coupling analysis for the $P \leq 1$ values, we want to highlight few aspects regarding the consistency of this approximation. The argument z_F of the fermionic loop depends on the asymptotic scaling property of the top-Yukawa coupling. In fact, while expanding the $l_0^{(\mathrm{F})}(z_\mathrm{F})$ loop for small z_F, we assume that h^2 scales proportional to g_s^2. Of course, the consistency of this assumption can be easily tested once a scaling solution for $f(x)$ is obtained. This check is important since the beta function of h^2 (cf. Eq. (4.31)) can differ from its DER limit due to a nontrivial g_s-dependence implicit in the minimum x_0. Indeed, in App. G.1, we will clarify in detail that the asymptotic scaling for the top-Yukawa coupling does change for $P < 1/4$, spoiling the existence of total AF trajectories for the general $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{c}$ model as well as for the $\mathbb{Z}_2$-Yukawa-QCD model. Same consistency checks also hold for the other loop argument; once we have a QFP solution for $f(x)$, it is important to very that $\omega_{\theta f}$, $\omega_{\mathrm{H}f}$ and z_W vanish too, as $g_\mathrm{s}^2 \to 0$.

A second consistency check has to be performed at the level of the anomalous dimension for the rescaled field x. In the DER $\eta_{x,\mathrm{s}}$ is proportional to g_s^2, such that it is always subleading with respect to the loop variables. This fact has to be verified as well, once a scaling solution for $f(x)$ is computed. It is found indeed that, for all $P \leq 1$, $\eta_{x,\mathrm{s}}$ is subleading such that the NLO contributions $\delta\beta_f$ are just given by a linear expansion of the threshold functions in their variables.

4.4.1. $\mathbf{P} \in (\mathbf{0}, \mathbf{1/2})$

For this range of P and for all models under consideration, only the scalar loops contribute to the first correction in the beta function for $f(x)$. In the general model, the NLO correction scales as g_s^{2P},

$$\delta\beta_f = -g_\mathrm{s}^{2P} \left[\mathcal{A}_\mathrm{H}(f' + 2xf'') + 3\mathcal{A}_\theta f'\right], \tag{4.94}$$

where we have distinguished the contributions coming from the Goldstone or radial modes with the labels θ and H, respectively. For the $\mathbb{Z}_2$-Yukawa-QCD model, the NLO correction can be recovered by simply setting $\mathcal{A}_\theta = 0$. On the other hand, for the non-Abelian Higgs model, $\delta\beta_f$ is the same as in the latter equation with the substitution $g_\mathrm{s}^2 \leftrightarrow g^2$.

By including the NLO correction in g_s^2, the QFP equation for $f(x)$ becomes a second order ordinary differential equation (ODE) which can be analytically solved and leads to two different solutions. The first one is given by a special case of the Kummer function which reduces to a quadratic polynomial, [29]

$$f(x) = c\left[x^2 - 3(\mathcal{A}_\mathrm{H} + \mathcal{A}_\theta) g_\mathrm{s}^{2P} x\right]. \tag{4.95}$$

The second solution grows exponentially for large field amplitudes, therefore we reject them since we are only interested in solutions that obey power-like scaling for $x \to \infty$. In fact, only in this latter case it is possible to define a scalar product in the space of eigenperturbations of these solutions [255, 257, 258].

By imposing the defining properties for the nontrivial minimum and the rescaled quartic scalar coupling, namely $f'(x_0) = 0$ and $f''(x_0) = \xi_2$ respectively, we find

$$x_0 = \frac{3}{2}(\mathcal{A}_\theta + \mathcal{A}_\mathrm{H}) g_\mathrm{s}^{2P}, \qquad \xi_2 = 2c. \tag{4.96}$$

Therefore we can infer that, for the general model as well as for the non-Abelian Higgs model, the condition for having a nontrivial minimum is $\mathcal{A}_\mathrm{H} > -\mathcal{A}_\theta$; whereas for the $\mathbb{Z}_2$-Yukawa-QCD model the latter expression becomes simply $\mathcal{A}_\mathrm{H} > 0$. As these conditions are satisfied for all admissible FRG regularization schemes, within the latter framework the existence of these solutions is a scheme-independent result. A particular limiting case is the one of $\overline{\mathrm{MS}}$, where these solutions are not present. In fact, from the definition of $\mathcal{A}_\Phi$ in Eq. (4.92), and the expression of the threshold functions in the $\overline{\mathrm{MS}}$ regularization scheme in Eq. (4.44), it is straightforward to realize that $\mathcal{A}_\mathrm{H} = \mathcal{A}_\theta = 0$. We provide an interpretation of this fact at the end of this section.

Let us now perform the consistency checks mentioned before, while describing the weak-coupling expansion. As a first test, we observe that, by inserting the solution of Eq. (4.95) back into the expression for the anomalous dimension $\eta_{x,\mathrm{s}}$, we find that there is no contribution to $\delta\beta_f$ coming from $\eta_{x,\mathrm{s}}$ for any $P < 1/2$. In fact, the leading terms in $\eta_{x,\mathrm{s}}$ scale as either g_s^2 or g_s^{8P}, being always subleading with respect to the scalar contributions proportional to g_s^{2P}. Same considerations can be drawn also for η_W in the non-Abelian Higgs model.

On the other hand, in App. G.1, we will show that the contributions in $\eta_{x,\mathrm{s}}$ proportional to g_s^{8P} modify the asymptotic UV behavior of the top-Yukawa coupling for $P < 1/4$, preventing the possibility of QFP solutions for h^2. Therefore, for $P < 1/4$ the current solution for $f(x)$ is no longer valid for both the $\mathbb{Z}_2$-Yukawa-QCD model as well as the general model, [30] while

[29] A standard solution of the Kummer equation $xf''(x) + (b-x)f'(x) - af(x) = 0$ is given by the confluent Hypergeometric function $U(a, b; x)$. In the particular case where $b - a - 1 = n \in \{0, 1, 2, \dots\}$, this special function assumes the polynomial expression $U(a, a+n+1; x) = x^{-a} \sum_{s=0}^{n} \binom{n}{s} a(a+1) \cdot \ldots \cdot (a+s-1) x^{-s}$.

[30] A different QFP for $P < 1/4$ might still be possible in Yukawa models, if κ and h^2 exhibit asymptotic scaling powers different from the ones discussed in this section. This behavior might require strong threshold phenomena and decoupling of some degrees of freedom. We leave this analysis for future

it survives only in the non-Abelian Higgs model since the Yukawa interaction is absent. For $P = 1/4$ the scaling relation $h^2 \propto g_\mathrm{s}^2$ still holds even if the QFP value $\hat{h}_*^2$ changes (cf. Eq. (G.2)). Therefore the QFP solution in Eq. (4.95) is valid for any P in the interval $1/4 \leq P < 1/2$.

4.4.2. $\mathbf{P = 1/2}$

For this particular value of P, the contributions from the scalar loops mix together with the contributions from the fermionic loop and the gauge boson loop. For the general SU(2)$_\mathrm{L}$ × SU(3)$_\mathrm{c}$ model, the NLO correction to $[\beta_f]_0$ is

$$\delta\beta_f = -g_\mathrm{s}\left[\mathcal{A}_\mathrm{H}(f' + 2xf'') + 3\mathcal{A}_\theta f' + \frac{9}{2}\mathcal{A}_\mathrm{W}\hat{g}_*^2 x - 12\mathcal{A}_\mathrm{F}\hat{h}_*^2 x\right], \tag{4.97}$$

where the QFP values $\hat{g}_*^2$ and $\hat{h}_*^2$ are given in Eq. (4.14). The equation $[\beta_f]_0 + \delta\beta_f = 0$ is a second-order linear ODE, whose solution is the sum of the general solution for the homogeneous part, cf. Eq. (4.95), plus a particular solution for the nonhomogeneous equation, which is a quadratic polynomial in x. We find therefore the QFP solution

$$f(x) = \frac{\xi_2}{2}x^2 - \frac{3g_\mathrm{s}}{4}\left[2\xi_2(\mathcal{A}_\theta + \mathcal{A}_\mathrm{H}) + 3\mathcal{A}_\mathrm{W}\hat{g}_*^2 - 8\mathcal{A}_\mathrm{F}\hat{h}_*^2\right]x, \tag{4.98}$$

where we have already implemented the consistency condition that $f''(x_0) = \xi_2$. The nontrivial minimum x_0 and the condition for its positivity are

$$x_0 = \frac{3g_\mathrm{s}}{4\xi_2}\left[2\xi_2(\mathcal{A}_\theta + \mathcal{A}_\mathrm{H}) + 3\mathcal{A}_\mathrm{W}\hat{g}_*^2 - 8\mathcal{A}_\mathrm{F}\hat{h}_*^2\right], \qquad \xi_2 > \frac{8\mathcal{A}_\mathrm{F}\hat{h}_*^2 - 3\mathcal{A}_\mathrm{W}\hat{g}_*^2}{2(\mathcal{A}_\theta + \mathcal{A}_\mathrm{H})}. \tag{4.99}$$

Let us now investigate the two limiting models. In the non-Abelian Higgs model, we have $\mathcal{A}_\mathrm{F} = 0$ and $\hat{g}_* = 1$ due to the replacement of g_s with g, implying that the QFP potential is

$$f(x) = \frac{\xi_2}{2}x^2 - \frac{3gx}{4}\left[3\mathcal{A}_\mathrm{W} + 2\xi_2(\mathcal{A}_\theta + \mathcal{A}_\mathrm{H})\right], \tag{4.100}$$

which has a positive nontrivial minimum at

$$x_0 = \frac{3g}{4\xi_2}\left[2\xi_2(\mathcal{A}_\theta + \mathcal{A}_\mathrm{H}) + 3\mathcal{A}_\mathrm{W}\right], \qquad \xi_2 > -\frac{3\mathcal{A}_\mathrm{W}}{2(\mathcal{A}_\theta + \mathcal{A}_\mathrm{H})}. \tag{4.101}$$

Also the results for the $\mathbb{Z}_2$-Yukawa-QCD model are easily recovered from the general model by setting $\mathcal{A}_{\mathrm{W},\theta} = 0$ and $\hat{h}_*^2 = \chi_\mathrm{s}^2$, where the QFP value χ_s^2 is given by Eq. (3.11). For example, the expression for the nontrivial minimum and the condition for its positivity

investigations.

become

$$x_0 = \frac{g_\mathrm{s}}{4\xi_2}\left(6\xi_2\mathcal{A}_\mathrm{H} - 24\mathcal{A}_\mathrm{F}\hat{h}_*^2\right), \qquad \xi_2 > \frac{4\mathcal{A}_\mathrm{F}\hat{h}_*^2}{\mathcal{A}_\mathrm{H}}. \tag{4.102}$$

We conclude that, in all models under consideration and for $P = 1/2$, the QFP equation $\partial_t f = 0$ admits scaling solutions which are in the SSB regime. For any FRG scheme, the scheme-dependent coefficients $\mathcal{A}_\Phi$ assume positive values and define a range of allowed values for ξ_2 which parameterizes the new AF solutions. The change of this range for different FRG schemes, corresponds to the expected mapping of the coupling space onto itself induced by a change of the regularization scheme.

Again the $\overline{\mathrm{MS}}$ scheme appears to be special. The vanishing of $\mathcal{A}_\Phi$ singles out the fact that the NLO contribution $\delta\beta_f$ should be quadratic in $f(x)$. As a consequence, at the present order of expansion, only the canonical scaling term survives and the QFP solution is purely classical. Therefore, for the $\overline{\mathrm{MS}}$ scheme and for $P = 1/2$, we find that there are no new AF scaling solutions at the NLO of the weak-coupling approximation, which is however in contrast with the EFT-like analysis provided in Sec. 4.3.2. In fact, within this latter approximation we found the existence of novel AF trajectories for the non-Abelian Higgs model at $P = 1/2$. This fact certainly represents a limitation of the present weak-coupling expansion when applied to the $\overline{\mathrm{MS}}$ scheme.

All the consistency checks for the present value $P = 1/2$ are passed. In fact we notice that the anomalous dimension $\eta_{x,\mathrm{s}}$ is subleading with respect to $\delta\beta_f$, and the QFP value for the top-Yukawa coupling is still given by Eq. (4.14) for the general model or Eq. (3.11) for the $\mathbb{Z}_2$-Yukawa-QCD model. This can be verified straightforwardly by substituting the QFP solution for $f(x)$ into the functional expressions for η_H and $\partial_t h^2$ given in Sec. 4.2.

4.4.3. $\mathbf{P \in (1/2, 1)}$

Differently from the previous cases, the NLO correction is given by the fermionic loop and/or the gauge boson loop, as the scalar contributions are subleading. For example, in the general $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{c}$ model, $\delta\beta_f$ is proportional to g_s^{2-2P} and reads

$$\delta\beta_f = g_\mathrm{s}^{2-2P}\left(-\frac{9}{2}\mathcal{A}_\mathrm{W}\hat{g}_*^2 + 12\mathcal{A}_\mathrm{F}\hat{h}_*^2\right)x, \tag{4.103}$$

where the QFP values $\hat{g}_*^2$ and $\hat{h}_*^2$ are given in Eq. (4.14). The FP equation $[\beta_f]_0 + \delta\beta_f = 0$ remains a first-order ODE whose analytical solution is

$$f(x) = \frac{\xi_2}{2}x^2 - \frac{3g_\mathrm{s}^{2-2P}}{4}\left(3\mathcal{A}_\mathrm{W}\hat{g}_*^2 - 8\mathcal{A}_\mathrm{F}\hat{h}_*^2\right)x, \tag{4.104}$$

where we have already implemented the defining condition for the rescaled quartic scalar coupling ξ_2, which remains yet an arbitrary positive parameter. The QFP potential in the latter equation admits a nontrivial minimum whose expression and positivity condition are

$$x_0 = \frac{3g_\mathrm{s}^{2-2P}}{4\xi_2}\left(3\mathcal{A}_\mathrm{W}\hat{g}_*^2 - 8\mathcal{A}_\mathrm{F}\hat{h}_*^2\right), \qquad \mathcal{A}_\mathrm{W} > \frac{8\mathcal{A}_\mathrm{F}\hat{h}_*^2}{3\hat{g}_*^2}. \tag{4.105}$$

It is interesting to notice that if we adopt the same scheme for both loops, such that $\mathcal{A}_\mathrm{F} = \mathcal{A}_\mathrm{W}$, the latter condition is not satisfied within the SM set of parameters since $3\hat{g}_*^2 - 8\hat{h}_*^2 < 0$.

If we consider the $\mathbb{Z}_2$-Yukawa-QCD model, the expression for the nontrivial minimum becomes

$$x_0 = -g_\mathrm{s}^{2-2P}\frac{6\mathcal{A}_\mathrm{F}\hat{h}_*^2}{\xi_2} < 0, \tag{4.106}$$

which would be negative for any positive value of ξ_2 and any admissible FRG scheme.

The opposite situation occurs, instead, for the non-Abelian Higgs model where the nontrivial minimum becomes

$$x_0 = g^{2-2P}\frac{9\mathcal{A}_\mathrm{W}}{4\xi_2}, \tag{4.107}$$

such that, for any positive value of $\mathcal{A}_\mathrm{W}$ and ξ_2, the scalar potential is in the broken regime and features a nontrivial minimum.

Let us notice that also for this range of P values, the consistency checks confirm that the anomalous dimension for the x field is subleading with respect to the NLO correction $\delta\beta_f$ and that the top-Yukawa coupling h^2 still scales as g_s^2 in the UV limit. In addition, let us emphasize that for all the solutions obtained for $P < 1$, the arguments of the threshold functions $\omega_{\theta f}, \omega_{\mathrm{H}f}$, z_F and z_W approach zero in the UV limit, thus verifying the last important point of the self-consistency check of our solutions.

We now want to discuss a few further aspects regarding the QFP solutions obtained for the values $P < 1$. Within the present weak-coupling approximation, all scaling solutions for $f(x)$ are analytic in x, in particular at $x = 0$. However, within the ϕ^4-dominance approximation discussed in Sec. 4.3.1, we have observed that the QFP solutions for $f(x)$ possess a singular $f''(0)$, whenever a nonzero anomalous dimension for x is involved. The same kind of singularity would also appear if some of the loop threshold contributions are proportional to x^2. Within the present weak-coupling expansion, we do not see this singularity since it is accompanied by a subleading power of g_s (or g) with respect to the NLO, for any $P < 1$.

Knowing about the presence of this singularity for any P at nonvanishing gauge couplings, we can accept the previous solutions only if $x_0 > 0$. This appears to be possible in all models under investigation (within the family of FRG schemes), even though for different range of P values: for the $\mathrm{SU(2)_L \times SU(3)_c}$ model the allowed range is $1/4 \leq P < 1$, for the $\mathbb{Z}_2$-Yukawa-QCD model the allowed range is $1/4 \leq P < 1/2$ and for the non-Abelian Higgs model the allowed range is $P < 1$.

As for the fate of these solutions in the $\overline{\mathrm{MS}}$ scheme, no new AF trajectories are visible for $P < 1$, within the weak-coupling analysis. However, this does not exclude the existence of scaling solutions in $\overline{\mathrm{MS}}$ altogether. In fact, as discussed in Sec. 4.3.2, an EFT-like analysis was able to reveal the $P = 1/2$ scaling solution fo the non-Abelian Higgs model.

A general observation that could be used to argue against the validity of the present approximation in the FRG framework, is that neglecting the nonlinear contributions in the beta functional $\partial_t f(x)$ might miss crucial terms and produce spurious or unphysical QFP solutions. In particular, part of the universal one-loop contribution, the one which arises from the Taylor expansion of the threshold functions to second order in $\omega_{\theta f}, \omega_{\mathrm{H}f}$, z_{F} and z_{W}, is not accounted for in the present discussion of the $P < 1$ scaling solutions. However, the inclusion of the latter contributions as well as of further nonlinearities, up to the full complexity of the FRG flow equations of Sec. 4.2, has been performed, with specific regulator choices, for the non-Abelian Higgs model [111, 112] as well as for the $\mathbb{Z}_2$-Yukawa-QCD model, in fact confirming the results of this leading-order weak-coupling expansion. For example, in App. F, we will show in details how the full one-loop nonlinear FRG flow equations for the $\mathbb{Z}_2$-Yukawa-QCD model can be accounted for with different numerical methods, confirming the analytic results provided here.

4.4.4. $\mathbf{P = 1}$

For this particular value of P, the LO beta function of $f(x)$ accounts for the full nonlinearity of the gauge boson and/or fermion loops. Indeed, the arguments z_{F} and z_{W} are finite and do not approach zero in the UV limit. As a matter of fact, the LO contribution for the $\mathrm{SU(2)_L \times SU(3)_c}$ model is

$$[\beta_f]_0 = -4f + 2xf' + \frac{1}{16\pi^2}\left[9 l_0^{(\mathrm{W})}(z_{\mathrm{W}}) - 12 l_0^{(\mathrm{F})}(z_{\mathrm{F}})\right]. \tag{4.108}$$

In addition, the RG flow equations for the gauge couplings and the top-Yukawa coupling cannot be treated as they were in the DER. Therefore, even though we still expect that $g^2 \propto g_{\mathrm{s}}^2$ and $h^2 \propto g_{\mathrm{s}}^2$ as $g_{\mathrm{s}}^2 \to 0$, the QFP values $\hat{g}_*^2$ and $\hat{h}_*^2$ will have different values from Eq. (4.14). In particular, $\hat{g}_*^2$ and $\hat{h}_*^2$ become dependent on the nontrivially minimum x_0.

The corresponding QFP equation $[\beta_f]_0 = 0$ can be solved analytically and leads to an integral solution

$$f(x) = c\, x^2 - \frac{9}{32\pi^2} z_{\mathrm{W}}^2 \int_1^{z_{\mathrm{W}}} \mathrm{d}y\, y^{-3} l_0^{(\mathrm{W})}(y) + \frac{3}{8\pi^2} z_{\mathrm{F}}^2 \int_1^{z_{\mathrm{F}}} \mathrm{d}y\, y^{-3} l_0^{(\mathrm{F})}(y), \tag{4.109}$$

where c is an arbitrary integration constant. For instance, within the FRG scheme and for the piecewise linear regulator (cf. Eq. (B.22b)), we would obtain a Coleman-Weinberg-like potential,

$$f(x) = c\,x^2 - \frac{9}{64\pi^2}\left[z_{\mathrm{W}} + z_{\mathrm{W}}^2 \log\left(\frac{z_{\mathrm{W}}}{1+z_{\mathrm{W}}}\right)\right] + \frac{3}{16\pi^2}\left[z_{\mathrm{F}} + z_{\mathrm{F}}^2 \log\left(\frac{z_{\mathrm{F}}}{1+z_{\mathrm{F}}}\right)\right] \tag{4.110}$$

which has a log-type singularity at the origin for $f''(0)$ due to the term $\sim x^2 \log x$.

This singularity is expected for any scheme. In fact by Taylor expanding the threshold functions $l_0^{(\mathrm{F})}(y)$ and $l_0^{(\mathrm{W})}(y)$ about $y = 0$, logarithmic divergences of the integral arise from the quadratic terms of this expansion. In other words, the appearance of this singular behavior is as universal as the one-loop beta function of the marginal couplings. Therefore we expect that this feature survives also in the full g_{s}^2-dependent solution.

The freedom of choosing the parameter c allows us to construct QFP solutions, which circumvent the problem of non analyticity at the origin by developing a nontrivial minimum away from the origin. The defining equation for the minimum $f'(x_0) = 0$, involves an integral of two arbitrary threshold functions and might be hard to solve analytically for x_0. Still, it can straightforwardly be used to express c as a function of x_0. From the point of view where the latter is the free parameter labeling the QFP solutions, the natural question then is, as to whether it can be chosen such that $f''(x_0) = \xi_2$ is positive and finite in the $g_{\mathrm{s}}^2 \to 0$ limit. The answer to this question involves some scheme dependence encoded in the coefficients

$$\mathcal{A}_\Phi(x_0) = -\frac{1}{16\pi^2} \lim_{x\to x_0} \left[\partial_z l_0^{(\Phi)}(z)\right]_{z=z_\Phi}, \qquad \Phi \in \{\mathrm{F}, \mathrm{W}\}. \tag{4.111}$$

For all FRG schemes, the coefficients $\mathcal{A}_\Phi(x_0)$ are similar to the ones defined in Eq. (4.92) and have the following integral representation

$$\mathcal{A}_\Phi(x_0) = \frac{1}{2k^2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\tilde{\partial}_t P_\Phi(p^2)}{\left[P_\Phi(p^2) + k^2 z_\Phi\right]^2}, \tag{4.112}$$

whose sign is still positive for any IR regularization scheme. For any threshold function, the expressions for the rescaled quartic scalar coupling and the condition in order to satisfy $\xi_2 > 0$ are

$$\xi_2 = \frac{9\hat{g}_*^2}{4x_0}\mathcal{A}_{\mathrm{W}}(x_0) - \frac{6\hat{h}_*^2}{x_0}\mathcal{A}_{\mathrm{F}}(x_0), \qquad 3\hat{g}_*^2 \mathcal{A}_{\mathrm{W}}(x_0) > 8\hat{h}_*^2 \mathcal{A}_{\mathrm{F}}(x_0). \tag{4.113}$$

For small values of x_0, the latter condition is the same as the one in Eq. (4.105).

Within the FRG framework, we have already observed that this condition is not fulfilled in the SM case, if the same regulator is chosen for both fields. We expect that this conclusion holds for any value of x_0, and for generic FRG schemes. In fact, let us suppose to take the limiting case of large x_0, then $\mathcal{A}_\Phi(x_0)$ would reduce to an x_0-independent value, and the condition $\xi_2 > 0$ would be again equivalent to that one in Eq. (4.105). As an example, for the piecewise linear regulator (cf. Eq. (B.22b)), the integration in Eq. (4.112) gives

$$\mathcal{A}_\Phi^{(\text{p.lin.})}(x_0) = \frac{1}{32\pi^2} \lim_{x\to x_0} \frac{1}{(1+z_\Phi)^2}, \qquad \Phi \in \{\text{F}, \text{W}\}, \tag{4.114}$$

such that the condition in Eq. (4.113) is not fulfilled for all $x_0 \geq 0$.

Let us now consider the two limiting cases. For the $\mathbb{Z}_2$-Yukawa-QCD model where the gauge-boson loop is absent, the condition $\xi_2 > 0$ implies $\mathcal{A}_\text{F}(x_0) < 0$, which cannot be fulfilled for any admissible IR regularization scheme. Conversely, for the non-Abelian Higgs model, the fermion loop is absent, resulting in the condition $\mathcal{A}_\text{W}(x_0) > 0$, which is satisfied by any admissible FRG regulators.

In the $\overline{\text{MS}}$ scheme these conclusions get twisted. In fact, from Eq. (4.44) we obtain

$$\mathcal{A}_\Phi^{(\overline{\text{MS}})}(x_0) = -\frac{1}{16\pi^2} \lim_{x\to x_0} z_\Phi < 0, \qquad \Phi \in \{\text{F}, \text{W}\}, \tag{4.115}$$

such that, for the $\mathbb{Z}_2$-Yukawa-QCD model the condition $\mathcal{A}_\text{F}^{(\overline{\text{MS}})}(x_0) < 0$ is fulfilled, [31] while for the non-Abelian Higgs model the condition $\mathcal{A}_\text{W}^{(\overline{\text{MS}})}(x_0) > 0$ is violated. Moreover, for the general $\text{SU}(2)_\text{L} \times \text{SU}(3)_\text{c}$ model, the condition given in Eq. (4.113) becomes $16\hat{h}_*^4 > 3\hat{g}_*^4$ which does indeed hold for the SM case.

Summary

We conclude this chapter by summarizing the results obtained within the weak-coupling approximation, as presented in the current section. For all models under investigation, we reveal the existence of a *two-parameter family* of novel AF trajectories within the SM matter-field content.

In a general FRG scheme based on an IR regulator, these solutions are effectively labeled by the power P, defining the approach of the quartic coupling λ_2 to the GFP, and by the rescaled quartic coupling ξ_2 itself. In the general $\text{SU}(2)_\text{L} \times \text{SU}(3)_\text{c}$ model, under the assumption that the same regulator is chosen for all fields, we have successfully constructed solutions with $P \in [1/4, 1/2]$. In case we leave the freedom of choosing different regulators for different fields, then also scaling solutions for $P \in (1/2, 1]$ become visible. For $P < 1/4$ we do not exclude *a priori* the possible existence of solutions for which h^2 exhibits an asymptotic scaling different from the one supported within the DER. For the $\mathbb{Z}_2$-Yukawa-QCD model, only scaling solutions in the window $P \in [1/4, 1/2]$ are admissible. For the non-Abelian Higgs model, we have recovered the $P \in [0, 1]$ scaling solutions already described in [111, 112].

For completeness, the $P > 1$ case is discussed in App. G.2, where we recover the known

[31] Further evidence for scaling solutions at $P = 1$ for the $\mathbb{Z}_2$-Yukawa-QCD model is given in Sec. 4.3.2.

	$\mathrm{SU}(2)_{\mathrm{L}}\times \mathrm{SU}(3)_{\mathrm{c}}$	$\mathbb{Z}_2$-Yukawa-QCD	non-Abelian Higgs
$\overline{\mathrm{MS}}$	$P = 1,\ Q, \hat{\kappa}$	$P = 1,\ Q, \hat{\kappa}$	$P = 1/2,\ Q, \breve{\kappa}$
FRG	$P \in [1/4, 1/2],\ \xi_2$	$P \in [1/4, 1/2],\ \xi_2$	$P \in (0, +\infty),\ \xi_2$

Tab. 4.1.: Summary of the family of new AF solutions constructed for the models under investigation in this thesis, and for the RG schemes we have analyzed. For each solution, we provide the value(s) of P for which they occur, and the (remaining) variables that parameterize the space of solutions. In the FRG schemes, we refer to the setup where the same regulator is used for all fields. Moreover we specify the SM matter content in the flow of the gauge and top-Yukawa couplings.

solutions for the non-Abelian Higgs model and we find no AF trajectories for the gauged-Yukawa models. Moreover, different IR regulators result in a change of range of the attainable values for the finite ratio $\xi_2 = \lambda_2 g_{\mathrm{s}}^{-4P}$.

The $\overline{\mathrm{MS}}$ scheme appears to correspond to a peculiar limit in the remapping of the allowed parameter ranges. In fact, in this case, the novel AF trajectories are labeled by the finite ratio $\hat{\kappa}$ and by the power Q, which describes the asymptotic scaling of the running dimensionless *vev* κ. The power P instead is fixed to $P = 1$ for the $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{c}}$ model as well as the $\mathbb{Z}_2$-Yukawa-QCD model, while for the non-Abelian Higgs model the rescaling power is fixed to $P = 1/2$.

These results are schematically summarized in Fig. 4.1. Here, for each model under investigation and for the different RG schemes, we list the two free parameters which play the role of "coordinates" in the space of AF scaling solutions: Q and $\hat{\kappa}$ for the $\overline{\mathrm{MS}}$ scheme, P and ξ_2 for any FRG schemes.

5. Conclusions

Ultraviolet (UV) complete quantum field theories (QFTs) represent, from a theoretical and mathematical point of view, optimal candidates to describe fundamental models of particle physics, due to the fact that they are valid and consistent up to arbitrary energy scales. Such UV-complete theories are described by the presence of a UV fixed point (FP) which can be either the free Gaußian fixed point (GFP) or a nontrivial interacting one. In the former case we say that the theory is asymptotically free (AF), whereas in the latter case the theory is asymptotically safe (AS). Identifying such QFTs hence provides information which can be crucial for our attempt at constructing a fundamental model which can overcome the difficulties of the Standard Model (SM) of particle physics.

Part of the SM including the Higgs-top sector exhibits, below a potential instability scale where the quartic scalar self-interaction λ_2 becomes negative, a behavior reminiscent to an AF trajectory: its couplings decrease and seem to approach the GFP. Stimulated by this observation we have taken a fresh look at *asymptotic freedom* for *gauged-Yukawa models* from a perspective that supersedes conventional studies within standard perturbation theory. In fact, our findings can be viewed as a generalization of previous investigations for non-Abelian Higgs models [111, 112] to systems which include a fermionic matter sector.

We then first focused our attention on a $\mathbb{Z}_2$-Yukawa-QCD model which features AF trajectories already in standard perturbation theory [77]. Using effective-field-theory (EFT) methods as well as various approximations based on the functional renormalization group (FRG) methods, we have discovered additional AF trajectories. Subsequently we have generalized the study to an $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{c}$ chiral model where the weak gauge group of the SM has been added to the previous theory. Also for this larger class of models we have discovered new routes to asymptotic freedom, beyond the Cheng–Eichten–Li solution.

These novel solutions have been found as *quasi-fixed-points* (QFTs) for the scalar potential and the Yukawa sector: all scalar couplings as well as the top-Yukawa coupling approach the GFP according to certain asymptotic scalings with respect to the gauge couplings. In the literature this special behavior has also been called "fixed flow". A common feature of the novel AF solutions is the presence of a scalar potential which is always in the spontaneously-symmetry-broken (SSB) regime, even in the UV limit of infinite energies. This peculiarity invalidates the often implicitly-made assumption of the deep Euclidean regime (DER), where running masses or a vacuum expectation value (*vev*) κ are neglected in the renormalization group (RG) analysis. Therefore, all our new solutions demonstrate that the inclusion of a nontrivial minimum κ is mandatory to reveal their existence. In this respect our approach deviates from standard perturbation theory.

As a matter of fact, the detection of these solutions requires an adequate renormalization scheme which consistently accounts for *mass-threshold effects*. This can be naturally accomplished by the FRG approach, or by the minimal subtraction scheme ($\overline{\mathrm{MS}}$) provided that higher-dimensional scalar operators $\lambda_{n\geq 3}$ as well as a nonvanishing *vev* are accounted for. As a matter of fact, the inclusion of mass-threshold effects into the RG flow of coupling constants induces a scheme dependence already in the one-loop beta functions. The universality of the one-loop coefficients is therefore lost as the latter is limited to the beta functions in the DER.

In order to probe the *scheme independence* of our results, we have investigated the flow equations of our gauged-Yukawa models using both the most widely used $\overline{\mathrm{MS}}$ scheme and the FRG framework. The latter one is best suited for addressing a general class of infrared (IR) regulator functions. On a line of constant physics, a change of the regularization scheme induces a map of the coupling space of initial conditions (given in terms of bare couplings at an initial UV scale Λ) onto itself. Our results show that such a *mapping* also involves the boundary conditions, i.e., the parameters which classify the AF trajectories.

In fact, these novel AF solutions are identified by a *two-parameters family* of solutions. Within the FRG framework, these parameters are such as the scaling power P, governing the UV behavior of λ_2 while approaching the GFP ($\lambda_2 \sim g_{\mathrm{s}}^{4P}$), and the finite rescaled value of λ_2 itself, e.g., $\hat{\lambda}_2 = \xi_2$. These parameters can in turn be related to the boundary conditions specified for the QFP potential. In fact, the requirement of a globally stable scalar potential leads to constraints on these parameters. For example P is confined to the values $P \in [1/4, 1/2]$ for the $\mathrm{SU(2)_L} \times \mathrm{SU(3)_c}$ model and the $\mathbb{Z}_2$-Yukawa-QCD model. This constraint is new in comparison with the non-Abelian Higgs model [111, 112], and may be indicative for the fact that further structures in the matter sector may lead to further constraints. Conversely, standard perturbation theory corresponds to an implicit choice of these boundary conditions, as, for example, the *Cheng–Eichten–Li* (CEL) solution [77] of the $\mathbb{Z}_2$-Yukawa-QCD model, corresponds to the particular choice ($P = 1/2, \kappa = 0$). Within the $\overline{\mathrm{MS}}$ scheme, instead, the two parameters are mapped into a different set, namely the scaling power Q, governing the asymptotic behavior of the *vev* ($\kappa \sim g_{\mathrm{s}}^{-2Q}$), and the position of its rescaled value $\hat{\kappa}$.

Due to the fact that our analyses involve *functional approaches* capable of including all possible scalar self-interactions, we were able to address the global stability properties of the QFP potentials. For example, we have provided direct evidence for the first time that the perturbative CEL potential is globally stable. While the CEL solution is regular for vanishing field amplitudes, all novel QFP solutions found in this work feature a logarithmic singularity for small scalar field amplitudes. However, the presence of a nonvanishing minimum κ at any scale guarantees that the couplings as well as the correlation functions remain well-defined to any order, as they usually involve field derivatives evaluated at this minimum. Together with the global stability, our scaling solutions satisfy also other important criteria which classify physical solutions in statistical physics [255, 257, 258], namely the finiteness of the scalar potential and its first derivative, and the polynomial boundedness of the scalar fluctuations.

The functional formalism is also well suited for the investigation of the effect of higher-dimensional operators on the perturbatively renormalizable couplings. From a naive power-counting perspective, higher-dimensional operators would be suspected to exhibit a high-energy enhancement inducing a loss of control on the UV properties of the theory. Yet, all the couplings $\lambda_{n\geq 3}$ also feature asymptotic freedom and their approach to the GFP is controlled by progressively higher powers of the gauge couplings, thus resulting in a stronger suppression at high energies. In this way, the naive power-counting argument that associates such operators to nonrenormalizable theories in perturbation theory is evaded.

Not only functional approaches have been used to study the present gauged-Yukawa models. The new findings have been substantiated also by means of EFT-like analyses including higher-dimensional operators and numerical techniques using pseudo-spectral methods (see App. F), which have allowed us to solve the full nonperturbative FP equation for the rescaled scalar potential. Among the functional approaches, particular emphasis has been given to a weak-gauge-coupling analysis which was extremely useful to corroborate the scheme (in)dependence of our results.

Let us finally summarize the key aspects which were needed in order to detect these novel total AF solutions.

1. A non-Abelian Yang-Mills sector is necessary in order to possess an AF gauge coupling which can serve as a rescaling parameter for the Yukawa coupling and all the self-interacting scalar couplings.

2. Generalized boundary conditions for the QFP potential, parameterizing its asymptotic UV behavior, need to be taken into account. These conditions are, for instance, the rescaling parameter P for the marginal quartic coupling or the scaling of higher-dimensional operators.

3. Mass-threshold effects need to be accounted for as they can invalidate the conventional naive analysis in the DER.

Standard perturbation theory usually does not fully account for (2) and (3) since it implicitly assumes the DER and special choices for the boundary conditions. We emphasize that point (1), e.g., the presence of an AF gauge sector, can be sufficient to seed asymptotic freedom in the complete model, provided that the generalizations (2) and (3) are included. This fact has been indeed verified in non-Abelian Higgs model [111, 112], and extensively used in the present context of gauged-Yukawa models.

Outlook

The machinery introduced in [111, 112], and further extended in this thesis to gauge-Yukawa models, can be used as a guiding principle to construct more realistic models. In fact, one pressing problem to address in the future is the inclusion of the abelian hypercharge $\mathrm{U}(1)_{\mathrm{Y}}$ gauge group of the SM. The hope is to provide a high-energy complete QFT description of the SM, where the triviality problem is finally solved. As a byproduct, also the metastability of the Higgs potential is expected to be overcome. In fact, the total AF trajectories discovered in this thesis are such that the quartic scalar self-interaction coupling approaches the GFP without never crossing an instability scale. We expect that the same mechanism would occur also for the full SM, *if* such total AF trajectories exists also for the full SM.

The scenario presented in this thesis is even more attractive in the light of our observation that, the QFP trajectories for these gauge-Yukawa models enhance the predictivity of the model due to their UV repulsive nature. This fact would indeed restrict the IR values for observable quantities such as the physical Higgs mass to top-quark mass ratio.

Therefore, another important investigation would be the study of the physical mass spectrum of these gauged-Yukawa models along the novel AF trajectories. To answer this question, a numerical integration of the full functional RG flow equations would be required.

In the spirit of the FRG program, another path we could follow in order to further generalize these models, is to include a Yukawa potential in the form of $h(\rho)$. This would generalize the single Yukawa coupling h which corresponds to the value of the Yukawa potential at the scalar minimum, i.e., $h(\rho = \kappa)$. In the FRG set-up, indeed, focusing only on the effective scalar potential $u(\rho)$ does not exhaust all possible operators in the theory space which may be important for identifying AF trajectories. The FRG methods are readily available to also deal with this additional layer of complexity [168, 169, 196, 239, 240, 259–261]. As further boundary conditions should be specified, it is an interesting open question as to whether the set of AF trajectories becomes more diverse or even more constrained.

Appendix A.

More about Perturbative Renormalizability

In this first appendix we complete the review of perturbatively renormalizable AF solutions allowed at one loop for the $\mathbb{Z}_2$-Yukawa-QCD model and the more general $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{c}$ model. We therefore restrict ourself only to the perturbative renormalizable operators defined in the Euclidean actions of Eqs. (3.1) and (4.1). Let us start first from the minimalistic gauged-Yukawa system where fermions are charged only under the $\mathrm{SU}(N_\mathrm{c})$ gauge group.

A.1. $\mathbb{Z}_2$-Yukawa-QCD Models

The following analysis is partly similar to that of [86], but we generalize it with the notion of QFPs. The flow in the (g_s^2, h^2) plane, provided by Eqs. (3.5) and (3.7), is best understood by direct analytic integration of the RG equations, and adopting g_s^2 as an (inverse) RG time. The solution of the flow reads:

$$h^2(g_\mathrm{s}^2) = \frac{g_\mathrm{s}^2}{c\, g_\mathrm{s}^{2(1-\gamma)} + 1/\chi_\mathrm{s}^2}, \qquad \gamma = \frac{18}{22N_\mathrm{c} - d_\gamma^\mathrm{c} N_\mathrm{f}^\mathrm{c}} \frac{N_\mathrm{c}^2 - 1}{N_\mathrm{c}}. \tag{A.1}$$

The QFP value χ_s^2 for the ratio h^2/g_s^2 is defined in Eq. (3.9) and c is an integration constant. Notice that γ is positive as long as g_s^2 is AF, according to Eq. (3.5). Also, the condition $\chi_\mathrm{s}^2 > 0$, which further restricts the viable field content as in Eq. (3.10), is equivalent to $\gamma > 1$. In fact, in the SM case where $N_\mathrm{c} = 3$, $N_\mathrm{f}^\mathrm{c} = 6$ and $d_\gamma^\mathrm{c} = 4$, γ assumes the value $\gamma = 8/7$. If one initializes the flow at some initial arbitrary RG scale Λ, with a gauge coupling $g_{\mathrm{s}0}^2$ and a Yukawa coupling h_0^2, then c is given by

$$c\, g_{\mathrm{s}0}^{2(1-\gamma)} = \frac{g_{\mathrm{s}0}^2}{h_0^2} - 1/\chi_\mathrm{s}^2. \tag{A.2}$$

There is only one trajectory along which h^2 exhibits an asymptotic scaling proportional to g_s^2, and this trajectory corresponds to the special case of $c = 0$ or equivalently to $h_0^2 = g_{\mathrm{s}0}^2\, \chi_\mathrm{s}^2$. If the initial condition is chosen in this way, the strong gauge coupling drives the top-Yukawa coupling to zero in the UV limit. If instead the initial condition is different, then $c \neq 0$ in Eq. (A.1) and the fate of the system depends on the sign of c. For $c < 0$, which corresponds to $\hat{h}_0^2 > \chi_{\rm s}^2$ according to Eq. (A.2), either $h^2 < 0$ for all $g_{\rm s}^2 < 1$, or $h_0^2 > 0$ and the Yukawa coupling hits a Landau pole in the UV, i.e., it diverges at a finite RG time. For $c > 0$, namely $\hat{h}_0^2 < \chi_{\rm s}^2$, there is no Landau pole and the trajectories are also AF, but with an asymptotic scaling that differs from the one defined by Eq. (3.8). In fact, in this case

$$h^2(g_{\rm s}^2) \underset{g_{\rm s}\to 0}{\sim} \frac{1}{c}\, g_{\rm s}^{2\gamma}, \tag{A.3}$$

for any $c > 0$, and $\gamma > 1$ due to the assumption that Eq. (3.10) holds. Also this scaling solution should be amenable to an interpretation as a QFP for the flow of a suitable ratio. Indeed, we could define the following rescaled coupling and its beta function

$$\hat{h}'^2 = \frac{h^2}{g_{\rm s}^{2\gamma}}, \qquad\qquad \partial_t \hat{h}'^2 = \frac{9 g_{\rm s}^{2\gamma}}{16\pi^2} \hat{h}'^4. \tag{A.4}$$

Notice that the term in Eq. (3.7) proportional to $g_{\rm s}^2$ has been canceled by the contribution $-\gamma\eta_{\rm G}\hat{h}'^2$ coming from the rescaling, due to the specific value of γ given in Eq. (A.1). While Eq. (A.4) does not vanish for any finite value of the strong gauge coupling $g_{\rm s}^2 \neq 0$, the fact that the would-be-leading contribution proportional to $g_{\rm s}^2$ vanishes for any $\hat{h}'^2$ signals the presence of a QFP with arbitrary value of $\hat{h}'^2$. This is only approximately realized at finite $g_{\rm s}^2 \neq 0$ and becomes exact in the $g_{\rm s}^2 \to 0$ limit.

Let us now address the stability properties of the AF trajectories plotted in Fig. 3.1 (left panel). From the previous discussion it is clear that an infinitesimal perturbation of a trajectory characterized by $c > 0$, along a direction which changes the value of the Yukawa coupling, i.e., c itself, results in a new trajectory which is still a scaling solution. Thus, one moves from a given $\hat{h}'^2$ to another $\hat{h}'^2 + \delta\hat{h}'^2$, and the distance between the two trajectories stays constant in RG time in the UV limit if measured in terms of the rescaled coupling $\hat{h}'^2$. Hence, we can call this a marginal perturbation. These QFP solutions are neither stable nor unstable. Yet, as it is clear from the left panel of Fig. 3.1, quantification of the distance between trajectories in terms of the unrescaled h^2 would lead to a different conclusion, since such a distance would decrease as $g_{\rm s}^2 \to 0$. The unique trajectory with $c = 0$ has a rather different behavior, as already discussed in Sec. 3.1.

The AF solutions of Eq. (A.3) in the Yukawa sector, translate into corresponding AF trajectories in the Higgs sector. As we did for the CEL solution, we inspect the running of the finite ratio $\hat{\lambda}_2$ defined in Eq. (3.13), and P still to be determined. We restrict h^2 such that the ratio in Eq. (A.4) attains an arbitrary finite value in the UV. In this case, the flow equation for $\hat{\lambda}_2$ becomes

$$\partial_t \hat{\lambda}_2 = \frac{9}{16\pi^2} \hat{\lambda}_2^2 g_{\mathrm{s}}^{4P} + \frac{N_{\mathrm{c}} \hat{\lambda}_2}{4\pi^2 c} g_{\mathrm{s}}^{2\gamma} + P \frac{11N_{\mathrm{c}} - 2N_{\mathrm{f}}^{\mathrm{c}}}{12\pi^2} \hat{\lambda}_2 g_{\mathrm{s}}^2 - \frac{N_{\mathrm{c}}}{4\pi^2 c^2} g_{\mathrm{s}}^{4(\gamma - P)}. \tag{A.5}$$

In order to have a QFP solution with a positive $\hat{\lambda}_2$, it is necessary that the last two contributions are leading in the small-g_{s}^2 limit. Therefore, we have to require

$$P = \gamma - \frac{1}{2}, \tag{A.6}$$

such that the positive QFP solution is

$$\hat{\lambda}_2 = \frac{3N_{\mathrm{c}}}{11N_{\mathrm{c}} - 2N_{\mathrm{f}}^{\mathrm{c}}} \frac{2}{(2\gamma - 1)c^2}. \tag{A.7}$$

The latter solution corresponds to a UV repulsive trajectory, meaning that for a chosen initialization value of $\hat{h}'^2$ (i.e., a value for c), there is only one AF trajectory for λ_2 approaching the GFP from positive values, and it corresponds to Eq. (A.7). Larger values of $\hat{\lambda}_2$ would result in a Landau pole, while smaller values would lead to negative λ_2 at high energy.

A.2. $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{c}}$ Models

Now we move to the more general model discussed in Chap. 4 and we summarize the analysis of the one-loop beta functions in the DER. Similar analyses have widely been discussed in the literature, see for example [85, 86]. In Sec. 4.1, we have presented the RG flow equation for the top-Yukawa coupling in the presence of the two gauge couplings g^2 and g_{s}^2, cf. Eq. (4.4), having the form

$$\partial_t h^2 = h^2 \left(a_h h^2 - a_g g^2 - a_{\mathrm{s}} g_{\mathrm{s}}^2 \right), \tag{A.8}$$

where the a's coefficients are all positive for the SM set of parameters, cf. Eq. (4.3):

$$a_h = \frac{9}{16\pi^2}, \qquad a_g = \frac{9}{32\pi^2}, \qquad a_{\mathrm{s}} = \frac{1}{\pi^2}. \tag{A.9}$$

Equation (A.8) can be integrated in the RG time t together with the beta functions for the gauge couplings, cf. Eqs. (3.5) and (4.2). The integration of the latter is straightforward, yielding

$$g(t)^2 = \frac{g_0^2}{1 + g_0^2\, \hat{\eta}_{\mathrm{W}} t}, \qquad g_{\mathrm{s}}(t)^2 = \frac{g_{\mathrm{s}0}^2}{1 + g_{\mathrm{s}0}^2\, \hat{\eta}_{\mathrm{G}} t}, \tag{A.10}$$

where g_0^2 and $g_{\mathrm{s}0}^2$ are the initial conditions at $t = 0$, or equivalently at some energy scale Λ. The ratio $\hat{\eta}_{\mathrm{G}}$ is defined in Eq. (3.14) and similarly for $\hat{\eta}_{\mathrm{W}} = \eta_{\mathrm{W}} g^{-2}$, cf. Eq. (4.2). The general solution of Eq. (A.8) reads

$$\frac{1}{h(t)^2} = \left(\frac{1}{h_0^2} - a_h I(t)\right) \left[\frac{g_0^2}{g^2(t)}\right]^{a_g/\hat{\eta}_{\rm W}} \left[\frac{g_{\rm s0}^2}{g_{\rm s}^2(t)}\right]^{a_{\rm s}/\hat{\eta}_{\rm G}}, \tag{A.11}$$

where the function $I(t)$ is defined as [86]

$$I(t) = \int_0^t d\tau \left[\frac{g^2(\tau)}{g_0^2}\right]^{a_g/\hat{\eta}_{\rm W}} \left[\frac{g_{\rm s}^2(\tau)}{g_{\rm s0}^2}\right]^{a_{\rm s}/\hat{\eta}_{\rm G}}. \tag{A.12}$$

The conditions for having total asymptotic freedom are

$$\chi \equiv \frac{a_g}{\hat{\eta}_{\rm W}} + \frac{a_{\rm s}}{\hat{\eta}_{\rm G}} - 1 > 0, \qquad h_0^2 \leq \frac{1}{a_h I(\infty)}. \tag{A.13}$$

The first requirement is necessary in order to provide for the existence of AF trajectories in the space of parameters $(h^2, g^2, g_{\rm s}^2)$. Incidentally, this criterion is fulfilled for the SM set of parameters. In fact, if this condition is not satisfied the integral $I(\infty)$ would diverge; consequently, there will be a finite RG time $t_{\rm critic}$ at which the right-hand side of Eq. (A.11) is zero and the top-Yukawa coupling hits a Landau pole, i.e., $I(t_{\rm critic}) = (h_0^2 a_h)^{-1}$. On the other hand, whenever the integral $I(\infty)$ converges, asymptotic freedom is guaranteed as long as the right-hand side of Eq. (A.11) remains positive. This is precisely the second inequality in Eq. (A.13).

By denoting $\Omega(g_{\rm s0}^2, g_0^2) = [a_h I(\infty)]^{-1}$, the equation $h_0^2 = \Omega(g_{\rm s0}^2, g_0^2)$ represents an upper critical surface for total asymptotic freedom in the $(h^2, g^2, g_{\rm s}^2)$ space. In fact, if $h_0^2 > \Omega(g_{\rm s0}^2, g_0^2)$, the negative one-loop contributions to $\partial_t h^2$ involving the gauge boson fluctuations are suppressed in the UV limit, and the dominant positive scalar fluctuations drive the top-Yukawa coupling towards a Landau pole. By contrast, all couplings approach the GFP towards the UV for all initial conditions such that $h_0^2 \leq \Omega(g_{\rm s0}^2, g_0^2)$.

Let us concentrate in more detail on the RG flow on the surface $\Omega:$ by virtue of its critical nature, Ω represents a UV repulsive surface along its orthogonal directions. Only for those initial conditions such that $h_0^2 = \Omega(g_{\rm s0}^2, g_0^2)$, the integrated RG trajectories will remain on the critical surface itself for all $t > 0$ and will approach the Gaußian FP in the UV limit $t \to \infty$. Next, we observe that the one-loop beta functions for the gauge couplings are independent of the Yukawa-coupling in the DER. As a consequence, the gauge-coupling-flow on each slice of constant h^2 looks the same as within the $(g_{\rm s}^2, g^2)$ plane, see Fig. A.1 (left panel).

Therefore, there will be a special AF trajectory also on the surface Ω along which the two gauge couplings are proportional to each other, representing a UV attractive trajectory. This special RG trajectory can be characterized more explicitly by introducing "rotated" gauge couplings

$$G_{\rm s}^2 = \cos\alpha\, g_{\rm s}^2 + \sin\alpha\, g^2, \qquad G^2 = -\sin\alpha\, g_{\rm s}^2 + \cos\alpha\, g^2. \tag{A.14}$$

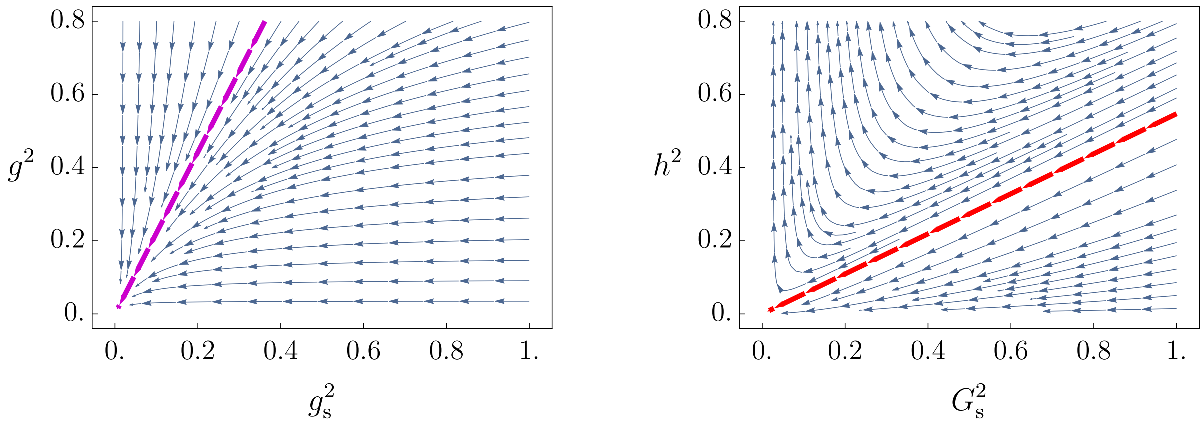


Fig. A.1.: *Left Panel*: phase diagram and flow of the gauge couplings in the $h^2 = 0$ plane for the $\mathrm{SU(2)_L \times SU(3)_c}$ model. Both couplings exhibit an AF flow to the Gaußian FP in the UV limit, being attracted by the trajectory where $g^2 = \hat{g}_*^2 g_s^2$ (magenta line). *Right Panel*: phase diagram and coupling flow using the rotated gauge coupling G_s (parametrizing the magenta-line trajectory on the left panel). AF trajectories require a sufficiently small initial Yukawa coupling $h_0^2 \leq \Omega$ with the upper bound on the critical surface Ω denoted by the red line. The latter represents the special trajectory satisfying Eq. (A.16).

Choosing $\tan\alpha = \hat{g}_*^2$ which corresponds to the proportionality factor of the gauge couplings, it suffices to study the flow of h^2 for $G^2 = 0$. For this particular choice, the beta functions for h^2 and G_s^2 are

$$\partial_t h^2 = h^2 \left[a_h h^2 - \cos\alpha\, (a_g \hat{g}_*^2 + a_s) G_s^2 \right], \qquad \partial_t G_s^2 = -\hat{\eta}_G \cos\alpha\, G_s^4, \tag{A.15}$$

whose RG phase diagram is plotted in Fig. A.1 (right panel). Here the highlighted red line represents the QFP trajectory along which the top-Yukawa coupling is proportional to G_s^2, and the corresponding QFP value satisfies

$$\left(\frac{h^2}{G_s^2}\right)_* = \left(\frac{h^2}{g_s^2}\right)_* \cos\alpha = \hat{h}_*^2 \cos\alpha. \tag{A.16}$$

We conclude that the special trajectory in Fig. A.1, along which all three perturbatively renormalizable couplings are proportional to each other, corresponds exactly to the QFP solution characterized in Eq. (4.14).

Appendix B.

Threshold Functions

For the application of the exact functional-RG equation (2.34), we choose a regulator R_k which is diagonal in field space. We keep the freedom to have different regulators, specified by corresponding sub- or super-scripts, for the Higgs scalar (H) and for the Goldstone bosons (θ), as this is possible in the SSB regime. Notice, however, that we do not distinguish between different runnings of the wave function renormalization for the radial excitation Z_{H} and the Goldstone modes Z_θ in the SSB regime, at the present level of our truncation. Since we are interested in projecting on the kinetic term of the radial scalar fluctuation, we will use Z_{H} as the collective wave function renormalization for all scalar degrees of freedom. Similarly, due to the choice of covariant gauges, we can have different regulators for the transverse gluons (or transverse W bosons) and for the longitudinal gluons (or longitudinal W bosons). However, in the Landau gauge used in this work, only transverse gauge bosons propagate. Finally, we account for independent regularizations of the left-handed (L) and right-handed (R) Weyl spinors. One of the left-handed Weyl spinors together with its right-handed partner (corresponding to the top-quark) becomes massive in the SSB regime, thus the contributions of the corresponding Dirac field are denoted with F. In the following of this section we will use the sub- or super-script (S) to refer to the scalar Higgs and Goldstone modes, (B) to indicate the bosonic degrees of freedom W and G, and (f) to refer to Dirac fermion as well as left- and right-handed Weyl spinor. The mass-like regulator R_k, which parametrizes the details of the momentum-shell integration, can be expressed in the term of dimensionless regulator shape functions $r_{(\dots)}(p^2/k^2)$. For the scalar and bosonic fields it reads

$$R_{\mathrm{S}} = Z_{\mathrm{H}} p^2 r_{\mathrm{S}}(p^2/k^2), \qquad R_{\mathrm{B}} = Z_{\mathrm{B}} p^2 r_{\mathrm{B}}(p^2/k^2), \qquad \begin{matrix} \mathrm{S} = \{\mathrm{H}, \theta\} \\ \mathrm{B} = \{\mathrm{W}, \mathrm{G}\} \end{matrix}, \tag{B.1}$$

while for fermions we have

$$R_{\mathrm{f}} = -Z_{\mathrm{f}} \not{p}\, r_{\mathrm{f}}(p^2/k^2), \qquad \mathrm{f} = \{\mathrm{F}, \mathrm{L}, \mathrm{R}\}. \tag{B.2}$$

Let us define also the different regularized kinetic terms in momentum space for scalars, bosons and fermion fields:

$$P_{\mathrm{S/B}}(p) = p^2(1 + r_{\mathrm{S/B}}) \tag{B.3}$$

$$P_{\mathrm{F}}(p) = p^2(1 + r_{\mathrm{L}})(1 + r_{\mathrm{R}}), \qquad P_{\mathrm{L/R}}(p) = p^2(1 + r_{\mathrm{L/R}})^2. \tag{B.4}$$

The loop momentum integrals appearing on the r.h.s. of the Wetterich equation (2.34) are classified by defining the corresponding threshold functions. All the threshold functions used in this work can be found also in [169, 194, 251]. However, since we have adopted in this thesis a different notation, for the sake of clarity we provide here the definition of the required loop integrals. We use the abbreviation $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2)$.

$$l_0^{(\mathrm{S})d}(\omega_{\mathrm{S}}, \eta_{\mathrm{H}}) = \frac{k^{-d}}{4v_d}\int_p \frac{Z_{\mathrm{H}}^{-1}\,\partial_t R_{\mathrm{S}}}{P_{\mathrm{S}} + k^2\omega_{\mathrm{S}}} \qquad \mathrm{S} = \{\mathrm{H}, \theta\}, \tag{B.5}$$

$$l_0^{(\mathrm{gh/G/W})d}(\omega_{\mathrm{B}}, \eta_{\mathrm{B}}) = \frac{k^{-d}}{4v_d}\int_p \frac{Z_{\mathrm{B}}^{-1}\,\partial_t R_{\mathrm{B}}}{P_{\mathrm{B}} + k^2\omega_{\mathrm{B}}} \qquad \mathrm{B} = \{u, \mathrm{G}, \mathrm{W}\}, \tag{B.6}$$

$$l_0^{(\mathrm{F})d}(\omega_{\mathrm{F}}, \eta_{\psi}) = \frac{1}{2v_d k^d}\int_p p^2(1 + r_{\mathrm{F}})\frac{Z_{\mathrm{F}}^{-1}\partial_t(Z_{\mathrm{F}} r_{\mathrm{F}})}{P_{\mathrm{F}} + k^2\omega_{\mathrm{F}}}, \tag{B.7}$$

$$l_{1,1}^{(\mathrm{F,S})d}(\omega_{\mathrm{F}}, \omega_{\mathrm{S}}; \eta_{\psi}, \eta_{\mathrm{H}}) = -\frac{k^{4-d}}{4v_d}\int_p \tilde{\partial}_t\left[\frac{1}{P_{\mathrm{F}} + k^2\omega_{\mathrm{F}}}\,\frac{1}{P_{\mathrm{S}} + k^2\omega_{\mathrm{S}}}\right] \qquad \mathrm{S} = \{\mathrm{H}, \theta\}, \tag{B.8}$$

$$l_{1,1}^{(\theta,\mathrm{W})d}(\omega_{\theta}, \omega_{\mathrm{W}}; \eta_{\mathrm{H}}, \eta_{\mathrm{W}}) = -\frac{k^{4-d}}{4v_d}\int_p \tilde{\partial}_t\left[\frac{1}{P_{\mathrm{H}} + k^2\omega_{\theta}}\,\frac{1}{P_{\mathrm{W}} + k^2\omega_{\mathrm{W}}}\right], \tag{B.9}$$

$$l_{1,1}^{(\mathrm{F,G})d}(\omega_{\mathrm{F}}, \omega_{\mathrm{G}}; \eta_{\psi}, \eta_{\mathrm{G}}) = -\frac{k^{4-d}}{4v_d}\int_p \tilde{\partial}_t\left[\frac{1}{P_{\mathrm{F}} + k^2\omega_{\mathrm{F}}}\,\frac{1}{P_{\mathrm{G}} + k^2\omega_{\mathrm{G}}}\right], \tag{B.10}$$

$$m_2^{(\mathrm{S})d}(\omega_{\mathrm{S}}, \eta_{\mathrm{H}}) = -\frac{k^{6-d}}{4v_d}\int_p p^2\tilde{\partial}_t\left[\frac{\partial_{(p^2)}P_{\mathrm{S}}}{(P_{\mathrm{S}} + k^2\omega_{\mathrm{S}})^2}\right]^2 \qquad \mathrm{S} = \{\mathrm{H}, \theta\}, \tag{B.11}$$

$$m_2^{(\mathrm{W})d}(\omega_{\mathrm{W}}, \eta_{\mathrm{W}}) = -\frac{k^{6-d}}{4v_d}\int_p p^2\tilde{\partial}_t\left[\frac{\partial_{(p^2)}P_{\mathrm{W}}}{(P_{\mathrm{W}} + k^2\omega_{\mathrm{W}})^2}\right]^2, \tag{B.12}$$

$$\widetilde{m}_2^{(\mathrm{W})d}(\omega_{\mathrm{W}}; \eta_{\mathrm{W}}) = -\frac{k^{6-d}}{16v_d}\int_p \frac{1}{p^2}\,\tilde{\partial}_t\frac{1}{(P_{\mathrm{W}} + k^2\omega_{\mathrm{W}})^2}, \tag{B.13}$$

$$m_2^{(\mathrm{F})d}(\omega_{\mathrm{F}}, \eta_{\psi}) = -\frac{k^{6-d}}{4v_d}\int_p p^2\tilde{\partial}_t\left[\frac{\partial_{(p^2)}P_{\mathrm{F}}}{(P_{\mathrm{F}} + \omega_{\mathrm{F}}k^2)^2}\right]^2, \tag{B.14}$$

$$m_4^{(\mathrm{F})d}(\omega_{\mathrm{F}}, \eta_{\psi}) = -\frac{k^{4-d}}{4v_d}\int_p p^4\tilde{\partial}_t\left[\partial_{(p^2)}\frac{1 + r_{\mathrm{F}}}{P_{\mathrm{F}} + \omega_{\mathrm{F}}k^2}\right]^2, \tag{B.15}$$

$$m_{1,2}^{(\mathrm{L,S})d}(\omega_{\mathrm{F}}, \omega_{\mathrm{S}}; \eta_{\psi}, \eta_{\mathrm{H}}) = -\frac{k^{4-d}}{4v_d}\int_p p^2\tilde{\partial}_t\left[\frac{1 + r_{\mathrm{R}}}{P_{\mathrm{F}} + \omega_{\mathrm{F}}k^2}\,\frac{\partial_{(p^2)}P_{\mathrm{S}}}{(P_{\mathrm{S}} + k^2\omega_{\mathrm{S}})^2}\right] \qquad \mathrm{S} = \{\mathrm{H}, \theta\}, \tag{B.16}$$

$$m_{1,2}^{(\mathrm{R,S})d}(\omega_{\mathrm{F}}, \omega_{\mathrm{S}}; \eta_{\psi}, \eta_{\mathrm{H}}) = -\frac{k^{4-d}}{4v_d}\int_p p^2\tilde{\partial}_t\left[\frac{1 + r_{\mathrm{L}}}{P_{\mathrm{F}} + \omega_{\mathrm{F}}k^2}\,\frac{\partial_{(p^2)}P_{\mathrm{S}}}{(P_{\mathrm{S}} + k^2\omega_{\mathrm{S}})^2}\right] \qquad \mathrm{S} = \{\mathrm{H}, \theta\} \tag{B.17}$$

$$m_{1,2}^{(\mathrm{F,H})d}(\omega_{\mathrm{F}}, \omega_{\mathrm{H}}; \eta_{\psi}, \eta_{\mathrm{H}}) = -\frac{k^{4-d}}{4v_d}\int_p p^2\tilde{\partial}_t\left[\frac{1 + r_{\mathrm{F}}}{P_{\mathrm{F}} + \omega_{\mathrm{F}}k^2}\,\frac{\partial_{(p^2)}P_{\mathrm{H}}}{(P_{\mathrm{H}} + k^2\omega_{\mathrm{H}})^2}\right], \tag{B.18}$$

$$m_{1,2}^{(\mathrm{f,B})d}(\omega_{\mathrm{F}},\omega_{\mathrm{B}};\eta_\psi,\eta_{\mathrm{B}}) = -\frac{k^{4-d}}{4v_d}\int_p p^2\tilde{\partial}_t\left[\frac{1+r_{\mathrm{f}}}{P_{\mathrm{F}}+\omega_{\mathrm{F}}k^2}\,\frac{\partial_{(p^2)}P_{\mathrm{B}}}{(P_{\mathrm{B}}+k^2\omega_{\mathrm{B}})^2}\right] \quad \begin{matrix}\mathrm{B}=\{\mathrm{W},\mathrm{G}\}\\ \mathrm{f}=\{\mathrm{F},\mathrm{L},\mathrm{R}\}\end{matrix}, \tag{B.19}$$

$$\widetilde{m}_{1,1}^{(\mathrm{f,B})d}(\omega_{\mathrm{F}},\omega_{\mathrm{B}};\eta_\psi,\eta_{\mathrm{B}}) = -\frac{k^{4-d}}{4v_d}\int_p \tilde{\partial}_t\left[\frac{1+r_{\mathrm{f}}}{P_{\mathrm{F}}+\omega_{\mathrm{F}}k^2}\,\frac{1}{P_{\mathrm{B}}+k^2\omega_{\mathrm{B}}}\right] \quad \begin{matrix}\mathrm{B}=\{\mathrm{W},\mathrm{G}\}\\ \mathrm{f}=\{\mathrm{F},\mathrm{L},\mathrm{R}\}\end{matrix}. \tag{B.20}$$

The operator $\tilde{\partial}_t$ denotes differentiation with respect to the RG-time $t=\log k$ acting, however, only on the regulators. It can be defined through the following expression

$$\tilde{\partial}_t = \sum_\Phi Z_\Phi^{-1}\partial_t(Z_\Phi r_\Phi)\cdot\frac{\delta}{\delta r_\Phi}, \tag{B.21}$$

where an integral over the momentum is understood and Φ is a collective index for all the degrees of freedom in the considered truncated effective average action. To be more explicitly, for the general $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(3)_{\mathrm{c}}$ model $\Phi\in\{\mathrm{H},\theta,\mathrm{L},\mathrm{R},\mathrm{W},\mathrm{G}\}$, for the $\mathbb{Z}_2$-Yukawa-QCD model $\Phi\in\{\mathrm{H},\mathrm{F},\mathrm{G}\}$ and for the non-Abelian Higgs model $\Phi\in\{\mathrm{H},\theta,\mathrm{W}\}$. Let us remind the reader that within our projection rules we have $Z_{\mathrm{H}}=Z_\theta$.

All throughout Chap. 3, we have used the piecewise linear regulator

$$r_{\mathrm{S/B}}(p^2/k^2) = (k^2/p^2-1)\,\theta(1-p^2/k^2), \tag{B.22a}$$

$$\left[1+r_{\mathrm{f}}(p^2/k^2)\right]^2 = 1+r_{\mathrm{S/B}}(p^2/k^2), \tag{B.22b}$$

which is optimized for the derivative expansion [192, 193]. Within this choice, the analytic integration of the threshold functions yield the following results

$$l_0^{(\mathrm{S})d}(\omega_{\mathrm{S}},\eta_{\mathrm{H}}) = \frac{2}{d}\frac{1-\frac{\eta_{\mathrm{H}}}{d+2}}{1+\omega_{\mathrm{S}}},$$

$$l_0^{(\mathrm{gh/G/W})d}(\omega_{\mathrm{B}},\eta_{\mathrm{B}}) = \frac{2}{d}\frac{1-\frac{\eta_{\mathrm{B}}}{d+2}}{1+\omega_{\mathrm{B}}},$$

$$l_0^{(\mathrm{F})d}(\omega_{\mathrm{F}},\eta_\psi) = \frac{2}{d}\frac{1-\frac{\eta_\psi}{d+1}}{1+\omega_{\mathrm{F}}},$$

$$l_{1,1}^{(\mathrm{F,S})d}(\omega_{\mathrm{F}},\omega_{\mathrm{S}};\eta_\psi,\eta_{\mathrm{H}}) = \frac{2}{d}\frac{1}{(1+\omega_{\mathrm{F}})(1+\omega_{\mathrm{S}})}\left[\frac{1}{1+\omega_{\mathrm{F}}}\left(1-\frac{\eta_\psi}{d+1}\right)+\frac{1}{1+\omega_{\mathrm{S}}}\left(1-\frac{\eta_{\mathrm{H}}}{d+2}\right)\right],$$

$$l_{1,1}^{(\mathrm{F,G})d}(\omega_{\mathrm{F}},\omega_{\mathrm{G}};\eta_\psi,\eta_{\mathrm{G}}) = \frac{2}{d}\frac{1}{(1+\omega_{\mathrm{F}})(1+\omega_{\mathrm{G}})}\left[\frac{1}{1+\omega_{\mathrm{F}}}\left(1-\frac{\eta_\psi}{d+1}\right)+\frac{1}{1+\omega_{\mathrm{G}}}\left(1-\frac{\eta_{\mathrm{G}}}{d+2}\right)\right],$$

$$l_{1,1}^{(\theta,\mathrm{W})}(\omega_\theta,\omega_{\mathrm{W}};\eta_{\mathrm{H}},\eta_{\mathrm{W}}) = \frac{2}{d}\frac{1}{(1+\omega_\theta)(1+\omega_{\mathrm{W}})}\left[\frac{1}{1+\omega_\theta}\left(1-\frac{\eta_{\mathrm{H}}}{d+2}\right)+\frac{1}{1+\omega_{\mathrm{W}}}\left(1-\frac{\eta_{\mathrm{W}}}{d+2}\right)\right],$$

$$m_2^{(\mathrm{S})d}(\omega_{\mathrm{S}},\eta_{\mathrm{H}}) = \frac{1}{(1+\omega_{\mathrm{S}})^4},$$

$$m_2^{(\mathrm{W})d}(\omega_{\mathrm{W}},\eta_{\mathrm{W}}) = \frac{1}{(1+\omega_{\mathrm{W}})^4},$$

$$\widetilde{m}_2^{(\mathrm{W})d}(\omega_{\mathrm{W}};\eta_{\mathrm{W}}) = \frac{1}{d-2}\frac{1-\frac{\eta_{\mathrm{W}}}{d}}{(1+\omega_{\mathrm{W}})^3},$$

$$
\begin{aligned}
m_2^{(\mathrm{F})d}(\omega_\mathrm{F},\eta_\psi) &= \frac{1}{(1+\omega_\mathrm{F})^4},\\
m_4^{(\mathrm{F})d}(\omega_\mathrm{F},\eta_\psi) &= \frac{1}{(1+\omega_\mathrm{F})^4} + \frac{1-\eta_\psi}{d-2}\frac{1}{(1+\omega_\mathrm{F})^3} - \left(\frac{1-\eta_\psi}{2(d-2)} + \frac{1}{4}\right)\frac{1}{(1+\omega_\mathrm{F})^2},\\
m_{1,2}^{(\mathrm{L,S})d}(\omega_\mathrm{F},\omega_\mathrm{S};\eta_\psi,\eta_\mathrm{H}) &= \frac{1-\frac{\eta_\mathrm{H}}{d+1}}{(1+\omega_\mathrm{F})(1+\omega_\mathrm{S})^2} = m_{1,2}^{(\mathrm{R,S})d}(\omega_\mathrm{F},\omega_\mathrm{S};\eta_\psi,\eta_\mathrm{H}),\\
m_{1,2}^{(\mathrm{F,H})d}(\omega_\mathrm{F},\omega_\mathrm{H};\eta_\psi,\eta_\mathrm{H}) &= \frac{1-\frac{\eta_\mathrm{H}}{d+1}}{(1+\omega_\mathrm{F})(1+\omega_\mathrm{H})^2},\\
m_{1,2}^{(\mathrm{f,B})d}(\omega_\mathrm{F},\omega_\mathrm{B};\eta_\psi,\eta_\mathrm{B}) &= \frac{1-\frac{\eta_\mathrm{B}}{d+1}}{(1+\omega_\mathrm{F})(1+\omega_\mathrm{B})^2},\\
\widetilde{m}_{1,1}^{(\mathrm{f,B})d}(\omega_\mathrm{F},\omega_\mathrm{B};\eta_\psi,\eta_\mathrm{B}) &= \frac{2}{d-1}\frac{1}{(1+\omega_\mathrm{F})(1+\omega_\mathrm{B})}\left[\frac{1-\frac{\eta_\mathrm{B}}{d+1}}{1+\omega_\mathrm{B}} + \frac{1-\frac{\eta_\psi}{d}}{1+\omega_\mathrm{F}} - \frac{1}{2} + \frac{\eta_\psi}{2d}\right].
\end{aligned}
$$

Appendix C.

Derivation of the FRG Flow Equations for the $\mathbb{Z}_2$-Yukawa-QCD Model

In this appendix we want to clarify how to project the Wetterich equation (2.34) onto the different operators which compose the truncated EAA of the $\mathbb{Z}_2$-Yukawa-QCD model. For the more general $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{c}$ model, the same procedure, modulo further complications due to the extra degrees of freedom, can be nevertheless straightforwardly derived. The first step is to evaluate the Hessian, or fluctuation matrix, for the chosen truncation in Eq. (3.17). Since it proves to be convenient to derive the flow equations in momentum space, we formulate our truncation in Fourier space. Applying the following conventions for the Fourier transformations

$$\varphi(x) = \int_p \mathrm{e}^{\mathrm{i}px}\, \varphi(p) \qquad \text{for} \qquad \varphi = \{H, \psi^A, G^I_\mu, u^I\}, \tag{C.1}$$

and

$$\bar\varphi(x) = \int_p \mathrm{e}^{-\mathrm{i}px}\, \bar\varphi(p) \qquad \text{for} \qquad \bar\varphi = \{\bar\psi^A, \bar u^I\}, \tag{C.2}$$

where we have defined $\int_p \equiv \int \frac{d^dp}{(2\pi)^d}$, we get

$$\begin{aligned}\Gamma_k = &\int_p\int_q \left\{ \frac{Z_\mathrm{H}}{2} H(q) p^2 H(p)\delta_{q,-p} - Z_\mathrm{F}\bar\psi^A(p)\left[\not{q}\,\delta^{AB}\delta_{p,q} + \bar g_\mathrm{s} \not{G}^I(p-q)T_I^{AB}\right]\psi^B(q)\right\}\\ &+ \int d^dx U\left(\textstyle\int_p \mathrm{e}^{ipx}\, H(p)\right) + \frac{\mathrm{i}\bar h}{\sqrt2}\int_p\int_q \bar\psi^A(p) H(p-q)\psi^A(q)\\ &+ \frac{Z_\mathrm{G}}{2}\int_p\int_q G^I_\mu(q)\left[p^2\delta_{\mu\nu} - p_\mu p_\nu\left(1-\frac{1}{\zeta_\mathrm{s}}\right)\right] G^I_\nu(p)\delta_{q,-p} + \Gamma_{k,G^3} + \Gamma_{k,G^4}\\ &+ Z_u \int_p\int_q \bar u^I(p)\left[-p^2\delta^{IJ}\delta_{q,p} + \bar g_\mathrm{s} f^{IJK} G^K_\mu(p-q)\right] u^J(q),\end{aligned} \tag{C.3}$$

where the Dirac-delta function in momentum space is defined as $\delta_{p,q} = (2\pi)^d\delta^d(p-q)$. The two terms Γ_{k,G^3} and Γ_{k,G^4} include respectively cubic and quartic interactions among the gluon gauge fields, which, by the way, are not displayed here because they are not relevant for our calculations. The fluctuation matrix is defined as

$$\Gamma_k^{(2)}(p,q) = \begin{pmatrix} \frac{\delta}{\delta H(-p)} \\ \frac{\delta}{\delta \psi^A(-p)} \\ \frac{\delta}{\delta \bar{\psi}^A(p)} \\ \frac{\delta}{\delta G^I_\mu(-p)} \\ \frac{\delta}{\delta u^I(-p)} \\ \frac{\delta}{\delta \bar{u}^I(p)} \end{pmatrix} \Gamma_k \begin{pmatrix} \frac{\delta}{\delta H(q)} & \frac{\delta}{\delta \psi^B(q)} & \frac{\delta}{\delta \bar{\psi}^B(-q)} & \frac{\delta}{\delta G^J_\nu(q)} & \frac{\delta}{\delta u^J(q)} & \frac{\delta}{\delta \bar{u}^J(-q)} \end{pmatrix}, \tag{C.4}$$

where the column of functional derivatives are acting from the left whereas the line of functional derivatives are acting from the right. The components of the Hessian can be calculated from Eq. (C.3) and they read

$$\Gamma_k^{(2)}(p,q) \equiv \begin{pmatrix} \Gamma_{\mathrm{HH}} & \Gamma_{\mathrm{H}\psi} & \Gamma_{\mathrm{H}\bar{\psi}} & 0 & 0 & 0 \\ \Gamma_{\psi\mathrm{H}} & 0 & \Gamma_{\psi\bar{\psi}} & \Gamma_{\psi\mathrm{G}} & 0 & 0 \\ \Gamma_{\bar{\psi}\mathrm{H}} & \Gamma_{\bar{\psi}\psi} & 0 & \Gamma_{\bar{\psi}\mathrm{G}} & 0 & 0 \\ 0 & \Gamma_{\mathrm{G}\psi} & \Gamma_{\mathrm{G}\bar{\psi}} & \Gamma_{\mathrm{GG}} & \Gamma_{\mathrm{G}u} & \Gamma_{\mathrm{G}\bar{u}} \\ 0 & 0 & 0 & \Gamma_{u\mathrm{G}} & 0 & \Gamma_{u\bar{u}} \\ 0 & 0 & 0 & \Gamma_{\bar{u}\mathrm{G}} & \Gamma_{\bar{u}u} & 0 \end{pmatrix}, \tag{C.5}$$

whit their explicit form

$$\begin{aligned}
\Gamma_{\mathrm{HH}} &= Z_{\mathrm{H}} p^2 \delta_{p,q} + \int d^d x U''\left(H(x)\right) \mathrm{e}^{\mathrm{i}(q-p)x}, \\
\Gamma_{\mathrm{H}\psi} &= \frac{\mathrm{i}\bar{h}}{\sqrt{2}} \bar{\psi}(q-p) = -\Gamma^{\mathrm{T}}_{\psi\mathrm{H}}, \\
\Gamma_{\bar{\psi}\mathrm{H}} &= \frac{\mathrm{i}\bar{h}}{\sqrt{2}} \psi(p-q) = -\Gamma^{\mathrm{T}}_{\mathrm{H}\bar{\psi}}, \\
\Gamma_{\bar{\psi}\psi} &= \frac{\mathrm{i}\bar{h}}{\sqrt{2}} H(p-q) - Z_{\mathrm{F}} \left[\delta_{p,q} \not{p} + \bar{g}_{\mathrm{s}} \not{G}^I(p-q) T_I\right], \\
\Gamma_{\psi\bar{\psi}} &= -\frac{\mathrm{i}\bar{h}}{\sqrt{2}} H(p-q) + Z_{\mathrm{F}} \gamma^{\mathrm{T}}_\mu \left[-\delta_{p,q} p_\mu + \bar{g}_{\mathrm{s}} G^I_\mu(p-q) T^{\mathrm{T}}_I\right], \\
\Gamma_{\bar{\psi}\mathrm{G}} &= -\bar{g}_{\mathrm{s}} Z_{\mathrm{F}} \gamma_\nu T_J \psi(p-q) = -\Gamma^{\mathrm{T}}_{G\bar{\psi}}, \\
\Gamma_{\mathrm{G}\psi} &= -\bar{g}_{\mathrm{s}} Z_{\mathrm{F}} \bar{\psi}(q-p) \gamma_\mu T_I = -\Gamma^{\mathrm{T}}_{\psi G}, \\
\Gamma_{\mathrm{GG}} &= Z_{\mathrm{G}} p^2 \left[\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\left(1 - \frac{1}{\zeta_{\mathrm{s}}}\right)\right] \delta^{IJ} \delta_{p,q} + \mathcal{O}\left(G\right), \\
\Gamma_{\mathrm{G}u} &= -\mathrm{i}\bar{g}_{\mathrm{s}} f^{IJK} \bar{u}^K (q-p)(q-p)_\mu, \\
\Gamma_{u\mathrm{G}} &= -\mathrm{i}\bar{g}_{\mathrm{s}} f^{IJK} \bar{u}^K (q-p)(q-p)_\nu, \\
\Gamma_{\mathrm{G}\bar{u}} &= \mathrm{i}\bar{g}_{\mathrm{s}} f^{IJK} u^K (p-q) q_\mu,
\end{aligned}$$

$$\begin{aligned}
\Gamma_{\bar{u}G} &= -\mathrm{i}\bar{g}_s f^{IJK} u^K(p-q)p_\nu ,\\
\Gamma_{\bar{u}u} &= Z_u\left[-\delta_{p,q}\delta^{IJ}p^2 + \mathrm{i}\bar{g}_s f^{IJK} G^K_\mu(p-q)p_\mu\right],\\
\Gamma_{u\bar{u}} &= Z_u\left[\delta_{p,q}\delta^{IJ}p^2 - \mathrm{i}\bar{g}_s f^{IJK} G^K_\mu(p-q)q_\mu\right],
\end{aligned}$$

where the transpose is respect with both the Dirac structure and the fundamental representation of $\mathrm{SU}(N_c)$. The regulator matrix is given by

$$R_k(p,q) = \delta_{p,q}\begin{pmatrix}
R_H(p) & 0 & 0 & 0 & 0 & 0\\
0 & 0 & R_F^T(p) & 0 & 0 & 0\\
0 & R_F(p) & 0 & 0 & 0 & 0\\
0 & 0 & 0 & R_{G,\mu\nu}(p) & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -R_u(p)\\
0 & 0 & 0 & 0 & R_u(p) & 0
\end{pmatrix}, \tag{C.6}$$

where

$$\begin{aligned}
R_H(p) &= Z_H p^2 r_H\left(\frac{p^2}{k^2}\right),\\
R_F(p) &= -Z_F \not{p}\, r_F\left(\frac{p^2}{k^2}\right)\delta^{AB},\\
R_u(p) &= -Z_u p^2 r_u\left(\frac{p^2}{k^2}\right)\delta^{IJ}\\
R_{G,\mu\nu}(p) &= Z_G p^2\left[\Pi_{T,\mu\nu} r_{GT}\left(\frac{p^2}{k^2}\right) + \frac{1}{\zeta_s}\Pi_{L,\mu\nu} r_{GL}\left(\frac{p^2}{k^2}\right)\right]\delta^{IJ},
\end{aligned}$$

and the Π's are the longitudinal and transverse projectors with respect to the momentum p_μ, namely

$$\Pi_{L,\mu\nu} = (p_\mu p_\nu)/(p^2), \qquad \Pi_{T,\mu\nu} = \delta_{\mu\nu} - (p_\mu p_\nu)/(p^2). \tag{C.7}$$

In principle we could choose different regulator shape functions r_{GL} and r_{GT} for the longitudinal and transverse modes, however, for the porpoises of this thesis, we have only considered the case where $r_{GL} = r_{GT}$.

C.1. Scalar Potential

We can extract the flow equation for the scalar potential by projecting the r.h.s of Eq. (2.34) onto the constant field configuration, $H(x) = \text{const}$, and setting all the other fields to zero

$$\partial_t\big|_H U_k(H) = \frac{1}{V}\frac{1}{2}\,\mathrm{STr}\left\{\frac{\partial_t R_k}{\Gamma_k^{(2)} + R_k}\right\}_{H=\text{const},\{\psi,\bar{\psi},G,\bar{u},u\}=0} \tag{C.8}$$

where V is the spacetime volume $\int d^dx$. With the simple substitution of constant scalar field, the regularized Hessian $\Gamma_k^{(2)} + R_k$ becomes block diagonal and straightforwardly invertible. The multiplication of its inverse with the regulator matrix R_k yields a diagonal matrix in all its indices except for the spinor indices, in fact

$$\left[\frac{\partial_t R_k}{\Gamma_k^{(2)} + R_k}\right](p,q) = \delta_{p,q}\mathrm{diag}\Bigg\{\mathfrak{a}(p,H)\partial_t R_{\mathrm{H}}, (\partial_t R_{\mathrm{F}})^{\mathrm{T}}\big[\mathfrak{b}(p,H) - \not{p}^{\mathrm{T}}\mathfrak{c}(p,H)\big], \\ -(\partial_t R_{\mathrm{F}})\big[\mathfrak{b}(p,H) + \not{p}\mathfrak{c}(p,H)\big], (\partial_t R_{\mathrm{G},\mu\nu})\frac{\mathbb{P}_{\nu\lambda}}{Z_{\mathrm{G}}}, -\frac{\partial_t R_u}{Z_u P_u}, -\frac{\partial_t R_u}{Z_u P_u}\Bigg\}, \tag{C.9}$$

where for convenience of space we have defined the momentum and field-dependent functions

$$\mathfrak{a}(p,H) = \frac{1}{Z_{\mathrm{H}}p^2(1+r_{\mathrm{H}}) + U''(H)}, \qquad \mathfrak{b}(p,H) = \frac{\frac{\mathrm{i}\bar{h}}{\sqrt{2}}H}{Z_{\mathrm{F}}^2 p^2(1+r_{\mathrm{F}})^2 + \frac{\bar{h}^2}{2}H^2}, \\ \mathfrak{c}(p,H) = \frac{Z_{\mathrm{F}}(1+r_{\mathrm{F}})}{Z_{\mathrm{F}}^2 p^2(1+r_{\mathrm{F}})^2 + \frac{\bar{h}^2}{2}H^2}. \tag{C.10}$$

The regularized propagator for the gluon fields reads

$$\mathbb{P}_{\mu\nu}(p) = \left[\frac{\Pi^{\mathrm{T}}_{\mu\nu}}{p^2(1+r_{\mathrm{GT}})} + \zeta_{\mathrm{s}}\frac{\Pi^{\mathrm{L}}_{\mu\nu}}{p^2(1+r_{\mathrm{GL}})}\right]\delta^{IJ} = \frac{\Pi^{\mathrm{T}}_{\mu\nu} + \zeta_{\mathrm{s}}\Pi^{\mathrm{L}}_{\mu\nu}}{p^2(1+r_{\mathrm{G}})}\delta^{IJ}, \tag{C.11}$$

where the last equality holds for equal regulator shape functions $r_{\mathrm{GL}} = r_{\mathrm{GT}}$.
We can now perform the STr over the internal indices, taking into account that it causes a minus sign in the fermionic and ghost sectors (due to their Graßman nature) and, using the standard identities to resolve the Dirac trace such as $\mathrm{tr}(\gamma_\mu) = 0$ and $\mathrm{tr}(\gamma_\mu\gamma_\nu) = \delta_{\mu\nu}d_\gamma^{\mathrm{c}}$, we obtain afterwards

$$\partial_t\big|_H U_k(H) = \frac{1}{2}\int_p \mathfrak{a}(p,H)p^2\partial_t(Z_{\mathrm{H}}r_{\mathrm{H}}) - d_\gamma^{\mathrm{c}}N_{\mathrm{c}}\int_p \mathfrak{c}(p,H)p^2\partial_t(Z_{\mathrm{F}}r_{\mathrm{F}}) \\ + (N_{\mathrm{c}}^2 - 1)\left[-\int_p \frac{Z_u^{-1}p^2\partial_t(Z_u r_u)}{P_u} + \frac{d+\zeta_{\mathrm{s}}-1}{2}\int_p \frac{Z_{\mathrm{G}}^{-1}p^2\partial_t(Z_{\mathrm{G}}r_{\mathrm{G}})}{P_{\mathrm{G}}}\right], \tag{C.12}$$

where the factor N_{c} is due to the trace over the indices of the fundamental representation of $\mathrm{SU}(N_{\mathrm{c}})$ and $N_{\mathrm{c}}^2 - 1$ is the dimension of the adjoint representation. The first and second momentum integrals in the latter equation correspond to the scalar and fermion loop threshold effects respectively, while the loop integrals in the second line correspond to ghost and gluon loops. Introducing the rescaled potential $u_k(\rho)$, defined via Eqs. (3.18) and (3.20), we have

$$\frac{\mathrm{d}u_k(\rho)}{\mathrm{d}t} = \partial_t\big|_\rho u_k(\rho) + u_k'(\rho)\partial_t\rho = \partial_t\big|_\rho u_k - (d-2+\eta_{\mathrm{H}})\rho u_k' \\ = -du_k + k^{-d}\partial_t\big|_H U_k(H), \tag{C.13}$$

where the anomalous dimension for the scalar field is given by Eq. (3.21). Collecting together the results in Eqs. (C.12) and (C.13) we finally obtain

$$\begin{aligned}\partial_t u_k(\rho) = -du_k + (d-2+\eta_{\mathrm{H}})\rho u_k'(\rho) + \frac{1}{2k^d}\int_p \frac{Z_{\mathrm{H}}^{-1}p^2\partial_t(Z_{\mathrm{H}}r_{\mathrm{H}})}{P_{\mathrm{H}}+k^2(u_k'+2\rho u_k'')} \\ - \frac{d_\gamma^{\mathrm{c}}N_{\mathrm{c}}}{k^d}\int_p \frac{Z_{\mathrm{F}}^{-1}p^2(1+r_{\mathrm{F}})\partial_t(Z_{\mathrm{F}}r_{\mathrm{F}})}{P_{\mathrm{F}}+k^2h^2\rho} - \frac{N_{\mathrm{c}}^2-1}{k^d}\int_p \frac{Z_u^{-1}p^2\partial_t(Z_u r_u)}{P_u} \\ + \frac{(N_{\mathrm{c}}^2-1)(d+\zeta_{\mathrm{s}}-1)}{2k^d}\int_p \frac{Z_{\mathrm{G}}^{-1}p^2\partial_t(Z_{\mathrm{G}}r_{\mathrm{G}})}{P_{\mathrm{G}}},\end{aligned} \tag{C.14}$$

where we have used the definition of the dimensionless top-Yukawa coupling as given in Eq. (3.19). By comparing the latter equation with the definition of the threshold functions $l_0^{(\mathrm{H})d}$ and $l_0^{(\mathrm{F})d}$ in App. B, we recover the same RG flow equation as given by Eq. (3.22) in the main text, apart from the last two terms. In fact, the last two loop integrals in Eq. (C.14) – proportional to the threshold functions $l_0^{(\mathrm{gh})d}(0,\eta_u)$ and $l_0^{(\mathrm{G})d}(0,\eta_{\mathrm{G}})$ respectively – are field independent and thus, they do not contribute to the running of the scalar potential apart from an overall vacuum constant shift.

C.2. Anomalous Dimension for the Scalar Field

In order to evaluate the anomalous dimension η_{H}, it should be clear that it is not possible to set the Higgs scalar field to a constant configuration, like in the previous section, inside the flow of the EAA. Indeed this would kill the kinetic term, which is precisely that operator carrying the information about the wave function renormalization. A straightforward way for taking into account the running of the scalar kinetic term, is to split the scalar field into a fixed constant configuration $\bar{v}$ plus fluctuations around it, such that the anomalous dimension becomes a function of this field configuration $\eta_{\mathrm{H}}(\bar{v})$. This fact represents a source of scheme dependency since the functional RG flow does not contain information about the original truncation. We can however solve this inconvenience by fixing $\bar{v}$ in a consistent way. A natural way of doing this is to choose $\bar{v}$ as precisely the ground state of the Higgs scalar field, since the physical particles are interpreted, in fact, as excitations around the ground state itself. We can thus write

$$H(x) = \bar{v} + \Delta H(x) \qquad \text{or} \qquad H(p) = \bar{v}\delta_{p,0} + \Delta H(p). \tag{C.15}$$

The substitution in the Hessian $\Gamma_k^{(2)}$ is straightforward, in particular, the second variation with respect the fluctuations ΔH becomes

$$\begin{aligned}\Gamma_{\mathrm{HH}} = & Z_{\mathrm{H}} p^2 \delta_{p,q} + U''(\bar{v})\delta_{p,q} + U'''(\bar{v})\Delta H(p-q) \\ & + \frac{1}{2}U''''(\bar{v})\int_l \Delta H(l)\Delta H(p-q-l) + \cdots .\end{aligned} \tag{C.16}$$

We can therefore state that, the correct projection rule onto the kinetic term of the Higgs scalar fluctuation fields reads

$$\partial_t Z_{\mathrm{H}} = \frac{1}{V}\frac{\partial}{\partial(p^2)}\bigg|_{p^2=0} \frac{\delta^2}{\delta\Delta H(-p)\delta\Delta H(p)}\bigg|_{\{\varphi,\bar{\varphi}\}=0} \partial_t\Gamma_k. \tag{C.17}$$

This time we are not free to take the limit of vanishing fields, $\varphi = 0 = \bar{\varphi}$, before taking the functional derivatives. Therefore we have to deal with a fluctuation matrix which is not anymore diagonal in its momentum indices. One way to proceed is to expand the operator under the STr of the Wetterich equation in powers of the fluctuation fields. To do so, we first write

$$\frac{\partial_t R_k}{\Gamma_k^{(2)} + R_k} = \tilde{\partial}_t \ln\left(\Gamma_k^{(2)} + R_k\right), \tag{C.18}$$

where the RG-time derivative $\tilde{\partial}_t$ acts only on the regulator part R_k, as defined in Eq. (B.21). Afterwards we decompose the operator $\Gamma_k^{(2)} + R_k$ into a kinetic part $\mathcal{K}$, independent from the fluctuation fields and invertible, and the remaining field-dependent part $\mathcal{F}(\varphi,\bar{\varphi})$. Performing this splitting allows us to expand the logarithm in the last equation by means of the Mercator series, such that the Wetterich equation can be written in a different form like

$$\partial_t\Gamma_k = \frac{1}{2}\mathrm{STr}\tilde{\partial}_t\left\{\ln\mathcal{K} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left[\mathcal{PF}(\varphi,\bar{\varphi})\right]^n\right\}, \tag{C.19}$$

where the $\mathcal{P}$ is the regularized propagator in momentum space $\mathcal{P} = \mathcal{K}^{-1}$. The building block of the latter expansion is therefore the matrix $(\mathcal{PF})$, which can be straightforwardly, though tediously, derived by the above expressions for $\Gamma_k^{(2)}$ and R_k. It reads

$$(\mathcal{PF})(p,q) = \begin{pmatrix} \mathcal{PF}_{\mathrm{HH}} & \mathcal{PF}_{\mathrm{H}\psi} & \mathcal{PF}_{\mathrm{H}\bar{\psi}} & 0 & 0 & 0 \\ \mathcal{PF}_{\psi\mathrm{H}} & \mathcal{PF}_{\psi\psi} & 0 & \mathcal{PF}_{\psi\mathrm{G}} & 0 & 0 \\ \mathcal{PF}_{\bar{\psi}\mathrm{H}} & 0 & \mathcal{PF}_{\bar{\psi}\bar{\psi}} & \mathcal{PF}_{\bar{\psi}\mathrm{G}} & 0 & 0 \\ 0 & \mathcal{PF}_{\mathrm{G}\psi} & \mathcal{PF}_{\mathrm{G}\bar{\psi}} & \mathcal{PF}_{\mathrm{GG}} & \mathcal{PF}_{\mathrm{G}u} & \mathcal{PF}_{\mathrm{G}\bar{u}} \\ 0 & 0 & 0 & \mathcal{PF}_{u\mathrm{G}} & \mathcal{PF}_{uu} & 0 \\ 0 & 0 & 0 & \mathcal{PF}_{\bar{u}\mathrm{G}} & 0 & \mathcal{PF}_{\bar{u}\bar{u}} \end{pmatrix}, \tag{C.20}$$

where

$$
\begin{aligned}
\mathcal{PF}_{\mathrm{HH}} &= \mathfrak{a}(p,\bar{v})\left[U'''(\bar{v})\Delta H(p-q) + \frac{1}{2}U''''(\bar{v})\int_l \Delta H(l)\Delta H(p-q-l) + \cdots\right],\\
\mathcal{PF}_{\mathrm{H}\psi} &= \frac{\mathrm{i}\bar{h}}{\sqrt{2}}\mathfrak{a}(p,\bar{v})\bar{\psi}(q-p),\\
\mathcal{PF}_{\mathrm{H}\bar{\psi}} &= -\frac{\mathrm{i}\bar{h}}{\sqrt{2}}\mathfrak{a}(p,\bar{v})\psi^{\mathrm{T}}(p-q),\\
\mathcal{PF}_{\psi\mathrm{H}} &= -\frac{\mathrm{i}\bar{h}}{\sqrt{2}}[\mathfrak{b}(p,\bar{v}) + \not{p}\mathfrak{c}(p,\bar{v})]\psi(p-q),\\
\mathcal{PF}_{\bar{\psi}\mathrm{H}} &= -\frac{\mathrm{i}\bar{h}}{\sqrt{2}}[\mathfrak{b}(p,\bar{v}) - \not{p}^{\mathrm{T}}\mathfrak{c}(p,\bar{v})]\bar{\psi}^{\mathrm{T}}(q-p),\\
\mathcal{PF}_{\psi\psi} &= [\mathfrak{b}(p,\bar{v}) + \not{p}\mathfrak{c}(p,\bar{v})]\left[-\frac{\mathrm{i}\bar{h}}{\sqrt{2}}\Delta H(p-q) + \bar{g}_{\mathrm{s}}Z_{\mathrm{F}}\not{G}(p-q)\right],\\
\mathcal{PF}_{\bar{\psi}\bar{\psi}} &= [\mathfrak{b}(p,\bar{v}) - \not{p}^{\mathrm{T}}\mathfrak{c}(p,\bar{v})]\left[-\frac{\mathrm{i}\bar{h}}{\sqrt{2}}\Delta H(p-q) + \bar{g}_{\mathrm{s}}Z_{\mathrm{F}}\not{G}^{\mathrm{T}}(p-q)\right],\\
\mathcal{PF}_{\psi\mathrm{G}} &= \bar{g}_{\mathrm{s}}Z_{\mathrm{F}}[\mathfrak{b}(p,\bar{v}) + \not{p}\mathfrak{c}(p,\bar{v})]\gamma_\nu T_J\psi(p-q),\\
\mathcal{PF}_{\bar{\psi}\mathrm{G}} &= \bar{g}_{\mathrm{s}}Z_{\mathrm{F}}[\mathfrak{b}(p,\bar{v}) - \not{p}^{\mathrm{T}}\mathfrak{c}(p,\bar{v})]\gamma_\nu^{\mathrm{T}}T_J^{\mathrm{T}}\bar{\psi}^{\mathrm{T}}(q-p),\\
\mathcal{PF}_{\mathrm{G}\psi} &= -\bar{g}_{\mathrm{s}}\frac{Z_{\mathrm{F}}}{Z_{\mathrm{G}}}\bar{\psi}(q-p)\mathbb{P}_{\mu\lambda}(p)\gamma_\lambda T_I,\\
\mathcal{PF}_{\mathrm{G}\bar{\psi}} &= \bar{g}_{\mathrm{s}}\frac{Z_{\mathrm{F}}}{Z_{\mathrm{G}}}\psi^{\mathrm{T}}(p-q)\mathbb{P}_{\mu\lambda}(p)\gamma_\lambda^{\mathrm{T}}T_I^{\mathrm{T}}.
\end{aligned}
$$

For completeness let us write down also the rest of the matrix elements for $\mathcal{PF}$, even though they do not contribute to the anomalous dimensions η_{H} and η_ψ or to the Yukawa coupling.

$$
\begin{aligned}
&\mathcal{PF}_{\mathrm{GG}} = Z_{\mathrm{G}}^{-1}\mathbb{P}_{\mu\nu}(p)\mathcal{O}(G), \qquad \mathcal{PF}_{\mathrm{G}u} = -\mathrm{i}\bar{g}_{\mathrm{s}}Z_{\mathrm{G}}^{-1}\mathbb{P}_{\mu\nu}(p)f^{IJK}\bar{u}^K(q-p)(q-p)_\mu,\\
&\mathcal{PF}_{\mathrm{G}\bar{u}} = \mathrm{i}\bar{g}_{\mathrm{s}}Z_{\mathrm{G}}^{-1}\mathbb{P}_{\mu\nu}(p)f^{IJK}\bar{u}^K(p-q)q_\mu, \qquad \mathcal{PF}_{u\mathrm{G}} = \mathrm{i}\bar{g}_{\mathrm{s}}\left[Z_uP_u(p)\right]^{-1}f^{IJK}u^K(p-q)p_\nu,\\
&\mathcal{PF}_{\bar{u}\mathrm{G}} = -\mathrm{i}\bar{g}_{\mathrm{s}}\left[Z_uP_u(p)\right]^{-1}f^{IJK}\bar{u}^K(q-p)(q-p)_\nu,\\
&\mathcal{PF}_{uu} = -\mathrm{i}\bar{g}_{\mathrm{s}}\left[P_u(p)\right]^{-1}f^{IJK}G_\mu^K(p-q)p_\mu, \qquad \mathcal{PF}_{\bar{u}\bar{u}} = -\mathrm{i}\bar{g}_{\mathrm{s}}\left[P_u(p)\right]^{-1}f^{IJK}G_\mu^K(p-q)q_\mu.
\end{aligned}
$$

For example, there are no contributions to η_{H}, η_ψ or $\partial_t h$ coming from the ghost fields because, actually, there are no vertices in the theory that couple the ghost fields with the scalar or fermionic fields, since they do only couple with the gluons. For this reason, ghost-contributions can appear only as threshold effects through the gluons anomalous dimension.

In order to proceed further, we have to identify those terms in the expansion of Eq. (C.19) which contain quadratic contributions in the scalar fluctuations ΔH. These are precisely $(\mathcal{PF})$ and $(\mathcal{PF})^2$. The linear term would contribute only with the quadratic part in the element $\mathcal{PF}_{\mathrm{HH}}$, which corresponds diagrammatically to a tadpole. However, since any tadpole is independent on the external momentum, it does not contribute to η_{H}, once we derive with respect to p^2. Therefore the projection rule onto the wave function renormalization simplifies to

$$\partial_t Z_{\rm H} = \frac{1}{V}\frac{\partial}{\partial(p^2)}\bigg|_{p^2=0} \frac{\delta^2}{\delta\Delta H(-p)\delta\Delta H(p)}\bigg|_{\{\varphi,\bar\varphi\}=0} \frac{1}{2}\,{\rm STr}\,\tilde\partial_t\left\{-\frac{1}{2}(\mathcal{P}\mathcal{F})^2\right\}, \tag{C.21}$$

and leads to the following expression

$$\partial_t Z_{\rm H} = \frac{\partial}{\partial(p^2)}\bigg|_{p^2=0} \tilde\partial_t\bigg\{-\frac{1}{2}[U'''(\bar v)]^2\int_\ell \mathfrak{a}(p+\ell,\bar v)\mathfrak{a}(\ell,\bar v) - \frac{1}{2}\bar h^2 N_{\rm c} d_\gamma^{\rm c}\int_\ell\Big[\mathfrak{b}(p+\ell,\bar v)\mathfrak{b}(\ell,\bar v) + \ell\cdot(\ell+p)\mathfrak{c}(p+\ell,\bar v)\mathfrak{c}(\ell,\bar v)\Big]\bigg\}. \tag{C.22}$$

It is useful to write the derivative with respect p^2 as $(2d)^{-1}\partial^2/\partial p_\mu\partial p_\mu$ and, integrating by part, the latter equation becomes

$$\partial_t Z_{\rm H} = \frac{[U'''(\bar v)]^2}{d}\tilde\partial_t\int_\ell \ell^2[\partial_{(\ell^2)}\mathfrak{a}(\ell,\bar v)]^2 - \frac{N_{\rm c} d_\gamma^{\rm c}\bar h^4}{d}\frac{\bar v^2}{2}\tilde\partial_t\int_\ell \ell^2[\partial_{(\ell^2)}\mathfrak{b}(\ell,\bar v)]^2 + \frac{N_{\rm c} d_\gamma^{\rm c}\bar h^2}{d}\tilde\partial_t\int_\ell \ell^4[\partial_{(\ell^2)}\mathfrak{c}(\ell,\bar v)]^2. \tag{C.23}$$

Translating this expression in terms of dimensionless and renormalized quantities as defined in Sec. 3.2, and taking into account the definitions of the threshold functions $m_2^{({\rm H})d}$, $m_2^{({\rm F})d}$ and $m_4^{({\rm F})d}$ – cf. App. B – we recover the expression for the scalar anomalous dimension $\eta_{\rm H}$ as given in main text in Eq. (3.27).

Let remind the reader that, from the definition of $\rho = \phi^2/2$, we have that

$$\partial_t\big|_H\phi = \partial_t\big|_H\left(\sqrt{Z_{\rm H}}k^{\frac{2-d}{2}}H\right) = -\frac{1}{2}(d-2+\eta_{\rm H})\phi, \tag{C.24}$$

such that the scaling of the renormalized dimensionless scalar field is $\phi \sim k^{-\frac{d-2+\eta_{\rm H}}{2}}H$. The RG-time derivative is taken at fixed dimensionfull fields, when applied to the effective average action Γ_k, but it is usually understood to be computed at fixed renormalized dimensionless quantities, when one looks at the beta function for $u(\rho)$ or h^2 for example. As a matter of fact, we can say that the anomalous dimension changes the scaling which one naively expects for the scalar field, indeed we can write

$$H \sim k^{\frac{d-2+\eta_{\rm H}}{2}}\phi, \tag{C.25}$$

where the field ϕ is considered as fixed.

C.3. Anomalous Dimension for the Fermion Field

The derivation of the flow for the fermionic wave function renormalization Z_{F} follows the same line of arguments as for the derivation of the flow for Z_{H}. The projection onto the fermionic kinetic term is given by

$$\partial_t Z_{\mathrm{F}} = -\frac{1}{N_{\mathrm{c}} d\, d_\gamma^{\mathrm{c}} V} \mathrm{tr}\gamma_\mu \frac{\partial}{\partial p_\mu} \frac{\overrightarrow{\delta}}{\delta \bar{\psi}^A(p)} \partial_t \Gamma_k \frac{\overleftarrow{\delta}}{\delta \psi^A(p)} \bigg|_{p=0,\{\varphi,\bar{\varphi}\}=0}, \tag{C.26}$$

where the trace is over the Dirac indices. Also in this case, it is possible to recognize that only the quadratic term in the expansion of $\ln(\mathcal{K}+\mathcal{F})$ in Eq. (C.19) survives the projection and, the flow equation for the fermionic wave function renormalization becomes

$$\begin{aligned}\partial_t Z_{\mathrm{F}} = \frac{1}{4d\, d_\gamma^{\mathrm{c}}} \mathrm{tr}\gamma_\mu \frac{\partial}{\partial p_\mu}\bigg|_{p=0} \tilde{\partial}_t \bigg\{ & \bar{h}^2 \int_\ell \mathfrak{a}(\ell,\bar{v})[\mathfrak{b}(\ell+p,\bar{v}) + (\not{\ell}+\not{p})\mathfrak{c}(\ell+p,\bar{v})] \\ & + \bar{h}^2 \int_\ell \mathfrak{a}(\ell+p,\bar{v})[\mathfrak{b}(\ell,\bar{v}) - \not{\ell}\mathfrak{c}(\ell,\bar{v})] \\ & - \bar{g}_{\mathrm{s}}^2 \frac{Z_{\mathrm{F}}^2}{Z_{\mathrm{G}}} \frac{N_{\mathrm{c}}^2-1}{2N_{\mathrm{c}}} \gamma_\nu \int_\ell \mathbb{P}_{\nu\lambda}(\ell)\, 4[\mathfrak{b}(\ell+p,\bar{v}) + (\not{\ell}+\not{p})\mathfrak{c}(\ell+p,\bar{v})]\gamma_\lambda \bigg\}.\end{aligned} \tag{C.27}$$

Expanding the integrands for small p and retaining only the linear terms in p_μ we get, after integrations by part,

$$\begin{aligned}\partial_t Z_{\mathrm{F}} = & -\frac{\bar{h}^2}{d} \tilde{\partial}_t \int_\ell \ell^2 \mathfrak{c}(\ell,\bar{v}) \partial_{(\ell^2)} \mathfrak{a}(\ell,\bar{v}) - \frac{2}{d} \frac{N_{\mathrm{c}}^2-1}{2N_{\mathrm{c}}} \frac{Z_{\mathrm{F}}^2}{Z_{\mathrm{G}}} \bar{g}_{\mathrm{s}}^2 \bigg[(1-\zeta_{\mathrm{s}})(d-1) \int_\ell \frac{\mathfrak{c}(\ell,\bar{v})}{P_{\mathrm{G}}(\ell)} \\ & - (d-1-\zeta_{\mathrm{s}}) \int_\ell \ell^2 \mathfrak{c}(\ell,\bar{v}) \frac{\partial_{(\ell^2)} P_{\mathrm{G}}(\ell)}{P_{\mathrm{G}}(\ell)^2} \bigg],\end{aligned} \tag{C.28}$$

where the following properties among the Dirac matrices have been used: $\mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_\lambda) = 0$ and $\mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma) = d_\gamma^{\mathrm{c}}(\delta_{\mu\nu}\delta_{\lambda\sigma} + \delta_{\mu\sigma}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\sigma})$. As it is clear, we have chosen the same regulator for both the transverse and longitudinal modes for the gluon fields, namely $r_{\mathrm{GL}} = r_{\mathrm{GT}}$, such that we can define $P_{\mathrm{G}}(\ell) = \ell^2[1 + r_{\mathrm{GL}}(\ell)]$. Comparing the latter equation with the definition for the threshold functions $m_{12}^{(\mathrm{FH})d}$, $m_{12}^{(\mathrm{FG})d}$ and $\tilde{m}_{11}^{(\mathrm{FG})d}$ – given in App. B – we recover the expression for the spinor anomalous dimension η_ψ as given in the main text in Eq. (3.28). Of course dimensionless and renormalized quantities have to be introduced.

C.4. Top-Yukawa Coupling

Given the separation of the scalar field into its *vev* plus fluctuations around it, as in Eq. (C.15), the RG flow of the Yukawa coupling can be obtained by projecting onto the operator $\sim \Delta H\bar{\psi}\psi$ rather then onto $\sim H\bar{\psi}\psi$. This distinction is obviously the same in the symmetric regime while it differs in the broken regime where the two operators flow differently. In fact, in the flow equation for the Yukawa coupling, obtained from the projection onto the Higgs scalar fluctuation, there are also contributions coming from higher-order Yukawa operators such as, for example, $\sim H^{2n}H\bar{\psi}\psi$. This operator is beyond our truncation but can contribute to the running of the operator $\bar{v}^{2n}\Delta H\bar{\psi}\psi$. Therefore we are indeed taking into account an effective operator of the form $\bar{h}(\bar{v})H\bar{\psi}\psi$ where $\bar{h}(\bar{v})$ is an arbitrary function of the nontrivial minimum $\bar{v}$ for the scalar potential. Therefore the projection rule onto the Yukawa term $\sim \Delta H\bar{\psi}\psi$ reads

$$\partial_t \bar{h} = \frac{1}{V}\frac{\sqrt{2}}{\mathrm{i}N_\mathrm{c}d_\gamma^\mathrm{c}}\frac{\delta}{\delta\Delta H(p-q)}\frac{\overrightarrow{\delta}}{\delta\bar{\psi}^A(p)}\partial_t\Gamma_k\frac{\overleftarrow{\delta}}{\delta\psi^A(q)}\bigg|_{\{\varphi,\bar{\varphi}\}=0,\{p,q\}=0}. \tag{C.29}$$

After having expanded the $\ln[\mathcal{K}(1+\mathcal{PF})]$ in a Taylor series, it is possible to see that only the cubic term $(\mathcal{PF})^3$ survives the projection such that the latter equation becomes

$$\partial_t \bar{h} = \frac{1}{V}\frac{\sqrt{2}}{\mathrm{i}N_\mathrm{c}d_\gamma^\mathrm{c}}\frac{\delta}{\delta\Delta H(p-q)}\frac{\overrightarrow{\delta}}{\delta\bar{\psi}^A(p)}\frac{1}{2}\,\mathrm{STr}\,\tilde{\partial}_t\left\{\frac{1}{3}(\mathcal{PF})^3\right\}\frac{\overleftarrow{\delta}}{\delta\psi^A(q)}\bigg|_{\{\varphi,\bar{\varphi}\}=0,\{p,q\}=0}. \tag{C.30}$$

Performing some tedious algebra leads to the following expression

$$\begin{aligned}\partial_t \bar{h} = \frac{\sqrt{2}}{\mathrm{i}}\frac{1}{6}\tilde{\partial}_t\bigg\{&U'''(\bar{v})\frac{\bar{h}^2}{2}\int_\ell 6\,\mathfrak{a}(\ell,\bar{v})^2\mathfrak{b}(\ell,\bar{v}) - \frac{\mathrm{i}\bar{h}^3}{2\sqrt{2}}\int_\ell 6\,\mathfrak{a}(\ell,\bar{v})[\mathfrak{b}(\ell,\bar{v})^2+\ell^2\mathfrak{c}(\ell,\bar{v})^2]\\ &+\frac{\mathrm{i}\bar{h}}{\sqrt{2}}\frac{Z_\mathrm{F}^2}{Z_\mathrm{G}}\bar{g}_\mathrm{s}^2\frac{N_\mathrm{c}^2-1}{2N_\mathrm{c}}\int_\ell 6\,\mathbb{P}_{\mu\mu}(\ell)[\mathfrak{b}(\ell,\bar{v})^2+\ell^2\mathfrak{c}(\ell,\bar{v})^2]\bigg\},\end{aligned} \tag{C.31}$$

where the trace of the regularized propagator for the gluons is $\mathbb{P}_{\mu\mu}(\ell)=(d+\zeta_\mathrm{s}-1)P_\mathrm{G}(\ell)^{-1}$, if the regularization scheme for both longitudinal and transverse modes are equal. Let remind the reader that the trace over the generators of the fundamental representation of $\mathrm{SU}(N_\mathrm{c})$ is $\mathrm{tr}(T_IT_J)=\frac{1}{2}\delta_{IJ}$, such that for $I=J$ we get $\delta_{II}=N_\mathrm{c}^2-1$ the number of components for the adjoint representation. Translating the latter equation in terms of the regularized dimensionless quantities, and taking into account the definition for the threshold functions $l_{n_1n_2}^{(\mathrm{FH})d}$ and $l_{n_1n_2}^{(\mathrm{FG})d}$ – as given in App. B – we obtain the following beta function [164]

$$\begin{aligned}\partial_t h^2 =\;& (d-4+\eta_\mathrm{H}+2\eta_\psi)h^2 - 4v_d h^4\rho(6u''+4\rho u''')l_{12}^{(\mathrm{FH})d}(\omega_\mathrm{F},\omega_\mathrm{H})\\ &-\frac{N_\mathrm{c}^2-1}{2N_\mathrm{c}}(d+\zeta_\mathrm{s}-1)8v_d h^2 g_\mathrm{s}^2\left[l_{11}^{(\mathrm{FG})d}(\omega_\mathrm{F},0;\eta_\psi,\eta_\mathrm{G})-2h^2\rho l_{21}^{(\mathrm{FG})d}(\omega_\mathrm{F},0;\eta_\psi,\eta_\mathrm{G})\right]\\ &+4v_d h^4\left[l_{11}^{(\mathrm{FH})d}(\omega_\mathrm{F},\omega_\mathrm{H};\eta_\psi,\eta_\mathrm{H})-2h^2\rho l_{21}^{(\mathrm{FH})d}(\omega_\mathrm{F},\omega_\mathrm{H};\eta_\psi,\eta_\mathrm{H})\right]\bigg|_{\rho=\kappa}.\end{aligned} \tag{C.32}$$

C.4.1. Generalized Yukawa Interaction

In Sec. 3.2 we have discussed two possible projection rules which can be used to derive the RG flow equation for a generalized Yukawa potential. In particular, we have emphasized the fact that the first one, given by Eq. (3.24), namely the projection onto the field-dependent two point function $\Gamma^{(2)}_{\bar\psi\psi}(H)$, presents a better convergence in the SSB regime once a finite truncation for the functional $\bar h(H)$ is chosen. Therefore we want to present here the main differences with respect to the calculation we have just presented above. Let us consider therefore a generalized Yukawa coupling $\bar h(H^2/2)$, function of the $\mathbb{Z}_2$ invariant field $H^2/2$, such that the original discrete chiral symmetry of the action is preserved. In order to keep arbitrary the dependence of $\bar h$ with respect to H, we consider the Higgs scalar field as a constant value, in a similar way as we did for deriving the flow of the scalar potential. In this way, the Yukawa interaction term reads, in momentum space,

$$\frac{\mathrm{i}}{\sqrt{2}}\bar h(H)H\int_p \bar\psi^A(p)\psi^A(p), \tag{C.33}$$

from which we can immediately read off the correct projection rule onto the generalized Yukawa potential

$$H\partial_t\big|_H \bar h(H) = \frac{\sqrt{2}}{\mathrm{i}}\,\frac{1}{VN_\mathrm{c}d_\gamma^\mathrm{c}}\,\frac{\overrightarrow{\delta}}{\delta\bar\psi^A(p)}\partial_t\Gamma_k\frac{\overleftarrow{\delta}}{\delta\psi^A(p)}\bigg|_{\{\varphi,\bar\varphi\}=0,p=0}. \tag{C.34}$$

To proceed further, we split again the regularized Hessian $\Gamma^{(2)}_k + R_k$ into a kinetic part plus a fluctuation part, and we expand the $\ln(1+\mathcal{PF})$ as in Eq. (C.19). The difference here from before is that, since H is considered as constant, we do not split the scalar field into its *vev* plus fluctuations around it. Due to this fact, the propagator matrix $\mathcal{P}$ remains essentially the same, [32] but the fluctuation matrix does change according to the generalized structure of the Yukawa coupling. Indeed the matrix elements which are different from the previous Eq. (C.20) are:

$$\mathcal{PF}_\mathrm{HH} = \frac{\mathrm{i}}{\sqrt{2}}\mathfrak{a}(p,H)\left[2\bar h'(H) + H\bar h''(H)\right]\int_\ell \bar\psi(\ell)\psi(\ell+p-q), \tag{C.35}$$

$$\mathcal{PF}_{\mathrm{H}\psi} = \frac{\mathrm{i}}{\sqrt{2}}\mathfrak{a}(p,H)\left[\bar h(H) + H\bar h'(H)\right]\bar\psi(q-p), \tag{C.36}$$

$$\mathcal{PF}_{\mathrm{H}\bar\psi} = -\frac{\mathrm{i}}{\sqrt{2}}\mathfrak{a}(p,H)\left[\bar h(H) + H\bar h'(H)\right]\psi^\mathrm{T}(p-q), \tag{C.37}$$

$$\mathcal{PF}_{\psi\mathrm{H}} = -\frac{\mathrm{i}}{\sqrt{2}}\left[\bar h(H) + H\bar h'(H)\right]\left[\mathfrak{b}(p,H) + \not{p}\,\mathfrak{c}(p,H)\right]\psi(p-q), \tag{C.38}$$

$$\mathcal{PF}_{\bar\psi\mathrm{H}} = -\frac{\mathrm{i}}{\sqrt{2}}\left[\bar h(H) + H\bar h'(H)\right]\left[\mathfrak{b}(p,H) - \not{p}^\mathrm{T}\mathfrak{c}(p,H)\right]\bar\psi^\mathrm{T}(q-p), \tag{C.39}$$

$$\mathcal{PF}_{\psi\psi} = \bar g_\mathrm{s}Z_\mathrm{G}\left[\mathfrak{b}(p,H) + \not{p}\,\mathfrak{c}(p,H)\right]\not{G}(p-q), \tag{C.40}$$

$$\mathcal{PF}_{\bar\psi\bar\psi} = \bar g_\mathrm{s}Z_\mathrm{G}\left[\mathfrak{b}(p,H) - \not{p}^\mathrm{T}\mathfrak{c}(p,H)\right]\not{G}^\mathrm{T}(p-q). \tag{C.41}$$

[32]The only substitution is that $\bar h v$ goes into $\bar h(H)H$ and the second derivative of the scalar potential, in the function $\mathfrak{a}(\ell,H)$, is not evaluated any more at the *vev* but is considered as a full function of H.

From the Taylor expansion given in Eq. (C.19), it is possible to see that only the linear and quadratic terms of the expansion survive the projection in Eq. (C.34). These terms lead to the following flow equation for the dimensionfull generalized Yukawa coupling

$$
\begin{aligned}
H\partial_t\big|_H \bar{h}(H) = \frac{1}{2}\left[2\bar{h}'(H) + H\bar{h}''(H)\right]\tilde{\partial}_t \int_\ell \mathfrak{a}(\ell, H) - \frac{1}{\mathrm{i}\sqrt{2}}\left[\bar{h}(H) + H\bar{h}'(H)\right]^2 \tilde{\partial}_t \int_\ell \mathfrak{a}(\ell, H)\mathfrak{b}(\ell, H) \\
+ \frac{\sqrt{2}}{\mathrm{i}}\bar{g}_\mathrm{s}^2 \frac{Z_\mathrm{F}^2}{Z_\mathrm{G}} \frac{N_\mathrm{c}^2 - 1}{2N_\mathrm{c}} \tilde{\partial}_t \int_\ell \mathfrak{b}(\ell, H)\mathbb{P}_{\mu\mu}(\ell). \qquad \text{(C.42)}
\end{aligned}
$$

From the expression for the dimensionless and renormalized Yukawa coupling we have that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}h(\rho) &= \partial_t\big|_\rho h(\rho) + (2 - d - \eta_\mathrm{H})\rho h'(\rho) \qquad \text{(C.43)} \\
&= \frac{1}{2}(\eta_\mathrm{H} + 2\eta_\psi + d - 4)h(\rho) + \frac{k^{\frac{d-4}{2}}}{\sqrt{Z_\mathrm{H}}Z_\mathrm{F}}\partial_t\big|_H \bar{h}(H), \qquad \text{(C.44)}
\end{aligned}
$$

such that the RG flow equation for $h(\rho)$ reads

$$
\partial_t\big|_\rho h(\rho) = \frac{1}{2}(\eta_\mathrm{H} + 2\eta_\psi + d - 4)h(\rho) + (d - 2 + \eta_\mathrm{H})\rho h'(\rho) + \frac{k^{\frac{d-4}{2}}}{\sqrt{Z_\mathrm{H}}Z_\mathrm{F}}\partial_t\big|_H \bar{h}(H), \qquad \text{(C.45)}
$$

where the last term is given by Eq. (C.42). The loop integrals can be expressed in terms of the threshold functions $l_1^{(\mathrm{H})d}$, $l_{11}^{(\mathrm{FH})}$ and $l_{11}^{(\mathrm{FG})}$ – as defined in Sec. B – such that we finally obtain

$$
\begin{aligned}
\partial_t h(\rho) = \frac{1}{2}(d - 4 + \eta_\mathrm{H} + 2\eta_\psi)h + (d - 2 + \eta_\mathrm{H})\rho h' - 2v_d\left(3h' + 2\rho h''\right) l_1^{(\mathrm{H})d}(\omega_\mathrm{H}, \eta_\mathrm{H}) \\
+ 2v_d\, h(h + 2\rho h')^2\, l_{11}^{(\mathrm{FH})d}(\omega_\mathrm{F}, \omega_\mathrm{H}; \eta_\psi, \eta_\mathrm{H}) \\
- 4v_d g_\mathrm{s}^2 h \frac{N_\mathrm{c}^2 - 1}{2N_\mathrm{c}}(d + \zeta_\mathrm{s} - 1) l_{11}^{(\mathrm{FG})d}(\omega_\mathrm{F}, 0; \eta_\psi, \eta_\mathrm{G}). \qquad \text{(C.46)}
\end{aligned}
$$

The RG flow equation given in the main text, cf. Eq. (3.26), can be obtained straightforwardly by specifying the generalized Yukawa potential $h(\rho)$ to be a k-scale dependent constant. Let us observe also that the two formulations presented in this appendix are related among each other. In fact Eq. (C.32) can be obtained from Eq. (C.46) by taking one derivative with respect to ρ before evaluating it at the nontrivial minimum $\rho = \kappa$. Vice versa, Eq. (C.46) can be obtained from Eq. (C.32) by setting to zero all the threshold functions which are explicitly multiplied by the nontrivial minimum κ.

Appendix D.

On the Stability of the full Scalar Potential for the $\mathbb{Z}_2$-Yukawa-QCD Model

D.1. Resummation of $u(\rho)$ in the Deep Euclidean Regime

For a study of the UV stability of the QFP potential $u(\rho)$, given by Eq. (3.49), it is necessary to address the combined limit $\rho \to \infty$ and $g_\mathrm{s}^2 \to 0$. However, a meaningful and consistent result requires to take these limites in an appropriate order while remaining in the outer or inner asymptotic region, defined respectively as the region where the variable $z = g_\mathrm{s}^2\rho$ appearing in the bosonic and fermionic threshold functions is either $z \gg 1$ or $z \ll 1$, as introduced at the end of Sec. 3.3.1.

Let us start with the outer region $z \gg 1$. If g_s^2 is small and finite we can address the asymptotic behavior by expanding the potential of Eq. (3.49) for $z \to \infty$. This gives the following result

$$
\begin{aligned}
u(\rho) = {} & \frac{\hat{\lambda}_2 z^2}{2g_\mathrm{s}^2} + \Gamma\left(\frac{2\eta_z}{2+\eta_z}\right)\left[\Gamma\left(\frac{2+3\eta_z}{2+\eta_z}\right)\right]^{-1} \frac{(3\hat{\lambda}_2 z)^2 - 12\left(\frac{2}{9}z\right)^2}{32\pi^2(2+\eta_z)} \\
& + \Gamma\left(\frac{4+4\eta_z}{2+\eta_z}\right)\Gamma\left(-\frac{2\eta_z}{2+\eta_z}\right)\frac{(3\hat{\lambda}_2 z)^{\frac{4}{2+\eta_z}} - 12\left(\frac{2}{9}z\right)^{\frac{4}{2+\eta_z}}}{32\pi^2(2+3\eta_z)} + \mathcal{O}(z)\Big|_{z=g_\mathrm{s}^2\rho} \\
\equiv {} & \left[\frac{\hat{\lambda}_2}{2g_\mathrm{s}^2} + c_1(g_\mathrm{s}^2)\right] z^2 + c_2(g_\mathrm{s}^2)\, z^{\frac{4}{2+\eta_z}} + \mathcal{O}(z)\Big|_{z=g_\mathrm{s}^2\rho}
\end{aligned}
\tag{D.1}
$$

where η_z is the anomalous dimension of z as given in Eq. (3.47) together with Eqs. (3.5) and (3.32). Now we can safely perform the limit $g_\mathrm{s}^2 \to 0$ in the outer asymptotic region where $z \gg 1$. We find that the leading order behavior is given only by the first term, as in Eq. (3.52). Indeed the noninteger power scaling $z^{4/(2+\eta_z)}$ behaves as z^2 for $g_\mathrm{s}^2 \to 0$ and the two coefficients c_1 and c_2 have simple poles in g_s^2 that cancel each other in the limit $g_\mathrm{s}^2 \to 0$.

Let us further consider the inner interval where $z \ll 1$. Expanding the potential in

Eq. (3.49) first for small g_{s}^2, i.e., small η_z, we obtain the following expression

$$u(\rho) = \frac{\hat{\lambda}_2 z^2}{2g_{\mathrm{s}}^2} + \frac{(3\hat{\lambda}_2)^3 - 12\left(\frac{2}{9}\right)^3}{64\pi^2} b_1(z) z^3 - \frac{3(3\hat{\lambda}_2)^3 - 36\left(\frac{2}{9}\right)^3}{128\pi^2} b_2(z)\eta_z z^3 + \mathcal{O}(\eta_z^2 z^3), \tag{D.2}$$

where the functions $b_{1,2}(z)$ has the property that $\lim_{z\to 0} b_{1,2}(z) = 1$. This expansion is valid only for those values of ρ such that $g_{\mathrm{s}}^2\rho \ll 1$. Therefore, in order to address the limit $\rho \to \infty$, it is also necessary to take the limit $g_{\mathrm{s}}^2 \to 0$ while keeping $z \ll 1$, in such a way that the expansion in Eq. (D.2) still holds. Doing so, we find that the leading term for the potential is again the one in Eq. (3.52). The same conclusion can be deduced by expanding the potential for small z and keeping g_{s}^2 fixed. Indeed the function $u(\rho)$ is analytic in z and its expansion reads

$$u(\rho) = \frac{\hat{\lambda}_2 z^2}{2g_{\mathrm{s}}^2} + \frac{(3\hat{\lambda}_2)^3 - 12\left(\frac{2}{9}\right)^3}{32\pi^2(2+3\eta_z)} z^3 + \mathcal{O}(z^4), \tag{D.3}$$

where again the $g_{\mathrm{s}}^2 \to 0$ limit with fixed $z \ll 1$ gives us the result in Eq. (3.52).

Let us emphasize once more that a consistent answer about the full stability of the potential $u(\rho)$ requires to take the two limits $\rho \to \infty$ and $g_{\mathrm{s}}^2 \to 0$ in such a way that the variable $z = g_{\mathrm{s}}^2\rho$ entering the bosonic and fermionic loops satisfies $z \gg 1$ for the outer region or $z \ll 1$ for the inner region. In these two asymptotic regions, the potential has the same positive asymptotic coefficient in front of the leading quadratic term. Therefore, we conclude that it is stable for any arbitrarily small values of the strong gauge coupling.

D.2. $f(x)$ in the ϕ^4-dominance Approximation

In Sec. 3.4, the same reasoning for taking the asymptotic limits as in the preceding section applies to the two loop-variables z_{H} and z_{F}, defined in Eq. (3.67). Let us start first by inspecting the potential $f(x)$ in the ϕ^4-dominance approximation in the outer region where $z_{\mathrm{H,F}} \gg 1$. For finite values of g_{s}^2, we can assume that the loop contributions are negligible for large field amplitudes and therefore expand the scalar and fermionic loops in powers of x^{-1}. In doing so we obtain

$$\partial_t f(x) = -4f(x) + d_{x,\mathrm{s}} x f'(x) + \frac{1}{32\pi^2}\sum_{n=1}^{\infty}(-)^{n+1}\left(\frac{1}{z_{\mathrm{H}}^n} - \frac{12}{z_{\mathrm{F}}^n}\right), \tag{D.4}$$

and its QFP solution then reads

$$f_{\mathrm{as}}(x) = C_{\mathrm{as}}\, x^{\frac{4}{2+\eta_{x,\mathrm{s}}}} - \frac{1}{32\pi^2}\sum_{n=1}^{\infty}(-)^n \frac{1}{4+n(2+\eta_{x,\mathrm{s}})}\left(\frac{1}{z_{\mathrm{H}}^n} - \frac{12}{z_{\mathrm{F}}^n}\right), \tag{D.5}$$

which can be resummed analytically

$$f_{\rm as}(x) = C_{\rm as}\, x^{\frac{4}{2+\eta_{x,\rm s}}} + \frac{1}{32\pi^2(6+\eta_{x,\rm s})}\left[\frac{1}{z_{\rm H}}\,{}_2F_1\left(1, \frac{6+\eta_{x,\rm s}}{2+\eta_{x,\rm s}}, \frac{8+2\eta_{x,\rm s}}{2+\eta_{x,\rm s}}; -\frac{1}{z_{\rm H}}\right)\right.$$
$$\left. - \frac{12}{z_{\rm F}}\,{}_2F_1\left(1, \frac{6+\eta_{x,\rm s}}{2+\eta_{x,\rm s}}, \frac{8+2\eta_{x,\rm s}}{2+\eta_{x,\rm s}}; -\frac{1}{z_{\rm F}}\right)\right]. \tag{D.6}$$

Using the following linear transformation among the hypergeometric functions

$$\frac{\sin\left[\pi(b-a)\right]}{\pi\Gamma(c)}\,{}_2F_1(a,b,c,z) = \frac{1}{(-z)^a}\frac{{}_2F_1\left(a, a-c+1, a-b+1; z^{-1}\right)}{\Gamma(b)\Gamma(c-a)\Gamma(a-b+1)}$$
$$- \frac{1}{(-z)^b}\frac{{}_2F_1\left(b, b-c+1, b-a+1; z^{-1}\right)}{\Gamma(a)\Gamma(c-b)\Gamma(b-a+1)}, \tag{D.7}$$

and the identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{D.8}$$

it is possible to rewrite the solution $f(x)$ (cf. Eq. (3.66)) into $f_{\rm as}(x)$. Indeed, this becomes clear from the relation between the two integration constants C_f and $C_{\rm as}$ which is

$$C_{\rm as} = C_f + \frac{1}{32\pi(2+\eta_{x,\rm s})}\left[\sin\left(\frac{4\pi}{2+\eta_{x,\rm s}}\right)\right]^{-1}\left[\left(3\xi_2 g_{\rm s}^{2P}\right)^{\frac{4}{2+\eta_{x,\rm s}}} - 12\left(\frac{2}{9}g_{\rm s}^{2-2P}\right)^{\frac{4}{2+\eta_{x,\rm s}}}\right]. \tag{D.9}$$

This mapping from $f_{\rm as}(x)$ to $f(x)$ tells us that the asymptotic behavior of the QFP solution in the outer region (where $z_{\rm H,F} \gg 1$) is determined only by the scaling terms in $\partial_t f(x) = 0$. In fact, this property can be inferred also by expanding the solution $f(x)$, instead of its beta function, for large $z_{\rm H,F}$

$$f(x) = C_{f,\infty}\, x^{\frac{4}{2+\eta_{x,\rm s}}} - \frac{\Gamma\left(-\frac{6+\eta_{x,\rm s}}{2+\eta_{x,\rm s}}\right)}{32\pi^2(2+\eta_{x,\rm s})\,\Gamma\left(-\frac{4}{2+\eta_{x,\rm s}}\right)}\left[\frac{1}{z_{\rm H}} - \frac{12}{z_{\rm F}} + \mathcal{O}(z_{\rm H}^{-2}) + \mathcal{O}(z_{\rm F}^{-2})\right]. \tag{D.10}$$

The coefficient in front of the scaling term is a function of C_f, ξ_2, $g_{\rm s}^2$, and P

$$C_{f,\infty} = C_f + \frac{\Gamma\left(\frac{-2+\eta_{x,\rm s}}{2+\eta_{x,\rm s}}\right)\Gamma\left(\frac{6+\eta_{x,\rm s}}{2+\eta_{x,\rm s}}\right)}{128\pi^2}\left[\left(3\xi_2 g_{\rm s}^{2P}\right)^{\frac{4}{2+\eta_{x,\rm s}}} - 12\left(\frac{2}{9}g_{\rm s}^{2-2P}\right)^{\frac{4}{2+\eta_{x,\rm s}}}\right]. \tag{D.11}$$

It is not surprising that this scaling factor is exactly the asymptotic coefficient $C_{\rm as}$. By using one of the defining properties of the Gamma function $\Gamma(1+z) = z\Gamma(z)$ as well as Eq. (D.8), we recover precisely the expression in Eq. (D.9), therefore

$$C_{f,\infty} = C_{\text{as}}. \tag{D.12}$$

As we are interested in the asymptotic behavior in the UV, it is convenient to expand Eq. (D.11) for small g_{s}^2 and keep only the leading terms,

$$C_{f,\infty} \underset{g_{\text{s}}^2\to 0}{\sim} C_f - \frac{1}{64\pi^2\hat{\eta}_{x,\text{s}}}\left[9\xi_2^2 g_{\text{s}}^{4P-2} - 12\left(\frac{2}{9}\right)^2 g_{\text{s}}^{2-4P}\right], \tag{D.13}$$

where $\hat{\eta}_{x,\text{s}} = g_{\text{s}}^{-2}\eta_{x,\text{s}}$ is a finite value in the $g_{\text{s}}^2 \to 0$ limit. Moreover, all the subleading terms in Eq. (D.10) of order $\mathcal{O}(z_{\text{H}}^{-1})$ and $\mathcal{O}(z_{\text{F}}^{-1})$ are regular in the $g_{\text{s}}^2 \to 0$ limit. We can thus conclude that the asymptotic property of the QFP potential is correctly described by Eq. (D.13) in the outer region, where both limits $x \to \infty$ and $g_{\text{s}}^2 \to 0$ are taken simultaneously while keeping $z_{\text{H,F}} \gg 1$.

Let us address now the situation in the inner region, where we can expand the potential $f(x)$ either for $z_{\text{H,F}} \ll 1$ or for $x \ll 1$ while keeping g_{s}^2 finite. In both cases the result is the same, and reads

$$f(x) = -\frac{11}{128\pi^2} + C_f x^{\frac{4}{2+\eta_{x,\text{s}}}} - \frac{z_{\text{H}} - 12z_{\text{F}}}{32\pi^2(2-\eta_{x,\text{s}})} - \frac{z_{\text{H}}^2 - 12z_{\text{F}}^2}{64\pi^2\eta_{x,\text{s}}} + \mathcal{O}(z_{\text{H}}^3) + \mathcal{O}(z_{\text{F}}^3). \tag{D.14}$$

In the UV limit, the inner region increases and thus allows to address the asymptotic behavior of the potential. Indeed this combined limit can be taken as long as $z_{\text{H,F}} \ll 1$ holds. From the latter equation we can deduce that

$$f(x) \underset{x\to\infty}{\sim} C_f x^2 - \frac{1}{64\pi^2\eta_{x,\text{s}}}\left[\left(3\xi_2 g_{\text{s}}^{2P}\right)^2 - 12\left(\frac{2}{9}g_{\text{s}}^{2-2P}\right)^2\right] x^2, \tag{D.15}$$

where the coefficient in front of the quadratic term coincides with Eq. (D.13). The same information is obtained also by expanding the hypergeometric functions in Eq. (3.66) for small strong gauge coupling, yielding

$$f(x) = C_f x^2 - \frac{z_{\text{H}}^2 - 12\,z_{\text{F}}^2}{64\pi^2\eta_{x,\text{s}}} + \mathcal{O}(g_{\text{s}}^0). \tag{D.16}$$

This is indeed in agreement with Eq. (D.15).

We finally conclude that it is possible to simultaneously take the $x \to \infty$ and the $g_{\text{s}}^2 \to 0$ limits in both the inner and outer asymptotic regions. This combined limit gives the same result in both cases, and can be summarized as

$$f(x) \underset{\substack{x\to\infty \\ g_s^2\to 0}}{\sim} g_s^{\pm(4P-2)}\, \hat{C}_{f,\infty}\, x^2, \tag{D.17}$$

where the $\pm$ sign is for $P \lessgtr 1/2$. The expression for $\hat{C}_{f,\infty}$ can be found in the main text in Eq. (3.70).

D.3. Comparison between $u(\rho)$ and $f(x)$

The potential $u(z)$ in Eq. (3.49), obtained within the DER, and the effective potential $f(x)$ in Eq. (3.66), derived in the ϕ^4-dominance approximation, can be related to each other by exploiting a general identity among the Gauß hypergeometric functions

$$\frac{zb}{c}\, {}_2F_1(a+1, b+1, c+1; z) = {}_2F_1(a+1, b, c; z) - {}_2F_1(a, b, c; z). \tag{D.18}$$

By setting $a = 0$ in the latter expression, we get the following relationship

$${}_2F_1(1, b, c; z) = 1 + \frac{zb}{c}\, {}_2F_1(1, b+1, c+1; z). \tag{D.19}$$

Iterating the latter relation tree times, we can rewrite the solution $f(x)$ as

$$\begin{aligned} f(x) = C_f\, x^{\frac{4}{2+\eta_{x,\mathrm{s}}}} + \frac{1}{32\pi^2}\Bigg[&-\frac{11}{4} + \frac{8g_s^{2-2P} - 9\xi_2 g_s^{2P}}{3(2-\eta_{x,\mathrm{s}})} x + \frac{16 g_s^{4-4P} x - 243\xi_2^2 g_s^{4P}}{54\eta_{x,\mathrm{s}}} x^2 \\ &+ \frac{(3\xi_2 g_s^{2P} x)^3}{2+3\eta_{x,\mathrm{s}}}\, {}_2F_1\left(1, \frac{2+3\eta_{x,\mathrm{s}}}{2+\eta_{x,\mathrm{s}}}, \frac{4+4\eta_{x,\mathrm{s}}}{2+\eta_{x,\mathrm{s}}}; -3\xi_2 g_s^{2P} x\right) \\ &- \frac{12\left(\frac{2}{9} g_s^{2-2P} x\right)^3}{2+3\eta_{x,\mathrm{s}}}\, {}_2F_1\left(1, \frac{2+3\eta_{x,\mathrm{s}}}{2+\eta_{x,\mathrm{s}}}, \frac{4+4\eta_{x,\mathrm{s}}}{2+\eta_{x,\mathrm{s}}}; -\frac{2}{9} g_s^{2-2P} x\right)\Bigg]. \end{aligned} \tag{D.20}$$

Let us consider the CEL potential corresponding to $P = 1/2$, $C_f = 0$, and $\xi_2 = \hat{\lambda}_2^+$. Working in the limit $g_s^2 \to 0$ where we can also take $\eta_{x,\mathrm{s}} \to \eta_z$, cf. Eqs. (3.47) and (3.65), the latter expression for $f(x)$ becomes

$$\lim_{\eta_{x,\mathrm{s}}\to\eta_z} f(x) = u(z)\Big|_{z=g_s x} - \frac{11}{128\pi^2} + \frac{8-9\xi_2}{96\pi^2(2-\eta_{x,\mathrm{s}})} g_s x. \tag{D.21}$$

We conclude that the solution $f(x)$ boils down to the solution $u(\rho)$ in the UV limit; apart from a linear term in the field variable that is, in fact, discarded by the definition of the DER approximation.

Appendix E.

The Unphysical $P > 1/2$ Values for the $\mathbb{Z}_2$-Yukawa-QCD Model

In the main text of Sec. 3.3, within the EFT-like approach, we have discussed only those P-values for the rescaled quartic scalar coupling corresponding to some physical solution for the scalar potential. Beyond the DER, the properties defining a physical solution are

- the existence of a nontrivial minimum κ for the scalar potential,
- the existence of a positive and finite QFP for the rescaled quartic scalar coupling $\hat{\lambda}_2$.

In this appendix we complete the EFT-like analysis, as presented in Sec. 3.3.2, and we illustrate that, for $P > 1/2$, these physical conditions are not fulfilled. By following the same prescription, we expand polynomially the scalar potential about the nontrivial minimum κ up to $N_\mathrm{p} = 2$, and we retain the coupling λ_3 as a free parameter. Afterwards we look for QFP solutions for the beta functions of the rescaled couplings $\hat{\kappa}$ and $\hat{\lambda}_2$.

$\mathbf{P} \in (\mathbf{1/2}, \mathbf{1})$

Inspired by the results of the non-Abelian Higgs model in [111, 112], we can make an educate guess and assume that, for $P > 1/2$, the nontrivial minimum goes to infinity according to some power of g_s^2. Therefore, we can assume that the scaling Q, as defined in Eq. (3.53), is positive. Choosing $Q = 2P - 1$ like for the non-Abelian Higgs model turns out to be the correct scaling also for the present system. However, we prefer to be more general and consider Q as an undetermined positive power in first place. It is possible to verify that, under the assumptions that $Q > 0$, $P_3 > 0$, and $P > 1/2$, the only terms that can contribute to the leading parts in the RG flow for $\hat{\lambda}_2$ and $\hat{\kappa}$ are

$$\begin{aligned}\partial_t\hat{\lambda}_2 =& \frac{\hat{\lambda}_3^2\hat{\kappa}\, g_s^{4P_3-8P-2Q}}{16\pi^2\hat{\lambda}_2(1+2\hat{\lambda}_2\hat{\kappa}g_s^{4P-2Q})^2} - \frac{g_s^{4-4P}}{27\pi^2(1+\frac{2}{9}g_s^{2-2Q}\hat{\kappa})^3} - \frac{\hat{\lambda}_3 g_s^{2-8P+2\gamma}}{12\pi^2\xi_2(1+\frac{2}{9}g_s^{2-2Q}\hat{\kappa})^2}\\ &+ \frac{\hat{\kappa}^2\hat{\lambda}_3^2\, g_s^{4P_3-4Q-4P}}{4\pi^2(1+2g_s^{4P-2Q}\hat{\lambda}_2\hat{\kappa})^3} + \frac{\hat{\kappa}^3\hat{\lambda}_2\hat{\lambda}_3^2\, g_s^{4P_3-6Q}}{2\pi^2(1+2g_s^{4P-2Q}\hat{\lambda}_2\hat{\kappa})^4},\end{aligned} \tag{E.1}$$

$$\partial_t\hat{\kappa} = -2\hat{\kappa} - \frac{\hat{\kappa}^4\hat{\lambda}_3^2\, g_s^{4P_3-6Q}}{4\pi^2(1+2g_s^{4P-2Q}\hat{\lambda}_2\hat{\kappa})^4} - \frac{g_s^{2-4P+2Q}}{12\pi^2\hat{\lambda}_2(1+\frac{2}{9}g_s^{2-2Q}\hat{\kappa})^2}. \tag{E.2}$$

By analyzing all the possible combinations among the three powers Q, P_3 and P, one has to take care that the two powers of g_s in the denominators, i.e., $4P-2Q$ and $2-2Q$, give different contributions to the beta functions, depending on whether they are positive or negative. Moreover, we have to keep in mind that, by definition, $\hat{\lambda}_2$ and $\hat{\kappa}$ have to approach their finite QFP values in the UV limit up to subleading corrections in some positive powers of g_s. Among the set of all possible configurations, there is only one real QFP solution corresponding to the case where the contribution arising from $\hat{\lambda}_3$ in Eq. (E.2) is subleading. We obtain therefore

$$\hat{\kappa} = \frac{1}{54\pi^2\hat{\lambda}_3}, \qquad \hat{\lambda}_2 = -\frac{9\hat{\lambda}_3}{4}, \qquad Q = 2P-1, \qquad P_3 = 2P+1, \tag{E.3}$$

where $\hat{\lambda}_3$ is a free parameter. We deduce that there are no reliable solutions for $P \in (1/2, 1)$ that fulfill our assumptions. Indeed it is not possible to simultaneously satisfy the two conditions that both the Higgs quartic coupling and the nontrivial minimum κ are positive.

P = 1

All throughout the analysis carried in the main text of Chap. 3, we have assumed that the QFP value χ_s^2, for the rescaled coupling $\hat{h}^2$, is given by Eq. (3.11). This value has been calculated by taking the DER limit of Eq. (3.26). However, for $P = 1$, the argument ω_F is finite and does not approach zero in the UV limit. Therefore the RG flow equation for the top-Yukawa coupling cannot be treated as being in the DER. As a consequence, we expect that h^2 still exhibits an asymptotic UV behavior proportional to g_s^2, but with a QFP solution which depends nontrivially on the minimum $\hat{\kappa}$ of the scalar potential. We can nevertheless still consider χ_s^2 as a finite ratio.

It is natural to assume that the rescaling powers Q and P_3 are continues functions of P such that, for $P = 1$, they assume the values $Q = 1$ and $P_3 = 3$. This configuration is indeed the only possible and leads to the following solution

$$\hat{\lambda}_3 = \frac{3\chi_s^4}{8\pi^2\hat{\kappa}(1+\hat{\kappa}\chi_s^2)^3}, \qquad \hat{\lambda}_2 = -\frac{3\chi_s^2}{16\pi^2\hat{\kappa}(1+\hat{\kappa}\chi_s^2)^2}, \qquad P_3 = 3, \qquad Q = 1, \tag{E.4}$$

obtained by solving the leading $g_{\rm s}^2$-dependence of the beta functions $\partial_t \hat{\kappa} = 0$ and $\partial_t \hat{\lambda}_2 = 0$. We observe once more that the latter solution does not corresponds to a physical once, since it is not possible to have simultaneously a positive $\hat{\kappa}$ and a positive $\hat{\lambda}_2$.

$\mathbf{P > 1}$

For these even more extreme values of P, the argument $\omega_{\rm F}$ diverges in the UV limit, corresponding to a complete decoupling of the fermion dynamic. Indeed the fermion fluctuation would acquire infinite mass and does not propagate anymore. In this case the scaling $h^2 \sim g_{\rm s}^2$ might be no longer valid. In fact, later on in App. G.2 while describing the even more general $\mathrm{SU}(2)_{\rm L} \times \mathrm{SU}(3)_{\rm c}$ model, we will give a detailed explanation of the reason why there are no such AF trajectories for $P > 1$.

Appendix F.

Numerical Scaling Solutions for the $\mathbb{Z}_2$-Yukawa-QCD Model

In this appendix, we test our analytical results for the $\mathbb{Z}_2$-Yukawa-QCD model, by integrating numerically the full one-loop nonlinear flow equation for $f(x)$. This is

$$\partial_t f = -4f + d_{x,\mathrm{s}} x f' + \frac{1}{32\pi^2}\frac{1}{1+\omega_{\mathrm{H}f}} - \frac{N_\mathrm{c}}{8\pi^2}\frac{1}{1+z_\mathrm{F}}, \tag{F.1}$$

where we have computed the threshold functions $l_0^{(\mathrm{B})}$ and $l_0^{(\mathrm{F})}$ of Eq. (3.22) by choosing the piece-wise linear regulator (see App. B). The arguments $\omega_{\mathrm{H}f}$ and z_F are defined in Eq. (4.86), and the quantum dimension of x in Eq. (3.65). We make a further approximation evaluating the anomalous dimensions η_G and η_H in the DER, leading to the expressions in Eqs. (3.6) and (3.12) respectively. We are moreover interested in the $P = 1/2$ case characterized by the existence of the perturbative CEL solution, regular at the origin, and a one-parameter family of AF trajectories, singular at the origin but featuring a nontrivial minimum $x_0 \neq 0$.

To address this numerical issue we exploit two different methods. First we study the global behavior of the CEL solution using pseudo-spectral methods. Second, we corroborate the existence of the new QFP family of solutions using shooting methods.

F.1. Pseudo-spectral Methods

Pesudo-spectral methods offer a powerful tool to numerically solve functional RG equations, provided the desired solution can be spanned by a suitable set of basis functions. Here, we are interested in a numerical construction of global properties of the QFP function $f(x)$. We follow the method presented in [262], as this approach has proven to be suited for this purpose. We refer the reader to [197, 263–265] for a variety of applications, and to [266] for earlier FRG implementations. A more general account of pseudo-spectral methods can be found in [267–270].

In order to solve the differential equation given by Eq. (F.1) globally on $\mathbb{R}_+$, the strategy is to decompose the potential $f(x)$ into two series of Chebyshev polynomials. The first series

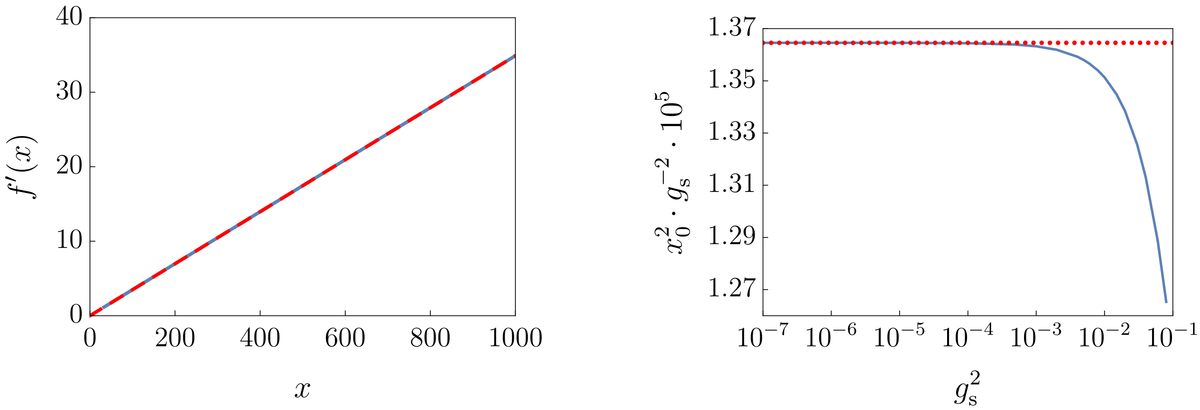


Fig. F.1.: *Left Panel*: Cheng–Eichten–Li solution. First derivative of the potential $f'(x)$ for $g_s^2 = 10^{-4}$, $P = 1/2$ and $\xi_2 = \hat{\lambda}_2^+$ as in Eq. (3.16). The numerical solution of the full one-loop flow equation, obtained from the pseudo-spectral method (solid blue line), lies exactly on top of the analytic solution, obtained within the ϕ^4-dominance approximation (red dashed line), see Eq. (3.66) at $C_f = 0$. *Right Panel*: Novel AF solution. The ratio x_0^2/g_s^2 as a function of g_s^2 for $P = 1/2$ and $\xi_2 = 4$. The solid line represents the numerical solution from the shooting from the minimum, whereas the dotted red line represents the analytic solution, see Eq. (3.79).

is defined over some domain $[0, x_\mathrm{M}]$ and is spanned in terms of Chebyshev polynomials of the first kind $T_i(z)$. The second series is defined over the remaining infinite domain $[x_\mathrm{M}, +\infty)$ and expressed in terms of rational Chebyshev polynomials $R_i(z)$. Moreover, to capture the correct asymptotic behavior of $f(x)$, the latter series is multiplied by the leading asymptotic term $x^{d/d_{x,\mathrm{s}}}$, which is in fact the solution of the homogeneous scaling part of Eq. (F.1). Finally the ansatz reads

$$f(x) = \begin{cases} \displaystyle\sum_{i=0}^{N_a} a_i\, T_i\left(\frac{2x}{x_\mathrm{M}} - 1\right), & x \leq x_\mathrm{M}, \\ \displaystyle x^{d/d_{x,\mathrm{s}}} \sum_{i=0}^{N_b} b_i\, R_i\big(x - x_\mathrm{M}\big), & x \geq x_\mathrm{M}. \end{cases} \tag{F.2}$$

We thus convert the initial equation into an algebraic set of $N_a + N_b + 2$ equations that can be solved applying the collocation method, for example by choosing the roots of T_{N_a+1} and R_{N_b+1}. At the matching point x_M, the continuity of $f(x)$ and $f'(x)$ must be taken into account. The solutions presented in the following are obtained by choosing $x_\mathrm{M} = 2$. We have further examined that the results do not change once x_M is varied.

In Fig. F.1 (left panel), we compare the first derivative $f'(x)$ obtained from this pseudo-spectral method and the analytical solution derived from the ϕ^4-dominance approximation, see Eq. (3.66), for a fixed value of $g_s^2 = 10^{-4}$ and $\xi_2 = \hat{\lambda}_2^+$. The two solutions lie perfectly on top of each other within the numerical error. Moreover, the coefficients a_i and b_i exhibit an exponential decay with increasing N_a and N_b – and thus indicate an exponentially small error of the numerical solution – until the algorithm hits machine precision.

The pseudo-spectral method thus allows us to provide clear numerical evidence for the global existence of the CEL solution within the full non-linear flow equation in the one-loop approximation. To our knowledge, this is the first time that results about global stability have been obtained for the scalar potential of this model.

We emphasize that the expansion around the origin in Chebyshev polynomials is an expansion over a set of basis functions that are in C^∞. Unfortunately, they do not form a suitable basis for the new QFP solutions parametrized by $C_f(\xi_2)$ as in Eq. (3.80), because of the presence of the log-type singularity at the origin. Naively applying the same pseudo-spectral methods to this case does, in fact, not lead to numerically stable results.

F.2. Shooting Method

Let us therefore use the shooting method that allows to deal with the presence of the log-singularity to some extent. For this, we integrate Eq. (F.1) starting from the minimum x_0 towards both the origin and infinity. The boundary conditions that have to be numerically fulfilled are precisely the definition of the minimum and the quartic self-interaction coupling

$$f'(x_0) = 0, \qquad\qquad f''(x_0) = \xi_2. \tag{F.3}$$

The set of parameters is (x_0, ξ_2, g_s^2). For the present type of equations, it is well known that the integration outwards $x \to +\infty$ is spoiled by the presence of a movable singularity s_+ [196, 210, 236, 257, 271, 272]. Here, the solution from shooting develops a peak of maximum value of $x = s_+$ only for a particular choice of the initial parameters. This peak is indicative for the solution that can be continuously integrated to asymptotic field values. In the (x_0, ξ_2, g_s^2) space, we therefore have a surface that can be parametrized, for example, by $x_0(\xi_2, g_s^2)$. In the ϕ^4-dominance approximation, we have seen that the leading contribution in g_s^2 to the nontrivial minimum x_0 is given by Eq. (3.79) for $P = 1/2$. Figure F.1 (right panel) shows how the full numerical solution converges to the analytical one in the $g_s^2 \to 0$ limit for the fixed value of $\xi_2 = 4$. Repeating the numerical analysis for different values of $\xi_2 > 8/9$, we find a similar agreement with the analytic solution in all studied cases.

Additionally, we have also seen in Sec. 3.4 that, the family of solutions with a nontrivial minimum are singular at the origin from the second derivative on. Very close to the origin this fixed singularity in $f''(x)$ may spoil standard integration algorithms and the numerical integration stops at some s_- value. This kind of feature has been studied also for non-Abelian Higgs model [112]. In principle, these singularities in higher derivatives could contradict asymptotic freedom if they persisted in the $g_s^2 \to 0$ limit. To verify that this is not the case, we first analyze the behavior of $f''(x)$ close to the origin and compare it to the analytic one. From Eq. (3.66), we know that the term responsible for the fixed singularity is the scaling

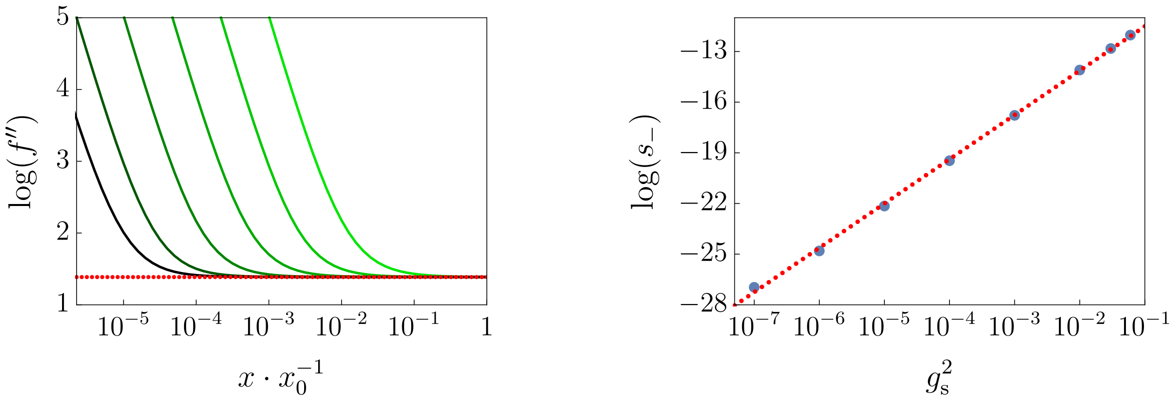


Fig. F.2.: *Left Panel:* $\log(f'')$ as a function of x/x_0 for $P = 1/2$ and fixed value of $\xi_2 = 4$. Solid green lines represent the numerical solutions from the shooting from x_0 for different values of the strong gauge coupling: $g_s^2 \in \{10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}\}$ from the darker (left) to the lighter (right). The dotted red line corresponds to the analytic solution in the ϕ^4-dominance approximation, cf., Eq. (3.66). *Right Panel:* Measure of the width of the singular region due to the presence of the fixed singularity in $x = 0$ as a function of g_s^2. The width is estimated by the criteria $f''(s_-) = 4$. For fixed $\xi_2 = 4$ we have found that the numerical points are well approximated by the power law (dotted red line) $s_- \sim g_s^{2a}$ where $a \simeq 1.13961$.

one $C_f x^{4/(2+\eta_{x,s})}$. Indeed, taking the log of the second derivative gives

$$\log f''(x) \underset{x\to 0}{\sim} -\frac{2\eta_{x,s}}{2+\eta_{x,s}} \log x. \tag{F.4}$$

In the left panel of Fig. F.2, we depict how the numerical solutions (green solid lines) deviate from this analytic one (dotted red line) close to the origin and for different values of g_s^2 at fixed $\xi_2 = 4$. This plot shows that the region of discrepancy progressively shrinks as g_s^2 gets smaller and smaller: indeed for smaller values of g_s^2 the point where the numerical solution deviates from the analytic one moves also towards smaller values. To measure this region, we have determined the onset of the singularity close to the origin as a function of g_s^2. Following the same idea as in [112], the criteria is to compute the position of s_- where $f''(s_-)$ assumes a sufficiently large value, let us say $\log f''(s_-) = 4$. An estimate of s_- is shown in the right panel of Fig. F.2 where a fit of the data confirms that the singular region shrinks to zero for $g_s^2 \to 0$. In fact, we have found a power law $s_- \sim g_s^{2a}$ with $a \simeq 1.13961$ for the present model.

We conclude that the existence of the new solutions is confirmed with the shooting method. We find satisfactory qualitative agreement with the solutions identified in the ϕ^4-dominance approximation, which are singular at the origin and show a nontrivial minimum.

Appendix G.

Subtleties on the Weak-coupling Expansion

We have already anticipated in the main text that, for certain values of the rescale power P, the beta function of the top-Yukawa coupling cannot be approximated with its DER limit. We finally give here an exhaustively explanation about this circumstance for the general $\mathrm{SU}(2)_\mathrm{L}\times\mathrm{SU}(3)_\mathrm{c}$ model, which comprises also the $\mathbb{Z}_2$-Yukawa-QCD model as a special limiting case.

G.1. P < 1/4

At the end of Sec. 4.4.1 we have observed that the UV behavior of the top-Yukawa coupling might change for $P \leq 1/4$, due to leading terms proportional to g_s^{8P} in the scalar anomalous dimension η_H. Even though this fact does not modify the QFP solution for the rescaled potential $f(x)$, the persistence of the QFP value of the top-Yukawa coupling is an important consistency check of our construction and approximations.

Let us consider the QFP solution of $f(x)$ for $P < 1/2$, as given in Eq. (4.95), and let us substitute it into the RG flow equation for h^2, cf. Eq. (4.31). By weak-coupling expanding the beta function, we recover the same expression as for the DER (cf. Eq. (4.4)), plus an extra term proportional to g_s^{8P}

$$\partial_t h^2 = \partial_t h^2\Big|_\mathrm{DER} + \frac{9(\mathcal{A}_\theta + \mathcal{A}_\mathrm{H})\xi_2}{8\pi^2} h^2 g_\mathrm{s}^{8P}. \tag{G.1}$$

The extra term arises from the scalar threshold functions $m_2^{(\mathrm{H})d}$ and $m_2^{(\theta)d}$ present in η_H, cf. Eq. (4.33), and contributes only in the SSB regime where $x_0 \neq 0$.

For $P = 1/4$, the extra term is of the same leading order as for the DER limit of $\partial_t h^2$. Therefore the QFP value for the ratio among the top-Yukawa coupling and the strong-gauge coupling becomes x_0-dependent and reads

$$\hat{h}_*^2 = \frac{1}{18}\left[4 - 12(\mathcal{A}_\theta + \mathcal{A}_{\mathrm{H}})\xi_2 + 9\hat{g}_*^2\right], \tag{G.2}$$

where $\hat{g}_*^2$ is given by Eq. (4.14). Moreover, the QFP value stays positive as long as

$$\mathcal{A}_\theta + \mathcal{A}_{\mathrm{H}} < \frac{4 + 9\hat{g}_*^2}{12\xi_2}. \tag{G.3}$$

The situation is different for $P < 1/4$ where the last term in Eq. (G.1) becomes leading. In order to capture the (in)existence of possible different scaling solutions for the top-Yukawa coupling with respect to the strong-gauge coupling, we look for QFP solutions for the ratio

$$\hat{h}^2 = \frac{h^2}{g_{\mathrm{s}}^{2E}} \tag{G.4}$$

with $E > 0$. With such a rescaling, the only possible QFP value is

$$\hat{h}_*^2 = -\frac{2\xi_2}{3}(\mathcal{A}_\theta + \mathcal{A}_{\mathrm{H}}), \qquad E = 4P. \tag{G.5}$$

For a positive nontrivial minimum, it is true that $\mathcal{A}_\theta + \mathcal{A}_{\mathrm{H}} > 0$ (see Eq. (4.96)), therefore the latter solution is unphysical. In other words, the presence of a nontrivial minimum for the scalar potential prevents the existence of scaling solutions for the top-Yukawa coupling for all $P < 1/4$. Nevertheless, scaling solutions for $f(x)$ do exist also for all $P < 1/4$ and do not depend on the asymptotic behavior of the top-Yukawa coupling.

G.2. P > 1

It is worthwhile to also discuss the possibility of new AF trajectories for the case $P > 1$. Previous studies found out that for the non-Abelian Higgs model such solutions indeed exist [111, 112], whereas no valid solutions have been found in the $\mathbb{Z}_2$-Yukawa-QCD model, as mentioned at the end of App. E. Since the $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{c}}$ model interpolates between the two limiting cases, a search for their scheme-independent (in)existence is particularly instructive. Our results confirm their existence in non-Abelian Higgs models as a special limiting case, whereas the general model does not feature the same mechanism.

For $P > 1$, the arguments z_{F} and z_{W} defined in Eq. (4.86) diverge in the $\{g_{\mathrm{s}}^2, g^2\} \to 0$ limit. Therefore, in order to capture the correct UV behavior, we are led to Taylor expand the threshold functions $l_0^{(\mathrm{F})}(z_{\mathrm{F}})$ and $l_0^{(\mathrm{W})}(z_{\mathrm{W}})$ about $z_{\mathrm{F,W}} = \infty$. Let us define the new scheme-dependent coefficients

$$\mathcal{B}_\Phi = \frac{1}{16\pi^2}\left[\partial_{(z^{-1})} l_0^{(\Phi)}(z)\right]_{(z)^{-1}=0} = \frac{1}{2k^6}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\tilde{\partial}_t P_\Phi(p^2), \qquad \Phi \in \{\mathrm{F}, \mathrm{W}\}, \tag{G.6}$$

such that $\mathcal{B}_\Phi > 0$ for general RG schemes providing an IR regularization. For instance, the piecewise linear regulator yields the positive value $\mathcal{B}_\Phi = 1/(32\pi^2)$. On the other hand, the bosonic thresholds associated to the radial Higgs fluctuation and the three Goldstone modes are always subleading in the UV for $P > 1$. Moreover, the anomalous dimension $\eta_{x,\mathrm{s}}$ can contribute to the LO in the beta function of $f(x)$ for these values of P. For example, it has been observed in the non-Abelian Higgs model that η_W becomes leading for $P > 2$, since it becomes proportional to g^2 [112]. The same conclusion holds also for the $\mathbb{Z}_2$-Yukawa-QCD model: the anomalous dimension η_G, being proportional to g_s^2, contributes to leading order for $P > 2$. In order to discuss also the possibility to have an anomalous dimension for the rescaled field, we solve the QFP differential equation $\partial_t f = 0$ retaining $\eta_{x,\mathrm{s}}$ as a parameter which becomes arbitrarily small in the UV limit.

Let us start by investigating the general $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{c}$ model. By keeping the terms linear in $(z_{\mathrm{F,W}})^{-1}$, the beta function for the rescaled potential becomes

$$\partial_t f = -4f + (2+\eta_{x,\mathrm{s}})xf' + \left(\frac{3\mathcal{B}_\mathrm{W}}{\hat{g}_*^2} - \frac{2\mathcal{B}_\mathrm{F}}{\hat{h}_*^2}\right)\frac{6g_\mathrm{s}^{2P-2}}{x}. \tag{G.7}$$

The presence of a singular term at the origin induces a corresponding pole in the QFP solution which is indeed

$$f(x) = c\, x^{\frac{4}{2+\eta_{x,\mathrm{s}}}} + \left[\frac{3\mathcal{B}_\mathrm{W}}{\hat{g}_*^2} - \frac{2\mathcal{B}_\mathrm{F}}{\hat{h}_*^2}\right]\frac{6g_\mathrm{s}^{2P-2}}{(6+\eta_{x,\mathrm{s}})x}, \tag{G.8}$$

where c is some integration constant of the first-order ODE. Additionally, there is also a log-type singularity in the second derivative at the origin. In fact, by Taylor expanding the scaling term for small $\eta_{x,\mathrm{s}}$, we obtain a contribution proportional to $x^2 \log x$. This singularity can be avoided if the potential admits a nontrivial minimum x_0 such that $f'(x_0) = 0$. The latter condition can be solved for c, yielding a function $c(g_\mathrm{s}^2, x_0)$, and by substituting it into the definition of the rescaled quartic scalar coupling $\xi_2 = f''(x_0)$ provides the following expression

$$\xi_2 = \left[\frac{3\mathcal{B}_\mathrm{W}}{\hat{g}_*^2} - \frac{2\mathcal{B}_\mathrm{F}}{\hat{h}_*^2}\right]\frac{6g_\mathrm{s}^{2P-2}}{x_0^3(2+\eta_{x,\mathrm{s}})}. \tag{G.9}$$

We observe that the QFP potential has a nontrivial minimum for positive ξ_2 only if

$$\mathcal{B}_\mathrm{W} > \frac{2\mathcal{B}_\mathrm{F}\hat{g}_*^2}{3\hat{h}_*^2}. \tag{G.10}$$

The anomalous dimension $\eta_{x,\mathrm{s}}$ depends nontrivially on x_0, but the consistency criterion holds that $\eta_{x,\mathrm{s}} \to 0$ in the UV limit. From this property we can infer that, for any finite value of ξ_2, the nontrivial minimum scales as $x_0 \sim g_\mathrm{s}^{2(P-1)/3}$ in the $g_\mathrm{s}^2 \to 0$ limit. By substituting this behavior inside the definitions for $z_{\mathrm{F,W}}$, we observe that $z_{\mathrm{F,W}} \to \infty$ in the UV limit, as stated above.

For the non-Abelian Higgs model, there are no fermion fluctuations, so the right-hand side of Eq. (G.10) vanishes and the criterion is satisfied in any scheme. The evidence found in [111, 112], for the existence of new AF trajectories, is thus confirmed in a scheme-independent manner. By contrast, there are no weak gauge contributions in the $\mathbb{Z}_2$-Yukawa-QCD model, implying that the left-hand side of Eq. (G.10) is zero. Hence, the criterion cannot be satisfied. For the $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{c}}$ model, a diagonal choice of regulators ($\mathcal{B}_{\mathrm{W}} = \mathcal{B}_{\mathrm{F}}$) inside Eq. (G.9) would result in $\xi_2 < 0$ for SM matter content. It seems that the Eq. (G.10) still leaves room for legitimate models in the general case. However, this is not the case as detailed in the following.

In writing Eq. (G.7), we have also assumed that h^2 and g^2 scale with g_{s}^2. This is true in the DER but has to be verified outside this regime. For $P > 1$, the arguments z_{F} and z_{W} diverge in the UV limit, corresponding to a divergence of the gauge-boson and the top-quark masses. Physically, this means that they decouple from the theory and do not propagate. In Sec. 4.2, we have seen that the anomalous dimensions η_{G} and η_{W}, within the decoupled regime, are still proportional to g_{s}^2 and g^2 respectively. Therefore, the scaling $g^2 \sim g_{\mathrm{s}}^2$ is still valid also outside the DER, but the constant of proportionality depends on the number of decoupled degrees of freedom. For the SM case, the QFP value for $\hat{g}^2$ is given by Eq. (4.42).

By contrast, the beta function for the top-Yukawa coupling changes drastically beyond the DER. Thus the scaling $h^2 \sim g_{\mathrm{s}}^2$ might no longer be valid outside the DER. As an example, let us assume that $z_{\mathrm{F,W}} \to \infty$ and $\omega_{\mathrm{H}f} \to 0$ in the UV limit. Since we are looking for solutions with a nontrivial minimum, we can set $\omega_{\theta f} = 0$. By expanding Eq. (4.31) and retaining only the leading terms in g_{s}^2, the beta function for the rescaled top-Yukawa coupling, defined in Eq. (G.4), reduces to

$$\partial_t \hat{h}^2 \simeq 2 v_d g_{\mathrm{s}}^{2E} \hat{h}^4 + \hat{h}^2 \left[-\eta_{\mathrm{G}} E - v_d \left(\frac{7}{x_0} g_{\mathrm{s}}^{2P} - 24 x_0 \xi_2^2 g_{\mathrm{s}}^{6P} - \frac{36}{\hat{g}_*^2 x_0^2} g_{\mathrm{s}}^{4P-2} \right) \right] + \frac{v_d}{3 x_0^2} \left(3 g_{\mathrm{s}}^{4P-2E} - 16 x_0 g_{\mathrm{s}}^{2P-2E+2} \right). \tag{G.11}$$

Let us assume the nontrivial minimum to scale with a power $(P - Q)$ of the strong gauge coupling,

$$x_0 = g_{\mathrm{s}}^{2(P-Q)} \hat{\kappa}, \quad \text{with} \quad P \leq Q \quad \text{or} \quad P > Q. \tag{G.12}$$

A careful analysis among all the possible combinations between the scaling powers $P > 1$, $E > 0$, and Q, leads to the conclusion that one or more of the assumptions above are violated for any combination which allows to have a QFP solution for $\hat{h}^2$.

As an example, let us consider the case where $P = E = Q$. The rescaled top-Yukawa coupling has a QFP solution depending on the nontrivial minimum which reads

$$\hat{h}_*^2 = -\frac{16 g_{\mathrm{s}}^2}{3\hat{\kappa}\eta_{\mathrm{G}} P}. \tag{G.13}$$

However this solution is not compatible with our assumptions, since z_{F} would stay finite and does not diverge in the UV limit. All other cases can be analyzed analogously.

We conclude that, the general $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{c}}$ model as well as the $\mathbb{Z}_2$-Yukawa-QCD model do not feature new AF trajectories for any $P > 1$, differently from the non-Abelian Higgs model which possesses these solutions in any scheme covered by our analysis.

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Die geltende Promotionsordnung der Physikalisch-Astronomischen Fakultät ist mir bekannt.

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Ort, Datum
Jena, 12.11.2020

Alessandro Ugolotti