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TANGENTIALS IN CUBIC STRUCTURES

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Abstract. In this paper we study geometric concepts in a general cubic structure. The well-known relationships on the cubic curve motivate us to introduce new concepts into a general cubic structure. We will define the concept of the tangential of a point in a general cubic structure and we will study tangentials of higher-order. The characterization of this concept will be also given by means of the associated totally symmetric quasigroup. We will introduce the concept of associated and corresponding points in a cubic structure, and discuss the number of mutually different corresponding points. The properties of the introduced geometric concepts will be investigated in a general cubic structure.

The cubic structure abstracts the properties of many geometric models, the most famous of which is the geometry on a cubic curve. In this model the terms tangentials, corresponding points and associated points appear. There is an abundance of literature on this topic, and we will use the classic Durége's book [1]. The theory of cubic structures is closely related to the theory of totally symmetric medial quasigroups, which has been exhaustively studied by Etherington ([2]). In this paper, the corresponding concepts are defined and studied in a general cubic structure. Although some theorems in certain models of a cubic structure could be proved algebraically by applying TSM-quasigroups, geometric proofs directly in a cubic structure (which is actually a "geometric" structure) can give a better insight into interrelationships between the statements in this structure or in its particular model. In addition, such a study of certain concepts and properties remains "purely geometric."

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1. Introduction

The cubic structure is defined in [5]. Let Q be a nonempty set, whose elements are called points, and let $[\,] \subseteq Q^3$ be a ternary relation on Q. Such a relation and the ordered pair $(Q, [\,])$ will be called a *cubic relation* and a *cubic structure*, respectively, if the following properties are satisfied:

C1. For any two points $a, b \in Q$ there is a unique point $c \in Q$ such that $[a, b, c]$, i.e., $(a, b, c) \in [\,]$.
C2. The relation $[\,]$ is totally symmetric, i.e., $[a, b, c]$ implies $[a, c, b]$, $[b, a, c]$, $[b, c, a]$, $[c, a, b]$ and $[c, b, a]$.
C3. $[a, b, c]$, $[d, e, f]$, $[g, h, i]$, $[a, d, g]$ and $[b, e, h]$ imply $[c, f, i]$, which can be clearly written in the form of the following table:

$$\begin{array}{cc|c|} a & b & c \\ d & e & f \\ g & h & i \end{array}$$

Throughout the paper we will use the property C2 without mentioning it explicitly.

Let Q be a nonempty set and $\cdot$ a binary operation on Q. The ordered pair $(Q, \cdot)$ is a *quasigroup* if for each $a, b \in Q$ there exist unique elements x and y such that $ax = b$ and $ya = b$. The quasigroup $(Q, \cdot)$ is *medial* if the identity $ab \cdot cd = ac \cdot bd$ is valid, and *totally symmetric* if it satisfies the identities $ab \cdot b = a$, $a \cdot ab = b$, where, e.g., $ab \cdot cd$ is the shorter notation for $(a \cdot b) \cdot (c \cdot d)$. A totally symmetric medial quasigroup will be called a TSM-quasigroup for short.

The following statement is proved in [5, Theorem 1]. If the ternary relation $[\,]$ and the binary operation $\cdot$ on the set Q are connected by the equivalence

$$[a, b, c] \Leftrightarrow ab = c,$$

then $(Q, [\,])$ is a cubic structure if and only if $(Q, \cdot)$ is a TSM-quasigroup. The properties of TSM-quasigroups have been studied in detail in [2]. In [5], a number of geometric examples of cubic structures are listed, the most important of which is perhaps the one in Example 2.1. Let Q be the set of all nonsingular points of a planar cubic curve Γ, and for three given points $a, b, c \in Q$, let the statement $ab = c$ mean that the points a, b, and c lie on the same line. Then $(Q, [\,])$ is a cubic structure.

In this paper, the well-known relationships on the cubic curve Γ will motivate us to introduce new concepts into a general cubic structure. The obtained results can easily be applied to other examples of cubic structures in [5].

2. Tangentials of elements of cubic structures

From now on, let $(Q, [\ ])$ be any cubic structure whose elements will be called *points*, and the triples of points $[a, b, c]$ will be called *lines*. We shall say that the point a' is the *tangential* of the point a if the statement $[a, a, a']$ holds. It is obvious that each point has one and only one tangential a'. In the associated TSM-quasigroup $(Q, \cdot)$ tangential of element a is the element $a' = aa$. If the point a' is the tangential of the point a, then we will also say that the point a is an *antecedent* of the point a'.

Theorem 2.1. *If a', b', and c' are the tangentials of points a, b, and c, then $[a, b, c]$ implies $[a', b', c']$.*

Proof. The proof follows applying the table

$$\begin{array}{cc|c|}\cline{3-3} a & a & a' \\ b & b & b' \\ c & c & c' \\ \cline{3-3}\end{array}.$$

∎

Theorem 2.2. *Let a_1, a_2, and a_3 be any three points. Let for each $i, j \in \{1, 2, 3\}$, $i \neq j$, $[a_i, a_j, a_{ij}]$ holds (obviously $a_{ij} = a_{ji}$), and let for each $i \in \{1, 2, 3\}$, $j, k \in \{1, 2, 3\} \setminus \{i\}$, $j \neq k$, $[a_{ij}, a_{ik}, b_i]$ holds. Then for each $i \in \{1, 2, 3\}$, $j, k \in \{1, 2, 3\} \setminus \{i\}, j \neq k$, $[a_{jk}, b_i, a_i']$ holds, where a_i' is the tangential of the point a_i.*

Proof.

$$\begin{array}{cc|c|}\cline{3-3} a_i & a_i & a_i' \\ a_j & a_k & a_{jk} \\ a_{ij} & a_{ik} & b_i \\ \cline{3-3}\end{array}.$$

∎

Theorem 2.3. *If a', b', and c' are the tangentials of the points a, b, and c, respectively, then $[b, c, a']$ and $[c, a, b']$ imply $[a, b, c']$.*

Proof.

$$\begin{array}{cc|c|}\cline{3-3} a' & a & a \\ b & b' & b \\ c & c & c' \\ \cline{3-3}\end{array}.$$

∎

Theorem 2.4. *If for the tangentials a', b', and c' of the points a, b, and c, $[a', b', c']$ holds true and if d, e, and f are points such that $[b, c, d]$, $[c, a, e]$ and $[a, b, f]$ then $[d, e, f]$ holds.*

Proof. Apply the following tables in succession

$$\begin{array}{cc|c|}\cline{3-3} a & b & f \\ a & b & f \\ a' & b' & c' \\ \cline{3-3}\end{array} \qquad \begin{array}{cc|c|}\cline{3-3} b & c & d \\ a & c & e \\ f & c' & f \\ \cline{3-3}\end{array}.$$

∎

For any integer n greater than 1, we define the n-th tangential of a point as the tangential of its $(n-1)$-tangential, with the first tangential of the point a being its tangential a'.

THEOREM 2.5. *If a' and a'' are the first and the second tangential of the point a, then $[a,b,c]$, $[a,d,e]$ and $[b,d,a']$ imply $[c,e,a'']$.*

PROOF.

$$\begin{array}{cc|c|} \hline a & b & c \\ a & d & e \\ a' & a' & a'' \\ \hline \end{array}.$$

□

THEOREM 2.6. *If b, c, and d are the first, second and the third tangential of the point a and if $[a,c,e]$, then $[b,d,e]$ is equivalent to the fact that the point a is the tangential of the point d, i.e., the point a itself is its fourth tangential.*

PROOF. Assuming $[b,d,e]$, then $[d,d,a]$ follows by applying the first table below, and assuming $[d,d,a]$ then $[b,d,e]$ follows from the second table

$$\begin{array}{cc|c|} \hline c & c & d \\ b & e & d \\ b & a & a \\ \hline \end{array} \qquad \begin{array}{cc|c|} \hline b & c & b \\ a & d & d \\ a & c & e \\ \hline \end{array}.$$

□

3. CORRESPONDING POINTS IN THE CUBIC STRUCTURE

Two points are said to be *corresponding* if they have the common tangential.

THEOREM 3.1. *Let a_1 and a_2 be corresponding elements with the common tangential a', o be any point, and let b_1, b_2 be points such that $[o,a_1,b_1]$ and $[o,a_2,b_2]$. Then b_1 and b_2 are corresponding points with the common tangential b' such that $[o',a',b']$, where o' is the tangential of the point o. In addition, there is a point c such that $[a_1,b_2,c]$ and $[a_2,b_1,c]$ hold and points o and c are corresponding.*

PROOF. Let b' be the point such that $[o',a',b']$. From the tables

$$\begin{array}{cc|c|} \hline o & a_1 & b_1 \\ o & a_1 & b_1 \\ o' & a' & b' \\ \hline \end{array} \qquad \begin{array}{cc|c|} \hline o & a_2 & b_2 \\ o & a_2 & b_2 \\ o' & a' & b' \\ \hline \end{array}$$

it follows that the point b' is the common tangential of points b_1 and b_2. Now let c be the point such that $[a_1,b_2,c]$ and let o' be the tangential of the point o. Then from the tables

$$\begin{array}{cc|c|} \hline o & b_2 & a_2 \\ b_1 & b' & b_1 \\ a_1 & b_2 & c \\ \hline \end{array} \qquad \begin{array}{cc|c|} \hline a_1 & b_2 & c \\ b_1 & a_2 & c \\ o & o & o' \\ \hline \end{array}$$

we acquire $[a_2, b_1, c]$, whence it follows that the point c has the tangential o' so the points o and c are corresponding. ∎

THEOREM 3.2. *If $[o, a_1, b_1]$ and $[o, a_2, b_2]$, and if there is a point c such that $[a_1, b_2, c]$ and $[a_2, b_1, c]$, then a_1, a_2 and b_1, b_2 are pairs of corresponding points.*

PROOF. Let a' and b' be the tangentials of points a_1 and b_1. From the tables

$$\begin{array}{cc|c|} \hline b_1 & c & a_2 \\ o & b_2 & a_2 \\ a_1 & a_1 & a' \\ \hline \end{array} \qquad \begin{array}{cc|c|} \hline a_1 & c & b_2 \\ o & a_2 & b_2 \\ b_1 & b_1 & b' \\ \hline \end{array}$$

it follows that points a_2 and b_2 also have tangentials a' and b', respectively; therefore, a_1, a_2 and b_1, b_2 are pairs of corresponding points. ∎

THEOREM 3.3. *If a_1 and a_2 are corresponding points with common tangential a', then the points a' and b are also corresponding, where b is the point such that $[a_1, a_2, b]$.*

PROOF. Let a'' be the tangential of the point a'. From the table

$$\begin{array}{cc|c|} \hline a_1 & a_2 & b \\ a_1 & a_2 & b \\ a' & a' & a'' \\ \hline \end{array}$$

we obtain that the point b has the tangential a'', so points a' and b are corresponding. ∎

THEOREM 3.4. *If $[a, b, c]$, $[a, e, f]$, $[b, f, d]$, and $[c, d, e]$, then a, d; b, e and c, f are pairs of corresponding points, and for the associated tangentials a', b', and c', $[a', b', c']$ holds true.*

PROOF. From the tables:

$$\begin{array}{cc|c|} \hline f & b & d \\ e & c & d \\ a & a & a' \\ \hline \end{array} \qquad \begin{array}{cc|c|} \hline f & a & e \\ d & c & e \\ b & b & b' \\ \hline \end{array} \qquad \begin{array}{cc|c|} \hline d & b & f \\ e & a & f \\ c & c & c' \\ \hline \end{array}$$

it follows that points d, e, and f have the tangentials a', b', and c', respectively, and therefore a, d; b, e and c, f are pairs of corresponding points. By Theorem 2.1, $[a, b, c]$ implies $[a', b', c']$. ∎

THEOREM 3.5. *If $[a, e, f]$, $[b, f, d]$, and $[c, d, e]$, and if a and d are corresponding points, then $[a, b, c]$ holds.*

Proof. Let a' be the common tangential of the points a and d. The assertion of the theorem follows from the table

$$\begin{array}{cc|c|} a & a' & a \\ f & d & b \\ e & d & c \end{array}.$$

∎

Theorem 3.6. *If the tangentials a', b', and c' of a, b, and c satisfy $[a', b', c']$, and if $[b, c, d]$, $[c, a, e]$ and $[a, b, f]$, then a, d; b, e and c, f are pairs of corresponding points and $[d, e, f]$ holds.*

Proof. From the tables:

$$\begin{array}{cc|c|} b & c & d \\ b & c & d \\ b' & c' & a' \end{array} \qquad \begin{array}{cc|c|} c & a & e \\ c & a & e \\ c' & a' & b' \end{array} \qquad \begin{array}{cc|c|} a & b & f \\ a & b & f \\ a' & b' & c' \end{array}$$

it follows that the points d, e, and f have the tangentials a', b', and c', respectively, so a, d; b, e and c, f are pairs of corresponding points. Now, the table

$$\begin{array}{cc|c|} b & c & d \\ a & c & e \\ f & c' & f \end{array}$$

proves $[d, e, f]$. ∎

In Theorems 3.4, 3.5 and 3.6, sextuples of the form a, b, c, d, e, f appear with the property that $[a, b, c]$, $[a, e, f]$, $[b, f, d]$ and $[c, d, e]$ hold. We say that a, d; b, e and c, f are pairs of *opposite vertices* of a *complete quadrilateral*.

In Theorems 3.1 and 3.2, a_1, a_2; b_1, b_2 and o, c are pairs of opposite vertices of a complete quadrilateral. From Theorem 3.4 we now get the following result.

Corollary 3.7. *The pairs of opposite vertices of a complete quadrilateral are pairs of corresponding points.*

Theorem 3.8. *If a, d is a pair of corresponding points and b is any point, then there are points c, e, and f such that a, d; b, e and c, f are pairs of opposite vertices of a complete quadrilateral, i.e., $[a, b, c]$, $[a, e, f]$, $[b, f, d]$ and $[c, d, e]$ hold.*

Proof. Let a' be the common tangential of the points a and d and let c, f, e be the points such that $[a, b, c]$, $[b, d, f]$ and $[a, f, e]$. Then $[c, d, e]$ follows from the table

$$\begin{array}{cc|c|} a & b & c \\ a' & d & d \\ a & f & e \end{array}.$$

∎

THEOREM 3.9. *If a, d; b_0, e_0 and c_0, f_0 are pairs of opposite vertices of a complete quadrilateral and if b is any point, then there are points c, e, and f such that a, d; b, e and c, f are pairs of opposite vertices of a complete quadrilateral.*

PROOF. By Corollary 3.7, points a and d are corresponding, and then the statement of the theorem follows from Theorem 3.8. ∎

THEOREM 3.10. *If a_1, a_2, a_3 are pairwise corresponding and if o and a_4 are points such that $[a_2, a_3, o]$ and $[a_1, o, a_4]$, then the point a_4 is also corresponding to each of the points a_1, a_2, and a_3.*

PROOF. Let a' be the common tangential of points a_1, a_2, and a_3 and let p and q be points such that $[a_1, a_2, p]$ and $[a_1, a_3, q]$. From the tables:

$$\begin{array}{cc|c|} a' & a_3 & a_3 \\ a_1 & o & a_4 \\ a_1 & a_2 & p \end{array} \qquad \begin{array}{cc|c|} a' & a_2 & a_2 \\ a_1 & o & a_4 \\ a_1 & a_3 & q \end{array}$$

we obtain $[a_3, a_4, p]$ and $[a_2, a_4, q]$. Then $[a', a_4, a_4]$ follows from the table

$$\begin{array}{cc|c|} a_1 & a_1 & a' \\ a_2 & q & a_4 \\ p & a_3 & a_4 \end{array}$$

and the point a_4 has also the tangential a'. ∎

COROLLARY 3.11. *If a point has at least three different antecedents, then it has at least four different antecedents.*

THEOREM 3.12. *Let $a_1, \ldots, a_n$ be mutually different points which are pairwise corresponding and have the common tangential a', let o be any point, and let $b_1, \ldots, b_n$ be points such that $[o, a_i, b_i]$, for $i = 1, \ldots, n$. Then $b_1, \ldots, b_n$ are mutually different pairwise corresponding points with the common tangential b' such that $[o', a', b']$, where o' is the tangential of the point o.*

PROOF. From $[o, a_i, b_i]$ and $[o, a_j, b_j]$, $a_i \neq a_j$, by C1, it follows that $b_i \neq b_j$. Other claims follow from the first assertion of Theorem 3.1. ∎

In the case of a cubic structure from the example with collinearity on the set of regular points of a cubic, the statements of several previous theorems can be found in the classic books [4] and [1].

4. ASSOCIATED POINTS IN THE CUBIC STRUCTURE

In the conditions of Theorem 3.12, the fact that the point a' has n different antecedents implies that the point b' has at least n different antecedents. Replacing the role of points a' and b', it follows that these points have an equal number of different antecedents. What about the number of possible different antecedents of individual points, i.e., the number of mutually different corresponding points in a cubic structure?

A third order plane curve can have a degree equal to 3, 4 or 6, i.e., from any point P of that plane 3, 4 or 6 tangents can be drawn to that curve. If the point P is on that curve, then besides the tangent at the very point P, which is counted as two tangents, we have 1, 2, or 4 remaining tangents to that curve from the point P. Therefore, the point of a cubic is tangential for one, two, or four other points of that curve.

In [2] Etherington proved that in general, in any TSM quasigroup, if the maximum number of elements having the common tangential is finite, then it is of the form 2^m, with a constant number $m \in \mathbf{N} \cup \{0\}$, and each element of that quasigroup has exactly that many antecedents. This means that in each cubic structure a maximum number of mutually different corresponding points is of the form $n = 2^m$ with a constant number $m \in \mathbf{N} \cup \{0\}$, and that each point has that many antecedents. In such a case, we shall say that mutually different points $a_1, \ldots, a_n$ with the common tangential are *associated*. The number m is called the *rank* of the observed cubic structure $(Q, [\ ])$. From Theorem 3.12 the corollary immediately follows.

Corollary 4.1. *If a_1, a_2, a_3, and a_4 are associated points with the common tangential a', and o is any point, and b_1, b_2, b_3, b_4 are points such that $[o, a_i, b_i]$, $i = 1, 2, 3, 4$, then b_1, b_2, b_3, and b_4 are associated points with the common tangential b' such that $[o', a', b']$, where o' is the tangential of the point o.*

The properties of associated points of rank $m = 1$, i.e., only pairs of points are associated, are covered by theorems proved in the previous section on corresponding points and other claims of the same form. Now we will prove several statements for rank $m = 2$, that is, when we have four associated points in a cubic structure. These statements are obtained by generalizing the properties of associated points on the sixth degree cubic curve. Many of such properties can be found in [1]. Cases of ranks $m \geq 3$ could be very interesting for future study, although we do not have specific geometric examples for them.

Theorem 4.2. *If a_1, a_2, a_3, and a_4 are associated points with the common first tangential a' and the second tangential a'', and if p and q are points such that $[a_1, a_2, p]$ and $[a_3, a_4, q]$ hold, and b is the point such that $[p, q, b]$, then a'' and b are corresponding points.*

Proof. From the tables:

$$\begin{array}{cc|c|} a_1 & a_2 & p \\ a_1 & a_2 & p \\ a' & a' & a'' \end{array} \qquad \begin{array}{cc|c|} a_3 & a_4 & q \\ a_3 & a_4 & q \\ a' & a' & a'' \end{array}$$

it follows that the points p and q have the common tangential a''. Let a''' be the tangential of the point a''. Then from the table

$$\begin{array}{cc|c|} p & q & b \\ p & q & b \\ a'' & a'' & a''' \end{array}$$

we get that the point b has the tangential a''', i.e., the points a'' and b are corresponding. □

THEOREM 4.3. *If a_1, a_2, a_3, and a_4 are associated points with the common tangential a', then there exist points p, q, and r such that $[a_1, a_2, p]$, $[a_3, a_4, p]$, $[a_1, a_3, q]$, $[a_2, a_4, q]$, $[a_1, a_4, r]$ and $[a_2, a_3, r]$ and the points a', p, q, and r are associated.*

PROOF. Let p, q, and r be points such that $[a_1, a_2, p]$, $[a_1, a_3, q]$ and $[a_1, a_4, r]$ hold. From the mutual difference of points a_2, a_3, and a_4, according to C1, the points p, q, and r are also mutually different. As the pairs of points a_1, a_2; a_1, a_3 and a_1, a_4 are corresponding, the first assertion of Theorem 3.1 implies that each of the points p, q, and r is corresponding with the point a'. Because of the correspondence of points a_2 and a_3, and $[a_1, a_2, p]$, $[a_1, a_3, q]$, according to the second statement of Theorem 3.1, there is a point o such that $[a_2, q, o]$, $[a_3, p, o]$ and that the points a_1 and o are corresponding. If $o = a_1$, then we would have $[a_1, a_2, p]$ and $[a_1, a_3, p]$, $a_2 \neq a_3$, which is impossible by C1. If $o = a_2$, then we would have $[a_1, a_2, p]$ and $[a_2, a_3, p]$, $a_1 \neq a_3$, which is again impossible. If $o = a_3$, then we would have $[a_1, a_3, q]$ and $[a_2, a_3, q]$, $a_1 \neq a_2$, which is impossible, too. All we have left is the possibility that $o = a_4$, and then we get $[a_3, a_4, p]$ and $[a_2, a_4, q]$. From the table

$$\begin{array}{cc|c|} a_2 & a' & a_2 \\ p & a_4 & a_3 \\ a_1 & a_4 & r \end{array}$$

follows $[a_2, a_3, r]$. It only remains to show that the points p, q, and r are different from the point a'. However, by C1, this follows from the difference of points a_2, a_3, and a_4 from the point a_1, and comparing $[a_1, a_2, p]$, $[a_1, a_3, q]$, $[a_1, a_4, r]$ with $[a_1, a_1, a']$. □

THEOREM 4.4. *If $[a', b', c']$ holds and a and b are antecedents of the points a' and b', respectively, and if c is the point such that $[a, b, c]$, then c is an antecedent of the point c'.*

PROOF. The points a' and b' are tangentials of points a and b. Let c_t be the tangential of the point c. Theorem 2.1 implies $[a', b', c_t]$ from $[a, b, c]$, which, together with $[a', b', c']$, yields $c_t = c'$, i.e., c' is the tangential of the point c. □

THEOREM 4.5. *Let* $[a', b', c']$ *hold, where* a', b' *and* c' *are different points. All different antecedents of points* a', b' *and* c' *can be denoted by* a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4 *and* c_1, c_2, c_3, c_4, *so that the following hold:*

$$\begin{array}{llll} [a_1,b_1,c_1], & [a_1,b_2,c_2], & [a_1,b_3,c_3], & [a_1,b_4,c_4], \\ [a_2,b_1,c_2], & [a_2,b_2,c_1], & [a_2,b_3,c_4], & [a_2,b_4,c_3], \\ [a_3,b_1,c_3], & [a_3,b_2,c_4], & [a_3,b_3,c_1], & [a_3,b_4,c_2], \\ [a_4,b_1,c_4], & [a_4,b_2,c_3], & [a_4,b_3,c_2], & [a_4,b_4,c_1]. \end{array}$$

PROOF. For each of the points a_1, a_2, a_3, a_4 and each of the points b_1, b_2, b_3, b_4 there is a line containing them, and thus we obtain 16 lines. On each of them, by Theorem 4.4, there are unique points c_1, c_2, c_3, and c_4 lying on these lines, and each of these four points lies on four such lines. We can select the indices of points c_1, c_2, c_3, and c_4 so that we have lines $[a_1, b_1, c_1]$, $[a_1, b_2, c_2]$, $[a_1, b_3, c_3]$ and $[a_1, b_4, c_4]$, where we have the option of choosing indices for points a_2, a_3, and a_4 freely. Since the points b_1, b_2 are corresponding and since we have lines $[a_1, b_1, c_1]$, $[a_1, b_2, c_2]$, then, by the second assertion of Theorem 3.1, there is a point corresponding to the point a_1, which completes the pairs b_1, c_2 and b_2, c_1 to lines. It cannot be the point a_1 and we denote that point by a_2. So we have the lines $[a_2, b_1, c_2]$ and $[a_2, b_2, c_1]$.

As the points b_1 and b_3 are corresponding and we have lines $[a_1, b_1, c_1]$, $[a_1, b_3, c_3]$, for the same reason, there is a point corresponding to the point a_1, which completes the pairs b_1, c_3 and b_3, c_1 to lines. This cannot be the point a_1 nor a_2, so let us denote it by a_3. Therefore we have the lines $[a_3, b_1, c_3]$ and $[a_3, b_3, c_1]$. In the same way, we conclude that the remaining point a_4, corresponding to a_1, belongs to the lines $[a_4, b_1, c_4]$ and $[a_4, b_4, c_1]$. As there are already lines $[a_2, b_2, c_1]$, $[a_2, b_1, c_2]$, $[a_1, b_3, c_3]$, the points a_2 and b_3 cannot be complemented by any of the points c_1, c_2, c_3, so we necessarily have the line $[a_2, b_3, c_4]$. From the existence of the lines $[a_2, b_1, c_2]$, $[a_2, b_2, c_1]$ and $[a_2, b_3, c_4]$ follows the existence of the line $[a_2, b_4, c_3]$. From the existence of the lines $[a_1, b_3, c_3]$, $[a_2, b_3, c_4]$ and $[a_3, b_3, c_1]$ we conclude that there is also the line $[a_4, b_3, c_2]$, and from the existence of lines $[a_1, b_4, c_4]$, $[a_2, b_4, c_3]$ and $[a_4, b_4, c_1]$, it follows that there is also the line $[a_3, b_4, c_2]$. Finally, as there are lines $[a_3, b_1, c_3]$, $[a_3, b_3, c_1]$ and $[a_3, b_4, c_2]$, there is also the line $[a_3, b_2, c_4]$, and as there are lines $[a_4, b_1.c_4]$, $[a_4, b_3, c_2]$ and $[a_4, b_4, c_1]$, there is also the line $[a_4, b_2, c_3]$. ∎

The proof of this theorem is essentially transcribed from pages 212–213 in Durége's book [1]. In [3] Hesse discovered a configuration of the type $(12_4, 16_3)$ of points and lines, which is today (obviously wrongly) called the Reye's configuration. As for the notation in the preceding theorem, it can be observed that for an arrangement of indices for the 16 obtained lines, the rule is that if any two indices are equal, then the third index is necessarily equal to 1, and if two indices are different from each other and different from 1, then all three indices are different from each other and different from 1.

THEOREM 4.6. *If a_1, a_2, a_3, and a_4 are associated points, i.e., different antecedents of a point a', and b_1, b_2, b_3, b_4 are different antecedents of the point a_1, then the indices of points a_2, a_3, and a_4 can be chosen such that*

$$[b_1, b_2, a_2], [b_3, b_4, a_2], [b_1, b_3, a_3], [b_2, b_4, a_3], [b_1, b_4, a_4], [b_2, b_3, a_4]. \tag{4.1}$$

PROOF. Let c be the point such that $[b_1, b_2, c]$, and then let d be the point such that $[b_3, c, d]$. By Theorem 3.1, it follows from $[b_1, b_2, c]$ and $[b_3, d, c]$ that b_3 and d are different corresponding points. The point d is different from points b_1 and b_2 because otherwise we would have $[b_3, b_1, c]$ or $[b_3, b_2, c]$, which is not possible by C1 since we already have $[b_1, b_2, c]$. Therefore, it is necessary that $d = b_4$, so we have $[b_3, b_4, c]$, that is, we have proved that there is a point c such that $[b_1, b_2, c]$ and $[b_3, b_4, c]$. Similarly, it can be proved that there are points e and f such that $[b_1, b_3, e]$, $[b_2, b_4, e]$ and $[b_1, b_4, f]$, $[b_2, b_3, f]$. From the tables:

$$\begin{array}{cc|c|} b_1 & b_2 & c \\ b_1 & b_2 & c \\ a_1 & a_1 & a' \end{array} \qquad \begin{array}{cc|c|} b_1 & b_3 & e \\ b_1 & b_3 & e \\ a_1 & a_1 & a' \end{array} \qquad \begin{array}{cc|c|} b_1 & b_4 & f \\ b_1 & b_4 & f \\ a_1 & a_1 & a' \end{array}$$

we obtain that points c, e, and f are corresponding to the point a_1. Owing to C1, points c, e, and f are mutually different because we have $[b_1, b_2, c]$, $[b_1, b_3, e]$ and $[b_1, b_4, f]$, and points b_2, b_3, and b_4 are different. Points c, e, and f are different from the point a_1 because otherwise we would have one of the statements $[a_1, b_1, b_2]$, $[a_1, b_1, b_3]$ or $[a_1, b_1, b_4]$, which is impossible by C1 because we have $[a_1, b_1, b_1]$. Accordingly, a_1, c, e, and f are mutually different corresponding points, and consequently, points c, e, f can be designated in the sequence as a_2, a_3, a_4. This proves the theorem. ∎

THEOREM 4.7. *Suppose $[a', b', c']$ holds, where a', b', c' are different points and let a, b, c be some of the antecedents of points a', b', c' such that $[a, b, c]$ is not valid. If d, e, and f are points such that $[b, c, d]$, $[c, a, e]$ and $[a, b, f]$, then $[d, e, f]$ holds and a, d; b, e; c, f are pairs of corresponding points.*

PROOF. Let a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4 be all different antecedents of the points a', b', c', respectively. By using Theorem 4.4, we conclude that the points a and d are some of the points a_1, a_2, a_3, and a_4, points b, e are some of the points b_1, b_2, b_3, b_4, and points c, f are some of the points c_1, c_2, c_3, c_4. Because of the rule stated after Theorem 4.5 about the arrangement of indices in the statements of that theorem, and since $[a, b, c]$ is not valid, it follows that the triple a, b, c has to be one of these four:

$$a_1, b_1, c_i;\ a_1, b_i, c_j;\ a_i, b_i, c_i;\ a_i, b_i, c_j,$$

or the triples derived from these by permuting of the letters a, b, c, which, without loss of generality, needs not to be studied. Hereafter, in this proof (i, j, k)

is always some permutation of $(2, 3, 4)$. In the first case, where the points a, b, c are a_1, b_1, c_i, respectively, according to the aforementioned rule, the lines $[b, c, d]$, $[c, a, e]$, $[a, b, f]$ are the lines $[b_1, c_i, a_i]$, $[c_i, a_1, b_i]$ and $[a_1, b_1, c_1]$, and therefore $d = a_i$, $e = b_i$, $f = c_1$, and the line $[a_i, b_i, c_1]$ is the required line $[d, e, f]$. In the second case, when $a = a_1$, $b = b_i$, $c = c_j$, the lines $[b, c, d]$, $[c, a, e]$, $[a, b, f]$ are $[b_i, c_j, a_k]$, $[c_j, a_1, b_j]$, $[a_1, b_i, c_i]$, respectively, hence $d = a_k$, $e = b_j$, $f = c_i$, and the line $[a_k, b_j, c_i]$ is the required line $[d, e, f]$. In the third case, when $a = a_i$, $b = b_i$, $c = c_i$, the lines $[b, c, d]$, $[c, a, e]$, $[a, b, f]$ are the lines $[b_i, c_i, a_1]$, $[c_i, a_i, b_1]$, $[a_i, b_i, c_1]$, respectively, so the line $[a_1, b_1, c_1]$ is the required line $[d, e, f]$. In the fourth case, when $a = a_i$, $b = b_i$, $c = c_j$, the lines $[b, c, d]$, $[c, a, e]$, $[a, b, f]$ are consecutively $[b_i, c_j, a_k]$, $[c_j, a_i, b_k]$, $[a_i, b_i, c_1]$, so the line $[a_k, b_k, c_1]$ is the required line $[d, e, f]$. We have proved $[d, e, f]$, and as $[b, c, d]$, $[c, a, e]$ and $[a, b, f]$ also hold, by Theorem 3.4 (with substitutions $a \leftrightarrow d, b \leftrightarrow e, c \leftrightarrow f$), it follows that a, d; b, e; c, f are pairs of corresponding points. □

The previous proof is also taken from [1], pp. 215–216.

THEOREM 4.8. *If claims* (4.1) *are valid, then* b_1, b_2, b_3 *and* b_4 *are associated points with a common tangential* a_1*, where the points* a_1*,* a_2*,* a_3*, and* a_4 *are associated.*

PROOF. If we write the first four statements (4.1) in the form $[a_2, b_1, b_2]$, $[a_2, b_4, b_3]$, $[b_1, b_3, a_3]$ and $[b_2, a_3, b_4]$, then by Theorem 3.4 it follows that the pairs of points a_2, a_3; b_1, b_4 and b_2, b_3 are corresponding. Similarly, if we write the last four statements (4.1) in the form $[a_3, b_1, b_3]$, $[a_3, b_2, b_4]$, $[b_1, b_4, a_4]$ and $[b_3, a_4, b_2]$, then by Theorem 3.4 it follows that the pairs of points a_3, a_4; b_1, b_2 and b_3, b_4 are corresponding. That is why points b_1, b_2, b_3, and b_4 are associated and have the common tangential, which we denote a_1, and we also know that points a_2, a_3, and a_4 are pairwise corresponding, so they have the common tangential a'. From the table

$$\begin{array}{cc|c|} b_1 & b_1 & a_1 \\ b_2 & b_2 & a_1 \\ a_2 & a_2 & a' \end{array}$$

it follows that the point a_1 has the tangential a', so the points a_1, a_2, a_3, and a_4 are associated. □

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