GLASNIK MATEMATIČKI
Vol. 55(75)(2020), 267 – 276

FURTHER RESULTS ON COMMON PROPERTIES OF THE PRODUCTS ac AND bd

Qingping Zeng, Kai Yan and Zhenying Wu
Fujian Agriculture and Forestry University, Fuzhou University, Fujian Normal University, P.R. China

Abstract. In this paper, we continue to investigate common properties of the products ac and bd in various categories under the assumption $acd = dbd$ and $dba = aca$. These properties include generalized strongly Drazin invertibility and generalized Hirano invertibility in rings, abstract index of Fredholm elements and B-Fredholm elements in the Banach algebra context, complementability of kernels and ranges for bounded linear operators on Banach spaces.

1. Introduction

Throughout this paper, $\mathcal{R}$ denotes an associative ring with unit 1. The classical Jacobson's lemma asserts that

$$1 - ab \text{ is invertible if and only if } 1 - ba \text{ is invertible} \tag{1.1}$$

for any $a, b \in \mathcal{R}$. In the last two decades, suitable analogues of Jacobson's lemma for Drazin inverse and generalized Drazin inverse have been found by many researchers around the world (see [6, 8, 14, 16, 17, 24]). Corach et al. in [7] generalized (1.1) and many of its relatives to the case that

$$aba = aca, \tag{1.2}$$

see also [20, 21, 22, 23]. Recently, it has been realized that there are proper counterparts of Jacobson's lemma for Drazin inverse and generalized Drazin inverse under the new condition

$$\begin{cases} acd = dbd, \\ dba = aca, \end{cases} \tag{1.3}$$

2020 *Mathematics Subject Classification.* 15A09, 16U99, 47A05, 47A53.
Key words and phrases. Jacobson's lemma, generalized inverse, complementability, index.

see [15, 18]. Obviously, the case "$a = d$" in (1.3) gives (1.2), the case "$b = c$" in (1.2) results in $aca = aca$.

This paper is a continuation of [15, 18]. In the presence of (1.3), common properties of the products ac and bd are further studied in various categories.

• In section 2, Jacobson's lemma for two new generalized inverses (i.e., generalized strong Drazin inverse and generalized Hirano inverse) are established in rings.

• In section 3, we derive the abstract index equality of Fredholm elements and B-Fredholm elements in the Banach algebra context.

• In section 4, we investigate the common complementability of kernels and ranges for bounded linear operators on Banach spaces.

2. Generalized inverses related to generalized Drazin inverse

For $a \in \mathcal{R}$, the commutant and double commutant of a are defined by $comm(a) = \{x \in \mathcal{R} : ax = xa\}$ and $comm^2(a) = \{x \in \mathcal{R} : xy = yx, \text{for all } y \in comm(a)\}$, respectively. We shall write $\mathcal{R}^{-1}$ and $\mathcal{R}^{nil}$ for the sets of all invertible and nilpotent elements of $\mathcal{R}$, respectively. An element $a \in \mathcal{R}$ is quasinilpotent ([12]) if $1 + ax \in \mathcal{R}^{-1}$ for all $x \in comm(a)$. The set of all quasinilpotent elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{qnil}$. Recall that $a \in \mathcal{R}$ is generalized Drazin invertible ([13]) if there exists $b \in \mathcal{R}$ such that

$$b \in comm^2(a), bab = b \text{ and } a - aba \in \mathcal{R}^{qnil}.$$

If such b exists, it is unique, and it is called the generalized Drazin inverse of a, denoted by a^{gD}. The set composed of generalized Drazin invertible elements in $\mathcal{R}$ will be denoted by $\mathcal{R}^{gD}$. In [18], the authors obtained the following analogue of Jacobson's lemma for generalized Drazin inverse under the assumption (1.3), which gives an affirmative answer to a conjecture of [15].

Lemma 2.1. *Suppose that $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$. Then $\beta = 1 - ac \in \mathcal{R}^{gD}$ if and only if $\alpha = 1 - bd \in \mathcal{R}^{gD}$. In this case, we have*

$$\beta^{gD} = (1 - d\alpha^{\pi}[1 - \alpha^{\pi}\alpha(1 + bd)]^{-1}bac)(1 + ac) + d\alpha^{gD}bac$$

and

$$\alpha^{gD} = (1 - bac\beta^{\pi}[1 - \beta^{\pi}\beta(1 + ac)]^{-1}d)(1 + bd) + bac\beta^{gD}d,$$

where $\alpha^{\pi} = 1 - \alpha\alpha^{gD}, \beta^{\pi} = 1 - \beta\beta^{gD}$.

If we replace the condition $a - aba \in \mathcal{R}^{qnil}$ in the definition of generalized Drazin inverse with $a - ab \in \mathcal{R}^{qnil}$, then a is said to be generalized strongly Drazin invertible and b is called the generalized strong Drazin inverse of a, denoted by a^{gsD} (see [11]). The set composed of generalized strongly Drazin invertible elements in $\mathcal{R}$ will be denoted by $\mathcal{R}^{gsD}$. According to [11, Corollary 3.3], $\mathcal{R}^{gsD} \subseteq \mathcal{R}^{gD}$.

THEOREM 2.2. *Suppose that $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$. Then $\beta = 1 - ac \in \mathcal{R}^{gsD}$ if and only if $\alpha = 1 - bd \in \mathcal{R}^{gsD}$. In this case, we have*

$$\beta^{gsD} = (1 - d\alpha^{\pi}[1 - \alpha^{\pi}\alpha(1 + bd)]^{-1}bac)(1 + ac) + d\alpha^{gsD}bac$$

and

$$\alpha^{gsD} = (1 - bac\beta^{\pi}[1 - \beta^{\pi}\beta(1 + ac)]^{-1}d)(1 + bd) + bac\beta^{gsD}d,$$

where $\alpha^{\pi} = 1 - \alpha\alpha^{gsD}, \beta^{\pi} = 1 - \beta\beta^{gsD}$.

PROOF. Write $p = \alpha^{\pi}$, $v = [1 - p\alpha(1 + bd)]^{-1}$ and $y = (1 - dpvbac)(1 + ac) + d\alpha^{gsD}bac$. By Lemma 2.1, y is a generalized Drazin inverse of β. To show $y \in \mathcal{R}^{gsD}$, we only need to show that $\beta - \beta y \in \mathcal{R}^{qnil}$. Noting $p = p(bd)^2v = pv(bd)^2$, we deduce that $\alpha - \alpha\alpha^{gsD} = p - bd = (pvbdb - b)d$. From the proof of [18, Theorem 3.3], we get $\beta y = 1 - dpvbac$. Hence $\beta - \beta y = 1 - ac - (1 - dpvbac) = dpvbac - ac = (dpvba - a)c$. Now we put $a' = dpvba - a$ and $b' = pvbdb - b$. Then a direct calculation shows that $a'cd = db'd$ and $db'a' = a'ca'$. Since $b'd = \alpha - \alpha\alpha^{gsD} \in \mathcal{R}^{qnil}$, by [19, Lemma 2.6], we conclude that $\beta - \beta y = a'c \in \mathcal{R}^{qnil}$, as required.

Conversely, set $q = \beta^{\pi}$, $u = [1 - q\beta(1 + ac)]^{-1}$ and $x = (1 - bacqud)(1 + bd) + bac\beta^{gsD}d$. By Lemma 2.1, it remains to prove that $\alpha - \alpha x \in \mathcal{R}^{qnil}$. Noting $q = q(ac)^2u = qu(ac)^2$, we get $\beta - \beta\beta^{gsD} = q - ac = (quaca - a)c$. Also, we obtain

$$\begin{aligned}
\alpha x &= (1 - bd)[(1 - bacqud)(1 + bd) + bac\beta^{gsD}d] \\
&= 1 - (bd)^2 - (1 - bd)bacqud(1 + bd) + (1 - bd)bac\beta^{gsD}d \\
&= 1 - [bacd - bac(1 - ac)\beta^{gsD}d] - (1 - bd)bacqud(1 + bd) \\
&= 1 - bacqd - bac(1 - ac)qud(1 + bd) \\
&= 1 - bacqd - bacqu(1 - ac)(1 + ac)d \\
&= 1 - bacqd - bacqu[1 - (ac)^2]d \\
&= 1 - bacqud,
\end{aligned}$$

whence $\alpha - \alpha x = bacqud - bd = (bacqu - b)d$. Now we write $a' = quaca - a$ and $b' = bacqu - b$, a direct calculation shows that $a'cd = db'd$ and $db'a' = a'ca'$. Since $a'c = \beta - \beta\beta^{gsD} \in \mathcal{R}^{qnil}$, the desired conclusion $\alpha - \alpha x = b'd \in \mathcal{R}^{qnil}$ then follows by [19, Lemma 2.6]. □

Recently, Abdolyousefi and Chen ([1]) introduced another subclass of generalized Drazin inverse, by replacing $a - aba \in \mathcal{R}^{qnil}$ with $a^2 - ab \in \mathcal{R}^{qnil}$ in the definition of generalized Drazin inverse. In this case, we say that a is generalized Hirano invertible and b is the generalized Hirano inverse of a, denoted by a^{gH}. We use $\mathcal{R}^{gH}$ to denote the set of all generalized Hirono invertible elements in $\mathcal{R}$. By [1, Theorem 2.2], $\mathcal{R}^{gH} \subseteq \mathcal{R}^{gD}$.

THEOREM 2.3. *Suppose that $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$. Then $\beta = 1 - ac \in \mathcal{R}^{gH}$ if and only if $\alpha = 1 - bd \in \mathcal{R}^{gH}$. In this case, we have*

$$\beta^{gH} = (1 - d\alpha^\pi[1 - \alpha^\pi\alpha(1 + bd)]^{-1}bac)(1 + ac) + d\alpha^{gH}bac$$

and

$$\alpha^{gH} = (1 - bac\beta^\pi[1 - \beta^\pi\beta(1 + ac)]^{-1}d)(1 + bd) + bac\beta^{gH}d,$$

where $\alpha^\pi = 1 - \alpha\alpha^{gH}, \beta^\pi = 1 - \beta\beta^{gH}$.

PROOF. Write $p = \alpha^\pi$, $v = [1 - p\alpha(1 + bd)]^{-1}$ and $y = (1 - dpvbac)(1 + ac) + d\alpha^{gH}bac$. By Lemma 2.1, y is a generalized Drazin inverse of β. To show $y \in \mathcal{R}^{gH}$, we only need to show that $\beta^2 - \beta y \in \mathcal{R}^{qnil}$. Noting $p = p(bd)^2v = pv(bd)^2$, we deduce that $\alpha^2 - \alpha\alpha^{gsD} = p - 2bd + bdbd = (pvbdb - 2b + bdb)d$. From the proof of [18, Theorem 3.3], we get $\beta y = 1 - dpvbac$. Hence $\beta^2 - \beta y = (1 - ac)^2 - (1 - dpvbac) = dpvbac - 2ac + acac = (dpvba - 2a + aca)c$. Now we put $a' = dpvba - 2a + aca$ and $b' = pvbdb - 2b + bdb$. Then a direct calculation shows that $a'cd = db'd$ and $db'a' = a'ca'$. Since $b'd = \alpha^2 - \alpha\alpha^{gsD} \in \mathcal{R}^{qnil}$, by [19, Lemma 2.6], we conclude that $\beta^2 - \beta y = a'c \in \mathcal{R}^{qnil}$, as required.

Conversely, put $q = \beta^\pi$, $u = [1 - q\beta(1 + ac)]^{-1}$ and $x = (1 - bacqud)(1 + bd) + bac\beta^{gsD}d$. According to Lemma 2.1, it remains to show that $\alpha^2 - \alpha x \in \mathcal{R}^{qnil}$. Since $q = q(ac)^2u = qu(ac)^2$, $\beta^2 - \beta\beta^{gsD} = q - 2ac + acac = (quaca - 2a + aca)c \in \mathcal{R}^{qnil}$. As in the proof of Theorem 2.2, we get $\alpha x = 1 - bacqud$, hence $\alpha^2 - \alpha x = bacqud - 2bd + bdbd = (bacqu - 2b + bdb)d$. Now we set $a' = quaca - 2a + aca$ and $b' = bacqu - 2b + bdb$, it is easy to verify that $a'cd = db'd$ and $db'a' = a'ca'$. Applying [19, Lemma 2.6], again, we get $\alpha^2 - \alpha x = b'd \in \mathcal{R}^{qnil}$ as required. □

3. ABSTRACT INDEX OF FREDHOLM AND B-FREDHOLM ELEMENTS

Following [9], an element $a \in \mathcal{R}$ is said to be Drazin invertible if there exist $b \in \mathcal{R}$ and $k \in \mathbb{N}$ such that

$$b \in comm(a), bab = b \text{ and } a^kba = a^k.$$

The element b above is unique if it exists. It is called the Drazin inverse of a and is denoted by a^D. The smallest k for which $a^kba = a^k$ is called the Drazin index of a, and is denoted by $\mathrm{i}(a)$. If $\mathrm{i}(a) \leq 1$, then a is called group invertible. An element $a \in \mathcal{R}$ is invertible precisely when a is Drazin invertible with $\mathrm{i}(a) = 0$. We use $\mathcal{R}^D$ and $\mathcal{R}^\sharp$ to denote all Drazin invertible elements and group invertible elements in $\mathcal{R}$, respectively. According to [15, Theorem 2.4], (see also [18, Theorem 3.1]), in the presence of (1.3), we have

$$1 - ac \text{ is Drazin invertible} \Longleftrightarrow 1 - bd \text{ is Drazin invertible,} \tag{3.1}$$

$$1 - ac \text{ is group invertible} \Longleftrightarrow 1 - bd \text{ is group invertible} \tag{3.2}$$

and

$$1-ac \text{ is invertible} \Longleftrightarrow 1-bd \text{ is invertible.} \tag{3.3}$$

Let $\mathcal{I}$ be an ideal of $\mathcal{R}$ and π the canonical homomorphism from $\mathcal{R}$ to $\mathcal{R}/\mathcal{I}$. Following [3] (resp., [4]), an element $r \in \mathcal{R}$ is called a Fredholm element (resp., generalized Fredholm element, B-Fredholm element) relative to $\mathcal{I}$ if $\pi(r) \in (\mathcal{R}/\mathcal{I})^{-1}$ (resp., $\pi(r) \in (\mathcal{R}/\mathcal{I})^{\sharp}$, $\pi(r) \in (\mathcal{R}/\mathcal{I})^{D}$). The set of all Fredholm elements, generalized Fredholm elements and B-Fredholm elements relative to $\mathcal{I}$ will be denoted by $\Phi(\mathcal{R},\mathcal{I})$, $g\Phi(\mathcal{R},\mathcal{I})$ and $B\Phi(\mathcal{R},\mathcal{I})$, respectively. Applying (3.1), (3.2) and (3.3) respectively to $\mathcal{R}/\mathcal{I}$, we get

$$1-ac \in B\Phi(\mathcal{R},\mathcal{I}) \Longleftrightarrow 1-bd \in B\Phi(\mathcal{R},\mathcal{I}), \tag{3.4}$$

$$1-ac \in g\Phi(\mathcal{R},\mathcal{I}) \Longleftrightarrow 1-bd \in g\Phi(\mathcal{R},\mathcal{I}) \tag{3.5}$$

and

$$1-ac \in \Phi(\mathcal{R},\mathcal{I}) \Longleftrightarrow 1-bd \in \Phi(\mathcal{R},\mathcal{I}), \tag{3.6}$$

provided that (1.3) holds.

Recall that a Banach algebra $\mathcal{A}$ is called semisimple if the radical $\mathrm{Rad}(\mathcal{A})$ of $\mathcal{A}$ is equal to $\{0\}$, and $\mathcal{A}$ is said to be primitive if $\{0\}$ is a primitive ideal of $\mathcal{A}$. Primitive Banach algebras are semisimple. Let $\mathcal{A}$ be a complex semisimple Banach algebra with unit 1 and let $\mathcal{I}$ be a trace ideal (i.e., an ideal on which a trace $\tau : \mathcal{I} \longrightarrow \mathbb{C}$ is defined, see [10, 5] for details) of $\mathcal{A}$. Following [10] (resp., [5]), the index of a Fredholm element (resp., B-Fredholm element) $a \in \mathcal{A}$ relative to trace ideal $\mathcal{I}$ is defined with the aid of the trace as $\iota(a) := \tau(aa_0 - a_0a)$, where $\pi(a_0)$ is an inverse (resp., a Drazin inverse) of $\pi(a)$ in $\mathcal{A}/\mathcal{I}$. The socle $\mathrm{soc}(\mathcal{A})$ of $\mathcal{A}$ is defined to be the sum of minimal ideals, and the set $\mathrm{kh}(\mathrm{soc}(\mathcal{A}))$ is defined by $\mathrm{kh}(\mathrm{soc}(\mathcal{A})) := \{a \in \mathcal{A} : a + \mathrm{soc}(\mathcal{A}) \in \mathrm{Rad}(\mathcal{A}/\mathrm{soc}(\mathcal{A}))\}$. In the following two results, we obtain the abstract index equality of Fredholm elements and B-Fredholm elements respectively in the Banach algebra context.

THEOREM 3.1. *Let $\mathcal{A}$ be a unital semisimple Banach algebra and let $\mathcal{I}$ be a trace ideal of $\mathcal{A}$ such that $soc(\mathcal{A}) \subseteq \mathcal{I} \subseteq kh(soc(\mathcal{A}))$. If $a,b,c,d \in \mathcal{A}$ satisfy $acd = dbd$ and $dba = aca$ and $1-ac$ is a Fredholm element relative to $\mathcal{I}$, then $\iota(1-ac) = \iota(1-bd)$.*

PROOF. By [10, Proposition 3.10 and Theorem 3.11], there exist idempotents p, q in $\mathrm{soc}(\mathcal{A})$ and $x \in \mathcal{A}$ such that $p(1-ac) = 0$, $(1-ac)q = 0$, $(1-ac)x = 1-p$, $x(1-ac) = 1-q$ and $\iota(1-ac) = \tau(q) - \tau(p)$. Now we take $y = 1 + bd + bacxd$. A direct calculation shows that $(1-bd)y = 1 - bacpd$ and $y(1-bd) = 1 - bacqd$, which implies that

$$\iota(1-bd) = \tau((1-bd)y - y(1-bd)) = \tau(bacqd - bacpd).$$

Since $p(1-ac)=0$, $\tau(bacpd)=\tau(pdbac)=\tau(pacac)=\tau(pac)=\tau(p)$. Analogously, $\tau(bacqd)=\tau(q)$. Therefore, $\iota(1-bd)=\tau(q)-\tau(p)=\iota(1-ac)$. ∎

Theorem 3.2. *Let $\mathcal{A}$ be a unital primitive Banach algebra and suppose that $a,b,c,d\in\mathcal{A}$ satisfy $acd=dbd$ and $dba=aca$.*

(1) *If $1-ac$ is a B-Fredholm element relative to soc$(\mathcal{A})$, then $\iota(1-ac)=\iota(1-bd)$.*
(2) *If ac is a B-Fredholm element relative to soc$(\mathcal{A})$, then $\iota(ac)=\iota(bd)$.*

Proof. (1) By the punctured neighborhood theorem for the index of B-Fredholm element (see [5, Theorem 3.1]), for nonzero λ with $|\lambda|$ small enough, we have

$$1-ac-\lambda\in\Phi(\mathcal{A},\mathrm{soc}(\mathcal{A})),\ 1-ba-\lambda\in\Phi(\mathcal{A},\mathrm{soc}(\mathcal{A}))$$

and

$$\iota(1-ac)=\iota(1-ac-\lambda),\ \iota(1-ba)=\iota(1-ba-\lambda).$$

Hence, the desired result follows by Theorem 3.1.

(2) The proof is analogous to that above. ∎

4. Complementability of kernels and ranges

Let $\mathcal{B}(X,Y)$ denote the set of all bounded linear operators from Banach space X to Banach space Y. For $T\in\mathcal{B}(X):=\mathcal{B}(X,X)$, let $\mathcal{N}(T)$ denote its kernel and $\mathcal{R}(T)$ its range. In this section, we discuss the complementability of kernels and ranges of $I-AC$ and $I-BD$ under the assumption $ACD=DBD$ and $DBA=ACA$. Recall that a closed subspace M of a Banach space X is complemented if there exists a (closed) subspace N of X such that $X=M\bigoplus N$. Equivalently, M is complemented in X if and only if there is a bounded projection P such that $\mathcal{R}(P)=M$.

Theorem 4.1. *Suppose that $A,D\in\mathcal{B}(X,Y)$ and $B,C\in\mathcal{B}(Y,X)$ satisfy $ACD=DBD$ and $DBA=ACA$. Then $\mathcal{N}(I-AC)$ is complemented in Y if and only if $\mathcal{N}(I-BD)$ is complemented in X.*

Proof. Assume that P is the projection onto $\mathcal{N}(I-AC)$. Then $(I-AC)P=0$, that is, $P=ACP$. Put $Q=BPACD$. From the fact $DBP=DBACP=ACACP=ACP=P$, it follows that

$$Q^2=(BPACD)(BPACD)=BPACPACD=BPACD=Q.$$

Noting that

$$(I-BD)Q=(I-BD)(BPACD)=BPACD-BDBPACD=0,$$

we have $\mathcal{R}(Q)\subseteq\mathcal{N}(I-BD)$. Let $x\in\mathcal{N}(I-BD)$. Then $Dx=DBDx=ACDx$, whence $Dx\in\mathcal{N}(I-AC)=\mathcal{R}(P)$. Thus $PDx=Dx$, and hence

$$Qx=BPACDx=BPACPDx=BPDx=BDx=x,$$

which implies that $\mathcal{N}(I-BD) \subseteq \mathcal{R}(Q)$. Consequently, Q is the projection onto $\mathcal{N}(I-BD)$.

Conversely, assume that U is the projection onto $\mathcal{N}(I-BD)$. Set $V = ACDUBACAC$. Noting that $BDU = U$, it follows

$$\begin{aligned} V^2 &= (ACDUBACAC)(ACDUBACAC) \\ &= ACDUBDBDBDBDUBACAC \\ &= ACDUBACAC \\ &= V. \end{aligned}$$

Since

$$\begin{aligned} (I-AC)V &= (I-AC)(ACDUBACAC) \\ &= ACDUBACAC - ACACDUBACAC \\ &= ACDUBACAC - ACDBDUBACAC \\ &= ACDUBACAC - ACDUBACAC \\ &= 0, \end{aligned}$$

$\mathcal{R}(V) \subseteq \mathcal{N}(I-AC)$. Let $x \in \mathcal{N}(I-AC)$. Then $x = ACx$. Since $BACx = BACACx = BDBACx$, $BACx \in \mathcal{N}(I-BD) = \mathcal{R}(U)$, and hence $UBACx = BACx$. Thus,

$$\begin{aligned} Vx &= ACDUBACACx = ACDUBACx \\ &= ACDBACx = ACACACx = x, \end{aligned}$$

which implies that $\mathcal{N}(I-AC) \subseteq \mathcal{R}(V)$. Consequently, V is the projection onto $\mathcal{N}(I-AC)$. ∎

THEOREM 4.2. *Suppose that $A, D \in \mathcal{B}(X,Y)$ and $B, C \in \mathcal{B}(Y,X)$ satisfy $ACD = DBD$ and $DBA = ACA$. Then $\mathcal{R}(I-AC)$ is complemented in Y if and only if $\mathcal{R}(I-BD)$ is complemented in X.*

PROOF. Assume that P is the projection onto $\mathcal{R}(I-AC)$. Set $Q = I - BAC(I-P)D$. Since $(I-P)(I-AC) = 0$, $(I-P)AC = I-P$. It follows that

$$\begin{aligned} [BAC(I-P)D][BAC(I-P)D] &= BAC(I-P)ACAC(I-P)D \\ &= BAC(I-P)D, \end{aligned}$$

and hence $Q^2 = Q$. Since $\mathcal{R}(P) = \mathcal{R}(I-AC)$,

$$\mathcal{R}(BACPD) \subseteq \mathcal{R}(BAC(I-AC)) = \mathcal{R}((I-BD)BAC) \subseteq \mathcal{R}(I-BD).$$

Noting that

$$\begin{aligned} Q &= I - BAC(I-P)D = I - BACD + BACPD \\ &= I - BDBD + BACPD = (I-BD)(I+BD) + BACPD, \end{aligned}$$

we get $\mathcal{R}(Q) \subseteq \mathcal{R}(I-BD)$. Let $x \in \mathcal{R}(I-BD)$. Then there is an $x_1 \in X$ such that $x = (I-BD)x_1$. Since $Dx = D(I-BD)x_1 = (I-AC)Dx_1 \in \mathcal{R}(P)$,

$$Qx = [I - BAC(I-P)D]x = x,$$

which deduces that $\mathcal{R}(I-BD) \subseteq \mathcal{R}(Q)$. Therefore, $\mathcal{R}(I-BD)$ is complemented in X.

Conversely, suppose that U is the projection onto $\mathcal{R}(I-BD)$ and put

$$V = I - ACD(I-U)BAC.$$

Next we will show that V is the associated projection onto $\mathcal{R}(I-AC)$. Since $(I-U)(I-BD) = 0$, $(I-U)BD = I-U$, and hence

$$\begin{aligned}[ACD(I-U)BAC]^2 &= ACD(I-U)BDBDBD(I-U)BAC \\ &= ACD(I-U)BAC,\end{aligned}$$

which implies that $V^2 = V$. Noting that

$$\begin{aligned}V &= I - ACD(I-U)BAC = I - ACDBAC + ACDUBAC \\ &= I - ACACAC + ACDUBAC,\end{aligned}$$

it follows

$$\begin{aligned}\mathcal{R}(V) &\subseteq \mathcal{R}(I - ACACAC + ACDUBAC) \\ &\subseteq \mathcal{R}[(I-AC)(I+AC+ACAC)] + \mathcal{R}[ACD(I-BD)] \\ &\subseteq \mathcal{R}(I-AC) + \mathcal{R}[(I-AC)ACD] \\ &\subseteq \mathcal{R}(I-AC).\end{aligned}$$

For any $y \in \mathcal{R}(I-AC)$, there exists an element $y_1 \in Y$ such that $y = (I-AC)y_1$. Thus $BACy = BAC(I-AC)y_1 = (I-BD)BACy_1 \in \mathcal{R}(U)$, and so

$$Vy = [I - ACD(I-U)BAC]y = y.$$

Hence $\mathcal{R}(I-AC) \subseteq \mathcal{R}(V)$. Consequently, $\mathcal{R}(I-AC)$ is complemented in Y. ∎

In the following we give an application of Theorem 4.1 and Theorem 4.2. Recall that an operator $T \in \mathcal{B}(X)$ is said to be relatively regular if there exists an operator $S \in \mathcal{B}(X)$ for which $TST = T$ and $STS = S$. Relatively regular operator plays a significant role in operator theory. We refer the reader to [2] for more details. It is known that $T \in \mathcal{B}(X)$ is relatively regular if and only if $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are complemented ([2, Theorem 3.88]). Thus it is easy to obtain the following conclusion about relatively regular operators from Theorem 4.1 and Theorem 4.2.

Corollary 4.3. *Suppose that $A, D \in \mathcal{B}(X,Y)$ and $B, C \in \mathcal{B}(Y,X)$ satisfy $ACD = DBD$ and $DBA = ACA$. Then $I-AC$ is relatively regular if and only if $I-BD$ is relatively regular.*

Acknowledgements.

The authors wish to thank the referees for their careful reading and valuable comments on the original draft. This work has been supported by National Natural Science Foundation of China (Grant Nos. 11971108 and 11901099), Natural Science Foundation of Fujian Province (Grant Nos. 2018J05004 and 2020J01569), the Educational Commission of Fujian Province (Grant No. JAT190070) and the Science and Technology Innovation Fund of Fujian Agriculture and Forestry University (Grant No. CXZX2018036).

References

[1] H.Y. Chen and M. Sheibani, *Generalized Hirano inverses in rings*, Comm. Algebra **47** (2019), 2967–2978.

[2] P. Aiena, Fredholm and local spectral theory, with applications to multipliers, Kluwer Academic Publishers, Dordrecht, 2004.

[3] B.A. Barnes, *The Fredholm elements of a ring*, Canadian J. Math. **21** (1969), 84–95.

[4] M. Berkani, *B-Fredholm elements in rings and algebras*, Publ. Math. Debrecen **92** (2018), 171–181.

[5] M. Berkani, *A trace formula for the index of B-Fredholm operators*, Proc. Edinb. Math. Soc. (2) **61** (2018), 1063–1068.

[6] N. Castro-González, C. Mendes-Araújo and P. Patricio, *Generalized inverses of a sum in rings*, Bull. Aust. Math. Soc. **82** (2010), 156–164.

[7] G. Corach, B.P. Duggal and R.E. Harte, *Extensions of Jacobson's lemma*, Comm. Algebra **41** (2013), 520–531.

[8] D. Cvetković-Ilić and R.E. Harte, *On Jacobson's lemma and Drazin invertibility*, Appl. Math. Lett. **23** (2010), 417–420.

[9] M.P. Drazin, *Pseudo-inverses in associative rings and semigroups*, Amer. Math. Monthly **65** (1958), 506–514.

[10] J.J. Grobler and H. Raubenheimer, *The index for Fredholm elements in a Banach algebra via a trace*, Studia Math. **187** (2008), 281–297.

[11] O. Gürgün, *Properties of generalized strongly Drazin invertible elements in general rings*, J. Algebra Appl. **16** (2017), 1750207, 13 pp.

[12] R.E. Harte, *On quasinilpotents in rings*, Panamer. Math. J. **1** (1991), 10–16.

[13] J.J. Koliha and P. Patrício, *Elements of rings with equal spectral idempotents*, J. Aust. Math. Soc. **72** (2002), 137–152.

[14] T.Y. Lam and P.P. Nielsen, *Jacobson's lemma for Drazin inverses*, in: Ring theory and its applications, AMS, Providence, 2014, 185–195.

[15] D. Mosić, *Extensions of Jacobson's lemma for Drazin inverses*, Aequationes Math. **91** (2017), 419–428.

[16] P. Patrício and R.E. Hartwig, *The link between regularity and strong-pi-regularity*, J. Aust. Math. Soc. **89** (2010), 17–22.

[17] P. Patrício and A. Veloso da Costa, *On the Drazin index of regular elements*, Cent. Eur. J. Math. **7** (2009), 200–205.

[18] K. Yan, Q.P. Zeng and Y.C. Zhu, *Generalized Jacobson's lemma for Drazin inverses and its applications*, Linear Multilinear Algebra **68** (2020), 81–93.

[19] Q.P. Zeng, Z.Y. Wu and Y.X. Wen, *New extensions of Cline's formula for generalized inverses*, Filomat **31** (2017), 1973–1980.

[20] Q.P. Zeng, K. Yan and S.F. Zhang, *New results on common properties of the products AC and BA, II*, Math. Nachr. **293** (2020), 1629–1635.

[21] Q.P. Zeng and H.J. Zhong, *Common properties of bounded linear operators AC and BA: spectral theory*, Math. Nachr. **287** (2014), 717–725.
[22] Q.P. Zeng and H.J. Zhong, *Common properties of bounded linear operators AC and BA: Local spectral theory*, J. Math. Anal. Appl. **414** (2014), 553–560.
[23] Q.P. Zeng and H.J. Zhong, *New results on common properties of the products AC and BA*, J. Math. Anal. Appl. **427** (2015), 830–840.
[24] G.F. Zhuang, J.L. Chen and J. Cui, *Jacobson's lemma for the generalized Drazin inverse*, Linear Algebra Appl. **436** (2012), 742–746.

Q. Zeng
College of Computer and Information Sciences, Institute of Applied Mathematics
Fujian Agriculture and Forestry University
350002 Fuzhou
P.R. China
E-mail: zqpping2003@163.com

K. Yan (Corresponding author)
College of Mathematics and Computer Science
Fuzhou University
350108 Fuzhou
P.R. China
E-mail: yklolxj@163.com

Z. Wu
College of Mathematics and Informatics
Fujian Normal University
350117 Fuzhou
P.R. China
E-mail: zhenyingwu2011@163.com

Received: 31.1.2020.
Revised: 20.5.2020.