GLASNIK MATEMATIČKI
Vol. 55(75)(2020), 203 – 235

ON REPRESENTATIONS OF REDUCTIVE p–ADIC GROUPS OVER $\mathbb{Q}$–ALGEBRAS

Goran Muić
University of Zagreb, Croatia

Abstract. In this paper we study certain category of smooth modules for reductive p–adic groups analogous to the usual smooth complex representations but with the field of complex numbers replaced by a $\mathbb{Q}$–algebra. We prove some fundamental results in these settings, and as an example we give a classification of admissible unramified irreducible representations using the reduction to the complex case.

1. Introduction

In this paper we define and study certain category of smooth modules for reductive p–adic groups analogous to the usual smooth complex representations ([1, 2, 3, 4, 8]). Nowadays there is an active current research in the field of complex representation theory as one can observe from the review articles [18, 19]. Representations in positive characteristic are also well understood thanks to the recent works of Henniart, Vignéras and others (see [12]). But the representations of reductive p-adic groups on the vector spaces over extensions of $\mathbb{Q}$ such as number fields are not well–understood beyond the study of fields of definition of complex representations ([17]). In this paper we start to consider such problems. On the example of a classification of unramified representations the reader will realize how rich and more interesting is this theory than the complex one (but it seems a lot more simpler than the case of positive characteristic ([11])). It is based on the description of $\mathbb{Z}$–structure of the Satake isomorphism due to Gross ([10]).

2020 *Mathematics Subject Classification.* 11E70, 22E50.
Key words and phrases. Reductive p–adic groups, $\mathbb{Q}$–admissible representations, Hecke algebras.
The author acknowledges Croatian Science Foundation grant no. 3628.

As with the approach in positive characteristic mentioned above, we use extensively Hecke algebra approach combined with the theory of semisimple algebras to reduce to the case of algebraically closed field. This is not new, basic ideas can be found already in the book by Curtis and Reiner ([9]). The theory in positive characteristic is more involved and it is based on a rather deep decomposition theorem ([12, Theorem I.1]). In our case, we just use very basic theory of semisimple rings ([13, Chapter XVII]) due to the fact that we work in characteristic zero. We expect application in the case of complex representations too but we leave it for another occasion.

In this paper rings are always associative commutative rings with $1 \neq 0$ (as in [14]). Also homomorphism of rings always send 1 onto 1. The identity of a subring S of a ring R is always the identity of R. Ring modules are always unital i.e., 1 acts as identity. We fix a non—Archimedean local field k. Let G be a reductive p–adic group which by abuse of notation is a group of k–points of a Zariski connected reductive group defined over k. As indicated at appropriate places, for some results in the paper we may assume that G is just an l–group (see [3]) but for introduction we stick with the assumption that G is a reductive p–adic group. I was informed by Casselman that new version of his classical book [8] would contain extensive theory of parabolic induction and Jacquet modules for smooth representations with coefficients in the rings (see Definition 1.1).

We continue with expected form of the definition of modules that we consider. The following Definitions 1.1 and 1.2 are essentially taken from ([1, 1.16]) but see also ([21, Chapter I]).

Definition 1.1. *Let $\mathcal{A}$ be a ring. An $(\mathcal{A},\ G)$–module is an $\mathcal{A}$–module V together with a homomorphism $G \longrightarrow GL_{\mathcal{A}}(V)$ such that the stabilizer of every element in V is open in G.*

The book by Vignéras ([21, Chapter I]) contains many basic results for such modules. Obviously, when $\mathcal{A} = \mathbb{C}$ we obtain usual smooth complex representation of G. More interesting example is when we use for $\mathcal{A}$ a center $\mathcal{Z}(G)$ of the category of smooth complex representations of G (see [1]).

Definition 1.1 implies that

$$V = \cup_L V^L \quad \text{(the union ranges over all open compact subgroups of } G\text{)},$$

and every

$$V^L \overset{def}{=} \{v \in V;\ \ l.v = v,\ l \in L\}$$

is an $\mathcal{A}$–module.

When $\mathcal{A} = \mathbb{C}$, the definition below gives us usual complex admissible representation of G.

Definition 1.2. *An $(\mathcal{A},\ G)$–module V is $\mathcal{A}$–admissible if V^L is finitely generated $\mathcal{A}$–module for all open–compact subgroups $L \subset G$.*

We consider category

$$\mathcal{C}\,(\mathcal{A},\ G)$$

of all $(\mathcal{A},\ G)$–modules. Obviously, $\mathcal{C}\,(\mathcal{A},\ G)$ is an Abelian category.

Now, we assume that $\mathcal{A}$ is $\mathbb{Q}$–algebra for the rest of the paper. Then, as expected, the functor $V \mapsto V^L$ from the category $\mathcal{C}\,(\mathcal{A},\ G)$ into category of $\mathcal{A}$–modules is exact, for all open compact subgroups $L \subset G$ (see Lemma 2.1). An important consequence of the fact that we work with rings is the following fundamental result (see Lemma 2.3).

Lemma 1.3. *Let $\mathfrak{a} \subset \mathcal{A}$ be an ideal of $\mathcal{A}$. Then, for any $(\mathcal{A},\ G)$–module V, and for any open compact subgroup $L \subset G$, we have the following:*

$$\left(\mathfrak{a}V\right)^L = \mathfrak{a}V^L.$$

Since we work with rings it is natural to consider the annihilator $\mathrm{Ann}_{\mathcal{A}}(V)$ in $\mathcal{A}$ of an $(\mathcal{A},\ G)$–module V. For irreducible but not $\mathcal{A}$–admissible modules V, the annihilator is just a prime ideal (see Lemma 2.4 and the example after the proof of that lemma). But when V is irreducible and $\mathcal{A}$–admissible, the situation is much more manageable as can be seen from the theorem that we recall below (see Theorem 2.5).

Theorem 1.4. *Assume that $\mathcal{A}$ is a $\mathbb{Q}$–algebra. Then, we have the following:*

(i) *For every irreducible $\mathcal{A}$–admissible $(\mathcal{A},\ G)$–module V, the annihilator of V is a maximal ideal. In fact, if we write $\mathfrak{m} = \mathrm{Ann}_{\mathcal{A}}(V)$, then V is an irreducible $\mathcal{A}/\mathfrak{m}$–admissible $(\mathcal{A}/\mathfrak{m},\ G)$–module.*

(ii) *Let $\mathrm{Irr}_{\mathfrak{m}}$ be the set of equivalence classes of irreducible $\mathcal{A}/\mathfrak{m}$–admissible $(\mathcal{A}/\mathfrak{m},\ G)$–modules. Then, the disjoint union*

$$\cup_{\mathfrak{m}}\ \mathrm{Irr}_{\mathfrak{m}} \quad (\mathfrak{m} \text{ ranges over maximal ideals of } \mathcal{A})$$

can be taken to be the set of equivalence classes of irreducible $\mathcal{A}$–admissible $(\mathcal{A},\ G)$–modules.

(iii) *Assume that $\mathcal{A}$ is a finitely generated $\mathbb{C}$–algebra. Let $\mathrm{Irr}(G)$ be the set of equivalence of complex irreducible admissible representations of G (see [3]). Let $Max(\mathcal{A})$ be the set of all maximal ideals in $\mathcal{A}$. Then, the set $\mathrm{Irr}(G) \times Max(\mathcal{A})$ parameterizes irreducible $\mathcal{A}$–admissible $(\mathcal{A},\ G)$–modules.*

Lemma 1.3 recalled above is of the fundamental importance in the proof of this theorem. Maintaining the notation of the theorem, the identity action of G on $\mathcal{A}/\mathfrak{m}$ is an example of irreducible $\mathcal{A}/\mathfrak{m}$–admissible $(\mathcal{A}/\mathfrak{m},\ G)$–module. We call it the trivial representation. Therefore, $\mathrm{Irr}_{\mathfrak{m}}$ is always non–empty. When G is a reductive p–adic group, we will prove the existence of other more complicated representation. But in the present generality, G could be the trivial group, and we can not do better.

Section 3 discusses the existence of irreducible $(\mathcal{A},\ G)$–modules via Hecke algebra adapted from the classical complex case ([3]). (See also [21, Chapter I] or [8].) Let $\mathcal{H}(G,\ \mathcal{A})$ be the Hecke algebra of $\mathcal{A}$–valued locally constant and compactly supported functions on G and $\mathcal{H}(G,\ L,\ \mathcal{A})$ its subalgebra of all L–biinvariant functions in $\mathcal{H}(G,\ \mathcal{A})$ for $L \subset G$ open compact. Usual relation between non–degenerate $\mathcal{H}(G,\ \mathcal{A})$–modules and smooth $(G,\ \mathcal{A})$–modules is valid as well as usual results for irreducible $(G,\ \mathcal{A})$–module regarding irreducibility of V^L. The main result of Section 3 is Theorem 3.3 in which we give very explicit construction of irreducible $(G,\ \mathcal{A})$–module V from the known irreducible module V^L for $\mathcal{H}(G,\ L,\ \mathcal{A})$. This is an improvement over the classical treatment in ([3], Proposition 2.10 c)) and it is needed for many results that follow in this paper such as the description of ring of endomorphisms in Theorem 4.1 which is the main result of Section 4, as well as the following fundamental result which is the main result of Section 5 (see Theorem 5.1).

THEOREM 1.5. *Assume that $\mathcal{A}$ is a field and hence an extension of $\mathbb{Q}$, since it is a $\mathbb{Q}$-algebra. Let $L \subset G$ be an open compact subgroup. Let V be an irreducible $(\mathcal{A},\ G)$–module such that $V^L \neq 0$ and $\mathcal{A}$–finite dimensional (i.e., V^L is an $\mathcal{A}$–admissible irreducible $\mathcal{H}(G,\ L,\ \mathcal{A})$–module). Then, for any field extension $\mathcal{A} \subset \mathcal{B}$, there exists irreducible $(\mathcal{B},\ G)$–modules $V_1, \ldots, V_t$ such that the following holds:*

(i) $V_i^L \neq 0$ *for all* $1 \leq i \leq t$.
(ii) V_i^L *are $\mathcal{B}$–admissible irreducible $\mathcal{H}(G,\ L,\ \mathcal{B})$–modules.*
(iii) $V_{\mathcal{B}} \overset{def}{=} \mathcal{B} \otimes_{\mathcal{A}} V \simeq V_1 \oplus \cdots \oplus V_t$ *as $(\mathcal{B},\ G)$–modules.*

We warn the reader that we do not assume that V is $\mathcal{A}$–admissible but that $V^L \neq 0$ is $\mathcal{A}$–admissible. On the level of L–invariants, the decomposition in (iii) is contained in Lemma 5.2 and it is based on some very simple facts from the theory of semi–simple rings ([13, Chapter XVII]). A more complicated case of positive characteristic requires more elaborated tools ([11, Theorem I.1]).

We warn the reader that because of Theorem 1.4, the assumption that $\mathcal{A}$ is a field is expected. Theorem 1.5 has many applications. They are contained in Section 6. We recall just the following one (see Corollary 6.1).

COROLLARY 1.6. *Assume that $\mathcal{A}$ is any subfield of $\mathbb{C}$. Let $L \subset G$ be an open compact subgroup. Let V be an irreducible $(\mathcal{A},\ G)$–module such that $V^L \neq 0$ and $\mathcal{A}$–finite dimensional. Then, V is $\mathcal{A}$–admissible (see Definition 1.2).*

This is proved reducing to the well–known result in the complex case via Theorem 1.5. We remind the reader that by a result of Jacquet ([16, Theorem VI.2.2]), every irreducible $(\mathbb{C},\ G)$–module is $\mathbb{C}$–admissible. But in the generality that we consider we are not sure that every irreducible $(\mathcal{A},\ G)$–module is $\mathcal{A}$–admissible without assumptions stated in Corollary 1.6. without additional and probably much deeper considerations which was kindly provided to us by Vignéras ([22]). In the present form Corollary 1.6 is quite useful since it fundamentally contributes to the construction of unramified irreducible representations (see Theorem 7.2 in Section 7).

THEOREM 1.7. *Let k be a non–Archimedean local field. Let $\mathcal{O} \subset k$ be its ring of integers, and let ϖ be a generator of the maximal ideal in $\mathcal{O}$. Let q be the number of elements in the residue field $\mathcal{O}/\varpi\mathcal{O}$. Assume that G is a k–split Zariski connected reductive group. Let A be its maximal k–split torus. Let W be the Weyl group of A in G. We write $\hat{A}$ for the complex torus dual to A. The orbit space*

$$X \stackrel{def}{=} \hat{A}/W$$

is an affine variety defined over $\mathbb{Q}$. Let $K = G(\mathcal{O})$ be a hyperspecial maximal compact subgroup of G. We normalize a Haar measure on G such that $\int_K dg = 1$ (see Section 3). Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$. Let $\mathcal{A}$ be any subfield of $\overline{\mathbb{Q}}$ if G is simply–connected, or an extension of $\mathbb{Q}(q^{1/2})$ in $\overline{\mathbb{Q}}$ otherwise. We define the (commutative) Hecke algebra $\mathcal{H}(G,\ K,\ \mathcal{A})$ with respect to above fixed Haar measures. Then, we have the following:

(i) *(Satake isomorphims over subfields of $\overline{\mathbb{Q}}$) Maximal ideals in $\mathcal{H}(G, K, \mathcal{A})$ are parame-terized by points in $X(\overline{\mathbb{Q}})$ such that points in $X(\overline{\mathbb{Q}})$ give the same maximal ideal if and only if they are $Gal(\overline{\mathbb{Q}}/\mathcal{A})$–conjugate: for $x \in X(\overline{\mathbb{Q}})$, we denote by $\mathfrak{m}_{x,\mathcal{A}}$ the corresponding maximal ideal. The corresponding quotient $\mathcal{H}(G,\ K,\ \mathcal{A})/\mathfrak{m}_{x,\mathcal{A}}$ is denoted by $F(x, \mathcal{A})$. It is a finite (field) extension of $\mathcal{A}$, and it is also naturally irreducible $\mathcal{A}$–admissible $\mathcal{H}(G,\ K,\ \mathcal{A})$–module. The map $Gal(\overline{\mathbb{Q}}/\mathcal{A}).x \mapsto F(x, \mathcal{A})$ is a bijection between $Gal(\overline{\mathbb{Q}}/\mathcal{A})$–orbits in $X(\overline{\mathbb{Q}})$, and the set of equivalence classes of irreducible $\mathcal{A}$–admissible irreducible $\mathcal{H}(G,\ K,\ \mathcal{A})$–modules.*

(ii) *For each $x \in X(\overline{\mathbb{Q}})$, the $(\mathcal{A},\ G)$–module (see Theorem 3.3 for the notation)*

$$\mathcal{V}(x, \mathcal{A}) \stackrel{def}{=} \mathcal{V}(\mathfrak{m}_x, K)$$

is an irreducible and $\mathcal{A}$–admissible $(\mathcal{A},\ G)$–module. We have

$$\mathcal{V}^K(x, \mathcal{A}) \simeq F(x, \mathcal{A})$$

as $\mathcal{H}(G,\ K,\ \mathcal{A})$–modules, and

$$\mathrm{End}_{(\mathcal{A},\ G)}(\mathcal{V}(x, \mathcal{A})) \simeq F(x, \mathcal{A}).$$

(iii) *$\mathcal{V}(x, \mathcal{A})$ is absolutely irreducible (see Corollary 6.3 for the standard definition of absolute irreducibility) if and only if $x \in X(\mathcal{A})$.*

(iv) *Let $x \in X(\overline{\mathbb{Q}})$. Then, for any Galois extension $\mathcal{A} \subset \mathcal{B}$ which contains $F(x, \mathcal{A})$, $\mathcal{V}(x, \mathcal{B})$ is absolutely irreducible. Moreover, there exist*

$t = \dim_{\mathcal{A}}\ F(x, \mathcal{A})$ mutually different elements (among them x) in $Gal(\overline{\mathbb{Q}}/\mathcal{B}).x$, say $x = y_1, y_2, \ldots, y_t$, such that we have the following:

$$(\mathcal{V}(x, \mathcal{A}))_{\mathcal{B}} \overset{def}{=} \mathcal{B} \otimes_{\mathcal{A}}\ \mathcal{V}(x, \mathcal{A}) \simeq \mathcal{V}(x, \mathcal{B}) \oplus \mathcal{V}(y_2, \mathcal{B}) \oplus \cdots \oplus \mathcal{V}(y_t, \mathcal{B}).$$

Furthermore, $V(x, \mathcal{B}), \mathcal{V}(y_2, \mathcal{B}), \ldots, \mathcal{V}(y_t, \mathcal{B})$ are mutually non–isomorphic $(\mathcal{B},\ G)$–modules.

(v) *(Classification of unramified admissible representations over subfields of $\overline{\mathbb{Q}}$) The map*

$$Gal(\overline{\mathbb{Q}}/\mathcal{A}).x \mapsto \mathcal{V}(x, \mathcal{A})$$

is a bijection between $Gal(\overline{\mathbb{Q}}/\mathcal{A})$–orbits in $X(\overline{\mathbb{Q}})$, and the set of equivalence classes of unramified $\mathcal{A}$–admissible irreducible $(\mathcal{A},\ G)$–modules.

Besides above mentioned result, the key point is the description of Satake isomorphism ([7]) over $\mathbb{Z}$ due to Gross ([10]) and a technical lemma about affine varieties proved in the Appendix (see Lemma A.1 in Section A).

The first ideas about the content of the paper were realized while the author visited the Hong Kong University of Science and Technology in January of 2018. The author would like to thank A. Moy and the Hong Kong University of Science and Technology for their hospitality. I would like to thank Marko Tadić for showing me the reference [17]. The discussions with Marie–France Vignéras and William Casselman were useful in the process of revision of the manuscript. Marie–France Vignéras kindly provided me with the references [11, 12, 21], and wrote the proof that every irreducible representation is admissible ([22]). We would like to thank two referees for suggestions on improvements of our exposition and several minor corrections.

2. Basic properties of $(\mathcal{A},\ G)$–modules

In this section we assume that G is an l–group ([3, 1.1]). This means that G is a topological group with Hausdorff topology such that there is a fundamental system of neighborhoods of the unit element consisting of open compact subgroups. We always assume that $\mathcal{A}$ is a $\mathbb{Q}$–algebra. In this section we prove basic properties of $(\mathcal{A},\ G)$–modules.

We start with the following result.

Lemma 2.1. *The functor $V \mapsto V^L$ from the category $\mathcal{C}\,(\mathcal{A},\ G)$ into category of $\mathcal{A}$–modules is exact.*

Proof. It is enough to show that if $V_1 \longrightarrow V_2 \longrightarrow V_3$ is an exact sequence in $\mathcal{C}\,(\mathcal{A},\ G)$, then $V_1^L \longrightarrow V_2^L \longrightarrow V_3^L$ is also exact. It is obvious that the image of V_1^L is contained in the kernel of $V_2^L \longrightarrow V_3^L$. Conversely, let v be an element in the kernel of $V_2^L \longrightarrow V_3^L$. Then, there exists $w \in V_1$ whose image is v under the map $V_1 \longrightarrow V_2$. Let $L' \subset L$ be an open compact subgroup such that $w \in V_1^{L'}$ and $v \in V_2^{L'}$. Let

$$w_0 = \frac{1}{\#\left(L/L'\right)} \left(\sum_{\gamma \in L/L'} \gamma.w \right).$$

Then, $w_0 \in V_1^L$, and v is image of w_0 under the map $V_1 \longrightarrow V_2$ since v is L–stable. ∎

LEMMA 2.2. *Assume that $\mathcal{A}$ is a Noetherian ring. Let $\mathcal{C}_{adm}\left(\mathcal{A},\ G\right)$ be a full subcategory of $\mathcal{C}\left(\mathcal{A},\ G\right)$ consisting of all $\mathcal{A}$–admissible modules (see Definition 1.2). Then, $\mathcal{C}_{adm}\left(\mathcal{A},\ G\right)$ is an Abelian category.*

PROOF. Let $L \subset G$ be an open–compact subgroup. Let V be an object in $\mathcal{C}_{adm}\left(\mathcal{A},\ G\right)$. Then, by definition V^L is finitely generated $\mathcal{A}$–module. If $W \subset V$ is a submodule, then $W^L \subset V^L$. Hence, W^L is finitely generated $\mathcal{A}$–module since $\mathcal{A}$ is a Noetherian ring. Next, if U is a quotient module of V. Then, U^L is a quotient module of V^L. Now, we apply Lemma 2.1 to prove that U^L is finitely generated $\mathcal{A}$–module. This shows that submodules and quotients belong to $\mathcal{C}_{adm}\left(\mathcal{A},\ G\right)$. This implies that category $\mathcal{C}_{adm}\left(\mathcal{A},\ G\right)$ is Abelian. ∎

The following lemma is one of the key technical results.

LEMMA 2.3. *Let $\mathfrak{a} \subset \mathcal{A}$ be an ideal of $\mathcal{A}$. Then, for any $\left(\mathcal{A},\ G\right)$–module V, and for any open compact subgroup $L \subset G$, we have the following:*

$$\left(\mathfrak{a}V\right)^L = \mathfrak{a}V^L.$$

PROOF. Obviously, we have

$$\mathfrak{a}V^L \subset \left(\mathfrak{a}V\right)^L,$$

for any open–compact subgroup L.

Let $v \in \left(\mathfrak{a}V\right)^L$. Then, there exists $v_1, \ldots, v_l \in V$, $a_1, \ldots, a_l \in \mathfrak{a}$ such that

$$v = \sum_{i=1}^{l} a_i v_i.$$

We select $L' \subset L$ an open compact subgroup such that $v_1 \ldots, v_l \in V^{L'}$. Then

$$\#\left(L/L'\right) \cdot v = \sum_{i=1}^{l} a_i \left(\sum_{\gamma \in L/L'} \gamma.v_i \right).$$

Obviously, we have

$$\sum_{\gamma \in L/L'} \gamma.v_i \in V^L.$$

Thus, we have

$$\#\,(L/L')\cdot v\in \mathfrak{a}V^L.$$

□

Now, we consider the ring of all endomorphims of $\mathrm{End}_{(\mathcal{A},\ G)}\,(V)$ of an irreducible $(\mathcal{A},\ G)$–module V. See also Theorem 4.1 where we relate to the Hecke algebras. We remark that when G is countable at infinity, and $\mathcal{A}=\mathbb{C}$, this ring is always isomorphic to $\mathbb{C}$ (see [3, Proposition 2.11]). In general, the situation is more interesting.

LEMMA 2.4. *Let V be an irreducible $(\mathcal{A},\ G)$–module. Then, the annihilator of V, denoted by $\mathrm{Ann}_{\mathcal{A}}(V)$, in $\mathcal{A}$ is a prime ideal. Moreover, if we let $\mathfrak{p}=\mathrm{Ann}_{\mathcal{A}}(V)$, then the module V extends naturally to an irreducible $(k(\mathfrak{p}),\ G)$–module, where $k(\mathfrak{p})$ is the field of fractions of $A/\mathfrak{p}$. The ring $\mathrm{End}_{(\mathcal{A},\ G)}\,(V)$ of all endomorphisms is a divison algebra naturally isomorphic to $\mathrm{End}_{(k(\mathfrak{p}),\ G)}\,(V)$, and therefore central over $k(\mathfrak{p})$.*

PROOF. By definition of a prime ideal, we need to show that $ab\in Ann_{\mathcal{A}}(V)$ implies $a\in Ann_{\mathcal{A}}(V)$ or $b\in Ann_{\mathcal{A}}(V)$. Indeed, if $b\notin Ann_{\mathcal{A}}(V)$, then bV is a non–zero $(\mathcal{A},\ G)$–submodule of V. Hence, $bV=V$ because V is irreducible. Hence,

$$aV=a\,(bV)=abV=0,$$

since $ab\in Ann_{\mathcal{A}}(V)$. This implies $a\in Ann_{\mathcal{A}}(V)$.

By Schur's lemma, $\mathrm{End}_{(\mathcal{A},\ G)}\,(V)$ is a divison algebra. Obviously, $\mathcal{A}/\mathfrak{p}$ embeddes into the center of $\mathrm{End}_{(\mathcal{A},\ G)}\,(V)$. The center is a field. Therefore, $k(\mathfrak{p})$ embeddes into the center. Now, V can be regarded as as a $(k(\mathfrak{p}),\ G)$–module. It is obviously irreducible since V was originally irreducible $(\mathcal{A},\ G)$–module. Next, it is clear that

$$\mathrm{End}_{(k(\mathfrak{p}),\ G)}\,(V)\subset \mathrm{End}_{(\mathcal{A},\ G)}\,(V)\,.$$

Finally, since $k(\mathfrak{p})$ belongs to the center of $\mathrm{End}_{(\mathcal{A},\ G)}\,(V)$, we have

$$\mathrm{End}_{(\mathcal{A},\ G)}\,(V)\subset \mathrm{End}_{(k(\mathfrak{p}),\ G)}\,(V)\,.$$

□

Here is an example for Lemma 2.4. The example shows that if an irreducible module is not $\mathcal{A}$–admissible, then the annihilator could be a prime ideal which is not maximal. Consider the ring of polynomials $\mathbb{Q}[T]$ over $\mathbb{Q}$. Then, we let $\mathcal{A}$ to be the localization of $\mathbb{Q}[T]$ at the prime ideal generated by T. Let $\mathcal{K}$ be the field of fractions of $\mathbb{Q}[T]$ and of $\mathcal{A}$. Then, $\mathcal{A}$ is a $\mathbb{Q}$–algebra and a local ring with a unique maximal ideal, say $\mathfrak{m}$, the one generated by T. We let $G=\mathcal{K}^\times$ and equip it with a discrete topology. In this way, we obtain an l–group. Let $V=\mathcal{K}$. Then, V is in an obvious way an irreducible $(\mathcal{A},\ G)$–module. Its annihilator is $\{0\}$ which is a prime ideal in $\mathcal{A}$. We remark that V is not $\mathcal{A}$–admissible since $\mathcal{K}$ is not finitely generated over $\mathcal{A}$. We remark also

$\mathfrak{m}V = V$, and $\mathrm{End}_{(\mathcal{A},\ G)}(V) = \mathcal{K}$. Finally, we remark that G is countable at infinity since it is a countable set.

The following theorem gives further description of irreducible $\mathcal{A}$–admissible modules and an improvement over Lemma 2.4.

Theorem 2.5. *Assume that $\mathcal{A}$ is a $\mathbb{Q}$–algebra, and G an l–group. Then, we have the following:*

(i) *For every irreducible $\mathcal{A}$–admissible $(\mathcal{A},\ G)$–module V, the annihilator of V is a maximal ideal. In fact, if we write $\mathfrak{m} = \mathrm{Ann}_{\mathcal{A}}(V)$, then V is an irreducible $\mathcal{A}/\mathfrak{m}$–admissible $(\mathcal{A}/\mathfrak{m},\ G)$–module.*

(ii) *Let $\mathrm{Irr}_{\mathfrak{m}}$ be the set of equivalence classes of irreducible $\mathcal{A}/\mathfrak{m}$–admissible $(\mathcal{A}/\mathfrak{m},\ G)$–modules. Then, the disjoint union*

$$\cup_{\mathfrak{m}}\ \mathrm{Irr}_{\mathfrak{m}} \quad (\mathfrak{m} \text{ ranges over maximal ideal of } \mathcal{A})$$

can be taken to be the set of equivalence classes of irreducible $\mathcal{A}$–admissible $(\mathcal{A},\ G)$–modules.

(iii) *Assume that $\mathcal{A}$ is a finitely generated $\mathbb{C}$–algebra. Let $\mathrm{Irr}(G)$ be the set of equivalence classes of complex irreducible admissible representations of G (see [3]). Let $Max(\mathcal{A})$ be the set of all maximal ideals in $\mathcal{A}$. Then, the set $\mathrm{Irr}(G) \times Max(\mathcal{A})$ parameterizes irreducible $\mathcal{A}$–admissible $(\mathcal{A},\ G)$–modules.*

Proof. We prove (i). Since V is irreducible, for each maximal ideal $\mathfrak{m} \subset \mathcal{A}$, we have $\mathfrak{m}V = 0$ or $\mathfrak{m}V = V$. Assume that $\mathfrak{m}V = V$ for all $\mathfrak{m}$. Then, for an open compact subgroup $L \subset G$, applying Lemma 2.3, we must have

$$V^L = (\mathfrak{m}V)^L = \mathfrak{m}V^L,$$

for all $\mathfrak{m}$. Then, because of Lemma 2.6, we must have $V^L = 0$. Since L is arbitrary, we obtain $V = 0$. This is a contradiction. Thus, there exists at least one maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}V = 0$. Then, $\mathfrak{m} \subset Ann_{\mathcal{A}}(V)$. Hence,

$$Ann_{\mathcal{A}}(V) = \mathfrak{m}.$$

It is is obvious that (ii) follows from (i) at once. Finally for (iii), we remark that by Nullstellensatz $\mathcal{A}/\mathfrak{m} = \mathbb{C}$ for each $\mathfrak{m} \in Max(\mathcal{A})$. Hence, (iii) is an obvious consequence of (ii). □

Maintaining the notation of the theorem, the identity action of G on $\mathcal{A}/\mathfrak{m}$ is an example of irreducible $\mathcal{A}/\mathfrak{m}$–admissible $(\mathcal{A}/\mathfrak{m},\ G)$–module. We call it the trivial module. Therefore, $\mathrm{Irr}_{\mathfrak{m}}$ is always non–empty. When G is a reductive p–adic group, we will prove the existence of other more complicated modules. But in the present generality, G could be the trivial group, and we can not do better. Section 3 discusses the existence of irreducible $(\mathcal{A},\ G)$–modules via Hecke algebra adapted from the classical complex case ([3]).

The following general result follows from ([14, Chapter 4, Theorems 4.6., 4.8]) and it is needed in the proof of Theorem 2.5.

Lemma 2.6. *Let V be a finitely generated unital module over a commutative ring R with identity. Then, if $\mathfrak{m}V = V$ for all maximal ideals $\mathfrak{m} \subset R$, then $V = 0$.*

Proof. We include the proof for the sake of completeness. Let $V_{\mathfrak{m}}$ be the localization of V at $\mathfrak{m}$. Then, by the assumption of the lemma and Nakayama's lemma, $V_{\mathfrak{m}} = 0$.

Let $v \in V$. Then, by above observation, there exists

$$s_{v,\mathfrak{m}} \in R - \mathfrak{m}$$

such that

$$s_{v,\mathfrak{m}}.v = 0 \ \text{ in } V,$$

for all maximal ideals $\mathfrak{m}$.

The collection of all $s_{v,\mathfrak{m}}$, where $\mathfrak{m}$ ranges over all maximal ideals of R, generates an ideal, say I, that is not contained in any $\mathfrak{m}$. But, then

$$I = R.$$

Thus, there exists $\mathfrak{m}_1, \dots, \mathfrak{m}_k$, and $r_1, \dots, r_l \in R$ such that

$$1_R = \sum_{i=1}^{l} r_i s_{v,\mathfrak{m}_i}.$$

Then, we have

$$v = 1_R.v = \sum_{i=1}^{l} r_i s_{v,\mathfrak{m}}.v = 0.$$

This proves $V = 0$. □

Let $\mathcal{A} \subset \mathcal{B}$ be an extension of rings. Then, for $(\mathcal{A},\ G)$–module V we can consider $(\mathcal{B},\ G)$–module defined as follows:

$$V_{\mathcal{B}} \overset{def}{=} V_{\mathcal{B}/\mathcal{A}} \overset{def}{=} \mathcal{B} \otimes_{\mathcal{A}} V.$$

Lemma 2.7. *Assume that $\mathcal{A}$ is a $\mathbb{Q}$–algebra. Then, under the above assumptions, we have the following:*

(i) *For each open compact subgroup $L \subset G$, we have the following:*

$$V_{\mathfrak{B}}^{L} = \mathcal{B} \otimes_{\mathcal{A}} V^{L}.$$

(ii) *The $V_{\mathcal{B}}$ is $\mathcal{B}$–admissible whenever V is $\mathcal{A}$–admissible.*

(iii) *The assignment $V \mapsto V_{\mathcal{B}}$ can be regarded as a functor $\mathcal{C}(\mathcal{A},\ G) \longrightarrow \mathcal{C}(\mathcal{B},\ G)$ and as a functor $\mathcal{C}_{adm}(\mathcal{A},\ G) \longrightarrow \mathcal{C}_{adm}(\mathcal{B},\ G)$.*

Proof. (i) has the proof similar to the proof of Lemma 2.3. (ii) follows from (i). Finally, the first functor in (iii) is obvious. The second one is well–defined because of (ii). □

Let V be an $(\mathcal{A},\ G)$–module. Let $\mathfrak{p} \subset \mathcal{A}$ be a prime ideal, and let $\mathcal{A}_\mathfrak{p}$ be the localization of $\mathcal{A}$ at $\mathfrak{p}$. Then, we write $V_\mathfrak{p}$ for the $(\mathcal{A}_\mathfrak{p},\ G)$–module $V_{\mathcal{A}_\mathfrak{p}}$.

Theorem 2.8. *Assume that $\mathcal{A}$ is a $\mathbb{Q}$–algebra, and G an l–group. Let V be an irreducible $\mathcal{A}$–admissible $(\mathcal{A},\ G)$–module. Then, for a prime ideal $\mathfrak{p} \subset \mathcal{A}$, we have the following:*

$$V_\mathfrak{p} = \begin{cases} \text{is } \mathcal{A}_\mathfrak{p}\text{–admissible irreducible } (\mathcal{A}_\mathfrak{p},\ G)\text{–module}, & \text{if } \mathfrak{p} = Ann_\mathcal{A}(V), \\ 0, & \text{if } \mathfrak{p} \neq Ann_\mathcal{A}(V). \end{cases}$$

Moreover, if $\mathfrak{p} = Ann_\mathcal{A}(V)$, then

$$Ann_{\mathcal{A}_\mathfrak{p}}(V_\mathfrak{p}) = \mathfrak{m}_\mathfrak{p},$$

where the right–hand side is the localization of $\mathfrak{p}$. Using the canonical isomorphism of localizations $\mathcal{A}/\mathfrak{p} \simeq \mathcal{A}_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$, $V_\mathfrak{p}$ is isomorphic to V as an $(\mathcal{A}/\mathfrak{p},\ G)$–module.

Proof. We recall that $Ann_\mathcal{A}(V)$ is a maximal ideal. Therefore, if $\mathfrak{p} \neq Ann_\mathcal{A}(V)$ is a prime ideal, then $Ann_\mathcal{A}(V) - \mathfrak{p} \neq \emptyset$. Select $x \in Ann_\mathcal{A}(V) - \mathfrak{p}$. Then $x/1 \in \mathcal{A}_\mathfrak{p}$ is invertible and it acts as zero on $V_\mathfrak{p}$. Thus, $V_\mathfrak{p}$ is zero.

Assume $\mathfrak{p} = Ann_\mathcal{A}(V)$. Then, the maximal ideal $\mathfrak{m}_\mathfrak{p}$, obtained by the localization of $\mathfrak{p}$, obviously annihilates $V_\mathfrak{p}$. None of the other elements in $\mathcal{A}_\mathfrak{p}$ can kill $V_\mathfrak{p}$ since by the properties of the localization and irreducibility of V would exist an $s \in \mathcal{A} - \mathfrak{p}$ which kills V which is not possible. This proves $Ann_{\mathcal{A}_\mathfrak{p}}(V) = \mathfrak{m}_\mathfrak{p}$.

Next, we may regard $V_\mathfrak{p}$ as $(\mathcal{A}/\mathfrak{p},\ G)$–module. Hence, the argument similar to the one used in the computation of the annihilator above shows that $V \longrightarrow V_\mathfrak{p}$, given by $v \mapsto 1 \otimes v$ is injective map of $(\mathcal{A}/\mathfrak{p},\ G)$–modules. Since, the usual properties of localization imply

$$\mathcal{A}/\mathfrak{p} \simeq \mathcal{A}_\mathfrak{p}/\mathfrak{m}_\mathfrak{p},$$

Hence, the map is an isomorphism of $(\mathcal{A}/\mathfrak{p},\ G)$–modules. Hence, $V_\mathfrak{p}$ is irreducible $(\mathcal{A}_\mathfrak{p},\ G)$–module. It is $\mathcal{A}_\mathfrak{p}$–admissible by Lemma 2.7 (ii). ∎

3. Existence of irreducible representations

In this section we assume that $\mathcal{A}$ is a $\mathbb{Q}$–algebra, and G an l–group. The goal of this section is to discuss existence of irreducible $(\mathcal{A},\ G)$–modules. As it may be expected, we use Hecke algebra adapted from the classical complex case ([3]) but there are some improvements of the classical complex case. The main result of this section is Theorem 3.3. We remark that the basic idea of the present approach to the construction of Hecke algebra over $\mathcal{A}$ was already well–known (see [15, 2.2], for the case of profinite groups).

Let $L \subset G$ be an open compact subgroup. Let $\mathcal{A}$ be a $\mathbb{Q}$–algebra. We consider the space $\mathcal{H}(G, L, \mathcal{A})$ of all functions $f : G \longrightarrow \mathcal{A}$ which are L–biinvariant and have compact support i.e., they are supported on finitely many double cosets LxL, where $x \in G$. If 1_T denotes the characteristic function of a subset $T \subset G$, then every function $f \in \mathcal{H}(G, L, \mathcal{A})$ can be written uniquely in the form:

$$f = \sum_{x \in L\backslash G/L} a_x \cdot 1_{LxL}, \text{ where } a_x \in \mathcal{A}, \text{ equal to zero for all but finitely many } x.$$

The Hecke algebra $\mathcal{H}(G, \mathcal{A})$ with coefficients in $\mathcal{A}$ is just the union of all $\mathcal{H}(G, L, \mathcal{A})$ when L ranges over all open compact subgroups of G.

When $\mathcal{A} = \mathbb{C}$, we obtain usual Hecke algebras ([8, 3]) The product is given by the convolution

$$f \star g(x) = \int_G f(xy^{-1})g(y)dy.$$

We recall that $\mathcal{H}(G, L, \mathbb{C})$ is associative $\mathbb{C}$–algebra with identity $1_L/vol(L)$. It is a subalgebra of $\mathcal{H}(G, \mathbb{C})$ for all L. As it is easy to see and also can be seen by inspecting the construction of Haar measure on G (see the proof of [3, Proposition 1.18]), we see that if we select an open compact subgroup and require that its volume is equal to one (a rational number!), then all volumes of all open compact subgroups are rational. Moreover, above defined convolution $\star$ makes $\mathcal{H}(G, \mathbb{Q})$ into an associative $\mathbb{Q}$–algebra (in general without identity), and $\mathcal{H}(G, L, \mathbb{Q})$ an associative $\mathbb{Q}$–algebra with identity $1_L/vol(L)$. Let us explain why $\mathcal{H}(G, \mathbb{Q})$ is closed under convolution. The reader can easily show that this boils down to showing that $1_{xL} \star 1_{yL} \in \mathcal{H}(G, \mathbb{Q})$ for all $x, y \in G$, and open compact subgroups $L \subset G$. Indeed, we have the following:

$$\begin{aligned} 1_{xL} \star 1_{yL}(z) &= \int_G 1_{xL}(zt^{-1}) 1_{yL}(t)dt \\ &= \int_{yL} 1_{xL}(zt^{-1})dt = vol\left(\left(Lx^{-1}z\right) \cap yL\right) \\ &= M(x, y, z) \cdot vol\left(L \cap yLy^{-1}\right) \in \mathbb{Q}, \end{aligned} \tag{3.1}$$

where $M(x, y, z)$ is the number of right cosets of the open compact subgroup $L \cap yLy^{-1}$ in which is decomposed $Lx^{-1}z \cap yL$. We remark that $Lx^{-1}z \cap yL \neq \emptyset$ is equivalent to $zL = xl_1yL$ for some $l_1 \in L$ determined uniquely modulo left coset $l_1'\left(L \cap yLy^{-1}\right)$. Also, we have the following:

$$Lx^{-1}z \cap yL = Ll_1y \cap yL = Ly \cap yL = \left(L \cap yLy^{-1}\right) \cdot y.$$

This implies that $M(x, y, z) = 1$ whenever $Lx^{-1}z \cap yL \neq \emptyset$.

An explicit computation using defining integral shows that $1_{xL} \star 1_{yL}$ is right–invariant under L. Thus, if we write

$$G = \cup_z zL \quad \text{(disjoint union)}, \tag{3.2}$$

then

$$1_{xL} \star 1_{yL} = \sum_z M(x, y, z) \cdot vol\left(L \cap yLy^{-1}\right) \cdot 1_{zL}. \tag{3.3}$$

The sum is of course finite since $Lx^{-1}z \cap yL \neq \emptyset$ implies that $x^{-1}z \in LyL$. This proves our claim about $\mathcal{H}(G, \mathbb{Q})$. We fix such choice of Haar measure and define $\star$ as we explained.

Now, it is obvious that as $\mathbb{Q}$–vector spaces

$$\mathcal{H}(G, L, \mathcal{A}) = \mathcal{H}(G, L, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathcal{A},$$
$$\mathcal{H}(G, \mathcal{A}) = \mathcal{H}(G, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathcal{A}.$$

This enables to define the structure of associative $\mathcal{A}$–algebra $\mathcal{H}(G, L, \mathcal{A})$ and $\mathcal{H}(G, \mathcal{A})$. Furthermore,

$$\epsilon_L = \epsilon_{L,\mathcal{A}} = \frac{1_L}{vol(L)} \otimes_{\mathbb{Q}} 1_{\mathcal{A}}. \tag{3.4}$$

is the identity of $\mathcal{H}(G, L, \mathcal{A})$. Furthermore, $\mathcal{H}(G, L, \mathcal{A})$ is a subalgebra of $\mathcal{H}(G, \mathcal{A})$, for all open compact subgroups L. We omit $\otimes 1_{\mathcal{A}}$ from the notation in this and similar situations in the text that follows.

Let V be an $(\mathcal{A},\ G)$–module. Then there exists a unique (subject to the choice of Haar measure above) homomorphism of $\mathcal{A}$–algebras $\mathcal{H}(G, \mathcal{A}) \longrightarrow \mathrm{End}_{\mathcal{A}}(V)$ defined as follows. For $f \in \mathcal{H}(G, \mathcal{A})$, and $v \in V$, we select an open compact subgroup $L \subset G$ such that f is right invariant by L, implying that we can write f as a finite sum $f = \sum_x f(x) 1_{xL}$, and $v \in V^L$. Then, we let $f.v = vol(L) \cdot \sum_x f(x) x.v$. This agrees with the usual definition $\int_G f(y) y.v dy$ when $\mathcal{A} = \mathbb{C}$ (see [3, 2.3]). Let us show that our definition is correct. Indeed, if $L' \subset G$ is another open compact subgroup such that f is right invariant by L', implying that we can write f as a finite sum $f = \sum_{x'} f(x') 1_{x'L'}$, and $v \in V^{L'}$. We decompose into disjoint unions of left cosets:

$$L = \cup_{l_1} l_1 L \cap L' \quad \text{and} \quad L' = \cup_{l'_1} l'_1 L \cap L'.$$

Then, we have

$$\begin{aligned}
vol(L') \cdot \sum_{x'} f(x') x'.v &= vol(L') \cdot \sum_{x'} \frac{1}{[L' : L \cap L']} \left(\sum_{l'_1} f(x' l'_1)\ x' l'_1.v \right) \\
&= vol(L \cap L') \cdot \sum_{x'} \sum_{l'_1} f(x' l'_1)\ x' l'_1.v \\
&= vol(L \cap L') \cdot \sum_{x} \sum_{l_1} f(x l_1)\ x l_1.v \\
&= vol(L) \cdot \sum_{x} f(x)\ x.v.
\end{aligned}$$

This shows that the action of elements of $\mathcal{H}(G, \mathcal{A})$ is well–defined. Next, we check that constructed map $\mathcal{H}(G, \mathcal{A}) \longrightarrow \mathrm{End}_{\mathcal{A}}(V)$ is a homomorphism of $\mathcal{A}$–algebras. Indeed, for an arbitrary open compact subgroup $L \subset G$, and

$x, y \in G$, we put $f = 1_{xL} \otimes 1_{\mathcal{A}}$ and $g = 1_{yL} \otimes 1_{\mathcal{A}}$. Then, for $v \in V^L$, we remark that

$$y.v \in V^{L \cap yLy^{-1}}.$$

If we write as a disjoint union

$$L = \cup_{l_1} \ l_1 \left(L \cap yLy^{-1}\right),$$

then by definition of the action

$$fg.v = f.\left(g.v\right) = vol(L) f.\left(y.v\right) = vol(L) vol\left(L \cap yLy^{-1}\right) \sum_{l_1} x l_1 y.v.$$

On the other hand using (3.2) and (3.3), by the definition of the action, we have the following:

$$\begin{aligned} f \star g.v &= vol(L) \cdot \sum_{\substack{z \quad \text{as in (3.2)} \\ Lx^{-1}z \cap yL \neq \emptyset}} vol\left(L \cap yLy^{-1}\right) z.v \\ &= vol(L) vol\left(L \cap yLy^{-1}\right) \sum_{l_1} x l_1 y.v \end{aligned}$$

This proves the claim that $\mathcal{H}(G, \mathcal{A}) \longrightarrow \mathrm{End}_{\mathcal{A}}(V)$ is a homomorphism of $\mathcal{A}$–algebras. Moreover, the constructed $\mathcal{H}(G, \mathcal{A})$–module V is non–degenerate (see [3, 2.5]) since for any $v \in V$ there exists an open compact subgroup $L \subset G$ such that (see (3.4))

$$\epsilon_L.v = v.$$

Furthermore, it easy to check that

$$x.\left(f.v\right) = \left(l_x f\right).v, \quad f \in \mathcal{H}(G, \mathcal{A}), \quad v \in V, \tag{3.5}$$

where l_x is the left translation $l_x f(y) = f(x^{-1}y)$.

Finally, it is easy to check the following standard result.

Lemma 3.1. *A non–degenerate $\mathcal{H}(G, \mathcal{A})$–module gives rise to a unique $(\mathcal{A},\ G)$–module such that (3.5) holds. The category of all $(\mathcal{A},\ G)$–modules can be identified with the category of all non–degenerate $\mathcal{H}(G, \mathcal{A})$–modules. In particular, an irreducible $\mathcal{H}(G, \mathcal{A})$–module is also irreducible $(\mathcal{A},\ G)$–module.*

The following lemma is also standard (see [3, Proposition 2.10]).

Lemma 3.2. (i) *For an irreducible $(\mathcal{A},\ G)$–module V, and an open compact subgroup $L \subset G$, $\mathcal{H}(G, L, \mathcal{A})$–module V^L is either 0 or irreducible.*

(ii) *Let $L \subset G$ be an open–compact subgroup. Assume that V_i, $i = 1, 2$, are irreducible $(\mathcal{A},\ G)$–modules such that $V_i^L \neq 0$, $i = 1, 2$. Then, V_1 is equivalent to V_2 as $(\mathcal{A},\ G)$–modules if and only if V_1^L is equivalent to V_2^L as $\mathcal{H}(G, L, \mathcal{A})$–modules.*

PROOF. We just sketch the proof. Let $L \subset G$ be an open–compact subgroup.

Then, ϵ_L defined in (3.4) is the identity of the associative algebra $\mathcal{H}(G, L, \mathcal{A})$. Moreover, we have the following:

$$\mathcal{H}(G, L, \mathcal{A}) = \epsilon_L \mathcal{H}(G, \mathcal{A}) \, \epsilon_L. \tag{3.6}$$

Now, we sketch the proof of (i). If $0 \subsetneqq W \subsetneqq V^L$ is a $\mathcal{H}(G, L, \mathcal{A})$–submodule of V^L. Then, $V_1 \overset{def}{=} \mathcal{H}(G, \mathcal{A}) W$ is an $(\mathcal{A},\ G)$–submodule of V such that $V_1^L = W$. Since V is irreducible and $V^L \neq W$ this a contradiction. For (ii), by adjusting the notation, we proceed as in the proof of b) in ([3, Proposition 2.10]). □

The following theorem is also standard. It is a part of ([3, Proposition 2.10 c)]) but we make it more explicit.

THEOREM 3.3. *Let $L \subset G$ be an open–compact subgroup. Then, for each maximal proper left ideal $I \subset \mathcal{H}(G, L, \mathcal{A})$, there exists a unique left ideal J' of $\mathcal{H}(G, \mathcal{A})$ such that the following three conditions hold:*

(i) $J' \subset \mathcal{H}(G, \mathcal{A}) \, \epsilon_L$
(ii) $I \subset J'$
(iii) $\mathcal{H}(G, \mathcal{A}) \, \epsilon_L / J'$ *is irreducible.*

The left ideal J' is a unique maximal proper left–ideal, denoted by $J_I = J_{I,L}$, in $\mathcal{H}(G, \mathcal{A}) \, \epsilon_L$ which contains I. It is a sum of all proper left ideals in $\mathcal{H}(G, \mathcal{A}) \, \epsilon_L$ which contain I. Moreover, $\epsilon_L \star J_{I,L} = I$.

(iv) *Regarding*
$$\mathcal{V}(I, L) \overset{def}{=} \mathcal{H}(G, \mathcal{A}) \, \epsilon_L / J_{I,L}$$
as an $(\mathcal{A},\ G)$–module, we have that its space of L–invariants is isomorphic to (irreducible module) $\mathcal{H}(G, L, \mathcal{A}) / I$ as a $\mathcal{H}(G, L, \mathcal{A})$–module. Up to isomorphism, $\mathcal{V}(I, L)$ is a unique irreducible $(\mathcal{A},\ G)$–module with this property.

(v) *The $(\mathcal{A},\ G)$–module*
$$\mathcal{W}(I, L) \overset{def}{=} \mathcal{H}(G, \mathcal{A}) \, \epsilon_L / \mathcal{H}(G, \mathcal{A}) \, I$$
has a unique maximal proper subrepresentation, and the corresponding quotient is $\mathcal{V}(I, L)$. The canonical projection $\mathcal{W}(I, L)^L \longrightarrow \mathcal{V}(I, L)^L$ is isomorphism of $\mathcal{H}(G, L, \mathcal{A})$–modules.

(vi) *If $f \in \mathcal{H}(G, L, \mathcal{A})$ does not belong to all maximal left ideals of $\mathcal{H}(G, L, \mathcal{A})$, then there exists an irreducible $(\mathcal{A},\ G)$–module such that f acts as a non–zero operator. More explicitly, if $f \notin I$, then f is not zero on $\mathcal{V}(I, L)$.*

(vii) *The ideal $I \cap \mathcal{A}\epsilon_L$ is a prime ideal in $\mathcal{A}\epsilon_L \equiv \mathcal{A}$. The ideal is maximal, if $\mathcal{A}$–module $\mathcal{H}(G, L, \mathcal{A}) / I$ is finite.*

Proof. If J is a proper left ideal contained in $\mathcal{H}(G,\mathcal{A})\,\epsilon_L$ which contains I, then $\epsilon_L J$ is a left ideal in $\mathcal{H}(G,L,\mathcal{A})$ which contains I. Since I is maximal proper left ideal, we must have $\epsilon_L J = I$ or $\epsilon_L J = \mathcal{H}(G,L,\mathcal{A})$. In the latter case, we have

$$J \supset \mathcal{H}(G,\mathcal{A})\,\epsilon_L J = \mathcal{H}(G,\mathcal{A})\,\mathcal{H}(G,L,\mathcal{A}) = \mathcal{H}(G,\mathcal{A})\,\epsilon_L.$$

Hence,

$$J = \mathcal{H}(G,\mathcal{A})\,\epsilon_L.$$

This is a contradiction. Therefore, if J_I denotes the sum of all proper left ideals J containing I, then

$$\epsilon_L J_I = I.$$

Obviously, J_I satisfies conditions (i)–(iii). The uniqueness is clear from its construction. Of course, we need to establish the existence of at least one such ideal J to be able to define J_I. This is easy. We just need to take $J = \mathcal{H}(G,\mathcal{A})\,I$.

For (iv), regarding them as $(\mathcal{A},\ G)$–modules and using Lemma 2.1, we have

$$\left(\mathcal{H}(G,\mathcal{A})\,\epsilon_L/J_I\right)^L = \mathcal{H}(G,L,\mathcal{A})\,/\epsilon_L J_I = \mathcal{H}(G,L,\mathcal{A})\,/I.$$

The uniqueness in the last part of (iv) follows from Lemma 3.2 (ii). Next, (v) is just the reformulation of maximality and uniqueness of J_I. (vi) is obvious. We remark that maximal left ideals of $\mathcal{H}(G,L,\mathcal{A})$ exist by Zorn's lemma. Finally, (vii) follows from Lemma 2.6 using simplified arguments of Lemma 2.4 and Theorem 2.5. □

Corollary 3.4. *Let $L \subset G$ be an open–compact subgroup. Then, for each irreducible $\mathcal{H}(G,L,\mathcal{A})$–module U there exists a unique up to an isomorphism irreducible $(\mathcal{A},\ G)$–module V such that its space of L–invariants is isomorphic to U as $\mathcal{H}(G,L,\mathcal{A})$–modules. Furthermore, the annihilator of an $\mathcal{A}$–module U is equal to the annihilator of V (see Lemma 2.4 for the definition of the annihilator). In addition, if U is $\mathcal{A}$–finite, then the annihilator of V is a maximal ideal.*

Proof. This first part is immediate from Lemma 3.2 and Theorem 3.3. Next, as in the proof of Lemma 2.4, the annihilator $\mathrm{Ann}_{\mathcal{A}}(U)$ is a prime ideal, say $\mathfrak{p}$. Now, the action of $\mathcal{H}(G,L,\mathcal{A})$ on U factors through the canonical map $\mathcal{H}(G,L,\mathcal{A}) \longrightarrow \mathcal{H}(G,L,\mathcal{A}/\mathfrak{p})$. In this way, we may regard U as a $\mathcal{H}(G,L,\mathcal{A}/\mathfrak{p})$–module. Now, Lemma 3.2 (ii) and Theorem 3.3 (iv) guarantee that there exists, unique up to an isomorphism, an irreducible $(\mathcal{A}/\mathfrak{p},\ G)$–module V_1 such that its space of L–invariants is isomorphic to U as $\mathcal{H}(G,L,\mathcal{A}/\mathfrak{p})$–modules. If we regard V_1 as an $(\mathcal{A},\ G)$–module, then we obtain an irreducible module with the space of L–invariants isomorphic to U as $\mathcal{H}(G,L,\mathcal{A})$–modules. Hence V_1 is isomorphic to V by Lemma 3.2 (ii). This clearly implies that the annihilator of V contains $\mathfrak{p}$. They are clearly equal. Otherwise, the annihilator of U would be larger. Finally, the last claim follows from Theorem 3.3 (vii). ☐

4. An application of Theorem 3.3

In this section we again assume that $\mathcal{A}$ is a $\mathbb{Q}$–algebra, and G an l–group. The goal of this section is to discuss

$$\mathrm{End}_{(\mathcal{A},\ G)}\left(V\right) = \mathrm{End}_{\mathcal{H}(G,\ \mathcal{A})}\left(V\right),$$

for an irreducible $(\mathcal{A},\ G)$–module V. We also consider

$$\mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{A})}\left(V^L\right),$$

for an open compact subgroup $L \subset G$ such that $V^L \neq 0$. It is obvious that the restriction map gives an embedding

$$\mathrm{End}_{(\mathcal{A},\ G)}\left(V\right) = \mathrm{End}_{\mathcal{H}(G,\ \mathcal{A})}\left(V\right) \hookrightarrow \mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{A})}\left(V^L\right).$$

In general, they are both division algebras central over the field of fractions $k(\mathfrak{p})$ of $A/\mathfrak{p}$ where $\mathfrak{p}$ is annihilator of V in $\mathcal{A}$. We have the following result (see [8, Proposition 2.2.2] for the proof of the similar result by different means).

Theorem 4.1. *Assume that V is an irreducible $(\mathcal{A},\ G)$–module. Then, the restriction map $\mathrm{End}_{(\mathcal{A},\ G)}\left(V\right) \longrightarrow \mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{A})}\left(V^L\right)$ induces an isomorphism of division algebras over $k(\mathfrak{p})$.*

Proof. We use Theorem 3.3. We select maximal proper left ideal $I \subset \mathcal{H}\left(G, L, \mathcal{A}\right)$ such that we have the following isomorphism of $(\mathcal{A},\ G)$–modules

$$V \simeq \mathcal{V}(I, L).$$

Then,

$$V^L \simeq \mathcal{H}\left(G, L, \mathcal{A}\right)/I$$

as $\mathcal{H}\left(G, L, \mathcal{A}\right)$–modules.

Now, we give elementary description of

$$\mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{A})}\left(\mathcal{H}\left(G, L, \mathcal{A}\right)/I\right).$$

First, let $f + I \in \mathcal{H}\left(G, L, \mathcal{A}\right)/I$ such that $I \star f \subset I$. Then, the map $h + I \mapsto h \star f + I$ belongs to $\mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{A})}\left(\mathcal{H}\left(G, L, \mathcal{A}\right)/I\right)$. We call this map φ_f. Conversely, let

$$\varphi \in \mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{A})}\left(\mathcal{H}\left(G, L, \mathcal{A}\right)/I\right).$$

If we put $f + I = \varphi(\epsilon_L + I)$, then

$$I \star f + I = I \star (f + I) = I \star \varphi(\epsilon_L + I) = \varphi(I \star \epsilon_L + I) = \varphi(I) = I.$$

Hence, $I \star f \subset I$. Also,

$$\varphi(h + I) = \varphi(h \star \epsilon_L + I) = h \star f + I, \quad h \in \mathcal{H}\left(G, L, \mathcal{A}\right).$$

Thus, $\varphi = \varphi_f$. This proves the following lemma.

Lemma 4.2. *$\mathcal{A}$–algebra with identity $\epsilon_L + I$ consisting of all $f + I$ such that $I \star f \subset I$ is anti–isomorphic to* $\mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{A})}\left(\mathcal{H}\left(G, L, \mathcal{A}\right)/I\right)$*: $f + I \mapsto \varphi_f$, $\varphi_f \varphi_g = \varphi = \varphi_{g \star f}$.*

Now, we prove the theorem. By the remark before the statement of the theorem it is enough to show that the restriction map is surjective. Let $\varphi \in \mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{A})}\left(\mathcal{H}\left(G, L, \mathcal{A}\right)/I\right)$. By Lemma 3.2, we can write $\varphi = \varphi_f$ for some $f \in \mathcal{H}\left(G,\ L,\ \mathcal{A}\right)$ such that $I \star f \subset I$. Using Theorem 3.3, we can write

$$\mathcal{V}(I, L) \overset{def}{=} \mathcal{H}\left(G, \mathcal{A}\right) \epsilon_L / \mathcal{H}\left(G, \mathcal{A}\right) J_{I,L},$$

where $J_{I,L}$ is a unique maximal proper left ideal in $\mathcal{H}\left(G, \mathcal{A}\right) \epsilon_L$ which contains I. Moreover,

$$\epsilon_L \star J_{I,L} = I.$$

After these preparations we define $\psi \in \mathrm{End}_{(\mathcal{A},\ G)}\left(\mathcal{V}(I, L)\right)$ by

$$\varphi(h + J_{I,L}) = h \star f + J_{I,L}, \quad h \in \mathcal{H}\left(G, \mathcal{A}\right) \epsilon_L.$$

First of all, this map is well–defined since $h - h' \in J_{I,L}$ implies that

$$(h - h') \star f \in J_{I,L} \star f.$$

We observe that $J_{I,L} \star f$ is left ideal in $\mathcal{H}\left(G, \mathcal{A}\right) \epsilon_L$. Also, we note that

$$\epsilon_L \star J_{I,L} \star f = I \star f \subset I.$$

Consequently, we have the following. The sum $J_{I,L} \star f + \mathcal{H}\left(G, \mathcal{A}\right) I$ is a left ideal in $\mathcal{H}\left(G, \mathcal{A}\right) \epsilon_L$ which contains I, and satisfies

$$\epsilon_L \star \left(J_{I,L} \star f + \mathcal{H}\left(G, \mathcal{A}\right) I\right) = I.$$

This shows that this ideal is proper ideal in $\mathcal{H}\left(G, \mathcal{A}\right) \epsilon_L$, and contains I. Thus, it is contained in $J_{I,L}$. In particular, we have $J_{I,L} \star f \subset J_{I,L}$. Hence, $(h - h') \star f \in J_{I,L}$. This shows that φ is well–defined. Obviously, it belongs to $\mathrm{End}_{(\mathcal{A},\ G)}\left(\mathcal{V}(I, L)\right)$. Finally, the space of L–invariants in $\mathcal{V}(I, L)$ is equal to

$$\epsilon_L \star \mathcal{V}(I, L) = \epsilon_L \mathcal{H}\left(G, \mathcal{A}\right) \epsilon_L / J_{I,L} \simeq \mathcal{H}\left(G, L, \mathcal{A}\right)/I.$$

The isomorphism is $h + J_{I,L} \mapsto h + I$, for $h \in H\left(G, L, \mathcal{A}\right)$, and it is an isomorphism of $\mathcal{H}\left(G, L, \mathcal{A}\right)$–modules. We transfer φ via that isomorphism to $\epsilon_L \star \mathcal{V}(I, L)$. As a result, we obtain the following map:

$$h + J_{I,L} \mapsto h \star f + J_{I,L},$$

which is clearly the restriction of ψ. □

5. Another application of Theorem 3.3

The aim of this section is to prove the following theorem.

Theorem 5.1. *Assume that $\mathcal{A}$ is a field and hence an extension of $\mathbb{Q}$, since it is a $\mathbb{Q}$-algebra. Let G be an l–group and $L \subset G$ an open compact subgroup. Let V be an irreducible $(\mathcal{A},\ G)$–module such that $V^L \neq 0$ and $\mathcal{A}$–finite dimensional (i.e., V^L is an $\mathcal{A}$–admissible irreducible $\mathcal{H}(G,\ L,\ \mathcal{A})$–module). Then, for any field extension $\mathcal{A} \subset \mathcal{B}$, there exists irreducible $(\mathcal{B},\ G)$–modules $V_1, \ldots, V_t$ such that the following holds:*

(i) *$V_i^L \neq 0$ for all $1 \leq i \leq t$.*
(ii) *V_i^L are $\mathcal{B}$–admissible irreducible $\mathcal{H}(G,\ L,\ \mathcal{B})$–modules.*
(iii) *$V_{\mathcal{B}} \overset{def}{=} \mathcal{B} \otimes_{\mathcal{A}} V \simeq V_1 \oplus \cdots \oplus V_t$ as $(\mathcal{B},\ G)$–modules.*

Proof. First, we recall that $\mathcal{H}(G,\ L,\ \mathcal{A})$ is an associative $\mathcal{A}$–algebra with identity $\epsilon_{L,\mathcal{A}}$ (see equation (3.4)). We can identify

$$\mathcal{H}(G,\ L,\ \mathcal{B}) = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{H}(G,\ L,\ \mathcal{A}),$$

and consequently

$$\epsilon_{L,\mathcal{B}} = 1 \otimes_{\mathcal{A}} \epsilon_{L,\mathcal{A}}.$$

Next, by Lemma 2.7 (i), we have

$$(\mathcal{B} \otimes_{\mathcal{A}} V)^L = \mathcal{B} \otimes_{\mathcal{A}} V^L.$$

Next, since V is irreducible and $V^L \neq 0$, we conclude that V^L is an irreducible $\mathcal{H}(G,\ L,\ \mathcal{A})$–module (see Lemma 3.2 (i)). Put

$$W = V^L,$$

and

$$W_{\mathcal{B}} = \mathcal{B} \otimes_{\mathcal{A}} W.$$

Obviously, the latter is a $\mathcal{B}$–admissible module for $\mathcal{H}(G,\ L,\ \mathcal{B})$. We write

$$\varphi_{\mathcal{A},W} : \mathcal{H}(G,\ L,\ \mathcal{A}) \longrightarrow \mathrm{End}_{\mathcal{A}}(W),$$

and

$$\varphi_{\mathcal{B},W_{\mathcal{B}}} : \mathcal{H}(G,\ L,\ \mathcal{B}) \longrightarrow \mathrm{End}_{\mathcal{B}}(W_{\mathcal{B}})$$

for the corresponding homomorphism of $\mathcal{A}$–algebras and $\mathcal{B}$–algebras, respectively. For example, $\varphi_{\mathcal{A},W}$ is the restriction of the homomorphism of $\mathcal{A}$–algebras $\mathcal{H}(G,\ \mathcal{A}) \longrightarrow \mathrm{End}_{\mathcal{A}}(V)$ constructed in Section 3 to a subalgebra $\mathcal{H}(G,\ L,\ \mathcal{A})$ which keeps $W = V^L$ invariant.

We let $\mathcal{H}_{\mathcal{A},W}$ be the image of $\varphi_{\mathcal{A},W}$. Similar notation we introduce for the field $\mathcal{B}$. Then, we have

$$\varphi_{\mathcal{B},W_{\mathcal{B}}} = id_{\mathcal{B}} \otimes_{\mathcal{B}} \varphi_{\mathcal{A},W}.$$

Next, by Schur's lemma, we have that

$$\mathcal{D} \overset{def}{=} \mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{A})}(W) \tag{5.1}$$

is a division algebra whose center contains $\mathcal{A}$. Since, by the assumption V^L is $\mathcal{A}$–finite dimensional, we conclude that $\mathcal{D}$ is finite dimensional over $\mathcal{A}$. Hence, we have the following standard result.

Lemma 5.2. *Maintaining above assumptions, we have the following:*

(i) $\mathcal{H}_{\mathcal{A},W}$ *is simple $\mathcal{A}$–algebra; its unique simple module up to an isomorphism is W.*
(ii) $\mathcal{H}_{\mathcal{A},W} = \mathrm{End}_{\mathcal{D}}(W)$.
(iii) $\mathcal{H}_{\mathcal{B},W_{\mathcal{B}}} = \mathcal{B} \otimes_{\mathcal{A}} \mathrm{End}_{\mathcal{D}}(W)$ *is a semisimple $\mathcal{B}$–algebra.*

Proof. (ii) is a consequence of Jacobson's density theorem (known as Wedderburn's theorem, see [13, Chapter XVII, Corollary 3.5]). (i) is well–known once we have (ii) (see [13, Chapter XVII, Theorem 5.5]). For (iii), we note that ([13, Chapter XVII, Theorem 6.2]) implies that $\mathcal{B} \otimes_{\mathcal{A}} \mathrm{End}_{\mathcal{D}}(W)$ is a semisimple $\mathcal{B}$–algebra. Finally, we have

$$\begin{aligned}\mathcal{H}_{\mathcal{B},W_{\mathcal{B}}} &= \varphi_{\mathcal{B},W_{\mathcal{B}}}\left(\mathcal{H}\left(G,\ L,\ \mathcal{B}\right)\right)\\ &= id_{\mathcal{B}} \otimes_{\mathcal{B}} \varphi_{\mathcal{A},W}\left(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{H}\left(G,\ L,\ \mathcal{A}\right)\right)\\ &= \mathcal{B} \otimes_{\mathcal{A}} \mathcal{H}_{\mathcal{A},W}\\ &= \mathcal{B} \otimes_{\mathcal{A}} \mathrm{End}_{\mathcal{D}}(W).\end{aligned}$$

This completes the proof of (iii). ∎

As a corollary of Lemma 5.2 (iii), there exists $\mathcal{B}$–admissible modules $W_1, W_2, \dots W_t$ of $\mathcal{H}_{\mathcal{B},W_{\mathcal{B}}}$ (and consequently of $\mathcal{H}\left(G,\ L,\ \mathcal{B}\right)$) such that

$$\mathcal{B} \otimes_{\mathcal{A}} V^L = \mathcal{B} \otimes_{\mathcal{A}} W = W_{\mathcal{B}} \simeq W_1 \oplus W_2 \oplus \cdots \oplus W_t \tag{5.2}$$

as $\mathcal{H}\left(G,\ L,\ \mathcal{B}\right)$–modules.

Now, we apply Theorem 3.3. Select $v \in V^L$, $v \neq 0$, and decompose it according to the decomposition in (5.2):

$$v = \sum_{i=1}^{t} w_i \quad w_i \in W_i. \tag{5.3}$$

We let

$$\begin{aligned}&I \overset{def}{=} \mathrm{Ann}_{\mathcal{H}(G,\ L,\ \mathcal{A})}(v), \quad V \simeq \mathcal{H}\left(G,\ L,\ \mathcal{A}\right)/I,\\ &I_i \overset{def}{=} \mathrm{Ann}_{\mathcal{H}(G,\ L,\ \mathcal{B})}(w_i), \quad W_i \simeq \mathcal{H}\left(G,\ L,\ \mathcal{B}\right)/I_i, \quad 1 \leq i \leq t.\end{aligned}$$

Remark 5.3. In what follows we use repeatedly the following elementary observation. Let X and Y be non–zero vector spaces over the field $\mathcal{A}$. Let $Z \subset X$, $Z \neq 0$, be a subspace. Then, if $\sum_{i=1}^{l} x_i \otimes y_i \in Z \otimes_{\mathcal{A}} Y$, with $\mathcal{A}$–linearly independent vectors $y_1, \dots, y_l$, then $x_1, \dots, x_l \in Z$. Indeed, if α is an $\mathcal{A}$–linear functional on Y, then there exists an $\mathcal{A}$–linear map $X \otimes_{\mathcal{A}} Y \longrightarrow X$

such that $x \otimes y \mapsto \alpha(y)x$. It maps $Z \otimes_{\mathcal{A}} Y$ into Z. Now, since $y_1, \dots, y_l$ are $\mathcal{A}$–linearly independent, there exists linear functionals $\alpha_1, \dots, \alpha_t$ on Y such that $\alpha_i(y_j) = \delta_{ij}$ (a Kronecker delta). Consequently, $\alpha_k \left(\sum_{i=1}^{l} x_i \otimes y_i\right) = x_k \in Z$.

Lemma 5.4. $\mathrm{Ann}_{\mathcal{H}(G,\ L,\ \mathcal{A})}(1 \otimes v) = \mathcal{B} \otimes_{\mathcal{A}} I = I_1 \cap I_2 \cap \cdots \cap I_t$.

Proof. $\mathrm{Ann}_{\mathcal{H}(G,\ L,\ \mathcal{A})}(1 \otimes v) = I_1 \cap I_2 \cap \cdots \cap I_t$ is obvious from (5.2) and (5.3). Also, $\mathcal{B} \otimes_{\mathcal{A}} I \subset \mathrm{Ann}_{\mathcal{H}(G,\ L,\ \mathcal{A})}(1 \otimes v)$ is obvious. The converse inclusion follows from elementary Remark 5.3. ∎

Now, following Theorem 3.3, we construct maximal left ideals

$$J \overset{def}{=} \sum_{\substack{J' \subset \mathcal{H}(G,\ \mathcal{A})\epsilon_{L,\mathcal{A}} \text{ a left ideal} \\ \epsilon_{L,\mathcal{A}} J' = I}} J' \subset \mathcal{H}\left(G,\ \mathcal{A}\right) \epsilon_{L,\mathcal{A}},$$

$$J_i \overset{def}{=} \sum_{\substack{J' \subset \mathcal{H}(G,\ \mathcal{B})\epsilon_{L,\mathcal{B}} \text{ a left ideal} \\ \epsilon_{L,\mathcal{B}} J' = I_i}} J' \subset \mathcal{H}\left(G,\ \mathcal{B}\right) \epsilon_{L,\mathcal{B}}, \quad 1 \leq i \leq t.$$

Then, we have (see Theorem 3.3 (iv))

$$V \simeq \mathcal{V}_{\mathcal{A}}(I, L) \overset{def}{=} \mathcal{H}\left(G,\ \mathcal{A}\right) \epsilon_{L,\mathcal{A}} / J.$$

Consequently, since $\mathcal{B}$ is a field, we have

$$\mathcal{B} \otimes_{\mathcal{A}} V \simeq \mathcal{H}\left(G,\ \mathcal{B}\right) \epsilon_{L,\mathcal{B}} / \mathcal{B} \otimes_{\mathcal{A}} J. \tag{5.4}$$

We also define irreducible $(\mathcal{B},\ G)$–modules using (Theorem 3.3 (iv))

$$V_i \overset{def}{=} \mathcal{H}\left(G,\ \mathcal{B}\right) \epsilon_{L,\mathcal{B}} / J_i, \quad 1 \leq i \leq t.$$

By Theorem 3.3 (iv), we have

$$V_i^L = \mathcal{H}\left(G,\ \mathcal{B}\right) / I_i \simeq W_i$$

as $\mathcal{H}\left(G,\ L,\ \mathcal{B}\right)$–modules for all $1 \leq i \leq t$. Thus, $V_1, V_2, \dots, V_t$ satisfies (i) and (ii) of the theorem. It remains to prove (iii). We need the following lemma.

Lemma 5.5. $\mathcal{B} \otimes_{\mathcal{A}} J = J_1 \cap J_2 \cap \cdots \cap J_t$.

Proof. We prove $\mathcal{B} \otimes_{\mathcal{A}} J \subset J_i$ for all $i = 1, \dots, t$. Indeed, let $J' \subset \mathcal{H}\left(G,\ L,\ \mathcal{A}\right) \epsilon_{L,\mathcal{A}}$ be a left ideal such that $\epsilon_{L,\mathcal{A}} J' = I$. Then, we define a left ideal in $\mathcal{H}\left(G,\ L,\ \mathcal{B}\right) \epsilon_{L,\mathcal{B}}$ as follows:

$$J''_i \overset{def}{=} \mathcal{H}\left(G,\ \mathcal{B}\right) \star I_i + \mathcal{B} \otimes_{\mathcal{A}} J'.$$

Then, applying Lemma 5.4, we obtain

$$\epsilon_{L,\mathcal{B}} J''_i = \epsilon_{L,\mathcal{B}} \left(\mathcal{H}\left(G,\ \mathcal{B}\right) \star I_i + \mathcal{B} \otimes_{\mathcal{A}} J'\right) = I_i + \mathcal{B} \otimes_{\mathcal{A}} \epsilon_{L,\mathcal{A}} J' = I_i + \mathcal{B} \otimes_{\mathcal{A}} I = I_i,$$

for all $1 \leq i \leq t$. Consequently, we have

$$\mathcal{B} \otimes_{\mathcal{A}} J' \subset J''_i \subset J_i, \quad 1 \leq i \leq t.$$

Since J' is arbitrary, we obtain

$$\mathcal{B} \otimes_{\mathcal{A}} J \subset J_i, \quad 1 \leq i \leq t.$$

This proves

$$\mathcal{B} \otimes_{\mathcal{A}} J \subset J_1 \cap J_2 \cap \cdots \cap J_t.$$

Conversely, let $f \in J_1 \cap J_2 \cap \cdots \cap J_t$. Then, we define a left ideal

$$J'' \overset{def}{=} \mathcal{H}(G,\ \mathcal{B})\, f \subset J_1 \cap J_2 \cap \cdots \cap J_t.$$

Then, for each i, we have

$$\epsilon_{L,\mathcal{B}} J'' \subset I_i,$$

by the definition of ideals J_i and an argument as above with J_i. Hence, by Lemma 5.4, we obtain

$$\epsilon_{L,\mathcal{B}} J'' \subset \mathcal{B} \otimes_{\mathcal{A}} I. \tag{5.5}$$

Now, we write

$$f = \sum_{i=1}^{l} b_i \otimes f_i, \quad f_i \in \mathcal{H}(G,\ \mathcal{A}), b_i \in \mathcal{B},$$

where $b_1, \ldots, b_l$ are $\mathcal{A}$—linearly independent. Then, (5.5) implies that

$$\sum_{i=1}^{l} b_i \otimes \epsilon_{L,\mathcal{B}} F \star f_i \in \mathcal{B} \otimes_{\mathcal{A}} I,$$

for any $F \in H(G,\ \mathcal{A})$. Applying now Remark 5.3 we obtain

$$\epsilon_{L,\mathcal{B}} F \star f_i \in I,$$

for all $F \in H(G,\ \mathcal{A})$ and all i. This implies that

$$f_i \in \mathcal{H}(G,\ \mathcal{A})\, f_i \subset J,$$

for all i. Consequently, we obtain that

$$f = \sum_{i=1}^{l} b_i \otimes f_i \in \mathcal{B} \otimes_{\mathcal{A}} J.$$

This proves that

$$J_1 \cap J_2 \cap \cdots \cap J_t \subset \mathcal{B} \otimes_{\mathcal{A}} J.$$

The proof of lemma is complete. ∎

Now, we are ready to prove (iii) in the theorem, and thus complete the proof of the theorem. By (5.4) and Lemma 5.5, we have the following inclusion of $(\mathcal{B},\ G)$–modules:

$$\mathcal{B} \otimes_{\mathcal{A}} V \hookrightarrow V_1 \oplus V_2 \oplus \cdots \oplus V_t.$$

But the map is surjective since the map is surjective on level of L–invariants by counting $\mathcal{A}$–dimensions (see (5.2)) which implies the following:

$$\mathcal{B} \otimes_{\mathcal{A}} V = \mathcal{H}(G,\ \mathcal{B}) \left(\mathcal{B} \otimes_{\mathcal{A}} V^L\right) = \sum_{i=1}^{t} \mathcal{H}(G,\ \mathcal{B})\, W_i = \oplus_{i=1}^{t} V_i.$$

This completes the proof of the theorem. ∎

6. Applications and improvements of Theorem 5.1

We start this section with the following application of Theorem 5.1.

Corollary 6.1. *Assume that $\mathcal{A}$ is any subfield of $\mathbb{C}$. Let G be a reductive p–adic group (i.e., a group of k–points of a reductive group over a local non–Archimedean field k). Let $L \subset G$ be an open compact subgroup. Let V be an irreducible $(\mathcal{A},\ G)$–module such that $V^L \neq 0$ and $\mathcal{A}$–finite dimensional. Then, V is $\mathcal{A}$–admissible (see Definition 1.2).*

Proof. We can select $\mathcal{B} = \mathbb{C}$ in Theorem 5.1 since $\mathcal{A} \subset \mathbb{C}$. Then all W_i, $1 \leq i \leq t$, are irreducible smooth complex representations of a reductive p–adic group G. Then, by a result of Jacquet ([16, Theorem VI.2.2]), every representation W_i is $\mathbb{C}$–admissible. This implies that $\mathbb{C} \otimes_{\mathcal{A}} V$ is. Hence, for every open compact subgroup $L_0 \subset G$, the complex vector space $(\mathbb{C} \otimes_{\mathcal{A}} V)^{L_0}$ is finite dimensional. But, by Lemma 2.7 (i), we have

$$\mathbb{C} \otimes_{\mathcal{A}} V^{L_0} \simeq (\mathbb{C} \otimes_{\mathcal{A}} V)^{L_0}.$$

But then

$$\dim_{\mathcal{A}} V^{L_0} = \dim_{\mathbb{C}} (\mathbb{C} \otimes_{\mathcal{A}} V)^{L_0},$$

proving the corollary. ∎

The following is analogue of the result for finite dimensional representations of associative algebras (see [9, Section 29]).

Corollary 6.2. *Assume that $\mathcal{A}$ is a field and hence an extension of $\mathbb{Q}$, since it is a $\mathbb{Q}$-algebra. Let G be an l–group and $L \subset G$ an open compact subgroup. Assume that V and U are $\mathcal{A}$–admissible irreducible $(\mathcal{A},\ G)$–modules such that $V^L \neq 0$, $U^L \neq 0$. Assume that there exists a field extension $\mathcal{A} \subset \mathcal{B}$ such that $V_{\mathcal{B}}$ and $U_{\mathcal{B}}$ have non–disjoint Jordan–Hölder series. Then, $V \simeq U$ as $(\mathcal{A},\ G)$–modules.*

Proof. By Theorem 5.1 (iii), both representations $V_{\mathcal{B}}$ and $U_{\mathcal{B}}$ are semisimple and have finite length. By Theorem 5.1 (i) and (ii), every irreducible composition factor has a non–zero and $\mathcal{B}$–finite dimensional space of L–invariants. Consequently, both $V_{\mathcal{B}}^L$ and $U_{\mathcal{B}}^L$ are semi–simple, and since $V_{\mathcal{B}}$ and $U_{\mathcal{B}}$ have a common irreducible factor, we obtain

$$\mathrm{Hom}_{\mathcal{H}(G,\ L,\ \mathcal{B})}(V_{\mathcal{B}}^L,\ U_{\mathcal{B}}^L) \neq 0.$$

But by the results that can be found in ([9, Section 29]):

$$\mathrm{Hom}_{\mathcal{H}(G,\ L,\ \mathcal{B})}(V^L_{\mathcal{B}},\ U^L_{\mathcal{B}}) \simeq \mathcal{B} \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{H}(G,\ L,\ \mathcal{A})}(V^L,\ U^L).$$

Thus, we obtain

$$\mathrm{Hom}_{\mathcal{H}(G,\ L,\ \mathcal{A})}(V^L,\ U^L) \neq 0.$$

Then, Lemma 3.2 (ii) implies that $V \simeq U$. □

Another application of Theorem 5.1 is the following corollary.

COROLLARY 6.3. *Assume that $\mathcal{A}$ is a field and hence an extension of $\mathbb{Q}$, since it is a $\mathbb{Q}$-algebra. Let G be an l–group and $L \subset G$ an open compact subgroup. Let V be an $\mathcal{A}$–admissible irreducible $(\mathcal{A},\ G)$–module such that $V^L \neq 0$. Then, V is absolutely irreducible (i.e., $V_{\mathcal{B}}$ is irreducible for all field extensions $\mathcal{A} \subset \mathcal{B}$, see* [8] *and* [17]*) if and only if* $\mathrm{End}_{(\mathcal{A},\ G)}(V) = \mathcal{A}$.

PROOF. Assume that $\mathrm{End}_{(\mathcal{A},\ G)}(V) = \mathcal{A}$. Then, using the notation of Lemma 5.2, $\mathcal{H}_{\mathcal{A},W} = \mathrm{End}_{\mathcal{A}}(W)$, where $W = V^L$. Thus, if $\mathcal{A} \subset \mathcal{B}$ is a field extension, using Lemma 5.2 (ii), we obtain

$$\mathcal{H}_{\mathcal{B},W_{\mathcal{B}}} = \mathcal{B} \otimes_{\mathcal{A}} \mathrm{End}_{\mathcal{A}}(W) = \mathrm{End}_{\mathcal{B}}(W_{\mathcal{B}}).$$

This implies that $W_{\mathcal{B}}$ is irreducible $\mathcal{H}(G,\ L,\ \mathcal{B})$–module. Applying Theorem 5.1 we conclude that $V_{\mathcal{B}}$ is irreducible.

Conversely, assume that V is absolutely irreducible. Then obviously $W = V^L$ is absolutely irreducible $\mathcal{A}$–admissible $\mathcal{H}(G,\ L,\ \mathcal{A})$–module (see Lemma 3.2). Now, we apply the following lemma ([9, Section 29]).

LEMMA 6.4. *Assume that U is an $\mathcal{A}$–admissible irreducible $\mathcal{H}(G,\ L,\ \mathcal{A})$–module. Then, U is absolutely irreducible if and only if* $\mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{A})}(U) = \mathcal{A}$.

Finally, Theorem 4.1 completes the proof. □

We remark that if V is absolutely irreducible, then $V_{\mathcal{B}}$ is also absolutely irreducible module for all field extensions $\mathcal{A} \subset \mathcal{B}$. One needs to apply Corollary 6.3 and the following observation:

$$\mathcal{C} \otimes_{\mathcal{B}} V_{\mathcal{B}} \simeq \mathcal{C} \otimes_{\mathcal{B}} (\mathcal{B} \otimes_{\mathcal{A}} V) \simeq \mathcal{C} \otimes_{\mathcal{A}} V = V_{\mathcal{C}}.$$

for field extensions $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$.

Finally, we give an improvement of Theorem 5.1.

COROLLARY 6.5. *Assume that $\mathcal{A}$ is a field and hence an extension of $\mathbb{Q}$, since it is a $\mathbb{Q}$-algebra. Let G be an l–group and $L \subset G$ an open compact subgroup. Let V be an irreducible $(\mathcal{A},\ G)$–module such that $W \overset{def}{=} V^L \neq 0$ and $\mathcal{A}$–admissible. Then, we can extend V to an obvious $(\mathcal{K},\ G)$–module, say V^{ext}, where $\mathcal{K}$ is the center of the division algebra $\mathrm{End}_{(\mathcal{A},\ G)}(V)$. Then, for any field extension $\mathcal{K} \subset \mathcal{B}$, there exists a unique irreducible $(\mathcal{B},\ G)$–module $V^{ext}(\mathcal{B})$ such that the following holds:*

(i) $V^{ext}(\mathcal{B})^L \neq 0$.

(ii) $V^{ext}(\mathcal{B})^L$ *is* $\mathcal{B}$*–admissible irreducible* $\mathcal{H}(G,\ L,\ \mathcal{B})$*–module.*

(iii) $V^{ext}_{\mathcal{B}} \overset{def}{=} \mathcal{B} \otimes_{\mathcal{A}} V^{ext}$ *is direct sum of finite number of copies of* $V^{ext}(\mathcal{B})$*.*

In addition, we define $\mathcal{D}$ *as before (see (5.1)). Then, if* $\mathcal{F}$ *is a maximal subfield of* $\mathcal{D}$ *(which must contain* $\mathcal{K}$*), then* $V^{ext}(\mathcal{F})$ *is absolutely irreducible.*

Proof. This follows from Theorem 5.1 and Corollary 6.3 but we need some preparation. We warn the reader that we use notation from the first part of the proof of Theorem 5.1 freely (see Lemma 5.2). Applying Theorem 4.1, we obtain (see (5.1)) the following isomorphism of $\mathcal{A}$–algebras (see (5.1)):

$$\mathrm{End}_{(\mathcal{A},\ G)}(V) \simeq \mathcal{D}.$$

In particular, $\mathcal{K}$ is the center of $\mathcal{D}$. We let

$$W^{ext} = \left(V^{ext}\right)^L.$$

Moreover, we have the following isomorphism:

$$\mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{K})}\left(W^{ext}\right) = \mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{A})}(W) = \mathcal{D}.$$

In difference to what we have in the proof of Theorem 5.1 (see the statement of Lemma 5.2), the simple algebra

$$\mathcal{H}_{\mathcal{K},W} = \mathrm{End}_{\mathcal{D}}(W^{ext})$$

has $\mathcal{K}$ as its center. Thus, by ([9, Section 68]) we obtain

$$\mathcal{H}_{\mathcal{B},W^{ext}_{\mathcal{B}}} = \mathcal{B} \otimes_{\mathcal{K}} \mathrm{End}_{\mathcal{D}}(W^{ext})$$

is a simple $\mathcal{B}$–algebra. This observation is responsible for the existence of unique $V^{ext}(\mathcal{B})$. We leave the details to the reader.

It remains to prove that $V^{ext}(\mathcal{F})$ is absolutely irreducible. We need the following lemma.

Lemma 6.6. *$\mathcal{F}$–algebra $\mathcal{F} \otimes_{\mathcal{K}} \mathcal{D}$ is isomorphic to the $\mathcal{F}$–algebra of all matrices of size $t \times t$ with coefficients in $\mathcal{F}$ where $t = \dim_{\mathcal{K}} \mathcal{D}$.*

Proof. This is a part of the standard theory of simple algebras (see [9, Section 68]). □

As in the proof of Corollary 6.2, by the results of ([9, Section 29]), we have

$$\mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{F})}\left(\left(V^{ext}_{\mathcal{F}}\right)^L\right) \simeq \mathcal{F} \otimes_{\mathcal{K}} \mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{K})}\left(\left(V^{ext}\right)^L\right) = \mathcal{F} \otimes_{\mathcal{K}} \mathcal{D}.$$

Thus, by Lemma 6.6, we see that

$$\mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{F})}\left(\left(V^{ext}_{\mathcal{F}}\right)^L\right)$$

is a matrix algebra of size $t \times t$ with coefficients in $\mathcal{F}$. Since, by already proved part (iii) of the corollary, the module $\left(V^{ext}_{\mathcal{F}}\right)^L$ is a direct sum of finite number of copies of $\left(V^{ext}(\mathcal{F})\right)^L$, we conclude that the number of copies is equal to t and

$$\mathrm{End}_{\mathcal{H}(G,\ L,\ \mathcal{F})}\left(\left(V^{ext}(\mathcal{F})\right)^L\right) = \mathcal{F}.$$

Finally, Theorem 4.1 and Corollary 6.3 complete the proof. ∎

7. An example: construction of unramified irreducible representations

Let k be a non–Archimedean local field. Let $\mathcal{O} \subset k$ be its ring of integers, and let ϖ be a generator of the maximal ideal in $\mathcal{O}$. Let q be the number of elements in the residue field $\mathcal{O}/\varpi\mathcal{O}$. Let G be a k–split Zariski connected reductive group. To simplify notation we write G for the group $G(k)$ of k–points. Similarly, we do for other algebraic subgroups defined over k.

Let

$$K \overset{def}{=} G(\mathcal{O})$$

be a hyperspecial maximal compact subgroup of G ([20, 3.9.1]). We normalize a Haar measure on G such that $\int_K dg = 1$ (see Section 3).

We recall the structure of the algebra

$$\mathcal{H}\left(G,\ K,\ \mathbb{C}\right)$$

is obtained via Satake isomorphism ([7]). In more detail, let A be a maximal k–split torus of G. Let $X^*(A)$ (resp., $X_*(A)$) be the group of k–rational characters (resp., cocharacters) of A. Let W be the the Weyl group of A in G. The group W acts on A and its complex dual torus $\hat{A}$. The Satake isomorphism enables us to identify $\mathcal{H}\left(G,\ K,\ \mathbb{C}\right)$ with the algebra of W–invariants

$$\mathbb{C}[X^*(\hat{A})]^W,$$

where

$$\mathbb{C}[X^*(\hat{A})]$$

is the $\mathbb{C}$-group algebra of finitely generated free Abelian group. This is also the algebra of regular functions on complex algebraic torus $\hat{A}$. The action of W on the torus is algebraic, and therefore

$$X \overset{def}{=} \hat{A}/W,$$

is the complex affine variety of W–orbits in $\hat{A}$. Its algebra of regular functions is

$$\mathbb{C}[X] = \mathbb{C}[X^*(\hat{A})]^W.$$

Thus, the Satake isomorphism identifies $\mathcal{H}\left(G,\ K,\ \mathbb{C}\right)$ with $\mathbb{C}[X]$ (it depends on the choice of a Borel subgroup $B = AU$ of G, where U is the unipotent radical).

By the standard Nullstellensatz, a point $x \in X$ defines a maximal ideal $\mathfrak{m}_x$ in $\mathcal{H}(G,\ K,\ \mathbb{C})$. Then, we apply Theorem 3.3 to construct irreducible (admissible) $(\mathbb{C},\ G)$–module module, denoted by $\mathcal{V}(\mathfrak{m}_x, K)$. We have

$$\mathcal{V}(\mathfrak{m}_x, K)^K \simeq \mathcal{H}(G,\ K,\ \mathbb{C})/\mathfrak{m}_x \simeq \mathbb{C},$$

the one dimensional module coming from the evaluation of $\mathbb{C}[X]$ at x. Therefore, $\mathcal{V}(\mathfrak{m}_x, K)$ is a complex unramified irreducible representation. Different $x \in X$ give rise to non–isomorphic $\mathcal{V}(\mathfrak{m}_x, K)$ $(\mathbb{C},\ G)$–modules. This completes the description of complex unramified representations in terms of Hecke algebra $\mathcal{H}(G,\ K,\ \mathbb{C})$.

By a careful analysis of $\mathbb{Z}$–structure of Satake isomorphim ([10]) due to Gross, we obtain the following.

Lemma 7.1. *Let $\mathcal{A}$ be field which is any extension of $\mathbb{Q}$ if G is simply–connected, or just an extension of $\mathbb{Q}(q^{1/2})$ otherwise. Then, we have the following isomorphism of $\mathcal{A}$–algebras.*

$$\mathcal{H}(G,\ K,\ \mathcal{A}) \simeq \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}[X^*(\hat{A})]^W = \mathcal{A}[X^*(\hat{A})]^W.$$

Since $\hat{A}$ is a split torus, it is defined over $\mathbb{Q}$ (and consequently all extensions of $\mathbb{Q}$) by considering the group algebra $\mathbb{Q}[X^*(\hat{A})]$. The action of W on $\hat{A}$ preserves $\mathbb{Q}[X^*(\hat{A})]$ and consequently it is defined over $\mathbb{Q}$. This implies that the variety X is defined over $\mathbb{Q}$ via $\mathbb{Q}[X^*(\hat{A})]^W$.

Now, we prove the main result of this section and of the paper.

Theorem 7.2. *Let k be a non–Archimedean local field. Let $\mathcal{O} \subset k$ be its ring of integers, and let ϖ be a generator of the maximal ideal in $\mathcal{O}$. Let q be the number of elements in the residue field $\mathcal{O}/\varpi\mathcal{O}$. Assume that is G is a k–split Zariski connected reductive group. Let A be its maximal k–split torus. Let W be the Weyl group of A in G. We write $\hat{A}$ for the complex torus dual to A. Let W be the Weyl group of A in G. The orbit space*

$$X \overset{def}{=} \hat{A}/W$$

is an affine variety defined over $\mathbb{Q}$. Let $K = G(\mathcal{O})$ be a hyperspecial maximal compact subgroup of G. We normalize a Haar measure on G such that $\int_K dg =$ 1 (see Section 3). Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$. Let $\mathcal{A}$ be any subfield of $\overline{\mathbb{Q}}$ if G is simply–connected, or an extension of $\mathbb{Q}(q^{1/2})$ in $\overline{\mathbb{Q}}$ otherwise. We define the (commutative) Hecke algebra $\mathcal{H}(G,\ K,\ \mathcal{A})$ with respect to above fixed Haar measures. Then, we have the following:

(i) *(Satake isomorphims over subfields of $\overline{\mathbb{Q}}$) Maximal ideals in $\mathcal{H}(G,\ K,\ \mathcal{A})$ are parame-terized by points in $X(\overline{\mathbb{Q}})$ such that points in $X(\overline{\mathbb{Q}})$ give the same maximal ideal if and only if they are $Gal(\overline{\mathbb{Q}}/\mathcal{A})$–conjugate: for $x \in X(\overline{\mathbb{Q}})$, we denote by $\mathfrak{m}_{x,\mathcal{A}}$ the corresponding maximal ideal. The corresponding quotient $\mathcal{H}(G,\ K,\ \mathcal{A})/\mathfrak{m}_{x,\mathcal{A}}$ is denoted by $F(x, \mathcal{A})$.*

It is a finite (field) extension of $\mathcal{A}$, and it is also naturally irreducible $\mathcal{A}$–admissible $\mathcal{H}(G,\ K,\ \mathcal{A})$–module. The map $Gal(\overline{\mathbb{Q}}/\mathcal{A}).x \mapsto F(x,\mathcal{A})$ is a bijection between $Gal(\overline{\mathbb{Q}}/\mathcal{A})$–orbits in $X(\overline{\mathbb{Q}})$, and the set of equivalence classes of irreducible $\mathcal{A}$–admissible irreducible $\mathcal{H}(G,\ K,\ \mathcal{A})$–modules.

(ii) *For each $x \in X(\overline{\mathbb{Q}})$, the $(\mathcal{A},\ G)$–module (see Theorem 3.3 for the notation)*

$$\mathcal{V}(x,\mathcal{A}) \stackrel{def}{=} \mathcal{V}(\mathfrak{m}_x, K)$$

is an irreducible and $\mathcal{A}$–admissible $(\mathcal{A},\ G)$–module. We have

$$\mathcal{V}^K(x,\mathcal{A}) \simeq F(x,\mathcal{A})$$

as $\mathcal{H}(G,\ K,\ \mathcal{A})$–modules, and

$$\mathrm{End}_{(\mathcal{A},\ G)}\left(\mathcal{V}(x,\mathcal{A})\right) \simeq F(x,\mathcal{A}).$$

(iii) *$\mathcal{V}(x,\mathcal{A})$ is absolutely irreducible if and only if $x \in X(\mathcal{A})$.*

(iv) *Let $x \in X(\overline{\mathbb{Q}})$. Then, for any Galois extension $\mathcal{A} \subset \mathcal{B}$ which contains $F(x,\mathcal{A})$, $\mathcal{V}(x,\mathcal{B})$ is absolutely irreducible. Moreover, there exist $t = \dim_{\mathcal{A}}\ F(x,\mathcal{A})$ mutually different elements (among them x) in $Gal(\overline{\mathbb{Q}}/\mathcal{B}).x$, say $x = y_1, y_2, \ldots, y_t$, such that we have the following:*

$$(\mathcal{V}(x,\mathcal{A}))_{\mathcal{B}} = \mathcal{B} \otimes_{\mathcal{A}}\ \mathcal{V}(x,\mathcal{A}) \simeq \mathcal{V}(x,\mathcal{B}) \oplus \mathcal{V}(y_2,\mathcal{B}) \oplus \cdots \oplus \mathcal{V}(y_t,\mathcal{B}).$$

Furthermore, $V(x,\mathcal{B}), \mathcal{V}(y_2,\mathcal{B}), \ldots, \mathcal{V}(y_t,\mathcal{B})$ are mutually non–isomorphic $(\mathcal{B},\ G)$–modules.

(v) *(Classification of unramified admissible representations over subfields of $\overline{\mathbb{Q}}$) The map*

$$Gal(\overline{\mathbb{Q}}/\mathcal{A}).x \mapsto \mathcal{V}(x,\mathcal{A})$$

is a bijection between $Gal(\overline{\mathbb{Q}}/\mathcal{A})$–orbits in $X(\overline{\mathbb{Q}})$, and the set of equivalence classes of unramified $\mathcal{A}$–admissible irreducible $(\mathcal{A},\ G)$–modules.

Proof. It is obvious that the algebraic closure of $\mathcal{A}$ is $\overline{\mathbb{Q}}$. This means that we can apply Lemma A.1 to any affine $\mathcal{A}$–variety. We apply it to X which has the structure of affine $\mathcal{A}$–variety by letting

$$\mathcal{A}[X] = \mathcal{A} \otimes_{\mathbb{Q}}\ \mathbb{Q}[X^*(\hat{A})]^W = \mathcal{A}[X^*(\hat{A})]^W.$$

We identify $\mathcal{H}(G,\ K,\ \mathcal{A})$ with $\mathcal{A}[X]$ via $\mathcal{A}$–algebras isomorphism given by Lemma 7.1.

By Lemma A.1, for each $x \in X(\overline{\mathbb{Q}})$, there exists a unique maximal ideal $\mathfrak{m}_{x,\mathcal{A}} \subset \mathcal{A}[X]$ such that $\mathfrak{m}_{x,\mathcal{A}}$ is the kernel of $\mathcal{A}$–algebra homomorphism $\mathcal{A}[X] \longrightarrow \overline{\mathbb{Q}}$ given by the evaluation at x; the image is a finite (field) extension, denoted by $F(x,\mathcal{A})$ of $\mathcal{A}$. Two points in $X(\overline{\mathbb{Q}})$ give the same maximal ideal in $\mathcal{A}[X]$ if and only if they are $Gal(\overline{\mathbb{Q}}/\mathcal{A})$–conjugate. Now, (i) easily follows.

In (ii), we use explicit construction of $\mathcal{V}(x, \mathcal{A}) \overset{def}{=} \mathcal{V}(\mathfrak{m}_x, K)$ from Theorem 3.3. The isomorphism $\mathcal{V}^K(x, \mathcal{A}) \simeq \mathcal{B}_{x,\mathcal{A}}$ as $\mathcal{H}(G,\ K,\ \mathcal{A})$–modules also follows from Theorem 3.3. The deep thing is the fact that $\mathcal{V}(x, \mathcal{A})$ is $\mathcal{A}$–admissible. This is a consequence of our assumption that $\mathcal{A} \subset \overline{\mathbb{Q}} \subset \mathbb{C}$ and Corollary 6.1 to Theorem 5.1. Next, by Theorem 4.1, we have

$$\mathrm{End}_{(\mathcal{A},\ G)}\left(\mathcal{V}(x, \mathcal{A})\right) \simeq \mathrm{End}_{\mathcal{H}(G,\ K,\ \mathcal{A})}\left(\mathcal{V}^K(x, \mathcal{A})\right).$$

But since

$$\mathcal{V}^K(x, \mathcal{A}) \simeq \mathcal{H}(G,\ K,\ \mathcal{A})/\mathfrak{m}_{x,\mathcal{A}} = F(x, \mathcal{A})$$

as a modules over $\mathcal{H}(G,\ K,\ \mathcal{A})$, we obtain

$$\mathcal{V}^K(x, \mathcal{A}) \simeq F(x, \mathcal{A})$$

as modules over $\mathcal{A}$–algebra

$$F(x, \mathcal{A}) = \mathcal{H}(G,\ K,\ \mathcal{A})/\mathfrak{m}_{x,\mathcal{A}}.$$

Thus, we have the following:

$$\begin{aligned}\mathrm{End}_{\mathcal{H}(G,\ K,\ \mathcal{A})}\left(\mathcal{V}^K(x, \mathcal{A})\right) &= \mathrm{End}_{F(x,\mathcal{A})}\left(\mathcal{V}^K(x, \mathcal{A})\right) \\ &\simeq \mathrm{End}_{F(x,\mathcal{A})}\left(F(x, \mathcal{A})\right) = F(x, \mathcal{A}).\end{aligned}$$

This proves (ii).

(iii) follows from the characterization of absolutely irreducible modules given by Corollary 6.3. Indeed, $\mathcal{V}(x, \mathcal{A})$ is absolutely irreducible if and only if

$$\mathrm{End}_{(\mathcal{A},\ G)}\left(\mathcal{V}(x, \mathcal{A})\right) \simeq \mathcal{A}.$$

By (ii), we must have

$$F(x, \mathcal{A}) = \mathcal{A}.$$

Using the notation from the beginning of the proof, we have

$$\mathcal{A}[x]/\mathfrak{m}_{x,\mathcal{A}} = F(x, \mathcal{A}) = \mathcal{A}.$$

This is equivalent to $x \in X(\mathcal{A})$ by the general theory of affine $\mathcal{A}$–varieties. This proves (iii). (v) follows from (i), (ii), and Lemma 3.2 (ii).

Finally, we prove (iv). By Theorem 5.1, there exists irreducible $(\mathcal{B},\ G)$–modules $V_1, \dots, V_t$ such that the following holds:

(i) $V_i^K \neq 0$ for all $1 \leq i \leq t$.

(ii) V_i^K are $\mathcal{B}$–admissible irreducible $\mathcal{H}(G,\ K,\ \mathcal{B})$–modules.

(iii) $V_{\mathcal{B}} \overset{def}{=} \mathcal{B} \otimes_{\mathcal{A}} V \simeq V_1 \oplus \cdots \oplus V_t$ as $(\mathcal{B},\ G)$–modules.

In order to identify modules V_i, an argument from the proof of Lemma A.1 regarding tensor product of fields implies

$$\mathcal{B} \otimes_{\mathcal{A}} F(x, \mathcal{A}) = \mathcal{B} \oplus \cdots \oplus \mathcal{B}, \quad (\dim_{\mathcal{A}}\ F(x, \mathcal{A})) \text{ copies.}$$

This can be considered as a decomposition of

$$\mathcal{H}(G,\ K,\ \mathcal{B}) = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{H}(G,\ K,\ \mathcal{A})$$

into irreducible modules. This implies that

$$t = \dim_{\mathcal{A}} \ F(x, \mathcal{A})$$

in (iii) above.

Since we have

$$\mathcal{B}[X] = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{A}[X],$$

and obviously

$$\mathfrak{m}_{x,\mathcal{A}}\mathcal{B} \subset \mathfrak{m}_{x,\mathcal{B}},$$

we see that evaluation at x for $\mathcal{B}$ i.e., $\mathcal{B}[X] \longrightarrow F(x, \mathcal{B})$ must come from an epimorphism

$$\mathcal{B} \oplus \cdots \oplus \mathcal{B} \longrightarrow F(x, \mathcal{A}).$$

Hence

$$F(x, \mathcal{A}) = \mathcal{B}.$$

This means that $x \in X(\mathcal{B})$. In particular, $\mathcal{V}(x, \mathcal{B})$ is absolutely irreducible by (iii). Moreover, each of $t = \dim_{\mathcal{A}} \ F(x, \mathcal{A})$ different projections $\mathcal{B} \oplus \cdots \oplus \mathcal{B} \longrightarrow \mathcal{B}$ give rise to the same number of different epimorphisms of $\mathcal{B}$–agebras $\mathcal{B}[X] \longrightarrow \mathcal{B}$ that factor through $\mathfrak{m}_{x,\mathcal{A}}\mathcal{B}$. This means that they must correspond to evaluations at mutually different

$$y_1, \ldots, y_t \in X(\overline{\mathbb{Q}})$$

which belong to $V(\mathfrak{m}_{x,\mathcal{A}})$ (see Lemma A.1 for the notation). One of them is x as we proved above. Hence, they must be mutually different elements (including x) in $Gal(\overline{\mathbb{Q}}/\mathcal{B}).x$ by Lemma A.1 (iii). Now, (iv) follows. We remark that $V(x, \mathcal{B}), \mathcal{V}(y_2, \mathcal{B}), \ldots, \mathcal{V}(y_t, \mathcal{B})$ are mutually non–isomorphic $(\mathcal{B},\ G)$–modules. Since all $x = y_1, y_2, \ldots, y_t \in X(\mathcal{B})$ because of the evaluation at them give $\mathcal{B}$ as an image. Then, $\gamma.y_i = y_i$, for all $\gamma \in Gal(\overline{\mathbb{Q}}/\mathcal{B})$, and $i = 1, \ldots, t$. Now, we apply (v). ∎

Appendix A. A result on affine varieties

We prove a simple general lemma which is an exercise to the exposition in ([13, IX, Section 1]).

Lemma A.1. *Let k be a field of characteristic zero. We fix an algebraic closure $\overline{k}$ of k. Assume that Z is (not necessarily irreducible) affine variety over k. We write $k[Z]$ for its algebra of k–regular functions, and $V(S)$ for Zariski closed set in Z given as a set of all common zeroes of elements of S for any subset $S \subset k[Z]$. Then, we have the following:*

(i) *If $z \in Z(\overline{k})$ is any point, then $k[z]$ is a field where by definition $k[z]$ is k–algebra inside $\overline{k}$ generated by all $f(z)$ where $f \in k[Z]$. Therefore, the kernel of the evaluation homomorphism $k[Z] \longrightarrow k[z]$ is a maximal ideal, say $\mathfrak{m}$. We have $z \in V(\mathfrak{m})$.*

(ii) *Conversely, let* $\mathfrak{m} \subset k[Z]$ *be a maximal ideal. Then,* $V(\mathfrak{m})$*, has a finite number of points. For each* $z \in V(\mathfrak{m})$*,* k*–algebra* $k[z] \subset \overline{k}$ *is a finite extension of* k*. The evaluation at* z *gives* $k[Z]/\mathfrak{m} \simeq k[z]$ *over* k*.*

(iii) *Let* $\mathfrak{m} \subset k[Z]$ *be a maximal ideal. Then,* $V(\mathfrak{m})$ *is defined over* k*. Moreover,* $V(\mathfrak{m})$ *is a single* $Gal(\overline{k}/k)$*–orbit. The set of* k*–points* $V(\mathfrak{m})(k)$ *of* $V(\mathfrak{m})$ *is not empty if and only if* $\mathfrak{m}$ *is the kernel of (a unique) evaluation at* $z \in Z(k)$*. If this is so, then* $V(\mathfrak{m}) = \{z\}$*.*

(iv) $Z(\overline{k})$ *is a disjoint union of all* $V(\mathfrak{m})$*, where* $\mathfrak{m}$ *ranges over all maximal ideals of* $k[Z]$*.*

Proof. We start with the following observation. The algebra $k[Z]$ is finitely generated k–algebra, say $f_1, \dots, f_t$ are generators. Let $z \in Z(\overline{k})$. Then, $f_i(z) \in \overline{k}$, and consequently $k[f_i(z)]$ is a finite (field) extension of k for all i. Hence, $k[z]$ is by definition equal to $k[f_1(z), f_2(z), \dots, f_t(z)]$. It is a field and finite extension of k by elementary field theory. This implies (i).

Let us prove (ii). We consider the ideal $I \subset \overline{k}[Z]$ defined by $I = \mathfrak{m} \cdot \overline{k}[Z]$. Then, obviously,

$$V(I) = V(\mathfrak{m}).$$

Now, by ([13, IX, Section 1, Theorem 1.5]), we have

$$V(\mathfrak{m}) \neq \emptyset.$$

Hence, I is proper ideal. Also, for $z \in V(\mathfrak{m})$, $k[Z]/\mathfrak{m} \simeq k[z]$ is a finite extension of k. Let us put

$$F = k[Z]/\mathfrak{m}.$$

Then, since $k \subset F$ is finite and separable extension (because k has characteristic zero), there exists $\alpha \in F$ such that

$$F = k(\alpha).$$

Let $P \in k[T]$ be a minimal polynomial of α, where T is a variable. Then, let

$$\alpha_1, \alpha_2, \dots, \alpha_u, \quad u = \deg(P),$$

be all zeroes of P in $\overline{k}$. They are all distinct. The reader may easily check that

$$\overline{k} \otimes_k F \simeq \overline{k} \oplus \cdots \oplus \overline{k} \ \text{ (a copy of } \overline{k} \text{ for each } \alpha_i.) \tag{A.1}$$

Indeed, we have the following elementary and well–known computation:

$$\begin{aligned}
\overline{k} \otimes_k F &\simeq \overline{k} \otimes_k (k[T]/k[T]P) \\
&\simeq \overline{k}[T]/\overline{k}[T]P \\
&= \overline{k}[T]/\overline{k}[T](T-\alpha_1)(T-\alpha_2)\cdots(T-\alpha_u) \\
&\simeq \oplus_{i=1}^u \overline{k}[T]/\overline{k}[T](T-\alpha_i) \\
&\simeq \oplus_{i=1}^u \overline{k}.
\end{aligned}$$

We observe that (A.1) implies

$$\overline{k}[Z]/I \simeq \overline{k} \otimes_k k[Z]/\overline{k} \otimes_k \mathfrak{m} \simeq \overline{k} \otimes_k (k[Z]/\mathfrak{m}) \simeq \oplus_{i=1}^{u} \overline{k}.$$

This shows that I is a radical ideal since the right–hand side has no nilpotent elements. Hence, $\overline{k}[Z]/I$ is algebra of regular functions on $V(\mathfrak{m})$. Now, the rest of (ii) is clear. Next, (iv) is obvious from (i) and (iii).

Now, we prove (iii). It is well–known that $V \overset{def}{=} V(\mathfrak{m})$ is defined over k. Indeed, this also follows from above considerations. We have shown $\overline{k}[V] = \overline{k}[Z]/I$. If we let, $k[V] = k[Z]/\mathfrak{m}$. Then, above isomorphism can be restated $\overline{k}[V] \simeq \overline{k} \otimes_k k[V]$, and it gives the k–structure on V.

To complete the proof of (iii), we observe that $V(\mathfrak{m})$ is a single $Gal(\overline{k}/k)$–orbit. Indeed, let $z \in V = V(\mathfrak{m})$. Then, for $\gamma \in Gal(\overline{k}/k)$, we have $\gamma.z \in V(\mathfrak{m})$ since by the definition of the Galois action on Z:

$$f(\gamma.z) = \gamma^{-1}(f(z)) = \gamma(0) = 0.$$

Conversely, if z and z' are in V. Then, the fields $k[z]$ and $k[z']$ are isomorphic to $k[V]$ over k. Thus, there exists $\gamma \in Gal(\overline{k}/k)$ such that $\gamma(k[z]) = k[z']$. Equivalently, $k[\gamma.z'] = k[z]$. This means that

$$f(\gamma.z') = f(z), \quad \text{for all } f \in k[Z].$$

Hence, we have

$$f(\gamma.z') = f(z), \quad \text{for all } f \in \overline{k}[Z].$$

This means that

$$\gamma.z' = z.$$

The rest of (iii) is clear. Finally, (iv) follows from (i) and (iii). □

References

[1] J. Bernstein, *Le 'centre' de Bernstein*, in: Représentations des groupes réductif sur un corps local, ed. P. Deligne, Herman, Paris, 1984, 1–32.

[2] J. Bernstein, *Draft of: Representations of p-adic groups*, lectures at Harvard University, 1992, written by Karl E. Rumelhart.

[3] J. Bernstein and A. V. Zelevinsky, *Representations of the group $GL(n, F)$, where F is a local non-Archimedean field* (Russian), Uspehi Mat. Nauk **31** (1976), 5–70.

[4] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p-adic groups I*, Ann. Sci. École Norm Sup. (4) **10** (1977), 441–472.

[5] J. N. Bernstein, P. Deligne and D. Kazhdan, *Trace Paley-Wiener theorem for reductive p-adic groups*, J. Analyse Math. **47** (1986), 180–192.

[6] A. Borel, Linear algebraic group, second edition, Graduate Texts in Mathematics **126**, Springer-Verlag, New York, 1991.

[7] P. Cartier, *Representations of p-adic groups: a survey*, in: Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, AMS, Providence, 1979, 111–155.

[8] W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, preprint.

[9] C. W. Curtis and I. Reiner, Representation theory of finite groups and associative algebras. Reprint of the 1962 original, AMS Chelsea Publishing, Providence, 2006.
[10] B. H. Gross, *On the Satake isomorphism,* preprint, `http://www.math.harvard.edu/~gross/preprints/sat.pdf`.
[11] G. Henniart and M.–F. Vignéras, *A Satake isomorphism for representations modulo p of reductive groups over local fields,* J. Reine Angew. Math. **701** (2015), 33–75.
[12] G. Henniart and M.–F. Vignéras, *Representations of a p-adic group in characteristic p,* in: Proceedings of Symposia in Pure Mathematics, Volume **101**, Providence, 2019.
[13] S. Lang, Algebra, Second edition, Addison-Wesley Publishing Company, Reading, 1984.
[14] H. Matsumura, Commutative ring theory, Translated from the Japanese by M. Reid, Second edition, Cambridge University Press, Cambridge, 1989.
[15] J. Newton and J. A. Thorne, *Torsion Galois representations over CM fields and Hecke algebras in the derived category,* Forum Math. Sigma **4** (2016), e21, 88 pp.
[16] D. Renard, *Représentations des groupes réductifs p-adiques*, Société Mathématique de France, Paris, 2010.
[17] A. Robert, *Modular representations of the group $GL(2)$ over a local field,* J. Algebra 22 (1972), 386–405.
[18] M. Tadić, *On interactions between harmonic analysis and the theory of automorphic forms*, Automorphic Representations and L-functions, Tata Inst. Fund. Res., Mumbai, 2013, 591–650.
[19] M. Tadić, *On unitarizability in the case of classical p-adic groups,* in: Geometric aspects of the trace formula, Simons Symp., Springer, Cham, 2018, 405–453.
[20] J. Tits, *Reductive groups over local fields*, in: Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, 1979, 29–69.
[21] M.–F. Vignéras, Représentations l–modulaires d'un groupe réductif p–adique avec $l \neq p$, Birkhäuser Boston, Inc., Boston, 1996.
[22] M.–F. Vignéras, *A letter to the author,* May 15, 2019.

G. Muić
Department of Mathematics, Faculty of Science
University of Zagreb
Bijenička cesta 30, 10000 Zagreb
Croatia
E-mail: `gmuic@math.hr`

Received: 18.12.2019.
Revised: 21.7.2020.