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A NOTE ON THE EXPONENTIAL DIOPHANTINE EQUATION $(A^2n)^x + (B^2n)^y = ((A^2 + B^2)n)^z$

Maohua Le and Gökhan Soydan

Lingnan Normal College, P. R. China and Bursa Uludağ University, Turkey

Abstract. Let A, B be positive integers such that $\min\{A, B\} > 1$, $\gcd(A, B) = 1$ and $2|B$. In this paper, using an upper bound for solutions of ternary purely exponential Diophantine equations due to R. Scott and R. Styer, we prove that, for any positive integer n, if $A > B^3/8$, then the equation $(A^2n)^x + (B^2n)^y = ((A^2 + B^2)n)^z$ has no positive integer solutions (x, y, z) with $x > z > y$; if $B > A^3/6$, then it has no solutions (x, y, z) with $y > z > x$. Thus, combining the above conclusion with some existing results, we can deduce that, for any positive integer n, if $B \equiv 2 \pmod 4$ and $A > B^3/8$, then this equation has only the positive integer solution $(x, y, z) = (1, 1, 1)$.

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let n be a positive integer, and let a, b be positive integers such that $\min\{a, b\} > 1$ and $\gcd(a, b) = 1$. Recently, P.-Z. Yuan and Q. Han ([9]) proposed the following conjecture:

Conjecture 1.1. *For any n, if $\min\{a, b\} \geq 4$, then the equation*

$$(an)^x + (bn)^y = ((a + b)n)^z,\ x, y, z \in \mathbb{N} \tag{1.1}$$

has only the solution $(x, y, z) = (1, 1, 1)$.

Since Conjecture 1.1 is much broader than Jeśmanowicz' conjecture concerning Pythagorean triples (see [2] and the survey paper on the conjectures of Jeśmanowicz and Terai which was published by G. Soydan, M. Demirci, I. N. Cangül and A. Togbé, ([5])), it is unlikely to be solved in the short term. There are only a few scattered results on Conjecture 1.1 at present (see [6]).

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Let A, B be positive integers such that $\min\{A, B\} > 1$, $\gcd(A, B) = 1$ and $2|B$. In the same paper, P.-Z. Yuan and Q. Han ([9]) deal with the solutions (x, y, z) of (1.1) for the case that $(a, b) = (A^2, B^2)$. Then (1.1) can be rewritten as

$$(A^2n)^x + (B^2n)^y = ((A^2 + B^2)n)^z, x, y, z \in \mathbb{N}. \tag{1.2}$$

For this equation, they proved that, for any $n > 1$, if $B \equiv 2 \pmod 4$, then (1.2) has no solutions (x, y, z) with $y > z > x$; in particular, if $B = 2$, then Conjecture 1.1 is true for any n.

In this paper, using an upper bound for solutions of ternary purely exponential Diophantine equations due to R. Scott and R. Styer ([4]), we prove a general result as follows:

Theorem 1.2. *For any n, if $A > B^3/8$, then* (1.2) *has no solutions (x, y, z) with $x > z > y$; if $B > A^3/6$, then* (1.2) *has no solutions (x, y, z) with $y > z > x$.*

Thus, combining Theorem 1.2 with the above mentioned results of [9], we can deduce the following corollary:

Corollary 1.3. *For any n, if $B \equiv 2 \pmod 4$ and $A > B^3/8$, then* (1.2) *has only the solution $(x, y, z) = (1, 1, 1)$.*

This implies that, for any fixed B with $B \equiv 2 \pmod 4$, then Conjecture 1.1 is true for $(a, b) = (A^2, B^2)$ except for finitely many values of A.

2. Lemmas

For any positive integer m, let $\mathrm{rad}(m)$ denote the product of all distinct prime divisors of m, and let $\mathrm{rad}(1) = 1$. Obviously, $\mathrm{rad}(m)$ is equal to the largest squarefree divisor of m.

Lemma 2.1 ([6, Theorem 1.1], [9, Proposition 3.1]). *Assume $n > 1$ in* (1.1) *and let (x, y, z) be a solution of* (1.1) *with $(x, y, z) \neq (1, 1, 1)$. If $\min\{a, b\} > 2$, then either*

$$x > z > y,\ \mathrm{rad}(n) \mid b,\ b = b_1b_2,\ b_1^y = n^{z-y},\ b_1, b_2 \in \mathbb{N},\ b_1 > 1,\ \gcd(b_1, b_2) = 1$$

or

$$y > z > x,\ \mathrm{rad}(n) \mid a,\ a = a_1a_2,\ a_1^x = n^{z-x},$$
$$a_1, a_2 \in \mathbb{N},\ a_1 > 1,\ \gcd(a_1, a_2) = 1.$$

Remark 2.2. Because when $\min\{a, b\} = 2$, there might be a solution (x, y, z) to (1.1) with $y > z = x$ (see [1, 3, 7, 8]), the condition $\min\{a, b\} > 2$ in Lemma 2.1 is necessary.

Lemma 2.3. *If $B \equiv 2 \pmod 4$ and $(x, y, z) \neq (1, 1, 1)$ is a solution to* (1.2)*, then $x > z > y$.*

PROOF. When $n > 1$, Lemma 2.1 shows that Lemma 2.3 is equivalent to [9, Theorem 1.3].

So we can assume $n = 1$. Suppose (1.2) has a solution $(x, y, z) \neq (1, 1, 1)$, so that

$$A^{2x} + B^{2y} = (A^2 + B^2)^z. \tag{2.1}$$

Clearly $(1, 1, 1)$ is the only possible solution to (1.2) with $z = 1$, so in (2.1) we have

$$z > 1. \tag{2.2}$$

Since $z \geq 2$, if $\max\{x, y\} \leq z$, then we have

$$(A^2 + B^2)^z = A^{2x} + B^{2y} \leq A^{2z} + B^{2z} < (A^2 + B^2)^z,$$

a contradiction from which we get

$$z < \max\{x, y\}. \tag{2.3}$$

Next, we show that $y < z$ using a straightforward approach which works when $n = 1$ (as well as when $n > 1$ as in [9]).

It is a familiar elementary result (see, for example, [9, Lemma 3.2]) that, if (2.1) holds, there are positive integers u and v such that $2 \mid v$, $u^2 + v^2 = A^2 + B^2$, $(u, v) = 1$, and

$$\pm(u \pm v\sqrt{-1})^z = A^x + B^y\sqrt{-1},$$

with

$$\nu_2(v) + \nu_2(z) = \nu_2(B^y) \tag{2.4}$$

where, for any positive integer m, $2^{\nu_2(m)} \,||\, m$. $2 \,||\, B$, so $A^2 + B^2 \equiv 5 \pmod 8$, so $2 \,||\, v$, so that (2.4) becomes

$$1 + \nu_2(z) = y,$$

so that

$$z \geq 2^{y-1} \geq y \tag{2.5}$$

and $z = y$ implies $y \leq 2$. Since $z > 1$ and $y = z = 2$ implies

$$A^{2x} = A^2(A^2 + 2B^2)$$

which contradicts $(A, 2B) = 1$, we must have

$$y < z. \tag{2.6}$$

(2.3) and (2.6) combine to give $y < z < x$. ∎

LEMMA 2.4 ([9, Theorem 1.4]). *For any n, if $B = 2$, then* (1.2) *has only the solution* $(x, y, z) = (1, 1, 1)$.

Lemma 2.5 ([4, Theorem 3]). *Let G, H, K be fixed positive integers with $\min\{G, H, K\} > 1$, $\gcd(G, H) = 1$ and $2 \nmid K$. Further, let PQ be the largest squarefree divisor of GH, with P and Q chosen so that $(GH/P)^{1/2}$ is an integer. If there exists a positive integer Z such that $G + H = K^Z$, then Z satisfies*

$$Z \begin{cases} \leq \frac{1}{2}Q, & \text{if} \quad P = 1, \\ \leq \frac{1}{2}(Q+1), & \text{if} \quad P = 2, \\ < \frac{1}{2}P^{1/2}Q\log P, & \text{if} \quad P \geq 3. \end{cases} \tag{2.7}$$

Lemma 2.6. *Under the assumptions of Lemma 2.5, we have*

$$Z \leq \frac{1}{2}PQ. \tag{2.8}$$

Proof. Obviously, by (2.7), (2.8) holds for $P \leq 2$. Let

$$f(t) = \frac{\log t}{t^{1/2}},\ t \geq 3. \tag{2.9}$$

Then we have

$$f'(t) = \frac{2 - \log t}{2t^{3/2}},\ t \geq 3, \tag{2.10}$$

where $f'(t)$ is the derivative of $f(t)$. We see from (2.9) and (2.10) that $f(e^2) = 2/e$ is the maximum value of $f(t)$. Therefore, if $P \geq 3$, then from (2.7) and (2.9) we get

$$\begin{aligned} Z < \frac{1}{2}P^{1/2}Q\log P &= \left(\frac{1}{2}PQ\right)\left(\frac{\log P}{P^{1/2}}\right) \\ &= \left(\frac{1}{2}PQ\right)(f(P)) \leq \left(\frac{1}{2}PQ\right)\left(\frac{2}{e}\right) < \frac{1}{2}PQ. \end{aligned}$$

This implies that (2.8) holds for $P \geq 3$. The lemma is proved. □

Lemma 2.7. *For any n, the solutions (x, y, z) of (1.2) satisfy $z \leq AB/2$.*

Proof. Since $AB/2 \geq 3$, the lemma holds for $(x, y, z) = (1, 1, 1)$. We now assume that (x, y, z) is a solution of (1.2) with $(x, y, z) \neq (1, 1, 1)$. Then, by Lemma 2.1, we have either $x > z > y$ or $y > z > x$.

Since $\min\{A^2, B^2\} \geq 4$, by Lemma 2.1, if $x > z > y$, then we have

$$B = B_1B_2,\ B_1, B_2 \in \mathbb{N},\ \gcd(B_1, B_2) = 1, \tag{2.11}$$

$$B_1^{2y} = n^{z-y} \tag{2.12}$$

and

$$A^{2x}n^{x-z} + B_2^{2y} = (A^2 + B^2)^z. \tag{2.13}$$

Take $G = A^{2x}n^{x-z}$, $H = B_2^{2y}$, $K = A^2 + B^2$ and $Z = z$. Let PQ be the largest squarefree divisor of GH. Since $\gcd(A, B) = 1$, by (2.11) and (2.12), we have

$$\begin{aligned} PQ &= \mathrm{rad}(GH) = \mathrm{rad}(A^{2x}n^{x-z}) \cdot \mathrm{rad}(B_2^{2y}) \\ &= \mathrm{rad}(AB_1) \cdot \mathrm{rad}(B_2) = \mathrm{rad}(AB) \le AB. \end{aligned} \tag{2.14}$$

Therefore, applying Lemma 2.6 to (2.13), we get from (2.14) that

$$z \le \frac{PQ}{2} \le \frac{AB}{2}. \tag{2.15}$$

Similarly, if $y > z > x$, then we have

$$A = A_1A_2,\ A_1, A_2 \in \mathbb{N},\ \gcd(A_1, A_2) = 1, \tag{2.16}$$

$$A_1^{2x} = n^{z-x} \tag{2.17}$$

and

$$A_2^{2x} + B^{2y}n^{y-z} = (A^2 + B^2)^z. \tag{2.18}$$

Take $G = A_2^{2x}$, $H = B^{2y}n^{y-z}$, $K = A^2 + B^2$ and $Z = z$. By (2.16) and (2.17), we have

$$\begin{aligned} PQ &= \mathrm{rad}(GH) = \mathrm{rad}(A_2^{2x}) \cdot \mathrm{rad}(B^{2y}n^{y-z}) \\ &= \mathrm{rad}(A_2) \cdot \mathrm{rad}(BA_1) = \mathrm{rad}(AB) \le AB, \end{aligned} \tag{2.19}$$

where PQ is the largest squarefree divisor of GH. Therefore, applying Lemma 2.6 to (2.18), we see from (2.19) that z satisfies (2.15). Thus, the lemma is proved. ∎

3. Proofs

Proof of Theorem 1.2. By Lemma 2.4, the theorem holds for $B = 2$. We may therefore assume that $B \ge 4$.

We now prove the first part of the theorem. Since $2 \nmid A$ and $A > B^3/8$, we have $A \ge 9$. Let (x, y, z) be a solution of (1.2) with $x > z > y$. By (2.13), we have $A^{2x}n^{x-z} < (A^2 + B^2)^z$, whence we get $(A^2n)^{x-z} < (1 + B^2/A^2)^z$ and

$$\log(A^2n) \le (x - z)\log(A^2n) < z \log\left(1 + \frac{B^2}{A^2}\right). \tag{3.1}$$

Since $\log(1 + t) < t$ for any $t > 0$, by (3.1), we have

$$\frac{A^2}{B^2}\log(A^2n) < z. \tag{3.2}$$

On the other hand, by Lemma 2.7, we have $z \le AB/2$. Hence, by (3.2), we get

$$\frac{A^2}{B^2}\log(A^2n) < \frac{AB}{2}. \tag{3.3}$$

Further, since $A > B^3/8$, we see from (3.3) that

$$\log(A^2 n) < 4. \tag{3.4}$$

But, since $A \geq 9$ and $n \geq 1$, (3.4) is false. Therefore, the first part of the theorem is proved.

Using the same method as before, we can easily prove the second part of the theorem. Since $2 \nmid A$ and $B > A^3/6$, we have $A \geq 3$ and $B \geq 6$. Let (x, y, z) be a solution of (1.2) with $y > z > x$. By (2.18), we have $B^{2y} n^{y-z} < (A^2 + B^2)^z$, whence we get

$$\frac{B^2}{A^2} \log(B^2 n) \leq \frac{B^2}{A^2}(y-z)\log(B^2 n) < z. \tag{3.5}$$

Further, by Lemma 2.7, we have $z \leq AB/2$. Hence, by (3.5), we get

$$\frac{B^2}{A^2} \log(B^2 n) < \frac{AB}{2}. \tag{3.6}$$

Furthermore, since $B > A^3/6$, we see from (3.6) that

$$\log(B^2 n) < 3. \tag{3.7}$$

But, since $B \geq 6$ and $n \geq 1$, (3.7) is false. Thus, the second part of the theorem is proved. The proof is complete. □

Proof of Corollary 1.3. Combining Theorem 1.2 with Lemma 2.3, we obtain the corollary immediately. □

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M.-H. Le
Institute of Mathematics, Lingnan Normal College
Guangdong, 524048 Zhangjiang
China
E-mail: lemaohua2008@163.com

G. Soydan
Department of Mathematics
Bursa Uludağ University
16059 Bursa
Turkey
E-mail: gsoydan@uludag.edu.tr

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