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Abstract

The Singapore Savings Bonds (SSB) is a unique investment program offered by the Singapore government whereby retail investors can earn risk-free tax-free step-up interest closely matched to Treasury bond rates for up to 10 years and can redeem on any business day prior to maturity without any early redemption penalty. This study analyses the unique design of the SSB and provides a valuation of the Bermudan option for early redemption that is embedded in the SSB. The Black-Derman-Toy model is used to build the interest rate tree, and an iterative method is employed to avoid arbitrary specification of the pre-determined short rate volatility function. This bespoke Bermudan option can have changing strike prices over time. It also has a novel characteristic whereby the value of exercise to a buyer need not equal to the cost of being exercised to a seller. Better understanding of embedded options within government savings bonds leads to innovative designs that may encourage effective citizens' savings.

Keywords: Bespoke Bermudan option, Singapore Savings Bonds, Iterative Black-Derman-Toy model, Spot rate model

JEL Classification: G13, G21, G28

1 Introduction

Singapore Savings Bonds (or SSB) are investment instruments for household or retail investors in Singapore whereby the investor buys the bonds for up to a maximum of S$200,000 per person and receives coupon interests on a semi-annual basis. Each

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The author is solely responsible for the view and the work done in this research paper. The idea was first mooted in 2015 when SS Bonds were introduced. All data are obtained from MAS publicly available website https://www.mas.gov.sg/bonds-and-bills. All errors are my sole responsibility.

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SSB bond with a par value of S$500 has a maturity of 10 years. The SSB is not transferable in the market. The pre-determined coupon interest rates of the SSB are stepped up each year. An investor will receive higher interest rates by holding onto the bond over a longer period. The step-up feature is unique and is created to incentivize investors who had bought the bond to continue holding the bond for the full term of 10 years. However, another key feature of the SSB is often overlooked. The investor also has the option to redeem or sell the bond back to the government represented by the Monetary Authority of Singapore (MAS) at any time before the end of 10 years, without any penalty, and will still be able to collect the accrued interest up to the point of redemption. This feature is unlike bank term deposits whereby the investor or depositor typically stands to forfeit all accrued interests if the bond or loan is redeemed before maturity. The SSB program, besides being described in Singapore Monetary Authority of Singapore (MAS) public websites (see for example https://www.mas.gov.sg/bonds-and-bills), is also discussed in the report of Singapore Accountant-General's Department (2016).

Our study analyses the unique design of the SSB and provides a valuation of the option for early redemption that is attached to the SSB. The early redemption option is a Bermudan option for exercise on any business day (effectively end of day) within 10 years after the start of investing in a SSB. The redemption pay-out of par value plus any accrued interest up to exercise date would occur at the end of the month. Assuming the investor has an original investment horizon of 10 years, the investor could choose to redeem an existing SSB investment and then re-invest in a Singapore Government Security (SGS) bond or Treasury bond with a maturity equal to the remaining period of the original 10 years when market interest rate levels have gone up. Otherwise the investor could re-invest in a newly issued SSB bond. If the investor keeps to the original 10-year horizon, he/she could re-invest in the new SSB for the remaining period of the original 10 years. SGS bonds are traded in the Singapore Exchange (SGX) or they can be bought OTC at local banks. However, for retail investors, the market for purchase or sale of SGS bonds is illiquid and there is sometimes a wide bid-ask spread. We shall defer the latter matter of market friction till a later discussion.

Generic Bermudan option had been discussed and modelled in Iwaki et al. (1995) and Schweizer (2002). Evaluation of upper bound on the price is discussed in Joshi (2007). However, unlike most existing research on Bermudan options, our study is in the context of a savings bond with step-up interests, and the embedded SSB Bermudan option can also have changing strike prices over time. It is also a unique bespoke option whereby the value of exercise to a buyer need not equal to the cost of being exercised to a seller. We provide a valuation of this embedded Bermudan option that arises due to possible early redemption of the SSB.

Since the launch of the SSB program in October 2015, the amount of investments in SSB has increased by March 2019 to a total of S$3.7 billion with 100,000 individual investors. Unlike several other countries that had issued savings bonds partly to fund government programs, the SSB program states clearly that the investors' monies would not be used to fund government expenses, but would be re-invested. The purpose of the program is to provide government-guaranteed AAA-rated long-term savings returns for retail investors mostly based in Singapore. The savings bonds with maximum 10-year maturity are issued once each month, and can be redeemed any business day within any month with no commission cost on purchase or redemption, and interests received are tax-free. The SSB interest rates are pegged to the yields of Singapore Government Securities (SGS) that consist of Treasury bills with less than a year maturity, benchmark 1-year, 2-year, 5-year, and 10-year Singapore Government bonds as well as yields of other n-year maturity SGS bonds transacted by the central bank and large banks. The latter non-benchmark yields are generally not observed in the market.

The first key design principle of the SSB step-up interests is that the cumulative risk-free return of S$1 invested in the SSB, if the SSB is held till maturity, should be equal to the cumulative risk-free return at maturity of S$1 invested in a 10-year SGS government bond. The second key design principle of the SSB, as a consequence of the step-up feature, is that in whichever nth year, for $n < 10$, that the investor redeems the SSB, the cumulative risk-free return on S$1 invested in the SSB up to nth year would be equivalent to or a little smaller than the cumulative return on S$1 invested in a SGS bond with maturity of n years. The equivalence is due to the ability to buy risk-free forward contracts to lock in the return at corresponding maturities.

In the remaining part of this introductory section, we show how the SSB step-up coupons are determined and how the embedded option provides value to the investor.

1.1 SSB step-up interests

The key design principles of the step-up interest rate fixing each month center on the yields of n-year maturity SGS bonds for n = 1, 2, 3, …, 10. We assume that a continuous yield curve exists up to 10 years with the yield-to-maturity (YTM) of a n-year government coupon bond as y_n. As government bonds pay fixed regular coupon interests, the YTM is related to the fixed per annum coupon rates c_n:

$$1 = \frac{c_n/2}{\left(1 + y_n/2\right)^1} + \frac{c_n/2}{\left(1 + y_n/2\right)^2} + \frac{c_n/2}{\left(1 + y_n/2\right)^3} + \frac{c_n/2}{\left(1 + y_n/2\right)^4} + \cdots + \frac{c_n/2}{\left(1 + y_n/2\right)^{2n-1}} + \frac{(1 + c_n/2)}{\left(1 + y_n/2\right)^{2n}} \quad (1)$$

for n = 1, 2, 3, …, 10. This is a statement about a default-free government bond purchased at price S$100 and returning interest S$$c_n$/2 interest every 6-month with redemption of principal[1] S$100 and final interest payment at the end of n-years. In Eq. (1), the solution is $y_n = c_n$. In practice, unobserved YTMs at intermediate maturities are often approximated by a monotonic curve linking two adjacent YTMs with a shorter and a longer maturity. In this study we approximate non-benchmark yields by linear interpolation.

Another set of key rates, s_i, relating to the yield curve or set of YTMs is given by:

$$1 = \frac{y_n}{(1 + s_1)} + \frac{y_n}{(1 + s_2)^2} + \frac{y_n}{(1 + s_3)^3} + \cdots + \frac{y_n}{(1 + s_{n-1})^{n-1}} + \frac{\left(1 + y_n\right)}{(1 + s_n)^n} \quad (2)$$

[1] We use a generic par of S$100. For SSB, par of S$500 means that where we report coupon interest rate c%, the interest on a generic par is (c/100) × S$100, but the actual interest on SSB is (c/100) × S$500. For algebraic equations, we use par of 1 (representing 100) for parsimony.

for n = 1, 2, 3, …, 10, whereby for simplicity, coupons are annual. Spot rate, s_i, is a per annum rate that is specific for a maturity i and generally not similar for other maturities. Given the YTMs $y_1, y_2, \ldots, y_{10}$, the spot rates $s_1, s_2, \ldots, s_{10}$ can be derived from Eq. (2). The solution process is sometimes called bootstrapping. For convenience of notation, we define $\frac{1}{(1+s_n)^n} \equiv D_n$ which is also called the n-year interest discount factor. Then $D_1 = 1/(1 + y_1)$, and the remaining D_n, where n = 2,3, …, 10, can be solved iteratively:

$$D_n = \frac{1 - y_n(D_1 + D_2 + D_3 + \cdots + D_{n-1})}{1 + y_n} \tag{3}$$

Unlike the YTM over n years which is some averaging of the spot rates over different maturities from 1 year up to n year, and is not an available borrowing or lending rate, an investor can borrow or lend at the spot rate s_n over a period of n years for any n = 1,2,3, …, 10. In lending over (0, n], the investor buys a n-year zero coupon bond[2] at price 1 and receives after n years the principal payback of $(1 + s_n)^n$. Or it can be lending at $1/(1 + s_n)^n$ and receiving 1 at year n. In Eq. (2), for the n-year bond, the investor lends $y_n/(1 + s_1)$ for 1 year, $y_n/(1 + s_2)^2$ for 2 years, and so on, as well as $(1 + y_n)/(1 + s_n)^n$ for n years. Then the investor receives a corresponding amount y_n for each year until n − 1, and finally $(1 + y_n)$ at year n. With no-arbitrage, this total amount of lending must equal to 1 on the LHS of Eq. (2), which is the price of an n-year bond with the same payments.

With borrowing institutions creating borrowing rates of 1, 2, 3, and so on up to n years, and investors willing to lend at those rates s_n, a general coupon bond can be priced presently at time t = 0 as

$$1 = \frac{C_1}{(1 + s_1)} + \frac{C_2}{(1 + s_2)^2} + \frac{C_3}{(1 + s_3)^3} + \cdots + \frac{C_{n-1}}{(1 + s_{n-1})^{n-1}} + \frac{(1 + C_n)}{(1 + s_n)^n} \tag{4}$$

where $C_1, C_2, \ldots, C_{n-1}, C_n$ are different coupon interest rates at year 1, 2, … , n. Equation (4) implies that the various coupon rates have to be set to ensure the present value in Eq. (4) equals 1. This is a generalization of the fixed coupon bond shown in Eq. (2).

Equation (4) is the basic setup for determining the various step-up interest rates in the SSB program. Solving for C_n, n = 1, 2, 3, …, 10 iteratively:

$$C_n = \frac{1 - (D_1C_1 + D_2C_2 + D_3C_3 + \cdots + D_{n-1}C_{n-1})}{D_n} - 1 \tag{5}$$

where D_n's are first found by Eq. (3) using the yield curves prevailing in the SGS market. The basic setup needs to be adjusted when the prevailing yield curve is convex

[2] Zero coupon bonds do not pay interim interest rates. The payments are accumulated and paid all at once at maturity. These types of bonds are typically sold at a deep discount and redeemed at a par of 1. They became highly popular for tax and other reasons when treasury bonds are stripped and the coupons are sold separately. The zero coupon bonds also augment the depth of the fixed income market and help to extend the maturity spectrum. These types of bonds are also available in the credit sector where corporate bonds with lower than AAA ratings are also stripped.

or when yields in adjoining maturities are too flat as the coupons C_n may not be monotonically increasing.

In order to derive smooth monotonically increasing coupon rates over time in the SSB, e_n for year n = 1, 2, …, 10, is defined as

$$e_n = 1 - \frac{1}{(1+s_n)^n} - \left(\frac{\alpha_1}{(1+s_1)} + \frac{\alpha_1 + \alpha_2}{(1+s_2)^2} + \cdots + \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{(1+s_n)^n} \right) \quad (6)$$

and $\sum_{n=1}^{10} e_n^2$ is minimized[3] using controls $\{\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_{10}\}$ subject to constraints $\alpha_1 = C_1$, $\alpha_1 + \alpha_2 = C_2$, $\alpha_1 + \alpha_2 + \alpha_3 = C_3$, …, $\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{10} = C_{10}$, all α_n's ≥ 0, and $e_{10} = 0$.

The e_n's above would be zeros if indeed $C_1 \leq C_2 \leq C_3 \leq \cdots \leq C_{10}$. Then Eqs. (4) and (5) would as well produce the SSB coupons. But where solutions in (4) and (5) do not produce monotonically increasing C_n's, then the optimization by α_n's ≥ 0 would lead to the shorter maturity coupons being adjusted lower than that in Eq. (5) while the later maturity coupons are adjusted higher. This implies that in Eq. (4), the value may be a little lower than 1 on the LHS for $n < 10$.

1.2 Early redemption as an embedded option

Suppose at $0 < t < 10$ years, the investor chooses to withdraw the cash in SSB and invest in a new SGS bond for the remaining period of $10 - t$ years. Without loss of generality, suppose the investor had invested S$100 at the start. This redemption decision would occur if at time t, the present value of the remaining stream of interests during $10 - t$ plus the final par payoff of S$100 is lesser than redemption value of S$100 at time t. This event could happen if the SGS treasury term structure or yield curve has risen in level or has become steeper since the start. The higher yields applied to the remaining step-up interests fixed at the start imply a reduced bond value below par of S$100. In other words, the retail investor could sell back the SSB to receive par S$100 and accrued interest at no penalty and re-invest a smaller amount, keeping the rest as profit, to generate the same set of step-up interests as if holding onto the original SSB.

We provide an example of how redemption or exercise occurred in the embedded put option. Suppose an investor purchased a SSB in October 2016. The 10-year coupons for that issue were respectively 0.84, 0.89, 1.28, 1.75, 2.00, 2.04, 2.13, 2.23, 2.38, 2.62% on annualized basis. After holding this SSB for 1½ years, in April 2018, the SGS yield curve increased in level and slope. The next coupon payments were due to be received in ½, 1, 1½, 2, …, 8½ years. At 8½ year, the par value redemption were also due to be received. The April 2018 benchmark yields were 1.5, 1.74, 1.95, 2.18, and 2.60% respectively for 3-month, 1-year, 2-year, 5-year, and 10-year maturities. We compute the yields for maturities ½, 1, 1½, 2, …, 8½ years, including market-unobserved intermediate yields, using interpolations. We also compute the corresponding discount factors and spot rates via Eq. (3). Using the spot rates, the investor could replicate

[3] See https://www.mas.gov.sg/-/media/MAS/SGS/SGS-Announcements-pdf/SSB-PDF/FAQ/20190201-SSB-Technical-specifications_SRS.pdf for the Monetary Authority's documentation.

the remaining 8½ years SSB coupon payments and the par redemption at 8½ year by buying a portfolio of ½, 1, 1½, 2, …, 8½ years SGS bonds. This portfolio cost S$96.50 per S$100 par of SSB bond. S$96.50 is the present value of the remaining SSB cash-flows using the spot rates to discount. This portfolio is in effect a synthetic SSB with remaining 8½ years. By buying this synthetic SSB for S$96.50 and at the same time exercising to redeem the October 2016 SSB for S$100, the investor received a profit of S$3.50 from exercise of this embedded put option.

The potential gain as explained above constitutes the potential payoff to the embedded option. The underlying stochastic price is the SSB bond price with remaining maturity 10 − t. The retail investor is de facto given a valuable put to sell the SSB back to the government at t at the price of S$100. This S$100 is also the strike price of the embedded put option.

The SSB bond price S$B(t) is random at t with remaining 10 − t years of step-up interests. The exercisable price of the embedded put is max (100 − B(t), 0). The value of the SSB to the retail investor at any time point t is the maximum of S$100 or S$B(t). This value is conditional upon exercise and is not affected per se by the fact that SSB is not transferable.

In Sect. 2 we discuss the approach and methods used in pricing the embedded Bermudan option. To price option based on underlying SSB bond prices, we need to specify the underlying interest rate processes. Section 3 shows the numerical method to compute the no-arbitrage prices of the Bermudan option. Section 4 considers other market factors that could affect the price of this bespoke Bermudan option, including transaction costs, liquidity, and inaccessibility to the SGS market by retail investors. Section 5 contains the conclusions.

2 Pricing model for embedded option

To value this Bermudan option, we construct an interest rate process that drives the stochastic bond price. There are several approaches in modelling interest rate processes including the use of short rates and use of lognormal forward LIBOR rates. As the problem in our study is pricing an embedded option on an underlying bond price process, it is inappropriate to use SIBOR (equivalent of S$-LIBOR) rates that are more suitable to price SIBOR-based derivatives. There are no commonly available market instruments on SIBOR-based derivatives. Besides, SIBOR is used as a mortgage-fixing rate in Singapore which causes systematic housing risk factors to affect that rate. It is also inconvenient to use forward rates as underlying as this leads to different forward measures for different maturities which is difficult to deal with when we have American exercise feature in our embedded option. It leads to drifts that are state-dependent and are difficult to compute numerically in a robust way. See Xiao (2011) for the latter issue.

We use short rate processes that are popular in the industry for pricing bond derivatives. Specifically we employ an iterative Black–Derman–Toy (BDT) model. See Black et al. (1990) and also the related Black–Karasinski model in Black and Karasinski (1991), both with non-negative interest rate processes. The BDT model satisfies the no-arbitrage condition, allows exogenous specification of non-constant non-anticipative volatility of short rates, provides for fitting to the current term structure of zero prices as in Ho and Lee (1986), always have positive interest rates in the dynamic process, and can be easily computed via stable numerical methods such as in a recombining binomial tree even though there is no exact analytical solution to options on this process.

For the BDT model, let the short rate process r(t) or the interest rate over an infinitesimal interval, follow the stochastic differential equation

$$d\ \ln(r(t)) = \left[\theta(t) + \frac{\dot{\sigma}(t)}{\sigma(t)}\ln(r(t))\right]dt + \sigma(t)dW^{Q}(t) \tag{7}$$

where $W^Q(t)$ is a Wiener process under equivalent martingale measure (EMM) Q that at the start of the process at time zero is a normal random variable with mean 0 and variance t. $\theta(t)$ is the deterministic-time mean reversion level to which ln(r(t)) will drift toward when $\dot{\sigma}(t) < 0$. The instantaneous volatility $\sigma(t)$ at future time t can be specified exogenously. We assume that the pre-determined or anticipated forward-looking volatility function $\sigma(t)$ holds for any SSB 10-year bonds issued at different months in our sample period from October 2015 till December 2018. For long maturity bonds, it is typical to expect mean reversion of short rates in Eq. (7) whereby $\frac{\partial\sigma(t)}{\partial t} < 0$.

Another way of representing the BDT short rate process instead of Eq. (7) is its integral form

$$r(t) = U(t)\exp\left(\sigma(t)W^{Q}(t)\right)$$

where U(t) and $\sigma(t)$ are deterministic functions of time t. Under the EMM Q, where r(t) is defined over the filtered probability space[4] $(\Omega, F, (F_t)_t, Q)$, no arbitrage occurs as

$$E_t^Q\left[\exp\left(-\int_t^{10} r(s)ds\right)\right] = B(t).$$

$W^Q(t)$ is a standard F_t-measurable Wiener process on this space.

The numerical computation of the BDT model interest rate process driving the stochastic bond price under measure Q involves construction of a lattice or binomial short rate tree. The constructions are described in Jamshidian (1991), Boyle et al. (2001), and Brigo and Mercurio (2001). By constructing a binomial short rate tree, we can test for exercise of the embedded put option at each interest rate node at each time t from t = 0 to t = 10 years. At each node, the maximum of either the exercise value or the value of the option at that point in time is assigned to the node. Working backward from the maturity end of the tree, we can arrive at the embedded put value at time zero or the start of any SSB issue.

In the lattice tree shown in Fig. 1 below, following usual constructions, the probabilities of short rate increase and short rate decrease are fixed at ½. In the BDT short

[4] The physical measure π is related to the Q-measure via the Girsanov theorem. Specifically, $dW_t^{\pi} = dW_t^{Q} - \lambda(r,t)\,dt$ where $\lambda(r,t) \equiv [½\,\sigma(r,t)^2 P_{rr} + \mu(r,t)P_r + B_t)/P - r(t)]\,dt/[\sigma(r,t)P_r/P]$ and P is a zero coupon bond price at t.

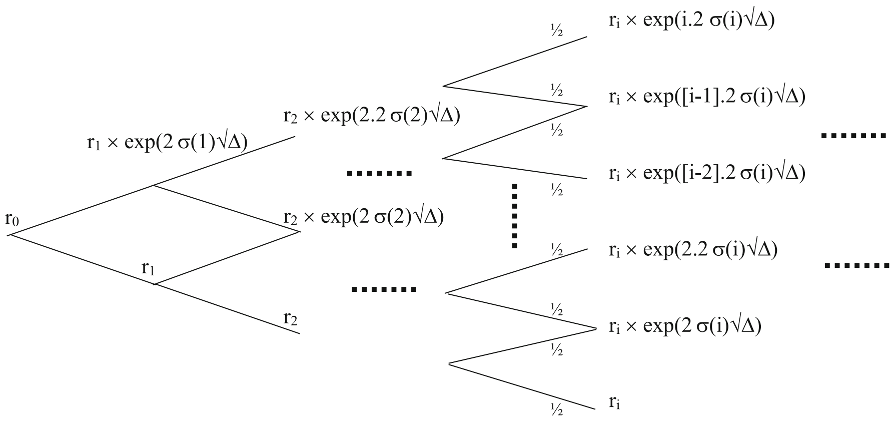


Fig. 1 BDT lattice tree at the ith interval

rate process of $r(t) = U(t)\exp\big(\sigma(t)W^{Q}(t)\big)$, a numerical discrete time representation is

$$r(i,j) = U(i)\exp\Big(\sigma(i)2j\sqrt{\Delta}\Big) \tag{8}$$

where i is an index referring to the ith interval such that $i\Delta = t$ years. Thus U(i)U(t) and σ(i)σ(t). And j is the node level or height at ith interval. The normally distributed $W^{Q}(t)$ is approximated by a binomial distribution resulting in short rates of $r_i \times e^{2j\sigma(i)\sqrt{\Delta}}$ at interval i for j = 0,1,2,3, …, i. The lowest node in the ith (for i = 1, 2, 3, …, 10/Δ) time interval where length of each interval is Δ year, has a short rate of r_i while the next higher node at the same interval has a short rate of $r_i \times e^{2\sigma(i)\sqrt{\Delta}}$. And the next higher node has a short rate of $r_i \times e^{4\sigma(i)\sqrt{\Delta}}$ and so on, where σ(i) is the anticipated volatility of short rate at interval i. We employ 2400 intervals, approximating a business day each, so the time interval Δ is 10/2400 year. Note that r_i is lowest node on the BDT binomial tree representing the short rate at time iΔ with time-to-maturity 10 − iΔ. r_i is different from the stochastic process r(t) that can take this value with EMM probability of 0.5^i at the ith interval.

Given the σ(i) function, r_i for each interval (or similarly, at t = iΔ) is determined and used to calibrate the lattice tree so that any zero price P(t) in the market, at time zero for zero-coupon bond at any maturity t, is correctly priced. In turn, σ(i) and r_i can determine U(i). Since r_i will be different for a different σ(i) given the zero prices at t = 0, iterating on different functions of σ(i) should provide a r_i that is consistent with the distributional characteristic of the short rate r(t) process defined over Δ.

The mean of r(t) at t, short rate over Δ at t, as viewed at time zero can be found as the forward rate at time zero. Denote the forward rate from t to t + Δ at time zero be F_t^0, i.e. $F_t^0 = E^{Q}(P(t)/P(t+\Delta)) - 1$. The lowest short rate at t on the BDT lattice tree is r_i for iΔ = t. Under the short rate process, therefore the cumulative normal distribution of $r_i - F_t^0$ has a value $\sigma(i)\sqrt{\Delta}Z_i$ where Z_i itself is the standard normal variable value with cumulative distribution $\Phi(Z_i) = 0.5^i$. $\Phi(Z_i)$ is the left-tail probability of Z_i.

For this study, all SSB coupon interest data and SGS 1-year, 2-year, 5-year, 10-year bond yield data are used over monthly periods from October 2015 to December 2018. Thus we have 39 monthly sample points with each sample point consisting of the set of yields and bond prices for that month within the span of October 2015 to December 2018 inclusive.

Our data show $r_i^m - F_t^m \approx \sigma(i)\sqrt{\Delta} Z_i$. We assume stationarity on the term structure and also a constant σ function for every sample point m = 1, 2, 3, …, 39, so we can estimate σ(i) using

$$\frac{1}{39}\sum_{m=1}^{39}\left[\left(r_i^m - F_t^m\right)\right]/[\sqrt{\Delta}Z_i]. \tag{9}$$

This approximate characterization is used by first iterating on a constant $\sigma = 0.0005$ for all i and then using the estimated function σ(i) in (9) iteratively until the estimated function σ(i) becomes stable or similar to within 1.0×10^{-6} or $o(\Delta^2)$ for any t. This approach avoids a totally exogenous volatility function input borne out of guess-work. The alternative of guess-work cannot be avoided as there are no other interest rate derivatives in this case such as caps, floors, or swaptions whereby volatility parameter or other additional parameters can be inferred. In this case, the characterization of approximately normal distribution of r_i and the additional assumption of stationarity of r(t) process underlying new SSB issues each month, provide some information with which to find an approximate σ(i) for each i. The approach provides more confidence in using an iterated volatility function. Thus the iterated BDT can be an improvement over the typical BDT with totally exogenous volatility function input when the underlying data are suitably close to normal.

All interest rates and SSB data used in this study are obtained from the publicly available MAS website https://www.mas.gov.sg/bonds-and-bills. A term structure of zero coupon bond prices from 1-day till maturity to 10 years till maturity, separated by each business trading day, is constructed using interpolation of the published government bond yield rates. Linear interpolation is used as nonlinear splines tried indicated trivial difference and may introduce bias especially during really flat term structures.

Table 1 shows the averages of SGS yields for 3-month, 1-year, 2-year, 5-year, and 10-year bills and bonds for different periods of 3-month up to 6-months from October 2015 till December 2018. It also shows the averages of step-up interests for 1-year, 2-year, up to 10-years for the different periods from October 2015 to December 2018.

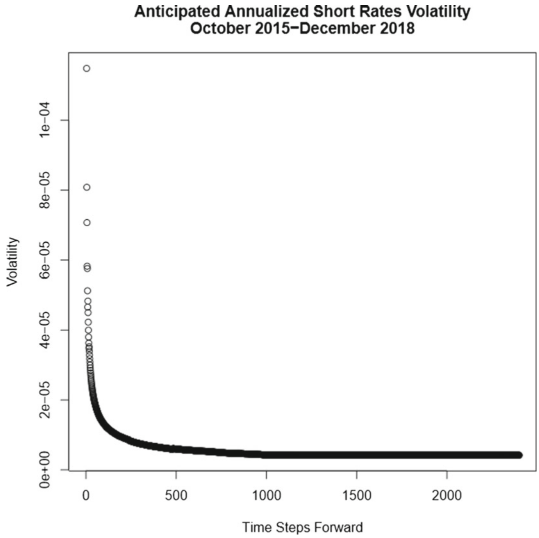
It is noted from Table 1 that between the period July 2016 and December 2016, the yield curves were on average at the lowest levels compared with the entire sample period of October 2015 to December 2018. The step-up interests for SSB issued between July 2016 and December 2016 were also the lowest (excepting 4th and 5th year interests) versus the period July–December 2017. Hence ex-post when the yield curves and interest rates picked up in 2017 and 2018, there would be higher numbers of redemptions of SSB issued between July 2016 and December 2016. This is indeed the context of the example shown in Sect. 1.

The zero coupon prices, together with iterated volatility function, are used to calibrate the EMM probabilities and the short rates at each branching out of time-level

Table 1 Averages of SGS yields and SSB step-up interest rates

% p.a.	Oct–Dec 2015	Jan–Jun 2016	Jul–Dec 2016	Jan–Jun 2017	Jul–Dec 2017	Jan–Jun 2018	Jul–Dec 2018
Average 3-mth T-bill yield	1.0267	0.7550	0.7717	0.8800	1.2617	1.4833	1.8217
Average 1-year T-bill yield	1.2033	1.0033	0.8750	1.0417	1.2417	1.6183	1.8633
Average 2-year T-bond yield	1.2333	1.0050	0.9433	1.2483	1.3333	1.7600	1.9783
Average 5-year T-bond yield	1.9667	1.7150	1.4367	1.6750	1.6033	2.0217	2.2017
Aver. 10-year T-bond yield	2.6355	2.3039	1.8950	2.3125	2.1242	2.2377	2.5124
Average SSB year 1 interest	1.0967	1.0350	0.8617	1.0067	1.1500	1.5067	1.7800
Average SSB year 2 interest	1.1467	1.1000	0.9950	1.4067	1.3883	1.7633	2.1217
Average SSB year 3 interest	1.8800	1.7333	1.3867	1.7367	1.5883	1.9133	2.3017
Average SSB year 4 interest	2.8133	2.5050	1.8350	2.0467	1.7833	2.0417	2.4367
Average SSB year 5 interest	3.1533	2.7350	2.0900	2.3167	2.0000	2.1850	2.5617
Average SSB year 6 interest	3.1533	2.7350	2.1483	2.5367	2.2367	2.3483	2.6733
Average SSB year 7 interest	3.1933	2.7500	2.2400	2.7350	2.2475	2.4917	2.7450
Average SSB year 8 interest	3.2433	2.7767	2.3467	2.9483	2.7117	2.6367	2.8233
Average SSB year 9 interest	3.3433	2.8483	2.5000	3.1933	2.9550	2.7917	2.9267
Average SSB year 10 interest	3.7233	3.1167	2.7467	3.4967	3.2017	2.9617	3.0800

Fig. 2 Estimated volatility function $\sigma(\tau)$ used as input



nodes on the BDT short rate tree. There is in theory an unlimited number of solutions for the calibration. We had employed EMM probabilities of ½ for both the interest level-up and interest level-down branches, to arrive at a unique solution of an EMM term structure for pricing the embedded option. At any time-level node (i, j), where $i\Delta = t$ on the lattice tree, a contingent bond price, yielding subsequent interests according to the SSB schedule, can be found, which is denoted B(t) in the last section. This is used to determine if exercise would occur. The maximum of next period expected option value and the possible non-zero exercise value is discounted by the risk-free short rate to arrive at an earlier period node value, and this process of backward propagation continues till the option value is found at $t = 0$.

We collect the step-up interest schedules month-by-month over 39 months from the start of the SSB program in October 2015 to December 2018. The approximated volatility function is shown in Fig. 2. The function used in our embedded option price estimation for each of 39 months from October 2015 to December 2018 is $\sigma(t) = 0.00015 \exp(-0.0072\, t/2400 + 0.0000022\, [t/2400]^2)$, denoting 1-trade day volatilities for forward t number of years. This can be re-stated as $\sigma(\tau) = 0.00015 \exp(-0.0072\, [T - \tau]/2400 + 0.0000022\, [(T - \tau)/2400]^2)$ when τ is expressed in terms of number of years remaining. This function is depicted in Fig. 2.

Figure 2 shows that volatility over a day in the short rate decreases rapidly in the first year and then reduces slowly afterward. The average volatility per period in the 10-year horizon is 0.00033.

3 Empirical results

To compute the Bermudan option which allows exercise at the end of each business day within the 10 year maturity horizon, we discretize the stochastic bond price process

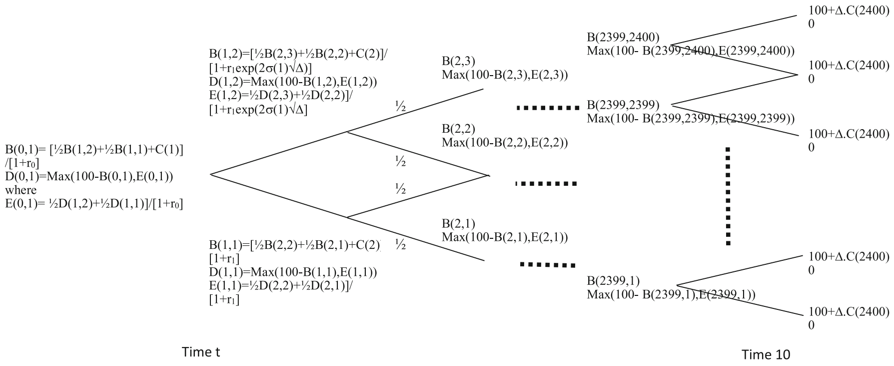


Fig. 3 Pricing algorithm of embedded put in 10-year SSB. SSB coupon at period i is C(i), par value of bond is 100, SSB bond price at ith interval and jth node is B(i, j) where time at i is $i\Delta$, $\Delta = 10/2400$ year, and $\max(j|i) = i$. The top number on the node indicates bond price and the bottom number indicates Bermudan put price D(i, j) at node (i, j). E(i, j) is the no-arbitrage discounted expected value of the next period put option at (i, j)

into 2400 periods within 10 years on the lattice tree. This maps approximately to the number of business days within the maturity horizon, excluding days without SGX price quotes. Thus it is seen that the Bermudan put option is not exactly an American put option which would have allowed continuous trading and exercise, and which would form an upper option value bound in our case. It is noted that carrying a bond entitles the owner to accrued interest on a continuous basis.

The input volatility function is estimated based on discussion in the previous section. Figure 3 shows the pricing algorithm of the embedded put in a 10-Year SSB.

At node before the 10-year expiry of the SSB, the investor holds the bond valued at \$B(2399, j) at node (2399, j). This is obtained by discounting $100 + \Delta \times C(2400)$ by $(1 + r_{2399} \times \exp(2j\,\sigma(2399)\sqrt{\Delta}))$. At node (2399, j), the European put option value is $E(2399, j) = [0.5 \times D(2400, j) + 0.5 \times D(2400, j + 1)]/(1 + r_{2399} \times \exp(2j\,\sigma(2399)\sqrt{\Delta}))$. The Bermudan put option value is then $D(2399, j) = \mathrm{Max}(100 - B(2399, j), E(2399, j))$.

The empirical results of the embedded put option value are shown below in Table 2. There are 3 cases involving zero transaction cost, 0.5 and 1.0% transaction costs when re-investing in SGS bonds after the SSB bond is redeemed at par of S\$100. For example, 0.5% means that when paying for the SGS bond with price S\$B(t), payment is S\$1.005 B(t). This implies that the exercise condition becomes harder, max $(100 - 1.005\,B(t), 0)$, and thus produces a less valuable embedded put option. Table 2 clearly shows monotonically decreasing value of the Bermudan option as transaction costs increase.

Table 2 shows that based on the market data between October 2015 and December 2018, when forward rates significantly exceeds SSB coupon interests most of the future periods, then the probability of exercise is higher (as forward rate increase indicates the short rates on average will increase, resulting in higher bond discounting, and hence cheaper bond with remaining maturity). Hence the values of the embedded put options are positive and significant. In more than 2/3 of the number of months, however, the

Table 2 SSB Bermudan put option S$ values (using 2400 daily intervals) per S$100 par value of the SSB bond

Transaction costs (%)	Oct 2015	Nov 2015	Dec 2015	Jan 2016	Feb 2016	Mar 2016	Apr 2016	May 2016	Jun 2016	Jul 2016
0.0	0.71	0.81	0.00	0.11	0.35	0.77	1.42	0.59	0.01	1.01
0.5	0.29	0.38	0.00	0.00	0.00	0.34	0.98	0.15	0.00	0.56
1.0	0.00	0.00	0.00	0.00	0.00	0.00	0.53	0.00	0.00	0.11
Transaction costs (%)	Aug 2016	Sep 2016	Oct 2016	Nov 2016	Dec 2016	Jan 2017	Feb 2017	Mar 2017	Apr 2017	May 2017
0.0	0.73	0.32	0.00	0.00	0.06	0.00	0.65	0.62	0.00	0.00
0.5	0.28	0.00	0.00	0.00	0.00	0.00	0.22	0.18	0.00	0.00
1.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Transaction costs (%)	Jun 2017	Jul 2017	Aug 2017	Sep 2017	Oct 2017	Nov 2017	Dec 2017	Jan 2018	Feb 2018	Mar 2018
0.0	0.19	0.00	0.00	0.16	0.00	0.00	0.00	0.00	0.09	0.00
0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Transaction costs (%)	Apr 2018	May 2018	Jun 2018	Jul 2018	Aug 2018	Sep 2018	Oct 2018	Nov 2018	Dec 2018	
0.0	0.00	0.00	0.00	0.15	0.12	0.07	0.00	0.00	0.80	
0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.37	
1.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	

embedded put option has zero or nearly zero values. The highest embedded put values are about 1.0–1.5% of the par value of the SSB bonds. This percentage is even smaller if there is transaction cost of ½ or 1% should the investor re-invest.

In the 12 months from April 2016 to March 2017, the average embedded Bermudan option values per monthly issue of SSB is S$0.45 per S$100 par. This is also the period when the yields were low but increasing afterward. Hence it corroborates the idea that the resulting higher forward rates would on average be larger than the step-up interests, leading to higher option value and higher ex-ante EMM probability of early redemption. This average option value is much larger than the S$0.04 per S$100 par on monthly issues in the 12 months from December 2017 to November 2018.

Of the various monthly tranches of SSB issued since October 2015, the tranches with relatively highest redemptions are reflected as those with the lowest % amount outstanding as of December 2018. See data in https://www.mas.gov.sg/bonds-and-bills/SGS-Bond-Statistics. Two periods stood out as having issued tranches with the lowest % outstanding as of December 2018. They are issues from May 2016 to March 2017 with 57–72% outstanding, and issues from June 2017 to March 2018 with 56–69% outstanding. The first period has some corroborating evidence with respect to the higher ex-ante probability of redemption indicated by higher option values. However, it should be noted that ex-post realization need not always be in line with ex-ante expectations.

The higher embedded option values for certain issues may be explained via several observations about the ensuing SGS yield curves, forward rate curves, and associated step-up coupon schedules on those issue months. Figure 4a–d provide typical characterizations of the yield curve, forward curve and step-up schedule in situations with high embedded option values (Fig. 4a, b) and in situations of low or zero embedded option values (Fig. 4c, d).

Figure 4a on the left corresponds to an embedded option of S$1.42, the highest in all the sample periods. The initial drop in yield in year 1 followed by steep yield increase of about 10% to year 10 maturity implied a very rapidly rising forward rate curve as well. The catch-up in the step-up interest occurred in the 2–6 year, but then lagged behind. Since the forward rate is the EMM-measure mean of the short rates, the high forward rate implied higher probabilities of having SGS bond prices dropping below the put exercise of S$100, and hence there is higher embedded value with higher exercise probabilities.

Figure 4b on the right on December 2016 has an embedded put option of only S$0.06 if exercisable into market investing. However, it is the only case (see Table 5) where even without market investing, there is a likelihood of switching into a new SSB since the coupon increase rate even at the beginning is faster than the increase rate toward the second half of the 10-year holding. The fast increase is due to the large forward rate premiums over yields in the first 5 years.

Figure 4c (left) and d (right) correspond to zero embedded option value or ex-ante zero probability of redemption. In these cases the yield curves' increases were more gentle at only about 5% over 10 years, unlike the case in Fig. 4a. The slower yield increase with maturity implied both forward rates and also the step-up rates grow into the future horizon more in sync and at a gentler pace. This situation implied future short rates were not likely to take extremely high values resulting in lower SGS bond

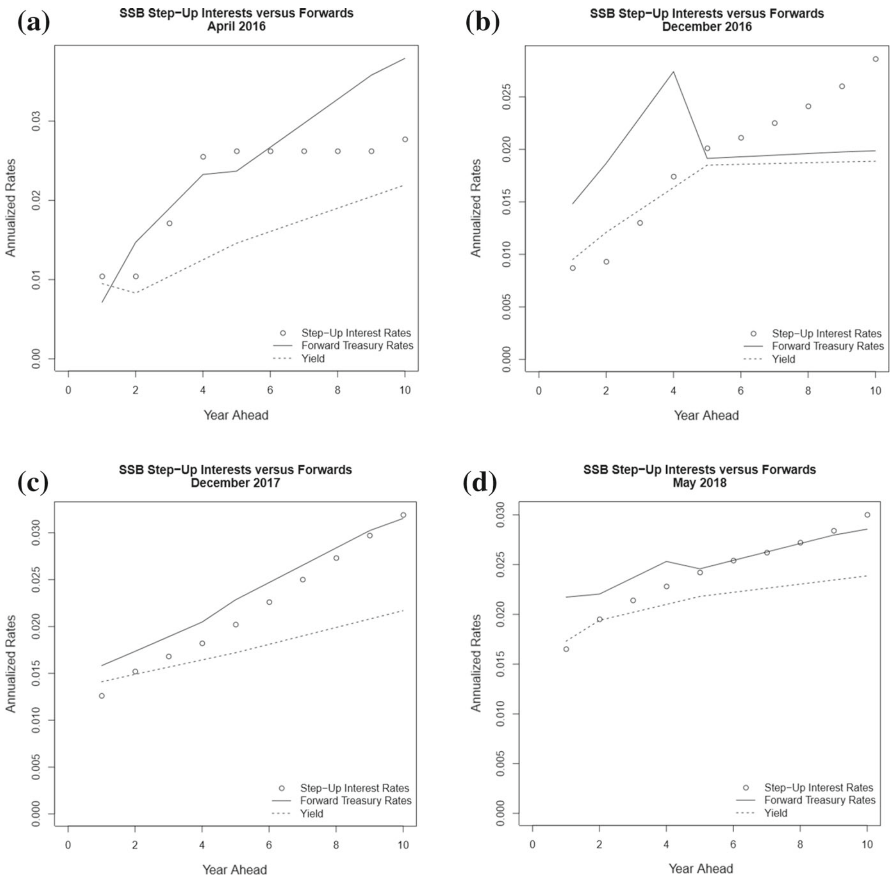


Fig. 4 **a**, **b** SGS yield curve, forward rate curve, and step-up interest schedule with high embedded option values. **c**, **d** SGS yield curve, forward rate curve, and step-up Interest schedule with low embedded option values

prices. Instead, the SGS bond prices would remain about close to and slightly above par, so that no redemption occurred.

To provide a robustness check on the use of the BDT model, we also use an alternative option pricing based on the Generalized Ho–Lee model that has similar advantage to the BDT model, but has a slightly different future volatility specification. The original Ho and Lee (1986) model specifies the short rate process as $dr = \gamma(t)\ dt + \sigma\ dW(t)$ where short-rate volatility σ is a constant. Ho and Lee (2004, 2007) extended the original model to a generalized model with changing yield volatilities over time. In Fig. 5, the binomial tree of this model shows that yield volatility at $i = 1$ is $½v_1$, at $i = 2$ is $½(v_1 + v_2)/2$, and at $i = 3$ is $½(v_1 + v_2 + v_3)/3$, and so on. Unlike the short rate volatility structure in BDT, the volatility structure here imposes yield volatility correlation over time.

The generalized Ho–Lee model similarly allows parameters within the model to be calibrated so that bond prices at the point of pricing fit with the initial term structure

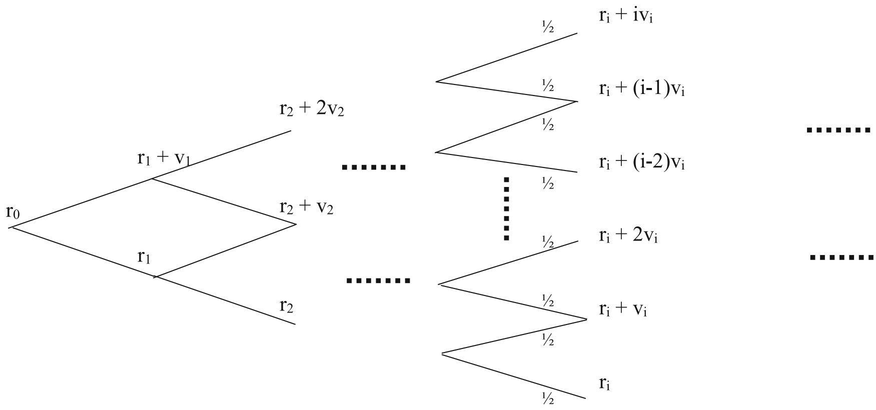


Fig. 5 Generalized Ho–Lee lattice tree at the ith interval

observed in the market. In this model, deterministic volatilities of the future yields can vary across time. The model also allows stable and fast price computations under recombining lattice trees. Applying the same lattice procedure gives rise to SSB bond price at (i, j) that is priced by B(i, j) = [0.5 × B(i + 1, j + 1) + 0.5 × B(i + 1, j)]/exp (r_i + jv_i). At node (i, j), the European put option value is E(i, j) = [0.5 × D(i, j + 1) + 0.5 × D(i, j)]/exp (r_i + jv_i). The Bermudan put option value is then D(i, j) = Max (100 − B(i, j), E(i, j)). By assuming for m = 1, 2, 3, …, 39, the yields y_n for n in the various periods, are stationary across m, we estimate the sample standard deviation of the various yields and use them to estimate v_1, v_2, v_3, …, v_{2399}. The r_i's are then calibrated to ensure the zero coupon prices are fitted with the observed values. The empirical pricing results of the embedded Bermudan put options are shown in Table 3.

Comparing Table 3 with Table 2, we see that the magnitude of the Bermudan put option prices are similar at about S$2 or less. In some cases the prices are close to zero. In April 2016 as in Table 2, the option price spiked up to over a dollar. Similarly in June 2016 to September 2016, the option prices were relatively high. The generalized Ho–Lee option prices however evidenced the volatility correlation effect as the prices clustered together. This may be an effect due to the model itself with a stronger volatility specification as pointed out in Chan et al. (1992). In general, we have shown robustness in the option price magnitudes.

4 Market frictions

Why is the SSB program valuable and helpful to retail investors within the country? In existing commercial bank term deposits, there would usually be a penalty or cost for early redemption of deposits, such as forfeiting of accumulated interests. This is the price to pay if investors want to exercise early redemption to recover the par value or the original deposit amount. It is also the cost the bank would recover to face the uncertainty of term structure risk such as having to find new source of financing

Table 3 SSB Bermudan put option S$ values (using 2400 daily intervals) per S$100 par value of the SSB bond computed using generalized Ho–Lee model

Transaction costs (%)	Oct 2015	Nov 2015	Dec 2015	Jan 2016	Feb 2016	Mar 2016	Apr 2016	May 2016	Jun 2016	Jul 2016
0.0	0.04	0.02	0.06	0.04	0.05	0.28	1.12	1.32	1.80	1.20
0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.67	0.87	1.34	0.75
1.0	0.00	0.00	0.00	0.00	0.00	0.00	0.23	0.42	0.89	0.30

Transaction costs (%)	Aug 2016	Sep 2016	Oct 2016	Nov 2016	Dec 2016	Jan 2017	Feb 2017	Mar 2017	Apr 2017	May 2017
0.0	0.97	2.21	1.83	1.47	1.20	0.22	0.08	0.13	0.07	0.03
0.5	0.52	1.75	1.37	1.01	0.74	0.00	0.00	0.00	0.00	0.00
1.0	0.09	1.29	0.91	0.55	0.29	0.00	0.00	0.00	0.00	0.00

Transaction costs (%)	Jun 2017	Jul 2017	Aug 2017	Sep 2017	Oct 2017	Nov 2017	Dec 2017	Jan 2018	Feb 2018	Mar 2018
0.0	0.47	0.49	0.60	0.26	0.34	0.61	0.05	0.65	1.26	1.01
0.5	0.06	0.07	0.16	0.00	0.01	0.17	0.04	0.21	0.80	0.56
1.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.35	0.13

Transaction costs (%)	Apr 2018	May 2018	Jun 2018	Jul 2018	Aug 2018	Sep 2018	Oct 2018	Nov 2018	Dec 2018
0.0	0.78	0.71	0.51	0.19	0.52	0.75	0.74	0.72	0.62
0.5	0.34	0.26	0.08	0.00	0.09	0.30	0.29	0.27	0.18
1.0	0.01	0.00	0.00	0.00	0.00	0.01	0.01	0.00	0.00

money for its loans to other businesses, when the deposits are redeemed at a time of higher interest rates. The SSB program provides retail investors with the option of early redemption, but takes away the cost of the term structure uncertainty which is absorbed by the government. This cost is also the value of the embedded option of early exercise, which is where investors do not have to pay any penalty for early redemption.

Some of the arbitrage reasons we utilize in the earlier section may not be feasible to the retail investors when there are market frictions such as liquidity and transaction costs. Retail or individual investors may buy or sell the SGS bonds on the Singapore Exchange (SGX). However, reported bid-ask price quotes on the SGX (see http://www.sgx.com/wps/portal/ sgxweb/home/marketinfo/fixed_income/sgs) show a large spread or illiquidity. The bid-ask spread for regularly traded bonds could be 1.40 for a 102 bid and 103.4 ask price e.g. on a 2012 issue with over 10 years to go. Many shorter maturity bonds do not appear to have ready quotes. Hence using a 0.5 and 1.0% transaction cost as shown in the last section is relevant in the consideration of market friction effect on the Bermudan option pricing.

When retail investors buying SSB have no access to the SGS market in terms of being able to buy and sell SGS bonds readily, then the option value computed in the last section will not be realized. In such circumstance, the investor can only roll over or re-invest in a newly issued SSB bond if the original SSB is to be redeemed. The investor will only exercise or redeem provided the present value of the stream of interests to come in the remaining time till the 10 year maturity is less than the present value of the stream of interests in a new SSB bond for the same remaining time till the 10 year maturity. The former is a changing strike price or condition and is what the investor will pay in terms of foregoing the remaining interests. The latter comprising interests from a new SSB is the underlying stochastic variable. We can see that the step-up feature in the SSB program makes it less likely for the exercise condition to happen. This is because the beginning months of the new SSB bond will have lower interest rates due to the step-up feature, so there is less probability the investor will exercise by giving up existing SSB interest streams that being older have been on the higher steps.

To price the embedded put option under this context of changing strike price, we use interest rate model in Eq. (7) but specialize it to a constant short rate volatility over each interval. For comparison with the case of changing volatility we price the embedded put option under various scenarios of increasing volatility, viz. $\sigma = 0.00005$, $\sigma = 0.00010$, and a higher $\sigma = 0.00020$. The results are shown in Table 4 below.

It is seen that the prices resemble the case in Table 2 with no transaction cost. The difference is typically in the 3rd decimal place except February 2018 when the difference is 0.01. For constant volatility, when it is doubled as in the case $\sigma = 0.00010$, the option price difference is also very small with difference in the 4th and occasionally 3rd decimal places, as seen below. Thus our results within the iterated BDT model appears to be robust. When volatility is quadrupled to $\sigma = 0.00020$, almost all cases differ marginally only in the 3rd decimal places. Not all cases show small increase in option price; the price in September 2016 and February–March 2018 show slight decreases.

Table 4 SSB Bermudan put option S$ cost (using 2400 daily intervals) per S$100 par value of the SSB bond under constant volatility of short rates σ and zero transaction cost

Short rate volatility σ	Oct 2015	Nov 2015	Dec 2015	Jan 2016	Feb 2016	Mar 2016	Apr 2016	May 2016	Jun 2016	Jul 2016
0.00005	0.713	0.807	0.000	0.115	0.351	0.768	1.418	0.585	0.007	1.007
0.00010	0.714	0.807	0.000	0.115	0.351	0.768	1.419	0.585	0.007	1.008
0.00020	0.716	0.809	0.000	0.116	0.352	0.770	1.420	0.586	0.008	1.010
Short rate volatility σ	Aug 2016	Sep 2016	Oct 2016	Nov 2016	Dec 2016	Jan 2017	Feb 2017	Mar 2017	Apr 2017	May 2017
0.00005	0.725	0.317	0.000	0.000	0.059	0.000	0.655	0.619	0.000	0.000
0.00010	0.727	0.318	0.000	0.000	0.060	0.000	0.655	0.619	0.000	0.000
0.00020	0.728	0.310	0.000	0.000	0.060	0.000	0.656	0.620	0.000	0.000
Short rate volatility σ	Jun 2017	Jul 2017	Aug 2017	Sep 2017	Oct 2017	Nov 2017	Dec 2017	Jan 2018	Feb 2018	Mar 2018
0.00005	0.187	0.000	0.000	0.163	0.000	0.000	0.000	0.000	0.154	0.117
0.00010	0.188	0.000	0.000	0.164	0.000	0.000	0.000	0.000	0.155	0.117
0.00020	0.190	0.000	0.000	0.166	0.000	0.000	0.000	0.000	0.100	0.000
Short rate volatility σ	Apr 2018	May 2018	Jun 2018	Jul 2018	Aug 2018	Sep 2018	Oct 2018	Nov 2018	Dec 2018
0.00005	0.000	0.000	0.000	0.154	0.117	0.065	0.000	0.000	0.804
0.00010	0.000	0.000	0.000	0.155	0.117	0.065	0.000	0.000	0.805
0.00020	0.000	0.000	0.000	0.156	0.118	0.066	0.000	0.000	0.806

From Eq. (7), for constant σ

$$d\ln(r(t)) = \theta(t)dt + \sigma dW(t) \tag{10}$$

then $r(t) = r_0 \exp\left(\int_0^t \theta(u)du + \sigma W(t)\right)$ where W(t) is distributed as N(0, t), and t in years is between 0 and T = 10. r(t) is the instantaneous short rate at t. r_0 is the value of r(t) at t = 0. In our lattice tree context r(t) is the (annualized) short rate over interval Δ at t. Now at any time-level node (i, j) where t* = i on the lattice tree, the short rate at the node is r(i, j) and the difference r(i, j) − r(0,1) = $\delta_{ij,}$ where r(0,1) is also the beginning short rate on the lattice, r_0. Suppose r(t) process starting at any future node at t* has the same representation as the current except for the mean shift over period (0, t*] by δ_{ij}. This is equivalent to one interval Δ shift of $\delta_{ij}/(240 \times t^*)$. Then at t*, the interval Δ forward rate t-period into the future is $F_{ij}(t^*) = F_0(t) + \delta_{ij}/(240 \times t^*)$. This is a characterization of the evolution of the short rate stochastic process itself over time.

At t = 0, suppose the SSB step-up interests were C_1, C_2, …, C_{10} for each of the 10 years into the future. These coupons were fixed with respect to the short rate process r_t for t ∈(0,10) (determined by the SGS yields and the volatility function). Another characterization of the coupon interest is with respect to interval forward rates $F_t \equiv F_0(t) = E_0[r(t)]$ where subscript 0 denotes expectation under EMM taken at start of original SSB investment, and $F_0(t)$ is forward rate over Δ at t>0 effective (t, t + Δ].

From Eq. (4), we can replace the spot rates with forward rates:

$$1 = \frac{C_1}{(1+F_1)} + \frac{C_2}{(1+F_1)(1+F_2)} + \frac{C_3}{(1+F_1)(1+F_2)(1+F_3)} + \cdots + \frac{(1+C_n)}{\prod_{k=1}^{n}(1+F_k)} \tag{11}$$

where an intuitive solution for Eq. (11) is $F_k = C_k$. We would see that if at t*, each $F_{ij}(t^*) = E_{t*}[r(t)]$ increases to $F_0(t) + \delta_{ij}/(240 \times t^*)$ per interval Δ at t*, then each $C_k/240$ per interval increases to $C_k/240 + \delta/(240 \times t^*)$.

At t*, the retail investor with a beginning investment horizon of 10 years would only have a remaining horizon of 10 − t* years or 240 (10 − t*) number of periods or business days. For easy explanation of what is going on, suppose t* is a whole number. At t*, continuing with the original SSB, the investor would receive C_{t*+1}, C_{t*+2}, …, $C_{t*+10.}$ However, a new SSB issued at t* would provide step-up interests of $C_1 + \delta_{ij}/t^*$, $C_2 + \delta_{ij}/t^*$, …, $C_{10} + \delta_{ij}/t^*$. If the investor switches to the new SSB, he/she would hold up to 10-t* years in order to maintain the original 10-year investment horizon.

Thus the retail investor would choose the new SSB and give up the original SSB provided the following condition holds such that the investor is better off, switching into a new SSB for the remainder of the original 10-year horizon: $\sum_{k=1}^{10-t*} \left(C_k + \frac{\delta_{ij}}{t}*\right) > \sum_{k=t*+1}^{10} C_k$, assuming the investor has no other risk-free investment opportunity outside the SSB program. For the investor, the early exercise of SSB is now a bespoke Bermudan call option with option strike price as $\sum_{k=t*+1}^{10} C_k$ that decreases with time t^*. The underlying is $\sum_{k=1}^{10-t*} \left(C_k + \frac{\delta_{ij}}{t}*\right)$ with stochastic δ taking values δ_{ij}. The investor receives the profit via exercise of the original SSB and then switches into the new SSB. This profit or value of exercise at t* of $\left(\sum_{k=1}^{10-\,t*} \left(C_k + \frac{\delta_{ij}}{t}*\right) \sum_{k=t*+1}^{10} C_k\right)$ to the investor is less than the cost of being exercised to the government or SSB seller.

The cost to the latter at t* is the same 100 − B(t*), as the government redeems at 100 but receives only redeemed SSB with a market value of only B(t*).

In standard option theory, a buyer's gain is equal to the seller's loss. However, in the SSB context, the retail investor or buyer may not enjoy the full value of the benefit of early redemption equal to the cost to the government which is the seller. The lesser or limited benefit to the investor is due to the limited access to the competitive prices in the SGS market which is usually available to institutions such as commercial banks. Thus the retail investor may not be able to construct the synthetic SSB. However, the banks can attract deposit by retail investors who redeem their SSB with a residual term deposit rate that is between the higher SGS market rate and the lower step-up rates of the new SSB. In this case, the difference between the limited embedded option value to retail investor and the cost of the option to the government can be earned by the commercial banks. This could lead to a cash leakage from SSB to bank deposits. Even without residual term deposits, the government in managing the gap, if by actually selling the excess term 10-t* bonds to the market, would in market equilibrium incur an additional cost relative to the case in which the retail investor does not exercise. The cost of the embedded option to the government when retail investors face market frictions is shown in Table 5. Note that this cost materializes only when the retail investor exercises early redemption. We include also the higher volatility case of $\sigma = 0.00020$ for robustness check on top of the standard case of $\sigma = 0.00005$.

Table 5 shows that there was only one case in December 2016 whereby the embedded call has non-zero value. This value, however, is very small and trivial as it is only 0.06% of the par of S$100. In all other issues, where the retail investor has no other market alternatives except to redeem and re-invest in a new SSB bond starting with initially low step-up interests, there is no incentive to redeem as the new low step-ups are mostly not as good as continuing with higher step-ups well into the horizon.

The cost of uncertain term structure to the government managing the SSB program is shown to be small, and is reduced to practically zero when the program has the step-up interest rate structure. The step-up feature in this regard is shown to be very effective in demotivating or de-incentivizing retail investors to redeem early when the investor has no free access to the SGS market and cannot replicate his or her own SGS bond. This of course would not be the case if retail investors are given opportunities by banks or hedge funds to re-invest redeemed SSB values with the same AAA-risk and with close to market rates of a horizon close to or exactly the same as the remaining 10-year horizon at the point of redemption.

5 Conclusions

The Singapore SSB investment program targets retail investors with limited or no access to borrowing and lending in the SGS market meant for institutional and larger investors. The same retail investors if requiring risk-free AAA-rated returns therefore have alternative access only to bank fixed deposit investing that provides low returns and that imposes a penalty of nullifying interest payments should there be an early redemption or withdrawal. The SSB program not only yields the equivalent of an institutional investor's return on a SGS 10-year bond provided the retail investor holds

Table 5 SSB Bermudan put option S$ cost to government (using 2400 daily intervals) per S$100 par value of the SSB bond

Short rate volatility σ	Oct 2015	Nov 2015	Dec 2015	Jan 2016	Feb 2016	Mar 2016	Apr 2016	May 2016	Jun 2016	Jul 2016
0.00005	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.00020	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Short rate volatility σ	Aug 2016	Sep 2016	Oct 2016	Nov 2016	Dec 2016	Jan 2017	Feb 2017	Mar 2017	Apr 2017	May 2017
0.00005	0.000	0.000	0.000	0.000	0.059	0.000	0.000	0.000	0.000	0.000
0.00020	0.000	0.000	0.000	0.000	0.060	0.000	0.000	0.000	0.000	0.000

Short rate volatility σ	Jun 2017	Jul 2017	Aug 2017	Sep 2017	Oct 2017	Nov 2017	Dec 2017	Jan 2018	Feb 2018	Mar 2018
0.00005	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.00020	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Short rate volatility σ	Apr 2018	May 2018	Jun 2018	Jul 2018	Aug 2018	Sep 2018	Oct 2018	Nov 2018	Dec 2018
0.00005	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.00020	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

over that horizon, but also affords a commission-free and tax-free return risk-free return. Moreover, should an early redemption or withdrawal occur, the investor does not suffer any loss of paid and accrued interests, and would obtain resulting interest nearly, if not, similar to holding a SGS bond with maturity up to the point of redemption. And even more so, the investor could redeem at par, and re-invest the par amount in any new market offerings which may include innovative hedge fund and bank products. The opportunity to redeem early, not due to behavioural or personal circumstantial reasons, leads to a redemption profit for the investor. This profit is basically the ability to redeem at par S$100 and then re-invest with a smaller amount to produce identical pay-outs for the remainder of the investor's original 10-year horizon. The present value of this possible profit is the value of an embedded put option in the SSB.

Our empirical results based on the market data between October 2015 and December 2018 show that only in months when forward rates significantly exceeds SSB coupon interests most of the future periods that the probability of exercise is higher, resulting in higher bond discounting, and hence cheaper bond with the remaining maturity. The values of the embedded put options would be correspondingly larger and significant. In most other months, however, the embedded put option has zero or nearly zero values. The highest embedded put values are about 1.0 to 1.5% of the par value of the SSB bonds. This percentage is even smaller if there is transaction cost of ½ or 1% should the investor re-invest. In the situation where there is market friction due to very high transaction costs or illiquid SGS market to prevent re-investing, where the investor can only roll over or re-invest in a newly issued SSB bond, then since the beginning months of the new SSB bond have lower interest rates due to the step-up feature, there is even less probability the investor will roll over. Thus the embedded option value, when only re-investing in SSB is possible, is negligible.

When SSB investors cannot access the SGS market to create synthetic SSB, they can only redeem the initial SSB and then invest in a new one if they wish. This case is interesting for two reasons. Firstly, the embedded option is now a call, and the strike price of the embedded call changes over each time period. This is because the strike value is no longer fixed but a function based on what next step-up interests would be given up. Only when the sum of future step-up interests based on remaining 10-year horizon and based on a new SSB exceeds the current bond's sum of remaining step-up interests would there then be a profit motive for the investor to redeem and re-invest in the new SSB. Secondly, the value of this embedded option to the investor is very small; but if exercised, this value to the investor need not equal to the cost of being exercised to the seller. In this context, the seller or government's loss may be larger as it has to manage change in different maturity pools of SGS bonds.

One very significant observation is that the step-up feature uniquely built into the SSB scheme proves very effective in almost nullifying the embedded option value, and hence also greatly reduce the probability of retail investors redeeming SSB bonds early. To diminish any trace of option values and de facto enhance investor adherence to the original investment, the current step-up formula could also consider additional constraints on slower ramp-up of the initial part of the step-up curve but add an incentive with a front-end higher coupon to attract investors from the draw of currently higher one-year bank time deposits, while keeping the 10-year yield commensurate with SGS yield of the same maturity. The added constraints could lower intermediate yields a little, increase cost of early redemption to an investor, and hence reduce the embedded option value. As a whole, the SSB is seen to offer more market opportunities to residents with restricted access to institutional market borrowing and lending rates.

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