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Chapter 1

Introduction

Integral equations and their solutions play a significant role in science and engineering. Many important physical problems can be modelled by using integral or differential equations. Only a few of them can be solved explicitly, so it is necessary to engage numerical methods to obtain approximate solutions. In general, those methods are sophisticated combinations of numerical integration, differentiation and approximations.

Integral equations arise in many scientific and engineering problems. The theory of integral equations is thoroughly considered in [11, 30, 48]. Typically integral equations can not be solved analytically. Hence there is a need for numerical solution of these equations. As a consequence, various methods for the numerical solution of integral equations have been developed by many researchers. In particular the collocation method is widely used for solving integral equations, for treatment of this method for integral equations see [12, 30, 47].

Modelling physical problems using integral equations with the exact parameters is often impossible in real problems. To handle this lack of information, one way is to use uncertainty measures such as fuzzy concept (Zadeh 1965 [49]). Instead of using deterministic models of integral equations, we can use fuzzy integral equations, where the values of functions may be fuzzy numbers. Hence there is a need to develop mathematical models and numerical procedures that would appropriately treat general fuzzy integral equations and solve them. The topics related to fuzzy integral equations have received particular attention from the research community during the last few decades [1, 9, 21, 22].

The main objects of study in the present thesis are the numerical solutions of fuzzy integral equations. Before discussing fuzzy integral equations and numerical algorithms for solving them, it is necessary to present a brief introduction to fuzzy numbers. A fuzzy number is a special case of the fuzzy set which is a function from the Euclidean space $\mathbb{R}$ to $[0,1]$ with a compact support, see details in [40]. Moreover, the set E of fuzzy numbers on $\mathbb{R}$ can be supplied with a metric D (see Definition 4) such that the space of fuzzy numbers is isometrically embedded as a convex cone in a real Banach space [27]. Fuzzy functions[1] were introduced by Zadeh [49]. Later, Dubois and Prade [19] presented an elementary framework for fuzzy calculus based on the extension principle. Alternative approaches were suggested by Goetschel and Voxman [23], Kaleva [26] and others. The concept of integration of fuzzy functions was introduced by Dubois and Prade [19], and investigated by Goetschel and Voxman [23]. It is common to use fuzzy functions in parametric form with upper and lower functions (see Theorem 1).

A fuzzy Volterra integral equation of the second kind (FVIE) is given by

$$g(t) = f(t) + \int_0^t K(t,s)g(s)ds, \quad t \in [0,T], \tag{1.1}$$

where $K\colon D_T \to \mathbb{R}$ is a function called the kernel of the integral equation with domain $D_T = \{(t,s)\colon 0 \leq s \leq t \leq T\}$ and $f(t)$ is a given fuzzy function of t. If $f(t)$ is a crisp[2] (non-fuzzy) function then equation (1.1) possesses a crisp solution and if $f(t)$ is a fuzzy function then the solution is fuzzy. Existence and uniqueness of solutions of fuzzy Volterra integral equations have been considered in [22, 35, 43]. Smoothness of solutions, to our knowledge, has not been considered before. We prove smoothness results for fuzzy Volterra interal equations in terms of the smoothness of upper and lower functions; this concept differs from being differentiable in the sense of fuzzy functions, but for obtaining convergence rates for numerical methods, smoothness of upper and lower functions is crucial. In some cases the smoothness results can be obtained from the corresponding results for crisp functions, but in the case when the kernel of the integral equation changes sign, it is more complicated. The smoothness results that we obtain are in some sense surprising, since when the fuzzy integral equation is converted to a system of ordinary integral equations, the kernels of the crisp equations are, in general (if the kernel of the original integral equation changes sign), not smooth.

Numerical solution of FVIEs is considered in [33, 41, 42, 43], but in many cases it is not proven that the approximate solution is a fuzzy function (in some cases it may be trivial, but it is not true in general). The convergence rates have not usually been considered. FVIEs with changing sign kernels were, to our knowledge, considered only in [41], but there only a trivial special case, when

[1] Throughout this thesis, a fuzzy function is a map from a set of real numbers to the set of fuzzy numbers on $\mathbb{R}$.

[2] Throughout this thesis, crisp means non-fuzzy.

the sign can only change on horizontal lines, is considered, the smoothness of the solution is not proven, and the convergence results are only valid under additional assumptions not mentioned in the paper. There are also a lot of papers which only describe some numerical method for solving fuzzy integral equations and give some numerical examples, but do not provide any analysis at all.

A fuzzy Volterra integral equation of the second kind with a weakly singular kernel (FVIEW) is given by the equation (1.1), where $K : D_T \to \mathbb{R}$ is a weakly singular kernel with domain $D_T = \{(t, s): 0 \leq s < t \leq T\}$, $T \in \mathbb{R}$, f is a given fuzzy function and g is an unknown fuzzy function. The kernel K may have some singularities at $t = s$. We will define weakly singular kernel in Chapter 4.

Integral equations with weakly singular kernels have received considerable interest in the mathematical literature, due to their applications in many fields of science such as the theory of elasticity, hydrodynamics, fractional differential equations and the physical problems with heredity and memory properties [11, 17, 25, 47].

Volterra integral equations with weakly singular kernels have been studied in wide variety of articles. We refer to [13, 29, 39, 28, 38, 47, 48, 50]. Especially [48] is devoted to the smoothness of the solutions of weakly singular integral equations of the second kind and the piecewise polynomial collocation method to solve such equations.

As far as we know, the fuzzy Volterra integral equation with weakly singular kernel has not yet been studied in the literature. The main achievement of this work is to study the fuzzy Volterra integral equation (1.1) with weakly singular kernel. First, we transform the fuzzy Volterra integral equation (1.1) with a weakly singular kernel to a system of Volterra integral equations with weakly singular kernels. We obtain the existence and uniqueness of solutions based on this transformation, and then we show that the corresponding solution is a fuzzy function which satisfies equation (1.1). When analysing the convergence of a numerical method for a given integral equation one needs information about the smoothness of the exact solution. We prove the smoothness of the solution, assuming that the sign of kernel can change only along the horizontal and vertical lines. Then we introduce collocation methods on piecewise polynomial spaces for solving the corresponding system of Volterra integral equations. We provide the conditions for fuzziness of the numerical solutions. Based on smoothness results we obtain the convergence analysis.

A fuzzy Fredholm integral equation of the second kind (FFIE) is given by

$$y(t) = f(t) + \int_0^T k(t,s)y(s)ds, t \in [0,T], \tag{1.2}$$

where k is a bivariate function with the domain $D = [0,T] \times [0,T]$, $T > 0$, and f is a given fuzzy valued (source) function. In the proposed contribution, we will be working with the fuzzy case. Existence and uniqueness of solutions of fuzzy Fredholm integral equations have been considered e.g. in [22].

Numerical methods for fuzzy Fredholm integral equations can be found in [1, 21, 36, 37]. These methods are focused on linear fuzzy Fredholm integral equations and use quadrature formulas. For example, in [9, 21], an iterative numerical method using the trapezoidal quadrature rule was proposed. In the subsequent papers, the convergence of this method was proved, but any error estimation was not given. In [9], the authors obtained a general quadrature rule for the Henstock integral of Lipschitz fuzzy functions and applied this rule for the construction of a numerical method for linear fuzzy Fredholm integral equations. Furthermore, they proposed a numerical algorithm and its error estimate.

We propose an approach based on the parametric form of the integral equation. We replace the original problem by a new one where all included functions are replaced by their approximations. The most tricky problem was to select a class of approximation functions that do not destroy the shape of fuzzy numbers. For this purpose, we used Chebyshev polynomials due to their good approximation properties and reasonable behavior near boundaries. Among various numerical methods that have been applied for solving fuzzy Fredholm integral equations, spectral methods using orthogonal polynomials have not been considered yet. We prove the convergence and fuzziness of the approximate solution.

In the following we briefly summarize the main results of the dissertation by chapters. This dissertation consists of six chapters.

Chapter 2 consists of some preliminary notions and presents some propositions and corollaries about fuzzy sets, fuzzy numbers, fuzzy functions and operation on fuzzy functions. At the end we have the definition of Chebyshev polynomials.

In Chapter 3 we consider fuzzy Volterra integral equations of the second kind whose kernel may change sign. We give conditions for smoothness of the upper and lower functions of the solution. For numerical solution we propose the collocation method with two different basis function sets: triangular and rectangular basis functions. The smoothness results allow us to obtain the convergence rates of the methods. The results about fuzzy Volterra integral equations in Chapter 3 are published in [2].

In Chapter 4 we present the existence and uniqueness theorem for fuzzy Volterra integral equations with a weakly singular kernel. A method of successive approximation and fuzziness of the approximate solution is the main tool in our analysis. For a numerical solution, we propose piecewise spline collocation methods with a graded mesh. By increasing the number of collocation points we show that the numerical solution exists and converges to the exact solution. We study the fuzziness of the approximate solution. The results of this chapter are intended to be published in [3].

In Chapter 5 we are focused on fuzzy Fredholm integral equations of the second kind. In the case of a smooth kernel, we approximate the kernel and the source function with Chebyshev polynomials and solve the integral equation with the degenerate kernel exactly. In case of smooth kernel the method will converge very quickly. We also prove fuzziness of the approximate solution. We discuss the existence and uniqueness of a solution. The results of Chapter 5 are published in [44].

The end of each chapter includes the numerical tests and figures which support our theoretical results. These results are in complete accordance with theory.

We will end with Chapter 6 as a conclusion and future work.

Chapter 2

Preliminary results

In this section, we review the fundamental notions of fuzzy numbers and fuzzy functions to be used throughout the thesis.

2.1 Fuzzy sets and numbers

In 1965 Zadeh [49] introduced the concept of fuzzy sets by defining them in terms of mappings from a set into the unit interval on the real line. Fuzzy sets were introduced to provide means to describe situations mathematically which give rise to ill-defined classes, i.e. collections of objects for which there is no precise criteria for membership. The fuzzy set theory presents the notion that membership in a given subset is a matter of degree rather than that of totally in or totally out.

Definition 1. [49] Let $\mathbb{X}$ be a set. A fuzzy set is characterized by a function called membership function and defined as

$$A(x) : \mathbb{X} \to [0,1], \quad \forall x \in \mathbb{X},$$

associating each element of $\mathbb{X}$ to a real number on $[0,1]$. The set of all fuzzy sets is denoted by $\mathbb{F}(\mathbb{X})$.

Fuzzy numbers are particular fuzzy sets on $\mathbb{R}$ (generally on $\mathbb{R}^n$, $n \geq 1$) that are identified with some additional properties.

Definition 2. [18] A fuzzy number is a mapping $u : \mathbb{R} \to [0,1]$ such that

1. u is normal, i.e. $\exists x_0 \in \mathbb{R}$ with $u(x_0) = 1$,
2. u is fuzzy convex, i.e.

$$u(tx + (1-t)y) \geq \min\{u(x), u(y)\},\ \forall t \in [0,1],\ x, y \in \mathbb{R},$$

3. u is upper semi-continuous,
4. u is compactly supported, i.e. $cl\{x \in \mathbb{R}: u(x) > 0\}$ is compact, where $cl(A)$ denotes the closure of the set A.

The set of all fuzzy numbers is denoted by E. Fuzzy numbers can also be represented in parametric form as follows.

Definition 3. For $0 \leq r \leq 1$, we denote $[u]_r = \{x \in \mathbb{R}: u(x) \geq r\}$, then $[u]_r$ will be called the r-cut of the fuzzy number u. We denote $[u]_0 = \overline{\{x \in \mathbb{R}: u(x) > 0\}}$. We call $[u]_0$ the support of fuzzy number u and denote it by supp(u). Fuzzy number u is called positive if supp $(u) \subset (0, \infty)$. We denote by E^+, the space of all positive fuzzy numbers.

The following couple of theorems [23] give another representation of a fuzzy number as a pair of functions that satisfy some properties. The representation of first theorem is called the LU (lower-upper) representation of a fuzzy number.

Theorem 1. *[23] Let u be a fuzzy number and let $[u]_r = [\underline{u}(r), \overline{u}(r)] = \{x \in \mathbb{R}: u(x) \geq r\}, 0 \leq r \leq 1$. The functions $\underline{u}(r), \overline{u}(r) : [0,1] \to \mathbb{R}$, defining the endpoints of the r-cuts, satisfy the following conditions:*

1. *$\underline{u}(r)$ is a bounded monotonically increasing, left-continuous function on $(0,1]$ and right continuous at 0;*
2. *$\overline{u}(r)$ is a bounded monotonically decreasing, left-continuous function on $(0,1]$ and right continuous at 0;*
3. *$\underline{u}(1) \leq \overline{u}(1)$.*

The reciprocal of the LU-representation is the Goetschel-Voxman characterization theorem.

Theorem 2. *(Goetschel-Voxman [23]) Let us consider the functions $\underline{u}(r), \overline{u}(r) : [0,1] \to \mathbb{R}$, that satisfy the following conditions:*

1. *$\underline{u}(r)$ is a bounded, non-decreasing, left continuous function in $(0,1]$ and it is right continuous at 0;*
2. *$\overline{u}(r)$ is a bounded, non-increasing, left continuous function in $(0,1]$ and it is right continuous at 0;*
3. *$\underline{u}(1) \leq \overline{u}(1)$.*

Then there is a fuzzy number $u \in E$ that has $\underline{u}(r), \overline{u}(r)$ as endpoints of it's r-cuts, $u(r)$.

For arbitrary $[u]_r = [\underline{u}(r), \overline{u}(r)], [v]_r = [\underline{v}(r), \overline{v}(r)]$ and $k \in \mathbb{R}$ we define addition and multiplication by k as

$[\underline{u+v}]_r = [\underline{u}]_r + [\underline{v}]_r$, $[\overline{u+v}]_r = [\overline{u}]_r + [\overline{v}]_r$,
$[\underline{ku}]_r = k[\underline{u}]_r$, $[\overline{ku}]_r = k[\overline{u}]_r$, if $k \geq 0$,
$[\overline{ku}]_r = k[\underline{u}]_r$, $[\underline{ku}]_r = k[\overline{u}]_r$, if $k < 0$.

Note that E is not a vector space, because $u + (-u) \neq 0$ in general. A crisp number is simply represented by $\underline{u}(r) = \overline{u}(r) = r$, $0 \leq r \leq 1$. Some special cases of fuzzy numbers are:

1. trapezoidal fuzzy numbers, where $\underline{u}(r), \overline{u}(r)$ are linear functions;
2. triangular fuzzy numbers, which are trapezoidal numbers with $\underline{u}(1) = \overline{u}(1)$;
3. interval numbers, where $\underline{u}(r), \overline{u}(r)$ are constants.

Example 1. Consider the fuzzy number with membership function as

$$u(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \leq x < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq x \leq 1, \\ -x+2, & 1 < x < 2, \\ 0, & x \geq 2. \end{cases}$$

The r-cuts are as follows:
$[u]_r = [r, 2-r], 0 < r < \frac{1}{2}$ and $[u]_r = [\frac{1}{2}, 2-r], \frac{1}{2} \leq r < 1$.

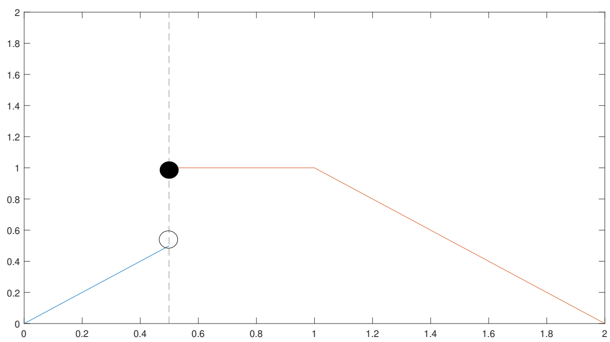


Figure 1: Membership function of Example 1

Next we will define the metric D in E.

Definition 4. For arbitrary fuzzy numbers u, v, we use the distance

$$D(u,v) = \sup_{0 \leq r \leq 1} \max\{|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|\}.$$

It is shown that (E, D) is a complete metric space [8]. Following Goetschel and Voxman [23] we define the integral of a fuzzy function using the Riemann integral concept.

Definition 5. Let $f : [a, b] \to E$. For each partition $P = \{t_0, ..., t_n\}$ of $[a, b]$ and for arbitrary $\xi_i \in [t_{i-1}, t_i], 1 \leq i \leq n$ suppose

$$R_P = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}), \quad \Delta := \max\{t_i - t_{i-1}, i = 1, ..., n\}.$$

The definite integral of $f(t)$ over $[a, b]$ is $\int_a^b f(t)dt = \lim_{\Delta \to 0} R_P$ provided this limit exists in metric D.

If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists and

$$\int_a^b f(t)dt = \left(\int_a^b \underline{f}(t, r)dt, \int_a^b \overline{f}(t, r)dt \right), \tag{2.1}$$

where $(\underline{f}(t, r), \overline{f}(t, r))$ is the parametric form of $f(t)$.

It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach [26]. Definition of the fuzzy integral using formula (2.1) is more convenient for numerical calculations.

The following theorem is known as the characterization theorem [23] which will be used in next sections.

Theorem 3. *If $u \in E$ is a fuzzy number and $[u]_r$, $r \in [0, 1]$ are its r-cuts, then:*

(i) $[u]_r$ *is a non-empty closed interval for any* $r \in [0, 1]$;

(ii) if $0 \leq r_1 \leq r_2 \leq 1$, *then* $[u]_{r_2} \subseteq [u]_{r_1}$;

(iii) for any sequence r_n *which converges from below to* $r \in [0, 1]$, *we have* $\bigcap_{n=1}^{\infty} [u]_{r_n} = [u]_r$;

(iv) for any sequence r_n *which converges from above to* 0, *we have* $\overline{\bigcup_{n=1}^{\infty} [u]_{r_n}} = [u]_0$.

Lemma 1. *[8] Let f be a continuous function from $R^+ \times R^+ \times R^+$ into R^+ and $u, v, w \in E$, then*

$$[f(u, v, w)]_r = f([u]_r, [v]_r, [w]_r), \; r \in [0, 1].$$

2.2 Chebyshev polynomials

Definition 6. [31] Let $x = \cos(\theta), \theta \in [0, \pi]$. Then the n-th degree Chebyshev polynomial $T_n(x)$, $n \in \mathbb{N} \cup \{0\}$, on $[-1, 1]$ is defined by the relation

$$T_n(x) = \cos(n\theta), \tag{2.2}$$

or explicitly,

$$T_n(x) = \cos(n \arccos(x)).$$

The Chebyshev polynomials are orthogonal with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ and the corresponding inner product

$$< f, g >= \int_{-1}^{1} w(x)g(x)f(x)dx, \quad \text{where} \quad f, g \in \mathcal{L}^2(-1, 1). \tag{2.3}$$

The well-known recursive formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \; n \in \mathbb{N}, \tag{2.4}$$

with $T_0(x) = 1$, $T_1(x) = x$ is important for numerical computation of these polynomials. Since it is more convenient to use range $[0, T]$ than $[-1, 1]$, we transform $[0, T]$ into $[-1, 1]$, using linear transformation $s = \dfrac{2}{T}x - 1$, where $x \in [0, T]$, $s \in [-1, 1]$. This leads to a shifted Chebyshev polynomial (of the first kind) $T_n^*(x)$ of degree n in x on $[0, T]$ given by

$$T_n^*(x) = T_n(\frac{2}{T}x - 1), \tag{2.5}$$

with the corresponding weight function $w^*(x) = w(\frac{2}{T}x - 1)$.

The discrete orthogonality of Chebyshev polynomials leads to the Clenshaw-Curtis formula [31]:

$$\int_{-1}^{1} w(x)f(x)dx \simeq \frac{\pi}{N+1} \sum_{k=1}^{N+1} f(x_k), \tag{2.6}$$

where f is defined on $[-1, 1]$, and x_k, $k = 1, \ldots, N+1$, are zeros of $T_{N+1}(x)$. Therefore, on $[0, T]$ we have

$$\int_{0}^{T} w^*(x)f(x)dx \simeq \frac{T\pi}{2(N+1)} \sum_{k=1}^{N+1} f(\frac{T}{2}(x_k + 1)). \tag{2.7}$$

Also, the induced norm of $T_n^*(x)$,

$$\gamma_n := \|T_n^*(x)\|^2 = \frac{T}{2}\begin{cases} \frac{\pi}{2}, & n > 0, \\ \pi, & n = 0. \end{cases}$$

will be used later.

Chapter 3

Collocation method for fuzzy Volterra integral equations of the second kind

3.1 Fuzzy Volterra integral equation

A fuzzy Volterra integral equation of the second kind (SFVIE) is given by

$$g(t) = f(t) + \int_0^t K(t,s)g(s)ds, \quad t \in [0,T], \tag{3.1}$$

where $K(t,s) : D_T \to \mathbb{R}$ is a function called the kernel of the integral equation with domain $D_T = \{(t,s); 0 \leq s \leq t \leq T\}$ and $f(t)$ is a given fuzzy function of t. If $f(t)$ is a crisp function then equation (3.1) possesses crisp solution and if $f(t)$ is a fuzzy function then the solution is fuzzy.

Existence and uniqueness of solution for fuzzy Volterra integral equation is proved in [35], where the result is given for a non-linear Volterra integral equation, whose kernel is Lipschitz with respect to the unknown function. Since our equation is linear, this condition is trivially satisfied. In addition, in [35] the existence of the solution is only obtained locally, but in the linear case the existence is global, i.e. in $[0,T]$. We get the following result from [35].

Theorem 4. *Let the kernel $K : D_T \to R$ and the fuzzy function $f : [0,T] \to E$ be continuous. Then equation (3.1) has a unique continuous fuzzy solution on $[0,T]$.*

3.2 Numerical methods

Several numerical techniques have been used successfully for fuzzy integral equations [1, 10, 22, 33, 41, 42, 43]. In many cases it is not proved that the approximate solution is a fuzzy function. Sometimes it follows from the construction, but whenever we have to solve a system of equations to find some unknown coefficients, it is not obvious at all. In this section we discuss in details the collocation method. The idea of collocation methods is the following: we look for solutions in a finite-dimensional approximation space X_N, where N is approximation parameter, usually connected with the dimension of the approximation space, and require that the equation is exactly satisfied at some collocation points. Different approximation spaces can be used, usually splines, polynomials or trigonometric polynomials are used. Here we use piecewise linear and piecewise constant splines with triangular and rectangular basis functions correspondingly. In these cases we prove that the approximate solution is always a fuzzy function.

3.2.1 Collocation method with triangular basis

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0, T]$ and let $h_k = t_k - t_{k-1}$, $k = 1, \ldots, N$.

Definition 7. The functions $\phi_0, \ldots, \phi_N$ defined by

$$\phi_0(t) = \begin{cases} 1 - \dfrac{t - t_0}{h_1}, & t_0 \le t \le t_1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_k(t) = \begin{cases} \dfrac{t - t_{k-1}}{h_k}, & t_{k-1} \le t \le t_k, \\ 1 - \dfrac{t - t_k}{h_{k+1}}, & t_k \le t \le t_{k+1}, \quad k = 1, ..., N-1 \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_N(t) = \begin{cases} \dfrac{t - t_{N-1}}{h_N}, & t_{N-1} \le t \le t_N, \\ 0, & \text{otherwise}, \end{cases}$$

are called triangular basis functions.

For the collocation points we use the partition points t_k, $k = 0, 1, \ldots, N$. Often the uniform mesh $t_k = kh$, $h = \dfrac{T}{N}$, $k = 0, \ldots, N$ is used, but sometimes non-uniform grids are useful, especially if the solution is not very smooth near some point. The theory works also for the general case. Let in the following $h = \max_{k=1,\dots,N} h_k$.

We look for solution of equation (3.1) in the form

$$g_N(t) = \sum_{k=0}^{N} c_k \phi_k(t), \tag{3.2}$$

where $\phi_k(t)$ are triangular basis functions and c_k, $k = 0, 1, \dots, N$ are fuzzy numbers. The collocation equations are

$$g_N(t_n) = f(t_n) + \int_0^{t_n} K(t_n, s) g_N(s) ds, \; n = 0, \dots, N. \tag{3.3}$$

Substituting (3.2) into these equations we get

$$c_n = \int_{t_{n-1}}^{t_n} c_n K(t_n, s)\phi_n(s)ds + f(t_n) + \sum_{k=0}^{n-1} \int_{t_{k-1}}^{t_{k+1}} c_k K(t_n, s)\phi_k(s)ds, \; n = 0, \dots, N, \tag{3.4}$$

where for simplicity we have denoted $t_{-1} = 0$.

Note that in general, if the kernel changes sign, one cannot take the fuzzy coefficients c_k in front of the integral sign.

We need to solve these linear equations to get the approximate solution. Note that if the coefficients c_n are fuzzy numbers then the approximate solution given by (3.2) is a fuzzy function.

3.2.2 Collocation method with rectangular basis

Let t_k, $k = 0, \dots, N$ and h_k, $k = 1, \dots, N$ be as defined above.

Definition 8. The functions ψ_k $k = 1, \dots, N$ defined by

$$\psi_k(t) = \begin{cases} 1, & t_{k-1} \le t \le t_k, \\ 0, & \text{otherwise}, \end{cases}$$

are called rectangular basis functions.

In the case of rectangular basis, the best collocation points are the midpoints of the subintervals $[t_{k-1}, t_k]$:

$$\tau_k = \frac{t_{k-1} + t_k}{2}, \quad k = 1, \dots, N.$$

We look for solution of equation (3.1) in the form

$$g_N(t) = \sum_{n=0}^{N} d_n \psi_n(t), \tag{3.5}$$

where d_n, $n = 1, \dots, N$ are fuzzy numbers. The collocation equations are

$$g_N(\tau_n) = f(\tau_n) + \int_0^{\tau_n} K(\tau_n, s) g_N(s) ds, \ n = 0, \dots, N. \tag{3.6}$$

Substituting (3.5) into these equations we get

$$d_n = \int_{t_{n-1}}^{\tau_n} d_n K(\tau_n, s) ds + f(\tau_n) + \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} d_k K(\tau_n, s) ds, \ n = 1, \dots, N. \tag{3.7}$$

Again, if d_n are fuzzy numbers then the approximate solution given by (3.5) is a fuzzy function.

3.2.3 Existence and uniqueness of the approximate solution

To show that equations (3.3) and (3.6) have a unique fuzzy solution we use the following lemma.

Lemma 2. *Consider equation*

$$ax = bx - dx + y, \tag{3.8}$$

where a, b are crisp coefficients, y is a given fuzzy number, $a > b + d$, $b, d \geq 0$. Then equation (3.8) has a unique fuzzy solution x.

Proof. By converting equation (3.8) to two crisp equations and solving these, we have

$$\underline{x} = \frac{\underline{y}(a-b) - d\overline{y}}{(a-b)^2 - d^2}, \quad \overline{x} = \frac{\overline{y}(a-b) - d\underline{y}}{(a-b)^2 - d^2}.$$

Since $a > b + d$, $\ b, d \geq 0$, then $a - b$ and $(a-b)^2 - d^2$ are positive. Also since $\underline{y}$ is non-decreasing (as a function of r), $-\overline{y}$ is non-decreasing, we conclude that $\underline{x}$ is non-decreasing. Similarly, since $\overline{y}$ is non-increasing, $-\underline{y}$ is non-increasing and by same reasoning as before we conclude that $\overline{x}$ is non-increasing. Since $\underline{y}$ and $\overline{y}$ are left continuous, $\underline{x}$ and $\overline{x}$ are left continuous as well. Finally $\underline{x} \leq \overline{x}$, since $\underline{y} \leq \overline{y}$, $-d\overline{y} \leq -d\underline{y}$ and denominators are positive. □

Remark 1. In Lemma 2 condition $b, d \geq 0$ is just a matter of notation. But the assumption $a > b + d$ is necessary: if this is not satisfied, then equation (3.8) does not have a fuzzy solution.

Now by using Lemma 2 we show that the collocation equation (3.3) has a unique approximate fuzzy solution g_N.

Theorem 5. *Let the kernel $K : D_T \to R$ and the fuzzy function $f : [0, T] \to E$ be continuous functions. If $h\|K\|_\infty < 1$ then the equation (3.3) has a unique approximate fuzzy solution g_N of the form* (3.2).

Proof. We use induction to show that the coefficients c_n determined by (3.4) are fuzzy numbers. For $n = 0$ equation (3.4) is $c_0 = f(0)$. Since $f(0)$ is a fuzzy number, c_0 is also a fuzzy number.

Assume that equation (3.4) has fuzzy solution for $n = 0, \ldots, m-1$ and $1 \leq m \leq N$. Let $n = m$, then (3.4) can be written as

$$\begin{aligned} c_m = c_m \int_{t_{m-1}}^{t_m} K_+(t_m, s)\phi_m(s)ds - c_m \int_{t_{m-1}}^{t_m} K_-(t_m, s)\phi_m(s)ds \\ + f(t_m) + \sum_{k=0}^{m-1} \int_{t_{k-1}}^{t_{k+1}} c_k K(t_m, s)\phi_k(s)ds, \end{aligned} \tag{3.9}$$

where $K_+(t, s) = \max\{K(t, s), 0\}$ and $K_-(t, s) = \max\{-K(t, s), 0\}$ are the positive and the negative parts of the kernel $K(t, s)$.

By induction assumption we know that $f(t_m) + \sum_{k=0}^{m-1} \int_{t_{k-1}}^{t_{k+1}} c_k K(t_m, s)\phi_k(s)ds$ is a fuzzy number. Hence (3.9) is an equation of form (3.8), where $a = 1$, $b = \int_{t_{m-1}}^{t_m} K_+(t_m, s)\phi_m(s)ds$ and $d = \int_{t_{m-1}}^{t_m} K_-(t_m, s)\phi_m(s)ds$. Since $b, d \geq 0$ and for h small enough, b, d are also small enough, we have $a > b + d$ for h small enough. So the assumptions of Lemma 2 are satisfied, therefore there exists a unique fuzzy solution.

Since c_n, $n = 0, \ldots, N$ are fuzzy numbers, the approximate solution (3.2) is a fuzzy function.

□

Similar result holds for the rectangular basis.

Theorem 6. *Let the kernel $K : D_T \to R$ and the fuzzy function $f : [0, T] \to E$ be continuous functions. If $h\|K\|_\infty < 1$ then the equation (3.6) has a unique approximate fuzzy solution g_N of the form (3.5).*

Proof. The proof is similar to the previous theorem. □

3.3 Smoothness of the solution

3.3.1 Parametric form of the equation

To prove regularity results and obtain the convergence rates of the collocation method we introduce parametric form of equation (1.1). Let $(\underline{f}(t,r), \overline{f}(t,r))$ and $(\underline{g}(t,r), \overline{g}(t,r))$ be parametric forms of $f(t)$ and $g(t)$. Then equation (3.1) is

$$\begin{aligned}
\underline{g}(t,r) &= \underline{f}(t,r) + \int_0^t \underline{K(t,s)g(t)}ds, \\
\overline{g}(t,r) &= \overline{f}(t,r) + \int_0^t \overline{K(t,s)g(t)}ds.
\end{aligned}$$

Denote $K_+(t,s) = \max\{K(t,s), 0\}$ and $K_-(t,s) = \max\{-K(t,s), 0\}$. Then equation (3.1) can be rewritten as system of two crisp integral equations

$$\begin{cases}
\underline{g}(t,r) = \underline{f}(t,r) + \int_0^t (K_+(t,s)\underline{g}(s,r) - K_-(t,s)\overline{g}(s,r))ds, \\
\overline{g}(t,r) = \overline{f}(t,r) + \int_0^t (K_+(t,s)\overline{g}(s,r) - K_-(t,s)\underline{g}(s,r))ds.
\end{cases} \tag{3.10}$$

We define the operators $\mathcal{K}_+, \mathcal{K}_- : \mathcal{C}[0,T] \to \mathcal{C}[0,T]$ by

$$(\mathcal{K}_+ y)(t) = \int_0^t K_+(t,s)y(s)ds, \ t \in [0,T],$$

$$(\mathcal{K}_- y)(t) = \int_0^t K_-(t,s)y(s)ds, \ t \in [0,T].$$

Then we can rewrite system (3.10) as

$$\begin{cases}
\underline{g} = \underline{f} + \mathcal{K}_+\underline{g} - \mathcal{K}_-\overline{g}, \\
\overline{g} = \overline{f} + \mathcal{K}_+\overline{g} - \mathcal{K}_-\underline{g}.
\end{cases} \tag{3.11}$$

3.3.2 Regularity properties

To derive the convergence rates of our numerical method, we need first to obtain some regularity results. We have to point out that we do not need fuzzy regularity here, we only need regularity of the crisp functions $\underline{g}(\cdot,r), \overline{g}(\cdot,r)$, where r can be considered as a parameter. So we consider the regularity of solutions of the system of integral equations (3.10). It is known that if the kernel and the right hand side of Volterra integral equation of the second kind are in $\mathcal{C}^m$, then the solution is also in $\mathcal{C}^m$ (see for example [11]), and this applies also for systems. However, if the kernel of the original integral equation (1.1) changes sign, then in our system (3.10) even for smooth K the kernels K_+ and K_- are only piecewise continuously differentiable. Still we can prove under quite general assumptions that the solution is at least piecewise twice continuously differentiable, and give some additional conditions under which it is twice continuously differentiable. So in this section we mainly deal with the non-trivial case when the kernel changes sign.

Since we consider r as a parameter and never differentiate with respect to r, we use in the following the notation $\underline{f}'$, $\overline{f}'$ for derivatives with respect to t. We also skip the parameter r inside the proof.

Theorem 7. *Let $K \in \mathcal{C}(D_T)$ and $f \in \mathcal{C}([0,T];E)$ be given. Let g be the solution of (1.1). Assume that K changes sign on continuous lines $s = s_i(t)$, $t \in [\alpha_i, \beta_i]$, $i = 1, \dots, n$ whose endpoints lie on the lines $s = t$, $s = 0$ or $t = T$. For simplicity assume also that at all intersection points of lines $s = s_i(t)$, $s = t$, $s = 0$ and $t = T$ only two of the lines are intersecting. Let $r \in [0,1]$ be fixed.*

1. *If $\underline{f}(\cdot,r), \overline{f}(\cdot,r) \in \mathcal{C}^1[0,T]$ and $\dfrac{\partial K}{\partial t} \in \mathcal{C}(D_T)$, then $\underline{g}(\cdot,r), \overline{g}(\cdot,r) \in \mathcal{C}^1[0,T]$.*
2. *If additionally $\underline{f}''(\cdot,r), \overline{f}''(\cdot,r), \dfrac{\partial^2 K}{\partial t^2}$ are piecewise continuous, and $t \mapsto K(t,t)$ and $s = s_i(t)$ are piecewise continuously differentiable, then $\underline{g}''(\cdot,r), \overline{g}''(\cdot,r)$ are piecewise continuous.*
3. *If additionally $\underline{f}(\cdot,r), \overline{f}(\cdot,r) \in \mathcal{C}^2[0,T]$, $\dfrac{\partial^2 K}{\partial t^2} \in \mathcal{C}(D_T)$, and $s = s_i(t)$ and $t \mapsto K(t,t)$ are continuously differentiable and*
   a) at points, where $s_i(t) = t \neq 0$, we have $\dfrac{dK(t,t)}{dt} = 0$ and either $s_i'(t) = 1$ or $\dfrac{\partial K}{\partial t}(t,t) = 0$;
   b) at points, where $s_i(t) = 0, t \neq 0$, we have either $s_i'(t) = 0$ or $\dfrac{\partial K}{\partial t}(t,0) = 0$,
   then $\underline{g}(\cdot,r), \overline{g}(\cdot,r) \in \mathcal{C}^2[0,T]$.

Proof. To establish the regularity of $\underline{g}, \overline{g}$, we differentiate equations (3.10). We have to examine the regularity of integrals of type

$$W(t) = \int_0^t U(t,s)ds, \tag{3.12}$$

where $U(t,s)$ is one of $K_+(t,s)\underline{g}(s)$, $K_+(t,s)\overline{g}(s)$, $K_-(t,s)\underline{g}(s)$ or $K_-(t,s)\overline{g}(s)$. Note that $U \in \mathcal{C}(D_T)$, since on lines of sign change of K we have $K(t,s) = 0$, but derivatives of K_+ and K_- have jumps on these lines.

Let $t \in (0,T)$ be fixed. If t does not correspond to any endpoints or intersection points of the lines of sign change, then we can renumber the lines in neighborhood of t in the order of increasing s and denote $s_0(t) = 0$ and $s_{n+1}(t) = t$. Assuming $\dfrac{\partial K}{\partial t} \in \mathcal{C}(D_T)$ we can differentiate (3.12):

$$W'(t) = U(t,t) + \int_0^t \frac{\partial U(t,s)}{\partial t} ds = U(t,t) + \sum_{i=0}^{n} \int_{s_i(t)}^{s_{i+1}(t)} \frac{\partial U(t,s)}{\partial t} ds. \qquad (3.13)$$

If $\dfrac{\partial K}{\partial t} \in \mathcal{C}(D_T)$, then $\dfrac{\partial U(t,s)}{\partial t}$ is continuous inside all integration regions and the limits of integration are also continuous. So all terms on right hand side are continuous at t.

Assuming K is (piecewise) twice differentiable with respect to t, we can differentiate (3.13) again:

$$\begin{aligned} W''(t) = \frac{dU(t,t)}{dt} &+ \sum_{i=0}^{n} \left(\left. \frac{\partial U(t,s)}{\partial t} \right|_{s_{i+1}(t)-} s'_{i+1}(t) - \left. \frac{\partial U(t,s)}{\partial t} \right|_{s_i(t)+} s'_i(t) \right) \\ &+ \sum_{i=0}^{n} \int_{s_i(t)}^{s_{i+1}(t)} \frac{\partial^2 U(t,s)}{\partial t^2} ds \end{aligned}$$

If $\dfrac{\partial^2 K}{\partial t^2}$ is (piecewise) continuous, then all terms here are (at least piecewise) continuous at t.

If $t = t_*$ corresponds to an endpoint or intersection point of the lines of sign change, then we have to consider one-sided limits of $W'(t)$ and $W''(t)$ as $t \to t_*$. We have three cases (they are not exclusive, so we may have several of them at the same time) as is shown in Figure 2.

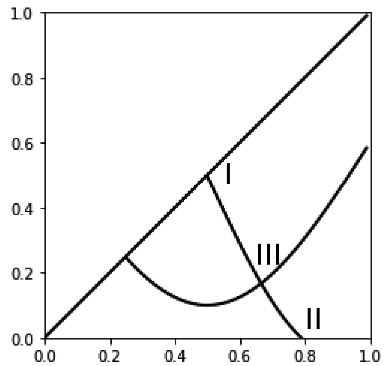


Figure 2: An example of three different intersections of lines of sign change of K.

Case I. Lines $s = s_i(t)$ and $s = t$ intersect at $t = t_*$. We can consider only a small neighborhood of point (t_*, t_*), where there are no other lines of sign change.

Assume the line $s = s_i(t)$ starts at $t = t_*$ (if it ends there, the argument is similar). Denote

$$W_\varepsilon(t) = \int_{t_*-\varepsilon}^{t} U(s,t)ds.$$

Then we have

$$\begin{aligned}
W_\varepsilon'(t) &= U(t,t) + \int_{t_*-\varepsilon}^{t} \frac{\partial U(t,s)}{\partial t} ds && \text{for } t < t_*,\\
W_\varepsilon'(t) &= U(t,t) + \int_{t_*-\varepsilon}^{s_i(t)} \frac{\partial U(t,s)}{\partial t} ds + \int_{s_i(t)}^{t} \frac{\partial U(t,s)}{\partial t} ds && \text{for } t > t_*.
\end{aligned}$$

Since $s_i(t) \to t_*$ as $t \to t_*+$, the one-sided limits of $W_\varepsilon'(t)$ at $t = t_*+$ are equal, if $\dfrac{\partial K}{\partial t} \in \mathcal{C}(D_T)$.

Assuming K is (piecewise) twice differentiable with respect to t, we have

$$\begin{aligned}
W_\varepsilon''(t) &= \frac{dU(t,t)}{dt} + \left.\frac{\partial U(t,s)}{\partial t}\right|_{s=t} + \int_{t_*-\varepsilon}^{t} \frac{\partial^2 U(t,s)}{\partial t^2} ds && \text{for } t < t_*,\\
W_\varepsilon''(t) &= \frac{dU(t,t)}{dt} + \left.\frac{\partial U(t,s)}{\partial t}\right|_{s_i(t)-} s_i'(t) - \left.\frac{\partial U(t,s)}{\partial t}\right|_{s_i(t)+} s_i'(t)\\
&\quad + \left.\frac{\partial U(t,s)}{\partial t}\right|_{s=t} \int_{t_*-\varepsilon}^{s_i(t)} \frac{\partial^2 U(t,s)}{\partial t^2} ds + \int_{s_i(t)}^{t} \frac{\partial^2 U(t,s)}{\partial t^2} ds && \text{for } t > t_*.
\end{aligned}$$

Now $\dfrac{dU(t,t)}{dt}$ is discontinuous at $t = t_*$ in general, unless $\dfrac{dK(t,t)}{dt} = 0$ at $t = t_*$. The one-sided limits of the integral terms are equal as $t \to t_*$. The remaining terms give the same limits if $s_i'(t) = 1$ or $\dfrac{\partial K(t,s)}{\partial t} = 0$ at $s = t = t_*$.

Case II. Lines $s = s_i(t)$ and $s = 0$ intersect at $t = t_*$. We can consider only a small neighborhood of point $(t_*, 0)$, where there are no other lines of sign change. Denote

$$W_\varepsilon(t) = \int_0^{\varepsilon} U(t,s)ds.$$

Assuming the line $s = s_i(t)$ ends at $t = t_*$,

$$\begin{aligned}
W_\varepsilon'(t) &= \int_0^{s_i(t)} \frac{\partial U(t,s)}{\partial t} ds + \int_{s_i(t)}^{\varepsilon} \frac{\partial U(t,s)}{\partial t} ds && \text{for } t < t_*,\\
W_\varepsilon'(t) &= \int_0^{\varepsilon} \frac{\partial U(t,s)}{\partial t} ds && \text{for } t > t_*.
\end{aligned}$$

Since $s_i(t) \to 0$ as $t \to t_*-$ the one-sided limits of $W_\varepsilon'(t)$ at $t = t_*-$ are equal.

For the second derivative we have

$$
\begin{aligned}
W_\varepsilon''(t) &= \left.\frac{\partial U(t,s)}{\partial t}\right|_{s\to s_i(t)^-} s_i'(t) + \int_0^{s_i(t)} \frac{\partial^2 U(t,s)}{\partial t^2} ds \\
&\quad - \left.\frac{\partial U(t,s)}{\partial t}\right|_{s\to s_i(t)^+} s_i'(t) + \int_{s_i(t)}^{\varepsilon} \frac{\partial^2 U(t,s)}{\partial t^2} ds \qquad \text{for } t < t_*, \\
W_\varepsilon''(t) &= \int_0^{\varepsilon} \frac{\partial^2 U(t,s)}{\partial t^2} ds \qquad \text{for } t > t_*.
\end{aligned}
$$

If K is piecewise twice differentiable then the one-sided limits of integrals are equal, since $s_i(t) \to 0$ as $t \to t_*^+$. The remaining terms give the same limits if $s_i'(t) = 0$ or $\dfrac{\partial K(t,s)}{\partial t} = 0$.

Case III. Lines of sign change intersect at $t = t_*$. Denote these lines by $s = s_i(t)$ and $s = s_j(t)$ so that for $t < t_*$ we have $s_j(t) < s_i(t)$ and for $t > t_*$ we have $s_i(t) < s_j(t)$. Consider only a small neighborhood of point (t_*, s_*), where $s_* = s_i(t_*) = s_j(t_*)$. Denote

$$
W_\varepsilon(t) = \int_{s_*-\varepsilon}^{s_*+\varepsilon} U(t,s) ds.
$$

Then

$$
\begin{aligned}
W_\varepsilon(t) &= \int_{s_*-\varepsilon}^{s_j(t)} U(t,s) ds + \int_{s_j(t)}^{s_i(t)} U(t,s) ds + \int_{s_i(t)}^{s_*+\varepsilon} U(t,s) ds \qquad \text{for } t < t_*, \\
W_\varepsilon(t) &= \int_{s_*-\varepsilon}^{s_i(t)} U(t,s) ds + \int_{s_i(t)}^{s_j(t)} U(t,s) ds + \int_{s_j(t)}^{s_*+\varepsilon} U(t,s) ds \qquad \text{for } t > t_*.
\end{aligned}
$$

If K is differentiable with respect to t we can take the derivative

$$
\begin{aligned}
W_\varepsilon'(t) &= \int_{s_*-\varepsilon}^{s_j(t)} \frac{\partial U(t,s)}{\partial t} ds + \int_{s_j(t)}^{s_i(t)} \frac{\partial U(t,s)}{\partial t} ds + \int_{s_i(t)}^{s_*+\varepsilon} \frac{\partial U(t,s)}{\partial t} ds \qquad \text{for } t < t_*, \\
W_\varepsilon'(t) &= \int_{s_*-\varepsilon}^{s_i(t)} \frac{\partial U(t,s)}{\partial t} ds + \int_{s_i(t)}^{s_j(t)} \frac{\partial U(t,s)}{\partial t} ds + \int_{s_j(t)}^{s_*+\varepsilon} \frac{\partial U(t,s)}{\partial t} ds \qquad \text{for } t > t_*.
\end{aligned}
$$

Since $\lim_{t\to t_*} s_i(t) = \lim_{t\to t_*} s_j(t) = s_*$, the one-sided limits are equal.

Assuming K is (piecewise) twice differentiable with respect to t we have

$$\begin{aligned}
W_\varepsilon''(t) &= \int_{s_*-\varepsilon}^{s_j(t)} \frac{\partial^2 U(t,s)}{\partial t^2} ds + \int_{s_j(t)}^{s_i(t)} \frac{\partial^2 U(t,s)}{\partial t^2} ds + \int_{s_i(t)}^{s_*+\varepsilon} \frac{\partial^2 U(t,s)}{\partial t^2} ds \\
&+ \left.\frac{\partial U(t,s)}{\partial t}\right|_{s_j(t)^-} s_j'(t) + \left.\frac{\partial U(t,s)}{\partial t}\right|_{s_i(t)^-} s_i'(t) \\
&- \left.\frac{\partial U(t,s)}{\partial t}\right|_{s_j(t)^+} s_j'(t) - \left.\frac{\partial U(t,s)}{\partial t}\right|_{s_i(t)^+} s_i'(t) \quad \text{for } t < t_*, \\
W_\varepsilon''(t) &= \int_{s_*-\varepsilon}^{s_i(t)} \frac{\partial^2 U(t,s)}{\partial t^2} ds + \int_{s_i(t)}^{s_j(t)} \frac{\partial^2 U(t,s)}{\partial t^2} ds + \int_{s_j(t)}^{s_*+\varepsilon} \frac{\partial^2 U(t,s)}{\partial t^2} ds \\
&+ \left.\frac{\partial U(t,s)}{\partial t}\right|_{s_i(t)^-} s_i'(t) + \left.\frac{\partial U(t,s)}{\partial t}\right|_{s_j(t)^-} s_j'(t) \\
&- \left.\frac{\partial U(t,s)}{\partial t}\right|_{s_i(t)^+} s_i'(t) - \left.\frac{\partial U(t,s)}{\partial t}\right|_{s_j(t)^+} s_j'(t) \quad \text{for } t > t_*.
\end{aligned}$$

Since two lines of sign change of K intersect at (t_*, s_*), it must be a saddle point of K, hence $\dfrac{\partial K(t,s)}{\partial t} = 0$ at (t_*, s_*), therefore all the terms outside the integral approach 0 as $t \to t_*$. The integral terms give the same limits as $t \to t_*$.

The smoothness of the solution depends on the solution of the integral terms, which we just investigated, and the smoothness of f. So assuming f is at least as smooth as the integral terms, the proof is completed. □

Remark 2. Theorem 7 does not cover all possible configurations of lines of sign changes of K, e.g. the case where three or more lines intersect at one point. Generally the smoothness of the solution can be investigated similarly in these cases. There are also cases when the first derivative of the solution may be discontinuous, if there is a vertical line of sign change or when the line of sign change is not a graph of a function (turns back).

For obtaining convergence rates for numerical methods we also need uniform boundedness of derivatives of $\underline{g}, \overline{g}$ with respect to r.

Lemma 3. *Let the assumptions of Theorem 7, except 2., 3. be satisfied. Assume additionally that there exists constant B such that*

$$|\underline{f}'(t,r)| \leq B, \quad |\overline{f}'(t,r)| \leq B \quad \forall t \in [0,T], r \in [0,1].$$

Then there exists constant C such that

$$|\underline{g}'(t,r)| \leq C, \quad |\overline{g}'(t,r)| \leq C \quad \forall t \in [0,T], r \in [0,1].$$

Proof. Continuity of g as a fuzzy function follows from Theorem 4; this implies uniform boundedness of $\underline{g}, \overline{g}$. Using expressions for derivatives of $\underline{g}, \overline{g}$ obtained in the proof of Theorem 7 we get an uniform bound for $\underline{g}', \overline{g}'$. $\square$

Lemma 4. *Let the assumptions of Theorem 7, except 3., be satisfied. Assume additionally that there exists constant B such that*

$$|\underline{f}''(t,r)| \leq B, \quad |\overline{f}''(t,r)| \leq B \quad \forall t \in [0,T], r \in [0,1].$$

Then there exists constant C such that

$$|\underline{g}''(t,r)| \leq C, \quad |\overline{g}''(t,r)| \leq C \quad \forall t \in [0,T], r \in [0,1].$$

Proof. Using Lemma 3 and expressions for second derivatives of $\underline{g}, \overline{g}$ in the proof of Theorem 7 we obtain an uniform bound for $\underline{g}'', \overline{g}''$. $\square$

3.4 Convergence of the collocation method

3.4.1 Parametric form of the approximate equation

To analyze the convergence we introduce the parametric form of the approximate equation. Consider the case of triangular basis. Let $c_n = (\underline{c_n}, \overline{c_n})$. Then equations (3.4) can be written as

$$\underline{c_n} = \underline{c_n}(\mathcal{K}_+^n \phi_n)(t_n) - \overline{c_n}(\mathcal{K}_-^n \phi_n)(t_n) + \underline{f(t_n)} + \sum_{k=0}^{n-1} (\underline{c_k}(\mathcal{K}_+ \phi_k)(t_n) - \overline{c_k}(\mathcal{K}_- \phi_k)(t_n)), \tag{3.14}$$

$$\overline{c_n} = \overline{c_n}(\mathcal{K}_+^n \phi_n)(t_n) - \underline{c_n}(\mathcal{K}_-^n \phi_n)(t_n) + \overline{f(t_n)} + \sum_{k=0}^{n-1} (\overline{c_k}(\mathcal{K}_+ \phi_k)(t_n) - \underline{c_k}(\mathcal{K}_- \phi_k)(t_n)). \tag{3.15}$$

Here

$$(\mathcal{K}_+^n \phi_n)(t_n) = \int_{t_{n-1}}^{t_n} K_+(t,s)\phi_n(s)ds, \ (\mathcal{K}_-^n \phi_n)(t_n) = \int_{t_{n-1}}^{t_n} K_-(t,s)\phi_n(s)ds$$

for $n = 1, \dots, N$. For $n = 0$ we can define $(\mathcal{K}_+^0 \phi_0)(t_0) = 0$, $(\mathcal{K}_-^0 \phi_0)(t_0) = 0$.

In the case of rectangular basis denote $d_n = (\underline{d_n}, \overline{d_n})$. Then the parametric form of equation (3.7) is

$$\underline{d_n} = \underline{d_n}(\mathcal{K}_+^n\psi_n)(\tau_n) - \overline{d_n}(\mathcal{K}_-^n\psi_n)(\tau_n) + \underline{f(\tau_n)} + \sum_{k=1}^{n-1}(\underline{d_k}(\mathcal{K}_+\psi_k)(\tau_n) - \overline{d_k}(\mathcal{K}_-\psi_k)(\tau_n)), \quad (3.16)$$

$$\overline{d_n} = \overline{d_n}(\mathcal{K}_+^n\psi_n)(\tau_n) - \underline{d_n}(\mathcal{K}_-^n\psi_n)(\tau_n) + \overline{f(\tau_n)} + \sum_{k=1}^{n-1}(\overline{d_k}(\mathcal{K}_+\psi_k)(\tau_n) - \underline{d_k}(\mathcal{K}_-\psi_k)(\tau_n)), \quad (3.17)$$

where

$$(\mathcal{K}_+^n\psi_n)(\tau_n) = \int_{t_{n-1}}^{\tau_n} K_+(t,s)\psi_n(s)ds,\ (\mathcal{K}_-^n\psi_n)(\tau_n) = \int_{t_{n-1}}^{\tau_n} K_-(t,s)\psi_n(s)ds.$$

3.4.2 Convergence

To prove the convergence of these methods with triangular and rectangular basis, we use Theorem 13.10 from [30].

Theorem 8. *Let X be Banach space and $X_N \subset X$ be a sequence of subspaces. Let $P_N : X \to X_N$ be projection operators. Assume that $A : X \to X$ is a compact linear operator and $I - A$ is injective. Assume that the projectors $P_N : X \to X_N$ satisfy*

$$\|P_N A - A\| \to 0, N \to \infty.$$

Then for sufficiently large N, the approximate equation

$$u_N - P_N A u_N = P_N f, \quad (3.18)$$

is uniquely solvable for all $f \in X$ and there holds an error estimate

$$\|u_N - u\| \leq M\|P_N u - u\|,$$

where u is the solution of $u - Au = f$ and the constant M depends only on A.

Let $X = \mathcal{C}[0,T] \times \mathcal{C}[0,T]$, $A = \begin{pmatrix} \mathcal{K}_+ & -\mathcal{K}_- \\ -\mathcal{K}_- & \mathcal{K}_+ \end{pmatrix}$. Let $u = \begin{pmatrix} \underline{g}(\cdot, r) \\ \overline{g}(\cdot, r) \end{pmatrix}$ for r fixed. It is known that A is compact and $I - A$ is injective (see Theorem 1.2.8 in [11]). For triangular basis we define $X_N = \text{span}\{\phi_n, n = 0, ..., N\}$ and P_N is then the interpolation projector onto X_N.

We use the standard estimate for $\|P_N u - u\|_\infty$ (e.g. Theorem 11.3 in [30]).

Lemma 5. *If $v \in W^{2,\infty}(0,T)$, then for the error in piecewise linear interpolation there holds*

$$\|P_N v - v\|_\infty \leq \frac{1}{8} h^2 \|v''\|_\infty.$$

Using Theorem 8 and Lemma 5 we get the error estimate for triangular basis as follows.

Theorem 9. *Let $K \in \mathcal{C}(D_T)$, $f \in \mathcal{C}([0,T]; E)$. Assume $h \to 0$ as $N \to \infty$. Then for sufficiently large N the approximate equation (3.3) has a unique solution g_N, which converges uniformly to the exact solution g of equation (3.1). If the assumptions of Lemma 4 are satisfied then the error estimate*

$$\sup_{t\in[0,T]} D(g_N(t), g(t)) \leq Mh^2$$

holds, where M is a constant not depending on N.

Proof. Let $r \in [0,1]$ be fixed. Since $Au \in X$, we have $\|P_N Au - Au\|_\infty \to 0$ as $N \to \infty$ for all $u \in X$. Since for compact operators, the pointwise convergence implies convergence in norm, we get

$$\|P_N A - A\|_\infty \to 0 \quad \text{as } N \to \infty.$$

By Theorem 8 we get the error estimate

$$\|\overline{g}_N(\cdot,r) - \overline{g}(\cdot,r)\|_\infty \leq M\|P_N\overline{g}(\cdot,r) - \overline{g}(\cdot,r)\|_\infty,$$
$$\|\underline{g}_N(\cdot,r) - \underline{g}(\cdot,r)\|_\infty \leq M\|P_N\underline{g}(\cdot,r) - \underline{g}(\cdot,r)\|_\infty,$$

where M does not depend on r. From $g \in \mathcal{C}([0,T]; E)$ it follows that $\underline{g}(\cdot,r), \overline{g}(\cdot,r)$ are equicontinuous with respect to r, hence the convergences

$$\|P_N\overline{g}(\cdot,r) - \overline{g}(\cdot,r)\| \to 0 \text{ and } \|P_N\underline{g}(\cdot,r) - \underline{g}(\cdot,r)\| \to 0 \text{ as } N \to \infty$$

are uniform in r. Consequently

$$\sup_{t\in[0,T]} D(g_N(t), g(t)) \leq \sup_{0\leq r\leq 1} \max\{\|\overline{g}_N - \overline{g}\|_\infty, \|\underline{g}_N - \underline{g}\|_\infty\} \to 0.$$

If the assumptions of Lemma 4 are satisfied, then by Lemma 5 and Lemma 4 we get the error estimate

$$\sup_{t\in[0,T]} D(g_N(t), g(t)) \leq \sup_{0\leq r\leq 1} \max\{\|\overline{g}_N - \overline{g}\|_\infty, \|\underline{g}_N - \underline{g}\|_\infty\} \leq Mh^2.$$

□

To get the convergence estimate for the collocation method with rectangular basis, we redefine $X_N = \operatorname{span}\{\psi_n, n = 0, ..., N\}$ and P_N is then the interpolation projector onto X_N with interpolation nodes τ_n. We use the following standard result for the error of piecewise constant interpolation.

Lemma 6. *Let $v \in W^{1,\infty}(0,T)$. Then for the error in piecewise constant interpolation there holds*

$$\|P_N v - v\|_\infty \leq \frac{1}{2} h \|v'\|_\infty.$$

In addition to the usual convergence result for rectangular basis, we also present a result about superconvergence at the collocation nodes.

Theorem 10. *Let $K \in \mathcal{C}(D_T)$, $f \in \mathcal{C}([0,T]; E)$. Assume $h \to 0$ as $N \to \infty$. Then for sufficiently large N the approximate equation (3.6) has a unique solution g_N which converges uniformly to the exact solution g of equation (3.1). If the assumptions of Lemma 3 are satisfied then the error estimate*

$$\sup_{t\in[0,T]} D(g_N(t), g(t)) \leq Ch$$

holds, where C is a constant not depending on N. Moreover if the assumptions of Lemma 4 are satisfied then error estimate at collocation nodes

$$\max_{k=1,...,N} D(g_N(\tau_k), g(\tau_k)) \leq Ch^2$$

holds, where C is a constant not depending on N.

Proof. The proof of the first part is similar to the proof of Theorem 9.

To prove the superconvergence, we subtract from equation (3.18) the projected equation $P_N u = P_N A u + P_N f$:

$$u_N - P_N u = P_N A(u_N - u) = P_N A((u_N - P_N u) + (P_N u - u)).$$

So

$$u_N - P_N u = (I - P_N A)^{-1} P_N A(P_N u - u), \tag{3.19}$$

where $(I - P_N A)^{-1}$ is a bounded operator in X_N. We have

$$A(P_N u - u) = \begin{pmatrix} \mathcal{K}_+(P_N \underline{g} - \underline{g}) - \mathcal{K}_-(P_N \overline{g} - \overline{g}) \\ -\mathcal{K}_-(P_N \underline{g} - \underline{g}) + \mathcal{K}_+(P_N \overline{g} - \overline{g}) \end{pmatrix}.$$

Since applying P_N to this result uses only the values at τ_k, we estimate one element of this vector at τ_k. The others are similar.

$$\begin{aligned} \mathcal{K}_+(P_N \underline{g} - \underline{g})(\tau_k) &= \int_0^{\tau_k} K_+(s, \tau_k)(P_N \underline{g}(s) - \underline{g}(s))ds \\ &= \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} K_+(s, \tau_k)(\underline{g}(\tau_i) - \underline{g}(s))ds + \int_{t_{k-1}}^{\tau_k} K_+(s, \tau_k)(\underline{g}(\tau_k) - \underline{g}(s))ds \end{aligned} \tag{3.20}$$

Using Taylor expansion at τ_i in each subinterval $[t_{i-1}, t_i]$ we have

$$\begin{aligned} \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} &(K_+(\tau_i, \tau_k) + O(h))((\tau_i - s)\underline{g}'(\tau_i) + O(h^2))ds + O(h^2) \\ &= \sum_{i=1}^{k-1} K_+(\tau_i, \tau_k)\underline{g}'(\tau_i) \int_{t_{i-1}}^{t_i} (\tau_i - s)ds + O(h^2). \end{aligned} \tag{3.21}$$

Since $\tau_i = \dfrac{t_{i-1} + t_i}{2}$, integrals are all zero, so we get the estimate $O(h^2)$. For other elements the calculation is the same. Since $(I - P_N A)^{-1}$ is bounded, then from equation (3.19) we get $\|u_N - P_N u\| = O(h^2)$. Now notice that all the constants in the estimates are either independent of r or contain first and second derivatives of $\underline{g}, \overline{g}$ which are uniformly bounded with respect to r by Lemmas 3 and 4. Hence

$$\max_{k=1,\ldots,N} D(g_N(\tau_k), g(\tau_k)) \leq Ch^2$$

holds, where C is a constant. □

Remark 3. In general one has to solve the equations for each $r \in [0, 1]$. In special cases, when $f(t)$ is a triangular, trapezoidal or interval fuzzy number for $t \in [0, T]$, then the solution is still of the same type, and it is enough to solve the equations only for $r = 0$ and $r = 1$.

3.5 Numerical examples

In this section we present some numerical results. We used the collocation method with triangular and rectangular bases to solve approximately four examples of fuzzy Volterra integral equations. In examples 2 and 4 the kernels are non-negative, in examples 3 and 5 they change sign. We use uniform mesh and take $N = 5, 10, 20, 40, 80, 160$. To estimate the error $\max_{t\in[0,T]} D(g_N(t), g(t))$ we calculate

$$\max_{k=0,\dots,3N} D(g_N(\tilde{t}_k), g(\tilde{t}_k)),$$

where $\tilde{t}_k = \frac{k}{3N}$, $k = 0, 1, \dots, 3N$. We also calculated the ratios of consecutive errors. If the convergence is of order $O(h^2)$ then the ratios should be approximately 4; if the convergence is $O(h)$ then the ratios should be 2.

Example 2. Consider the fuzzy Volterra integral equation (FVIE)

$$\underline{f}(t, r) = (t^3 - \frac{t^6}{5})(r^2 + r), \quad \overline{f}(t, r) = (t^3 - \frac{t^6}{5})(4 - r^3 - r)$$

and the kernel

$$K(t, s) = st, \quad 0 \le s \le t \le 1.$$

The exact solution is given by

$$\underline{g}(t, r) = t^3(r^2 + r), \quad \overline{g}(t, r) = t^3(4 - r^3 - r),$$

The results are given in Table 1. The errors given in the table are fuzzy distances between the approximate and the exact solutions.

N	error (triangular)	ratio	error (rectangular)	ratio	error at τ_k	ratio
5	$1.2491e-01$		1.0591		$1.9792e-02$	
10	$3.2793e-02$	3.8091	$5.6263e-01$	1.8825	$6.2641e-03$	3.1595
20	$8.4129e-03$	3.8979	$2.9036e-01$	1.9377	$1.7567e-03$	3.5658
40	$2.1316e-03$	3.9467	$1.4755e-01$	1.9679	$4.6492e-04$	3.7786
80	$5.3656e-04$	3.9727	$7.4382e-02$	1.9837	$1.1958e-04$	3.8881
160	$1.3460e-04$	3.9862	$3.7345e-02$	1.9918	$3.0320e-05$	3.9437

Table 1 Comparison of numerical results for Example 2

We see that for triangular basis the convergence is of order $O(h^2)$. For rectangular basis the convergence is $O(h)$ but at collocation points the convergence is $O(h^2)$. In fact, when we have better convergence at collocation points, then using these values we can construct a better approximate solution as well.

Example 3. Consider the FVIE with

$$\underline{f}(t, r) = tr - \begin{cases} \frac{t^4}{4}(1 - 2t)^3 r, & t \le \frac{1}{2}, \\ \frac{1}{64}(1 - 2t)^3 r + (\frac{t^4}{4} - \frac{1}{64})(1 - 2t)^3(2 - r), & t \ge \frac{1}{2}. \end{cases}$$

and

$$\overline{f}(t,r) = t(2-r) - \begin{cases} \frac{t^4}{4}(1-2t)^3(2-r), & t \le \frac{1}{2}, \\ \frac{1}{64}(1-2t)^3(2-r) + (\frac{t^4}{4} - \frac{1}{64})(1-2t)^3 r, & t \ge \frac{1}{2}. \end{cases}$$

The kernel is

$$K(t,s) = s^2(1-2t)^3, \quad 0 \le s \le t \le 1,$$

with

$$K_+(t,s) = \begin{cases} s^2(1-2t)^3, & t \le \frac{1}{2}, \\ 0, & t > \frac{1}{2}, \end{cases}$$

$$K_-(t,s) = \begin{cases} 0, & t < \frac{1}{2}, \\ -s^2(1-2t)^3, & t \ge \frac{1}{2}. \end{cases}$$

and the exact solution is given by

$$\underline{g}(t,r) = t^3 r, \quad \overline{g}(t,r) = t^3(2-r).$$

In this case the kernel changes sign on the line $t = \frac{1}{2}$, but two derivatives with respect to t are also zero on this line, so the kernels K_+ and K_- are smooth (they have discontinuous third derivatives). Theoretically the solution might also have discontinuous third derivatives, but instead in our case $\underline{f}$ and $\overline{f}$ have discontinuous third derivatives which compensate the singularities in the solution. The results are given in Table 2. Again we can see that the theoretical convergence rates coincide with the real convergence.

N	error (triangular)	ratio	error (rectangular)	ratio	error at τ_k	ratio
5	$5.0752e-02$		$5.4180e-01$		$3.4160e-03$	
10	$1.3088e-02$	3.8778	$2.8514e-01$	1.9001	$1.5224e-03$	2.2438
20	$3.3231e-03$	3.9384	$1.4624e-01$	1.9498	$4.9619e-04$	3.0682
40	$8.3757e-04$	3.9676	$7.4053e-02$	1.9748	$1.4086e-04$	3.5226
80	$2.1029e-04$	3.9830	$3.7262e-02$	1.9873	$3.7476e-05$	3.7587
160	$5.2687e-05$	3.9912	$1.8690e-02$	1.9937	$9.6618e-06$	3.8787

Table 2 Comparison of numerical results for Example 3

Example 4. [43] Consider the FVIE with

$$\underline{f}(t,r) = (1-t-\frac{t^2}{2})r, \quad \overline{f}(t,r) = (1-t-\frac{t^2}{2})(2-r).$$

The kernel is

$$K(t,s) = t - s, \quad 0 \le s \le t \le \frac{1}{2},$$

and the exact solution is given by

$$\underline{g}(t,r) = (1 - \mathrm{sinh\,t})\mathrm{r}, \quad \overline{\mathrm{g}}(\mathrm{t},\mathrm{r}) = (1 - \mathrm{sinh\,t})(2 - \mathrm{r}).$$

We used $T = \frac{1}{2}$ here, because in $[0,1]$ the function f is not a fuzzy function. The results are given in Table 3. In this example neither f nor g is Hukuhara differentiable, but as emphasized before, we only need differentiability of $\underline{f}$, $\overline{f}$ and $\underline{g}$, $\overline{g}$ to get the convergence results.

N	error (triangular)	ratio	error (rectangular)	ratio	error at τ_k	ratio
5	$1.1106e-03$		$1.1072e-01$		$7.8944e-04$	
10	$2.8822e-04$	3.8532	$5.5882e-02$	1.9823	$2.0931e-05$	3.7716
20	$7.3339e-05$	3.9300	$2.8056e-02$	1.9907	$5.3851e-05$	3.8869
40	$1.8493e-05$	3.9658	$1.4061e-02$	1.9953	$1.3655e-05$	3.9437
80	$4.6429e-06$	3.9831	$7.0391e-03$	1.9976	$3.4378e-06$	3.9719
160	$1.1632e-06$	3.9916	$3.5217e-03$	1.9988	$8.6249e-07$	3.9860

Table 3 Comparison of numerical results for Example 4

Example 5. Consider the FVIE with

$$\underline{f}(t,r) = (t^3 - \frac{t^5}{320})r - \frac{49t^5}{320}(r-2), \quad \overline{f}(t,r) = \frac{49t^5}{320}r - (t^3 - \frac{t^5}{320})(r-2)$$

and the kernel

$$K(t,s) = t - 2s, \quad 0 \le s \le t \le 1.$$

The exact solution is given by

$$\underline{g}(t,r) = t^3 r, \quad \overline{g}(t,r) = t^3(2-r).$$

In this case there is a sign change of the kernel along the line $s = \frac{t}{2}$. Since this line does not have any endpoints or intersection points with the line $s = t$ inside $[0,1]$, the solution is smooth. The results are given in Table 4.

N	error (triangular)	ratio	error (rectangular)	ratio	error at τ_k	ratio
5	$1.7384e-03$		$4.6071e-01$		$6.6514e-03$	
10	$4.5297e-04$	3.8377	$2.4240e-01$	1.9006	$2.0008e-03$	3.3244
20	$1.1391e-04$	3.9765	$1.2432e-01$	1.9498	$5.4034e-04$	3.7028
40	$2.8520e-05$	3.9941	$6.2951e-02$	1.9749	$1.4021e-04$	3.8538
80	$7.1327e-06$	3.9985	$3.1675e-02$	1.9874	$3.5699e-05$	3.9275
160	$1.7833e-06$	3.9996	$1.5887e-02$	1.9937	$9.0062e-06$	3.9639

Table 4 Comparison of numerical results for Example 5

Chapter 4

Fuzzy Volterra integral equation with weakly singular kernel

4.1 Fuzzy Volterra integral equation of the second kind with weakly singular kernels

A fuzzy Volterra integral equation of the second kind with weakly singular kernel (FVIEW) is given by

$$g(t) = f(t) + \int_0^t K(t,s)g(s)ds, \quad t \in [0,T], \tag{4.1}$$

where $K\colon D_T \to \mathbb{R}$ is a weakly singular kernel with domain $D_T = \{(t,s)\colon 0 \leq s < t \leq T\}$, $T \in \mathbb{R}$, f is a given fuzzy function and g is an unknown fuzzy function. The kernel K may have some singularities at $t = s$.

In the literature, weak singularity of the kernel K may have different definitions. We follow here the definition of [48], where it was introduced for Fredholm integral equations.

Definition 9. For given $m \in \mathbb{N}_0$, denote by $S^{m,\alpha} = S^{m,\alpha}(D_T)$ the set of m times continuously differentiable functions K on D_T that satisfy there for all $j,l \in \mathbb{N}_0$, $j + l \leq m$, the inequality

$$|(\frac{\partial}{\partial t})^j(\frac{\partial}{\partial t} + \frac{\partial}{\partial s})^l K(t,s)| \leq C_{K,m} \begin{cases} 1 & \text{if} \quad j+\alpha < 0, \\ 1 + |\log(t-s)| & \text{if} \quad j+\alpha = 0, \\ (t-s)^{-j-\alpha} & \text{if} \quad j+\alpha > 0. \end{cases} \tag{4.2}$$

A kernel $K \in S^{m,\alpha}$ is called weakly singular if $\alpha < 1$.

For example, kernels of the type

$$K(t,s) = a(t,s)(t-s)^{-\alpha},$$

where $a \in C^m\left(\overline{D_T}\right)$ and $\alpha < 1$, $\alpha \neq 0$ are weakly singular and belong to $S^{m,\alpha}$. For $\alpha = 0$ and $a \in C^m\left(\overline{D_T}\right)$ the kernel

$$K(t,s) = a(t,s)\log(t-s)$$

belongs to $S^{m,0}$. In fact, $C^m\left(\overline{D_T}\right) \subset S^{m,\alpha}\left(D_T\right)$, but usually one does not call smooth kernels weakly singular.

To describe the smoothness of the solution of (4.1) we need the following space of functions.

Definition 10. [48] For $m \in \mathbb{N}_0$, $\alpha < 1$, denote by $\mathcal{C}^{m,\alpha}(0,T]$ the space of functions $v \in \mathcal{C}^m(0,T]$, that satisfy the inequalities

$$|v^{(i)}(t)| \leq c \begin{cases} 1 & \text{if} \quad i < 1-\alpha, \\ 1 + |\log(t)| & \text{if} \quad i = 1-\alpha, \\ t^{1-\alpha-i} & \text{if} \quad i > 1-\alpha, \end{cases} \tag{4.3}$$

where $c = c(v)$, for all $t \in (0,T]$ and $i = 0,\ldots,m$.

For $\alpha \in \mathbb{R}$ we define the weight function

$$|\omega_\alpha(t)| = \begin{cases} 1 & \text{if} \quad \alpha < 0, \\ (1 + |\log(t)|)^{-1} & \text{if} \quad \alpha = 0, \\ t^\alpha & \text{if} \quad \alpha > 0. \end{cases} \tag{4.4}$$

Then $\mathcal{C}^{m,\alpha}(0,T]$, equipped with the norm

$$||v||_{m,\alpha} := \sum_{k=0}^{m} \sup_{0<t\leq T} \omega_{k-1+\alpha}(t)|v^{(k)}(t)|,$$

becomes a Banach space and for $m \geq 1$

$$\mathcal{C}^m[0,T] \subset \mathcal{C}^{m,\alpha}(0,T] \subset \mathcal{C}[0,T]. \tag{4.5}$$

For $m = 0$ we have $\mathcal{C}^{0,\alpha}(0,T] = \mathcal{BC}(0,T]$, i.e. the space of bounded continuous functions on $(0,T]$.

4.2 Parametric and operator form of the integral equation

Let $(\underline{f}(t,r), \overline{f}(t,r))$ and $(\underline{g}(t,r), \overline{g}(t,r))$, $(t,r) \in [0,T] \times [0,1]$ be parametric forms of $f(t)$ and $g(t)$. Then equation (4.1) can be rewritten as a system of Volterra integral equations:

$$\begin{cases} \underline{g}(t,r) = \underline{f}(t,r) + \displaystyle\int_0^t (K_+(t,s)\underline{g}(s,r) - K_-(t,s)\overline{g}(s,r)ds), \\ \\ \overline{g}(t,r) = \overline{f}(t,r) + \displaystyle\int_0^t (K_+(t,s)\overline{g}(s,r) - K_-(t,s)\underline{g}(s,r)ds), \end{cases} \tag{4.6}$$

where

$$K_+(t,s) = \begin{cases} K(t,s), & K(t,s) \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$K_-(t,s) = \begin{cases} -K(t,s), & K(t,s) \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

We must solve system (4.6) provided it has a solution. We define the operators $\mathcal{K}_{\alpha_+}, \mathcal{K}_{\alpha_-} : \mathcal{C}[0,T] \to \mathcal{C}[0,T]$ by

$$(\mathcal{K}_{\alpha_+}y)(t) = \int_0^t K_+(t,s)y(s)ds,$$

$$(\mathcal{K}_{\alpha_-}y)(t) = \int_0^t K_-(t,s)y(s)ds.$$

Then we can rewrite system (4.6) as

$$\begin{cases} \underline{g} = \underline{f} + \mathcal{K}_{\alpha_+}\underline{g} - \mathcal{K}_{\alpha_-}\overline{g}, \\ \overline{g} = \overline{f} + \mathcal{K}_{\alpha_+}\overline{g} - \mathcal{K}_{\alpha_-}\underline{g}. \end{cases} \tag{4.7}$$

We can also write this system as

$$G = F + \mathcal{K}G, \tag{4.8}$$

where $G = [g_1, g_2]^T$, $g_1 = \underline{g}$, $g_2 = \overline{g}$, $F = [f_1, f_2]^T$, $f_1 = \underline{f}$, $f_2 = \overline{f}$ and

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_{\alpha_+} & -\mathcal{K}_{\alpha_-} \\ -\mathcal{K}_{\alpha_-} & \mathcal{K}_{\alpha_+} \end{pmatrix}. \tag{4.9}$$

We also use the notation

$$\mathcal{K}G = \int_0^t \mathbf{K}(t,s)G(s,r)ds, \tag{4.10}$$

where

$$\mathbf{K}(t,s) = \begin{pmatrix} K_+(t,s) & -K_-(t,s) \\ -K_-(t,s) & K_+(t,s) \end{pmatrix}.$$

We call the vector G a fuzzy function if (g_1, g_2) is a fuzzy function.

4.3 Existence, uniqueness and smoothness of the solution

4.3.1 Existence and uniqueness of the solution

To prove existence of solutions we need to recall some results for weakly singular integral operators. For $k \in S^{m,\alpha}$, define the Volterra integral operator H by

$$Hu(t) = \int_0^t k(t,s)u(s)ds, \quad t \in [0,T].$$

Then the following compactness result is true (see [48]).

Theorem 11. *Let $k(x,y) \in S^{m,\alpha}$, $m \geq 0$, $\alpha < 1$. Then the Volterra integral operator H maps $\mathcal{C}^{m.\alpha}(0,T]$ into itself and $H\colon \mathcal{C}^{m,\alpha}(0,T] \to \mathcal{C}^{m,\alpha}(0,T]$ is compact. Moreover, $H\colon L^\infty(0,T) \to \mathcal{C}(0,T]$ is compact.*

Next we extend the previous result for the system of equations.

Theorem 12. *Let $K \in S^{m,\alpha}$, $m \geq 0$, $\alpha < 1$. Then the matrix Volterra integral operator $\mathcal{K}$ defined by (4.9) is a compact operator $\mathcal{K}\colon (\mathcal{L}^\infty(0,T))^2 \to (\mathcal{C}(0,T])^2$, hence also a compact operator in $(\mathcal{L}^\infty(0,T))^2$ and in $(\mathcal{C}(0,T])^2$.*

Proof. Since $\mathcal{K}$ is a matrix operator with elements $\mathcal{K}_{\alpha+}$ and $\mathcal{K}_{\alpha-}$ and the integral operators $\mathcal{K}_{\alpha+}$ and $\mathcal{K}_{\alpha-}$ are compact from $\mathcal{L}^\infty(0,T)$ to $\mathcal{C}(0,T]$, the operator $\mathcal{K}\colon (\mathcal{L}^\infty(0,T))^2 \to (\mathcal{C}(0,T])^2$ is also compact. □

To prove uniqueness of the solution, we need Gronwall's inequality and its generalization (Lemmas 1.2.17 and 1.3.13 of [11]).

Lemma 7. *Suppose that $q \in \mathcal{C}([0,T])$ is a non-decreasing function and $q(t) \geq 0$ for all $t \in [0,T]$. Let the non-negative continuous function z satisfy*

$$z(t) \leq q(t) + \int_0^t Mz(s)ds, \quad t \in [0,T]$$

for some $M > 0$ and $\beta < 0$. Then

$$z(t) \leq q(t) + \int_0^t Mq(s)\exp(M(t-s))ds \quad \forall t \in [0,T].$$

If q is non-decreasing on $[0,T]$, the inequality reduces to

$$z(t) \leq \exp(Mt)q(t) \quad \forall t \in [0,T].$$

Lemma 8. *Suppose that $q \in \mathcal{C}([0,T])$ is a non-decreasing function and $q(t) \geq 0$ for all $t \in [0,T]$. Let the non-negative continuous function z satisfy*

$$z(t) \leq q(t) + M\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}z(s)ds, \quad t \in [0,T]$$

for some $M > 0$ and $0 < \beta < 1$. Then

$$z(t) \leq E_\beta(Mt^\beta)q(t),$$

where E_β is the one-parameter Mittag-Leffler function [32] defined by

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k+1)}, \quad z \in \mathbb{C}, \beta > 0.$$

Next we prove the uniqueness of the trivial solution.

Lemma 9. *Suppose that $K \in S^{m,\alpha}$, $m \geq 0$, $\alpha < 1$ and $f = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then equation (4.8) has only the trivial solution in $(\mathcal{L}^\infty(0,T))^2$.*

Proof. Suppose that G is a solution of (4.8) in $(\mathcal{L}^\infty(0,T))^2$. Since $\mathcal{K}$ maps $(\mathcal{L}^\infty(0,T))^2$ into $(\mathcal{C}(0,T))^2$, we have $G \in (\mathcal{C}(0,T))^2$. By defining

$$|[g_1(t), g_2(t)]^T| := \max\{|g_1(t)|, |g_2(t)|\},$$

we have

$$|G(t,r)| \leq \int_0^t |\mathbf{K}(G(s,r)|ds \leq C_K \begin{cases} \int_0^t |G(s,r)|\, ds & \text{if} \quad \alpha < 0, \\ \int_0^t |G(s,r)|(1+|\log(t-s)|)\, ds & \text{if} \quad \alpha = 0, \\ \int_0^t |G(s,r)|(t-s)^{-\alpha} ds & \text{if} \quad \alpha > 0. \end{cases} \tag{4.11}$$

If $\alpha < 0$ then Lemma 7 gives

$$|G(t,r)| \leq 0, \quad t \in [0,T],\ r \in [0,1]. \tag{4.12}$$

If $\alpha = 0$ then for any $\beta \in (0,1)$ there exists $M > 0$ such that

$$1 + |\log(t-s)| \leq \frac{M}{(t-s)^\beta} \text{ for } 0 \leq s < t \leq T.$$

Now Lemma 8 gives (4.12).

If $0 < \alpha < 1$ we use Lemma 8 with $\beta = 1 - \alpha$ to get (4.12).

Hence in all cases we get that equation (4.8) has only the trivial solution in $(\mathcal{L}^\infty(0,T))^2$. $\square$

Now we can prove existence and uniqueness of solution of (4.8).

Theorem 13. *Suppose that $K \in S^{m,\alpha}$, $m \geq 0$, $\alpha < 1$ and $F \in (\mathcal{C}[0,T])^2$. Then equation (4.8) has a unique solution G in $(\mathcal{L}^\infty(0,T))^2$ and $G \in (\mathcal{C}[0,T])^2$.*

Proof. Since $\mathcal{C}[0,T] \subset \mathcal{L}^\infty(0,T)$, uniqueness in $(\mathcal{L}^\infty(0,T))^2$ implies uniqueness in $(\mathcal{C}[0,T])^2$. Hence $N(I - \mathcal{K}) = \{0\}$, where I is a identity matrix and $N(I - \mathcal{K})$ is the null-space of the operator $I - \mathcal{K}$ in $(\mathcal{C}[0,T])^2$. Now by Theorem 12, the operator $\mathcal{K}$ is compact in $(\mathcal{C}[0,T])^2$ and by Fredholm Alternative Theorem, equation (4.8) has a solution in $(\mathcal{C}[0,T])^2$ which is unique in $(\mathcal{L}^\infty(0,T))^2$. $\square$

4.3.2 Smoothness of the solution

To prove the smoothness of the solution, smoothness of the kernel K is not enough, because in our system of integral equations (4.6) the kernels are K_+ and K_-. If K does not change sign in D_T, then the smoothness of K_+ and K_- is the same as the smoothness of K, but in general the derivatives of K_+ and K_- are discontinuous at lines where K changes sign. We first provide the smoothness results under assumptions that K_+ and K_- are smooth.

Theorem 14. *Let $K_+, K_- \in S^{m,\alpha}$, $m \geq 1$, $\alpha < 1$. Then the matrix Volterra integral operator $\mathcal{K}$ is compact in $(\mathcal{C}^{m,\alpha}(0,T])^2$. If $F \in (\mathcal{C}^{m,\alpha}(0,T])^2$ then equation (4.8) has a unique solution $G \in (\mathcal{C}^{m,\alpha}(0,T])^2$.*

Proof. Since under the assumptions the integral operators $\mathcal{K}_{\alpha,+}$ and $\mathcal{K}_{\alpha,-}$ are compact in $\mathcal{C}^{m,\alpha}(0,T])$, then $\mathcal{K}$ is compact in $(\mathcal{C}^{m,\alpha}(0,T])^2$.

Rest of the proof is similar to the proof of Theorem 13. $\square$

The next proposition about smoothness of K_+ and K_- is straightforward.

Proposition 1. *If $K \in \mathcal{C}(D_T)$, then $K_+, K_- \in \mathcal{C}(D_T)$. Let us consider the set $\Gamma = \{(t,s) \in D_T\colon K(t,s) = 0\}$. If $K \in S^{m,\alpha}$ and for each $(t^*, s^*) \in \Gamma$ and $|j+l| \leq m$, $\dfrac{\partial^{j+k}}{\partial t^j \partial s^k} K(t^*, s^*) = 0$, then $K_+, K_- \in S^{m,\alpha}$.*

However, the assumptions of this proposition are very restrictive, especially if m is large. In general K_+ and K_- have discontinuous first derivatives, so we have to consider weakly singular kernels with discontinuous derivatives. Usually the sign of the kernel K changes along some lines in D_T. Under general configuration of the lines of sign change the smoothness results for weakly singular kernels are very complicated. For smooth kernels some results of smoothness of solution were provided in subsection 3.3.2 of this thesis. Here we provide some results for the case when the lines of sign change can only be vertical and/or horizontal lines.

Suppose the kernel changes sign along the vertical and/or horizontal lines $s = a_i$ and/or $t = a_i$, $i = 1, ..., n$, $0 < a_1 < a_2 < ... < a_n < T$. Denote $a_0 = 0, a_{n+1} = T$ and $D_{\{a_1,...,a_n\}} = D_T \setminus \cup_{i=1}^{n}(\{s = a_i\} \cup \{t = a_i\})$. Define $S^{m,\alpha}(D_{\{a_1,...,a_n\}})$ as the collection of m times continuously differentiable functions K on $D_{\{a_1,...,a_n\}}$ that satisfy inequality (4.2) for all $j, l \in \{0\} \cup \mathbb{N}, j + l \leq m$ and $(t,s) \in D_{\{a_1,...,a_n\}}$.

Without loss of generality we can assume there is only one vertical and/or horizontal line of sign change of K. Denote $d = a_1$ and $D_d = D_T \setminus (\{s = d\} \cup \{t = d\})$. We recall some definitions and theorems from [39], where similar results were obtained for weakly singular Fredholm integral equations. For $\alpha \in \mathbb{R}$, define the following weight functions on $(0,T)$:

$$|\omega_\alpha^{(0,T)}(t)| = \begin{cases} 1 & \text{if} \quad \alpha < 0, \\ (1 + |\log(\rho_{(0,T)}|)^{-1} & \text{if} \quad \alpha = 0, \\ \rho_{(0,T)}(t)^\alpha & \text{if} \quad \alpha > 0, \end{cases} \tag{4.13}$$

where $\rho_{(0,T)} = \min\{t, T-t\}$ is the distance from $t \in (0,T)$ to the boundary of the interval $(0,T)$. Let $\mathcal{G}_d = (0,T) \setminus \{d\}, 0 < d < T$. Introduce a cutting function $e \in \mathcal{C}[0,T]$ such that $0 \leq e(t) \leq 1$ for $0 \leq t \leq T, e(t) = 1$ in the vicinity of 0 and

T, and $e(t) = 0$ in the vicinity of d. In order to characterize the growth rates of the derivatives of the function $u(t)$ as $t \to d$, Pedas et al. [39] introduced also the weight functions

$$|\omega_\alpha^{(d)}(t)| = \begin{cases} 1 & \text{if} \quad \alpha < 0, \\ (1 + |\log(\rho_d)|)^{-1} & \text{if} \quad \alpha = 0, \\ \rho_d(t)^\alpha & \text{if} \quad \alpha > 0, \end{cases} \tag{4.14}$$

where $t \in \mathcal{G}_d$ and $\rho_d = |t - d|$. For $m, p \in \mathbb{N}$, $p \le m$, $\alpha \in \mathbb{R}$, $\alpha < 1$, denote by $\mathcal{C}^{m,\alpha,p}(\mathcal{G}_d)$ the Banach space of functions $g \in \mathcal{C}^m(\mathcal{G}_d) \cap \mathcal{C}^p(0, T)$ such that

$$\|u\|_{m,\alpha,p} = \sum_{j=0}^{m} \sup_{t \in \mathcal{G}_d} e(t) \omega_{j+\alpha-1}^{(0,T)}(t) |u^{(j)}(t)| + \sum_{j=0}^{m} \sup_{t \in \mathcal{G}_d} (1 - e(t)) \omega_{j+\alpha-1-p}^{d}(t) |u^{(j)}(t)| < \infty. \tag{4.15}$$

We can consider the Volterra integral equation as a special case of Fredholm integral equation if we extend the kernel above the diagonal by zero. Therefore we can use the theorems about the smoothness of solution from [39].

We state first the smoothness result for a system of Volterra integral equations which follows directly from the results for Fredholm equations.

Proposition 2. *Let $K_+, K_- \in S^{m,\alpha}(D_d) \cap \mathcal{C}^{p-1}(D_T)$ where $m, p \in \mathbb{N}, p \le m, \alpha \in \mathbb{R}, \alpha < 1$. Then $\mathcal{K} : (\mathcal{C}^{m,\alpha,p}(\mathcal{G}_d))^2 \to (\mathcal{C}^{m,\alpha,p}(\mathcal{G}_d))^2$ is compact and equation (4.6) has a unique solution in $(\mathcal{C}^{m,\alpha,p}(\mathcal{G}_d))^2$.*

Proof. It follows from Theorem 9, 10 of [39], if we extend kernel by zero above the diagonal. We can extend the results for the system of equations since the operators are compact and uniqueness follows from Theorem 13. □

For Volterra integral equation we can actually prove a stronger result. Solutions of Fredholm integral equations generally have singularities at both ends of the interval $(0, T)$ and at both sides of d. On the other hand, solutions of Volterra integral equations do not have singularities at T and when approaching d from left. Therefore we define $\mathcal{C}_d^{m,\alpha,p}(0, T]$ similarly to the space $\mathcal{C}^{m,\alpha,p}(\mathcal{G}_d)$, but functions in this space don't have singularity when approaching d, T from left side. We denote by $\mathcal{C}_d^{m,\alpha,p}(0, T]$ the Banach space of functions $u \in \mathcal{C}^m((0, T] \setminus \{d\}) \cap \mathcal{C}^p(0, T]$ such that

$$
\begin{aligned}
\|u\|_{d,m,\alpha,p} = \sum_{j=0}^{m} \sup_{t\in\mathcal{G}_d} e(t)\omega_{j+\alpha-1}(t)|u^{(j)}(t)| + \\
\sum_{j=0}^{m} \sup_{t\in\mathcal{G}_d} (1-e(t))\omega_{j+\alpha-1-p}(t-d)|u^{(j)}(t)| < \infty, \quad (4.16)
\end{aligned}
$$

where ω is defined in (4.4).

Theorem 15. *Let the assumption of Proposition 2 be fulfilled. Then the equation (4.8) has a unique solution in* $(\mathcal{C}_d^{m,\alpha,p}(0,T])^2$.

Proof. Without loss of generality let us assume that the sign of kernel is positive in regions I and III and negative in II as it shown in Figure 3. The other cases are

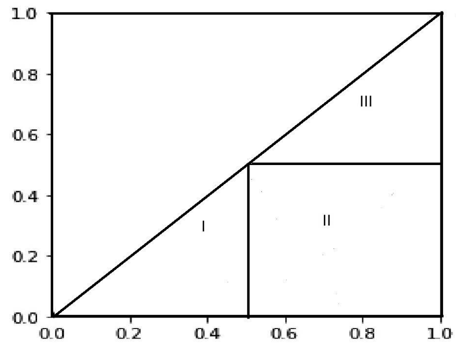


Figure 3: Regions of positivity and negativity of the kernel K.

similar and we skip them. Let r be fixed and denote $u(s) = g_1(s,r)$, $v(s) = g_2(s,r)$. We can write the first component of $\mathcal{K}G$ as follows:

$$
\begin{cases}
\displaystyle\int_0^t K_+(t,s)u(s)ds, \quad t \le d \\[2ex]
\displaystyle\int_0^d K_-(t,s)v(s)ds + \int_d^t K_+(t,s)u(s)ds, \quad t > d.
\end{cases} \tag{4.17}
$$

Here $\int_0^d K_-(t,s)v(s)ds$ is a Fredholm integral where the kernel has singular point outside the integration interval. The other integral operators in (4.17) are Volterra integral operators.

For $t \le d$ we have Volterra integral equation where the kernel doesn't change the sign. Therefore, we can use the smoothness result of Theorem 14 to conclude that the solution does not have a singularity as $t \to d^-$. In second part where $t > d$

and when $t \to d^+$, the singularity of the solution is described in Proposition 2. For $t \to T^-$, take $\varepsilon > 0$ such that $d < T-\varepsilon$. Then $K_- \in \mathcal{C}^m([T-\varepsilon, T] \times [0, d])$ and $v \in \mathcal{C}^{m,\alpha}(0, d]$. Thus we can differentiate the integral $\int_0^d K_-(t, s)v(s)ds$ m times under the integral sign and it belongs to $\mathcal{C}^m[T-\varepsilon, T]$. Note that when we are solving the integral equation in $(d, T]$ we can consider the integral $\int_d^t K_+(t, s)u(s)ds$ as given. So in $[T-\varepsilon, T]$ we have Volterra integral equation for u where the source function has no singularity at T. Hence we can use the result of Theorem 13, which implies that solution doesn't have singularity at T. Consequently the solution is in $(\mathcal{C}_d^{m,\alpha,p}(0, T])^2$. $\square$

4.3.3 Fuzziness of the exact solution

Fuzziness of the exact solution is proved for integral equations with continuous kernels. The idea of proof is similar in weakly singular case.

Theorem 16. *Let $K \in S^{m,\alpha}$ with $m \in \mathbb{N}_0$ and $\alpha > 1$. Let f be a fuzzy function such that $\underline{f}, \overline{f} \in C[0, T]$. Assume in addition that $\underline{f}, \overline{f}$ are continuous with respect to r. Then the solution $G = [\underline{g}, \overline{g}]$ of (4.6) is a fuzzy function.*

Proof. We use in the proof the equation (4.8) as the operator form of (4.6). It is well-known that if G is a fuzzy function then $\mathcal{K}G$ is a fuzzy function. Also, the components of $\mathcal{K}G$ inherit the continuity of G with respect to their variables. We prove that $G = [g_1, g_2]^T$ satisfies the conditions (1)-(3) of Theorem 2. By using the recursion formula

$$G_0 = F, \quad G_n = F + \mathcal{K}G_{n-1}, \ n = 1, 2, \dots \tag{4.18}$$

and by standard argument for Volterra equation one can say G_n converges uniformly to the solution $G = [g_1, g_2]^T$. Hence G is continuous both with respect to t and r. Let $r_1 < r_2$ be two arbitrary real numbers in $[0, 1]$. The components of $G_n = [g_{n1}, g_{n2}]^T$ compose fuzzy function $G_n = [g_{n1}, g_{n2}]^T$, hence

$$g_{n1}(t, r_1) - g_{n1}(t, r_2) \leq 0$$

for each $t \in [0, T]$. Now, for fixed t we can take the limit as $n \to \infty$ to get $g_1(t, r_1) \leq g_1(t, r_2)$. Therefore g_1 is a monotonically increasing function with respect to r. Similarly, g_2 is a monotonically decreasing function with respect to r and $g_1(t, r) \leq g_2(t, r)$ for $(t, r) \in [0, T] \times [0, 1]$ which proves the fuzziness of the vector function G. $\square$

4.4 Collocation methods

4.4.1 Collocation method on the discontinuous piecewise polynomial spaces

Define a mesh on $[0, T]$ by

$$\triangle_h := \{t_n : 0 = t_0 < t_1 < ... < t_N = T\}.$$

Let $\sigma_n := (t_n, t_{n+1}]$, $\overline{\sigma}_n := [t_n, t_{n+1}]$, $h_n = t_{n+1} - t_n$ $(n = 0, 1, ..., N-1)$ and let the diameter of the mesh be $h = \max\{h_n : 0 \leq n \leq N-1\}$. In the following we mainly use the graded mesh, where the mesh points are defined by

$$t_j := T\left(\frac{j}{N}\right)^{\rho}, \quad j = 0, \ldots, N. \tag{4.19}$$

Here $\rho \geq 1$ is called the grading parameter.

Define the piecewise polynomial space which we use in this thesis as follows:

$$\mathcal{S}_{m-1}^{(-1)}(\triangle_h) := \{v : v|_{\sigma_n} \in \pi_{m-1} \ (n = 0, 1, ..., N-1)\},$$

where π_{m-1} are polynomials of degree not exceeding $m-1$. Any $v_N \in (\mathcal{S}_{m-1}^{(-1)}(\triangle_h))$ can be determined by

$$v_N(t)|_{\sigma_n} \ = \ \sum_{j=1}^{m} L_j(\tau) V_{n,j}, \quad t = t_n + \tau h_n, \tag{4.20}$$

where $V_{n,i} := v_N(t_{n,i})$, $0 \leq c_1 < \ldots < c_m \leq 1$ are the collocation parameters, $t_{n,i} := t_n + c_i h_n$ for $n = 0, \cdots, N-1$, and

$$L_j(\tau) := \prod_{k=1,\ldots,m, k\neq j} \frac{\tau - c_k}{c_j - c_k}, \quad j = 1, \ldots, m, \ \tau \in [0, 1],$$

are the Lagrange fundamental polynomials on $[0, 1]$.

For fixed $r \in [0, 1]$ we look for approximate solution of equation (4.8) as a spline $u_N \in (\mathcal{S}_{m-1}^{(-1)}(\triangle_h))^2$. We require that the equation is exactly satisfied at collocation points $t_{n,i}$. Then we get the linear system of equations

$$u_N(t_{n,i}, r) = F(t_{n,i}, r) + (\mathcal{K}u_N)(t_{n,i}, r). \tag{4.21}$$

for determining $u_N(t_{n,i}, r)$. By partitioning the integration interval in equation

(4.8) we obtain

$$\begin{aligned} u_N(t_{n,i},r) = F(t_{n,i},r) + \sum_{l=0}^{n-1} h_l \int_0^1 \mathbf{K}(t_{n,i}, t_l + zh_l) u_N(t_l + zh_l, r) dz \\ + h_n \int_0^{c_i} \mathbf{K}(t_{n,i}, t_n + zh_n) u_N(t_n + zh_n, r) dz. \end{aligned} \tag{4.22}$$

Denote $U_{n,i}(r) = u_N(t_{n,i},r)$. Note that we can solve the equations on each interval σ_n separately, so when solving for $U_{n,i}(r)$ for fixed n, we can consider $U_{l,j}$ with $l < n$ as known. By substituting (4.20) into (4.22), we obtain

$$\begin{aligned} U_{n,i}(r) = h_n \sum_{j=1}^{m} \int_0^{c_i} \mathbf{K}(t_{n,i}, t_n + zh_n) L_j(z) dz\, U_{n,j}(r) \\ + F(t_{n,i},r) + \sum_{l=0}^{n-1} \sum_{j=1}^{m} h_l \int_0^1 \mathbf{K}(t_{n,i}, t_l + zh_l) L_j(z) dz\, U_{l,j}(r). \end{aligned} \tag{4.23}$$

For fixed n we obtain a linear system of $2m$ equations and $2m$ unknowns

$$\begin{aligned} U_{n,i}(r) = h_n \sum_{j=1}^{m} Q_{n,i,j} U_{n,j}(r) \\ + F(t_{n,i},r) + \sum_{l=0}^{n-1} \sum_{j=1}^{m} h_l R_{n,l,i,j} U_{l,j}(r), \quad i = 1, \dots, m, \end{aligned} \tag{4.24}$$

where

$$Q_{n,i,j} = \int_0^{c_i} \mathbf{K}(t_{n,i}, t_n + zh_n) L_j(z)\, dz \tag{4.25}$$

and

$$R_{n,l,i,j} = \int_0^1 \mathbf{K}(t_{n,i}, t_l + zh_l) L_j(z)\, dz. \tag{4.26}$$

4.4.2 The fully discretized collocation method

To describe the fully discretized collocation method we make an additional assumption:

$$K(t,s) = k(t,s) p_\alpha(t-s), \quad p_\alpha(t) = \begin{cases} t^{-\alpha}, & \text{for } \alpha < 1,\ \alpha \neq 0, \\ \log(t), & \text{for } \alpha = 0, \end{cases} \tag{4.27}$$

where $k \in \mathcal{C}^m(\overline{D_T})$ and $\alpha < 1$. Then $K \in S^{m,\alpha}$. Assume that the lines of sign change of K are only at horizontal and/or vertical lines $t = t_j$ or $s = t_j$ for some

$j \in \{1, \ldots, N-1\}$. Then all integrals in (4.22) are either zero or of the form

$$\int_0^a k(t_{n,i}, t_l + zh_l) p_\alpha(t_{n,i} - (t_l + zh_l)) v(t_l + zh_l)\, dz$$

with $a = 1$ or $a = c_i$. We approximate these integrals by product quadrature rule with the mesh $\{c_1, \ldots, c_m\}$ as follows

$$\int_0^{c_i} k(t_{n,i}, t_n + zh_n) p_\alpha(t_{n,i} - (t_n + zh_n)) v(t_n + zh_n)\, dz \approx \sum_{j=1}^m w_{n,i,j} k(t_{n,i}, t_n + c_j h_n) V_{n,j}$$

and

$$\int_0^1 k(t_{n,i}, t_l + zh_l) p_\alpha(t_{n,i} - (t_l + zh_l)) v(t_l + zh_l)\, dz \approx \sum_{j=1}^m w_{n,l,i,j} k(t_{n,i}, t_l + c_j h_l) V_{l,j},$$

where

$$w_{n,i,j} := \int_0^{c_i} p_\alpha((c_i - z)h_n) L_j(z) dz$$

and

$$w_{n,l,i,j} := \int_0^1 p_\alpha(t_{n,i} - (t_l + zh_l)) L_j(z) dz.$$

For fixed r we look for approximate solution of equation (4.8) as a spline $\widehat{u}_N(\cdot, r) \in (\mathcal{S}_{m-1}^{(-1)}(\triangle_h))^2$ and denote $\widehat{U}_{n,i}(r) = \widehat{u}_N(t_{n,i}, r)$. Then for determining $\widehat{U}_{n,i}(r)$ we get the following linear system of equations

$$\begin{aligned} \widehat{U}_{n,i}(r) = {} & h_n \sum_{j=1}^m \widehat{Q}_{n,i,j} \widehat{U}_{n,j}(r) \\ & + F(t_n + c_i h_n, r) + \sum_{l=0}^{n-1} \sum_{j=1}^m h_l \widehat{R}_{n,l,i,j} \widehat{U}_{l,j}(r), \quad i = 1, \ldots, m, \end{aligned} \tag{4.28}$$

where

$$\widehat{Q}_{n,i,j} = \mathbf{k}(t_n + c_i h_n, t_n + c_j h_n) w_{n,i,j}$$

and

$$\widehat{R}_{n,l,i,j} = \mathbf{k}(t_n + c_i h_n, t_l + c_j h_l) w_{n,l,i,j},$$

where

$$\mathbf{k}(t,s) = \begin{pmatrix} k(t,s) & 0 \\ 0 & k(t,s) \end{pmatrix} \quad \text{or} \quad \mathbf{k}(t,s) = \begin{pmatrix} 0 & -k(t,s) \\ -k(t,s) & 0 \end{pmatrix},$$

depending on whether $K(t,s)$ is positive or negative in $\sigma_n \times \sigma_l$. Then the approximate solution can be written as

$$\widehat{u}_N(t,r)|_{\sigma_n} = \sum_{j=1}^{m} L_j(z)\widehat{U}_{n,j}(r), \quad t = t_n + zh_n, \quad z \in (0,1]. \tag{4.29}$$

4.5 Convergence

4.5.1 Convergence estimates for the collocation method

We denote by p_N the interpolation projector onto the set of all piecewise polynomial functions on $[0,T]$ which are real polynomials of degree not exceeding $m-1$ on every interval $[t_j, t_{j+1}]$, $0 \le j \le N-1$, where the interpolation points are defined by $t_{n,i} = t_n + c_i h_n$, $0 \le c_1 < \cdots < c_m \le 1$. The approximation properties of $p_N u$ on graded mesh (4.19) are considered in [29, 38, 47]. These results can be summarized as follows.

Lemma 10. *[38] Assume that $u \in \mathcal{C}^{m,\alpha}(0,T]$ and the graded mesh (4.19) with grading parameter ρ is used. Then the following estimates hold where the constant C does not depend on N:*

$$\max_{t\in[0,T]} |u(t) - (p_N u)(t)| \le C\|u\|_{m,\alpha} E(N,m,\rho,\alpha), \tag{4.30}$$

where

$$E(N,m,\rho,\alpha) = \begin{cases} N^{-m}, & \text{for } m < 1-\alpha,\ \rho \ge 1, \\ N^{-m}(|\log N|+1), & \text{for } m = 1-\alpha,\ \rho = 1, \\ N^{-m}, & \text{for } m = 1-\alpha,\ \rho > 1, \\ N^{-\rho(1-\alpha)}, & \text{for } m > 1-\alpha,\ 1 \le \rho < \frac{m}{1-\alpha}, \\ N^{-m}, & \text{for } m > 1-\alpha,\ \rho \ge \frac{m}{1-\alpha}. \end{cases} \tag{4.31}$$

Remark 4. If $u \in \mathcal{C}_d^{m,\alpha,p}(0,T]$, then we can use different graded meshes on $[0,d]$ and $[d,T]$, possibly with different grading parameters, and use Lemma 10 separately on these intervals.

In the consequent theorems we present the convergence result for fuzzy weakly singular integral equation.

Define an interpolation projector

$$\mathbf{p}_N : (\mathcal{C}[0,T])^2 \mapsto (\mathcal{S}_{m-1}^{(-1)}(\triangle_h))^2, \quad m, N \in \mathbb{N},$$

by

$$(\mathbf{p}_N v)(t_{n,i}) = v(t_{n,i}), \quad i = 1, \ldots, m, \quad n = 0, \ldots, N, \tag{4.32}$$

for any continuous function $v \in (\mathcal{C}(0,T])^2$.

Let $r \in [0,1]$ be fixed. Then the system (4.24) can be replaced by an operator equation of the form

$$u_N(t,r) = \mathbf{p}_N F(t,r) + \mathbf{p}_N \mathcal{K} u_N(t,r). \tag{4.33}$$

Proposition 3. *Assume that $F(.,r) \in (\mathcal{C}^{m,\alpha}(0,T])^2$ for all $r \in [0,1]$ and $\|F(\cdot,r)\|_{m,\alpha}$ are uniformly bounded with respect to r. Let $K_+, K_- \in S^{m,\alpha}$ with $m \geq 1$, $\alpha < 1$. Let G be the unique solution of the system* (4.8). *Let the graded mesh (4.19) with grading parameter ρ be used. Then*

$$\sup_{r\in[0,1]} \|G(\cdot,r) - \mathbf{p}_N G(\cdot,r)\|_\infty \leq \mathrm{const} E(N,m,\rho,\alpha). \tag{4.34}$$

Proof. By Theorem 14 the operator $(I - \mathcal{K})$ is invertible in $(\mathcal{C}^{m,\alpha}(0,T])^2$. Since $F(.,r) \in (\mathcal{C}^{m,\alpha}(0,T])^2$, we get $\|G(.,r)\|_{m,\alpha} \leq \|(I-\mathcal{K})^{-1}\| \, \|F(.,r)\|_{m,\alpha}$. Then by Lemma 10 we get (4.34). □

In general case when the kernel K changes sign, the assumptions of the previous proposition may be too restrictive. Then we have the following result.

Proposition 4. *Assume that $F(.,r) \in (\mathcal{C}^{m,\alpha}(0,T])^2$ for all $r \in [0,1]$ and $\|F(\cdot,r)\|_{m,\alpha}$ are uniformly bounded with respect to r. Let $K_+, K_- \in S^{m,\alpha}(D_d) \cap \mathcal{C}^{p-1}(D_T)$ with $m \geq 1$, $\alpha < 1$, $1 \leq p \leq m$. Let G be the unique solution of the system* (4.8). *Let two graded meshes on $[0,d]$ and on $[d,T]$ with numbers of intervals N_1 and N_2 and grading parameters ρ_1 and ρ_2 be used. Then*

$$\sup_{r\in[0,1]} \|G(\cdot,r) - \mathbf{p}_N G(\cdot,r)\|_\infty \leq \mathrm{const}\max\{E(N_1,m,\rho_1,\alpha), E(N_1,m,\rho_2,\alpha-p)\}. \tag{4.35}$$

Proof. The proof is similar to the proof of the previous proposition, only we use Theorem 15 and the space is $(\mathcal{C}_d^{m,\alpha,p}(0,T])^2$. □

Remark 5. Similar results hold when there are more lines of sign change of K.

We use the following general theorem about interpolation operator p_N.

Lemma 11. *[14] Let $\mathcal{T}\colon \mathcal{L}^\infty(0,T) \to \mathcal{C}[0,T]$ be a linear compact operator. Let $p_N\colon \mathcal{C}[0,T] \to \mathcal{S}_{m-1}^{(-1)}(\triangle_h)$ be the interpolation operator with graded mesh (4.19).*

Then

$$\|\mathcal{T} - p_N\mathcal{T}\|_{\mathcal{L}(\mathcal{L}^\infty(0,T),\mathcal{L}^\infty(0,T))} \to 0, \text{ as } N \to \infty.$$

To establish convergence order we can use the following theorem.

Theorem 17. *Let X, X' be Banach spaces and $X' \subset X$. Assume $\mathcal{T}\colon X \to X'$ is bounded and $I - \mathcal{T}\colon X \to X$ is a bijective operator. Further, assume*

$$\|\mathcal{T} - p_N\mathcal{T}\| \to 0, \text{ as } N \to \infty,$$

where $p_N\colon X' \to X, N = 1, 2, \ldots$ are bounded linear operators. Then for all sufficiently large N (say $N > N_0$) the operator $I - p_N\mathcal{T}$ is invertible in X and

$$\sup_{N > N_0} \|(I - p_N\mathcal{T})^{-1}\| < \infty.$$

For the solutions of $x_N = p_N\mathcal{T}x_N + p_N f$ and $x = \mathcal{T}x + f$,

$$c_1\|x - p_N x\| \leq \|x - x_N\| \leq c_2\|x - p_N x\|,$$

where c_1 and c_2 are positive constants.

Proof. This theorem is a generalization of Theorem 12.1.2 in [5] and the proof is similar.

□

Theorem 18. *Assume that $F(., r) \in (\mathcal{C}^{m,\alpha}(0, T])^2$ for all $r \in [0, 1]$ and $\|F(\cdot, r)\|_{m,\alpha}$ are uniformly bounded with respect to r. Assume $K_+, K_- \in \mathcal{S}^{m,\alpha}$. Let G be the unique solution of the system (4.8). Assume that the collocation method (4.33) with collocation parameters $0 \leq c_1 < \ldots < c_m \leq 1$, $m \in \mathbb{N}$ and with grading parameter $\rho \geq 1$ are used. Then there exists an integer N_0 such that for all $N \geq N_0$, operator equation (4.33) possesses a unique solution $u_N(., r) \in (\mathcal{S}_{m-1}^{(-1)}(\triangle_h))^2$ and*

$$\sup_{r \in [0,1]} \|G(., r) - u_N(., r)\|_\infty \to 0, \qquad N \to \infty.$$

Furthermore, for $N \geq N_0$ the following error estimates hold:

$$\sup_{r \in [0,1]} \|G(., r) - u_N(., r)\|_\infty \leq \text{const} E(N, m, \rho, \alpha). \tag{4.36}$$

Proof. The conditions of Theorem 17 are satisfied with $X = (\mathcal{L}^\infty(0, T))^2$, $X' =$

$(\mathcal{C}[0,T])^2$, $\mathcal{T}=\mathcal{K}$, $x=G(.,r)$, $x_N=u_N$ and $p_N=\mathbf{p}_N$. Thus

$$\|G(.,r)-u_N(.,r)\|_\infty \leq c\|G(.,r)-\mathbf{p}_N G(.,r)\|_\infty.$$

Now Proposition 3 completes the proof. $\square$

We state separately the case when the kernel K changes sign.

Theorem 19. *Assume that $F(.,r)\in(\mathcal{C}^{m,\alpha}(0,T])^2$ for all $r\in[0,1]$ and $\|F(\cdot,r)\|_{m,\alpha}$ are uniformly bounded with respect to r. Let $K_+,K_-\in S^{m,\alpha}(D_d)\cap\mathcal{C}^{p-1}(D_T)$ with $m\geq 1$, $\alpha<1$, $1\leq p\leq m$. Let G be the unique solution of the system* (4.8)*. Let the collocation method (4.33) with collocation parameters $0\leq c_1<\ldots<c_m\leq 1$, $m\in\mathbb{N}$ and with two graded meshes on $[0,d]$ and on $[d,T]$ with numbers of intervals N_1 and N_2 and grading parameters ρ_1 and ρ_2 be used. Then there exists an integer N_0 such that for all $N\geq N_0$, operator equation (4.33) possesses a unique solution $u_N(.,r)\in(\mathcal{S}^{(-1)}_{m-1}(\triangle_h))^2$ and*

$$\sup_{r\in[0,1]}\|G(.,r)-u_N(.,r)\|_\infty \to 0,\qquad N\to\infty.$$

Furthermore, for $N\geq N_0$ the following error estimates hold:

$$\sup_{r\in[0,1]}\|G(\cdot,r)-u_N(.,r)\|_\infty \leq \mathrm{const}\max\{E(N_1,m,\rho_1,\alpha),E(N_1,m,\rho_2,\alpha-p)\}. \tag{4.37}$$

4.5.2 Convergence estimates for the fully discretized collocation method

In deriving the fully dicretized collocation method, we approximated the integrals by the product quadrature rule. The quadrature rule was obtained by substituting the smooth part under the integral sign by its interpolation polynomial. The next lemma estimates the error of the quadrature rule.

Lemma 12. *Let p_α be defined as in* (4.27)*. Let p_N be the interpolation projector to spline space $S^{-1}_{m-1}(\Delta_N)$. Then the following estimates hold.*

i) If $f\in\mathcal{C}^m[0,T]$ then

$$\left|\int_0^t f(s)p_\alpha(s)ds-\int_0^t(p_N f)(s)p_\alpha(s)ds\right|\leq C\max_{s\in[0,t]}|f^{(m)}(s)|h^m. \tag{4.38}$$

ii) If $f \in C^{m,\alpha}(0,T]$ and the graded mesh with the grading parameter ρ is used, then

$$\left| \int_0^t f(s)p_\alpha(s)ds - \int_0^t (p_N f)(s)p_\alpha(s)ds \right| \leq CE(N,m,\rho,\alpha). \tag{4.39}$$

iii) If $k \in \mathcal{C}^m(\overline{D_T})$ and $v_N(s) \in S_{m-1}^{-1}(\Delta_h)$, then

$$\left| \int_0^t p_\alpha(t,s)[k(t,s)v_N(s) - p_N(kv)(t,s)]ds \right| \leq Ch\|v\|_\infty. \tag{4.40}$$

Proof. We use the standard estimate of the interpolation error:

$$\|f - p_N f\|_\infty \leq C \max_{s\in[0,T]} |f^{(m)}(s)| h^m.$$

Then

$$\begin{aligned} \left| \int_0^t f(s)p_\alpha(s)ds - \int_0^t (p_N f)(s)p_\alpha(s)ds \right| &\leq \int_0^t p_\alpha(s)ds \|f - p_N f\|_\infty \\ &\leq C \max_{s\in[0,t]} |f^{(m)}(s)| h^m. \end{aligned}$$

This proves the first estimate. Proof of the second estimate is similar, using Lemma 10.

For the proof of assertion (iii) we start as before:

$$\begin{aligned} &\left| \int_0^t p_\alpha(t,s)[k(t,s)v_N(s) - p_N(kv)(t,s)]ds \right| \\ &\qquad \leq \int_0^t p_\alpha(t-s)ds \max_{s\in[0,t]} \left| k(t,s)v_N(s) - p_N(kv)(t,s) \right|. \end{aligned} \tag{4.41}$$

We estimate the interpolation error in each subinterval $[t_l, t_{l+1}]$ separately. Since v_N is a polynomial of order $m-1$ in each subinterval, we have

$$\max_{s\in[t_l,t_{l+1}]} \left| v_N^{(j)}(s) \right| \leq Ch_l^{-j} \|v\|_\infty, \quad j = 0, 1, \ldots, m-1$$

and $v_N^{(m)}(s) = 0$. Hence we can estimate

$$\max_{s\in[t_l,t_{l+1}]} |k(t_{n,i},s)v_N(s) - p_N(kv)(t,s)|$$
$$\leq C \max_{s\in[t_l,t_{l+1}]} \left| \frac{\partial^m}{\partial s^m} \left(k(t,s)v_N(s)\right) \right| h_l^m \leq C h_l^{-m+1} \cdot h_l^m \leq C h_l \leq Ch. \quad (4.42)$$

Now using the fact that p_α is integrable, (4.41) and (4.42) give the desired estimate. □

To prove the convergence of the fully discretized collocation method we use the following general theorem about convergence of projection methods, when the operator is first approximated. It is similar to Corollary 13.11 of [30], but since one of the assumptions is not satisfied in our case, we give a new proof. Similar results have also been proved in [12], but our results are more general.

Theorem 20. *Let X, X' be Banach spaces and $X' \subset X$. Assume $\mathcal{T}: X \to X'$ is bounded and $I - \mathcal{T}: X \to X$ is injective operator. Assume*

$$\|\mathcal{T} - p_N\mathcal{T}\| \to 0, \text{ as } N \to \infty,$$

where $p_N: X' \to X, N = 1, 2, \ldots$ are bounded linear operators. Let $X_N = p_N(X')$. Let $\mathcal{T}_N : X_N \to X$ be an approximation of $\mathcal{T}$ such that

$$\sup_{v_N\in X_N,\ \|v_N\|=1} \|(p_N\mathcal{T}_N - p_N\mathcal{T})v_N\| \to 0, \quad N \to \infty.$$

Then for all sufficiently large N the operator $I - p_N\mathcal{T}_N$ is invertible in X_N and

$$\sup_{N>N_0} \|(I - p_N\mathcal{T}_N)^{-1}\|_{X_N} < \infty.$$

For the solutions of $\widehat{x}_N = p_N\mathcal{T}_N\widehat{x}_N + p_Nf_N$ and $x = \mathcal{T}x + f$ we have the estimate

$$\|x - \widehat{x}_N\| \leq C \left(\|x - p_Nx\| + \|p_N(\mathcal{T}_Np_N - \mathcal{T})x\| + \|p_N(f_N - f)\|\right), \quad (4.43)$$

where C is a positive constant.

Proof. Since $I - p_N\mathcal{T}_N = (I - p_N\mathcal{T}) + (p_N\mathcal{T} - p_N\mathcal{T}_N)$, $I - p_N\mathcal{T}$ is invertible in X_N (see Theorem 17) and $\|p_N\mathcal{T}_N - p_N\mathcal{T}\|_{X_N} \to 0$, invertibility of $I - p_N\mathcal{T}_N$ in X_N and uniform boundedness of the inverse operators follows.

Let x_N be the solution of $x_N = p_N\mathcal{T}x_N + p_Nf$ (note that the assumptions of

Theorem 17 are satisfied). Subtracting the equations for x_N and $\widehat{x}_N$ we get

$$x_N - \widehat{x}_N = p_N \mathcal{T}_N (x_N - \widehat{x}_N) - (p_N \mathcal{T}_N - p_N \mathcal{T}) x_N + p_N f - p_N f_N,$$

hence

$$\|x_N - \widehat{x}_N\| \leq \|(I - p_N \mathcal{T}_N)^{-1}\| (\|(p_N \mathcal{T}_N - p_N \mathcal{T}) x_N\| + \|p_N f - p_N f_N\|) . \quad (4.44)$$

We can estimate $\|(p_N \mathcal{T}_N - p_N \mathcal{T}) x_N\|$ as follows, using Theorem 17:

$$\begin{aligned}
\|(p_N \mathcal{T}_N - p_N \mathcal{T}) x_N\| &\leq \|(p_N \mathcal{T}_N - p_N \mathcal{T})(x_N - p_N x)\| \\
&\quad + \|p_N \mathcal{T}(x - p_N x)\| + \|(p_N \mathcal{T}_N p_N - p_N \mathcal{T}) x\| \\
&\leq \|p_N \mathcal{T}_N - p_N \mathcal{T}\|_{X_N} \|x_N - p_N x\| + \|p_N \mathcal{T}\| \, \|x - p_N x\| + \|(p_N \mathcal{T}_N p_N - p_N \mathcal{T}) x\| \\
&\leq C \|p_N x - x\| + \|(p_N \mathcal{T}_N p_N - p_N \mathcal{T}) x\|
\end{aligned}$$

Estimate (4.43) now follows from (4.44), the inequality

$$\|x - \widehat{x}_N\| \leq \|x_N - \widehat{x}_N\| + \|x_N - x\|$$

and Theorem 17. □

Now we can state the convergence result for fully discretized collocation method.

Theorem 21. *Assume that $F(.,r) \in (\mathcal{C}^{m,\alpha}(0,T])^2$ for all $r \in [0,1]$ and $\|F(\cdot,r)\|_{m,\alpha}$ are uniformly bounded with respect to r. Assume $K_+, K_- \in \mathcal{S}^{m,\alpha}$. Let G be the unique solution of the system* (4.8)*. Assume that the fully discretized collocation method (4.28) with collocation parameters $0 \leq c_1 < \ldots < c_m \leq 1$, $m \in \mathbb{N}$ and with grading parameter $\rho \geq 1$ is used. Then there exists an integer N_0 such that for all $N \geq N_0$, operator equation (4.33) possesses a unique solution $\widehat{u}_N(.,r) \in (\mathcal{S}^{(-1)}_{m-1}(\triangle_h))^2$ and*

$$\sup_{r \in [0,1]} \|G(.,r) - \widehat{u}_N\|_\infty \to 0, \qquad N \to \infty.$$

Furthermore, for $N \geq N_0$ the following error estimates hold:

$$\sup_{r \in [0,1]} \|G(.,r) - \widehat{u}_N\|_\infty \leq \text{const} E(N, m, \rho, \alpha). \quad (4.45)$$

Proof. The conditions of Theorem 20 are satisfied with $X = (\mathcal{L}^\infty(0,T))^2$, $X' = (\mathcal{C}[0,T])^2$, $\mathcal{T} = \mathcal{K}$, $x = G(.,r)$, $\widehat{x}_N = \widehat{u}_N$, $p_N = \mathbf{p}_N$ and $\mathcal{T}_N$ the approximation of

$\mathcal{K}$ by using the product quadrature rule introduced in Section 4.4.2. Furthermore the assumption of Theorem 20 about $\mathcal{T}_N$ is satisfied by iii) of Lemma 12. The estimate for the first term in right hand side of (4.43) is given in (4.34). By ii) of Lemma 12 we get the estimate of second right hand side term of (4.43). The last term is zero, because in our method $f_N = f$. $\square$

Similar results also holds when the kernel changes sign on vertical and/or horizontal lines and different graded meshes are used on subintervals where the kernel does not change sign.

4.5.3 Fuzziness of the approximate solution

The main question is whether the approximate solution is fuzzy. In this section, we propose sufficient conditions which guarantee fuzziness of the approximate solution.

Definition 11. Suppose $F = [\underline{f}, \overline{f}]^T$ is a vector function. We say F is a strictly fuzzy function if $[\underline{f}, \overline{f}]$ is a fuzzy function and there is $\delta > 0$ such that

1. $\dfrac{\underline{f}(t, r_2) - \underline{f}(t, r_1)}{r_2 - r_1} > \delta$ for all $t \in [0, T]$ and $0 \leq r_1 < r_2 \leq 1$.
2. $\dfrac{\overline{f}(t, r_2) - \overline{f}(t, r_1)}{r_2 - r_1} < -\delta$ for all $t \in [0, T]$ and $0 \leq r_1 < r_2 \leq 1$.
3. $\underline{f}(t, 1) < \overline{f}(t, 1), \quad t \in [0, T]$.

Now, it is possible to prove the fuzziness of $\mathbf{u}_N = (\underline{u}_N, \overline{u}_N)$ for those F that are strictly fuzzy functions.

In the following theorem by adding some more assumptions on F we guarantee the fuzziness of approximate solution.

Theorem 22. *Suppose that F is a strictly fuzzy vector function. Let for any $r \in [0, 1]$, $F(., r) \in (\mathcal{C}^{m,\alpha}(0, T])^2$ and $K_+, K_- \in S^{m,\alpha}$. Then the system (4.8) has a unique solution $G = [\underline{g}, \overline{g}]$, $G(\cdot, r) \in (\mathcal{C}^{m,\alpha}(0, T])^2$, and G is a strictly fuzzy function. Assume that a collocation method of the form (4.33) with collocation points $0 \leq c_1 < \ldots < c_m \leq 1$, $m \in \mathbb{N}$ and with grading parameter $\rho \geq 1$ is used. Let*

$$\left\| \frac{F(\cdot, r_2) - F(\cdot, r_1)}{r_2 - r_1} \right\|_{1,\alpha} \leq \text{const}, \quad 0 \leq r_1 < r_2 \leq 1,$$

where the constant does not depend on N and r. Then there exists an integer N_0 such that for all $N \geq N_0$, operator equation (4.33) possesses a unique fuzzy solution u_N.

Proof. Fuzziness of the exact solution is proved in Theorem 16. First we prove that if F is a strictly fuzzy function, then the exact solution G is also a strictly fuzzy function. Let $0 \le r_1 < r_2 \le 1$. Then by the first equation of the system (4.6)

$$\frac{\underline{g}(t, r_2) - \underline{g}(t, r_1)}{r_2 - r_1} > \frac{\underline{f}(t, r_2) - \underline{f}(t, r_1)}{r_2 - r_1} > \delta.$$

The second condition of strictly fuzziness follows similarly. To prove the third condition we take $r = 1$ in system (4.6) and subtract the two equations.

Next we prove the monotonicity of approximate solution (condition 1. of Theorem 2). Let $0 \le r_1 < r_2 \le 1$. By Theorem 18

$$\begin{aligned} &\|\frac{\underline{g}(\cdot, r_2) - \underline{g}(\cdot, r_1)}{r_2 - r_1} - \frac{\underline{u_N}(\cdot, r_2) - \underline{u_N}(\cdot, r_1)}{r_2 - r_1}\|_\infty \\ &\qquad\qquad \le \|\frac{\underline{F}(\cdot, r_2) - \underline{F}(\cdot, r_1)}{r_2 - r_1}\|_{1,\alpha} E(N, 1, \rho, \alpha) \quad (4.46) \end{aligned}$$

By assumption, $\|\frac{\underline{F}(\cdot, r_2) - \underline{F}(\cdot, r_1)}{r_2 - r_1}\|_{1,\alpha} \le$ const, where the constant does not depend on N and r. Then for sufficiently large N the right hand side of (4.46) is less than $\delta/2$ and since $\dfrac{\underline{g}(\cdot, r_2) - \underline{g}(\cdot, r_1)}{r_2 - r_1} > \delta$, we get $\underline{u_N}(\cdot, r_2) - \underline{u_N}(\cdot, r_1) > \delta/2$. Similarly we can prove that $\overline{u_N}(\cdot, r_2) - \overline{u_N}(\cdot, r_1) < -\delta/2$.

Similarly we get

$$\|\overline{g}(\cdot, 1) - \underline{g}(\cdot, 1) - (\overline{u}_N(\cdot, 1) - \underline{u}_N(\cdot, 1))\| \le E(N, m, \rho, \alpha),$$

therefore for sufficiently large N, $\underline{u}_N(t, 1) < \overline{u}_N(t, 1)$. Hence all conditions of Theorem 2 are satisfied. In fact we have proved that for N large enough, $\mathbf{u}_N$ is a strictly fuzzy function. $\square$

Similar result holds also for the fully discretized collocation method.

If F does not satisfy condition 3 of Definition 11, we cannot guarantee that the approximate solution $\mathbf{u}_N$ satisfies condition 3 of Theorem 2. In this case we can modify our approximate solution to make it fuzzy without spoiling the rate of convergence. Assume that for sufficiently large N, $\underline{u}_N(t, 1) > \overline{u}_N(t, 1)$ for some $t \in [0, T]$. Let $r_N = \inf\{r \in [0, 1] \colon \underline{u}_N(t, r) > \overline{u}_N(t, r)\}$. In this case, we propose to use new forms of approximating functions:

$$\underline{u}_N^{\text{new}}(t, r) = \begin{cases} \underline{u}_N(t, r), & \text{if } 0 \le r < r_N, \\ \dfrac{\underline{u}_N(t, r_N) + \overline{u}_N(t, r_N)}{2}, & \text{if } r_N \le r \le 1 \end{cases}$$

and

$$\overline{u}_N^{\text{new}}(t,r) = \begin{cases} \overline{u}_N(t,r), & \text{if } 0 \le r < r_N, \\ \dfrac{\underline{u}_N(t,r_N) + \overline{u}_N(t,r_N)}{2}, & \text{if } r_N \le r \le 1. \end{cases}$$

Obviously $\mathbf{u}_N^{\text{new}}$ is a fuzzy function.

Let $t \in [0,T]$ be such that $\underline{u}_N(t,1) > \overline{u}_N(t,1)$ for N large enough and let $r > r_N$. Then $\overline{u}_N(t,r) \le \underline{u}_N(t,r)$, hence

$$\overline{u}_N(t,r) - \overline{g}(t,r) \le \overline{u}_N(t,r) - \underline{g}(t,r) \le \underline{u}_N(t,r) - \underline{g}(t,r).$$

Therefore $|\overline{u}_N(t,r) - \underline{g}(t,r)| \le \max\{|\underline{u}_N(t,r) - \underline{g}(t,r)|, |\overline{u}_N(t,r) - \overline{g}(t,r)|\}$. Hence if N is large enough so that the convergence estimate of Theorem 18 (or Theorem 21) holds, we have for $r \in [r_N, 1]$

$$\begin{aligned} \left|\underline{u}_N^{\text{new}}(t,r) - \underline{g}(t,r)\right| &= \left|\frac{\underline{u}_N(t,r) + \overline{u}_N(t,r)}{2} - \underline{g}(t,r)\right| \\ &\le \left|\frac{\underline{u}_N(t,r) - \underline{g}(t,r)}{2}\right| + \left|\frac{\overline{u}_N(t,r) - \underline{g}(t,r)}{2}\right| \le C\,E(N,m,\rho,\alpha). \end{aligned}$$

Similar estimate holds also for $\overline{u}_N^{\text{new}}(t,r)$ and the proof is also similar. Consequently the convergence estimates also hold for the modified solution.

4.6 Numerical examples

In this section, we illustrate the convergence of the fully discretized collocation method by some selected examples. In examples 6 and 7 kernels are non-negative, in example 8 the kernel changes the sign. Here we use the following approximations for errors

$$\underline{E}_N = \max_{\eta_k \in [0,T]} \{|\underline{u}_N(\eta_k, r) - \underline{g}(\eta_k, r)|\}$$

and

$$\overline{E}_N = \max_{\eta_k \in [0,T]} \{|\overline{u}_N(\eta_k, r) - \overline{g}(\eta_k, r)|\},$$

where $(\underline{g}, \overline{g})$ and $(\underline{u}_N, \overline{u}_N)$ (for $N \in \mathbb{N}$) are exact and numerical solutions of the system (4.6), respectively, and $\eta_k = \frac{k}{10N}$, $k = 0, \ldots, 10N$. The approximate order of the convergence can be obtained by using the formula

$$O_N = \log_2 \frac{E_N}{E_{2N}}.$$

Example 6. Consider the system of fuzzy Volterra integral equation with weakly singular kernel (FVIEW) on $[0,1]$ with

$$K(t,s) = \frac{(t-s)^{2.5}+1}{(t-s)^{0.5}},$$

$$\underline{f}(t,r) = (t^{0.5} - \frac{\pi}{8}t^2 - \frac{\pi}{2}t)(r^2+r),$$

$$\overline{f}(t,r) = (t^{0.5} - \frac{\pi}{8}t^2 - \frac{\pi}{2}t)(4-r^3-r).$$

The exact solution is

$$g(t,r) = \left(t^{0.5}(r^2+r), t^{0.5}(4-r^3-r)\right).$$

Here $K \in S^{m,0.5}$, $f \in \mathcal{C}^{m,0.5}(0,1]$ and according to Theorem 14, $g \in \mathcal{C}^{m,0.5}(0,1]$ for any $m \in \mathbb{N}$. We used fully discretized collocation method with discontinuous linear splines with two collocation points, and with piecewise constant splines, and a graded mesh with grading parameter ρ.

Method (1) $m = 2$, $c_1 = 0.6$, $c_2 = 0.8$.

Method (2) $m = 1$, $c_1 = 0.4$.

By our convergence results we expect the order of convergence to be $\rho/2$ for $\rho < 2m$, and m for $\rho \geq 2m$. In Tables 5–9 we illustrate the error and order of convergence by applying Method (1), Method (2) on $r = 0.9$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	9.2334e-02	1.2803e-01		
128	6.4451e-02	8.9365e-02	0.51	0.51
256	4.5191e-02	6.2659e-02	0.51	0.51
512	3.1774e-02	4.4056e-02	0.51	0.51
1024	2.2381e-02	3.1032e-02	0.51	0.51

Table 5 The errors and orders of Example 6 by Method (1) for $\rho = 1$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	1.1140e-02	1.5446e-02		
128	5.5576e-03	7.7059e-03	1.00	1.00
256	2.7758e-03	3.8488e-03	1.00	1.00
512	1.3871e-03	1.9233e-03	1.00	1.00
1024	6.9338e-04	9.6141e-04	1.00	1.00

Table 6 The errors and orders of Example 6 by Method (1) for $\rho = 2$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	1.5303e-02	2.12118e-02		
128	3.8052e-03	5.2760e-03	2.00	2.00
256	9.4905e-04	1.3159e-03	2.00	2.00
512	2.3718e-04	3.2886e-04	1.98	1.98
1024	6.0028e-05	8.3232e-05	1.98	1.98

Table 7 The errors and orders of Example 6 by Method (1) for $\rho = 4$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	1.5871e-01	2.20050e-01		
128	1.0411e-01	1.4435e-01	0.60	0.60
256	6.8837e-02	9.5446e-02	0.51	0.51
512	3.4095e-02	4.7275e-05	0.51	0.51
1024	1.4045e-02	3.3340e-02	0.50	0.50

Table 8 The errors and orders of Example 6 by Method (2) for $\rho = 1$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	1.1202e-01	1.5532e-01		
128	6.2853e-02	8.7148e-02	0.91	0.91
256	3.3659e-02	4.6532e-02	0.94	0.94
512	1.7434e-03	2.4173e-02	0.97	0.97
1024	8.9182e-03	1.2365e-03	0.98	0.98

Table 9 The errors and orders of Example 6 by Method (2) for $\rho = 2$.

Example 7. Consider a FVIEW of the form (4.1) with

$$K(t,s) = \frac{1}{(t-s)^{0.5}},$$

$$\underline{f}(t,r) = (\frac{\sin(t)}{\sqrt{t}} - \pi \sin(\frac{t}{2}) J_0(\frac{t}{2}))(r),$$

$$\overline{f}(t,r) = (\frac{\sin(t)}{\sqrt{t}} - \pi \sin(\frac{t}{2}) J_0(\frac{t}{2}))(2-r).$$

Here $J_\nu(z)$ is a Bessel function defined by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(-\frac{z}{4}\right)^k}{k!\,\Gamma(k+v+1)}.$$

The exact solution of this system is $g(t,r) = (\underline{g}(t,r), \overline{g}(t,r))$ with

$$\underline{g}(t,r) = \frac{\sin(t)}{\sqrt{t}}(r), \quad \overline{g}(t,r) = \frac{\sin(t)}{\sqrt{t}}(2-r).$$

Again $K \in S^{m,0.5}$, $f \in \mathcal{C}^{m,0.5}(0,1]$ and $g \in \mathcal{C}^{m,0.5}(0,1]$ for any $m \in \mathbb{N}$. We used fully discretized collocation method with discontinuous linear splines with different choices of the collocation points, and a graded mesh with grading parameter ρ. For this example we use the following methods:

Method (1) $m = 2$ with $c_1 = 0.5$, $c_2 = 1$.

Method (2) $m = 2$ with $c_1 = \frac{3-\sqrt{3}}{6}$, $c_2 = \frac{3+\sqrt{3}}{6}$ (the roots of shifted Legendre polynomial of degree 2).

In Tables 10–15, we illustrate the error and order of convergence by applying Method (1) and Method (2) with $r = 0.9$. Convergence rates are the same when using different collocation points, but the roots of shifted Legendre polynomial (Gauss points) give better results, since the approximation of the integrals is better.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	1.9364e-02	2.3667e-02		
128	1.3290e-02	1.6244e-02	0.54	0.54
256	9.2114e-03	1.1258e-02	0.52	0.52
512	6.4252e-03	7.8530e-03	0.51	0.51
1024	4.5008e-03	5.5010e-03	0.51	0.51

Table 10 The errors and orders of Example 7 by Method (1) for $\rho = 1$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	2.3215e-03	2.8374e-03		
128	1.1507e-03	1.4064e-03	1,01	1.01
256	5.7291e-04	7.0022e-04	1.00	1.00
512	2.8585e-04	3.4937e-04	1.00	1.00
1024	1.4278e-04	1.7450e-04	1.00	1.00

Table 11 The errors and orders of Example 7 by Method (1) for $\rho = 2$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	2.6126e-03	3.1932e-03		
128	6.8685e-04	8.3948e-04	1.92	1.92
256	1.7534e-04	2.1431e-04	1.96	1.96
512	4.4197e-05	5.4018e-05	1.98	1.98
1024	1.1079e-05	1.3541e-05	1.99	1.99

Table 12 The errors and orders of Example 7 by Method (1) for $\rho = 4$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	7.6802e-03	9.3869e-03		
128	5.2434e-03	6.4086e-03	0.55	0.55
256	3.6196e-03	4.4240e-03	0.53	0.53
512	2.5173e-03	3.0767e-03	0.52	0.52
1024	1.7595e-03	2.1505e-03	0.51	0.51

Table 13 The errors and orders of Example 7 by Method (2) for $\rho = 1$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	1.0624e-03	1.2985e-03		
128	5.2787e-04	6.4517e-04	1.00	1.00
256	2.6312e-04	3.2159e-04	1.00	1.00
512	1.3136e-04	1.6055e-04	1.00	1.00
1024	6.5628e-05	8.0211e-05	1.00	1.00

Table 14 The errors and orders of Example 7 by Method (2) for $\rho = 2$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	2.6425e-04	3.2297e-04		
128	6.2779e-05	7.6730e-05	2.07	2.07
256	1.5120e-05	1.8480e-05	2.05	2.05
512	3.6791e-06	4.4966e-06	2.03	2.03
1024	9.0214e-07	1.1026e-06	2.07	2.07

Table 15 The errors and orders of Example 7 by Method (2) for $\rho = 4$.

Example 8. Let us consider a FVIEW of the form (4.1) on $[0, 2]$ with

$$K(t, s) = \frac{t-1}{(t-s)^{\frac{1}{3}}},$$

$$\underline{f}(t, r) = (1 + t^4)(r^2 + r),$$

$$\overline{f}(t, r) = (1 + t^4)(4 - r^3 - r).$$

In this example $\alpha = \frac{1}{3}$. This time the kernel changes sign along the line $t = 1$. Here the exact solution is not known. By Theorem 15 the exact solution belongs to $(\mathcal{C}_d^{m,\alpha,p}(0, T])^2$. In this case we should use graded meshes with different grading parameters on $[0, 1]$ and on $[1, 2]$.

We use the fully discretized collocation method with $m = 2$ and $c_1 = 0.5$, $c_2 = 1$ for numerical approximation of the solution. The optimal grading parameters which give the convergence order $O(h^2)$, are $\rho = 3$ on $[0, 1]$ and $\rho = \frac{6}{5}$ on $[1, 2]$. Our numerical result should provide fuzzy numbers for every t. Since we do not have the exact solution, we use the difference of the approximate solutions with

N and $2N$ as an error estimation. We report the numerical solutions on $r = 0.5$ with various values of N. In Table 16 the first columns show the estimated errors of the method and the last columns show the order of convergence which is 2.

N	$\lvert \underline{U}_N - \underline{U}_{2N} \rvert$	$\lvert \overline{U}_N - \overline{U}_{2N} \rvert$	$\underline{O}_N$	$\overline{O}_N$
32	-	-	-	-
64	3.2350e-02	1.4558e-01	-	-
128	8.1722e-03	3.6775e-02	1.98	1.98
256	2.0536e-03	9.2411e-03	1.99	1.99
512	5.1490e-04	2.3171e-03	1.99	1.99
1024	1.2895e-04	5.8030e-04	1.99	1.99

Table 16 The errors and orders of Example 8 for the first and second components.

Chapter 5

Classical approximation for fuzzy Fredholm integral equation

Fuzzy Fredholm integral equation of second kind (FFIE) is given by

$$y(t) = f(t) + \int_0^T k(t,s)y(s)ds, \tag{5.1}$$

where k is a bivariate function with the domain $D = [0,T] \times [0,T]$, $T > 0$, and f is a given fuzzy (source) function. We observe that when f is an ordinary function, then under some conditions (if 1 is not an eigenvalue of the integral operator) equation (5.1) possesses a crisp solution. On the other hand, if f is a fuzzy function then the solution y is a fuzzy function as well. In the proposed contribution, we will be working with the fuzzy case.

5.1 Function approximation

In this section, we analyze the approximation of an ordinary function by its finite expansion using Chebyshev polynomials. Depending on the smoothness of the function and the selected approximation space, we give lower and upper estimates of the quality of approximation. These estimates will be further used in our analysis of the linear fuzzy Fredholm integral equation.

Due to the weighted orthogonality of Chebyshev polynomials, a function f, which is defined on the interval $[0,T]$, can be approximately expanded as follows:

$$f(t) \simeq \sum_{m=0}^{N} c_m T_m^*(t) = C^T \Psi(t), \ \ N \in \mathbb{N}, \tag{5.2}$$

where C and Ψ are the matrices of size $(N+1)\times 1$

$$\begin{aligned} C^T &= [c_0, \cdots, c_N], \\ \Psi(t) &= [T_0^*(t), \ldots, T_N^*(t)]^T, \end{aligned} \tag{5.3}$$

with the elements of matrix C as follows:

$$\begin{aligned} c_i &= \frac{1}{\gamma_i}\int_0^T w^*(x)f(x)T_i^*(x)dx \\ &= \frac{1}{\gamma_i}\int_0^T w(\frac{2}{T}x-1)f(x)T_i(\frac{2}{T}x-1)dx \\ &= \frac{T}{2\gamma_i}\int_{-1}^1 w(t)f(\frac{T}{2}(t+1))T_i(t)dt \\ &\simeq \frac{T\pi}{2\gamma_i(N+1)}\sum_{k=1}^{N+1} f(\frac{T}{2}(x_k+1))T_i(x_k), \quad i=0,\ldots,N. \end{aligned} \tag{5.4}$$

The polynomial in the right hand side of (5.2) is the orthogonal projection (orthogonality is with respect to (2.3)) of f on the span of orthogonal polynomials $T_0^*(t),\ldots,T_N^*(t)$. Let us denote this projection by p_N where $p_N : \mathcal{C}[0,T] \mapsto \pi_N$, and π_N is the space of polynomials with the degree not exceeding N. In detail:

$$p_N(f) = \sum_{m=0}^{N} c_m T_m^*(t), \tag{5.5}$$

where coefficients c_m, $m = 0,\ldots,N$ are as above.

Let $u(x,y)$ be a bivariate function defined on $[0,T_1]\times[0,T_2]$. In the similar way, it can be expanded using Chebyshev polynomials as follows

$$u(x,y) \simeq p_{N,M}(u)(x,y) = \sum_{n=0}^{N}\sum_{m=0}^{M} u_{n,m}T_n^*(x)\widehat{T}_m^*(y) = \Psi(x)^T U\widehat{\Psi}(y), \tag{5.6}$$

where $p_{N,M} : \mathcal{C}([0,T]\times[0,T]) \mapsto \pi_N\times\pi_M$, $(N,M\in\mathbb{N})$ is an orthogonal projection and we use $\widehat{}$ to distinguish the shifted Chebyshev polynomials corresponding to different intervals. Here, $U = (u_{i,j})$ is a matrix of size $(N+1)\times(M+1)$ with the elements

$$\begin{aligned} u_{i,j} &= \frac{1}{\gamma_i\widehat{\gamma}_j}\int_0^{T_1}\int_0^{T_2} w^*(x)\widehat{w}^*(y)u(x,y)T_i^*(x)\widehat{T}_j^*(x)dxdy \qquad (5.7) \\ &\simeq \frac{T_1T_2\pi^2}{4\gamma_i\widehat{\gamma}_j(N+1)^2}\sum_{r=1}^{N+1}\sum_{s=1}^{M+1} u(\frac{T_1}{2}(x_r+1),\frac{T_2}{2}(x_s+1))T_i(x_r)T_j(x_s). \end{aligned}$$

Theorem 23. *Let $\Psi(x)$ be the vector of shifted Chebyshev polynomials defined in* (5.3)*. Let the $(N+1)\times(N+1)$ matrix P be defined by*

$$P := \int_0^T \Psi(s)\Psi(s)^T ds. \tag{5.8}$$

Then the elements of this matrix are

$$p_{00} = T,\ p_{11} = \frac{T}{3},\ p_{10} = p_{01} = 0,$$

$$p_{ij} = \frac{T}{4}\left(\frac{-1-(-1)^{i+j}}{(i+j-1)(i+j+1)}\right) \quad \text{for } j = i+1, i-1,\ i \in \{1,\dots,N\},$$

$$p_{ij} = \frac{T}{4}\left(\frac{-1-(-1)^{i+j}}{(i+j-1)(i+j+1)} + \frac{-1-(-1)^{|i-j|}}{(|i-j|-1)(|i-j|+1)}\right) \quad \text{for other } i,j.$$

Proof. We note that

$$P = \int_0^T \Psi(s)\Psi(s)^T ds = \begin{pmatrix} \int_0^T T_0^*(s)T_0^*(s)ds & \dots & \int_0^T T_0^*(s)T_N^*(s)ds \\ \vdots & \ddots & \vdots \\ \int_0^T T_N^*(s)T_0^*(s)ds & \dots & \int_0^T T_N^*(s)T_N^*(s)ds \end{pmatrix}.$$

The elements of this matrix can be computed as

$$\int_0^T T_i^*(s)T_j^*(s)ds = \frac{T}{4}\int_{-1}^1 T_{i+j}(s) + T_{|i-j|}(s)ds.$$

From [31] we know that

$$\int_{-1}^1 T_n(s)ds = \frac{-1-(-1)^n}{(n-1)(n+1)}, \quad n > 1,$$

and

$$\int_{-1}^1 T_0(s)ds = 2,\ \int_{-1}^1 T_1(s)ds = 0.$$

The rest is straightforward. □

The following error estimate for Dini-Lipschitz continuous function f provides the uniform convergence of approximation by Chebyshev polynomials.

Theorem 24. *[31](Theorem 5.7) Let $g \in \mathcal{C}[0,T]$ satisfy the Dini-Lipschitz condition, i.e.*

$$\omega(\delta, g)\log(\delta) \to 0, \quad \text{provided that} \quad \delta \to 0, \tag{5.9}$$

where $\omega(\delta, g)$ is the modulus of continuity of g with respect to δ. Then $\|g - p_N(g)\|_\infty \to 0$ as $N \to \infty$, where $p_N(g)$ is the corresponding to g Chebyshev polynomial, determined in (5.5).

The similar error estimate is true for the Clenshaw-Curtis quadrature.

Theorem 25. *Let $f \in \mathcal{C}[0, T]$ satisfy the Dini-Lipschitz condition. Then*

$$|I(f) - I_N(f)| < 4 \|f - p_N(f)\|_\infty ,$$

where $I(f) = \int_{-1}^{1} w(x) f(x) dx$ and $I_N(f) = \frac{\pi}{N+1} \sum_{k=1}^{N+1} f(x_k)$.

Proof. It follows from Theorem 4.1 of [46]. Also, see [15]. □

For Lebesgue spaces the uniform convergence with $\mathcal{L}^2$ norm is guaranteed by the following theorem.

Theorem 26. *[31](Theorem 5.2) Let $g \in \mathcal{L}^2[0, T]$. Then $\|g - p_N(g)\|_{\mathcal{L}^2} \to 0$ as $N \to \infty$.*

There is another useful error estimate in Sobolev spaces $\mathcal{H}^s$ $(s > 0)$.

Theorem 27. *[5] Let $g \in \mathcal{H}^s[0, T]$ with $s > 0$. Then*

$$\|g - p_N(g)\|_{\mathcal{L}^2} \leq c N^{-s} \|g\|_{\mathcal{H}^s}$$

and

$$\|g - p_N(g)\|_{\mathcal{H}^1} \leq c N^{\frac{3}{2} - s} \|g\|_{\mathcal{H}^s},$$

where c is a constant.

Theorem 28. *([31] Section 5.7) Let $f \in \mathcal{C}^4[a, b]$. Then*

$$\|((p_N(f))' - f'\|_\infty \to 0 \quad as \quad N \to \infty.$$

Proof. First, we recall the Peano's theorem ([31] Section 5.7):

> *Let $\mathfrak{L}$ be a bounded linear functional in the space $\mathcal{C}^{m+1}[a, b]$ such that $\mathfrak{L} p_m = 0$ for every polynomials $p_m \in \pi_m, m \geq 0$. Then for all $f \in \mathcal{C}^{m+1}[a, b]$*
>
> $$\mathfrak{L}(f) = \int_a^b \frac{f^{m+1}(t)}{m!} \mathfrak{L}((x - t)_+^m) dt, \qquad x \in [a, b].$$

Now, let $\mathfrak{L}_N(f) = (p_N(f))' - f'$, $f \in \mathcal{C}^{m+1}[-1,1]$, and $N \geq m$. By the Peano's theorem, we have

$$(p_N(f))'(x) - f'(x) = \int_{-1}^{1} \frac{f^{m+1}(t)}{m!}((p_N(x-t)_+^m)' - ((x-t)_+^m)')dt.$$

Note that

$$(x-t)_+^m = \sum_{k=0}^{\infty} c_{km}T_k(x) = p_N((x-t)_+^m) + \sum_{k=N+1}^{\infty} c_{km}T_k(x), \tag{5.10}$$

where for $k > 0$

$$c_{km} = \frac{2}{\pi}\int_t^1 w(x)T_k(x)(x-t)^m dx \leq ck^{-m-1} \quad \text{as } k \to \infty$$

and

$$c_{0m} = \frac{1}{\pi}\int_t^1 w(x)(x-t)^m dx.$$

Let $U_k := \frac{1}{k+1}T'_{k+1}(x)$ for $k = 0, 1, \cdots$ be Chebyshev polynomials of second kind. It is known that $\|U_k\|\infty \leq k+1$, (see [31]). Therefore, $\|c_{km}kU_{k-1}(x)\|< ck^{-m+1}$ as $k \to \infty$ and for $m \geq 3$ the series

$$\sum_{k=N+1}^{\infty} c_{km}kU_{k-1}(x)$$

is uniformly convergent because of the convergent majorant numerical series. Therefore, we can differentiate both sides of equation (5.10) and obtain the following equality

$$((x-t)_+^m)' - (p_N((x-t)_+^m))' = \sum_{k=N+1}^{\infty} c_{km}kU_{k-1}(x), \tag{5.11}$$

where the right hand side series is uniformly convergent.

Consequently,

$$\|((x-t)_+^m)' - (p_N((x-t)_+^m))'\|_\infty \to 0.$$

Therefore, when $N \to \infty$,

$$\|(p_N(f))'(x) - f'(x)\|_\infty \to 0.$$

$\square$

Corollary 1. *For all* $f \in \mathcal{C}^4[a,b]$ *we have*

$$\|(p_N(f)) - f\|_{\infty,1} \to 0 \quad \text{as} \quad N \to \infty,$$

where

$$\|f\|_{\infty,1} = \|f\|_\infty + \|f'\|_\infty.$$

5.2 General scheme of the proposed method

In this section, we give a detailed description of the proposed method focused on the numerical solution of fuzzy Fredholm integral equation (5.1). The scheme is as follows: obtain a parametric form of (5.1), replace functional components by their polynomial approximations using the results of the preceding section, reduce the integral equation to the algebraic system of linear equations.

We start with the parametric form of fuzzy Fredholm integral equation (5.1) of the second kind where the function f is fuzzy valued. Let $(\underline{f}(t,r), \overline{f}(t,r))$ and $(\underline{y}(t,r), \overline{y}(t,r))$ on $(t,r) \in [0,T] \times [0,1]$ be parametric forms of f and y, respectively. Then equation (5.1) can be rewritten as follows:

$$\underline{y}(t,r) = \underline{f}(t,r) + \int_0^T (k_+(t,t)\underline{y}(s,r) - k_-(t,s)\overline{y}(s,r))ds, \tag{5.12}$$

$$\overline{y}(t,r) = \overline{f}(t,r) + \int_0^T (k_+(t,s)\overline{y}(s,r) - k_-(t,s)\underline{y}(s,r))ds, \tag{5.13}$$

where

$$k_+(t,s) = \begin{cases} k(t,s), & k(t,s) \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$k_-(t,s) = \begin{cases} -k(t,s), & k(t,s) \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The system (5.12)-(5.13) can be written as

$$y(t,r) = f(t,r) + \int_0^T \mathbf{k}(t,s)y(s,r)ds, \tag{5.14}$$

where $y(t,r)=[\underline{y}(t,r),\overline{y}(t,r)]^T$ and $f(t,r)=[\underline{f}(t,r),\overline{f}(t,r)]^T$

$$\mathbf{k}(t,s)=\begin{pmatrix} k_+(t,s) & -k_-(t,s) \\ -k_-(t,s) & k_+(t,s) \end{pmatrix}.$$

Using (5.6), we obtain the following approximations for y, f, k_+ and k_-, respectively:

$$y(t,r)\simeq[\Psi(t)^T\underline{U}\widehat{\Psi}(r),\Psi(t)^T\overline{U}\widehat{\Psi}(r)]^T \quad on\ [0,T]\times[0,1],$$

$$f(t,r)\simeq[\Psi(t)^T\underline{F}\widehat{\Psi}(r),\Psi(t)^T\overline{F}\widehat{\Psi}(r)]^T \quad on\ [0,T]\times[0,1],$$

$$k_+(t,s)\simeq\Psi(t)^TK_1\Psi(s) \quad on\ [0,T]\times[0,T],$$

and

$$k_-(t,s)\simeq\Psi(t)^TK_2\Psi(s) \quad on\ [0,T]\times[0,T],$$

where $\underline{U}$, $\overline{U}$, $\underline{F}$, $\overline{F}$ are $(N+1)\times(M+1)$ real matrices, and K_1, K_2 are $(N+1)\times(N+1)$ real matrices.

Substituting these approximations into (5.14) we obtain

$$\begin{aligned}\begin{pmatrix}\Psi(t)^T\underline{U}\widehat{\Psi}(r)\\ \Psi(t)^T\overline{U}\widehat{\Psi}(r)\end{pmatrix}&=\begin{pmatrix}\Psi(t)^T\underline{F}\widehat{\Psi}(r)\\ \Psi(t)^T\overline{F}\widehat{\Psi}(r)\end{pmatrix}\\ &+\int_0^T\begin{pmatrix}\Psi(t)^TK_1\Psi(s) & -\Psi(t)^TK_2\Psi(s)\\ -\Psi(t)^TK_2\Psi(s) & \Psi(t)^TK_1\Psi(s)\end{pmatrix}\begin{pmatrix}\Psi(s)^T\underline{U}\widehat{\Psi}(r)\\ \Psi(s)^T\overline{U}\widehat{\Psi}(r)\end{pmatrix}ds.\end{aligned} \tag{5.15}$$

Multiplying each equation in (5.15) by $w^*(t)\Psi(t)$ and then integrating, we can delete the $\Psi(t)$ on the basis of the orthogonality property of polynomials in $\Psi(t)$. Similarly, the orthogonality property of polynomials in $\widehat{\Psi}(r)$ makes it easy to delete this term from the right hand side of (5.15). Then

$$\begin{pmatrix}\underline{U}\\ \overline{U}\end{pmatrix}=\begin{pmatrix}\underline{F}\\ \overline{F}\end{pmatrix}+\begin{pmatrix}K_1\int_0^T\Psi(s)\Psi(s)^Tds\underline{U}-K_2\int_0^T\Psi(s)\Psi(s)^Tds\overline{U}\\ -K_2\int_0^T\Psi(s)\Psi(s)^Tds\underline{U}+K_1\int_0^T\Psi(s)\Psi(s)^Tds\overline{U}\end{pmatrix}. \tag{5.16}$$

The above given integrals can be computed using matrix P, which has been introduced in Theorem 23. Thus,

$$\begin{pmatrix}\underline{U}\\ \overline{U}\end{pmatrix}=\begin{pmatrix}\underline{F}\\ \overline{F}\end{pmatrix}+\begin{pmatrix}K_1P\underline{U}-K_2P\overline{U}\\ -K_2P\underline{U}+K_1P\overline{U}\end{pmatrix}. \tag{5.17}$$

By rearranging the terms in Equation (5.17), we finally obtain the following algebraic system of $2(N+1)(M+1)$ linear equations:

$$\begin{pmatrix} I - K_1P & +K_2P \\ +K_2P & I - K_1P \end{pmatrix} \begin{pmatrix} \underline{U} \\ \overline{U} \end{pmatrix} = \begin{pmatrix} \underline{F} \\ \overline{F} \end{pmatrix}, \tag{5.18}$$

considered with respect to unknown components in $\underline{U}$ and $\overline{U}$. Solving system (5.18) (see the next section where we discuss the solvability), we come to the numerical approximation of the solution y to the fuzzy Fredholm integral equation (5.1) in the parametric form, i.e.

$$\underline{y}(t,r) \simeq \Psi(t)^T \underline{U} \widehat{\Psi}(r),$$

$$\overline{y}(t,r) \simeq \Psi(t)^T \overline{U} \widehat{\Psi}(r),$$

so that

$$y(t,r) = [\underline{y}(t,r), \overline{y}(t,r)]^T.$$

5.3 Existence of the unique solution

In the previous section, we observed that the solution of a fuzzy Fredholm integral equation of the second kind satisfies the system (5.14) of (non-fuzzy) Fredholm integral equations of the second kind, i.e.

$$(\mathcal{I} - \mathfrak{K})y(t,r) = f(t,r), \tag{5.19}$$

where $\mathcal{I} : (\mathcal{C}_{DL}[0,T])^2 \to (\mathcal{C}_{DL}[0,T])^2$ is the identity operator and the operator $\mathfrak{K} : (\mathcal{C}_{DL}[0,T])^2 \to (\mathcal{C}_{DL}[0,T])^2$ is defined by

$$\mathfrak{K}(y(t,r)) := \int_0^T \mathbf{k}(t,s)y(s,r)ds.$$

Here the vector space $(\mathcal{C}_{DL}[0,T])^2$ is the space of Dini-Lipschitz continuous functions defined by

$$\mathcal{C}_{DL}[0,T] = \{f \in \mathcal{C}[0,T] | \ f \text{ satisfies } (5.9)\}$$

with

$$\|\mathbf{f}\| = \max\{\|f_1\|_\infty, \|f_2\|_\infty\},$$

where $\mathbf{f} = [f_1, f_2] \in (\mathcal{C}_{DL}[0,T])^2$. By the Geometric series theorem [5], this system has a unique solution provided that $\mathfrak{K}$ is a bounded operator and satisfies

$$\|\mathfrak{K}\| < 1. \tag{5.20}$$

As a consequence, the operator $(\mathfrak{I} - \mathfrak{K})^{-1}$ exists and is bounded. This fact justifies the existence of a unique solution to (5.19).

The important question is whether the solution is a fuzzy function such that its values are fuzzy numbers that fulfill conditions (1)-(3) of Theorem 2. The answer is positive, and the explanation is as follows: by Geometric series expansion, we have

$$y(t,r) = f(t,r) + \mathfrak{K}(f(t,r)) + \mathfrak{K}^2(f(t,r)) + \dots .$$

It is obvious that each term of this expansion satisfies conditions (1)-(3) of Theorem 2. Consequently, the whole sum satisfies conditions (1)-(3) and the exact solution is an fuzzy function where all its summands are fuzzy functions as well.

5.4 Existence of unique fuzzy approximate solution and convergence analysis

The above given theoretical justification of solvability of (5.14) is a combination of the projection and degenerate kernel methods for integral equations. Below we discuss some practical results that justify existence of approximate solutions explained in Section 5.2. Moreover, we give an estimation of the quality of approximate solutions. We keep the denotation of Section 5.2 and additionally, we denote

$$\mathfrak{K}_N(y(t,r)) := \int_0^T \begin{pmatrix} \Psi(t)^T K_1 \Psi(s) & -\Psi(t)^T K_2 \Psi(s) \\ -\Psi(t)^T K_2 \Psi(s) & \Psi(t)^T K_1 \Psi(s) \end{pmatrix} \begin{pmatrix} \underline{y}(s,r) \\ \overline{y}(s,r) \end{pmatrix} ds.$$

We examine the following approximate form of the system (5.14) of (non-fuzzy) Fredholm integral equations of the second kind

$$(\mathfrak{I} - \mathfrak{K}_N)(U_N) = p_{N,N}(f), \tag{5.21}$$

where $U_N = [\underline{U}_N, \overline{U}_N]^T$, $\underline{U}_N, \overline{U}_N \in \pi_N$ and coefficients of polynomials $\underline{U}_N, \overline{U}_N$ are solutions of (5.18). Our purpose is to prove solvability of (5.21) and estimate the difference between its solution U_N and the exact solution y to the system (5.14).

We recall the following general fact known from the theory of linear operators [6].

Theorem 29. *([6], page 24) Let $\mathfrak{K} : X \to X$ be a bounded linear operator in a Banach space X and let $\mathfrak{I} - \mathfrak{K}$ be injective. Assume $\{\mathfrak{K}_N\}$ is a sequence of bounded operators with*

$$\|\mathfrak{K} - \mathfrak{K}_N\| \to 0$$

as $N \to \infty$. Then for all sufficiently large $N > \mathbf{N}$, the inverse operators $(\mathcal{I} - \mathfrak{K}_N)^{-1}$ exist and are bounded in accordance with

$$\|(\mathcal{I} - \mathfrak{K}_N)^{-1}\| \leq \frac{\|(\mathcal{I} - \mathfrak{K})^{-1}\|}{1 - \|(\mathcal{I} - \mathfrak{K})^{-1}(\mathfrak{K} - \mathfrak{K}_N)\|}. \tag{5.22}$$

Let us apply Theorem 29 to our particular case and show that for every sufficiently large $N > \mathbf{N}$, there exists unique approximate solution to the system (5.21). By (5.22) and the discussion in the previous section, we will be focused on the three spaces, introduced above (see Theorems 25 - 27). We will study the space of Dini-Lipschitz continuous functions with the supremum norm $\|.\|_\infty$. The study of other spaces is similar.

Let $k \in \mathcal{C}_{DL}(([0,T] \times [0,T])^2)$. Then it is straightforward to show that k_+ and k_- are in $\mathcal{C}_{DL}([0,T] \times [0,T])$. Therefore,

$$M_{1,N} := \sup_{(t,s) \in [0,T] \times [0,T]} |\Psi(t)^T K_1 \Psi(s) - k_+(t,s)| \to 0$$

and

$$M_{2,N} := \sup_{(t,s) \in [0,T] \times [0,T]} |\Psi(t)^T K_2 \Psi(s) - k_-(t,s)| \to 0$$

as $N \to \infty$. Hence

$$\begin{aligned}
&\|\mathfrak{K} - \mathfrak{K}_N\|_\infty = \sup_{\|y\|_\infty \leq 1} \|(\mathfrak{K} - \mathfrak{K}_N)y\| \\
&= \left\| \int_0^T \begin{pmatrix} \Psi(t)^T K_1 \Psi(s) - k_+ & -\Psi(t)^T K_2 \Psi(s) + k_- \\ -\Psi(t)^T K_2 \Psi(s) + k_- & \Psi(t)^T K_1 \Psi(s) - k_+ \end{pmatrix} \begin{pmatrix} \underline{y}(s,r) \\ \overline{y}(s,r) \end{pmatrix} ds \right\|_\infty \\
&\leq (M_{1,N} + M_{2,N}) T \|y\|_\infty \to 0, N \to \infty.
\end{aligned} \tag{5.23}$$

Therefore,

$$\begin{aligned}
y - U_N &= (\mathcal{I} - \mathfrak{K})^{-1} f - (\mathcal{I} - \mathfrak{K}_N)^{-1} p_{N,N}(f) \\
&= (\mathcal{I} - \mathfrak{K})^{-1} f - (\mathcal{I} - \mathfrak{K}_N)^{-1} f + (\mathcal{I} - \mathfrak{K}_N)^{-1}(f - p_{N,N}(f)) \\
&= (\mathcal{I} - \mathfrak{K}_N)^{-1}(\mathfrak{K} - \mathfrak{K}_N)(\mathcal{I} - \mathfrak{K})^{-1} f + (\mathcal{I} - \mathfrak{K}_N)^{-1}(f - p_{N,N}(f)),
\end{aligned}$$

and thus

$$\begin{aligned}
&\|y - U_N\|_\infty \\
&\leq \|(\mathcal{I} - \mathfrak{K}_N)^{-1}\|_\infty \big(\|\mathfrak{K} - \mathfrak{K}_N\|_\infty \|(\mathcal{I} - \mathfrak{K})^{-1}\|_\infty \|f\|_\infty + \|f - p_{N,N}(f)\|_\infty\big) .
\end{aligned}$$

Taking into account (5.22), there exists a constant $c > 0$ such that

$$\|y - U_N\|_\infty \leq c(\|\mathfrak{K} - \mathfrak{K}_N\|_\infty + \|f - p_{N,N}(f)\|_\infty), \tag{5.24}$$

which converges to zero as $N \to \infty$, provided that both k and f are Dini-Lipschitz continuous functions with respect to all their variables. The above given analysis is taken as a justification of the following theorem:

Theorem 30. *Let k and f be Dini-Lipschitz continuous functions with respect to all their variables. Let $\|\mathfrak{K}\|_\infty < 1$. Then the approximate solution U_N obtained by solving system* (5.18)*, exists and is unique for a sufficiently large $N > \mathbf{N}$. The corresponding sequence of approximate solutions converges to the exact solution and the rate of the convergence depends on $\|\mathfrak{K} - \mathfrak{K}_N\|_\infty$ and $\|f - p_{N,N}(f)\|_\infty$, and can be estimated by the inequality in* (5.24)*.*

Theorem 31. *Let $\mathbf{k} \in (\mathcal{C}^4([0,T] \times [0,T]))^{2\times 2}$ and $f \in (\mathcal{C}^4([0,T] \times [0,1])^2$. Let $\|\mathfrak{K}\|_{\infty,1} < 1$. Then the system (5.18) has for sufficiently large N the unique solution U_N. This approximate solution converges to the exact solution as $N \to \infty$ and*

$$\|y - U_N\|_{\infty,1} \leq c(\|\mathfrak{K} - \mathfrak{K}_N\|_{\infty,1} + \|f - p_{N,N}(f)\|_{\infty,1}), \tag{5.25}$$

where c is a constant.

Proof. The conditions of Theorem 29 hold with $\mathcal{X} = (\mathcal{C}^1)^2$, hence $(\mathcal{I} - \mathfrak{K}_N)^{-1}$ is a bounded operator in $(\mathcal{C}^1)^2$. The rest of the proof is similar to the proof of Theorem 30. $\square$

Again, the important question is whether the approximate solution is a fuzzy function. Below, we propose sufficient conditions that guarantee that the discussed above approximate solution can be a fuzzy function. Let $\mathbf{k} \in (\mathcal{C}^4([0,T] \times [0,T]))^{2\times 2}$

and $F(t,r) = [\underline{f}(t,r), \overline{f}(t,r)]^T \in (\mathcal{C}^4([0,T] \times [0,1]))^2$. Let $\underline{f}(t,r)$ and $\overline{f}(t,r)$ be strictly increasing and respectively, strictly decreasing functions with respect to variable r. Then by the similar argumentation to that in the previous section we conclude that the exact solution $y(t,r) = [\underline{y}(t,r), \overline{y}(t,r)]^T \in (\mathcal{C}^1([0,T] \times [0,1]))^2$ and both of its components $\underline{y}(t,r)$ and $\overline{y}(t,r)$ are strictly monotone functions with respect to variable r. We know that $U_N = [\underline{U}_N(t,r), \overline{U}_N(t,r)]^T \in (\mathcal{C}^\infty([0,T] \times [0,1]))^2$ since they are polynomials. By Theorem 31 the convergence of U_N to y and of $\frac{\partial U_N(t,r)}{\partial r}$ to $\frac{\partial y(t,r)}{\partial r}$ is assured. Since $\frac{\underline{y}(t,r)}{\partial r} > 0$ and $\frac{\overline{y}(t,r)}{\partial r} < 0$, then for sufficiently large N we have $\frac{\underline{U}_N(t,r)}{\partial r} \geq 0$ and $\frac{\overline{U}_N(t,r)}{\partial r} \leq 0$. Therefore, $\underline{U}_N(t,r)(\overline{U}_N(t,r))$ is a monotonically increasing (decreasing) function. Consequently, both conditions (1) and (2) of Theorem 2 hold.

The analysis of condition (3) can be split into three cases. At first, if $\underline{y}(t,1) < \overline{y}(t,1)$, then $\underline{U}_N(t,1) \leq \overline{U}_N(t,1)$ for sufficiently large N. At second, if $\underline{y}(t,1) = \overline{y}(t,1)$, and N, $\underline{U}_N(t,1) \leq \overline{U}_N(t,1)$ for sufficiently large N, then condition (3) is fulfilled. Finally, assume that $\underline{y}(t,1) = \overline{y}(t,1)$, and $\underline{U}_N(t,1) > \overline{U}_N(t,1)$ for sufficiently large N. Let r_N be the infimum r such that $\underline{U}_N(t,r_N) > \overline{U}_N(t,r_N)$. We are speaking about the situation where the exact solution y (it is a fuzzy function) is unimodal at the moment t. Let such t be fixed. Then if starting from some r_N the sequence of approximating polynomials does not fulfill the requirement of being a fuzzy function, we shall "repair" it. In this case, we propose to use new forms of approximating polynomials:

$$\underline{U}_N^{\text{new}}(t,r) = \begin{cases} \underline{U}_N(t,r), & \text{if } 0 \leq r < r_N, \\ \frac{\underline{U}_N(t,r_N) + \overline{U}_N(t,r_N)}{2}, & \text{if } r_N \leq r \leq 1, \end{cases}$$

and

$$\overline{U}_N^{\text{new}}(t,r) = \begin{cases} \overline{U}_N(t,r), & \text{if } 0 \leq r < r_N, \\ \frac{\underline{U}_N(t,r_N) + \overline{U}_N(t,r_N)}{2}, & \text{if } r_N \leq r \leq 1. \end{cases}$$

However, $r_N \to 1$ as $N \to \infty$, because the r_N is determined by U_N and the latter converges to the unimodal function $y(t,r)$.

5.5 Numerical examples

In the following examples we illustrate our theoretical considerations and show the maximal values of error, using the following estimators:

$$\underline{E}(N,M) = \max_{(t,r) \in D_{100}} |\underline{U}_{NM}(t,r) - \underline{y}(x,y)|,$$

and

$$\overline{E}(N,M) = \max_{(t,r)\in D_{100}} |\overline{U}_{NM}(t,r) - \overline{y}(x,y)|,$$

where

$$D_M = \{(t_i, r_j)|\ t_i = ik,\ r_j = jh,\ i,j = 0,\dots,\ M,\ h = \frac{1}{M},\ \ k = \frac{T}{M}\}.$$

The selection of examples is complete within the class of solvable FFIEs: we consider various kernels (from smooth to sharp monotone or oscillating), source functions, and lengths of the time intervals. We see that the proposed approximation has satisfactory quality for all considered cases. In some of them, approximation coincides with the exact solution.

Example 9. Consider a class of fuzzy Fredholm integral equation of second kind (FFIE) with the strictly monotone kernel

$$k(t,s) = t^{\gamma}s^{\lambda},\ 0 \le s,t \le 1$$

and the fuzzy function f in its parametric form $f = (\underline{f}, \overline{f})$, where

$$\underline{f}(t,r) = (t^n - \frac{T^{n+\lambda+1}}{n+\lambda+1}t^{\gamma})2r,\ \ \overline{f}(t,r) = (t^n - \frac{T^{n+\lambda+1}}{n+\lambda+1}t^{\gamma})(3-r),$$

with free parameters n, γ and λ. The parametric form $y = (\underline{y}, \overline{y})$ of the exact solution is given by

$$\underline{y}(t,r) = (t^n)2r,\ \ \overline{y}(t,r) = (t^n)(3-r).$$

For various selection of the free parameters and degrees of approximating polynomials, we obtain approximate solutions and compare them with the exact one.

1. Let us specify the free parameters: $n = 1, \gamma = 1,\ \lambda = 1$. Then for $N = 1$ (the degree of an approximating polynomial) we have

$$K_1 = \begin{pmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{pmatrix},\quad K_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\underline{F} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ \ \overline{F} = \frac{1}{3}\begin{pmatrix} \frac{5}{6} & \frac{-1}{6} \\ \frac{5}{6} & \frac{-1}{6} \end{pmatrix}.$$

We obtain

$$\underline{U} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \overline{U} = \frac{1}{3}\begin{pmatrix} \frac{5}{4} & \frac{-1}{4} \\ \frac{5}{4} & \frac{-1}{4} \end{pmatrix}.$$

Hence the approximate solution $U_N = [\underline{U}_N, \overline{U}_N]^T$ is obtained by

$$\underline{U}_N(t,r) = \frac{1}{2}[1, 2t-1]\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2r-1 \end{pmatrix} = 2tr,$$

and similarly

$$\overline{U}_N(t,r) = t(3-r).$$

It is easy to see that this approximation coincides with the exact solution for the considered case $n = 1$.

2. Now, we select $\lambda = 2$ and examine for various choices of other two free parameters n, γ the error of the approximate solution for different values of N (degree of approximating polynomials). Tables 17 and 18 show that when $N > n$, the approximate solutions are almost exact (up to the floating-point relative accuracy of the MATLAB software).

Table 17 $\underline{E}_N$ for $N = 1, \ldots, 6$, with $\lambda = 2$

N	$n=2,\ \gamma=2$	$n=3,\ \gamma=2$	$n=4,\ \gamma=2$	$n=5,\ \gamma=2$
1	$1.6134e^{-00}$	$1.6762e^{-00}$	$1.7251e^{-00}$	$1.7670e^{-00}$
2	$2.5000e^{-01}$	$4.3066e^{-01}$	$5.6250e^{-01}$	$6.6791e^{-01}$
3	$4.4409e^{-16}$	$6.2500e^{-02}$	$1.3969e^{-01}$	$2.1582e^{-01}$
4	$1.1102e^{-15}$	$8.8818e^{-16}$	$1.5625e^{-02}$	$4.2852e^{-02}$
5	$1.9984e^{-15}$	$1.7764e^{-15}$	$1.7764e^{-15}$	$3.9062e^{-03}$
6	$4.4409e^{-16}$	$6.6613e^{-16}$	$6.6613e^{-16}$	$8.8818e^{-16}$

Table 18 $\overline{E}_N$ for $N = 1, \ldots, 6$, with $\lambda = 2$

N	$n=2,\ \gamma=2$	$n=3,\ \gamma=2$	$n=4,\ \gamma=2$	$n=5,\ \gamma=2$
1	$2.0336e^{-00}$	$2.1904e^{-00}$	$2.3127e^{-00}$	$2.4175e^{-00}$
2	$3.7500e^{-01}$	$6.4600e^{-01}$	$8.4375e^{-01}$	$1.0019e^{-00}$
3	$1.7764e^{-15}$	$9.3750e^{-02}$	$2.0954e^{-01}$	$3.2373e^{-01}$
4	$2.6645e^{-15}$	$2.2204e^{-15}$	$2.3438e^{-02}$	$6.4279e^{-02}$
5	$3.1086e^{-15}$	$2.6645e^{-15}$	$3.1086e^{-15}$	$5.8594e^{-03}$
6	$4.8850e^{-15}$	$4.8850e^{-15}$	$5.3291e^{-15}$	$5.3291e^{-15}$

Example 10. Consider a second kind fuzzy Fredholm integral equation with the

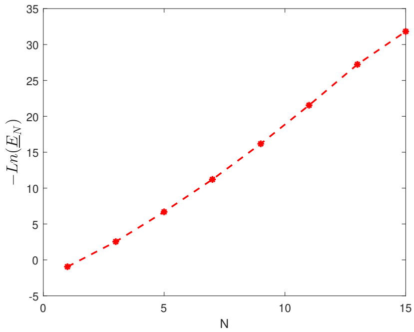


Figure 4: The negative logarithm of the maximal error versus N in Example 10.

Table 19 The maximal error estimate in Example 10 for various N.

N	4	6	8	10	12	14
$\underline{E}_N$	$5.8191e^{-2}$	$1.2064e^{-3}$	$1.3504e^{-5}$	$9.3500e^{-8}$	$4.4019e^{-10}$	$1.5041e^{-12}$
$\overline{E}_N$	$1.1638e^{-1}$	$2.4128e^{-3}$	$2.7007e^{-5}$	$1.8700e^{-7}$	$8.8040e^{-10}$	$2.9924e^{-12}$

smooth, non-monotone kernel

$$k(t,s) = \sin(s)\sin(t), \ 0 \le s,t \le \frac{\pi}{2},$$

and the fuzzy function f in its parametric form $f = (\underline{f}, \overline{f})$, where

$$\underline{f}(t,r) = (1-\frac{\pi}{4})(\sin(t))(r^2+r), \quad \overline{f}(t,r) = (1-\frac{\pi}{4})(\sin(t))(4-r^3-r).$$

The parametric form $y = (\underline{y}, \overline{y})$ of the exact solution is given by

$$\underline{y}(t,r) = \sin(t)(r^2+r), \quad \overline{y}(t,r) = \sin(t)(4-r^3-r).$$

We compute the approximate solutions for various N. The results in Table 19 and Figure 4 confirm Theorem 29. In Figure 5(a-d), the numerical and exact solutions for some particular values of t and r are exhibited.

Example 11. Consider FFIE with the *smooth and monotone* kernel

$$k(t,s) = s+t, \ 0 \le s,t \le \frac{1}{2}$$

and the fuzzy function f in its parametric form $f = (\underline{f}, \overline{f})$, where

$$\underline{f}(t,r) = \begin{cases} e^{tr} - \frac{1}{r^2}(re^r - e^r + 1) - \frac{t}{r}(e^r - 1), & r \in (0,1], \\ 0.5 - t, & r = 0, \end{cases}$$

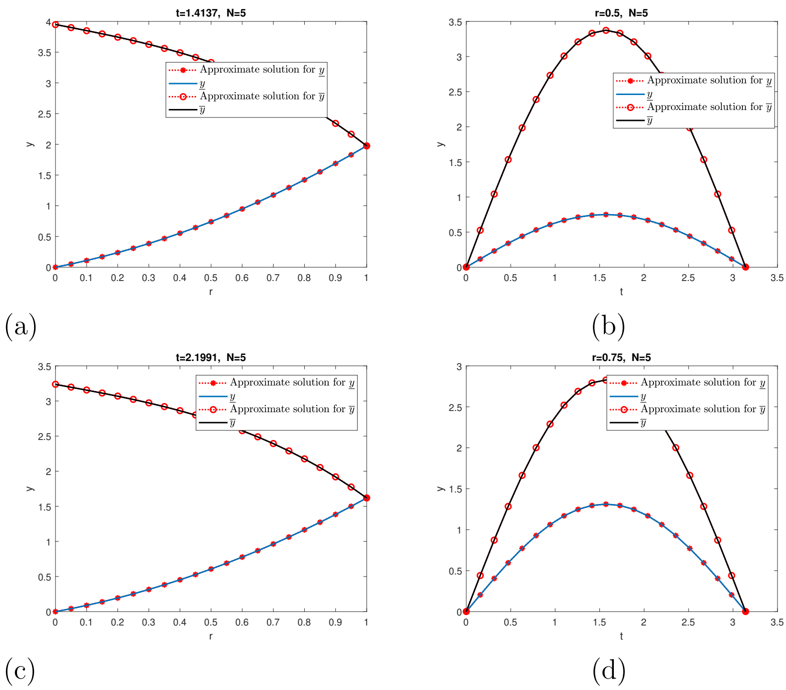


Figure 5: Exact and approximate solutions of Example 10.

$$\overline{f}(t,r) = 4^t - r^2 + 1 - t - \frac{3t}{2\ln(2)} + r^2 t + \frac{r^2}{2} - \frac{\ln(4) - 3/4}{\ln(2)^2} - \frac{1}{2}.$$

The parametric form $y = (\underline{y}, \overline{y})$ of the exact solution is given by

$$\underline{y}(t,r) = e^{tr}, \quad \overline{y}(t,r) = 4^t - r^2 + 1.$$

The results of Table 20 and Figure 6 confirm the theoretical results. Figures 7(a-b) show the numerical and exact solution for some values of t and r.

Table 20 The maximal error estimate in Example 11.

N	2	4	6	8	10
$\underline{E}_N$	$2.2301e^{-01}$	$1.1475e^{-03}$	$2.3469e^{-06}$	$2.5883e^{-09}$	$2.3991e^{-11}$
$\overline{E}_N$	$4.7598e^{-01}$	$2.7354e^{-03}$	$1.0415e^{-05}$	$2.1869e^{-08}$	$2.8856e^{-11}$

Example 12. Consider FFIE with the *low-amplitude oscillating* kernel $k(t,s) = 0.1\sin(t)\sin(s)$ on an extended domain $0 \leq s,t \leq 2\pi$ and the fuzzy function $f = (\underline{f}, \overline{f})$, where

$$\underline{f}(t,r) = \frac{1}{15}(13(r^2 + r) + 2(4 - r^3 - r))\sin(\frac{t}{2}),$$

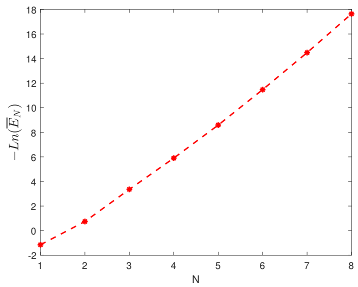
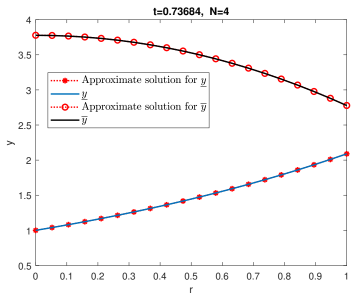
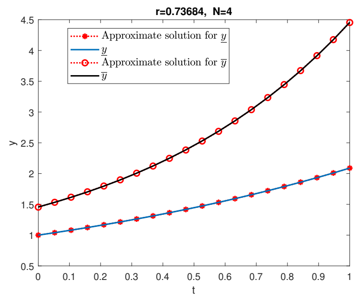


Figure 6: The negative logarithm of the maximal error versus N in Example 11.



(a)



(b)

Figure 7: Exact and approximate solutions of Example 11.

$$\overline{f}(t,r) = \frac{1}{15}(2(r^2+r)+13(4-r^3-r))\sin(\frac{t}{2}).$$

The parametric form $y = (\underline{y}, \overline{y})$ of the exact solution is given by

$$\underline{y}(t,r) = \sin(\frac{t}{2})(r^2+r), \quad \overline{y}(t,r) = \sin(\frac{t}{2})(4-r^3-r).$$

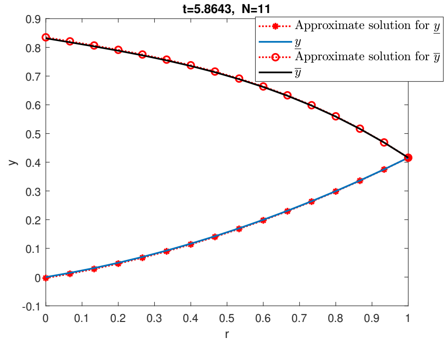
In Table 21, we report the maximal error for various values of N. Figures 8(a-b) show the numerical and exact solution for some values of t and r.

Table 21 The maximal error estimate in Example 12.

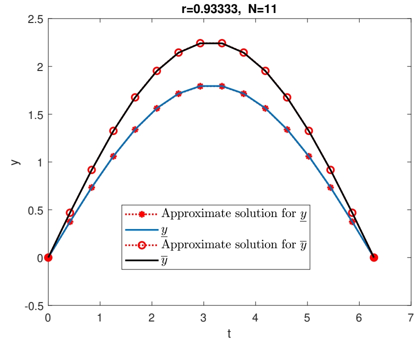
N	2	12	22	32	52
$\underline{E}_N$	$1.1669e^{-01}$	$1.1669e^{-02}$	$3.8712e^{-03}$	$1.9166e^{-03}$	$7.5692e^{-04}$
$\overline{E}_N$	$2.6085e^{-01}$	$1.1669e^{-02}$	$3.8712e^{-03}$	$1.9166e^{-03}$	$7.5692e^{-04}$

Example 13. Consider FFIE with the *sharp oscillating* kernel $k(t,s) = \frac{1}{2}e^s \sin t$, $0 \leq s,t \leq 1$ and the fuzzy $f = (\underline{f}, \overline{f})$, where

$$\underline{f}(t,r) = (r+1)(e^{-t}+t-\sin(t)),$$



(a)



(b)

Figure 8: Exact and approximate solutions of Example 12.

$$\overline{f}(t,r) = (3-r)(e^{-t} + t - \sin(t)).$$

This problem has been solved by Adomian Decomposition Methods in [4]. The exact solution is given by

$$\underline{y}(t,r) = (r+1)(e^{-t} + t), \quad \overline{y}(t,r) = (3-r)(e^{-t} + t).$$

The maximal error for the first five numbers of N is reported in Table 22. As expected, the results show efficiency of the proposed method.

Table 22 The maximal error estimate in Example 13.

N	1	2	3	4	5
$\underline{E}_N$	$9.8133e^{-01}$	$8.4259e^{-02}$	$6.8395e^{-03}$	$4.2068e^{-04}$	$2.0812e^{-05}$
$\overline{E}_N$	$1.1796e^{+00}$	$1.2639e^{-01}$	$1.0259e^{-02}$	$6.3102e^{-04}$	$3.1218e^{-05}$

When observing and analyzing the obtained numerical results, we see that the lowest errors correspond to the cases where kernels are smooth and do not abruptly change their behavior, i.e. do not oscillate or are strictly monotone. Another parameter that influences the error range is the length of the time interval: the shorter - the better.

Chapter 6

Conclusion

In this thesis, we proved a regularity result for solution of fuzzy Volterra integral equations. If the kernel changes sign, then the solution is not smooth in general. We proposed collocation method with triangular and rectangular basis functions for solving these equations. The advantage of these methods is simplicity of use and robustness, i.e. they do not require high regularity of the solution. If the solution is not smooth, then many other methods are not applicable, especially those which use Taylor expansions or high order polynomials to approximate the solution. Using the regularity result we estimated the order of convergence of these methods.

We also investigated fuzzy Volterra integral equations with weakly singular kernels. The existence, regularity and the fuzziness of the exact solution is studied. Collocation methods on discontinuous piecewise polynomial space are proposed. A convergence analysis is given. The fuzziness of the approximate solution is investigated. Both the analysis and numerical methods show that graded mesh is better than uniform mesh for these problems.

We proposed a new numerical method for solving fuzzy Fredholm integral equations of the second kind. This method is based on approximation of all functions involved by Chebyshev polynomials. We analyzed the existence and uniqueness of both exact and approximate fuzzy solutions. We proved the convergence and fuzziness of the approximate solution.

Bibliography

[1] S. Abbasbandy, E. Babolian, M Alavi, Numerical method for solving linear Fredholm fuzzy integral equations of the second kind. Chaos, Solitons and Fractals, 31(1) (2007) 138-146.

[2] Z. Alijani, U. Kangro, Collocation method for fuzzy Volterra integral equations of the second kind, Mathematical Modelling and Analysis, 25(1) (2020) 146-166.

[3] Z. Alijani, U. Kangro, Fuzzy Volterra integral equation with weakly singular kernel, (to be submitted).

[4] M. Amawi, N. Qatanani, Analytical methods for solving fuzzy Fredholm integral equation of the second kind, Research Journal of Applied Science, 11 (2016) 1159 - 1568.

[5] K. Atkinson, W. Han, Theoretical numerical analysis, Springer 39, 2009.

[6] K. Atkinson, The numerical solution of integral equations of the second kind, Cambridge university press, 1997.

[7] D. Baleanu, B. Shiri, H. M. Srivastava, M. Al Qurashi, A Chebyshev spectral method based on operational matrix for fractional differential equations involving non-singular Mittag-Leffler kernel, Advances in Difference Equations, 353 (2018) 1-23.

[8] B. Bede, Mathematics of Fuzzy Sets and Fuzzy Logic, volume 295 of Studies in Fuzziness and Soft Computing, Springer, 2013.

[9] B. Bede and S. G. Gal, Quadrature rules for integral of fuzzy number-valued functions, Fuzzy Sets and System, 145 (2004) 359-380.

[10] A. M. Bica, Error estimation in the approximation of the solution of nonlinear fuzzy Fredholm integral equations, Information Sciences, 178(5) (2008) 1279-1292.

[11] H. Brunner, Volterra integral equations, An introduction to theory and applications, Cambridge Monographs on Applied Computational Mathematics, Cambridge University Press, 2017.

[12] H. Brunner, Collocation methods for Volterra integral and related functional differential equations, Cambridge University Press, 2004.

[13] H. Brunner, A. Pedas, G. Vainikko, The piecewise polynomial collocation methods for nonlinear weakly singular Volterra equations, Mathematics of Computing, 68 (1999) 1079-1095.

[14] H. Brunner, A. Pedas, G. Vainikko, Piecewise polynomial collocation methods for linear Volterra integro-differential equations with weakly singular kernels, SIAM Journal on Numerical Analysis, 39 (2001) 957-982.

[15] M. M. Chawla, Error estimates for the Clenshaw-Curtis quadrature, Mathematics of Computation, 22 (1986) 651-656.

[16] J.M. Cushing, A strong ergodic theorem for some nonlinear matrix models for the dynamics of structured populations, Natural Resource Modelling, 3 (1989) 331–357.

[17] C. M. Dafermos, An abstract Volterra equation with application to linear viscoelasticity, Journal of Differential Equation, 7 (1970) 554-589.

[18] P. Diamond, P. Kloeden, Metric topology of fuzzy numbers and fuzzy analysis. In: Dubois, D., Prade, H., et al. (eds.) Handbook Fuzzy Sets Ser., vol. 7, pp. 583–641. Kluwer Academic Publishers, Dordrecht (2000).

[19] D. Dubois and H. Prade, Operations on fuzzy numbers, International Journal of Systems Science, 9 (1978) 613-626.

[20] D. Dubois, H. Prade, Towards fuzzy differential calculus, Fuzzy Sets and Systems, 8 (1982) 1-7.

[21] M. Friedman, Ma Ming, A. Kandel, Solution to the fuzzy integral equations with arbitrary kernels, International Journal of Approximate Reasoning, 20 (1999) 249-262.

[22] M. Friedman, Ma Ming, A. Kandel, Numerical solutions of fuzzy differential and integral equations, Fuzzy sets and Systems, 106 (1) (1999) 35-48.

[23] R. Goetschel, W. Voxman, Elementary calculus, Fuzzy Sets and Systems, 18 (1986) 31-43.

[24] M. Hukuhara, Integration des applications measurables dont la valeur est un compact convexe, Funkc. Ekvacioj 10 (1967) 205–223.

[25] J. Janno, L. Von Wolfersdorf, Identification of weakly singular kernels in heat conduction, Journal of Applied Mathematics and Mechanics, 77 (1997) 141-164.

[26] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems, 24 (1987) 301-317.

[27] O. Kaleva, The Cauchy problem for fuzzy differential equations, Fuzzy Sets and Systems, 35 (1990) 389-396.

[28] M. Kolk, A. Pedas, Numerical solution of Volterra integral equations with singularities, Frontiers of Mathematics in China, 8 (2) (2013) 239-259.

[29] M. Kolk, A. Pedas, G. Vainikko, High order methods for Volterra integral equations with general weak singularities, Numerical Functional Analysis and Optimization, 30 (2009) 1002-1024.

[30] R. Kress, Linear integral equations, Applied Mathematical Sciences, Springer, 2014.

[31] J. C. Mason, D. C. Handscomb, Chebyshev polynomials, CRC Press, 2002.

[32] G. Mittag-Leffler, Sur la representation analytiqie d'une fonction monogene cinquieme note, Acta Math. 1905, 29 (1) 10-181.

[33] M. Mosleh, M. Otadi, Solution of fuzzy Volterra integral equations in a Bernstein polynomial basis. Journal of Advances in Information Technology, 4(3)(2013)148- 155.

[34] J. M. Ortega, Numerical analysis: A second course, New York, Academic press, 1972.

[35] J. Y. Park, H. K. Han. Existence and uniqueness theorem for a solution of fuzzy Volterra integral equations, Fuzzy Sets and Systems, 105 (3) (1999) 481-488.

[36] J. Y. Park, Y. C. Kwun, J. U. Jeong, Existence of solutions of fuzzy integral equations in Banach spaces, Fuzzy Sets and Systems, 72 (1995) 373-378.

[37] J. Y. Park, S. Y. Lee, J. U. Jeong, The approximate solutions of fuzzy functional integral equations, Fuzzy Sets and Systems, 110 (2000) 79-90.

[38] A. Pedas, G. Vainikko, Superconvergence of piecewise polynomial collocation for nonlinear weakly singular integral equations, Journal of Integral Equations and Applications, 9 (1997) 379-406.

[39] A. Pedas, G. Vainikko, On the regularity of solutions to integral equations with nonsmooth kernel on union of open intervals, Journal of Computational and Applied Mathematics, 229 (2009) 440-451.

[40] M. L. Puri, D.A. Ralescu, Differentials for fuzzy functions, Journal of Mathematical Analysis and Applications, 91 (1983) 552 - 558.

[41] F. Saberidad, S. M. Karbassi, M. Heydari, Numerical solution of nonlinear fuzzy Volterra integral equations of the second kind for changing kernels, Soft Computing, 21 (2019).

[42] P. K. Sahu, S. Saha Ray, A new Bernoulli wavelet method for accurate solutions of nonlinear fuzzy Hammerstein-Volterra delay integral equations. Fuzzy Sets and Systems, 309 (2017) 131-144.

[43] S. Salahshou, T. Allahviranloo, Application of fuzzy differential transform method for solving fuzzy Volterra integral equations, Applied Mathematical Modelling, 37 (3) (2013) 1016-1027.

[44] B. Shiri, I. Perfilieva, Z. Alijani, Classical approximation for fuzzy Fredholm integral equation, Fuzzy Sets and Systems, 404 (2021) 159-177.

[45] S. Tomasiello, J. E. Macías-Díaz, A. Khastan, Z. Alijani, New sinusoidal basis functions and a neural network approach to solve nonlinear Volterra–Fredholm integral equations, Neural Computing and Applications, 31(2019) 4865–4878.

[46] L. N. Trefethen, Is Gauss quadrature better than Clenshaw-Curtis, SIAM review, 50 (2008) 67-87.

[47] G. Vainikko, Multidimensional weakly singular integral equations, Springer, 1993.

[48] G. Vainikko, Weakly singular integral equations, Lecture notes, HUT, UT, 2006-2007, http://kodu.ut.ee/ gen/WSIElecturesSIAM.pdf.

[49] L. A. Zadeh, Fuzzy sets, Information and control, 8 (1965) 338-353.

[50] L. Zhu, Y. Wang, Numerical solutions of Volterra integral equation with weakly singular kernel using SCW method, Applied Mathematics and Computation, 260 (2015) 63-70.

Sisukokkuvõte

Hägusad teist liiki integraalvõrrandid

Paljude erinevate teadusalade mudelid on kirjeldatavad integraalvõrrandite abil. Mudelites esinevad tihti parameetrid, mis on teada ainult ligikaudu. Üks võimalus seda väljendada on kasutada hägusaid arve tavaliste reaalarvude asemel. Hägusad arvud on erijuht hägusatest hulkadest. Hägusate arvude hulga jaoks saab defineerida tehted ning meetrika, kuid liitmine ja korrutamine pole üldjuhul pööratavad. Hägusateks funktsioonideks nimetatakse funktsioone, mille väärtused on hägusad arvud. Hägusaid funktsioone saab esitada ka parameetrilisel kujul, ülemise ja alumise funktsiooni kaudu, mis on reaalväärtustega kahe muutuja funktsioonid. Käesolevas doktoritöös uurime integraalvõrrandeid, milles esinevad hägusad funktsioonid.

Võrrandit

$$g(t) = f(t) + \int_0^t K(t,s)g(s)ds, \quad t \in [0,T], \tag{6.1}$$

kus antud on piirkonnal $D_T = \{(t,s)\colon 0 \leq s \leq t \leq T\}$ määratud integraalvõrrandi tuum $K\colon D_T \to \mathbb{R}$ ja hägus funktsioon f, nimetatakse teist liiki hägusaks Volterra integraalvõrrandiks (HVIV). Siin g on otsitav funktsioon. Kui f on tavaline reaalväärtustega (mitte hägus) funktsioon, siis võrrandil (6.1) on tavaline lahend g, kui aga f on hägus funktsioon, siis lahend g on hägus. Pideva tuumaga HVIV korral on teada pideva hägusa lahendi olemasolu ja ühesus. Seni pole lahendi siledust eraldi uuritud. Juhul kui integraalvõrrandi tuum säilitab märki, saab lahendi sileduse (s.t. ülemise ja alumise funktsiooni sileduse) tulemused järeldada tavalise Volterra integraalvõrrandi lahendi sileduse tulemustest, kuid märki muutva tuuma korral on olukord keerulisem ning seda on uuritud käesolevas töös.

Numbrilisi meetodeid HVIV jaoks on vaadeldud paljudes artiklites, kuid paljudel juhtudel ei ole tõestatud, et ligikaudne lahend on hägus funktsioon. Mõnel juhul võib see olla triviaalne, kuid üldiselt see ei pruugi kehtida. Koonduvuskiirust pole tavaliselt tõestatud. Märki muutva tuumaga HVIV puhul on vaadeldud vaid kitsast erijuhtu ning lahendi siledust ega meetodi koonduvuskiirust pole tõestatud.

Peatükis 3 anname tingimused hägusa lahendi alumise ja ülemise funktsiooni sileduseks. Võrrandi ligikaudseks lahendamiseks vaatleme kollokatsioonimeetodit kasutades nii kolmnurkseid kui ka ristkülikukujulisi baasfunktsioone. Lahendi sileduse tulemused võimaldavad leida meetodite koonduvuskiirused.

Peatükis 4 vaatleme nõrgalt singulaarse tuumaga teist liiki hägusat Volterra integraalvõrrandit (NSHVIV) kujul (6.1), kus K on piirkonnal $D_T = \{(t,s)\colon 0 \leq s < t \leq T\}$ määratud nõrgalt singulaarne tuum, millel võib olla iseärasus joonel $t = s$. Esitame selle võrrandi hägusa lahendi olemasolu ja ühesuse teoreemi, samuti tulemused selle sileduse kohta, mis on meie teada uued. Uurime juhtu, kus tuum muudab märki horisontaalsetel ja/või vertikaalsetel joontel. Ligikaudse meetodina vaatleme katkevate splainidega kollokatsioonimeetodit ebaühtlasel võrgul. Näitame, et kollokatsioonisõlmede arvu suurenemisel lähislahendid koonduvad võrrandi lahendiks ja tõestame tulemused koonduvuskiiruse kohta. Eraldi tähelepanu pööratakse lähislahendi hägususe tõestamisele.

Võrrandit

$$y(t) = f(t) + \int_0^T K(t,s)y(s)ds, \quad t \in [0,T],$$

kus $K\colon [0,T] \times [0,T] \to \mathbb{R}, T > 0$ ja f on hägus funktsioon, nimetatakse teist liiki hägusaks Fredholmi integraalvõrrandiks (HFIV). Sileda tuuma korral lähendame tuuma ning vabaliiget Tšebõsovi polünoomidega ning lahendame saadud kõdunud tuumaga integraalvõrrandi täpselt. Kui tuum on sile ning ei muuda märki, siis meetod koondub väga kiiresti. Need tulemused on esitatud peatükis 5.

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Z. Alijani, Fuzzy integral equations, University of Ostrava, Institute for research and application of fuzzy modelling, IRAFM, February 2020.

Z. Alijani, A Chebyshev based numerical methods for solving fractional fuzzy differential equations, International conference on mathematical modelling in applied sciences, ICMMAS'19, Belgorod, Russia, August 2019.

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A. Khastan, Z. Alijani, I. Perfilieva, Fuzzy transform to approximate solution of two-point boundary value problems, Mathematical and computational modelling in science and technology, Izmir, Turkey, August 2015.

List of Publications

1 Z. Alijani, U. Kangro, Fuzzy Volterra integral equation with weakly singular kernel, (to be submitted).

2 B. Shiri, I. Perfilieva, Z. Alijani, Classical approximation for fuzzy Fredholm integral equation, Fuzzy Sets and Systems, 404, 159-177, 2021.

3 Z. Alijani, U. Kangro, Collocation Method for Fuzzy Volterra Integral Equations of the Second Kind, Mathematical Modelling and Analysis, 25(1), 146-166, 2020.

4 Z. Alijani, D. Baleanu, B. Shiri, Spline collocation methods for systems of fuzzy fractional differential equations, Chaos, Solitons and Fractals, G. C. Wu, Chaos, Solitons and Fractals, 131, 109510, 2020.

5 S. Tomasiello, J. E. Macias-Diaz, A. Khastan, Z. Alijani, New sinusoidal basis functions and a neural network approach to solve nonlinear Volterra-Fredholm integral equations, Neural Computing and Applications, 31 (9), 4865–4878, 2019.

6 A. Khastan, Z. Alijani, On the new solutions to the fuzzy difference equation $x_{n+1} = A + \frac{B}{x_n}$, Fuzzy Sets and Systems, 358, 64-83, 2019.

7 Z. Alijani, A. Khastan, S. K. Khattri, S. Tomasiello, Fuzzy transform to approximate solution of boundary value problems via optimal coefficients, Publisher IEEE, 466-471, 2017.

8 A. Khastan, Z. Alijani, I. Perfilieva, Fuzzy transform to approximate solution of two-point boundary value problems, Mathematical Methods in the Applied Sciences 40, 6147-6154, 2017.

9 A. Khastan, I. Perfilieva, Z. Alijani, A new fuzzy approximation method to Cauchy problems by fuzzy transform, Fuzzy Sets and Systems, 288, 75-95, 2016.

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