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LINEAR PREDICTION APPROACH TO THE ROBUST PARAMETER ESTIMATION FOR THE DAMPED SINUSOIDS

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ABSTRACT

In this paper, we focus on the problem of parameter estimation for the damped sinusoids, which are corrupted by impulsive noise. To provide a robust initial guess for the current parameter estimators, the robust weighted linear prediction (RWLP) estimator is developed, where the parameter estimates are obtained by minimizing the weighted ℓ_p-norm of the linear prediction (LP) error vector. The Markov optimum weighting matrix is derived, and an iteratively reweighted least-squares (IRLS) procedure is devised to calculate the LP coefficient estimates. The simulation results demonstrate the robustness of the RWLP estimator by comparing with the ℓ_2-norm based counterpart, and the computational efficiency by comparing with the ℓ_p-MUSIC algorithm.

Index Terms— Robust parameter estimation, damped sinusoids, weighted ℓ_p-norm, Markov optimum weighting, impulsive noise.

1. INTRODUCTION

Spectral analysis of sinusoidal signals [1] has been a classical but ever active topic in the signal processing community, finding its applications in a wide range of areas. For example, in music and voiced speech signal processing, the measured signals from the microphone array can be modeled as a two-dimensional (2-D) harmonic signal [2], which is characterized by the temporal fundamental frequencies and directions-of-arrival (DoAs). The accurate acquisition of these parameters is crucial to the signal enhancement and source localization. In the biomedical engineering, the free induction decay (FID) signal, which is measured using the spectroscopic methods such as the nuclear magnetic resonance (NMR) and nuclear quadrupole resonance (NQR), may be modeled as a sum of exponentially damped sinusoids well [3]. With the exact quantification of these damped sinusoids, it is advantageous to extract the useful biomedical information for diagnosis.

The work of spectral analysis includes two aspects: 1) the detection of the signal order, that is the number of the signal's sinusoids; and 2) the estimation of the frequencies and/or damping factors. In this paper, we focus on the parameter estimation for the damped sinusoidal signals. During the past decades, there has been published a lot of literature on this problem. It is known that the maximum likelihood estimator (MLE) is statistically optimal [4], whereas it requires enormous computational cost in the multi-dimensional search. To lower the computational complexity, several kinds of computationally efficient techniques have been developed for the parameter estimation, such as the subspace-based algorithms [5–7] and the linear prediction (LP)-based methods [8–10]. However, in most of the above work, the background noise is assumed as white Gaussian.

Impulsive noise (sometimes named "outlier") is an important class of disturbance in the signal measurement [11]. It follows a non-Gaussian and heavy-tailed distribution, and occurs randomly with a value several times larger than the standard deviation of the background noise. To the best of our knowledge, the conventional parameter estimation methods, which are designed for the white Gaussian noise, perform worse in the impulsive noise environment.

In the recent years, there appear several schemes for the spectral analysis, which are based on the dictionary learning [12, 13]. However, there is a lack of the solid theoretical foundation to construct the dictionary for the damped sinusoidal signals. Up to now, there are mainly three categories of methods, which have the potential to estimate the parameters of the damped sinusoids in the impulsive noise environment. The first category is to perform the robust estimation based on the fractional lower-order statistics (FLOS). To overcome the vulnerability of the second-order covariance matrix to the impulsive noise, the FLOS such as the robust covariation matrix (RCM) [14], fractional lower order moment (FLOM) [15], sign covariance matrix (SCM), Kendall's tau covariance matrix (TCM) [16], etc., has been utilized to develop the robust versions of the MUSIC-based parameter estimation algorithms. However, the reliability of these methods depends on the large sample size [14], which is normally not available in the time-varying systems. In the second category of methods [11,17–19], the second-order covariance matrix is firstly estimated in a robust way. Based on this, the parameter estimates are solved with the common subspace-based algorithms. For example, the ℓ_p-MUSIC algorithm is proposed in [19], where the signal subspace is extracted directly from the observation by means of the robust singular value decomposition. In the third category, the sinusoidal parameter estimates are found by minimizing the ℓ_p-norm ($1 \leq p < 2$) of the fitting error between the signal model and observation [20–22]. Since the ℓ_p-norm is less sensitive to the outliers than the ℓ_2-norm, the parameter estimation is expected to be more resistant to the impulsive noise.

The extensive simulation results show the robustness and accuracy of the ℓ_p-norm based parameter estimation [19–22] in impulsive noise environments. Nevertheless, the high computational burden is a critical problem of these methods, especially for the damped sinusoids. For example, in [19], the low-rank decomposition is employed to extract the signal subspace in an iterative way. At each iteration, it is necessary to solve two convex optimization subproblems with the dimension of $\mathcal{O}(N)$ (with N being the data length). In addition, the damped sinusoids are characterized by the two-dimensional parameters, that is the damping factor and frequency. Thus, it is challenging to find a robust and accurate initial guess quickly for the ℓ_p-norm based methods such as [19–22].

In this work, we address this issue by means of the LP technique. With good computational efficiency, the LP-based estimators are extensively applied in the spectral analysis [10]. However, in the conventional LP-based estimators, it is aimed to estimate the LP coefficients by minimizing the ℓ_2-norm of the LP error vector, which is vulnerable to the impulsive noise. To alleviate this difficulty, we propose to estimate the LP coefficients by minimizing the weighted ℓ_p-norm of the LP error vector. Since the weighting matrix is dependent on the unknown sinusoidal parameters, and there is no closed-form solution to the LP coefficient estimates, an iteratively reweighted least-squares (IRLS) procedure is devised. In this procedure, the estimates of the LP coefficients are calculated just from several successive least-squares solutions. We term this approach as the robust weighted LP (RWLP) estimator.

The rest of this paper is organized as follows. The signal model and problem formulation are introduced in Section 2. Then, the RWLP estimator is developed in Section 3. The theoretical performance analysis of the RWLP estimator is also provided. The simulation results are presented in Section 4 to evaluate the performance of the proposed parameter estimator. Finally, the conclusion is drawn in Section 5.

2. PROBLEM FORMULATION

Consider the damped sinusoids, which are corrupted by impulsive noise, as follows:

$$y(n) = x(n) + v(n) = \sum_{m=1}^{M} \rho_m z_m^n + v(n), \tag{1}$$

for $n = 1, \cdots, N$, where $z_m = e^{-\beta_m + j\omega_m}$ represents the mth sinusoidal pole, with $\omega_m \in [0,\ 2\pi)$, $\beta_m \geq 0$, and $\rho_m \in \mathbb{C}$ being the frequency, damping factor, and complex-valued amplitude of z_m, respectively; and M represents the signal order, that is the number of the sinusoidal poles. Note that the signal model of (1) also covers the undamped sinusoids with $\beta_m = 0$. Here, $v(n)$ is the additive impulsive noise, and is assumed as independent and identically distributed (i.i.d.).

The purpose of this paper is to estimate the sinusoidal parameters, that is $\{(\omega_m, \beta_m)\}_{m=1}^{M}$, of the damped sinusoids $x(n)$ from the observation $y(n)$. In the parameter estimation, it is a critical issue to overcome the outliers in a computationally efficient way.

3. ALGORITHM DEVELOPMENT

In this section, the RWLP estimator is developed for the damped sinusoids. Firstly, we construct the LP error vector, and derive the Markov optimum weighting matrix. The LP coefficients are estimated by minimizing the weighted ℓ_p-norm of the LP error vector. Since there is no closed-form solution to this optimization problem, an IRLS procedure is devised. The detail is illustrated as follows.

3.1. Sinusoidal Parameter Estimation with the RWLP

First of all, the following LP equation is established, which is based on the LP property of the sinusoidal signals [8–10]:

$$x(n) + \sum_{m=1}^{M} a_m x(n-m) = 0, \tag{2}$$

for $n = M+1, \cdots, N$, where $\{a_m\}_{m=1}^{M}$ are the LP coefficients. When the signal is contaminated with noise, the LP equation is not satisfied exactly, and there exist the LP errors of:

$$\mathbf{e} = \mathbf{Y}\mathbf{a} - \mathbf{b}, \tag{3}$$

where $\mathbf{a} = [a_1,\ \cdots,\ a_M]^T$ is the LP coefficient vector, and

$$\mathbf{Y} = \text{Toeplitz}\big(\big[y(M),\quad y(M+1),\quad \cdots,\quad y(N-1)\big]^T, \big[y(M),\quad y(M-1),\quad \cdots,\quad y(1)\big] \big), \tag{4}$$

$$\mathbf{b} = -\big[y(M+1),\quad y(M+2),\quad \cdots,\quad y(N)\big]^T, \tag{5}$$

with $\text{Toeplitz}(\mathbf{c}_1, \mathbf{c}_2^T)$ denoting the Toeplitz matrix with $\mathbf{c}_1$ and $\mathbf{c}_2^T$ as the first column and first row, respectively.

Under the assumption of the white-Gaussian noise, $\mathbf{a}$ is estimated normally by minimizing the weighted ℓ_2-norm of the LP error vector [10]:

$$\hat{\mathbf{a}} = \arg\min_{\mathbf{a}} \mathbf{e}^H \mathbf{W} \mathbf{e}, \tag{6}$$

where $\mathbf{W}$ is the Markov optimum weighting matrix defined as [23]:

$$\mathbf{W} = \sigma_v^2 \cdot \left[E\left\{ \mathbf{e}\mathbf{e}^H \right\} \right]^{-1} = \left(\mathbf{A}_0 \mathbf{A}_0^H \right)^{-1}, \tag{7}$$

with σ_v^2 being the noise variance of $v(n)$, and

$$\mathbf{A}_0 = \mathrm{Hankel}\big([\mathbf{0}_{N-M-1}^T,\ 1]^T, \\ [1,\ a_1,\ \cdots,\ a_M,\ \mathbf{0}_{N-M-1}^T] \big).$$

Here, $\mathrm{Hankel}(\mathbf{c}_1, \mathbf{c}_2^T)$ denotes the Hankel matrix with $\mathbf{c}_1$ and $\mathbf{c}_2^T$ as the first column and last row, respectively.

It is known that the ℓ_2-norm based weighted linear prediction (WLP) estimator is not resistant to the impulsive noise [19]. To enhance the robustness of the WLP estimator, it is proposed to substitute the ℓ_p-norm ($1 \le p < 2$) for the ℓ_2-norm as [20–22]. Accordingly, the LP coefficient vector $\mathbf{a}$ is estimated by:

$$\hat{\mathbf{a}} = \arg\min_{\mathbf{a}} \|\mathbf{e}\|_{\mathbf{W},p}^p, \tag{8}$$

where the weighted ℓ_p-norm of $\mathbf{e}$ is defined as (see (9) at the bottom of the next page):
Here, $\overline{[\mathbf{Ya}-\mathbf{b}]_i}$ stands for the complex conjugate of $[\mathbf{Ya}-\mathbf{b}]_i$, and $\mathbf{P}$ is computed as

$$\mathbf{P} = \mathrm{diag}\Big(\big[\, |[\mathbf{Ya}-\mathbf{b}]_1|^{p-2}, \cdots, \\ |[\mathbf{Ya}-\mathbf{b}]_{N-M}|^{p-2} \,\big]^T \Big). \tag{10}$$

Given $\mathbf{W}$ and $\mathbf{P}$, the solution to (8) is:

$$\hat{\mathbf{a}} = \left(\mathbf{Y}^H \mathbf{P} \mathbf{W} \mathbf{Y} \right)^{-1} \cdot \left(\mathbf{Y}^H \mathbf{P} \mathbf{W} \mathbf{b} \right). \tag{11}$$

Nevertheless, the LP coefficient vector $\mathbf{a}$ is unknown *a priori*, and the matrices $\mathbf{P}$ and $\mathbf{W}$ are unavailable before the estimation. Instead, we devise the IRLS procedure to solve $\hat{\mathbf{a}}$ of (8), which is detailed as follows:

- Step 1. Determine the initial estimate of $\mathbf{a}$, denoted by $\hat{\mathbf{a}}$, by setting $\mathbf{P}$ and $\mathbf{W}$ as the identity matrix $\mathbf{I}_{N-M}$.
- Step 2. Construct the matrices $\mathbf{P}$ and $\mathbf{W}$ using $\mathbf{a} = \hat{\mathbf{a}}$.
- Step 3. Update $\hat{\mathbf{a}}$ according to (11).
- Step 4. Repeat Step 2 - Step 3 until the ℓ_2-norm of the difference of $\hat{\mathbf{a}}$ between successive iterations is smaller than 10^{-13}, and then the final estimate $\hat{\mathbf{a}}$ is obtained.

Having obtained the estimate $\hat{\mathbf{a}}$, the existing sinusoidal poles, that is $\{z_m\}_{m=1}^M$ of (1), are estimated as the M roots of the following LP equation:

$$z^M + \sum_{m=1}^{M} \hat{a}_m z^{M-m} = 0, \tag{12}$$

and the corresponding estimates of the frequency and damping factor of the mth sinusoidal pole z_m, are calculated as:

$$\hat{\omega}_m = \angle \hat{z}_m, \quad \hat{\beta}_m = -\log|\hat{z}_m|. \tag{13}$$

3.2. Computational Complexity of the RWLP Estimator

In the RWLP parameter estimation, the main computational complexity (taking only the multiplications into account) in one iteration consists of three main parts according to (11): 1) the matrix multiplication of $\mathbf{Y}^H\mathbf{PWY}$ and $\mathbf{Y}^H\mathbf{PWb}$, 2) the matrix inversion of $\mathbf{Y}^H\mathbf{PWY}$, 3) the construction of the matrices $\mathbf{W}$ and $\mathbf{P}$, among which the third part occupies most of the computation, and requires FLOPs of $\mathcal{O}(N^3)$.

3.3. Accuracy of the RWLP Estimator

Mostly, the accuracy of one parameter estimator is evaluated in terms of the mean square error (MSE). Applying the MSE formula for the unconstrained optimization problems at the sufficiently small noise conditions, it is derived that the MSE of the RWLP estimate of the mth sinusoidal pole, $\hat{z}_m$, $m = 1, \cdots, M$, is:

$$\begin{aligned} \mathrm{MSE}(\hat{z}_m) &= E\{|\hat{z}_m - z_m|^2\} \\ &= \frac{1}{|\beta_m|^2} \boldsymbol{\mu}_m^H \mathbf{C}_{\hat{\mathbf{a}}} \boldsymbol{\mu}_m, \end{aligned} \tag{14}$$

where $\mathbf{C}_{\hat{\mathbf{a}}}$ is the covariance matrix of the LP coefficient estimates, that is $\hat{\mathbf{a}}$ of (8), and

$$\begin{aligned} \mathbf{C}_{\hat{\mathbf{a}}} &= E\left\{ (\hat{\mathbf{a}} - \mathbf{a})(\hat{\mathbf{a}} - \mathbf{a})^H \right\} \\ &= \sigma_v^2 \left(\mathbf{X}^H \left(\mathbf{A}_0 \mathbf{A}_0^H \right)^{-1} \mathbf{X} \right)^{-1}, \end{aligned} \tag{15}$$

with $\mathbf{X}$ being the noiseless part of $\mathbf{Y}$. In addition,

$$\beta_m = M z_m^{M-1} + \sum_{i=1}^{M-1} (M-i) a_i z_m^{M-i-1}, \tag{16}$$

$$\boldsymbol{\mu}_m = \begin{bmatrix} z_m^{M-1} & \cdots & z_m & 1 \end{bmatrix}^T. \tag{17}$$

Accordingly, the MSEs of the frequency ω_m and damping factor β_m estimates are expressed as [24]:

$$\mathrm{MSE}(\hat{\omega}_m) = \mathrm{MSE}(\hat{\beta}_m) = \frac{1}{2e^{-2\beta_m}} \mathrm{MSE}(\hat{z}_m). \tag{18}$$

Note that, the MSEs of (18) are independent of the p's value in the weighted ℓ_p-norm (9), which means that the p

plays a minor role in the estimation accuracy of the RWLP estimator. The different choice of the p will influence the threshold performance of the estimation. In addition, as shown in (14), when $\sigma_v^2 \rightarrow 0$, $\mathrm{MSE}(\hat{z}_m) \rightarrow 0$. This means that the RWLP-based estimation provides the asymptotically consistent parameter estimates, that is the RWLP estimates of the frequencies and damping factors converge to their respective true values when the noise level approaches zero.

4. SIMULATION RESULTS

In this section, we investigate the performance of the proposed parameter estimation method with respect to signal-to-noise ratio (SNR). Here, the SNR is explicitly defined as the average signal power divided by the noise variance. The performance is evaluated in terms of the empirical mean square errors (EMSEs):

$$\mathrm{EMSE}_\omega = \frac{1}{MS}\sum_{m=1}^{M}\sum_{s=1}^{S}\left(\hat{\omega}_m^{(s)} - \omega_m\right)^2, \quad (19)$$

$$\mathrm{EMSE}_\beta = \frac{1}{MS}\sum_{m=1}^{M}\sum_{s=1}^{S}\left(\hat{\beta}_m^{(s)} - \beta_m\right)^2, \quad (20)$$

with (ω_m, β_m) and $(\hat{\omega}_m^{(s)}, \hat{\beta}_m^{(s)})$ being the true values of the frequencies, damping factors and their estimates at the sth trial, respectively, and S being the number of trials. All the results provided are the averages of 1000 independent runs, which are conducted on a PC with an Intel(R) Core(TM) i7-8750H CPU @ 2.20 GHz, with 16.0 GB of installed memory.

In this study, we consider the two-tone damped sinusoidal signal: $s(n) = A_1 e^{(-\beta_1 + j\omega_1)n + j\phi_1} + A_2 e^{(-\beta_2 + j\omega_2)n + j\phi_2}$, where $\omega_1 = 0.3$, $\beta_1 = 0.01$, $A_1 = 1$, $\phi_1 = 1$, and $\omega_2 = 0.7$, $\beta_2 = 0.02$, $A_2 = 1$, $\phi_2 = 2$. The p of the weighted ℓ_p-norm in (9) is set as $p = 1.5$, which is found empirically to result in good performance. To show the robustness of the RWLP estimator, the results of the WLP approach, where the weighted ℓ_2-norm is adopted instead, are provided. Besides, the results of the ℓ_p-MUSIC algorithm [19], with the estimates of the RWLP and WLP as the initial guess, are provided, respectively. Here, we consider the impulsive noise model of the Gaussian mixture model (GMM). The PDF of the circular GMM noise, v, is:

$$p_v(v) = \sum_{i=1}^{2}\frac{\rho_i}{\pi\sigma_i^2}\exp\left(-\frac{|v|^2}{\sigma_i^2}\right), \quad (21)$$

which consists of two terms of Gaussian distributions with the different variances σ_i^2, $i = 1, 2$. Here, it is set that $c_1 = 0.1$, $c_2 = 0.9$, and $\sigma_1^2 = 100\sigma_2^2$, which means that the outliers come present with the probability of 10% and with the variance 100 times that of the background noise. Correspondingly, the total variance of the impulsive noise v is $\sigma_v^2 = 10.9\sigma_2^2$.

Figs. 1 and 2 show the estimation performance for the damped sinusoids corrupted by the GMM noise, with the data length set as $N = 50$ and $N = 100$, respectively. It is observed that, the EMSEs of all the methods are asymptotically linear with respect to the SNR. At the sufficiently large SNR, the EMSEs of the RWLP and WLP estimators are almost equal to their theoretical expression of (18). Further on, when $N = 50$, the RWLP estimator falls into the asymptotic region at $\mathrm{SNR} \geq 5$ dB, whereas the threshold SNR of the WLP is 7 dB. This means that the RWLP estimator is more resistant to the outliers, with the threshold SNR advantage over the WLP by 2 dB, or equivalently, with the power saving by 37%. When $N = 100$, the RWLP estimator bears the threshold SNR advantage over the WLP by 1 dB, or equivalently, saves the power by 21%. It is also noted that the threshold performance of the ℓ_p-MUSIC algorithm accords with its initial guess. Fed on the results of the RWLP estimator, the ℓ_p-MUSIC algorithm has a lower threshold SNR.

In addition, when $N = 50$, there exist the gaps between the EMSEs of the RWLP, WLP, and ℓ_p-MUSIC estimators and their corresponding CRLBs by 9.8 dB, 9.8 dB, and 3.9 dB, respectively. This means that the ℓ_p-MUSIC algorithm improves the estimation accuracy by 5.9 dB. When $N = 100$, the asymptotic EMSEs of the ℓ_p-MUSIC algorithm are smaller than those of the RWLP and WLP estimators by 6.0 dB. Table 1 shows the CPU time for each run of the parameter estimation at $\mathrm{SNR} = 10$ dB. From this table, it is seen that the RWLP and WLP estimators bear the similar computational efficiency; and the ℓ_p-MUSIC algorithm provides the accuracy improvement at the cost of vast computational overhead. In detail, the CPU time of the ℓ_p-MUSIC algorithm is longer than those of the LP-based estimators by about 430 times and 330 times, for $N = 50$ and $N = 100$, respectively.

Table 1: CPU Time for Sinusoidal Parameter Estimation (ms)

N	RWLP	WLP	ℓ_p-MUSIC
50	4.6	4.4	1941
100	13.4	14.0	4646

$$\|\mathbf{e}\|_{\mathbf{W},p}^p = \sum_{i=1}^{N-M}\sum_{j=1}^{N-M}[\mathbf{W}]_{i,j}\cdot\overline{[\mathbf{Ya}-\mathbf{b}]_i}\cdot\left|[\mathbf{Ya}-\mathbf{b}]_i\right|^{p-2}\cdot[\mathbf{Ya}-\mathbf{b}]_j = (\mathbf{Ya}-\mathbf{b})^H\mathbf{PW}(\mathbf{Ya}-\mathbf{b}). \quad (9)$$

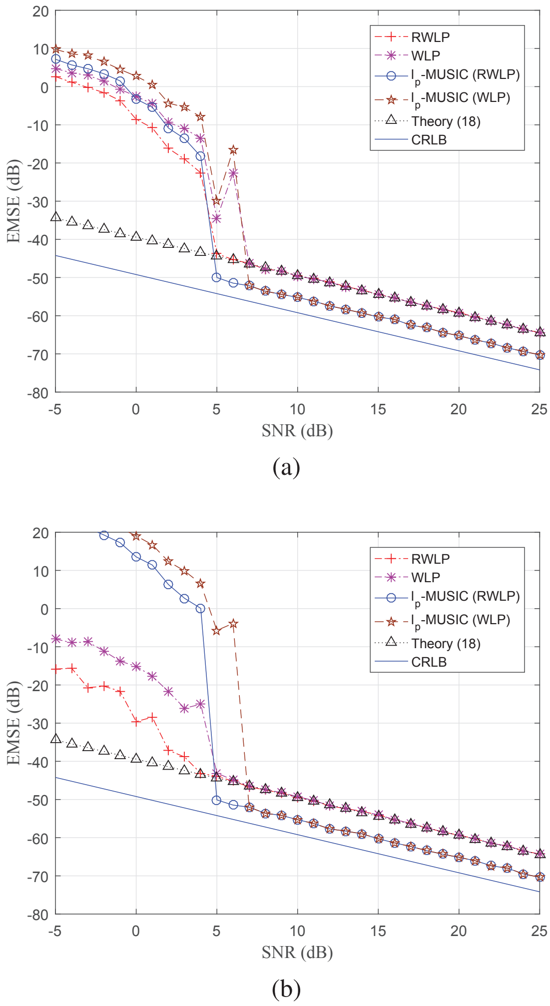

Fig. 1: EMSEs for the GMM noise corrupted damped sinusoids when $N = 50$: (a) frequency and (b) damping factor.

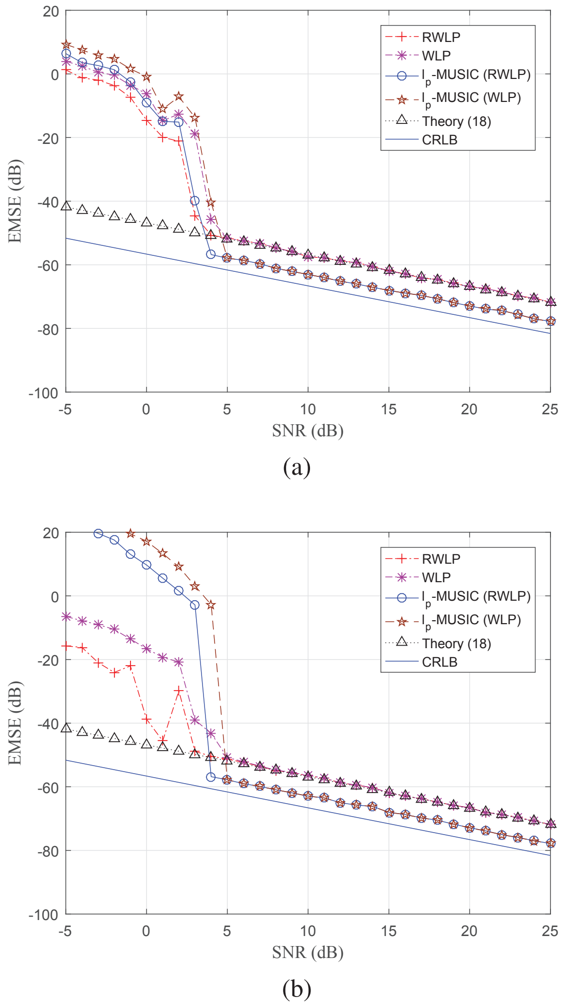

Fig. 2: EMSEs for the GMM noise corrupted damped sinusoids when $N = 100$: (a) frequency and (b) damping factor.

5. CONCLUSION

In this work, the RWLP estimator is developed for the parameter estimation of the damped sinusoidal signals, which are corrupted by the impulsive noise. We derive the Markov optimum weighting matrix, and propose to estimate the LP coefficients by minimizing the weighted ℓ_p-norm of the LP error vector. Furthermore, an IRLS procedure is devised to calculate the LP coefficient estimates. Theoretical analysis and simulation results demonstrate the computational efficiency of the proposed estimator. By comparing with the ℓ_2-norm based counterpart, it is observed that the ℓ_p-norm minimization brings the robustness to the RWLP estimator in terms of the threshold performance.

Future works include the extension of the RWLP estimator to the more general scenarios such as the multi-channel and multi-dimensional sinusoidal signals, and its application in source localization [2], biomedical signal analysis [3], wireless communication [25], and so on.

6. ACKNOWLEDGMENT

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