# Hybrid Equity Warrants Pricing Formulation under Stochastic Dynamics

Teh Raihana Nazirah Roslan, Siti Zulaiha Ibrahim, Sharmila Karim

*Abstract*—A warrant is a financial contract that confers the right but not the obligation, to buy or sell a security at a certain price before expiration. The standard procedure to value equity warrants using call option pricing models such as the Black–Scholes model had been proven to contain many flaws, such as the assumption of constant interest rate and constant volatility. In fact, existing alternative models were found focusing more on demonstrating techniques for pricing, rather than empirical testing. Therefore, a mathematical model for pricing and analyzing equity warrants which comprises stochastic interest rate and stochastic volatility is essential to incorporate the dynamic relationships between the identified variables and illustrate the real market. Here, the aim is to develop dynamic pricing formulations for hybrid equity warrants by incorporating stochastic interest rates from the Cox-Ingersoll-Ross (CIR) model, along with stochastic volatility from the Heston model. The development of the model involves the derivations of stochastic differential equations that govern the model dynamics. The resulting equations which involve Cauchy problem and heat equations are then solved using partial differential equation approaches. The analytical pricing formulas obtained in this study comply with the form of analytical expressions embedded in the Black-Scholes model and other existing pricing models for equity warrants. This facilitates the practicality of this proposed formula for comparison purposes and further empirical study.

*Keywords*—Cox-Ingersoll-Ross model, equity warrants, Heston model, hybrid models, stochastic.

## I. INTRODUCTION

WARRANTS enable the buyer to purchase an asset at a certain price on a predetermined date. Theoretically, equity warrants and call options are quite analogous in which these two derivatives require the buyer to purchase the assets at a certain price prior to expiry. The general formula for equity warrants is given by:

$$W(t) = \frac{1}{N+Mk}(kS(t) - NG)^{+},$$

where $W$ is the price of warrant for a firm supported by $N$ common stocks, $M$ is the sum of outstanding equity warrants and $S$ is the current value of underlying asset. When the payment of $G$ is released, the holder of the warrant shall receive $k$ shares at time $T$ for each warrant. The + symbol marks the difference between call option and put option for warrants.

Considering the similar definition between the concept of equity warrants and options, it is apparent to value equity warrants utilizing options pricing models, including the Black Scholes model which is a famous traditional model in 1973 [1]. Black and Scholes [1] achieved a breakthrough by proposing an innovative model with an analytic method for European options in accordance to geometric Brownian motion (gBm). Nonetheless, some basic presumptions were made for its analytical simplicity and tractability which were found unsuitable and could lead to pricing errors [2]. As reported by [3], the Black Scholes model would cause errors in prices if there exist stochastic motions in volatility. Furthermore, this well-known model adopted a gBm with constant volatility, thus the inconsistency between the model and real data was typically present [4]. Due to this fact, [5] declared that pricing models for options under stochastic volatility had enhanced market performance. However, the models of stochastic volatility are too complex and not straightforward enough for implementation. Additionally, the Black Scholes model also presumed constant interest rates for all maturities, with no trading cost. Further, [4] observed that the claim of constant interest rate in the Black Scholes model was impractical and inconsistent with empirical results. In fact, the rate of interest for assets and liabilities in the real market are not always constant. Clearly, stochastic interest rates take into account market fluctuations [6]. Since market fluctuations can drastically affect an investor's currency, a completely different result might be expected under constant interest rates assumption.

According to [7], a major development occurred in 1993 when Heston [8] outlined the volatility using the CIR process. The Heston model had better empirical results compared to other stochastic volatility models [9] and this model takes into account the mean-reverting property. However, this model is inadequate in explaining the dynamics of the underlying price, which results in further exploration of suitable and practical models that suit the market. Incorporating stochastic volatility with stochastic interest rates is one of the most common approaches to construct a hybrid model [2], [10], [11]. For example, [10] came out with a hybrid model to price European options where the price of underlying is based on the stochastic volatility of the Heston model, whereas the rate of interest rate is determined by the CIR model.

The implementation of stochastic volatility along with stochastic interest rates has significantly improved existing pricing formulas in various financial derivatives [14], [15]. In this paper, an analytical pricing formula for equity warrants is

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World Academy of Science, Engineering and Technology
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derived according to the Heston stochastic volatility model, as well as on the rate of interest as an additional stochastic variable under the CIR model. Recently, [11] utilized the change of measure technique and generalized Fourier transform to derive the characteristic function of hybrid equity warrants. In this paper, [12] is extended where Cauchy problem and heat equations are employed specifically for the Heston-CIR hybrid model, which are then solved using partial differential equation approaches. This fills the gap in the literature of warrant pricing involving hybrid models, particularly consisting stochastic volatility and stochastic interest rates.

## II. THE HYBRID MODEL DYNAMICS

This section presents the setup of the model which involves the hybridization of Heston stochastic volatility model with the CIR stochastic interest rate model. Generally referred as the Heston-CIR model, it is represented as:

$$dS(t) = r(t)S(t)dt + \sqrt{v(t)}S(t)d\widetilde{w}_1(t)$$
$$dv(t) = k^*(\theta^* - v(t))dt + \sigma\sqrt{v(t)}d\widetilde{w}_2(t)$$
$$dr(t) = \alpha^*(\beta^* - r(t))dt + \eta\sqrt{r(t)}d\widetilde{w}_3(t) \quad (1)$$

Here, $S(t), v(t)$ and $r(t)$ represent the underlying price, volatility and interest rate respectively. We denote $S(t)$ as the price of asset driven by the drift $r(t)$, and $v(t)$ represents its volatility. For the instantaneous variance process of $v(t)$, $k^*$ represents its mean-reversion process, $\theta^*$ represents the long term-mean and $\sigma$ represents its volatility. In addition, $r(t)$ is defined as the instantaneous interest rate in which $\alpha^*$ symbolizes its mean-reversion speed, $\beta^*$ represents the long-term mean of the interest rate and $\eta$ monitors the volatility of the interest rate. The correlation in the model are determined by $(d\widetilde{w}_1(t), d\widetilde{w}_2(t)) = \rho dt, (d\widetilde{w}_1(t), d\widetilde{w}_3(t)) = 0$, and $(d\widetilde{w}_2(t), d\widetilde{w}_3(t)) = 0$, with $-1 < \rho < 1$ and $0 \le t \le T$. Additionally, the conditions $2k^*\theta^* \ge \sigma^2$ and $2\alpha^*\beta^* \ge \eta^2$ are required to make sure that the processes of square root are constantly positive. Moreover, if the risk-free interest rate is compatible with the dynamics of the CIR model, therefore $P(r,t,T)$ which is the zero-coupon bond price at time $t$ which matures at $T$ is given as

$$P(r,t,T) = A(t,T)e^{-B(t,T)r(t)}, \quad (2)$$

where the expressions of $A(t,T)$ and $B(t,T)$ are

$$A(t,T) = \left( \frac{2\left( e^{\left(\alpha^* + \sqrt{(\alpha^*)^2 + 2\eta^2}\right)\frac{(T-t)}{2}}\right)\left(\sqrt{(\alpha^*)^2 + 2\eta^2}\right)}{\left(2\sqrt{(\alpha^*)^2 + 2\eta^2}\right) + \left(\alpha^* + \sqrt{(\alpha^*)^2 + 2\eta^2}\right)\left(e^{(T-t)\sqrt{(\alpha^*)^2 + 2\eta^2}} - 1\right)} \right)^{\frac{2\alpha^*\beta^*}{\eta^2}}$$

and

$$B(t,T) = \frac{2\left( e^{(T-t)\sqrt{(\alpha^*)^2 + 2\eta^2}} - 1\right)}{\left(2\sqrt{(\alpha^*)^2 + 2\eta^2}\right) + \left(\alpha^* + \sqrt{(\alpha^*)^2 + 2\eta^2}\right)\left(e^{(T-t)\sqrt{(\alpha^*)^2 + 2\eta^2}} - 1\right)}.$$

## III. PRICING FORMULA FOR HYBRID EQUITY WARRANTS

We denote $W(t)$ as the appraisal of the equity warrant. At time $t \in [0,T]$, the appraisal of the equity warrant satisfies the following partial differential equation (PDE) which associates to the derivative value

$$\frac{\partial W}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 W}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 W}{\partial v^2} + \frac{1}{2}\eta^2 r\frac{\partial^2 W}{\partial r^2} + rS\frac{\partial W}{\partial S} + k^*(\theta^* - v)\frac{\partial W}{\partial v} + [\alpha^*\beta^* - (\alpha^* + B(t,T)\eta^2)r]\frac{\partial W}{\partial r} + \rho\sigma vS\frac{\partial^2 W}{\partial S\partial v} = 0 \quad (3)$$

with the boundary condition

$$W(t) = \frac{1}{N+Mk}(kS(t) - NG)^+. \quad (4)$$

The PDE in (3) is obtained by performing the Feynman Kac theorem on the stochastic differential equations as in [13].

### A. Theorem 1

Let $W(t)$ be the appraisal of the equity warrant. The appraisal of the equity warrant at time $t \in [0,T]$ is

$$W(t)(S(t),T,t,G,\sigma,v,r,k,N,M) = \frac{1}{N+Mk}\left[kS(t)\phi(d_1) - NGe^{-r(T-t)}\phi(d_2)\right] \quad (5)$$

where

$$d_1 = \frac{ln\frac{kS}{NG} - ln\, P(r,t,T) + \frac{1}{2}L(T-t) + \frac{1}{2}Q}{\sqrt{L(T-t)+Q}},$$

$$Q = \eta^2 r \int_t^T \left( \frac{2(e^{2R} - 2e^R + 1)}{2((\alpha^*)^2 + 2\eta^2) + (e^R - 1)(C)} \right) ds,$$

$$R = (T-s)\sqrt{(\alpha^*)^2 + 2\eta^2},$$

$$C = \left( \alpha^*\sqrt{(\alpha^*)^2 + 2\eta^2} + (\alpha^*)^2 + 3\eta^2 + (e^R)\left((\alpha^*)^2 + \alpha^*\sqrt{(\alpha^*)^2 + 2\eta^2} + \eta^2\right)\right),$$

$$d_2 = d_1 - \sqrt{L(T-t) + \eta^2 r \int_t^T B^2(s,T)ds},$$

and $\Phi(\cdot)$ represents the cumulative Gaussian distribution function.

Clearly, (3) is a parabolic PDE with variable coefficients. This requires the transformation of (3) into a case of Cauchy to find its solution. By working on the following coordinate transforms of $y = \frac{S}{P(r,t,T)}$, $\widehat{W}(y,t,L) = \frac{W(S,v,r,T)}{P(r,t,T)}$ and $L = v$, the following expressions are obtained:

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:14, No:11, 2020

$$\begin{cases} \frac{\partial W}{\partial t} = \widehat{W}\frac{\partial P}{\partial t} + P\frac{\partial \widehat{W}}{\partial t} + P\frac{\partial \widehat{W}}{\partial y}\frac{\partial y}{\partial t} = \widehat{W}\frac{\partial P}{\partial t} + P\frac{\partial \widehat{W}}{\partial t} - y\frac{\partial \widehat{W}}{\partial y}\frac{\partial P}{\partial t}, \\ \frac{\partial W}{\partial s} = P\frac{\partial \widehat{W}}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial \widehat{W}}{\partial y}, \\ \frac{\partial^2 W}{\partial S^2} = \frac{\partial^2 \widehat{W}}{\partial y^2}\frac{\partial y}{\partial s} = \frac{\partial^2 \widehat{W}}{\partial y^2}\left(\frac{1}{P}\right), \\ \frac{\partial W}{\partial v} = P\frac{\partial \widehat{W}}{\partial L}\frac{\partial L}{\partial v} = P\frac{\partial \widehat{W}}{\partial L}, \\ \frac{\partial^2 W}{\partial v^2} = P\frac{\partial^2 \widehat{W}}{\partial L^2}\frac{\partial L}{\partial v} = P\left(\frac{\partial^2 \widehat{W}}{\partial L^2}\right), \\ \frac{\partial W}{\partial r} = \widehat{W}\frac{\partial P}{\partial r} + P\left(\frac{\partial \widehat{W}}{\partial y}\left(-\frac{y}{P}\frac{\partial P}{\partial r}\right)\right) = \widehat{W}\frac{\partial P}{\partial r} - y\frac{\partial \widehat{W}}{\partial y}\frac{\partial P}{\partial r}, \\ \frac{\partial^2 W}{\partial r^2} = \widehat{W}\frac{\partial^2 P}{\partial r^2} - y\frac{\partial \widehat{W}}{\partial y}\frac{\partial^2 P}{\partial r^2} + \frac{y^2}{P}\frac{\partial^2 \widehat{W}}{\partial y^2}\left(\frac{\partial P}{\partial r}\right)^2, \\ \frac{\partial^2 W}{\partial s\partial v} = \frac{\partial^2 \widehat{W}}{\partial s\partial L}. \end{cases} \quad (6)$$

Note that $\frac{\partial P}{\partial r} = -P(r,t,T)B(t,T), \frac{\partial^2 P}{\partial r^2} = P(r,t,T)B^2(t,T)$, and $S^2 = y^2 P^2$. Applying (6) in (3), (3) can be transformed as:

$$\frac{\partial \widehat{W}}{\partial t} + \frac{1}{2}y^2 L\left(\frac{\partial^2 \widehat{W}}{\partial y^2}\right) + \frac{1}{2}\sigma^2 L\left(\frac{\partial^2 \widehat{W}}{\partial L^2}\right) + \frac{1}{2}\eta^2 r$$
$$\left((yB(t,T))^2 \frac{\partial^2 \widehat{W}}{\partial y^2}\right) = 0. \quad (7)$$

Let $x = \ln y$. Then (7) can be transformed into

$$\frac{\partial \widehat{W}}{\partial t} - \frac{1}{2}L\frac{\partial \widehat{W}}{\partial x} + \frac{1}{2}L\frac{\partial^2 \widehat{W}}{\partial x^2} + \frac{1}{2}\sigma^2 L\left(\frac{\partial^2 \widehat{W}}{\partial L^2}\right) - \frac{1}{2}\eta^2 rB^2(t,T)\frac{\partial \widehat{W}}{\partial x} + \frac{1}{2}\eta^2 rB^2(t,T)\frac{\partial^2 \widehat{W}}{\partial x^2} = 0. \quad (8)$$

Next, let $\widehat{W}(y,t,L) = u(\hat{\eta}, \tau, \hat{\lambda})$, $\hat{\eta} = x + \hat{\alpha}(t)$ where $\hat{\alpha}(T) = \omega(T) = 0$, $\tau = \omega(t)$, and $\hat{\lambda} = L + h(t)$. In the present case, it is easy to see that

$$\frac{\partial \widehat{W}}{\partial t} = \frac{\partial u}{\partial \hat{\eta}}\hat{\alpha}'(t) + \frac{\partial u}{\partial \tau}\omega'(t) + \frac{\partial u}{\partial \hat{\lambda}}h'(t),$$
$$\frac{\partial \widehat{W}}{\partial x} = \frac{\partial u}{\partial \hat{\eta}}\left(\frac{\partial \hat{\eta}}{\partial x}\right) = \frac{\partial u}{\partial \hat{\eta}},$$
$$\frac{\partial^2 \widehat{W}}{\partial x^2} = \frac{\partial^2 u}{\partial \hat{\eta}^2}, \quad (9)$$
$$\frac{\partial \widehat{W}}{\partial L} = \frac{\partial u}{\partial \hat{\lambda}}\left(\frac{\partial \hat{\lambda}}{\partial L}\right) = \frac{\partial u}{\partial \hat{\lambda}},$$

and

$$\frac{\partial^2 \widehat{W}}{\partial L^2} = \frac{\partial^2 u}{\partial \hat{\lambda}^2}.$$

Based on (9) and (8), (8) can be written as

$$\frac{\partial u}{\partial \hat{\eta}}\left[\hat{\alpha}'(t) - \frac{1}{2}L - \frac{1}{2}\eta^2 rB^2(t,T)\right] + \frac{\partial^2 u}{\partial \hat{\eta}^2}\left[\frac{1}{2}L + \frac{1}{2}\eta^2 rB^2(t,T)\right] + \frac{\partial u}{\partial \tau}\omega'(t) + \frac{\partial u}{\partial \hat{\lambda}}h'(t) + \frac{1}{2}\sigma^2 L\left(\frac{\partial^2 u}{\partial \hat{\lambda}^2}\right) = 0. \quad (10)$$

Moreover, let

$$\begin{cases} \hat{\alpha}'(t) = \frac{1}{2}L + \frac{1}{2}\eta^2 rB^2(t,T), \\ \omega'(t) = -\frac{1}{2}L - \frac{1}{2}\eta^2 rB^2(t,T), \\ h'(t) = -\frac{1}{2}\sigma^2 L. \end{cases} \quad (11)$$

Performing integration on (11) results in:

$$\begin{cases} \hat{\alpha}(t) = \int_t^T \left(\frac{1}{2}L + \frac{1}{2}\eta^2 rB^2(s,T)\right)ds, \\ \omega(t) = -\int_t^T \left(\frac{1}{2}L + \frac{1}{2}\eta^2 rB^2(s,T)\right)ds, \\ h(t) = -\int_t^T \left(\frac{1}{2}\sigma^2 L\right)ds. \end{cases} \quad (12)$$

Based on the form of expressions in (12), (10) can be further deduced as

$$\frac{\partial^2 u}{\partial \hat{\eta}^2} = \frac{\partial u}{\partial \tau} \text{ and } \frac{\partial^2 \widehat{W}}{\partial \hat{\lambda}^2} = \frac{\partial u}{\partial \hat{\lambda}}, \quad (13)$$

with the final condition

$$u(\hat{\eta}, 0, \hat{\lambda}) = \frac{1}{N+Mk}(ke^{\hat{\eta}} - NG)^+.$$

Note that (13) is an explicit solution of a one-dimensional heat equation given as

$$u(\hat{\eta}, \tau, \hat{\lambda}) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{+\infty}\frac{1}{N+Mk}(ke^{\xi} - NG)^+ e^{-\frac{(\hat{\eta}-\xi)^2}{4\tau}}d\xi.$$

As a result,

$$\widehat{W} = u(\hat{\eta}, \tau, \hat{\lambda}) = \frac{1}{2\sqrt{\pi\tau}}\int_{\ln\frac{NG}{k}}^{+\infty}\frac{1}{N+Mk}(ke^{\xi} - NG)e^{-\frac{(\hat{\eta}-\xi)^2}{4\tau}}d\xi,$$
$$= \frac{k}{2\sqrt{\pi\tau}}\int_{\ln\frac{NG}{k}}^{+\infty}\frac{e^{\xi}e^{-\frac{(\hat{\eta}-\xi)^2}{4\tau}}}{N+Mk}d\xi - \frac{NG}{2\sqrt{\pi\tau}}\int_{\ln\frac{NG}{k}}^{+\infty}\frac{e^{-\frac{(\hat{\eta}-\xi)^2}{4\tau}}}{N+Mk}d\xi, := I_1 - I_2 \quad (14)$$

Here, the computations on $I_2$ is first being worked out. Let $z_2 = \frac{\hat{\eta}-\xi}{\sqrt{2\tau}}$. Then $-\sqrt{2\tau}dz_2 = d\xi$. A standard calculation is performed as:

$$I_2 = \frac{NG}{2\sqrt{\pi\tau}}\int_{\ln\frac{NG}{k}}^{+\infty}\frac{e^{-\frac{(\hat{\eta}-\xi)^2}{4\tau}}}{N+Mk}d\xi = \frac{NG}{(N+Mk)}\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\frac{\hat{\eta}-\ln\frac{NG}{k}}{\sqrt{2\tau}}}e^{-\frac{z_2^2}{2}}(-\sqrt{2\tau})dz_2 = \frac{NG}{N+Mk}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\hat{\eta}-\ln\frac{NG}{k}}{\sqrt{2\tau}}}e^{-\frac{z_2^2}{2}}d_2 = \frac{NG}{N+Mk}\phi(d_2) \quad (15)$$

where

$$d_2 = \frac{\ln\frac{kS}{NG} - \ln P(r,t,T) - \frac{1}{2}L(T-t) - \frac{1}{2}Q}{\sqrt{L(T-t)+Q}},$$
$$Q = \eta^2 r\int_t^T\left(\frac{2(e^{2R} - 2e^R + 1)}{2((\alpha^*)^2 + 2\eta^2) + (e^R-1)(C)}\right)ds,$$
$$R = (T-s)\sqrt{(\alpha^*)^2 + 2\eta^2},$$

And

$$C = \left(\alpha^*\sqrt{(\alpha^*)^2 + 2\eta^2} + (\alpha^*)^2 + 3\eta^2 + (e^R)((\alpha^*)^2 + \alpha^*\sqrt{(\alpha^*)^2 + 2\eta^2} + \eta^2)\right).$$

Next, focus is given on the derivations for the expressions in $I_1$. Let $z_1 = \frac{\hat{\eta}-\xi+2\tau}{\sqrt{2\tau}}$. Subsequently $-\sqrt{2\tau}dz_1 = d\xi$. Employing the same computation as in $I_2$ results in

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:14, No:11, 2020

$$I_1 = \frac{1}{2\sqrt{\pi\tau}} \int_{ln\frac{NG}{k}}^{+\infty} \frac{ke^\xi}{N+Mk} e^{-\frac{(\hat{\eta}-\xi)^2}{4\tau}} d\xi =$$

$$\frac{ke^{\hat{\eta}+\tau}}{(N+Mk)2\sqrt{\pi\tau}} \int_{-\infty}^{\frac{-ln\frac{NG}{k}+\hat{\eta}+2\tau}{\sqrt{2\tau}}} e^{-\frac{z_1^2}{2}} (-\sqrt{2\tau}) dz_1 = \frac{kS(t)}{N+Mk} \frac{1}{P(r,t,T)} \phi(d_1), (16)$$

where

$$d_1 = \frac{ln\frac{kS}{NG} - ln P(r,t,T) + \frac{1}{2}L(T-t) + \frac{1}{2}Q}{\sqrt{L(T-t)+Q}},$$

$$Q = \eta^2 r \int_t^T \left( \frac{2(e^{2R} - 2e^R + 1)}{2((\alpha^*)^2 + 2\eta^2) + (e^R - 1)(C)} \right) ds,$$

$$R = (T-s)\sqrt{(\alpha^*)^2 + 2\eta^2},$$

and

$$C = \left( \alpha^*\sqrt{(\alpha^*)^2 + 2\eta^2} + (\alpha^*)^2 + 3\eta^2 + (e^R)\left((\alpha^*)^2 + \alpha^*\sqrt{(\alpha^*)^2 + 2\eta^2} + \eta^2\right) \right).$$

In the end, the desired result as in Theorem 1 is achieved through the integration of (14)-16), along with the relationship $W(S, v, r, T) = \widehat{W}(y, t, L)P(r, t, T)$.

## IV. CONCLUSION

This paper considers the Heston-CIR hybrid model for the formulation of hybrid equity warrants under stochastic volatility and stochastic interest rates. The development of the model involves the derivations of stochastic differential equations involving Cauchy problem and heat equations, which are then solved using PDE approaches. Since the analytical pricing formulas in this paper comply with the form of analytical expressions in the Black-Scholes model and other existing equity warrants pricing models, future extension includes empirical study for the model formulations' practicality.

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