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# Isoperimetric stability in lattices

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## Abstract

We obtain isoperimetric stability theorems for general Cayley digraphs on  $\mathbb{Z}^d$ . For any fixed  $B$  that generates  $\mathbb{Z}^d$  over  $\mathbb{Z}$ , we characterise the approximate structure of large sets  $A$  that are approximately isoperimetric in the Cayley digraph of  $B$ : we show that  $A$  must be close to a set of the form  $kZ \cap \mathbb{Z}^d$ , where for the vertex boundary  $Z$  is the conical hull of  $B$ , and for the edge boundary  $Z$  is the zonotope generated by  $B$ .

## 1 Introduction

An important theme at the interface of Geometry, Analysis and Combinatorics is understanding the structure of approximate minimisers to isoperimetric problems. These problems take the general form of minimising surface area of sets with a fixed volume, for various meanings of ‘area’ and ‘volume’. The usual meanings give the Euclidean Isoperimetric Problem considered since the ancient Greek mathematicians, where balls are the measurable subsets of  $\mathbb{R}^d$  with a given volume which minimize the surface area. There is a large literature on its stability, i.e. understanding the structure of approximate minimisers, culminating in the sharp quantitative isoperimetric inequality of Fusco, Maggi and Pratelli [9].

In the discrete setting, isoperimetric problems form a broad area that is widely studied within Combinatorics (see the surveys [2, 15]) and as part of the Concentration of Measure phenomenon (see [16, 26]). Furthermore, certain particular settings have been intensively studied due to their applications; for example, there has been considerable recent progress (see [12, 13, 14, 23]) on isoperimetric stability in the discrete cube  $\{0, 1\}^n$ , which is intimately connected to the Analysis of Boolean Functions (see [21]) and the Kahn–Kalai Conjecture (see [11]) on thresholds for monotone properties (see [8] for the recent solution of Talagrand’s

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fractional version). This paper concerns the setting of integer lattices, which is widely studied in Additive Combinatorics, where the Polynomial Freiman–Ruzsa Conjecture (see [10]) predicts the structure of sets with small doubling.

For an isoperimetric problem on a digraph (directed graph)  $G$ , we measure the ‘volume’ of  $A \subseteq V(G)$  by its size  $|A|$ , and its ‘surface area’ either by the *edge boundary*  $\partial_{e,G}(A)$ , which is the number of edges  $\overrightarrow{xy} \in E(G)$  with  $x \in A$  and  $y \in V(G) \setminus A$ , or by the *vertex boundary*  $\partial_{v,G}(A)$ , which is the number of vertices  $y \in V(G) \setminus A$  such that  $\overrightarrow{xy} \in E(G)$  for some  $x \in A$ . Here we consider Cayley digraphs: given a generating set  $B$  of  $\mathbb{Z}^d$ , we write  $G_B$  for the digraph on  $\mathbb{Z}^d$  with edges  $E(G_B) = \{(u, v) : v - u \in B\}$ .

It is an open problem to determine the minimum possible value of  $\partial_{v,G_B}(A)$  or  $\partial_{e,G_B}(A)$  for  $A \subseteq \mathbb{Z}^d$  of given size, let alone any structural properties of (approximate) minimisers; exact results are only known for a few instances of  $B$  (see [3, 4, 27, 24]). It is therefore natural to seek asymptotics. For the vertex boundary, this was achieved by Ruzsa [25], who showed that for  $A \subseteq \mathbb{Z}^d$  of size  $n \rightarrow \infty$ , the minimum value  $\partial_{v,G_B}(A)$  is asymptotic to that achieved by a set of the form  $kC(B) \cap \mathbb{Z}^d$ , where  $C(B)$  is the conical hull of  $B$ , i.e. the convex hull of  $B \cup \{0\}$ . A corresponding result for the edge boundary was obtained in [1]: the minimum value  $\partial_{e,G_B}(A)$  is asymptotic to that achieved by a set of the form  $kZ(B) \cap \mathbb{Z}^d$ , where  $Z(B) = \{\sum_{b \in B} bx_b : x \in [0, 1]^B\}$  is the zonotope generated by  $B$ .

We will prove stability versions of both these results, describing the approximate structure of asymptotic minimisers for both the vertex and edge isoperimetric problems in  $G_B$ . The statements require the following notation. Given  $B \subseteq \mathbb{Z}^d$  and  $n \in \mathbb{N}$  we write  $B_n = (\kappa_B(n)C(B)) \cap \mathbb{Z}^d$ , where  $\kappa_B(n) = \min\{k \in \mathbb{N} : |kC(B) \cap \mathbb{Z}^d| \geq n\}$ . We also write  $Z(B) = C([B])$ , where  $[B] = \{\sum B' : B' \subseteq B\}$ . We use  $\mu$  to denote Lebesgue measure.

**Theorem 1.1.** *Let  $B$  be a generating set of  $\mathbb{Z}^d$  with  $d \geq 2$ . Then there is  $K \in \mathbb{N}$  so that for any  $A \subseteq \mathbb{Z}^d$  with  $|A| = n \geq K$  and  $\partial_{v,G_B}(A) \leq d\mu(C(B))^{1/d}n^{1-1/d}(1 + \varepsilon)$ , where  $Kn^{-1/2d} < \varepsilon < K^{-1}$ , there exists  $v \in \mathbb{Z}^d$  with  $|A \Delta (v + B_n)| < Kn\sqrt{\varepsilon}$ .*

**Theorem 1.2.** *Let  $B$  be a generating set of  $\mathbb{Z}^d$  with  $d \geq 2$ . For any  $\delta > 0$  there are  $\varepsilon > 0$  and  $K \in \mathbb{N}$  so that for any  $A \subseteq \mathbb{Z}^d$  with  $|A| = n \geq K$  and  $\partial_{e,G_B}(A) \leq d\mu(C([B]))^{1/d}n^{1-1/d}(1 + \varepsilon)$ , there exists  $v \in \mathbb{Z}^d$  with  $|A \Delta (v + [B]_n)| < \delta n$ .*

We remark that the ‘square root dependence’ in Theorem 1.1 constitutes a tight quantitative stability result for the vertex isoperimetric inequality in  $G_B$ , as may be seen from an example where  $B$  consists of the corners of a cube and  $A$  is an appropriate cuboid. We only give a qualitative statement of our stability result in Theorem 1.2 for the edge isoperimetric inequality in  $G_B$ , as our proof does not provide good quantitative bounds. On the other hand, we do obtain a tight quantitative stability result for certain  $B$ , namely those for which  $G_B$  is equivalent to the  $\ell^1$ -grid (see Theorem 3.1 below), thus giving a new proof of a result of Ellis, Friedgut, Kindler and Yehudayoff [6].

In proving our results, besides drawing on the methods of [25] (particularly Plünnecke’s inequality for sumsets) and [1] (a probabilistic reduction to [25]), the most significant new contribution of our paper is a technique for transforming discrete problems to a continuous setting where one can apply results from Geometric Measure Theory. We will employ the

sharp estimate on asymmetric index in terms of anisotropic perimeter with respect to any convex set  $K$  due to Figalli, Maggi and Pratelli [7] (building on the case when  $K$  is a ball, established in [9]). We consider vertex isoperimetry in the next section and then edge isoperimetry in the following section. We conclude the paper by discussing some potential directions for further research.

## 2 Vertex isoperimetry

This section contains the proof of our sharp tight quantitative stability result for the vertex isoperimetric inequality in general Cayley digraphs. We start in the first subsection with a summary of Ruzsa’s approach in [25], during which we record some key lemmas on sumsets and fundamental domains of lattices that we will also use in our proof. In the second subsection we state the Geometric Measure Theory result of [7] (in a simplified setting that suffices for our purposes). The third subsection contains a technical lemma in elementary Real Analysis. We conclude in the final subsection by proving Theorem 1.1.

### 2.1 Ruzsa’s approach

The sumset of  $A, B \in \mathbb{Z}^d$  is defined by  $A + B := \{a + b : a \in A, b \in B\}$ . The vertex isoperimetric problem in the Cayley digraph  $G_B$  is equivalent to finding the minimum of  $|A + B|$  over all sets  $A$  of given size. The following result of Ruzsa [25, Theorem 2] implies an asymptotic for this minimum.

**Theorem 2.1.** *Let  $B$  be a generating set of  $\mathbb{Z}^d$  with  $d \geq 2$ . Then for any  $A \subseteq \mathbb{Z}^d$  with  $|A| = n$  large we have  $|A + B| \geq d\mu(C(B))^{1/d}n^{1-1/d}(1 - O(n^{-1/2d}))$ .*

A key ingredient of Ruzsa’s proof is the following well-known inequality of Plünnecke [22] (see [25, Statement 6.2]). We use the notation  $\Sigma_k(A)$  for the  $k$ -fold sumset of  $A$  rather than the commonly used  $kA$ , which in this paper denotes the dilate of  $A$  by factor  $k$ .

**Theorem 2.2.** *Let  $k \in \mathbb{N}$  and  $A, B \subseteq \mathbb{Z}^d$  with  $|A| = n$  and  $|A + B| = \alpha n$ . Then there is  $\emptyset \neq A' \subseteq A$  with  $|A' + \Sigma_k(B)| \leq \alpha^k |A'|$ .*

Given  $A$  of size  $n$  with  $|A + B|$  small and any  $k' \in \mathbb{N}$ , Theorem 2.2 provides  $A' \subseteq A$  such that  $|A' + \Sigma_{k'}(B)|$  is small. The next step is to convert this into smallness of a Minkowski sum  $U + V = \{u + v : u \in U, v \in V\}$  of two bodies  $U, V$  in  $\mathbb{R}^d$ . The final bound in Theorem 2.1 will then follow from the Brunn–Minkowski inequality  $\mu(U + V)^{1/d} \geq \mu(U)^{1/d} + \mu(V)^{1/d}$  (in the form due to Lusternik [17]).

The required  $U$  and  $V$  are obtained as  $U = A' + Q$  and  $V = kC(B)$ , taking  $k' = k + p$  with  $p$  and  $Q$  as in the next lemma, where  $k$  is a free parameter that can be optimised to obtain the  $O(n^{-1/2d})$  error term in Theorem 2.1. This lemma, which is [25, Lemma 11.2], provides a certain set  $Q \subseteq \mathbb{R}^d$  with four key properties. Firstly, it is *fundamental*, i.e. any  $x \in \mathbb{R}^d$  has a unique representation as  $x = y + z$  with  $y \in Q$  and  $z \in \mathbb{Z}^d$ , and so  $\mu(X + Q) = |X|$  for any  $X \subseteq \mathbb{Z}^d$ . Secondly, the conclusion of the lemma shows that  $U + V$  as above is small

when  $A' + \Sigma_{k'}(B)$  is small. The remaining two properties are not explicitly stated in [25], but are clear from the proof; we have  $Q \subseteq Z(B)$ , and a certain technical condition, which for brevity we will call *nice*:  $Q$  is a finite union of bounded convex polytopes. (In fact, one can take  $Q$  to be a parallelepiped; we omit the proof of this fact, as we do not use it.)

**Lemma 2.3.** *Let  $B$  be a generating set of  $\mathbb{Z}^d$  with  $d \geq 2$  and  $0 \in B$ . Then there are  $p \in \mathbb{N}$ ,  $z \in \mathbb{Z}^d$  and a nice fundamental set  $Q \subseteq Z(B)$  such that  $kC(B) + Q + z \subseteq \Sigma_{k+p}(B) + Q$  for any  $k \in \mathbb{N}$ .*

## 2.2 Some Geometric Measure Theory

Here we give a brief account of the quantitative isoperimetric stability result of Figalli, Maggi and Pratelli [7]. We adopt simplified definitions that suffice for sets that are nice, as defined in the previous subsection (see [18, 19] for the general setting of sets of finite perimeter).

For a closed convex polytope  $K \subseteq \mathbb{R}^d$  and a union  $E$  of disjoint (possibly non-convex) closed polytopes, the perimeter of  $E$  with respect to  $K$  is given by

$$\text{Per}_K(E) = \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(E + \varepsilon K) - \mu(E)}{\varepsilon}. \quad (2.1)$$

In our setting given a nice set  $A$ , for all  $r \geq 0$ , the measure of  $A + rK$  and its closure  $\overline{A + rK}$  are the same, that is  $\mu(A + rK) = \mu(\overline{A + rK})$ . Thus for all  $r \geq 0$ , (2.1) gives

$$\text{Per}_K(\overline{A + rK}) = \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(A + (r + \varepsilon)K) - \mu(A + rK)}{\varepsilon}. \quad (2.2)$$

The anisotropic isoperimetric problem was posed in 1901 by Wulff [28], who conjectured that minimisers of  $\text{Per}_K$  up to null sets are homothetic copies of  $K$ , giving  $\text{Per}_K(E) \geq d\mu(K)^{1/d}\mu(E)^{1-1/d}$ . This was established for sets  $E$  with continuous boundary by Dinghas [5] and for general sets  $E$  of finite perimeter by Gromov [20]. It is equivalent to non-negativity of the *isoperimetric deficit*  $\delta_K(E)$  of  $E$  with respect to  $K$ , defined by

$$\delta_K(E) := \frac{\text{Per}_K(E)}{d\mu(K)^{1/d}\mu(E)^{1-1/d}} - 1.$$

We quantify the structural similarity between  $K$  and  $E$  via the *asymmetric index* (also known as Fraenkel asymmetry) of  $E$  with respect to  $K$ , which is given by

$$\mathcal{A}_K(E) = \inf \left\{ \frac{\mu(E \triangle (x_0 + rK))}{\mu(E)} : x_0 \in \mathbb{R}^d \text{ and } r^d \mu(K) = \mu(E) \right\}.$$

The following is [7, Theorem 1.1].

**Theorem 2.4.** *For any  $d \in \mathbb{N}$  there exists  $D = D(d)$  such that for any bounded convex open set  $K \subseteq \mathbb{R}^d$  and  $E \subseteq \mathbb{R}^d$  of finite perimeter we have*

$$\mathcal{A}_K(E) \leq D\sqrt{\delta_K(E)}.$$

## 2.3 Some Real Analysis

In this subsection we establish the following technical lemma in elementary Real Analysis, which will allow us to pass to the setting of perimeters in Ruzsa's approach as described in the first subsection, so that we can apply the result of the second subsection. We presume the result is well-known, but we include a proof for completeness.

**Lemma 2.5.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous and right differentiable. Then for any  $\varepsilon > 0$  there is  $x \in [a, b]$  with  $f'_+(x) \leq \frac{f(b)-f(a)}{b-a} + \varepsilon$ .*

*Proof.* Without loss of generality we may assume  $a = 0$ ,  $b = 1$  and  $f(0) = f(1) = 0$ . Suppose for contradiction we have  $f'_+(x) \geq \varepsilon > 0$  for all  $x \in [0, 1]$ . Let  $B = \{x : f(x) \geq \varepsilon x/2\}$ . As  $f(1) = 0$  we have  $1 \notin B$ . As  $f$  is continuous,  $B$  is closed. The required contradiction will thus follow if we show that  $B$  is open to the right, i.e. for any  $x \in B$  there is  $\delta > 0$  such that  $(x, x + \delta) \subseteq B$ . To see this, note that for small enough  $\delta$ , by definition of  $f'_+(x)$  we have  $f(y) \geq f(x) + (y - x)\varepsilon/2 \geq \varepsilon y/2$  for any  $y \in (x, x + \delta)$ , so  $y \in B$ .  $\square$

## 2.4 Stability

In this final subsection we prove our theorem on stability for vertex isoperimetry in  $G_B$ .

For convenience we work with sumsets, which is an equivalent setting via the identity  $\partial_{v, G_B}(n) = |A + (B \cup \{0\})| - |A|$  for  $A, B \subseteq \mathbb{Z}^d$ .

*Proof of Theorem 1.1.* Let  $B$  be a generating set of  $\mathbb{Z}^d$  with  $d \geq 2$ , and assume without loss of generality that  $0 \in B$ . Suppose  $K = K(B, d)$  is sufficiently large and  $A \subseteq \mathbb{Z}^d$  is such that  $|A| = n \geq K$  and  $|A + B| \leq \alpha|A|$ , where

$$\alpha = 1 + (1 + \varepsilon)\beta n^{-1/d}, \quad \text{with } \beta = d\mu(C(B))^{1/d} \text{ and } Kn^{-1/2d} < \varepsilon < K^{-1}.$$

We need to find  $v \in \mathbb{Z}^d$  with  $|A \Delta (v + B_n)| < Kn\sqrt{\varepsilon}$ .

By Lemma 2.3, there are  $p \in \mathbb{N}$ ,  $z \in \mathbb{Z}^d$  and a nice fundamental set  $Q \subseteq Z(B)$  such that  $kC(B) + Q + z \subseteq \Sigma_{k+p}(B) + Q$ , where we choose  $k = \lceil n^{1/2d} \rceil$ . By Lemma 2.2, there is  $\emptyset \neq A' \subseteq A$  with  $|A' + \Sigma_{k+p}(B)| \leq \alpha^{k+p}|A'|$ . It now suffices to prove the following claim.

**Claim 2.6.** We have  $|A'| \geq (1 + 2\varepsilon)^{-d}n$  and  $|A' \Delta (v + B_n)| \leq \frac{1}{2}Kn\sqrt{\varepsilon}$  for some  $v \in \mathbb{Z}^d$ .

To see the bound on  $|A'|$ , we use the choice of  $Q$  and Brunn–Minkowski to get

$$\begin{aligned} \alpha^{k+p}|A'| &\geq |A' + \Sigma_{k+p}(B)| = \mu(A' + \Sigma_{k+p}(B) + Q) \geq \mu(A' + kC(B) + Q) \\ &\geq (\mu(A' + Q)^{1/d} + \mu(kC(B))^{1/d})^d = (|A'|^{1/d} + k\mu(C(B))^{1/d})^d. \end{aligned}$$

Expanding the last expression and dividing throughout by  $|A'|$  then gives

$$1 + k\beta|A'|^{-1/d} \leq \alpha^{k+p} = 1 + (1 + \varepsilon)k\beta n^{-1/d} + O(n^{-1/d}) < 1 + (1 + 2\varepsilon)k\beta n^{-1/d},$$

from which the first part of the claim follows.

For the second part of the claim, let  $A_r := \overline{A' + Q + rC(B)}$  and  $f(r) = \mu(A_r)$ . Since  $Q$ , and so  $A_r$ , is nice, by (2.2) we have  $\text{Per}_{C(B)}(A_r) = f'_+(r)$  for all  $r \geq 0$ .

Now  $f(k) - f(0) < (\alpha^{k+p} - 1)|A'| < (1 + 2\varepsilon)k\beta n^{-1/d}|A'|$ , so by Lemma 2.5 with  $\varepsilon = 1$  there is an  $r \in [0, k)$  such that

$$\text{Per}_{C(B)}(A_r) \leq \frac{f(k) - f(0)}{k} + 1 < (1 + 3\varepsilon)\beta n^{-1/d}|A'|.$$

Then, by Theorem 2.4, the asymmetric index  $\mathcal{A}_{C(B)}(A_r)$  of  $A_r$  with respect to  $C(B)$  is at most  $D\sqrt{3\varepsilon}$ . Thus, there is  $t \in \mathbb{R}^d$  such that

$$\mu(A_r \Delta (t + r'C(B))) \leq D\sqrt{3\varepsilon}\mu(A_r),$$

where  $r' = (\mu(A_r)/\mu(C(B)))^{1/d}$ . As  $\mu(A_r) \leq \mu(A_k) \leq \alpha^{k+p}|A'|$ , we have

$$r' < q := (\alpha^{k+p}|A'|/\mu(C(B)))^{1/d}.$$

Since  $Q \subseteq Z(B)$ , by increasing  $D$  if necessary we may assume that  $Q \subseteq DC(B)$ . Let  $v$  be the unique lattice point in  $t + Q$ . We will show that  $v$  satisfies the claim.

To see this, we start by applying the triangle inequality to get

$$|A' \Delta (v + B_n)| \leq \mu((A' + Q) \Delta (t + qC(B))) + \mu((t + qC(B)) \Delta (v + B_n + Q)). \quad (2.3)$$

We will use the inequality  $\mu(X \Delta Y) \leq 2\mu(X \setminus Y) + |\mu(X) - \mu(Y)|$  for any measurable  $X$  and  $Y$  to estimate both terms on the right of (2.3). Using

$$\mu((A' + Q) \setminus (t + qC(B))) \leq \mu(A_r \setminus (t + r'C(B))) \leq D\sqrt{3\varepsilon}\alpha^{k+p}|A'|,$$

we bound the first term as

$$\begin{aligned} \mu((A' + Q) \Delta (t + qC(B))) &\leq 2\mu((A' + Q) \setminus (t + qC(B))) + |\mu(qC(B)) - |A'|| \\ &\leq (2D\sqrt{3\varepsilon}\alpha^{k+p} + \alpha^{k+p} - 1)|A'| \leq 4D\sqrt{\varepsilon}n, \end{aligned}$$

since  $\alpha^{k+p} = 1 + O(n^{-1/2d})$ . For the second, we observe that

$$\begin{aligned} \mu((v + B_n + Q) \setminus (t + qC(B))) &\leq \mu((t + B_n + 2Q) \setminus (t + qC(B))) \\ &\leq \mu((\kappa_B(n) + 2D)C(B) \setminus qC(B)), \end{aligned}$$

so the second term on the right of (2.3) is  $O(n^{1-1/2d})$ , as  $|B_n|$ ,  $\mu((\kappa_B(n) + 2D)C(B))$  and  $\mu(qC(B))$  are all  $n + O(n^{1-1/2d})$ . This proves the claim, and so the lemma.  $\square$

### 3 Edge isoperimetry

In this short section we deduce our stability result for edge isoperimetry from our stability result for vertex isoperimetry proved in the previous section. We use the reduction in [1].

*Proof of Theorem 1.2.* Let  $B$  be a generating set of  $\mathbb{Z}^d$  with  $d \geq 2$ . We adopt a parameter hierarchy  $n^{-1} \ll \varepsilon \ll s^{-1} \ll \gamma \ll \delta \ll |B|^{-1}$ , i.e. let  $\delta$  be small given  $|B|$ , let  $\gamma$  be small given  $\delta$ , let  $s$  be large given  $\gamma$ , let  $\varepsilon$  be small given  $s$ , and let  $n$  be large given  $\varepsilon$ . Suppose  $A \subseteq \mathbb{Z}^d$  is such that  $|A| = n$  and  $\partial_{e, G_B}(A) \leq \beta n^{1-1/d}(1 + \varepsilon)$ , where  $\beta = d\mu(Z)^{1/d}$  with  $Z = Z(B) = C([B])$ . We need to find  $v \in \mathbb{Z}^d$  with  $|A \Delta (v + [B]_n)| < \delta n$ .

From the proof of [1, Theorem 1], identifying our parameters with theirs via  $\eta = \gamma/2$  and  $s = (1 - \eta)t/k$ , we have  $|A + \Sigma_s([B])| - |A| < (1 + \gamma)s\beta n^{1-1/d}$ . As

$$|A + \Sigma_s([B])| - |A| = \sum_{j=0}^{s-1} |A + \Sigma_{j+1}([B])| - |A + \Sigma_j([B])|,$$

we can therefore fix  $A_+ = A + \Sigma_j([B])$  for some  $j < s$  such that

$$|A_+ + [B]| - |A_+| < (1 + \gamma)\beta n_+^{1-1/d} \leq (1 + \gamma)\beta n_+^{1-1/d},$$

where  $n_+ = |A_+| \leq n + O(n^{1-1/d})$ . By Theorem 1.1 we have  $|A_+ \Delta (v + B_{n_+})| < \frac{1}{2}\delta n_+$  for some  $v \in \mathbb{Z}^d$ . Now  $|A \Delta (v + [B]_n)| < |A_+ \Delta (v + [B]_{n_+})| + |A_+ \setminus A| + |\mathbb{Z} \cap [B]_{n_+}| - |\mathbb{Z} \cap [B]_n| < \delta n$ .  $\square$

Our proof of Theorem 1.2 does not give a tight quantitative result, but we will now demonstrate a simple trick that provides such a result when the generating set  $B$  takes the form  $\{\pm v : v \in \mathcal{B}\}$  for some integral basis  $\mathcal{B}$  of  $\mathbb{Z}^d$  (which may as well be the standard basis  $\{e_1, \dots, e_d\}$ ).

**Theorem 3.1.** *Let  $B = \{\pm e_i : i \in [d]\} \subseteq \mathbb{Z}^d$  with  $d \geq 2$ . Then there is  $K \in \mathbb{N}$  so that for any  $A \subseteq \mathbb{Z}^d$  such that  $|A| = n \geq K$  and  $\partial_{e, G_B}(A) \leq 2dn^{1-1/d}(1 + \varepsilon)$ , where  $Kn^{-1/2d} < \varepsilon < K^{-1}$ , there exists  $v \in \mathbb{Z}^d$  with  $|A \Delta (v + [B]_n)| < Kn\sqrt{\varepsilon}$ .*

*Proof.* Let  $A' = A + [-1/2, 1/2]^d$ . Then the edges of  $G_B$  counted by  $\partial_{e, G_B}(A)$  are in bijection with those  $(d - 1)$ -cubes that occur exactly once as  $x + C$  with  $x \in A$  and  $C$  a facet of  $[-1/2, 1/2]^d$ . Thus  $\partial_{e, G_B}(A) = \text{Per}_Z(A')$ , where  $Z = Z(B) = [-1, 1]^d$ . By Theorem 2.4 we have  $\mathcal{A}_Z(A') \leq D\sqrt{\varepsilon}$ , i.e. there is  $x \in \mathbb{R}^d$  with  $\mu(A' \Delta (x + rZ)) \leq nD\sqrt{\varepsilon}$ , where  $\mu(rZ) = (2r)^d = n$ . We fix  $v \in \mathbb{Z}^d$  with  $x + (r + 1)Z \subseteq v + (r + 2)Z$ . As

$$(A \setminus (x + (r + 1)Z)) + [-1/2, 1/2]^d \subseteq A' \setminus (x + rZ)$$

we have  $|A \setminus (v + (r + 2)Z)| \leq nD\sqrt{\varepsilon}$ . The theorem now follows from

$$A \Delta (v + [B]_n) = 2|A \setminus (v + (r + 2)Z)| + O(n^{1-1/d}) < Kn\sqrt{\varepsilon}. \quad \square$$

We remark that by considering  $A' = A + [-1/2, 1/2]^d$  as in the previous proof one can obtain a bound for the edge isoperimetric problem in  $G_B$  that is tight in some cases. Indeed, by the anisotropic isoperimetric inequality,  $\partial_{e, G_B}(A) = \text{Per}_Z(A') \geq d\mu(Z)^{1/d}\mu(A')^{1-1/d} = 2dn^{1-1/d}$ , which is tight whenever  $n = k^d$  for some  $k \in \mathbb{N}$ . Bollobás and Leader [4] gave a tight result for general  $n$ , although they used compression techniques that alter structure, so it is interesting that exact results can also be obtained by geometric methods, and moreover in two ways: by the argument above, or by the Loomis-Whitney inequality as in [6].



## 4 Concluding Remarks

As mentioned in the introduction, there are several challenging and important open problems in isoperimetric stability, such as the Kahn–Kalai Conjecture and the Polynomial Freiman–Ruzsa Conjecture. We therefore find it rather striking that in this short paper we have been able to characterise isoperimetric stability for general Cayley graphs in lattices. Of course, the brevity of our paper masks the fact that we have greatly relied on previous work, particularly an analogous stability result of [7] in Geometric Measure Theory. This naturally suggests that further investigation of transformations between the discrete and continuous settings may be fruitful in future research.

For the isoperimetric problems considered in this paper, it is natural to ask if one can obtain tighter estimates than those in the asymptotic results of [1, 25], particularly for the edge boundary, where the probabilistic reduction in [1] introduces error terms that are presumably far from optimal. Does our improved estimate for the edge isoperimetric inequality in the remark following Theorem 3.1 hold for general  $B$ , i.e. do we always have  $\partial_{e,G_B}(A) \geq d\mu(Z(B))^{1/d}|A|^{1-1/d}$ ? Do we always have  $\partial_{v,G_B}(A) \geq d\mu(C(B))^{1/d}|A|^{1-1/d}$ ?

It would also be interesting to quantify the dependence on the dimension  $d$  of the constants  $K$  in our theorems. Our use of [7] gives  $K = O(d^7)$ , whereas in the case of the  $\ell^1$ -grid the authors of [6] show  $K = O(d^{5/2})$  and conjecture  $K = O(\sqrt{d})$ .

Finally, a challenging direction for further research is to understand the structure of large sets  $A$  for which  $|A + B| - |A|$  is within a multiplicative  $O(1)$  factor of its minimum value. We conjecture that any such  $A$  can be covered by  $O(1)$  homothetic copies of  $C(B)$  with total volume  $O(|A|)$ .

More generally, in the spirit of the Kahn–Kalai and Polynomial Freiman–Ruzsa Conjectures, we pose the following (somewhat vague) question for any (natural) isoperimetric problem: can any set with boundary  $O(\cdot)$  of the minimum possible be covered by  $O(1)$  ‘canonical examples’ of size  $O(|A|)$ , perhaps even with polynomially-related constants?

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