Initial ideals, Veronese subrings, and rates of algebras

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Abstract

Let S be a polynomial ring over an infinite field and let I be a homogeneous ideal of S. Let T_d be a polynomial ring whose variables correspond to the monomials of degree d in S. We study the initial ideals of the ideals $V_d(I) \subset T_d$ that define the Veronese subrings of S/I. In suitable orders, they are easily deduced from the initial ideal of I. We show that $in(V_d(I))$ is generated in degree $\leq max(\lceil reg(I)/d \rceil, 2)$, where $reg(I)$ is the regularity of the ideal I. (In other words, the d^{th} Veronese subring of any commutative graded ring S/I has a Gröbner basis of degree $\leq max(\lceil reg(I)/d \rceil, 2)$.) We also give bounds on the regularity of I in terms of the degrees of the generators of $in(I)$ and some combinatorial data. This implies a version of Backelin's Theorem that high Veronese subrings of any ring are homogeneous Koszul algebras in the sense of Priddy [Pr70].

We also give a general obstruction for a homogeneous ideal $I \subset S$ to have an initial ideal $in(I)$ that is generated by quadrics, beyond the obvious requirement that I itself should be generated by quadrics, and the stronger statement that S/I is Koszul. We use the obstruction to show that in certain dimensions, a generic complete intersection of quadrics cannot have an initial ideal that is generated by quadrics.

For the application to Backelin's Theorem, we require a result of Backelin whose proof has never appeared. We give a simple proof of a sharpened version, bounding the rate of growth of the degrees of generators for syzygies of any multihomogeneous module over a polynomial ring modulo an ideal generated by monomials, following a method of Bruns and Herzog.

Notation: Throughout this paper we write $S = k[x_1, \ldots, x_r]$ for the graded polynomial ring in r variables over an infinite field k. We will generally deal with a monomial order $>$ on S. We always suppose $x_1 > \cdots > x_r$; by the initial term $in_>(p)$ of a polynomial p we mean the term with the largest monomial. Similarly,

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by the **initial ideal** $in_>(I)$ of I, we mean the ideal generated by the initial terms of all polynomials in I:

$$in_>(I) = \langle in_>(p) \,|\, p \in I \rangle.$$

We write T_d for the polynomial ring over k whose variables correspond to the monomials of degree d in S; there is a natural map $\phi_d : T_d \to S$ sending each variable of T_d to the corresponding monomial in S. If $I \subset S$ is an ideal, we write $V_d(I)$ for the preimage of I in T_d, and $A_{(d)}$ for $T_d/V_d(I)$.

Furthermore, we make the following definitions:

Definition: For a homogeneous ideal $I \subset S$, the **minimal generators** of I are the homogeneous elements of I not in $(x_1, \ldots, x_r)I$. Let $\delta(I)$ be the maximum of the degrees of minimal generators of I, and let $\Delta(I)$ be the minimum, over all choices of variables and of monomial orderings of S of the maximum of the degrees of minimal generators of the initial ideal of I.

All ideals and rings that appear will be graded.

1 Introduction and motivating results

Given a homogeneous ideal $I \subset S$ it is a matter of both computational and theoretical interest to know how low the degree of $in_>(I)$ can be made by choosing variables and monomial order on S in an appropriate way. In particular, one may ask which ideals I admit *quadratic* initial ideals; that is, for which I are there choices of variables and order such that $in_>(I)$ can be generated by monomials of degree 2?

One of the theoretical reasons for interest in this question is that if I admits a quadratic initial ideal then, by a result of Fröberg and a deformation argument noticed by Kempf and others, $A := S/I$ is a (homogeneous) Koszul algebra in the sense of Priddy [Pr70]; that is, the residue field k of A admits an A-free resolution whose maps are given by matrices of linear forms. Using complex arguments about a lattice of ideals derived from a presentation of A as a quotient of a free noncommutative algebra, Backelin [Ba86] proved that for any graded ring A as above the d^{th} **Veronese subring**

$$A_{(d)} := \bigoplus_{i=1}^{\infty} A_{di},$$

is Koszul for all sufficiently large d. Our work started from a request by George Kempf for a simpler proof. In this paper we shall prove the stronger result that $A_{(d)}$ admits a quadratic initial ideal for all sufficiently large d.

To make these results more quantitative, we define a measure of the rate of growth of the degrees of the syzygies in a minimal free resolution:

Definition: Let $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ be a graded ring. For any finitely generated graded A-module M, set $t_i^A(M) = max\{j \mid Tor_i^A(k, M)_j \neq 0\}$, where $Tor_i^A(k, M)_j$ denotes the j^{th} graded piece of $Tor_i^A(k, M)$. The **rate** of A is defined by Backelin [Ba86] to be

$$rate(A) = sup\{(t_i^A(k) - 1)/(i - 1) | i \geq 2\}.$$

For example, A is Koszul iff $rate(A) = 1$. It turns out that the rate of any graded algebra is finite (see for example [An86]) and Backelin actually proves

Theorem 1 *([Ba86]): $rate(A_{(d)}) \leq max(1, rate(A)/d)$.*

One should compare this with the rather trivial result (Proposition 5 below) that if a homogeneous ideal I can be generated by forms of degree m, then the ideal $V_d(I)$ defining $A_{(d)}$ can be generated by forms of degree $\leq max(2, \lceil m/d \rceil)$. In the notation introduced above, $\delta(V_d(I)) \leq max(2, \lceil \delta(I)/d \rceil)$. A similar result with Δ (the minimum, over all choices of variables and of monomial orderings $>$, of the maximum degree of a minimal generator of $in_>(I)$) would lead to a bound on the rate by virtue of Proposition 3. Unfortunately, as we show in Example 3 below, it is *not* true that if some initial ideal of I can be generated by monomials of degree m then $V_d(I)$ admits an initial ideal generated by forms of degree $\leq max(2, \lceil m/d \rceil)$. But there *is* a replacement for m that makes such a formula true, and this is the main result of this paper.

Recall that the **Castelnuovo-Mumford regularity** of I is defined as follows:
Definition: For $I \subset S$, the **regularity** of I is defined as

$$reg(I) = max\{t_i^S(I) - i | i \geq 0\}.$$

Since $t_0^S(I) = \delta(I) \leq reg(I)$, the regularity is $\geq$ the maximal degree of the generators of I. One may think of the regularity as a more stable measure of the size of the generators of I. Our main result is that we may replace the degree of the generators of I by the regularity and get a bound on the degrees of the initial ideals of Veronese powers:

Theorem 2

$$\Delta(V_d(I)) \leq max(2, \lceil reg(I)/d \rceil).$$

In particular, if $d \geq reg(I)/2$ then $\Delta(V_d(I)) = 2$.

In section 5 we explain how to generalize this result to Segre products of Veronese embeddings.

To deduce a version of Backelin's Theorem, one needs a result strengthening the theorem of Fröberg mentioned above. Such a result was stated without proof by Backelin ([Ba86, pp98ff]):

Proposition 3 *If $A = S/I$ with I a homogeneous ideal, then $rate(A) \leq \Delta(I) - 1$. In particular, if $\Delta(I) = 2$ then A is Koszul.*

We will give a simple proof of this proposition (and something more general) in section 4 following ideas of Bruns, Herzog, and Vetter [BrHeVe].

Unfortunately the converse of this result is not true: in particular, the algebra A may be Koszul without I admitting a quadratic initial ideal. In section 6, we formulate another obstruction for an ideal $I \subset S$ to have an quadratic initial ideal.

An easily stated part of Theorem 19 is that if I admits a quadratic initial ideal then I contains far more quadrics of low rank than would a generic subspace of quadrics. We may make this quantitative as follows:

Corollary 4 *If $I \subset S$ admits a quadratic initial ideal, and $dim(S/I) = n$, then I contains an m-dimensional space of quadrics of rank $\leq 2(n+m)-1$ for every $m \leq codim(I)$.*

The obstruction shows that in certain dimensions, a generic complete intersection of quadrics has no initial ideal $in(I)$ which is generated by quadrics, even though every complete intersection of quadrics is a Koszul algebra. There seems no reason to believe that this obstruction, even with the Koszul condition, is enough to guarantee that an ideal admits a quadratic initial ideal, so we pose as a problem the question raised at the beginning of the introduction :

Find necessary and sufficient conditions for an ideal to admit a quadratic initial ideal.

A noncommutative analogue of some of the results on initial ideals, in which T_d is replaced by a free noncommutative algebra mapping onto S, is given in [Ei].

2 Initial ideals for Veronese subrings

As above, let

$$T_d = k[\{z_m\}] \text{ where } m \text{ is a monomial of } S \text{ of degree } d,$$

and let $\phi_d : T_d \to S$ be the map sending z_m to m. If $J \subset S$ is a homogeneous ideal, let $V_d(J)$ denote the preimage of J in T_d. It is easy to see that $V_d(J)$ is generated by the kernel of ϕ_d and, for each generator g of J in degree e, the preimages of the elements of degree nd in $(x_1, \ldots, x_r)^{nd-e}g$, where nd is the smallest multiple of n that is $\geq e$. These elements have degree n in T_d. Since $\ker(\phi_d)$ is generated by forms of degree 2 it follows that $V_d(J)$ is generated by forms of degree $\leq max(\lceil \delta(J)/d \rceil, 2)$.

This gives a proof of the following well-known Proposition.

Proposition 5

$$\delta(V_d(I)) \leq max(2, \lceil \delta(I)/d \rceil).$$

In particular, if $d \geq \delta(I)/2$ then $V_d(I)$ is generated by quadrics.

This Proposition is mentioned by Mumford in [Mu70] (in a slightly different form), though it is surely much older.

We extend the given monomial order on S to a monomial order on T_d as follows: If a, b are monomials in T_d, then $a > b$ if $\phi(a) > \phi(b)$ or $\phi(a) = \phi(b)$ but a is bigger than b in the reverse lexicographic order: that is, given two monomials in T_d of the same degree having the same image in S we order the factors of each in decreasing order, and take as larger the monomial with the smaller factor in the last place where the two differ. Here the order of the variables z_m is defined to be the same as the order in S on the monomials m. We first compute the initial ideal of $\ker(\phi_d)$.

Proposition 6 *With notation as above, $in(\ker(\phi_d)) \subset T_d$ is generated by quadratic forms for every d.*

Remark: Barcanescu and Manolache [BaMa82] proved that the Veronese rings are Koszul – which is a corollary of Proposition 6.

Proof: The ideal $\ker(\phi_d)$ is generated by quadratic forms, each a difference of two monomials that go to the same monomial under ϕ_d. Let J be the monomial ideal generated by the initial terms of these quadratic elements of $\ker(\phi_d)$. We have $J \subset in(\ker(\phi_d))$, and we wish to prove equality. We will show that distinct monomials of T_d not in J map by ϕ_d to distinct monomials of S. It will follow that, for each e,

$$\begin{aligned} dim S_{de} &\geq dim(T_d/J)_e \\ &\geq dim(T_d/(in(\ker(\phi_d))))_e \\ &= dim(T_d/(\ker(\phi_d)))_e \\ &= dim(S_{de}). \end{aligned}$$

Thus $dim(T_d/J)_e = dim(T_d/(in(\ker(\phi_d))))_e$, and $J = in(\ker(\phi_d))$ as desired.

Call the monomials not in J "standard", and say that a product of monomials of degree d in S is standard if its factors correspond to the factors of a standard monomial in T_d. Since $J \subset in(\ker(\phi_d))$, any monomial of S_{de} may be written as a standard product of e monomials of degree d. We must show that if $m \in S$ is a monomial of degree de, then there is a unique way of writing m as a standard product $m_1 \cdots m_e$ of monomials of degree d. We claim that this unique product is obtained by writing out the de factors of m in decreasing order

$$m = x_1 \cdots x_1 x_2 \cdots x_2 \cdots x_r \cdots x_r,$$

and taking m_1 to be the product of the first d factors, m_2 to be the product of the next d factors, and so on.

First we prove the claim for a standard product $m_1 m_2$ with just two factors. Suppose the sequences of indices of the two factors are

$$i_1 = (i_{11} \leq \cdots \leq i_{1d}),\ i_2 = (i_{21} \leq \cdots \leq i_{2d})$$

and $m_1 > m_2$. We must show that $i_{1d} \leq i_{21}$. The product of the monomials m_1' and m_2' obtained from m_1 and m_2 by interchanging the factors $x_{i_{1d}}$ and $x_{i_{21}}$ represents the same element of S. The difference of these products represents a quadratic element of $\ker(\phi_d)$. If $i_{1d} > i_{21}$ then $x_{i_{1d}} < x_{i_{21}}$ and thus $m_2' < m_2$. Consequently the leading term of this quadratic element of $\ker(\phi_d)$ would correspond to $m_1 m_2$, and the product $m_1 m_2$ would not be standard.

Now suppose that $m_1 \cdot \cdots \cdot m_e$ is a standard product of degree de, with e arbitrary, and $m_1 \geq \cdots \geq m_e$. Let $i_{j1} \leq \cdots \leq \alpha_{jd}$ be the indices of the variables in m_j. If $i_{jd} > i_{j+1,1}$ for some j then the product $m_j m_{j+1}$ would not be standard, contradicting the standardness of the entire product. ∎

Note: Even when $r = 3$ and $d = 2$, the initial terms of the minimal system of generators for the ideal $\ker(\phi_2)$ do not generate $in_{>}(\ker(\phi_2))$ under all possible orders $>$. In this case, the ideal is minimally generated by the 2×2 minors of the symmetric 3×3 matrix with entries z_{ij}, $i \leq j$. There are 29 different initial ideals (depending upon the order chosen), 23 of which are generated entirely in degree 2. The other 6 each require an additional generator of degree three.

It would be interesting to characterize the orders $>$ for which the ideal $in_>(\ker(\phi_d))$ is generated by elements of degree 2.

Following the proof of Proposition 6 we say a monomial of T_d is standard if it is not in $in(\ker(\phi_d))$. Let $\sigma : S \to T_d$ be the k-linear map that takes each monomial to its unique standard representative. Because of the way we have defined the order on T_d we have $in(\sigma(p)) = \sigma(in(p))$. Also, σ takes J into $V_d(J)$. As a consequence we have:

Lemma 7 *If J is a homogeneous ideal of S, then $in(V_d(J))$ is the ideal K generated by $in(\ker(\phi_d))$ and the monomials $\sigma(m)$ for $m \in in(J) \cap (im(\phi_d))$.*

Proof: For each degree e it is clear that $dim_k(T_d/K)_e \leq dim_k(S/J)_{de}$, so it is enough to show that $K \subset in(V_d(J))$. Let $p \in J$ be a form with initial term m. Clearly $\sigma(p) \in V_d(J)$, and $in(\sigma(p)) = \sigma(m)$. ∎

Definition: If $m \in S$ is a monomial, then following Eliahou and Kervaire [ElKe90] we write $max(m)$ for the largest index i such that x_i divides m. We call a monomial ideal I **(combinatorially) stable** if for every monomial $m \in I$ and $j < max(m)$, the monomial $(x_j/x_{max(m)})m \in I$.

Theorem 8 *If J is a homogeneous ideal of S such that $in(J)$ is combinatorially stable, then $in(V_d(J))$ is generated by $in(\ker(\phi_d))$ and the monomials $\sigma(m)$ where m runs over the minimal generators of $in(J) \cap im(\phi_d)$. Thus if $\delta(in(J)) \leq u$, then $\delta(in(V_d(J))) \leq max(2, \lceil u/d \rceil)$.*

Proof: Let $n \in in(J) \cap im(\phi_d)$. By Lemma 7 it suffices to show that $\sigma(n)$ is divisible by some $\sigma(m)$, where m is a minimal generator of $in(J) \cap im(\phi_d)$. Let m' be an element of $in(J) \cap im(\phi_d)$ of minimal degree among those dividing n, and write $n = x_{i_1} \cdot \cdots \cdot x_{i_{sd}}$ with $i_1 \leq \cdots \leq i_{sd}$. Say $deg\, m' = de$. Since $in(J)$ is combinatorially stable, it follows that $m := x_{i_1} \cdot \cdots \cdot x_{i_{de}} \in in(J)$, and since m has degree de, we have $m \in im(\phi_d)$ as well. If m were not a minimal generator of $in(J) \cap im(\phi_d)$, then some proper divisor of it would be in $in(J) \cap (im(\phi_d))$ and would divide n, contradicting our choice of m'. As $\sigma(m)$ divides $\sigma(n)$, we are done. ∎

Example 1: The hypothesis of stability cannot be dropped. For example, if $r = 3$ and $J = (x_1x_2x_3)$, and we take the lexicographic or reverse lexicographic order on S, then the initial ideal $in(V_d(J))$ (defined using the order on T_d we associate to the given order on S) requires a cubic generator for all d.

To obtain Theorem 2 as given in the introduction, we need to recall the notion of **Castelnuovo-Mumford regularity**.

Definition: For $I \subset S$, the **regularity** of I is defined as

$$reg(I) = max\{t_i^S(I) - i | i \geq 0\}.$$

Note: $t_0^S(I) = \delta(I) \leq reg(I)$.

Bayer and Stillman [BaSt87] give the following criterion for an ideal to be m-regular, assuming (as we do throughout this paper) that the field k is infinite.

Theorem 9 (BaSt87) *Let $I \subset S$ be an ideal generated in degrees $\leq e$. The following conditions are equivalent:*

1. *I is e-regular,*

2. (a) *For some $j \geq 0$ and for some linear forms $h_1, \ldots, h_j \in S_1$ we have*

$$((I, h_1, \ldots, h_{i-1}) : h_i)_e = (I, h_1, \ldots, h_{i-1})_e$$

   for $i = 1, \ldots, j$, and

   (b)
$$(I, h_1, \ldots, h_j)_e = S_e.$$

3. *Conditions 2a) and 2b) hold for some $j \geq 0$ and for* generic *linear forms $h_1, \ldots, h_j \in S_1$.*

Note: Let g be generically chosen in the Borel group B, the subgroup of $Gl(r)$ consisting of the upper triangular matrices. Then $\langle gx_r, \ldots, gx_{r-j+1} \rangle$ is a generic linear subspace for I. Since gx_i is a generic linear form with respect to $(I, gx_r, \ldots, gx_{i+1})$, x_i is a generic linear form with respect to $(g^{-1}I, x_r, \ldots, x_{i+1})$. If I is Borel-fixed, then $g^{-1}I = I$, and hence we can replace $h_1, \ldots, h_j$ by $x_r, \ldots, x_{r-j+1}$ in the statement of Theorem 9 in this case.

Proposition 10 *Let $I \subset S$ be a Borel-fixed monomial ideal generated in degrees $\leq e$. Then I is e-regular if and only if I_e is combinatorially stable.*

Proof: By the note following the statement of Theorem 9, we may replace $h_1, \ldots, h_j$ by $x_r, \ldots, x_{r-j+1}$ in the statement of the theorem.

Suppose I is e-regular. Then 2b) of Theorem 9 implies that I includes all monomials in $x_1, \ldots, x_{r-j}$ of degree e. And 2a) of the same theorem implies that for every monomial $m \in I$ of degree e with $max(m) > r - j$, I also contains $x_k / x_{max(m)} \cdot m$ for every k with $1 \leq k \leq max(m)$. Taken together, these two statements imply I_e is combinatorially stable.

Conversely, suppose I_e is combinatorially stable, and let j be the smallest integer such that I contains a power of x_{r-j}. It follows that $x_{r-j}^e \in I_e$, and by stability, $(x_1, \ldots, x_{r-j})^e \subset I$, and hence 2b) holds. Let $m \in ((I, x_r, \ldots, x_{i+1}) : x_i)_e$ for some $r - j + 1 \leq i \leq r$; since I is a monomial ideal, we can assume that m is a monomial in proving 2a). If m is divisible by x_k for some $i + 1 \leq k \leq r$, then it is clear that $m \in (I, x_r, \ldots, x_{i+1})$. Thus we may assume m is not divisible by $x_{i+1}, \ldots, x_r$. Since the monomial mx_i belongs to $(I, x_r, \ldots, x_{i+1})$, it must belong to I. Since it has degree $e + 1$, there must be a monomial $m' \in I_e$ and an l such that $mx_i = m'x_l$. Clearly $l \leq i$. Since $m = (x_l / x_i)m'$, combinatorial stability of I_e implies that $m \in I_e$. Thus 2a) of Theorem 9 holds as well, and I is e-regular. ∎

Proof of Theorem 3: If I is an ideal in generic coordinates which is e-regular, then by Theorem 9, $in_>(I)$ is generated in degrees $\leq e$, where $<$ is the reverse lexicographic order. By the above proposition, $in(I_e)$ with respect to reverse lexicographic order is combinatorially stable, and hence by Theorem 8 we have:

$$\begin{aligned}\Delta(V_d(I)) &\leq \delta(in_{>'}(V_d(gI))) \\ &\leq max(2, \lceil reg(gI))/d\rceil) \\ &= max(2, \lceil reg(I)/d\rceil)\end{aligned}$$

(where g is a "general" choice of coordinates, $>$ is reverse lexicographic order, and $>'$ is the induced order), proving Theorem 2.

3 Comments on the main theorem

We have proved that for any homogeneous ideal $I \subset S$, we have $\Delta(V_d(I)) \leq max(2, \lceil reg(I)/d\rceil)$. In particular, for suitable coordinates and order on T_d, the Veronese ideal $V_d(I)$ has quadratic initial ideal for $d \geq reg(I)/2$, and it follows that the Veronese subring $T_d/V_d(I) \subset S/I$ is Koszul for $d \geq reg(I)/2$. In this section we will estimate the regularity of I in order to bound $\Delta(V_d(I))$ in terms of other invariants of I such as $\Delta(I)$. These results can probably be improved, but we will give an example to show that the most optimistic hopes are false.

Theorem 11 *Let r be the number of generators of the polynomial ring S. For any homogeneous ideal $I \subset S$, $\Delta(V_d(I)) \leq max(2, \lceil (r\Delta(I) - r + 1)/d\rceil)$. In particular, for suitable coordinates and order on T_d, the Veronese ideal $V_d(I)$ has quadratic initial ideal for $d \geq (r\Delta(I) - r + 1)/2$.*

Proof: The Taylor resolution [Ta60] gives an upper bound on $reg(I)$, specifically:

$$reg(I) \leq r\Delta(I) - r + 1.$$

With Theorem 2, this gives the result. It is worth mentioning that, by Bayer and Stillman's Theorem 9, there are actually upper and lower bounds relating $reg(I)$ and $\Delta(I)$:

$$\Delta(I) \leq reg(I) \leq r\Delta(I) - r + 1. \blacksquare$$

The assumption $d \geq \Delta(I)/2$ is not enough to imply that $V_d(I)$ has quadratic initial ideal, by the example at the end of this section. We do not know the best estimate for $\Delta(V_d(I))$ in terms of $\Delta(I)$. The problem is combinatorial in the sense that it suffices to consider monomial ideals I.

We have been assuming that the field k is infinite. For arbitrary (in particular, finite) fields k, we have a slightly weaker version of Theorem 11: there is an order on T_d such that $V_d(I)$ has quadratic initial ideal for $d \geq r\lceil\Delta(I)/2\rceil$. We omit the proof, which is not too difficult given a definition of the correct order. The ordering which yields this result is defined as follows:

Definition: For each monomial m in S of degree d, we produce a vector

$$\nu(m) = (\nu_{11}(m), \nu_{12}(m), \ldots, \nu_{1r}(m), \nu_{21}(m), \ldots, \nu_{2r}(m), \nu_{31}(m) \ldots),$$

where $\nu_{ij}(m) = \begin{cases} 0 & \text{if } x_j^i | m \\ 1 & \text{else} \end{cases}$. The order on monomials in S of degree d is then defined by $m > n$ if $\nu(m) > \nu(n)$ in lexicographic order. We define the order of the variables in T_d using the above order on S. Specifically, $z_m > z_n$ if $m > n$ in the order on S defined above. Given this ordering on the variables in T_d, let the order on the monomials in T_d be reverse lexicographic order. ∎

For some monomial ideals I, we can improve the Taylor bound on the regularity of I. First, since one direction of the proof of Proposition 10 does not use the Borel-fixed hypothesis, we have:

Proposition 12 *If $in(I)$ is generated in degrees $\leq u$ and $in(I)_u$ is combinatorially stable, then*

$$reg(I) \leq u.$$

Next we generalize the definition of combinatorial stability.

Definition: Let q be an integer. A monomial ideal I is **q-combinatorially stable**, if for every $m \in I$ and for each $j < max(m)$ there exists an integer s with $1 \leq s \leq q$ such that $x_j^s/x_{max(m)}^s m \in I$.

Proposition 13 *Let I be an ideal in generic coordinates, and let $e = \delta(in_>(I))$, where $>$ is reverse lexicographic order. If I is q-combinatorially stable, then $reg(I) \leq e + (r-1)(q-1)$.*

Proof: Let $t = e + (r-1)(q-1)$. By Proposition 12, we need only show that $J := in(I)_t$ is combinatorially stable. Let $m \in J$. $m = x_1^{b_1+c_1} \cdots x_r^{b_r+c_r}$, where $l := x_1^{b_1} \cdots x_r^{b_r} \in in(I)_e$, and set $n := x_1^{c_1} \cdots x_r^{c_r}$. We have $\sum_{i=1}^r c_i = (r-1)(q-1)$. If $max(m) = max(n)$, then $(x_i/x_{max(m)})m \in I_t$ for all i with $1 \leq i \leq x_{max(m)}$ because n is divisible by $x_{max(m)}$. If $max(m) > max(n)$, then $max(m) = max(l)$, and either there exists some index k such that $c_k \geq q$, or else $c_j = q-1$ for all $j = 1, \ldots, n$. In the first case, we can rewrite m as $l'n'$, where $l' = (x_k^s/x_{max(l)}^s)l$ and $n' = (x_{max(l)}^s/x_k^s)n$, for some $1 \leq s \leq q$. After doing so, $max(m) = max(n)$ and we conclude as before. In the second case, the degree of x_i in $x_i/x_{max(m)}m$ is $b_i + c_i + 1$, and $c_i + 1 = q$. We may rewrite $(x_i/x_{max(m)})m$ as $(x_i^s/x_{max(l)}^s)ln'$, where $n' = (x_{max(l)}^{s-1}/x_i^{s-1})n$, and $1 \leq s \leq q$ is chosen so that $(x_i^s/x_{max(l)}^s)l \in I$. Thus $(x_i/x_{max(m)})m$ is in I_t. ∎

If $char\ k = 0$, then every ideal I in generic coordinates is Borel-fixed and hence 1-combinatorially stable. In this case, the proposition above yields $reg(I) \leq e$, and in fact, equality holds, as Bayer and Stillman proved in [BaSt87]. In $char\ k = p$, every ideal I in generic coordinates has a q-combinatorially stable initial ideal for some q that is a power of $p \leq \delta(in(I))$ [Pa], but even if q is chosen to be as small as possible, $reg(I)$ can be strictly less than $e + (r-1)(q-1)$. An example is the ideal $I = \{a^6, a^2b^4, a^2c^4, b^8, c^8\}$, which is 8-combinatorially stable (implying a bound of 22 on the regularity), but has regularity 16. Also, I has a quadratic initial ideal in the Veronese embedding of degree 5, which is strictly less than the degree of 7 given by Theorem 8.

As noted above, in characteristic 0 and generic coordinates, the regularity of I is equal to $\delta(in(I))$, where the initial ideal is with respect to reverse lexicographic order. In characteristic p, we cannot replace $reg(I)$ with $\delta(in(I))$ in the statement of Theorem 2, as the following example illustrates.

Example 2: A Borel-fixed ideal $I \subset k[a,b]$, with *char* $k = 2$, $e := \delta(I) = 6$, such that the algebra $T_3/V_3(I)$ is not Koszul. It follows that the initial ideal of the Veronese embedding of degree $\lceil e/2 \rceil = 3$ is not generated in degree 2 under any order and any generators for the graded algebra T_3. In fact the ideal defined below has the same properties for k of characteristic 0, except that it is not Borel-fixed in characteristic 0.

Let $I = (a^6, a^2b^4)$, and consider the embedding in degree $3 = \lceil 6/2 \rceil$. Let $A = T_3/V_3(I)$. Thus, in the obvious coordinates $y_i = a^{3-i}b^i$,

$$A = k[y_0, y_1, y_2, y_3]/(y_0^2 = 0, y_0y_2 = y_1^2, y_0y_3 = y_1y_2, y_1y_3 = y_2^2 = 0).$$

The graded vector space $Tor_3^A(k,k)$ is not entirely in degree 3: it has dimension 26 in degree 3 and dimension 2 in degree 4. So A is not Koszul.

In fact, under the induced order used throughout this paper, $in(V_3(I))$ requires 2 cubic generators. However, $in(V_4(I))$ is generated in degree 2. The regularity of I is 9.

4 Resolution of multihomogeneous modules

Fundamental to the discussion of rates above is the estimate of the rate for a monomial ideal given (without proof) by Backelin in [Ba86]. The case of quadratic monomials follows at once from the more precise result of Fröberg in [Fr75]. Fröberg's result was recently reexamined and reproved by Bruns, Herzog and Vetter [BrHeVe] using a different method. Jürgen Herzog has pointed out to us that their method actually proves the entire result claimed by Backelin, in a somewhat strengthened form, and we now present this argument.

Let $S = k[x_1, \ldots, x_r]$ be a polynomial ring over a field k. We will regard S as a Z^r-graded ring, graded by the monomials. Suppose that I is a monomial ideal of S, and set $A := S/I$; the ring A is again Z^r-graded. If M is a finitely generated Z^r-graded module over A, then M has a Z^r-graded minimal free resolution over A. The vector spaces $\mathrm{Tor}_i^A(k,M)$ are Z^r-graded.

For the purpose of bounding degrees it is convenient to turn these multigradings into single gradings. Rather than simply using the total degree, we get a more refined result by defining weights, as follows: Let $w_1, \ldots, w_r$ be non-negative real numbers. For any monomial $m = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ define the *weight* of m to be $w(m) = \sum w_i\alpha_i$. Generalizing the definition of $t_i^A(M)$ used above we define $t_i^A(w,M)$ to be the maximal weight, with respect to w, of a nonzero vector in $Tor_i^A(k,M)$.

We can estimate the $t_i^A(w,M)$ as follows. Given an ordered set $\{g_1, \ldots, g_s\}$ of generators of M we get a filtration

$$Ag_1 \subset Ag_1 + Ag_2 \subset \cdots \subset Ag_1 + \cdots + Ag_s = M$$

of M with quotients the cyclic modules A/J_i where $J_i = ((g_1, \ldots, g_{i-1}) : g_i)$. The set of generators $\{g_i\}$ also gives rise to a surjection ϕ of a free A-module A^s to M sending the i^{th} basis element to g_i. It is easy to show that the kernel of ϕ has a filtration whose successive quotients are the ideals J_i (see the proof of Theorem 15). Thus the weights of the generators of the J_i added to the weights of the g_i give a bound for $t_1^A(w, M)$ (we get a bound and not an exact result because the set of generators for the first syzygy of M produced from sets of generators for the J_i may not be minimal). Moreover, if we have a method for bounding the weights of syzygies of the J_i, we may continue this process. The following Lemma provides what we require:

Lemma 14 *Let $S = k[x_1, \ldots, x_r]$ be a polynomial ring over a field k. Suppose that I is an ideal of S, generated by monomials $n_1, \ldots, n_t$, and set $A := S/I$.*

If $J = (m_1, \ldots, m_s) \subset A$ is an ideal generated by the images m_i of monomials m'_i of S, then the quotient $((m_1, \ldots, m_{s-1}) :_A m_s)$ is generated by the images in A of divisors of the monomials $m'_1, \ldots, m'_{s-1}$ and proper divisors of the monomials $n_1, \ldots, n_t$.

Proof: The quotient is the image in A of $((n_1, \ldots, n_t, m_1, \ldots, m_{s-1}) :_S m_s)$ and is thus generated by divisors of the monomials $n_1, \ldots, n_t, m_1, \ldots, m_{s-1}$. The divisors of the n_i that are not proper go to zero in A. ∎

Using Lemma 14 with the idea above we obtain:

Theorem 15 *Let $S = k[x_1, \ldots, x_r]$ be a polynomial ring over a field k, and let w be a weight function on S as above. Suppose that I is an ideal of S, generated by monomials, and set $A := S/I$. Let M be a Z^r-graded A-module with Z^r-homogeneous generators $\{g_1, \ldots, g_s\}$, of weights $\leq d$, and set $J_i = (Ag_1 + \ldots + Ag_{i-1} \; : \; _A \, g_i)$. If the J_i are generated by elements of weight $\leq e$, and both these elements and the proper divisors of the generators of I have weights $\leq f$, then for each integer $i \geq 1$ we have*

$$t_i^A(w, M) \leq d + e + (i-1)f.$$

Proof: We will inductively construct a (not necessarily minimal) free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

such that the generators of F_0 have weights $\leq d$, the generators of F_1 have weights $\leq d+e$, and for $i \geq 2$ the weights of the generators of F_i are $\leq d+e+(i-1)f$. Since the minimal (multigraded) free resolution is a summand of any free resolution, it follows that the weights of the i^{th} free module in the minimal resolution are also $\leq d + e + (i-1)f$, proving the desired inequality.

Let F_0 be Z^r-graded free A-module with s generators whose degrees match those of the g_i, so that the surjection $\phi_0 : F_0 \to M$ sending the i^{th} basis vector of F_0 to g_i is multihomogeneous. The weights of the generators of F_0 are $\leq d$.

We will prove by induction on s that the module $\ker(\phi_0)$ admits a filtration with successive quotients isomorphic, up to a shift in multidegree, to the ideals J_i, and that the weight of the generators of this kernel are $\leq d + e$. If we define F'_0 to be F_0 modulo the first basis element, and define M' by the short exact sequence

$$0 \to Ag_1 \to M \to M' \to 0$$

then by the snake lemma we get a short exact sequence

$$0 \to J_1 \to \ker(\phi_0) \to \ker(\phi_0') \to 0,$$

where $\phi_0' : F_0' \to M'$ is the induced map. By induction $\ker(\phi_0')$ has a filtration with quotients $J_2, \ldots, J_s$, and generators of weights $\leq d+e$. This gives the desired filtration of $\ker(\phi_0)$. The weights of the generators of the copy of J_1 in the kernel are the weights of the monomials generating the ideal J_1 plus the weight of g_1, so they are also $\leq d+e$, and we are done.

Using this filtration of $\ker(\phi_0)$, we define a free module F_1 whose generators have weights $\leq d+e$ and a map $\phi_1 : F_1 \to F_0$ sending the generators of F_1 to representatives h_l in $\ker(\phi_0)$ of the generators of the successive quotients J_i.

We now repeat the argument, replacing M by $\ker(\phi_0)$ and ϕ_0 by ϕ_1. Lemma 14 applied to the ideals J_i implies that the argument works as before if we replace e by f: that is, $\ker(\phi_1)$ has a filtration with successive quotients isomorphic (up to a shift in mult-degree) to ideals with generators of weight $\leq f$. This allows us to construct F_2 with generators of degrees $\leq d+e+f$ that map onto generators h_i of $\ker(\phi_0)$ such that the ideals $(Ah_1 + \ldots + Ag_{l-1} :_A g_l)$ have generators of weight $\leq f$.

We may continue to repeat the argument, using the bound f from the second step on, and constructing the desired resolution. ∎

In the special case of the resolution of the residue class field k, we may take J_1 to be the maximal ideal, and we get Backelin's result referred to above:

Corollary 16 *Let $S = k[x_1, \ldots, x_r]$ be a polynomial ring over a field k. Suppose that I is an ideal of S, generated by monomials of degree $\leq f$, and set $A := S/I$. We have*

$$t_i^A(k) \leq 1 + (i-1)(f-1).$$

Note that Theorem 15 does *not* prove much about the entries of the matrices in even a non-minimal resolution.

With the ring A as in the Theorem, it would be interesting to know whether there is a minimal free resolution (say of the residue class field), with bases for the free modules occurring, such that the entries of the matrices representing the maps of the resolution with respect to the given bases all have low multidegree (or low weight). The rate bound given in the above theorem, together with the requirement that the resolution be minimal (so that each syzygy has weight at least 1) implies that the individual entries of the i^{th} syzygy matrix must have weights bounded by $d+e+(i-1)f-i+1$. We do not know whether there always exists a free resolution with bases — even non-minimal — such that the entries appearing in the matrices are all proper divisors of the generators of I, or even of the least common multiple of the generators of I.

5 Segre Products of Veronese embeddings

The proofs of Proposition 6 and Theorem 8 can be easily generalized to the Segre-Veronese case. Below are the definitions and statements we can make in this case.

Let

$$S := k[x_{11}, \ldots, x_{1r_1}, \ldots, x_{s1}, \ldots, x_{sr_s}]$$

be the coordinate ring of $\mathbf{P}_1^r \times \cdots \times \mathbf{P}^{r_s}$, where $x_i = (x_{i1}, \ldots, x_{ir_i})$ are the homogeneous coordinates on $\mathbf{P}^{r_i}$. And let

$$T := k[\{z_m\}, m \text{ a monomial of } S \text{ of multi-degree } (d_1, \ldots, d_s)]$$

be the coordinate ring of $\mathbf{P}^N$.

Definition: $\phi : T \longrightarrow S$ by $\phi(z_m) = m = m_1 \cdots m_s$, where $m_i = x_{i1}^{\alpha_{i1}} \cdots x_{is}^{\alpha_{is}}$.

$\ker(\phi)$ is generated by the quadratic binomials of the form $z_m z_n - z_{m'} z_{n'}$, where $m \cdot n = m' \cdot n'$ in S. As in section 2, if a, b are monomials in T, $a > b$ if $\phi(a) > \phi(b)$, or $\phi(a) = \phi(b)$ and $a > b$ in reverse lexicographic order.

Proposition 17 *In reverse lexicographic order, the initial terms of the binomials* $z_m z_n - z_{m'} z_{n'}$ *generate* $in(\ker(\phi))$.

Definition: The **stabilization** $\{I\}$ of an ideal I is defined to be the ideal generated by

$$\{(x_j / x_{max(m)}) m \; | m \in I, \; j = 1, \ldots, max(m)\}.$$

Definition: Call a multi-homogeneous monomial ideal I *combinatorially stable* if it is combinatorially stable in each set of variables independently. That is, given

$$x_1^{\alpha_1} \cdots x_s^{\alpha_s} \in I,$$

we must have

$$\{x_1^{\alpha_1}\} \cdot \{x_2^{\alpha_2}\} \cdot \cdots \cdot \{x_s^{\alpha_s}\} \subset I,$$

where $\{x_i^{\alpha_i}\}$ is the set of all monomials necessary for an ideal in $k[x_{i0}, \ldots, x_{ir_i}]$ containing $x_i^{\alpha_i}$ to be combinatorially stable, and where the above product is the outer product, i.e. all possible products of elements taken one from each set.

Theorem 18 *If I is a multi-homogeneous ideal whose initial ideal is combinatorially stable, then $in(\sigma(I)) = \sigma(in(I))$ (where the initial terms on the left are computed with respect to the induced order with ties broken by reverse lex). Thus if $in(I)$ is generated in degrees $\leq (u_1, \ldots, u_s)$ (u_i = maximum degree of any generator with respect to the i^{th} set of variables), then $in(V(I))$ is generated in degrees $\leq max(2, \lceil u_1/d_1 \rceil, \ldots, \lceil u_s/d_s \rceil)$.*

6 Another obstruction to having a quadratic initial ideal

In this section, we formulate a general obstruction to the existence of a quadratic initial ideal for a given polynomial ideal, beyond the obvious requirement that the ideal must be generated by quadratic polynomials, and even beyond the stronger requirement that the quotient ring must be a Koszul algebra. We use the obstruction to show that in certain dimensions, the ideal of a generic complete intersection of quadrics has no quadratic initial ideal, although every complete intersection of quadrics is a Koszul algebra [BaFr85].

Note. In discussing the existence of a quadratic initial ideal for a homogeneous ideal I in a polynomial ring $S = k[x_1, \ldots, x_r]$, we are asking whether there exists a set of coordinates $x_1', \ldots, x_r'$ (linear combinations of $x_1, \ldots, x_r \in S_1$) and a monomial order with respect to which $in(I)$ is generated by quadratic polynomials.

Theorem 19 *Let I be a homogeneous ideal in a polynomial ring S. Consider the Krull dimensions $r = dim(S)$, $n = dim(S/I)$, $e = r - n$. (Thus, if $n \geq 1$, n is one more than the dimension of the projective variety defined by I.) If there are coordinates and a monomial order such that the initial ideal $in(I)$ has quadratic generators, then the ideal I contains e linearly independent quadratic elements of the form:*

$$\begin{aligned} c_1 x_1^2 &= x_2 L_{1,2} + \cdots + x_{e+n} L_{1,e+n} \\ &\vdots \\ c_e x_e^2 &= x_{e+1} L_{e,e+1} + \cdots + x_{e+n} L_{e,e+n} \end{aligned}$$

for some basis $x_1, \ldots, x_{e+n}$ for S_1 and some $c_i \in k$ and linear forms $L_{ij} \in S_1$.

In particular, I contains an m-dimensional space of quadrics of rank $\leq 2(n + m) - 1$ for every $m \leq codim(I)$.

We recall that the rank of a quadratic form Q over a field k is the rank of a symmetric matrix representing the form.

Proof. We are given that there is a basis $x_1, \ldots, x_{e+n}$ for the vector space S_1 and an ordering of the x-monomials, such that the resulting initial ideal $in(I)$ is generated by $in(I)_2$. We can assume that $x_1 > \cdots > x_{e+n}$ in the monomial ordering.

We observe that for $i = 1, \ldots, e$, there must be at least i quadratic monomials $x_j x_k$ with $e - i + 1 \leq j, k \leq e + n$ which are not allowable. Otherwise the Hilbert series of S/I would be at least equal to the Hilbert series of an algebra $k[x_{e-i+1}, \ldots x_{e+n}]/(< i \text{ relations})$, so the dimension of S/I would be at least that of the latter ring, which is greater than n; this contradicts $dim(S/I) = n$.

Thus, for $i = 1, \ldots, e$, there are i monomials $x_j x_k$, $e - i + 1 \leq j, k \leq e + n$, which are linear combinations of earlier monomials $x_l x_m$. No matter what monomial ordering we are using, at least one of l and m must be $> e - i + 1$ in this situation. So, for all $1 \leq i \leq e$, R satisfies i independent relations of the form:

$$\begin{aligned} & b x_{e-i+1}^2 + (a_{e-i+2,1} x_{e-i+2} x_1 + \cdots + a_{e-i+2,e+n} x_{e-i+2} x_{e+n}) \\ & + (a_{e-i+3,1} x_{e-i+3} x_1 + \cdots + a_{e-i+3,e+n} x_{e-i+3} x_{e+n}) + \cdots \\ & \qquad + (a_{e+n,1} x_{e+n} x_1 + \cdots + a_{e+n,e+n} x_{e+n}^2) = 0. \end{aligned}$$

This implies the statement of the lemma. ∎

Corollary 20 *Let k be an infinite field, and let I be an ideal in $S = k[x_1, \ldots, x_{e+n}]$ generated by e generic quadratic forms defined over k. We assume that $n \geq 0$. (If $n \geq 1$, S/I is the homogeneous coordinate ring of an $(n-1)$-dimensional complete intersection of quadrics in $\mathbf{P}^{e+n-1}$.) If*

$$n < \frac{(e-1)(e-2)}{6}$$

then generic complete intersection ideals I as above do not admit any quadratic initial ideal.

Proof. Any n-dimensional complete intersection of homogeneous quadrics in affine $(e+n)$-space can be described by a point of the Grassmannian of e-dimensional subspaces of S^2V, $\mathrm{Gr}(X_e \subset S^2V)$, where we let $V = S_1$, a vector space of dimension $e+n$ over k; conversely, a nonempty open subset of this Grassmannian corresponds to complete intersections.

Those e-dimensional linear spaces of quadrics which generate an ideal which admits a quadratic initial ideal can, by Lemma 19, be written in the form:

$$\begin{aligned} c_1x_1^2 &= x_2L_{1,2} + \cdots + x_{e+n}L_{1,e+n} \\ &\vdots \qquad\qquad (*) \\ c_ex_e^2 &= x_{e+1}L_{e,e+1} + \cdots + x_{e+n}L_{e,e+n} \end{aligned}$$

for some basis $x_1, \ldots, x_{e+n}$ for V and some $c_i \in k$ and linear forms $L_{ij} \in V$. We want an upper bound for the dimension of the space of e-dimensional linear subspaces of S^2V which can be written in this form. If our bound is less than the dimension of the whole Grassmannian $\mathrm{Gr}(X_e \subset S^2V)$, then we will know that generic complete intersections of quadrics in this dimension do not have any quadratic initial ideal.

Our dimension estimate (which is not always sharp) is based on the following observation. The basis $x_1, \ldots, x_{e+n}$ for V is not important, only the flag $\langle x_{e+n}, \ldots, x_{e+1}\rangle \subset \langle x_{e+n}, \ldots, x_e\rangle \subset \cdots \subset \langle x_{e+n}, \ldots, x_1\rangle = V$. That is, if a linear system of quadrics has the form $(*)$ for one basis $x_1, \ldots, x_{e+n}$ of V, then it has the same form for any basis which gives the same flag $\langle x_{e+1}, \ldots, x_{e+n}\rangle \subset \cdots$.

For any flag $V_n \subset V_{n+1} \subset \cdots \subset V_{e+n} = V$, we consider an associated flag $W_{a_1} \subset W_{a_2} \subset \cdots \subset W_{a_e} = S^2V$ defined by

$$W_{a_i} = (V_{n+i-1} \cdot V) + (V_{n+i} \cdot V_{n+i}) \subset S^2V.$$

(That is, in terms of any basis $x_1, \ldots, x_{e+n}$ adapted to the flag $V_n \subset \cdots$, W_{a_i} is the space of quadrics of the form $c_{e-i+1}x_{e-i+1}^2 = x_{e-i+2}L_{e-i+2} + \cdots + x_{e+n}L_{e+n}$, $L_j \in V$.)

Note. We always use the notation V_i, W_i, etc. to denote i-dimensional vector spaces.

Then, also associated to any flag $V_n \subset V_{n+1} \subset \cdots \subset V_{e+n} = V$, we can consider the space of flags $X_1 \subset \cdots \subset X_e \subset V$ such that $X_i \subset W_{a_i}$ for $i = 1, \ldots, e$. Let $Q_{n,e}$ be the space of flags $V_n \subset V_{n+1} \subset \cdots \subset V_{e+n} = V$ and $X_1 \subset \cdots \subset X_e \subset S^2V$ such that $X_i \subset W_{a_i}$ for $i = 1, \ldots, e$. Then the image of the map

$$\begin{aligned} Q_{n,e} &\to \mathrm{Gr}(X_e \subset S^2V) \\ (V_i, X_i) &\mapsto X_e \end{aligned}$$

contains the set of complete intersections of e homogeneous quadrics in $(e+n)$-space which have a quadratic initial ideal.

Moreover, it is easy to compute the dimension of $Q_{n,e}$, which is an iterated projective bundle over the flag manifold $\mathrm{Fl}(V_n \subset V_{n+1} \subset \cdots \subset V_{e+n} = V)$: given a flag $V_n \subset \cdots$ and hence the associated subspaces W_{a_i}, we first choose the line $X_1 \subset W_{a_1}$, then a line $X_2/X_1 \subset W_{a_2}/X_1$, and so on.

The dimension of the vector space W_{a_i}, for $i = 1, \ldots, e$, is

$$\begin{aligned} a_i &= (e+n) + (e+n-1) + \cdots + (e-i-2) + 1 \\ &= (e+n)(e+n+1)/2 - (e-i+1)(e-i+2)/2 + 1. \end{aligned}$$

Using this, we compute the dimension of the variety $Q_{n,e}$:

$$\begin{aligned} \dim Q_{n,e} &= \dim \mathrm{Fl}(V_n \subset \cdots \subset V_{e+n} = V) + \sum_{i=1}^{e} \dim \mathbf{P}(W_{a_i}/X_{i-1}) \\ &= en + \frac{e(e-1)}{2} + \sum_{i=1}^{e}[\frac{(e+n)(e+n+1)}{2} - \frac{(e-i+1)(e-i+2)}{2} + 1 - i] \\ &= en + \frac{e(e-1)}{2} + \frac{e(e+n)(e+n+1)}{2} - (\sum_{j=1}^{e} \frac{j(j+1)}{2}) + e - \frac{e(e+1)}{2} \\ &= e(n + \frac{(e+n)(e+n+1)}{2} - \frac{(e+1)(e+2)}{6}). \end{aligned}$$

So

$$\begin{aligned} & \dim Q_{n,e} < \dim \mathrm{Gr}(X_e \subset S^2V) \\ \iff & e(n + (e+n)(e+n+1)/2 - (e+1)(e+2)/6) < e((e+n)(e+n+1)/2 - e) \\ \iff & n < (e+1)(e+2)/6 - e \\ \iff & n < (e-1)(e-2)/6 \end{aligned}$$

Thus, if $n < (e-1)(e-2)/6$, then the image of the map $Q_{n,e} \to \mathrm{Gr}(X_e \subset S^2V)$ has dimension less than the dimension of $\mathrm{Gr}(X_e \subset S^2V)$. Thus for $n < (e-1)(e-2)/6$, the ideal generated by e generic quadratic forms in $e+n$ variables has no quadratic initial ideal. ∎

For example, the ideal generated by 3 generic quadratic forms in 3 variables has no quadratic initial ideal. Similarly for a generic complete intersection of 5 quadrics in $\mathbf{P}^5$, or a generic complete intersection of 6 quadrics in $\mathbf{P}^7$. (These last examples are smooth curves of genus 129.)

Explicitly, say over a field k of characteristic 0, the ideal

$$I = (x(x+y), y(y+z), z(z+x)) \subset k[x, y, z] = S$$

is a complete intersection of quadrics, and so S/I is a Koszul algebra [BaFr85], but one can check using Lemma 19 that it has no quadratic initial ideal, for any coordinates and any monomial order. (One has to check that no nonzero linear combination of the relations $x(x+y)$, $y(y+z)$, $z(z+x)$ is the square of a linear form, which is easy.)

References

[An86] D. Anick: On the homology of associative algebras, *Trans. Am. Math. Soc.* **296** (1986), 641-659.

[Ba86] J. Backelin: On the rates of growth of the homologies of Veronese subrings, in *Algebra, Algebraic Topology, and Their Interactions*, ed. J.-E. Roos, Springer Lect. Notes in Math. **1183** (1986), 79-100.

[BaFr85] J. Backelin and R. Fröberg: Koszul algebras, Veronese subrings, and rings with linear resolutions, *Rev. Roum. Math. Pures Appl.* **30** (1985), 85-97.

[BaMa82] S. Barcanescu and N. Manolache: Betti numbers of Segré-Veronese singularities, *Rev. Roum. Math. Pures Appl.* **26** (1982) 549-565.

[BaSt87] D. Bayer and M. Stillman: A theorem on refining division orders by the reverse lexicographic order, *Duke Math. J.* **55** (1987), 321-328.

[BrHeVe] W. Bruns, J. Herzog, and U. Vetter: Syzygies and walks, preprint.

[Ei] D. Eisenbud: Noncommutative Gröbner bases for commutative algebras (to appear).

[EiKoSt88] D. Eisenbud, J. H. Koh, and M. Stillman: Determinantal equations for curves of high degree, *Am. J. Math.* **110** (1988), 513-539.

[ElKe90] S. Eliahou and M. Kervaire: Minimal resolutions of some monomial ideals, *J. Alg.* **129** (1990), 1-25.

[Fr75] R. Fröberg: Determination of a class of Poincaré series. *Math. Scand.* **37** (1975), 29-39.

[Ga79] A. Galligo: Théorème de division et stabilité, *Ann. Inst. Fourier (Grenoble)* **24**(1979), 107-184.

[Ke90] G. Kempf: Some wonderful rings in algebraic geometry, *J. Alg.* **134** (1990), 222-224.

[La90] R. Lazarsfeld: Linear series on algebraic varieties, pp. 715-723 in *Proc. of the International Congress of Mathematicians, Kyoto, 1990*, Springer-Verlag, New York.

[Mu70] D. Mumford: Varieties defined by quadratic equations, pp. 29-100 in *Questions on Algebraic Varieties*, Cremonese, Rome, 1970.

[Pa] K. Pardue: Thesis, Brandeis University (in preparation).

[Pa93] G. Pareschi: Koszul algebras associated to adjunction bundles, *J. Alg.* **157** (1993), 161-169.

[Pr70] S. Priddy: Koszul resolutions, *Trans. Am. Math. Soc.* **152** (1970), 39-60.

[Ta60] D. Taylor: Ideals generated by monomials in an R-sequence, Thesis, University of Chicago, 1960.