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The octonionic eigenvalue problem

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Abstract

By using a real matrix translation, we propose a coupled eigenvalue problem for octonionic operators. In view of possible applications in quantum mechanics, we also discuss the hermiticity of such operators. Previous difficulties in formulating a consistent octonionic Hilbert space are solved by using the new coupled eigenvalue problem and introducing an appropriate scalar product for the probability amplitudes.

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1. Introduction

Developments in quaternionic matrix theory produced interesting and important results in approaching and solving quaternionic eigenvalue problems [1, 2] and linear differential equations with quaternionic coefficients [3, 4]. This renewed the interest in studying quaternionic formulations of quantum mechanics [5]. Previous discussions on quaternionic diffusion and tunneling phenomena [6–11] and recent analysis of confined states [12] are now based on a more solid mathematical understanding of the quaternionic structures involved in such physical problems. Consequently, many of the previous hidden aspects of theory have been clarified and more convincing proposals of quaternionic deviations from complex theory can now be formulated. If quaternions, due to their non-commutativity, represent a challenge for mathematicians and physicists, the use of octonions to formulate quantum theories, due to their non-commutativity and non-associativity, seems a very hard challenge. It will soon become clear that describing the physical world in terms of octonionic mathematical structures involves many conceptual (algebraic and analytic) difficulties that the lack of associativity inevitably conjures up [13, 14]. In this spirit, the paper was intended as an attempt to motivate and stimulate the study of octonionic mathematical problems in view of possible applications in physical theory. The main difficulty in carrying out octonionic formulations of quantum mechanics is related to the appropriate definition of octonionic Hilbert spaces and scalar products [5, 15, 16]. In the following section, we shall come back to this point and present a detailed discussion on octonionic operators and complex geometry. We shall

see that the use of basic concepts in quantum mechanics, such as the correct choice of the number system to appropriately define amplitudes of probabilities, suggests the introduction of coupled octonionic eigenvalue equations where two real parameters play the role that complex eigenvalues do in complex and quaternionic quantum mechanics.

To proceed with the main topic of our paper, i.e. the octonionic eigenvalue problem in quantum mechanics, let us set up notation and terminology, and study a simple eigenvalue problem by using complex, quaternionic and octonionic algebras. This shall elucidate in a practical way some of the difficulties due to loss of commutativity and associativity.

Let \mathbb{O} be the octonionic division algebra. A generic element in this algebra will be represented by

$$o = r_0 + \sum_{m=1}^7 r_m e_m, \quad r_{0,1,\dots,7} \in \mathbb{R}, \quad (1)$$

where e_m are the octonionic imaginary units obeying the following non-commutative and non-associative algebraic rules

$$e_m e_n = -\delta_{mn} + \epsilon_{mnp} e_p, \quad (2)$$

with $m, n, p = 1, 2, \dots, 7$ and ϵ_{mnp} which is a totally antisymmetric tensor equal to the unit for the seven (quaternionic) combinations

$$123, 145, 176, 246, 257, 347 \text{ and } 365. \quad (3)$$

The conjugate, the norm and the inverse of an octonion are respectively defined by

$$\begin{aligned} o^\dagger &= r_0 - \sum_{m=1}^7 r_m e_m, \\ N(o) &= \sqrt{o^\dagger o} = \sqrt{o o^\dagger} = \sqrt{r_0^2 + \dots + r_7^2}, \\ o^{-1} &= o^\dagger / N(o) (0 \neq 0). \end{aligned} \quad (4)$$

Let us now analyze a particular eigenvalue problem in four different contexts. Given the Hermitian matrix

$$M = \begin{pmatrix} 1 & e_1 \\ -e_1 & 1 \end{pmatrix}, \quad (5)$$

we aim to find its complex, quaternionic and octonionic eigenvalues. In this paper, e_1 represents the imaginary unit of the complex field, \mathbb{C} , and e_1, e_2 and e_3 the imaginary units of the quaternionic field \mathbb{H} .

(i) The complex eigenvalue problem (CEP),

$$\text{CEP: } M \Psi = \lambda \Psi, \quad \Psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \quad \psi_{a,b} \text{ and } \lambda \in \mathbb{C}, \quad (6)$$

can be easily solved by standard calculations. Its eigenvectors and eigenvalues are

$$\{\psi_a, \psi_b; \lambda\}_{1,2} = \{z, e_1 z; 0\}_1 \text{ and } \{z, -e_1 z; 2\}_2, \quad z \in \mathbb{C}.$$

(ii) In exactly the same way, we can establish the quaternionic version of equation (6). Due to the non-commutativity of quaternions, we have to introduce left and right eigenvalue problems. Let us first discuss the quaternionic left eigenvalue problem (QLEP),

$$\text{QLEP: } M \Psi = \lambda \Psi, \quad \Psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \quad \psi_{a,b} \text{ and } \lambda \in \mathbb{H}. \quad (7)$$

After some algebraic manipulations, we find the following eigenvectors and eigenvalues:

$$\{\psi_a, \psi_b; \lambda\} = \{q, (\alpha e_1 + \beta e_2 + \gamma e_3)q; 1 - \alpha + \beta e_3 - \gamma e_2\},$$

where

$$\alpha^2 + \beta^2 + \gamma^2 = 1, \quad \alpha, \beta, \gamma \in \mathbb{R} \quad \text{and} \quad q \in \mathbb{H}.$$

In addition to the complex result ($\beta = \gamma = 0$ and $q = z$), we find a surprising result, that is, the Hermitian matrix M has *quaternionic left* eigenvalues. This apparent paradox is soon explained by observing that, due to the non-commutativity of the quaternionic algebra, the standard proof used to show that complex Hermitian matrices have *real* eigenvalues

$$\Psi^\dagger(M\Psi) = (M\Psi)^\dagger\Psi \Rightarrow \Psi^\dagger\lambda\Psi = (\lambda\Psi)^\dagger\Psi [= \Psi^\dagger\lambda^\dagger\Psi]$$

fails for quaternions. In fact, due to the position of the quaternionic eigenvalue λ ,

$$\Psi^\dagger\lambda\Psi = \Psi^\dagger\lambda^\dagger\Psi \not\Rightarrow \lambda = \lambda^\dagger.$$

- (iii) To overcome this difficulty, we change the position of the eigenvalue in equation (7), and introduce the quaternionic right eigenvalues problem (QREP),

$$\text{QREP :} \quad M\Psi = \Psi\lambda, \quad \Psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \quad \psi_{a,b} \text{ and } \lambda \in \mathbb{H}. \quad (8)$$

Simple calculations show that the QREP has the same eigenvalues of the CEP, i.e.

$$\{\psi_a, \psi_b; \lambda\}_{1,2} = \{q, e_1q; 0\}_1 \text{ and } \{q, -e_1q; 2\}_2.$$

The position of the quaternionic eigenvalue plays a fundamental role to guarantee real eigenvalues for Hermitian matrices,

$$\Psi^\dagger(M\Psi) = (M\Psi)^\dagger\Psi \Rightarrow \Psi^\dagger\Psi\lambda = (\Psi\lambda)^\dagger\Psi [= \lambda^\dagger\Psi^\dagger\Psi] \Rightarrow \lambda = \lambda^\dagger.$$

- (iv) What happens for quaternions suggests to consider an octonionic right-eigenvalue problem (OREP),

$$\text{OREP :} \quad M\Psi = \Psi\lambda, \quad \Psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \quad \psi_{a,b} \text{ and } \lambda \in \mathbb{O}. \quad (9)$$

The matrix equation (9) implies

$$-\psi_b^{-1}(e_1\psi_a) = \psi_a^{-1}(e_1\psi_b) \text{ and } \lambda = 1 + \psi_a^{-1}(e_1\psi_b).$$

Restricting ourselves to quaternionic sub-algebras containing e_1 , i.e.

$$e_1e_2e_3, e_1e_4e_5 \text{ and } e_1e_7e_6,$$

we immediately find the following solutions:

$$\begin{aligned} \{\psi_a, \psi_b; \lambda\}_{1,2} = & \{e_2, e_3; 0\}_1 \quad \text{and} \quad \{e_2, -e_3; 2\}_2, \\ & \{e_4, e_5; 0\}_1 \quad \{e_4, -e_5; 2\}_2, \\ & \{e_7, e_6; 0\}_1 \quad \{e_7, -e_6; 2\}_2. \end{aligned}$$

To simplify our discussion, in this example we choose the quaternionic sub-algebra $e_1e_2e_3$. It is rather surprising that, notwithstanding the right position of the octonionic eigenvalue, we find, as for the case of the QLEP, new solutions characterized by *not* real eigenvalues:

$$\begin{aligned} \{\psi_a, \psi_b; \lambda\}_{1,10} = & \{e_2, e_3; 0\}_1 \quad \text{and} \quad \{e_2, -e_3; 2\}_2, \\ & \{e_2, e_4; 1 - e_7\}_3 \quad \{e_2, -e_4; 1 + e_7\}_4, \\ & \{e_2, e_5; 1 + e_6\}_5 \quad \{e_2, -e_5; 1 - e_6\}_6, \\ & \{e_2, e_6; 1 - e_5\}_7 \quad \{e_2, -e_6; 1 + e_5\}_8, \\ & \{e_2, e_7; 1 + e_4\}_9 \quad \{e_2, -e_7; 1 - e_4\}_{10}. \end{aligned}$$

This result is well known in the literature. For a deeper discussion of similar problems encountered in studying octonionic eigenvalue equations for Hermitian matrices and for mathematical techniques to find the new *not* real eigenvalues, the best references are [17–21]. In this paper, we are interested in presenting a conversion method, the main quite practical, to find real eigenvalues of octonionic operators. The matrix representations for these operators [22, 23] could be very useful in view of possible applications in quantum mechanics. Before elaborating the core of our paper, let us briefly explain where the proof given for the CEP and for the QREP is lacking in validity. Due to the non-associativity of the octonionic algebra

$$\Psi^\dagger(M\Psi) \neq (M\Psi)^\dagger\Psi [= (\Psi^\dagger M)\Psi].$$

Consequently, the position of the octonionic eigenvalue in the OREP is *not* sufficient to guarantee the real nature of Hermitian matrix eigenvalues.

The results of this preliminary study are instructive for several reasons. Firstly, they show that using a quaternionic algebra, a great simplification can be obtained by making an initial assumption concerning the position of the quaternionic eigenvalue. Secondly, the possibility of applying a similarity transformation ($u\lambda\bar{u} = z$) opens the door to translating the QREP in its complex or real counterpart. This method is frequently employed to circumvent the difficulties attendant upon the non-commutativity of quaternions [24]. What we aim to prove in this paper is a stronger result. The existence of conversion rules for octonionic operators allows us to give a practical method to solve the coupled octonionic eigenvalue problem (cOEP) which reduces to the QREP in the quaternionic limit [1].

2. Octonionic eigenvalue problem

2.1. Real matrix conversion

The non-associativity of octonions seems to suggest the impossibility of obtaining a real matrix representations (with the standard matrix multiplication rules) for octonionic operators [23]. Nevertheless, the use of left/right barred octonionic operators allows us to reproduce the $GL(8, \mathbb{R})$ group [22]. For the convenience of the reader and to make our exposition as self-contained as possible, we repeat part of the material exposed in [22].

It is well known that octonions are a non-commutative algebra, so we must distinguish between left and right actions of the octonionic imaginary units e_m , by introducing operators L_{o_1} and R_{o_2} whose action on octonionic functions of a real variable, ψ , $\psi : \mathbb{R} \rightarrow \mathbb{O}$, gives

$$L_{o_1}\psi = o_1\psi \quad \text{and} \quad R_{o_2}\psi = \psi o_2, \quad (10)$$

But octonions are also a non-associative algebra that means

$$o_1(\psi o_2) \neq (o_1\psi)o_2,$$

so $L_{o_1}R_{o_2}\psi$ must be calculated, necessarily, in the order the operations appear, that is,

$$L_{o_1}R_{o_2}\psi = L_{o_1}(R_{o_2}\psi) = o_1(\psi o_2)$$

and

$$R_{o_2}L_{o_1}\psi = R_{o_2}(L_{o_1}\psi) = (o_1\psi)o_2.$$

Naturally, the same holds for

$$L_{o_1}L_{o_2}\psi = L_{o_1}(L_{o_2}\psi) = o_1(o_2\psi)$$

and

$$R_{o_1} R_{o_2} \psi = R_{o_1} (R_{o_2} \psi) = R_{o_1} (\psi o_2) = (\psi o_2) o_1.$$

In order to write the more general octonionic operator it is enough to describe all the possible actions of the imaginary units e_m . So, computing the operators just described for the imaginary units, we find

$$\begin{aligned} 1, L_m, R_m & \quad 15 \text{ elements} \\ L_m R_m = R_m L_m & \quad 7 \text{ elements} \\ L_m R_n \quad m \neq n & \quad 42 \text{ elements} \\ R_n L_m \quad m \neq n & \quad 42 \text{ elements} \end{aligned} \tag{11}$$

for $m, n = 1, \dots, 7$, which totals 106 operators. However, it is possible to prove that $L_m R_n$ can be expressed by a suitable combination of $R_n L_m$ operators, reducing the previous 106 elements to 64 [22]. For example, we have

$$L_m R_n + L_n R_m = R_n L_m + R_m L_n.$$

As explicitly shown in appendix A of [22], it is possible to prove the linear independence of these 64 elements which represent the most general octonionic operator

$$o_0 + \sum_{m=1}^7 R_m L_{o_m}, \tag{12}$$

with $o_0, \dots, o_7 \in \mathbb{O}$. This shows the correspondence between our generalized octonions, equation (12) and $GL(8, \mathbb{R})$.

In [23], the authors give representations of octonions and other non-associative algebras by special matrices, which are endowed with special multiplication rules. The introduction of left/right octonionic operators allows us to establish the isomorphism between themselves and $GL(8, \mathbb{R})$ with the standard multiplication rules. In order to explain the idea of conversion, let us look explicitly at the action of the operators R_1 and L_2 on a generic octonionic function $\psi : \mathbb{R} \rightarrow \mathbb{O}$:

$$\psi(x) = \psi_0(x) + e_1 \psi_1(x) + e_2 \psi_2(x) + e_3 \psi_3(x) + e_4 \psi_4(x) + e_5 \psi_5(x) + e_6 \psi_6(x) + e_7 \psi_7(x)$$

with $\psi_{0,\dots,7} : \mathbb{R} \rightarrow \mathbb{R}$. In order to simplify our notation, we omit x variable. So, let us calculate $R_1 \psi$ that gives

$$R_1 \psi = \psi e_1 = e_1 \psi_0 - \psi_1 - e_3 \psi_2 + e_2 \psi_3 - e_5 \psi_4 + e_4 \psi_5 + e_7 \psi_6 - e_6 \psi_7$$

and now, we calculate

$$L_2 \psi = e_2 \psi = e_2 \psi_0 - e_3 \psi_1 - \psi_2 + e_1 \psi_3 + e_6 \psi_4 + e_7 \psi_5 - e_4 \psi_6 - e_5 \psi_7.$$

If we represent the octonionic function ψ by a real column vector 8×1

$$\psi \leftrightarrow \Psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \end{pmatrix}, \tag{13}$$

we can rewrite ψe_1 and $e_2\psi$, respectively, as

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \end{pmatrix} = \begin{pmatrix} -\psi_1 \\ \psi_0 \\ \psi_3 \\ -\psi_2 \\ \psi_5 \\ -\psi_4 \\ -\psi_7 \\ \psi_6 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \end{pmatrix} = \begin{pmatrix} -\psi_2 \\ \psi_3 \\ \psi_0 \\ -\psi_1 \\ -\psi_6 \\ -\psi_7 \\ \psi_4 \\ \psi_5 \end{pmatrix}.$$

Following this procedure, we can construct the complete set of conversion rules for the imaginary units operators L_m and R_m . Observe that if we multiply the matrices that represent L_1 and L_2 , we will find a new matrix that is different from L_3 , that is, $L_1L_2 \neq L_3$, while $e_1e_2 = e_3$. This bluff is soon explained. In deducing our conversion rules, we understand octonions as operators, and so they must be applied to a certain octonionic function, ψ . If we have the octonionic relation

$$(e_1e_2)\psi = e_3\psi,$$

the matrix counterpart will be

$$L_3\psi,$$

since the matrix counterparts are defined by their action upon a function and not upon another operator. On the other hand,

$$e_1(e_2\psi) \neq e_3\psi$$

will be translated to

$$L_1L_2\psi = L_1(L_2\psi) \neq L_3\psi.$$

We have to differentiate between two kinds of multiplication, one, for octonions, denoted by a middle dot, ‘ \cdot ’, and the other, for octonionic operators, denoted by the usual multiplication sign ‘ \times ’. At the level of octonions, one has

$$e_1 \cdot e_2 = e_3,$$

but at the level of octonionic operators

$$L_1 \times L_2 \neq L_3$$

but

$$L_1 \times L_2 = L_3 + R_2L_1 - L_1R_2.$$

Observe that the matricial representations of our operators enable us to reproduce the octonion non-associativity by the matrix algebra. Consider, for example,

$$R_1L_3\psi = R_1(L_3\psi) = (e_3\psi)e_1 = e_2\psi_0 - e_3\psi_1 + \psi_2 - e_1\psi_3 - e_6\psi_4 - e_7\psi_5 + e_4\psi_6 + e_5\psi_7 \tag{14}$$

which gives, in matricial representation

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \end{pmatrix} = \begin{pmatrix} \psi_2 \\ -\psi_3 \\ \psi_0 \\ -\psi_1 \\ \psi_6 \\ \psi_7 \\ -\psi_4 \\ -\psi_5 \end{pmatrix},$$

whereas

$$L_3 R_1 \psi = L_3 (R_1 \psi) = e_3 (\psi e_1) = e_2 \psi_0 - e_3 \psi_1 + \psi_2 - e_1 \psi_3 + e_6 \psi_4 + e_7 \psi_5 - e_4 \psi_6 - e_5 \psi_7 \tag{15}$$

which gives, in matricial representation

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \end{pmatrix} = \begin{pmatrix} \psi_2 \\ -\psi_3 \\ \psi_0 \\ -\psi_1 \\ -\psi_6 \\ -\psi_7 \\ \psi_4 \\ \psi_5 \end{pmatrix}.$$

From the 106 elements given in (11) that we can rewrite in the matricial form, we can extract two different bases for $GL(8, \mathbb{R})$ those are

$$1, L_m, R_n, R_n L_m$$

or

$$1, L_m, R_n, L_m R_n.$$

We now remark some difficulties deriving from octonion non-associativity. When we translate from generalized octonions to 8×8 real matrices, there is no problem. For example, in the octonionic object

$$e_4 \{ [(e_6 \psi) e_1] e_5 \},$$

we quickly recognize the operators

$$L_4 R_5 \text{ and } R_1 L_6.$$

Thus, rewriting the previous object, we have

$$L_4 R_5 R_1 L_6 \psi.$$

In going from 8×8 real matrices to octonions, we should be careful in ordering. For example,

$$AB\psi$$

can be understood as

$$(AB)\psi \text{ or } A(B\psi).$$

For example, by choosing $A = L_1$ and $B = L_2$, we have two possible different translations

$$(L_1 L_2)\psi \rightarrow (e_1 e_2)\psi = e_3 \psi \text{ or } L_1(L_2 \psi) \rightarrow e_1(e_2 \psi).$$

Which is the right equation? The second translation is the right one, observe that $L_1 L_2 \neq L_3$. So, when we find multiple multiplications, to translate correctly in the octonionic formalism, we have to use the following ordering rule:

$$ABC, \dots, Z\psi = A(B(\dots, (Z\psi))).$$

2.2. Coupled eigenvalue problem

Due to the non-commutativity of quaternions, they have represented a challenge when mathematicians tried to extend the well-known complex eigenvalue problem to the quaternionic field. The eigenvalue problem for \mathbb{H} -linear quaternionic operators, which means that only the left action of quaternionic imaginary units are present, and an extension to \mathbb{C} - and \mathbb{R} -linear quaternionic operators that are, respectively, operators which contain, besides the left action of all units, the right action of only one imaginary unit and the right action of all quaternionic imaginary units, have been recently discussed and can be found in [1, 2]. The quaternionic eigenvalue problem allowed the first steps in the theory of quaternionic differential equations [3] and, the latter made possible the study of the Schrödinger equation in the presence of a quaternionic potential in view of analyzing the quaternionic tunneling effect [12]. It is worth pointing out that the eigenvalue problem for \mathbb{R} -linear quaternionic operators provides a remarkable version of the eigenvalue problem dividing it into two coupled equations. For the octonionic eigenvalue problem, we resort to this same method that gives a coupled eigenvalue problem.

Consider the octonionic real eigenvalue problem

$$\mathcal{O}\psi = \psi r, \quad r \in \mathbb{R},$$

where matricial conversion is given by

$$M_o \Psi_{8 \times 1} = r \Psi_{8 \times 1}, \tag{16}$$

where M_o is a real square matrix of order 8. Note that the eigenvalue must be real and this condition is too restrictive. The matrix $M_o \in M_8[\mathbb{R}]$ will probably have complex eigenvalues and, consequently, complex eigenvectors, but when this occurs the conversion backward to the octonionic formalism is not possible. This is exactly the same problem when studying the eigenvalue problem for real linear quaternionic operators. For a deeper discussion of this subject, we refer the reader to [2]. We will use the same technique to study the octonionic eigenvalue problem, but here this is done in a much more rough way. In order to differentiate between octonionic and complex formalisms, after the conversion from octonion to matrices, the complex unit will be denoted by i . So, accepting that M_o has complex eigenvalues, we can rewrite the eigenvalue problem (16) as

$$M_o \Psi = z \Psi, \quad z \in \mathbb{C}. \tag{17}$$

Introducing $\Psi = \xi + i\eta$, where ξ and η are real column vectors 8×1 and $z = a + ib$, with $a, b \in \mathbb{R}$ in (17), we have

$$M_o(\xi + i\eta) = (a + ib)(\xi + i\eta).$$

Separating the real from the imaginary part, we have a coupled equation given by

$$\begin{aligned} M_o \xi &= a\xi - b\eta \\ M_o \eta &= a\eta + b\xi. \end{aligned} \tag{18}$$

Now, all the elements present in (18) are real which allows the conversion to the octonionic formalism. Let us see a one-dimensional example. Consider the problem

$$e_4 \psi = \psi \lambda.$$

We want to find ξ, η and $\lambda \in \mathbb{C}$ that satisfies (18). The first step is to translate the problem in the matricial form which is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \end{pmatrix} = z \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \end{pmatrix}. \tag{19}$$

The eigenvalues and eigenvectors of problem (19) are, respectively,

eigenvalues:

$$-i \quad -i \quad -i \quad -i \quad i \quad i \quad i \quad i$$

eigenvectors:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ i \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ i \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ i \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -i \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -i \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -i \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -i \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} i \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Consider $\lambda = -i$ and the respective eigenvector $\Psi = (0, 0, 0, i, 0, 0, 0, 1)^t$. We have $a = 0$ and $b = -1$, and separating the real from the imaginary part of the eigenvector,

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ i \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

we obtain

$$\xi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \eta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Note that all elements a, b, ξ and η are real, so we can transform each of them in octonionic numbers. This procedure gives

$$\xi = e_7 \text{ and } \eta = e_3$$

and the coupled equations, already in the octonionic formalism, became

$$\begin{aligned} e_4(e_7) &= 0e_3 + 1e_3 = e_3 \\ e_4(e_3) &= -e_7 + 0e_3 = -e_7. \end{aligned}$$

Following the same procedure, consider the 2×2 octonionic matrix

$$\begin{pmatrix} 1 & e_4 \\ 0 & e_5 \end{pmatrix}.$$

Its conversion leads us to a 16×16 real matrix whose eigenvalues are i , $-i$ and 1 . The algebraic multiplicity of λ is equal to its geometric multiplicity. In this case, the algebraic multiplicity of i and $-i$ is equal to 4 and the algebraic multiplicity of 1 is equal to 8. By taking the eigenvalue $-i$, we have $a = 0$ and $b = -1$, and the corresponding real vectors (ξ and η) are given by $\xi = (0, 0, 0, -1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 2)^t$ and $\eta = (0, 0, 0, -1, 0, 0, 1, 0, 0, 0, 2, 0, 0, 0, 0, 0)^t$ which, in octonionic formalism, are written as

$$\xi = \begin{pmatrix} -e_3 + e_6 \\ 2e_7 \end{pmatrix} \text{ and } \eta = \begin{pmatrix} e_3 + e_6 \\ 2e_2 \end{pmatrix},$$

and the coupled equation becomes

$$\begin{aligned} \begin{pmatrix} 1 & e_4 \\ 0 & e_5 \end{pmatrix} \begin{pmatrix} -e_3 + e_6 \\ 2e_7 \end{pmatrix} &= 0 \begin{pmatrix} -e_3 + e_6 \\ 2e_7 \end{pmatrix} - (-1) \begin{pmatrix} e_3 + e_6 \\ 2e_2 \end{pmatrix}, \\ \begin{pmatrix} 1 & e_4 \\ 0 & e_5 \end{pmatrix} \begin{pmatrix} e_3 + e_6 \\ 2e_2 \end{pmatrix} &= - \begin{pmatrix} -e_3 + e_6 \\ 2e_7 \end{pmatrix} + 0 \begin{pmatrix} e_3 + e_6 \\ 2e_2 \end{pmatrix}. \end{aligned}$$

It is interesting to note that when the eigenvalue problem is considered for a \mathbb{H} -linear quaternionic operator, we have

$$Q\psi = \psi\lambda = \psi a + \psi ib. \tag{20}$$

Multiplying the expression above by i , we obtain

$$(Q\psi)i = (\psi a)i + (\psi ib)i,$$

but quaternions are associative which means that

$$Q(\psi i) = (\psi i)a - \psi b.$$

The eigenvectors ψi and ψ play the role of ξ and η , respectively, in the coupled equations. Thus,

$$\begin{aligned} Q\xi &= Q(\psi i) = (\psi i)a - \psi b = \xi a - \eta b \\ Q\eta &= Q\psi = \psi a + (\psi i)b = \eta a + \xi b, \end{aligned} \tag{21}$$

which means that it is possible to associate any pair of functions ξ and η that solves the system (21) with a corresponding eigenvector of (20).

2.3. Complexified eigenvalue problem

An alternative way to solve octonionic eigenvalue problems is to consider complexified octonions, $\mathbb{C}(1, i) \times \mathbb{O}$, which allow us to immediately translate equation (17) as follows:

$$\mathcal{O}\Phi = \Phi z,$$

where \mathcal{O} is the octonionic matrix, $\Psi \in \mathbb{C}(1, i) \times \mathbb{O}$ and $z \in \mathbb{C}(1, i)$. Introducing $\Phi = \phi_1 + i\phi_2$ and $z = a + ib$, we have

$$\mathcal{O}(\phi_1 + i\phi_2) = (\phi_1 + i\phi_2)(a + ib), \quad \mathcal{O}, \phi_1, \phi_2 \in \mathbb{O}, \quad a, b \in \mathbb{R}. \quad (22)$$

Algebraic manipulations lead us to the solution of the problem. For example, consider the same problem present in section 2.2, which is

$$e_4(\phi_1 + i\phi_2) = (\phi_1 + i\phi_2)(a + ib). \quad (23)$$

Now, multiplying (23) by e_4 from the left, we find

$$e_4[e_4(\phi_1 + i\phi_2)] = e_4[(\phi_1 + i\phi_2)(a + ib)],$$

which gives

$$-(\phi_1 + i\phi_2) = [e_4(\phi_1 + i\phi_2)](a + ib) = (a + ib)^2(\phi_1 + i\phi_2),$$

since e_4 commutes with $a + ib$. The equation above gives the very simple system

$$a^2 - b^2 = -1 \text{ and } 2ab = 0,$$

whose solution is $a = 0$ and $b = \pm 1$. So, the eigenvalue is $\pm i$. Now, re-introducing $-i$ in the eigenvalue equation, we find the eigenvectors. Then, for $z = -i$, we have

$$e_4(\phi_1 + i\phi_2) = (\phi_1 + i\phi_2)(-i) = -i\phi_1 + \phi_2 \Rightarrow e_4\phi_1 = \phi_2, e_4\phi_2 = -\phi_1.$$

Remembering that the octonionic imaginary units obey the relation

$$123, 145, 176, 246, 257, 347 \text{ and } 365, \quad (24)$$

it is easy to see that ϕ_1 can assume the values e_4, e_5, e_6 and e_7 and ϕ_2 can assume, respectively, the values $-1, e_1, e_2$ and e_3 . Thus,

$$e_4(e_4 - i) = (e_4 - i)(-i),$$

$$e_4(e_5 + e_1) = (e_5 + e_1)(-i),$$

$$e_4(e_6 + e_2) = (e_6 + e_2)(-i),$$

$$e_4(e_7 + e_3) = (e_7 + e_3)(-i).$$

Let us consider the complexified octonion formalism to solve another example from section 2.2. Consider

$$\begin{pmatrix} 1 & e_4 \\ 0 & e_5 \end{pmatrix}.$$

We want to find the eigenvalues $\lambda = a + ib, a, b \in \mathbb{R}$ and the respective eigenvectors which have the following form:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + i \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \phi_{1,2}, \psi_{1,2} \in \mathbb{O}.$$

The eigenvalue problem is

$$\begin{pmatrix} 1 & e_4 \\ 0 & e_5 \end{pmatrix} \begin{pmatrix} \phi_1 + i\psi_1 \\ \phi_2 + i\psi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 + i\psi_1 \\ \phi_2 + i\psi_2 \end{pmatrix} (a + ib).$$

Performing the matrix multiplication, we have

$$e_4(\phi_2 + i\psi_2) = (\phi_1 + i\psi_1)(a - 1 + ib), \quad (25)$$

$$e_5(\phi_2 + i\psi_2) = (\phi_2 + i\psi_2)(a + ib). \quad (26)$$

The procedure to solve equation (26) is almost the same as the previous one. So, repeating it and fixing the eigenvalue $-i$, we find

$$\phi_2 + i\psi_2 \in \{1 + ie_5, e_1 + ie_4, e_3 + ie_6, e_7 + ie_2\}.$$

In order to find ϕ_1 and ψ_1 , we have to fix an eigenvalue on the set above which introduced in equation (25) gives the result. To find all the possibilities, we have to do this for all elements of the set. To illustrate this, we choose $1 + ie_5$. Remember that we are working with $\lambda = -i$, so

$$e_4(1 + ie_5) = (\phi_1 + i\psi_1)(-1 - i) \Rightarrow e_4 + ie_1 = -\phi_1 - i\phi_1 - i\psi_1 + \psi_1.$$

Solving the system of equations, we find $\phi_1 = -(e_1 + e_4)/2$ and $\psi_1 = (e_4 - e_1)/2$, and we have

$$\begin{pmatrix} 1 & e_4 \\ 0 & e_5 \end{pmatrix} \left[\begin{pmatrix} -e_1 - e_4 \\ 2 \end{pmatrix} + i \begin{pmatrix} e_4 - e_1 \\ 2e_5 \end{pmatrix} \right] = \left[\begin{pmatrix} -e_1 - e_4 \\ 2 \end{pmatrix} + i \begin{pmatrix} e_4 - e_1 \\ 2e_5 \end{pmatrix} \right] (-i).$$

Repeating the procedure for the remaining three cases, always for $\lambda = -i$, we have

$$\begin{aligned} \begin{pmatrix} 1 & e_4 \\ 0 & e_5 \end{pmatrix} \left[\begin{pmatrix} 1 + e_5 \\ 2e_1 \end{pmatrix} + i \begin{pmatrix} 1 - e_5 \\ 2e_4 \end{pmatrix} \right] &= \left[\begin{pmatrix} 1 + e_5 \\ 2e_1 \end{pmatrix} + i \begin{pmatrix} 1 - e_5 \\ 2e_4 \end{pmatrix} \right] (-i), \\ \begin{pmatrix} 1 & e_4 \\ 0 & e_5 \end{pmatrix} \left[\begin{pmatrix} e_2 - e_7 \\ -2e_3 \end{pmatrix} + i \begin{pmatrix} e_2 + e_7 \\ -2e_6 \end{pmatrix} \right] &= \left[\begin{pmatrix} e_2 - e_7 \\ -2e_3 \end{pmatrix} + i \begin{pmatrix} e_2 + e_7 \\ -2e_6 \end{pmatrix} \right] (-i), \\ \begin{pmatrix} 1 & e_4 \\ 0 & e_5 \end{pmatrix} \left[\begin{pmatrix} e_6 - e_3 \\ 2e_7 \end{pmatrix} + i \begin{pmatrix} e_3 + e_6 \\ 2e_2 \end{pmatrix} \right] &= \left[\begin{pmatrix} e_6 - e_3 \\ 2e_7 \end{pmatrix} + i \begin{pmatrix} e_3 + e_6 \\ 2e_2 \end{pmatrix} \right] (-i). \end{aligned}$$

Observe that what is missing when comparing the procedure, for a matrix of order 2, just presented with the previous case is that, in addition, from equation (26) we have

$$\phi_2 + i\psi_2 = 0 \Rightarrow \phi_2 = \psi_2 = 0.$$

Introducing $\phi_2 + i\psi_2 = 0$ in equation (25), we obtain

$$(\phi_1 + i\psi_1)(a - 1 + ib) = 0 \Rightarrow a = 1, b = 0.$$

So, another eigenvalue of the octonionic problem discussed is $\lambda = 1$ whose eigenvectors are

$$\phi_1 + i\psi_1, \forall \phi_1, \psi_1 \in \mathbb{O}.$$

2.4. The group $GL(8, \mathbb{C})$

In the previous subsections, we have presented two equivalent methods to solve the octonionic eigenvalue problem. The translation of octonionic left/right operators by 8×8 real matrices, discussed in subsection 2.1, gives us the possibility of introducing a coupled eigenvalue equation characterized by *two* real numbers a and b and two real eigenvectors ξ and η , see equation (18). This real coupled eigenvalue problem can then be translated by using the equivalence between the group $GL(8, \mathbb{R})$ and the real linear octonionic operators in a coupled real eigenvalue problem for octonionic operators,

$$\begin{aligned} \mathcal{O}_{\mathbb{R}} \psi_{\xi} &= a \psi_{\xi} - b \psi_{\eta}, \\ \mathcal{O}_{\mathbb{R}} \psi_{\eta} &= a \psi_{\eta} + b \psi_{\xi}, \end{aligned} \tag{27}$$

with ψ_{ξ} and $\psi_{\eta} \in \mathbb{O}$, a and $b \in \mathbb{R}$ and $\mathcal{O}_{\mathbb{R}}$ represented by a real linear left/right octonionic operator. The presence of two real coupled eigenvalues suggests, see subsection 2.3, the possibility of re-writing such a coupled real eigenvalue problem as a ‘complex’ eigenvalue

problem. Indeed, by introducing a *new* complex imaginary unit i which commutes with the imaginary units of the octonionic field $e_{1,2,\dots,7}$, we find

$$\mathcal{O}(\underbrace{\psi_\xi + i\psi_\eta}_{\Psi \in \mathbb{O} \times \mathbb{C}}) = \underbrace{(a + ib)}_{z \in \mathbb{C}} (\psi_\xi + i\psi_\eta). \quad (28)$$

The introduction of this complex imaginary unit also suggests us to extend the matrix translation for real linear octonionic operators,

$$\mathcal{O}_{\mathbb{R}} \leftrightarrow GL(8, \mathbb{R}),$$

introduced in [22] e re-presented in subsection 2.1, to *complex* linear octonionic operators

$$\mathcal{O}_{\mathbb{C}} \leftrightarrow GL(8, \mathbb{C}).$$

Thus, the two equivalent methods to solve the eigenvalue problem for a real linear octonionic operators, presented in subsections 2.2 and 2.3, can be used to solve the eigenvalue problem for complex linear octonionic operators. Observing that a complex linear octonionic operator, $\mathcal{O}_{\mathbb{C}}$, is characterized by two real linear octonionic operators ($\mathcal{O}_{1\mathbb{R}}$ and $\mathcal{O}_{2\mathbb{R}}$ —real and complex parts of the complex linear octonionic operator $\mathcal{O}_{\mathbb{C}}$), we can use the matrix translation ($\mathcal{O}_{1\mathbb{R}} \rightarrow M_1$ and $\mathcal{O}_{2\mathbb{R}} \rightarrow M_2$) to represent the complex linear operator by the 8×8 complex matrix $M := M_1 + iM_2$. By solving the eigenvalue problem for the complex matrix M , we find its eigenvectors $\psi \in \mathbb{C}_{8 \times 1}$ and eigenvalues $z \in \mathbb{C}$. Then, we translate back to the octonionic formalism

$$\mathcal{O}_{\mathbb{C}} \Psi = z \Psi, \quad (29)$$

with $\Psi \in \mathbb{O} \times \mathbb{C}$ and $z \in \mathbb{C}$. After a discussion of the hermiticity of octonionic operators and the appropriate use of scalar product to define probability amplitudes, we shall give an explicit example of a physical problem where this translation can be applied.

3. On the hermiticity of octonionic matrices and operators

An important step toward a generalization of standard quantum theories is the use of complex scalar products or complex geometry as referred to by Rembieliński [25].

In quantum mechanics, it is well known that an anti-Hermitian operator obeys the following rule:

$$\langle \psi, \mathcal{O}\varphi \rangle = -\langle \mathcal{O}\psi, \varphi \rangle \Rightarrow \int \psi^\dagger (\mathcal{O}\varphi) dx = - \int (\mathcal{O}\psi)^\dagger \varphi dx = - \int (\psi^\dagger \mathcal{O}^\dagger) \varphi dx. \quad (30)$$

Nevertheless, while in complex and quaternionic quantum mechanics, we can define a direct correspondence between Hermitian matrices and Hermitian operators; in octonionic quantum mechanics this is not possible. For example, we shall see that the matricial representation of e_m is an anti-Hermitian matrix, but no imaginary unit e_m represents an anti-Hermitian operator [22]. In fact, given $\psi : \mathbb{R} \rightarrow \mathbb{O}$ and $\varphi : \mathbb{R} \rightarrow \mathbb{O}$, octonionic functions of a real variable x ,

$$\psi(x) = \psi_0(x) + \sum_{n=1}^7 \psi_n(x)e_n \text{ and } \varphi(x) = \varphi_0(x) + \sum_{n=1}^7 \varphi_n(x)e_n,$$

where each ψ_n and each φ_n are real-valued functions; the non-associativity of the octonionic algebra implies that

$$\int \psi^\dagger (e_m \varphi) dx = \langle \psi, e_m \varphi \rangle \neq -\langle e_m \psi, \varphi \rangle = \int (\psi^\dagger e_m) \varphi dx. \quad (31)$$

This is in contrast to the situation within complex and quaternionic quantum mechanics. Such a difficulty is overcome by using a complex projection of the scalar product (complex

geometry) with respect to one of our imaginary units. We break the symmetry between the seven imaginary units e_1, \dots, e_7 and choose as a projection plane, the one characterized by $(1, e_1)$. The new scalar product is quickly obtained by performing, in the standard definition, the following substitution:

$$\int F(x) dx \rightarrow \int_c F(x) dx \equiv \frac{1}{2} \left[\int F(x) dx - e_1 \left(\int F(x) dx \right) e_1 \right]. \quad (32)$$

Working in octonionic quantum mechanics with complex geometry, e_m represents an anti-Hermitian operator. In order to simplify the proof for the imaginary unit e_1 , we write the octonionic functions ψ and φ as follows:

$$\begin{aligned} \psi &= \tilde{\psi}_1 + e_2 \tilde{\psi}_2 + e_4 \tilde{\psi}_3 + e_6 \tilde{\psi}_4 \\ \varphi &= \tilde{\varphi}_1 + e_2 \tilde{\varphi}_2 + e_4 \tilde{\varphi}_3 + e_6 \tilde{\varphi}_4, \end{aligned} \quad (33)$$

where $\tilde{\psi}_m$ and $\tilde{\varphi}_m$, $m = 1, 2, 3, 4$ are complex-valued functions. The anti-hermiticity of e_1 is recovered if

$$\int_c \psi^\dagger (e_1 \varphi) dx = - \int_c (e_1 \psi)^\dagger \varphi dx \quad (34)$$

holds. Algebraic manipulation shows that, after the complex projection, the only non-vanishing terms are represented by *diagonal* terms, that is, terms that contain the functions $\tilde{\psi}_1^\dagger \tilde{\varphi}_1$, $\tilde{\psi}_2^\dagger \tilde{\varphi}_2$, $\tilde{\psi}_3^\dagger \tilde{\varphi}_3$ and $\tilde{\psi}_4^\dagger \tilde{\varphi}_4$. In fact, terms such as $\tilde{\psi}_1^\dagger e_5 \tilde{\varphi}_3$ and $\tilde{\psi}_3^\dagger \tilde{\varphi}_4$ are annulled by complex projection,

$$\tilde{\psi}_1^\dagger (e_5 \tilde{\varphi}_3) = (\alpha_0 - e_1 \alpha_1) (e_5 \beta_0 + e_4 \beta_1) = \rho e_4 + \sigma e_5, \quad (35)$$

$$(\tilde{\psi}_3^\dagger e_4) (-e_7 \tilde{\varphi}_4) = -(\gamma_0 e_4 - \gamma_1 e_5) (e_7 \delta_0 - e_6 \delta_1) = \varrho e_2 + \varsigma e_3, \quad (36)$$

with $\alpha_{0,1}, \beta_{0,1}, \gamma_{0,1}, \delta_{0,1}, \rho, \sigma, \varrho, \varsigma$ being the real numbers. The diagonal terms give

$$\begin{aligned} \int_c \psi^\dagger (e_1 \varphi) dx &= \int_c \tilde{\psi}_1^\dagger (e_1 \tilde{\varphi}_1) - (\tilde{\psi}_2^\dagger e_2) [e_1 (e_2 \tilde{\varphi}_2)] - (\tilde{\psi}_3^\dagger e_4) [e_1 (e_4 \tilde{\varphi}_3)] - (\tilde{\psi}_4^\dagger e_6) [e_1 (e_6 \tilde{\varphi}_4)] dx \\ &- \int_c (e_1 \psi)^\dagger \varphi dx = \int_c (\tilde{\psi}_1^\dagger e_1) \tilde{\varphi}_1 - [(\tilde{\psi}_2^\dagger e_2) e_1] (e_2 \tilde{\varphi}_2) - [(\tilde{\psi}_3^\dagger e_4) e_1] (e_4 \tilde{\varphi}_3) \\ &- [(\tilde{\psi}_4^\dagger e_6) e_1] (e_6 \tilde{\varphi}_4) dx. \end{aligned}$$

The parentheses above are not relevant since the first term $\tilde{\psi}_1^\dagger e_1 \tilde{\varphi}_1$ is a complex number and the three remaining terms, respectively, $\tilde{\psi}_2^\dagger e_2 e_1 e_2 \tilde{\varphi}_2$ (subalgebra 123), $\tilde{\psi}_3^\dagger e_4 e_1 e_4 \tilde{\varphi}_3$ (subalgebra 145) and $\tilde{\psi}_4^\dagger e_6 e_1 e_6 \tilde{\varphi}_4$ (subalgebra 167) are quaternions. This proves that equation (34) holds. Let us discuss the octonionic Hermitian operator. Following the well-known definition, we have \mathcal{O} as a Hermitian operator, if

$$\langle \psi, \mathcal{O} \varphi \rangle = \langle \mathcal{O} \psi, \varphi \rangle \Rightarrow \int \psi^\dagger (\mathcal{O} \varphi) dx = \int (\mathcal{O} \psi)^\dagger \varphi dx = \int (\psi^\dagger \mathcal{O}^\dagger) \varphi dx. \quad (37)$$

Now, suppose \mathcal{O} is an octonionic Hermitian operator and

$$\mathcal{O} \psi = \psi \lambda, \quad \lambda \in \mathbb{O}. \quad (38)$$

Applying the definition of hermiticity is easy to prove that λ must be real. In fact, let $\psi = \varphi$, then

$$\langle \psi, \mathcal{O} \psi \rangle = \langle \mathcal{O} \psi, \psi \rangle \Rightarrow \psi^\dagger (\mathcal{O} \psi) = (\mathcal{O} \psi)^\dagger \psi = (\psi^\dagger \mathcal{O}^\dagger) \psi \quad (39)$$

which gives, after using equation (38),

$$\psi^\dagger(\psi\lambda) = (\lambda^*\psi^\dagger)\psi. \quad (40)$$

But octonionic numbers, $o_1, o_2 \in \mathbb{O}$, satisfy the property

$$o_1^\dagger(o_1o_2) = (o_1^\dagger o_1)o_2 = (o_2o_1^\dagger)o_1. \quad (41)$$

So, returning to equation (40) and using the property above, we have

$$\psi^\dagger(\psi\lambda) = (\psi^\dagger\psi)\lambda = |\psi|^2\lambda \quad (42)$$

and

$$(\lambda^\dagger\psi^\dagger)\psi = \psi^\dagger(\psi\lambda^\dagger) = (\psi^\dagger\psi)\lambda^\dagger = |\psi|^2\lambda^\dagger. \quad (43)$$

Since $|\psi|^2$ is real, we obtain

$$\lambda = \lambda^\dagger \Rightarrow \lambda \in \mathbb{R}. \quad (44)$$

This means that even in the octonionic formalism, given a Hermitian operator, its eigenvalues must be real. Now, consider the example given by Dray *et al* in [20]. Given a Hermitian matrix, M ,

$$M = \begin{pmatrix} 1 & e_4 \\ -e_4 & 1 \end{pmatrix},$$

we find

$$\begin{pmatrix} 1 & e_4 \\ -e_4 & 1 \end{pmatrix} \begin{pmatrix} e_5 \\ e_7 \end{pmatrix} = \begin{pmatrix} e_5 \\ e_7 \end{pmatrix} (1 - e_6). \quad (45)$$

So, the Hermitian matrix M has an eigenvalue given by $1 - e_6 \in \mathbb{O}$, and $(e_5e_7)^t$ is an eigenvector associated with it. Since the matrix is Hermitian, we should expect *real* eigenvalues, but this does not happen. How can we prove that the matrix M does not represent a Hermitian operator? Consider example (45) and let $\psi = \varphi$. So, we have

$$\begin{aligned} \psi^\dagger(\mathcal{O}\psi) &= \psi^\dagger(\psi\lambda) = (-e_5 - e_7) \left[\begin{pmatrix} e_5 \\ e_7 \end{pmatrix} (1 - e_6) \right] \\ &= (-e_5 - e_7) \begin{pmatrix} e_5 + e_3 \\ e_7 - e_1 \end{pmatrix} = -e_5(e_5 + e_3) - e_7(e_7 - e_1) \\ &= 1 - e_6 + 1 - e_6 = 2 - 2e_6. \end{aligned} \quad (46)$$

Now, calculating $(\mathcal{O}\psi)^\dagger\psi$, we find

$$\begin{aligned} (\mathcal{O}\psi)^\dagger\psi &= \begin{pmatrix} e_5 + e_3 \\ e_7 - e_1 \end{pmatrix}^\dagger \begin{pmatrix} e_5 \\ e_7 \end{pmatrix} = (-e_5 - e_3 - e_7 + e_1) \begin{pmatrix} e_5 \\ e_7 \end{pmatrix} \\ &= -(e_5 + e_3)e_5 + (e_1 - e_7)e_7 = 1 + e_6 + e_6 + 1 = 2 + 2e_6. \end{aligned} \quad (47)$$

Thus, according to definition (37) the operator associated with the Hermitian matrix M is not a Hermitian operator since $\psi^\dagger(\mathcal{O}\psi) \neq (\psi^\dagger\mathcal{O}^\dagger)\psi$.

At this point, we can assert that octonionic Hermitian matrices are not associated with the Hermitian operators, so it is natural to find non-real eigenvalues for the matrices. We conclude that octonionic Hermitian operators necessarily have real eigenvalues. Furthermore, a way to re-obtain the relation between Hermitian matrices and Hermitian operators is by using the complex projection. It is noteworthy to observe that the complex projection of the internal product gives the same result. In the example just given

$$[\langle\psi, \mathcal{O}\psi\rangle]_{\mathbb{C}} = [(\mathcal{O}\psi, \psi)]_{\mathbb{C}} = 2. \quad (48)$$

4. Conclusions

It is well known that amplitudes of probability have to be defined in associative division algebras [5]. Amplitudes of probabilities defined in non-division algebras fail to satisfy the requirement that, in the absence of quantum interference effects, probability amplitude superposition should reduce to probability superposition. The associative law of multiplication is needed to satisfy the completeness formula. Amidst these constraints, how can it be possible to formulate quantum theories by using wavefunctions defined in non-division or non-associative algebras? The answer is very simple. The constraints concern the inner product and *not* the Hilbert space in which we define our wavefunctions. The amplitudes of probability have to be defined in \mathbb{C} or \mathbb{H} , but the vectors in the Hilbert space have *no* constraint. The choice of the quaternionic inner product seems to be best adapted to investigate deviations from the standard complex theory in quantum mechanics and quantum fields [5].

In this paper, we have suggested a solution for the octonionic eigenvalue problem by a formulation based on a coupled equation with two real parameters which play the same role of the complex eigenvalue in complex and quaternionic quantum mechanics. The complex eigenvalue problem in complex and quaternionic quantum mechanics can be obviously solved by a real coupled problem and this should represent the complex and quaternionic limits of the octonionic eigenvalue problem proposed in this paper.

An intriguing result obtained from our investigation was that octonionic Hermitian matrices do *not* necessarily represent octonionic Hermitian operators. This is essentially due to the fact that octonionic Hermitian matrices can have *not* real eigenvalues. To overcome this problem, we need to introduce a complex geometry, i.e. complex inner products. The natural choice is represented by the use of the imaginary unit i which commutes with the octonionic imaginary units $e_{1,2,\dots,7}$.

An interesting application of the material presented in this work can be immediately found in the octonionic formulation of the Dirac equation. The Dirac Hamiltonian [26],

$$\mathcal{H} = \alpha \cdot \mathbf{p} c + \beta m c^2, \quad \mathbf{p} = -i \hbar \nabla, \tag{49}$$

which describes the temporal behavior of relativistic particles is given in terms of 4×4 complex matrices, α and β , satisfying the Dirac algebra

$$\{\alpha, \beta\} = 0, \quad \{\alpha_m, \alpha_n\} = 0 \text{ for } m \neq n, \quad \alpha_n^2 = \beta^2 = I_{4 \times 4}.$$

By using the quaternionic sub-algebra $e_{1,2,3}$, we can immediately find a complexified quaternionic representation for the α matrices,

$$\alpha = i \mathbf{e} = (i e_1, i e_2, i e_3). \tag{50}$$

The matrix β can be represented by using, for example, the octonionic imaginary unit e_4 ,

$$\beta = i e_4. \tag{51}$$

An octonionic representation for the Dirac Hamiltonian is thus given by

$$\mathcal{H} \rightarrow \hbar c \mathbf{e} \cdot \nabla + i m c^2 e_4. \tag{52}$$

The complexified octonionic solution

$$\underbrace{\psi_0 + \mathbf{e} \cdot \boldsymbol{\psi}}_{\Psi \in \mathbb{H} \times \mathbb{C}} + e_4 \underbrace{(\phi_0 + \mathbf{e} \cdot \boldsymbol{\phi})}_{\Phi \in \mathbb{H} \times \mathbb{C}}, \quad \psi_0, \boldsymbol{\psi}, \phi_0, \boldsymbol{\phi} \in \mathbb{C}(1, i),$$

by using \mathbb{C} inner products contains in this formulation two orthogonal spinorial solutions, Ψ and Φ , each one with its four complex degrees of freedom represents a Dirac particle. This suggests a natural e simple one-dimensional octonionic formulation of the standard model, where two orthogonal spinorial solutions are needed to represent the leptonic and quark doublets [27].

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