LOOPS AND THE LAGRANGE PROPERTY

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Abstract. Let $\mathcal{F}$ be a family of finite loops closed under subloops and factor loops. Then every loop in $\mathcal{F}$ has the strong Lagrange property if and only if every simple loop in $\mathcal{F}$ has the weak Lagrange property. We exhibit several such families, and indicate how the Lagrange property enters into the problem of existence of finite simple loops.

The two most important open problems in loop theory, namely the existence of a finite simple Bol loop and the Lagrange property for Moufang loops, have been around for more than 40 years. While we did not solve these problems, we show that they are closely related. Some of the ideas developed here have been present in the loop-theoretical community, however, in a rather vague form. We thus felt the need to express them more precisely and in a more definite way.

We assume only basic familiarity with loops, not reaching beyond the introductory chapters of [Pflugfelder(1990)]. All loops mentioned below are finite.

Let us get started with the crucial notion: the Lagrange property. A loop L is said to have the *weak Lagrange property* if $|K|$ divides $|L|$, for each subloop K of L. It has the *strong Lagrange property* if every subloop K of L has the weak Lagrange property.

A loop may have the weak Lagrange property but not the strong Lagrange property [Pflugfelder(1990), p. 13].

Our main result depends on the following lemma, which is a restatement of [Bruck (1958), Lemma 2.1].

Lemma 1. *Let L be a loop with a normal subloop N such that*

(i) *N has the weak (resp. strong) Lagrange property, and*
(ii) *L/N has the weak (resp. strong) Lagrange property.*

Then L has the weak (resp. strong) Lagrange property.

There are some classes of loops studied in the literature to which the lemma applies directly. For each of these, the normal subloop in question is associative.

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Before we have a look at these examples, allow us to recall a few definitions. Let L be a loop and $x \in L$. When x has a two-sided inverse, we denote it by x^{-1}. A loop L has the *automorphic inverse property* if $x^{-1}y^{-1} = (xy)^{-1}$ holds for every x, $y \in L$. A loop L is (*right*) *Bol* (resp. *left Bol*), if $((xy)z)y = x((yz)y)$ (resp. $(x(yx))z = x(y(xz))$) holds for every x, y, $z \in L$. *Moufang loops* are loops that are both right Bol and left Bol. Since the concepts of right Bol loop and left Bol loop are anti-isomorphic to each other, right Bol and left Bol loops share the same algebraic properties. Thus, in what follows, when we refer to Bol loops, the reader may think of left Bol or right Bol as he or she sees fit. A Bol loop which has the automorphic inverse property is called a *Bruck loop.* Bruck loops of odd exponent are called *B-loops*[Glauberman(1964), p. 376]. An *(*A-loop) is a loop all of whose inner mappings are automorphisms. Finally, an M_k *loop* is a Moufang loop L for which $L/\mathrm{Nuc}(L)$ has exponent $k-1$ and no smaller exponent, where $\mathrm{Nuc}(L)$ is the nucleus of L.

Example 1. For every M_k loop with k even, $K = \mathrm{Nuc}(L)$ is an associative normal subloop such that L/K is of odd exponent, $k-1$, and so L/K must be of odd order. By [Glauberman(1968), Thm. 2], L/K has the strong Lagrange property. By the lemma, so does L.

In particular, every commutative Moufang loop L is a Moufang A-loop [Bruck and Paige(1956)], thus an M_4 loop [Kinyon, Kunen and Phillips, Cor. 2]. Thus, every commutative Moufang loop and every Moufang A-loop has the strong Lagrange property.

Example 2. Alternatively, a commutative Moufang loop of odd order is obviously a B-loop. By [Glauberman(1964), Cor. 4], every B-loop has the strong Lagrange property.

That every commutative Moufang loop has the strong Lagrange property is a well-known folk result. It also follows directly from the lemma and induction on the nilpotence class.

Example 3. The lemma also applies directly to those loops L for which the derived subloop L' (i.e., the smallest subloop L' such that L/L' is an abelian group) has the strong Lagrange property. For instance, let L be a "central" Bol loop in the terminology of Kreuzer [Kreuzer(1997)], i.e., a Bol loop L such that L' is contained in the center. By the lemma, every central Bol loop has the strong Lagrange property.

We now come to our main result—the connection between simple loops and loops satisfying the Lagrange property.

Theorem 1. *Let $\mathcal{F}$ be a nonempty family of finite loops such that*

(1) *If $L \in \mathcal{F}$ and $N \lhd L$, then $N \in \mathcal{F}$;*
(2) *If $L \in \mathcal{F}$ and $N \lhd L$, then $L/N \in \mathcal{F}$;*
(3) *Every simple loop in $\mathcal{F}$ has the weak Lagrange property.*

Then every loop in $\mathcal{F}$ has the weak Lagrange property.

Proof. We proceed by induction on the order of loops in $\mathcal{F}$. Note that (1) implies that $\mathcal{F}$ contains the trivial loop $\langle 1 \rangle$, for which the desired conclusion is trivial. Now fix $L \in \mathcal{F}$ and assume that the result holds for all loops in $\mathcal{F}$ of order less than $|L|$. If L is simple, we are finished by (3). Thus assume that L is not simple, so that L has a nontrivial proper normal subloop N. Since $|N| < |L|$, (1) and the induction hypothesis imply that N has the weak Lagrange property. Since $|L/N| < |L|$, (2) and the induction hypothesis imply that L/N has the weak Lagrange property. By Lemma 1, L has the weak Lagrange property. □

Corollary 1. *Let $\mathcal{F}$ be a nonempty family of finite loops such that*

(1') *If $L \in \mathcal{F}$ and $K \leq L$, then $K \in \mathcal{F}$;*
(2) *If $L \in \mathcal{F}$ and $N \lhd L$, then $L/N \in \mathcal{F}$;*
(3) *Every simple loop in $\mathcal{F}$ has the weak Lagrange property.*

Then every loop in $\mathcal{F}$ has the strong Lagrange property.

Proof. Since (1') implies (1), Theorem 1 implies that every loop in $\mathcal{F}$ has the weak Lagrange property. But then (1') yields the desired result. □

Corollary 2. *Let $\mathcal{V}$ be a variety of loops such that every simple loop in $\mathcal{V}$ has the weak Lagrange property. Then every loop in $\mathcal{V}$ has the strong Lagrange property.*

Corollary 2 is of particular interest for those varieties of loops for which there exists a classification of all simple loops. A prominent example is the variety of Moufang loops, where it is known that every simple nonassociative Moufang loop is isomorphic to a Paige loop (cf. [Paige(1956)], [Liebeck(1987)]). It follows from Corollary 2 that if the weak Lagrange property can be established for each of the Paige loops, then every Moufang loop will have the strong Lagrange property.

There is one Paige loop for every finite field $GF(q)$; its order is $q^3(q^4 - 1)$ when q is even, and $q^3(q^4 - 1)/2$ when q is odd [Paige(1956)]. The weak Lagrange property for the smallest 120-element Paige loop has been established in [Guiliani and Milies(2000)] and [Vojtěchovský(2001)].

To the authors' knowledge, the weak Lagrange property has not been established for the next smallest Paige loop, which has order 1080. Nevertheless, we can still state this result:

Corollary 3. *Every Moufang loop of order less than* 1080 *has the strong Lagrange property.*

Proof. A simple Moufang loop of order less than 1080 is a group or the smallest Paige loop. The result follows from Corollary 1. □

Since no Paige loop is commutative, it follows at once from Corollary 2 that every commutative Moufang loop has the strong Lagrange property.

Altogether, we have demonstrated that a loop L has the strong Lagrange property whenever there is $K \lhd L$ such that both K and L/K belong to one of the (not necessarily equal) classes of loops: Moufang loops with an associative normal subloop of odd index, B-loops, Bruck loops of odd order, loops with the derived subloop having the strong Lagrange property, Moufang loops of order less than 1080.

We conclude this paper with a couple of remarks on a potential application for Corollary 2 to the existence of finite simple nonMoufang Bol loops. This can be split into two problems: the existence of a finite simple Bruck loop and the existence of a finite simple proper (nonMoufang, nonBruck) Bol loop.

First, to establish the existence of a finite simple Bruck loop, it would be sufficient to find a Bruck loop which violated the weak Lagrange property, for then Corollary 2 would imply the existence of a simple Bruck loop which is not a cyclic group.

On the other hand, Corollary 2 might apply to the problem of the existence of finite simple proper Bol loops if the weak Lagrange property is established for all Paige loops. It would then be sufficient to find a Bol loop which violated the weak Lagrange property, for by Corollary 2, there would exist a simple Bol loop which violated that property. If all simple Moufang loops have the weak Lagrange property, then the simple Bol loop in question will not be Moufang.

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