Option Pricing Driven by Lévy Processes

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Nov 2020

A thesis submitted in partial fulfilment of the requirements for the degree of Bachelor of Finance, Economics and Statistics with Honours in Statistics at the Australian National University.

Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and, to the best of my knowledge and belief, contains no material published or written by another person, except where due reference is made in the thesis.

Leo Xiang

Acknowledgements

Just as the motto of ANU goes, ”Naturam Primum Cognoscere Rerum”, I want to be among the top group of people to learn the nature of statistics and this is the most important reason that I chose to study honours in statistics.

The honours year has been the most challenging and fruitful year in my four-year university life. First of all, I would like to express my deepest gratitude to my supervisor Prof. Boris Buchmann. I would not have successfully finished my honours study without his patience and support. Every time I did not know which direction to go, it is Boris who gave me encouragement and confidence to overcome any obstacle and continued my research. His research sparkled my interests in the stochastic process and its general application in financial derivatives.

I would like to extend my gratitude to Dr. Browyn Loong for her effort and patience in guiding me through course selections, thesis writing and advice for further study in US.

Big thanks to another honours cohort Charles for his help and encouragement during the most difficult time of the year.

It goes without saying that my parents have always encouraged me especially through the most difficult period. Every phone call they told me that do not push myself too hard. They gave me the most helpful advice for my study and career seeking. Without their support, I would not have reached where I am today.

I wish to thank all my other friends who made my 4-year university life meaningful. Finally, I would like to acknowledge the financial support provided by Australian National University, Research School of Finance, Actuarial Studies and Statistics Honours Scholarship.

Abstract

The methodology of pricing financial derivatives, particularly stock options, was first introduced by Bachelier and developed by Black, Scholes and Merton, who gave the explicit formula for option pricing. Recent developed models such as jump-diffusion, Heston and Variance Gamma are also widely studied within the quantitative finance field and are proven to be applicable to a certain degree in real markets.

A brief understanding of option pricing with stochastic processes is given in this thesis. Risk neutral valuation and notion of finding an equivalent martingale measure provide a framework under which derivatives are priced. Basic procedures of constructing a Brownian motion and stochastic integral from fundamental blocks are introduced. Infinitely divisible distributions and Lévy processes are detailedly discussed, including Lévy-Itô decomposition and the notion of subordination. Exponential-Lévy model and Fourier transform methods are presented to illustrate different approaches to option pricing. Simulation of AAPL stock prices based on estimated parameters from historical data under jump diffusion model is compared with empirical data to test the fitness of the model. Stock prices by minimal measure and Esscher transform measure are computed under geometric Lévy processes. Finally, univariate Variance Gamma process model is extended to Sato's two factor model for multivariate option pricing.

The focus of this thesis is to give a detailed analysis of different option pricing models using mathematical and statistical concepts and theories, accompanied with simulations and empirical data to test the fitness of models. Extensions to numerous popular models are also discussed.

Contents

Chapter 1

Discrete time models for asset pricing

The main focus of this chapter is the discrete time models in pricing of financial assets. Price formation in financial markets may be explained in terms of fundamentals, so-called rational expectation model or in a relative manner explaining the prices of some assets in terms of other given and observable asset prices.

A derivative security, or contingent claim, is a financial contract whose value at expiration date T is determined by the price of underlying financial instruments at time T. A natural question hereby exists: What are appropriate mathematical objects that would allow us to model asset price dynamics? Since asset prices evolve randomly over time period, in the early financial theory, probability distributions were often used to describe price movements.

Today, the stochastic process provides a general mathematical framework that allows us to build and evaluate models that involve Brownian motion and other various complicated families of random variables. It is useful to distinguish between various classes of stochastic process according to their specific properties. This is important for application in finance since we need to find classes of stochastic processes that we can use as basis for realistic market model. Hence, it allows us to establish fast and efficient methods to calculate important characteristics as stock price, option price or other financial instruments. In this chapter, we start with introducing notion of filtration and end up with multinomial asset pricing model as an extension to CRR binomial model.

1.1 Discrete time processes

One of the fundamental concepts in modern finance is the notion of a martingale. This is a stochastic process that, with its last observed value, provides the best forecast for its future values. Intuitively, the financial quantities, such as asset prices are driven by information. Access to full, accurate, up-to-date information is clearly essential to people engaged in financial service or trading. As time passes, new information becomes available to all participants, who continually update their information. What we need is a mathematical language to model the information flow with respect to time. Therefore the idea of filtration will be introduced in this section.

Definition 1.1.1 *A **(discrete time) filtration** on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a sequence of σ-algebras on Ω, $(\mathcal{F}_n)_{n\in\mathbb{N}}$, with property that*

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}$$

We use symbol $\mathbb{F} := (\mathcal{F}_n)_{n\in\mathbb{N}}$ to denote filtration.

Definition 1.1.2 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\mathbb{F} := (\mathcal{F}_n)_{n\in\mathbb{N}}$. Then $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called a **filtered probability space**. A sequence of random variables $(X_n)_{n\in\mathbb{N}}$ is said to be **adapted** to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if for each $n \in \mathbb{N}$, X_n is $\mathcal{F}_n$-measurable.*

Definition 1.1.3 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Let $(M_n)_{n\in\mathbb{N}}$ be an adapted stochastic process with $M_n \in L^2(\Omega, \mathcal{F}_n, \mathbb{P})$ for each $n \in \mathbb{N}$.*

(i) *$(M_n)_{n\in\mathbb{N}}$ is called a **martingale** with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if $M_n = \mathbb{E}[M_{n+1}|\mathcal{F}_n]$ for all $n \in \mathbb{N}$.*

(ii) *$(M_n)_{n\in\mathbb{N}}$ is called a **submartingale** with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if $M_n \leq \mathbb{E}[M_{n+1}|\mathcal{F}_n]$ for all $n \in \mathbb{N}$.*

(iii) *$(M_n)_{n\in\mathbb{N}}$ is called a **supermartingale** with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if $M_n \geq \mathbb{E}[M_{n+1}|\mathcal{F}_n]$ for all $n \in \mathbb{N}$.*

Remark 1.1.1 *Let $(M_n)_{n\in\mathbb{N}}$ be an adapted stochastic process with $M_n \in L(\Omega, \mathcal{F}_n, \mathbb{P})$ for each $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$,*

$$\mathbb{E}[M_{n+1} - M_n|\mathcal{F}_n] = \mathbb{E}[M_{n+1}|\mathcal{F}_n] - \mathbb{E}[M_n|\mathcal{F}_n] = \mathbb{E}[M_{n+1}|\mathcal{F}_n] - M_n$$

This shows that $(M_n)_{n\in\mathbb{N}}$ will be a martingale if and only if $\mathbb{E}[M_{n+1} - M_n|\mathcal{F}_n] = 0$ for all $n \in \mathbb{N}$.

Martingales were studied by Paul Lévy from 1934 on and Joseph L. Doob from 1940 on. The first systematic exposition was Doob 1953 [17]. Martingales have a useful interpretation in terms of dynamic games: a martingale is ‘constant on average’, and models a fair game.

1.2 Mathematical finance in discrete time

In this section we consider mathematical finance model in discrete time based on Bingham's book [5], particularly which will lead to the Cox-Ross-Rubinstein binomial model (CRR-model). Firstly, we will introduce basic concepts of arbitrage, equivalent martingale and risk neutral pricing formula in order to build the CRR-model. Following the approach of Harrison and Pliska (1981) [30], it suffices to proceed to work with a finite probability space.

We will assume the market contains $d+1$ financial assets with one risk-free asset denoted 0, and d risky assets denoted 1 to d.

Definition 1.2.1 *A **numéraire** is a price process $(X_t)_{t=0}^T$, which is strictly positive for all $t = 0, 1, ..., T$.*

For the standard approach the risk-free asset price process (bond) is used as numéraire. In this case, we just use S_0 without further specification as a numéraire. We take $S_{0,0} = 1$ (the initial value as numéraire), and define $\beta := \dfrac{1}{S_{0,t}}$ as a discount factor.

Definition 1.2.2 *A **trading strategy** φ is a $\mathbb{N}^{d+1}$ vector stochastic process $\varphi = (\varphi_t)_{t=1}^T = ((\varphi_{0,t}, \varphi_{1,t}, ..., \varphi_{d,t})')_{t=1}^T$ which is predictable: each $\varphi_{i,t}$ is $\mathcal{F}_{t-1}$-measurable for $t \geq 1$.*
Remark: $\varphi_{i,t}$ denotes the number of shares of asset i held in the portfolio at time t-to be determined on the basis of information available before time t. The investor selects his time t portfolio after observing the prices at $t-1$.

One thing needs to highlight that the portfolio φ_t must be established before and held until after, announcement of the prices S_t.

Definition 1.2.3 *The value of the portfolio at time t is the scalar product*

$$V_{\varphi,t} = \varphi_t \cdot S_t := \sum_{i=0}^{d} \varphi_{i,t} S_{i,t}, \quad \textit{for } t = 1, 2, ..., T \textit{ and } V_{\varphi,0} = \varphi_1 \cdot S_0$$

*The process $V_{\varphi,t}$ is called the **wealth process** of the trading strategy.*

Definition 1.2.4 *The **gains process** G_φ of a trading strategy φ is given by*

$$G_{\varphi,t} := \sum_{\tau=1}^{t} \varphi_\tau (S_\tau - S_{\tau-1}) = \sum_{\tau=1}^{t} \varphi_\tau \Delta S_\tau, \quad (t = 1, 2, ..., T)$$

Define $\widetilde{S}_t = (1, \beta_t S_{1,t}, ..., \beta_t S_{d,t})'$, the vector of discounted stock prices, and consider the discounted value process

$$\widetilde{V}_{\varphi,t} = \beta_t(\varphi_t \cdot S_t) = \varphi_t \cdot \widetilde{S}_t, \quad (t = 1, 2, ..., T)$$

and the discounted gains process

$$\widetilde{G}_{\varphi,t} := \sum_{\tau=1}^{t} \varphi_\tau \cdot (\widetilde{S}_\tau - \widetilde{S}_{\tau-1}) = \sum_{\tau=1}^{t} \varphi_\tau \cdot \Delta \widetilde{S}_\tau, \quad (t = 1, 2, ..., T)$$

The discounted gains process reflects gains from trading with assets 1 to d only, which in the case of standard model are d risky assets.

Definition 1.2.5 *The strategy φ is **self-financing** if*

$$\varphi_t \cdot S_t = \varphi_{t+1} \cdot S_t, \quad \textit{for } t = 1, 2, ..., T-1$$

This formula explains that when new prices S_t are observed at time t, the investor adjusts the portfolio from φ_t to φ_{t+1}, without consuming any wealth, i.e. the new portfolio allocation does not change the overall value from what it was at time $t-1$.

We then start modelling of derivative instruments under the current framework and we can look into concepts of equivalent martingale measure.

Proposition 1.2.1 *(Numeraire Invariance). Let X_t be a numeraire. A trading strategy φ is self-financing with respect to S_t if and only if φ is self-financing with respect to $X_t^{-1}S_t$.*

Proof: Since X_t is strictly positive for all $t = 0, 1, 2..., T$, we have the following equivalence:

$$\varphi_t \cdot S_t = \varphi_{t+1} \cdot S_t \Leftrightarrow \varphi_t \cdot X_t^{-1} S_t = \varphi_{t+1} \cdot X_t^{-1} S_t, \quad t = 0, 1, 2..., T-1. \quad \square$$

Corollary 1.2.1 *A trading strategy φ is self-financing with respect to S_t if and only if φ is self-financing with respect to $\tilde{S}_t$.*

Definition 1.2.6 *A contingent claim X with maturity date T is an arbitrary $\mathcal{F}_T-$ **measurable** random variable, which is by the finiteness of the probability space bounded. We denote the class of all contingent claims by $L^0 = L^0(\Omega, \mathcal{F}, \mathbb{P})$.*

A typical example of a contingent claim X is an option on some underlying asset S; we then have a functional relation $X = f(S)$ with some function f (e.g. $X = (S_T - K)^+$).

1.3 Equivalent martingale measure

The central condition in single period case is the absence of arbitrage opportunities. We now define the mathematical counter part of this economic principle in our current setting.

Existence of equivalent martingale measures provide a basic condition for the CRR-model (non-arbitrage condition). To further illustrate this concept, we first look at the non-arbitrage condition.

Definition 1.3.1 *Let ϕ be a set of self-financing strategies. A strategy ϕ is called an **arbitrage opportunity** with respect to ϕ if $\mathbb{P}\{V_\varphi(0) = 0\} = 1$, and the terminal wealth of ϕ satisfies*

$$\mathbb{P}\{V_\varphi(T) \geq 0\} = 1 \quad \textit{and} \quad \mathbb{P}\{V_\varphi(T) \geq 0\} > 0.$$

Therefore, an arbitrage is a self-financing strategy with zero initial value. This produces a non-negative final value with probability one and a positive probability of a positive final value.

Definition 1.3.2 *We say that a financial market $\mathcal{M}$ is **arbitrage-free** if there are no arbitrage opportunities in the class ϕ of trading strategies.*

One of the basic assumptions in finance is that markets are free of arbitrage possibilities. Since arbitrage implies the creation of wealth out of nothing, it seems obvious that such possibilities should be rare in financial markets. Thus in the most of financial pricing models, we assume that markets are arbitrage-free.

Definition 1.3.3 *A probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbb{P}$ is called a **martingale measure** for discounted price process $\tilde{S}$ if the process $\tilde{S}$ follows a $\mathbb{P}^*$-martingale with respect to the filtration $\mathbb{F}$. We denote by $\mathcal{P}(\tilde{S})$ the class of equivalent martingale measures.*

From a finance perspective, equivalent martingale measures is a probability distribution that shows possible expected payouts from an investment adjusted for an investor's degree of risk aversion. In an efficient market, this present value calculation should be equal to the price at which the security is currently trading. EMMs are most commonly used in the pricing of financial derivative, since this is the most common case of security type which has numerous discrete, contingent payouts.

Proposition 1.3.1 *Let $\mathbb{P}^* \in \mathcal{P}(\tilde{S})$ be an equivalent martingale measure and $\varphi \in \phi$ any self-financing strategy. Then the wealth process $\widetilde{V}_\varphi(t)$ is a $\mathbb{P}^*$-martingale with respect to the filtration $\mathbb{F}$.*

Proof: By the self-financing property of φ, we have

$$\widetilde{V}_{\varphi,t} = V_{\varphi,0} + \widetilde{G}_{\varphi,t}, \quad (t = 0, 1, ..., T)$$

Hence,

$$\widetilde{V}_{\varphi,t+1} - \widetilde{V}_{\varphi,t} = \widetilde{G}_{\varphi,t+1} - \widetilde{G}_{\varphi,t} = \varphi_{t+1} \cdot (\widetilde{S}_{t+1} - \widetilde{S}_t).$$

$\widetilde{V}_{\varphi,t}$ is the martingale transform of the $\mathbb{P}^*$ martingale $\tilde{S}$ by φ and a $\mathbb{P}^*$ martingale itself. $\square$

Proposition 1.3.2 *If an equivalent martingale measure exists-that is, if $\mathcal{P}(\tilde{S}) \neq \emptyset$–then the market $\mathcal{M}$ is arbitrage-free.*

Proof: Assume such a $\mathbb{P}^*$ exists. $\forall$ self-financing strategy φ, we have

$$\widetilde{V}_{\varphi,t} = V_{\varphi,0} + \sum_{\tau=1}^{t} \varphi_\tau \cdot \Delta\widetilde{S}_\tau.$$

By Proposition (1.3.1), $\widetilde{S}_t$ a $\mathbb{P}^*$-martingale implies $\widetilde{V}_{\varphi,t}$ is a $\mathbb{P}^*$-martingale. Therefore the initial and final $\mathbb{P}^*$-martingale are the same,

$$\mathbb{E}^*[\widetilde{V}_{\varphi,T}] = \mathbb{E}^*[\widetilde{V}_{\varphi,0}].$$

If the strategy is an arbitrage opportunity its initial value, RHS of the above equation, is zero. Therefore the LHS of the equation is zero, but $\widetilde{V}_{\varphi,T} \geq 0$ by definition. Therefore no arbitrage is possible. $\square$

Theorem 1.3.1 *(No-arbitrage Theorem). The market $\mathcal{M}$ is arbitrage free if and only if there exists a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which the discounted d-dimensional asset price process $\tilde{S}$ is a $\mathbb{P}^*$-martingale.*

Summary

These definitions and theorems are known as the Fundamental Theorem of Finance and lead to discovery of risk-neutral pricing. The earlier option pricing results of Black, Scholes and Merton 1973 [6,52] were the catalyst for much of the work. The central part of the Fundamental Theorem, that the absence of arbitrage is equivalent to the existence of a positive linear pricing operator, first appeared in Ross (1973), and the first statement of risk neutral pricing appeared in Cox and Ross (1976). The Fundamental Theorem was extended in Harrison and Kreps (1979) [31], who described risk-neutral pricing as a martingale expectation. Dybvig and Ross (1987) coined the terms *Fundamental Theorem* to describe these basic results and *Representation Theorem* to describe the principal equivalent forms for the pricing operator.

1.4 Risk neutral pricing

We now focus on building the main pricing theorem of the financial derivatives on the ground of the introduced concepts. As in Section 1.2, we have already reproduced the cash flow of a contingent claim in terms of a portfolio of the underlying assets. On the other hand, Section 1.3 gives the equivalence of the no-arbitrage condition and the existence of risk-neutral probability measures imply that risk-neutral pricing may exist. We will explore the risk neutral pricing theorem in further details in this section.

Definition 1.4.1 *A contingent claim is **attainable** if there exists a replicating strategy $\varphi \in \Phi$ such that*

$$V_{\varphi,T} = X \textit{(payoff of the claim).}$$

Working with discounted values, we find

$$\beta_T X = \widetilde{V}_{\varphi,T} = V_0 + \widetilde{G}_{\varphi,T},$$

where β is the discount factor.

In an efficient financial market, we expect that the law of one price holds true, i.e. identical securities should sell for identical prices. Otherwise, arbitrageurs would make riskless profit. Therefore, the no-arbitrage condition implies that for an attainable contingent claim its price at time t must be the same as the value of any replicating strategy.

Proposition 1.4.1 *Suppose the market $\mathcal{M}$ is arbitrage-free. Then any attainable contingent claim X is uniquely replicated in $\mathcal{M}$.*

Proof: See [5] Proposition 4.2.4.

Definition 1.4.2 *Suppose the market is arbitrage-free. Let X be any attainable contingent claim with time T maturity. Then the **arbitrage price process** $\pi_X(t), 0 \leq t \leq T$ is given by the value process of any replicating strategy φ for X.*

The hedging strategy that replicates the outcome of a contingent claim is fundamental of financial derivatives pricing. The famous arbitrage valuation models, Black-Scholes, depend on the idea that a financial instrument, such as option, can be perfectly hedging using the underlying asset, which makes possible to create a portfolio that replicates the instrument exactly.

Next, we will introduce an important proposition from the definitions above that is the risk-neutral pricing formula, which is the central idea used in CRR binomial model.

Proposition 1.4.2 *The arbitrage price process of any attainable contingent claim X is given by the risk-neutral valuation formula*

$$f_X(t) = \beta(t)^{-1}\mathbb{E}^*[X\beta(T)|\mathcal{F}_t], \quad \forall t = 0, 1, ..., T,$$

where $\mathbb{E}^*$ is the expectation with respect to an equivalent martingale $\mathbb{P}^*$.

Proof: Given the market is arbitrage-free, there exists an equivalent martingale measure $\mathbb{P}^*$. By Proposition (1.3.1), the discounted value process $\widetilde{V}_\varphi$ of any self-financing strategy φ is a $\mathbb{P}^*$-martingale. Therefore, $\forall$ contingent claim X with maturity T and $\forall$ replicating trading strategy φ we have for each $t = 0, 1, ..., T$,

$$\begin{aligned} f_{X,t} = V_{\varphi,t} &= \beta_t^{-1} \widetilde{V}_{\varphi,t} \\ &= \beta_t^{-1} \mathbb{E}^*(\beta_T V_{\varphi,T} \mid \mathcal{F}_t) \\ &= \beta_t^{-1} \mathbb{E}^*(\beta_T X \mid \mathcal{F}_t) \end{aligned}$$

as required.

Until now, we have shown that attainable contingent claim can be priced with respect to an equivalent martingale measure. In this part, we will examine the question that if we assume all contingent claims are attainable. Then we are able to solve the pricing question completely. We will start with the definition of complete market.

Definition 1.4.3 *A market $\mathcal{M}$ is **complete** if every contingent claim is attainable,i.e. for every $\mathcal{F}_t$-measurable random variable $X \in L^0$ there exists a replicating self-financing strategy $\varphi \in \Phi$ such that $V_\varphi(T) = X$.*

Theorem 1.4.1 *(Completeness Theorem) An arbitrage-free market $\mathcal{M}$ is complete if and only if there exists a unique probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which the discounted asset prices are martingales.*

Proof: See [5] Theorem 4.3.1.

To summarise, combing no-arbitrage theorem and completeness theorem, we obtain one of the most important results in asset pricing- Fundamental Theorem of Asset Pricing.

Theorem 1.4.2 *(Fundamental Theorem of Asset Pricing). In an arbitrage-free complete market $\mathcal{M}$, there exists a unique equivalent martingale measure $\mathbb{P}^*$.*

Assume now that $\mathcal{M}$ is an arbitrage-free complete market and let φ is a self-financing strategy replicating contingent claim X. Then $V_{\varphi,T} = X$.

As $\widetilde{V}_{\varphi,t}$ is the martingale transform of the $\mathbb{P}^*$-martingale $\widetilde{S}_t$, $\widetilde{V}_{\varphi,t}$ is a $\mathbb{P}^*$-martingale. Therefore,

$$V_{\varphi,0} = \mathbb{E}^*(\widetilde{V}_{\varphi,T}) = \mathbb{E}^*(\beta_T X),$$

giving us the risk-neutral pricing formula

$$V_{\varphi,0} = \mathbb{E}^*(\beta_T X).$$

Theorem 1.4.3 *(Risk-neutral Pricing Formula). In an arbitrage-free complete market $\mathcal{M}$, arbitrage prices of contingent claims are the discounted expected values under the risk-neutral or equivalent martingale measure $\mathbb{P}^*$.*

1.5 Cox-Ross-Rubinstein binomial model

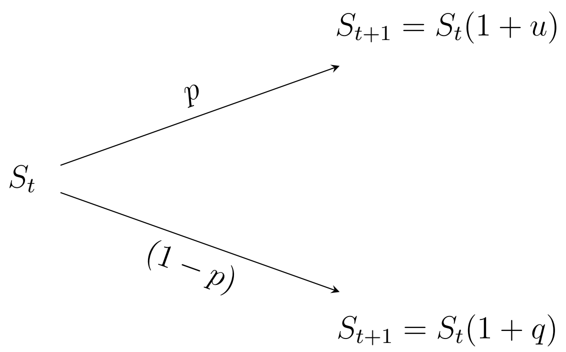
We construct the model following [15]. Take $d = 1$, that is, the model consists of one risky asset and one risk-free asset. Set time horizon T and the set of time period is $t = 0, 1, ..., T$. Assume that the first asset is a risk-free bond B, which yields a risk-free rate of return $r > 0$ in each time interval $[t, t+1]$.Therefore,

$$B_{t+1} = B_t(1+r), \quad B_0 = 1.$$

So the price process is $B_t = (1+r)^t, t = 0, 1, ..., T$. More, we have a risky asset-stock S with price process

$$S_{t+1} = \begin{cases} S_t(1+u) & \text{with probability} \quad p, \\ S_t(1+q) & \text{with probability} \quad 1-p, \end{cases}$$

See Figure below:



with $-1 < q < u, t = 0, 1, ..., T-1, S_0 \in \mathbb{R}^+$. u is the factor by which the price rises and q is the factor by which the price falls.

To make the model more straightforward, we consider return process $Z_t := \frac{S_t}{S_{t-1}} - 1, t = 0, 1, ..., T$ as random variables defined on probability spaces, where we define:

$$Z_{t,u} = u, Z_{t,q} = q, \quad t = 0, 1, ..., T.$$

Therefore we can write the stock price as

$$S_t = S_0 \prod_{\tau=1}^{t}(1 + Z_\tau), \quad t = 0, 1, ..., T.$$

If we define the $C_t, t = 1, ..., T$ as random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ as

$$C_{t,\omega} = C_{t,\omega_t}.$$

where we use a different symbol C_t to represent a return process which is different from Z_t. $C_1, C_2, ..., C_T$ are i.i.d with

$$\mathbb{P}(C_t = u) = p$$
$$\mathbb{P}(C_t = d) = 1 - p$$

We model the flow of information in the market we use filtration:

$$\mathcal{F}_0 = \{\emptyset, \Omega\},$$
$$\mathcal{F}_t = \sigma(C_1, ..., C_t) = \sigma(S_1, ..., S_t),$$
$$\mathcal{F}_T = \mathcal{P}(\Omega)$$

Proposition 1.5.1 *(i) A martingale measure Q for the discounted stock price* $\tilde{S} = \frac{S_t}{B_t}$ *exists if and only if*

$$q < r < u.$$

(ii) If above condition holds true, then there is a unique such measure in $\mathbb{P}$ *characterised by*

$$\hat{p} = \frac{r - q}{u - q}$$

Corollary 1.5.1 *The Cox-Ross-Rubinstein model is arbitrage free and complete.*

Proposition 1.5.2 *The arbitrage price process of a contingent claim* X *in the CRR model is given by*

$$f_{X,t} = B_t \mathbb{E}^*[X/B_T | \mathcal{F}_t], \quad \forall t = 0, 1, ..., T,$$

where $\mathbb{E}$ *is the expectation with respect to unique equivalent martingale measure* $\mathbb{P}^*$ *characterised by* $\hat{p} = \frac{r - q}{u - q}$.

According to this formula, suppose the contingent claim is stock itself, which has a price process $S_t, t = 0, 1, ..., T$, hence we obtain:

$$\mathbb{E}^*(S_T) = S_0 B(T) = S_0 e^{rT}$$

That is, stocks grow at risk-free rate under risk-neutral distribution.

Based above model and concepts, if a market is complete then for a contingent claim, there is a unique price is equal to the expectation of the discounted payoff at maturity under the new measure. This is called the fair price of claim. For the CRR-model and Black-Scholes model, the market is complete. later, the market is incomplete. We can still price contingent claim by taking the expectation of the discounted payoff at maturity under the EMM, but we cannot guarantee that this price is unique fair. For details in pricing over-the-counter derivative securities under incomplete markets, please refer to Jeremy Staum's notes on incomplete markets.

1.6 Multinomial models

The CRR binomial model is generalized to the multinomial case. This is discussed in Madan, Milne and Shefrin 1990 [55]. Limits are investigated and shown to yield the Black-Scholes formula in the case of continuous sample paths for a wide variety of complete market structures. The main idea behind their model is to derive a multinomial option pricing formula consistent with an Arrow-Debreu complete markets equilibrium. Economic uncertainty is modelled as evolving on an $(d+1)$-ary tree with branching occurring during a short period of time where there is no trading. In this section, we will introduce the intuition and simplified version of multinomial model.

We now construct an arbitrage-free, complete market model with $d > 2$ assets following the information rule of allowing as many different states of the world as we have assets to trade in. We start with the single-period model with $d = 2$ case as the case $d > 2$ follows by the same procedure. B represents the risk-free bank account, with risk-free rate of return r and two risky assets S_1, S_2.

Let

$$S_{1,1} = S_{1,0}Z_1 \quad \text{and} \quad S_{2,1} = S_{2,0}Z_2,$$

where $S_{i,t}$ means the stock price of asset i at time t, $i \in \{1, 2, ..., d\}$ and $t \in \{0, 1, ..., T\}$. In this case $d = 2$ and $T = 1$.

Set

$$\mathbb{P}(Z_1 = u_{11}, Z_2 = u_{21}) = p_1; \mathbb{P}(Z_1 = u_{12}, Z_2 = u_{22}) = p_2; \mathbb{P}(Z_1 = u_{13}, Z_2 = u_{23}) = p_3.$$

where u_{ij} are chosen so that Z_1 and Z_2 are not independent. The discounted stock price processes $\widetilde{S}_{i,t}$ are martingales under risk neutral probabilities $\hat{p}_1, \hat{p}_2, \hat{p}_3$. The martingales conditions yield two equations:

$$\begin{aligned}
\mathbb{E}[\widetilde{S}_{1,1}] = \widetilde{S}_{1,0} \Rightarrow u_{11}\hat{p}_1 + u_{12}\hat{p}_2 + u_{13}\hat{p}_3 = 1 + r, \\
\mathbb{E}[\widetilde{S}_{2,1}] = \widetilde{S}_{2,0} \Rightarrow u_{21}\hat{p}_1 + u_{22}\hat{p}_2 + u_{23}\hat{p}_3 = 1 + r.
\end{aligned}$$

plus the sum of risk neutral probabilities is 1: $\hat{p}_1 + \hat{p}_2 + \hat{p}_3 = 1$.

Solve for the system of equations, we get solutions:

$$\begin{aligned}
\hat{p}_1 &= \frac{(1 + r - u_{13})(u_{22} - u_{23}) - (1 + r - u_{23})(u_{12} - u_{13})}{(u_{11} - u_{13})(u_{22} - u_{23}) - (u_{12} - u_{13})(u_{21} - u_{23})}, \\
\hat{p}_2 &= \frac{1 + r - u_{13}}{u_{12} - u_{13}} - \frac{u_{11} - u_{13}}{u_{12} - u_{13}}\hat{p}_1, \\
\hat{p}_3 &= 1 - \hat{p}_1 - \hat{p}_2.
\end{aligned}$$

Hence, we get a unique solution of the system of the equations above, and hence an arbitrage-free, complete model.

In the multi-period setting with time horizon T and the set of trading dates given by $0 = t_0 < t_1 < ... < t_n = T$ of equally time interval Δt. The stock price process is modelled by

$$S_{i,t_m} = S_{i,0} \prod_{j=1}^{k} Z_{ij}, \quad m = 0, 1, ..., n, i = 1, 2.$$

with a sequence of independent random vectors $(Z^j)_{1 \leq j \leq n}$ such that Z_1^j, Z_2^j are uncorrelated and

$$\mathbb{P}(Z_1^j = u_{1q}^j, Z_2^j = u_{2q}^j) = p_q^j, \quad q = 1, 2, 3.$$

For each j the random vector Z^j can be in three possible states and this model applies for the multi-period market that is arbitrage-free and complete.

The paper generalises CRR model to the multinomial case. Limits were investigated and shown to yield the Black-Scholes formula in the case of continuous sample paths for a wide variety of complete market structures. In the discontinuous a Merton-type formula was shown to result, provided jump probabilities were replaced by their corresponding Arrow-Debreu prices.

Chapter 2

Brownian motion and stochastic integral

Up to now, we have only introduced the discrete time processes. Now lets turn to the continuous case. The well known Brownian motion named after the botanist Robert Brown, who first described the phenomenon in 1827, while looking through a microscope at pollen of the plant immersed in water when he observed minute particles, ejected by pollen grains, executing a jittery motion. The first person to describe the mathematics behind Brownian motion was Thorvald N. Thiele in a paper on method of least squares published in 1880 [70]. This was followed by Louis Bachelier in 1900 proposed a model of Brownian motion while deriving the dynamic behaviour of the Pairs stock market [1].

The theory of stochastic processes, at least in terms of its application to physics, started with Albert Einstein's work on the theory of Brownian motion in 1905 [19] and in a series of additional papers published in the period 1905-1906. In 1923, Norbert Wiener used the ideas of measure theory to construct a measure on the path space of continuous functions, giving the canonical path projection process the distribution of Brownian motion [71]. Wiener and others proved many properties of the paths of Brownian motion, which still continues to today.

The next step in the groundwork for stochastic integration lay with A.N. Kolmogorov. The beginnings of the theory of stochastic integration, from the non-finance perspective, were motivated and intertwined with the theory of Markov processes. Then we turn to Kiyosi Itô, the father of stochastic integration. In Itô's paper [35], he had a powerful analytic method to study the transition probabilities of the process, namely Kolmogorov's parabolic equation and its extension by Feller. Itô's first paper on stochastic integration was published in 1944 [37], the same year that Kakutani published two brief notes connecting Brownian motion and harmonic functions. The much later insights of Black-Scholes formula derived by Black, Scholes, and Merton, relating prices of options to perfect hedging strategies, is not our concern for this thesis. The final precursor to the Black, Scholes and Merton option pricing formulas can be found in the paper of Samuelson and Merton [60].

2.1 Brownian motion

Definition 2.1.1 *A **stochastic process** on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of random variables, $\{X_t\}_{t\in[0,T]}$, where the indexing set is continuous.*

Definition 2.1.2 *A family of σ-algebras $(\mathcal{F}_t)_{t\geq 0}$ is called a **(continuous time) filtration** for the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if for any $t, s \geq 0$ with $t \geq s$ we have*

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$$

Definition 2.1.3 *A process $(X_t)_{t\geq 0}$ is **adapted** if X_t is $\mathcal{F}_t$-measurable for each $t \geq 0$.*

Definition 2.1.4 *A continuous stochastic process $(W_t)_{t\in[0,\infty)}$ is called a **standard Brownian motion** on $[0, \infty)$ if it has the following properties:*

(i) $W_0 = 0$*, a.s.*

(ii) For any set of finite times, $0 \leq t_1 < t_2 < \cdots < t_n < T$, the increments of the process $(W_t)_{t\in[0,T)}$,

$$W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, \cdots, W_{t_n} - W_{t_{n-1}}$$

are all independent;

(iii) $W_t - W_s \sim \mathcal{N}(0, \sigma^2_{t-s})$ for any $s, t \in [0, T)$ with $s \leq t$.

(iv) The paths of W_t are almost surely continuous,

$$\exists\tilde{\Omega} \in \mathcal{F}, \mathbb{P}(\tilde{\Omega}) = 1, t \mapsto W_t(\omega) \text{ is continuous for } \omega \in \tilde{\Omega}.$$

Definition 2.1.5 *Let $(W_s)_{s\geq 0}$ be a standard Brownian motion on $[0, \infty)$. Define, for each $t \geq$ 0,*

$$\mathcal{F}_t^W := \sigma(W_s : s \in [0, t])$$

and

$$\mathcal{N}_t := \{A \in \mathcal{F} : A \subset B \text{ for some } B \in \mathcal{F}_t \text{ with } P(B) = 0\}.$$

*The **standard Brownian filtration**, $(\mathcal{G}_t^W)_{t\geq 0}$, is defined through*

$$(\mathcal{G}_t^W) := \sigma(\mathcal{F}_t \cup \mathcal{N}_t), \forall t \geq 0.$$

In our construction of Brownian motion, we will use two sequences of functions that have been studied for years. Both sequences are examples of wavelets. To define these functions, we will first consider a function that can serve as a "mother wavelet" and we call this function Haar function proposed by Alfréd Haar in 1910 [28].

Definition 2.1.6 *(Haar functions). Let $H : \mathbb{R} \to \mathbb{R}$ be the function*

$$H(t) = \begin{cases} 1 & \textit{if} \quad t \in [0, 1/2) \\ -1 & \textit{if} \quad t \in [1/2, 1) \\ 0 & \textit{otherwise} \end{cases}$$

*The **Haar functions** are defined through*

$$H_0(t) := 1, \quad \forall t \in [0, 1],$$

and for any $k \in \mathbb{N}$ and $i = 0, \cdots, 2^k - 1$

$$H_{2^k+i}(t) := 2^{k/2} H(2^k t - i), \quad \forall t \in [0, 1].$$

The next step of the plan is to obtain a good representation for the integrals of the H_n, which is the key elements of representation for Brownian motion. These integrals turn out to have an expression in terms of another wavelet sequence that is generated by another mother wavelet, which this time is given by the triangle function:

$$\Delta(t) = \begin{cases} 2t & \text{if} \quad t \in [0, 1/2) \\ 2(1-t) & \text{if} \quad t \in [1/2, 1) \\ 0 & \text{otherwise} \end{cases}$$

Next, for $n \geq 1$, we use internal scaling and translating of the mother wavelet to define the sequence

$$\Delta_n(t) = \Delta(2^k t - i) \text{ for } n = 2^k + i, \quad \text{where } k \geq 0 \text{ and } 0 \leq i < 2^k$$

and for $n = 0$ we simply take $\Delta_0(t) = t$. The function $\Delta_n, 0 \leq n < \infty$ will serve as the fundamental building blocks in our representation of Brownian motion.

Since the mother wavelet $\Delta(t)$ is the integral of the mother wavelet $H(t)$, there is a close connection between the integrals of the H_n and the Δ_n,

$$\int_0^t H_n(u) du = \lambda_n \Delta_n(t),$$

where $\lambda_0 = 1$ and for $n \geq 1$ we have

$$\lambda_n = \tfrac{1}{2} \cdot 2^{-j/2} \text{ where } n \geq 1 \text{ and } n = 2^k + i \text{ with } 0 \leq i < 2^k.$$

The following lemma will aid in the proof of the existence of such a process.

Lemma 2.1.1 *Let $(Z_n)_{n \in \mathbb{N}}$ be a sequence of independent Gaussian variables with $Z_n \sim \mathcal{N}(0, 1)$. For almost every $\omega \in \Omega$, $\exists$ a constant $C(\omega) > 0$, for which $|Z_n(\omega)| \leq C(\omega)\sqrt{\ln(n)}, \quad \forall n \geq 2$.*

Proof: As variables Z_n are normally distributed, $\forall x \geq 1$,

$$\mathbb{P}(|Z_n| \geq x) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{y^2}{2}} dy \leq \sqrt{\frac{2}{\pi}} \int_x^\infty y e^{-\frac{y^2}{2}} dy = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}$$

for any $n \in \mathbb{N}$. This implies $\forall \alpha > 1$ and $n \geq 2$,

$$\mathbb{P}(|Z_n| \geq \sqrt{2\alpha \ln(n)}) \leq \sqrt{\frac{2}{\pi}} e^{-\alpha \ln(n)} = \sqrt{\frac{2}{\pi}} n^{-\alpha},$$

and therefore

$$\sum_{n=2}^{\infty} \mathbb{P}(|Z_n| \geq \sqrt{2\alpha \ln(n)}) < \infty.$$

The Borel-Cantelli lemma can then be applied to obtain

$$\mathbb{P}(|Z_n| \geq \sqrt{2\alpha \ln(n)} \text{ for infinitely many } n \in \mathbb{N}) = 0.$$

Equivalently, for almost every $\omega \in \Omega$, the bound

$$\frac{|Z_n(\omega)|}{\sqrt{\ln(n)}} \leq \sqrt{2\alpha}$$

for finitely many $n \in \mathbb{N}$. This implies that for almost every $\omega \in \Omega$,

$$C(\omega) := \sup_{n \geq 2} \frac{|Z_n(\omega)|}{\sqrt{\ln(n)}}$$

is finite. □

The following theorem asserts the existence of the Brownian motion process.

Theorem 2.1.1 *(Wavelet Representation of Brownian Motion). If $Z_n : 0 \leq n < \infty$ is a sequence of independent Gaussian random variables with mean 0 and variance 1, then the series defined by*

$$X_t = \sum_{n=0}^{\infty} \lambda_n Z_n \Delta_n(t)$$

converges uniformly on $[0, 1]$ with probability one. Moreover, the process X_t defined by the limit is a standard Brownian motion for $0 \leq t \leq 1$.

Proof: See [65].

2.2 Stochastic integral

2.2.1 Gains from trade as stochastic integral

One of the most fundamental notions in finance is that of gains from trade. In stochastic calculus this corresponds exactly to the notion of a stochastic integral, the Itô's integral, which is highly relevant in finance. In this section, we will follow Platen and Heath [58], to briefly cover the concepts of stochastic integral.

Consider an investor who holds during the time period $[0, T]$ a constant number $\xi(0)$ of units of an asset with price process $X = X_t, t \in [0, T]$. The investor's allocation strategy $\xi = \{\xi_t \equiv \xi_0, 0 \leq t \leq T\}$, characterised by the number of units of the asset held, is assumed to be constant in this case. Then the investor's gains from trade over the period $[0, t]$ equals

$$I_{\xi,X}(t) = \xi_0\{X_t - X_0\},$$

for $t \in [0, T]$. This provides the first step towards an appropriate definition of a stochastic integral, which we shall call later Itô integral. Formally, we interpret the above gains from trade $I_{\xi,X}(t)$ as an **Itô integral** of the **integrand** ξ with respect to the **integrator** X over the time interval [0,t], and use the following notation:

$$I_{\xi,X}(t) = \int_0^t \xi_s dX_s.$$

In this section, it is sufficient in many applications to use a Brownian motion as integrator. The Itô integral exhibits a number of important properties and characteristics that are essential in stochastic calculus and hence for many applications in quantitative finance, especially pricing financial derivatives. But first, let's have a look at the formal definition of Itô integral.

2.2.2 Itô integral

Fix $T > 0$ and let $(W_t)_{t\in[0,T]}$ be a standard Brownian motion on $[0, T]$. Let $(\mathcal{F}_t)_{t\geq 0} := (\mathcal{G}_t^W)_{t\geq 0}$ be the standard Brownian filtration as defined in (2.1.5). Let $\mathcal{B}([0, T])$ denote the Borel σ-algebra on $[0, T]$.

Definition 2.2.1 *Consider the product σ-algebra*

$$\mathcal{B}([0, T]) \times \mathcal{F}_t := \sigma(W \times A_t : W \in \mathcal{B}([0, T]) \text{ and } A_t \in \mathcal{F}_t).$$

*A process $f : [0, T] \times \Omega \to \mathbb{R}$ is called **measurable** if it is $\mathcal{B}([0, T]) \times \mathcal{F}_t$ measurable. It is said to be **adapted** if for each $t \in [0, T]$, $f(t, \cdot)$ is $\mathcal{F}_t$-measurable.*

To give a complete definition of Itô integral, we need to introduce the concept of Hilbert space and Harr basis.

Theorem 2.2.1 *$(H_n)_{n\in\mathbb{N}}$ is an orthonormal basis for the Hilbert space $L^2([0,1))$. It is called the Harr basis.*

Proof: See [69].

Let H^2 denote the collection of all $f \in L^2([0,T]\times\Omega))$ that are adapted and H_0^2 the subset of H^2 that are consists of f of the form

$$f(t,\omega) = \sum_{i=0}^{n-1} a_i(\omega)\mathbb{1}_{(t_i,t_{i+1}]}(t)$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ and $a_i \in L^2(\Omega, \mathcal{F}_{t_i}, \mathbb{P})$ for $i = 0, \cdots, n-1$.

The Itô integral will be constructed on the set of processes H^2. This will be done by first defining the integral on the simple processes of H_0^2. In analogy to the Lebesgue integral, the Itô integral can then be extended to H^2 through a suitable limiting process. In this manner, define $I : H_0^2 \to L^2(\Omega)$ by

$$I(f) := \sum_{i=0}^{n-1} a_i(W_{t_{i+1}} - W_{t_i})$$

for any f of the form defined above.

We would now like to show that we can extend the domain of I from H_0^2 to all of H^2, and to complete this extension we need to know that $I : H_0^2 \to L^2(dP)$ is a continuous mapping. This is indeed the case based on the following fundamental lemma.

Lemma 2.2.1 *(Itô's Isometry on H_0^2). For $f \in H_0^2$ we have*

$$||I(f)||_{L^2(\Omega)} = ||f||_{L^2([0,T]\times\Omega)}$$

Proof:

$$\begin{aligned}||I(f)||^2_{L^2(\Omega)} &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_\Omega a_i a_j (W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j}) \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\mathbb{E}[a_i a_j (W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})].\end{aligned}$$

For $i < j$, since a_i, a_j and $W_{t_{i+1}} - W_{t_i}$ are $\mathcal{F}_{t_j}$-measurable,

$$\begin{aligned}\mathbb{E}[a_i a_j (W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})] &= \mathbb{E}[\mathbb{E}[a_i a_j (W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j}) \mid \mathcal{F}_{t_j}]] \\ &= \mathbb{E}[a_i a_j (W_{t_{i+1}} - W_{t_i})\mathbb{E}[W_{t_{j+1}} - W_{t_j} \mid \mathcal{F}_{t_j}] \\ &= \mathbb{E}[a_i a_j (W_{t_{i+1}} - W_{t_i})\mathbb{E}[W_{t_{j+1}} - W_{t_j}]] \\ &= 0. \quad \text{(Given } W_{t_{j+1}} - W_{t_j} \text{ is independent of } \mathcal{F}_{t_j}\text{)}\end{aligned}$$

Identical argument follows for $j < i$. This implies:

$$\begin{aligned}||I(f)||_{L^2(\Omega)} &= \sum_{i=1}^{n-1} \mathbb{E}[a_i^2 (W_{t_{i+1}} - W_{t_i})^2] \\ &= \sum_{i=1}^{n-1} \mathbb{E}[a_i^2](t_{i+1} - t_i).\end{aligned}$$

We also have that

$$\begin{aligned}||f||_{L^2([0,T]\times\Omega)} &= \sum_{i=0}^{n-1} \mathbb{E}[a_i^2] \int_0^T \mathbb{1}^2_{(t_i, t_{i+1}]} dt \\ &= \sum_{i=1}^{n-1} \mathbb{E}[a_i^2](t_{i+1} - t_i).\end{aligned}$$

as required. □

Theorem 2.2.2 *H_0^2 is dense in H^2 in the $L^2([0,T] \times \Omega)$-norm. That is, for any $f \in H^2$, there exists a sequence $(f_n)_{n\in\mathbb{N}} \in (H_0^2)^{\mathbb{N}}$ such that $||f - f_n||_2 \to 0$ as $n \to \infty$.*

A few preliminary results will be required in the proof of this theorem. In this case, we will skip the proof and use the result directly. This theorem together with the Itô isometry, allows us to extend the definition of Itô integral to H^2.

Definition 2.2.2 *The **Itô integral** on H^2 is the map $I : H^2 \to L^2(\Omega)$ defined by*

$$I(f) := \lim_{n\to\infty}^{L^2} I(f_n)$$

for $f \in H^2$, where $(f_n)_{n\in\mathbb{N}}$ is a sequence in H_0^2 that converges to f in L^2.

Remark 2.2.1 *The Itô integral is well-defined on H^2 since I sends Cauchy sequence. This directly from the Itô isometry.*

As it is currently defined, the Itô integral maps stochastic processes in H^2 to random variable in $L^2(\Omega)$. An important question to ask whether we can define the Itô integral as a map from processes and processes. That is, can we define $\tilde{I}(f)$ as a stochastic process $f \in H^2$. This is analogous to constructing the indefinite integral for functions out of the definite one.

Theorem 2.2.3 *Let $f \in H^2$. There exists a continuous martingale with respect to the standard Brownian filtration, $(X_t)_{t\in[0,T]}$, such that*

$$\mathbb{P}(X_t = I(\mathbb{1}_{[0,t]} \cdot f)) = 1, \quad \forall t \in [0,T].$$

2.3 Quadratic variation

The notion of *quadratic variation* of a stochastic process X plays a fundamental role in stochastic calculus and as well as in finance. It is a characteristic of the fluctuating part of a stochastic process and can be easily observed. Hence it is useful for measuring locally in time the risk of an asset price.

To introduce this notion in a simple way, we consider an **equidistant time discretization**

$$\{t_k = kh : k \in \{0, 1, ...\}\},$$

with small time steps of lengths $h > 0$, such that $0 = t_0 < t_1 < t_2 < ...$

Definition 2.3.1 *For a given stochastic process X the* ***quadratic variation process*** $[X] = \{[X]_t, t \in [0, \infty)\}$ *is defined as the limit in probability as $h \to 0$ of the sums of squared increments of the process X, provided this limit exists and is unique. At time t the quadratic variation*

$$[X]_t \stackrel{\mathrm{P}}{=} \lim_{h \to 0} [X]_{h,t}, \tag{2.1}$$

where the ***approximate quadratic variation*** $[X]_{h,t}$ *is given by the sum*

$$[X]_{h,t} = \sum_{k=1}^{i_t} (X_{t_k} - X_{t_{k-1}})^2. \tag{2.2}$$

Here i_t denotes the integer

$$i_t = \max\{k \in \mathbb{N} : t_k \leq t\}$$

of last discretization point before or including $t \in [0, \infty)$.

For more details we refer to Jacod & Shiryaev (2003) [39] and Protter (2004) [58].

We can use program R to find approximate quadratic variation for a standard Wiener process. For example, by using R, quadratic variation of approximation of Wiener process paths with 1000 scaled random steps with 200 partition intervals is 0.904.

Theoretically, it can be shown, refer to Karatzas & Shreve (1991) [40], that the value of the quadratic variation process $[W] = \{[W]_t, t \in [0, \infty)\}$ at time t for a standard Wiener process W is given by the relation

$$[W]_t = t$$

for $t \in [0, \infty)$. Thus, for finer time discretizations, the approximate quadratiic variation becomes almost a perfect straight line.

2.3.1 Application of quadratic variation

The quadratic variation turns out to be one of the most important characteristics of a martingale. The standard asset pricing model is given by Black-Scholes model, given by a geometric Brownian motion, which we write in the form

$$X_t = X_0 \exp\{L_t\}$$

where $L_t = gt + \sigma W_t, \quad t \in [0, \infty)$.

Note that the quadratic variation is not linear. However, if we take the quadratic variation of the logarithm $ln(X_t)$ of X_t, then we can obtain that the quadratic variation is an almost perfect straight line. This can be seen from the following identity:

$$[ln(X)]_t = [L]_t = \sigma^2 [W]_t = \sigma^2 t \tag{2.3}$$

for $t \in [0, \infty)$. These relations hold because $L_t = \ln(X_t)$ forms a linearly transformed Wiener process and we can use the fact that $[W]_t = t$.

2.3.2 Volatility

The key parameter used in Black-Scholes model is the volatility. We noticed that in (2.3) under the BS model the variance is the time derivative of the quadratic variation of the logarithm of the asset price:

$$\sigma^2 = \frac{d}{dt}[\ln(X)]_t.$$

To be more precise, we define the historical volatility $\mathrm{Vol}_X(t)$ at a given time $t \in [0, \infty)$ of a given continuous asset price process X as:

$$\mathrm{Vol}_X(t) = \sqrt{\frac{d}{dt}[\ln(X)]_t}.$$

2.3.3 Quadratic covariation

Definition 2.3.2 *Given two semimartingales X, Y, the quadratic covariation process $[X, Y]$ is the semimartingale defined by*

$$[X, Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t X_{s-} dY_s - \int_0^t Y_{s-} dX_s$$

.

Example 2.3.1 *(Quadratic covariation of correlated Brownian motions) If $Z_t^1 = \sigma^1 W_t^1$ and $Z_t^2 = \sigma^2 W_t^2$, where W^1, W^2 are standard Brownian motions with correlation ρ then $[Z^1, Z^2]_t = \rho \sigma_1 \sigma_2 t$.*

Chapter 3

Lévy processes

Lévy processes have become increasingly common for modelling market fluctuations, for pricing and risk management purposes. In particular, the literature associated with Lévy processes are quite technical and difficult for people not specialised in stochastic analysis. Majority of the applications of Lévy processes in financial modelling use sophisticated probabilistic and analytical tools.

In this chapter, we briefly prepare the basic concepts and mathematical tools which are necessary for stochastic calculus with jumps throughout the thesis. We consider Lévy processes and Itô calculus, as these concepts are extremely important such that will be later used in option pricing model.

Similarly to random walks, sum of independent identically distributed random variables, provide basic examples of stochastic processes in discrete time. In continuous cases, processes with independent stationary increments are called Lévy processes. The Brownian motion we discussed in Chapter 2 is a simple example of Lévy process.

We will begin with Lévy processes and discuss some of the properties. Then we introduce two important theoretical tools: the Lévy-Khinchin formula, which allows us to study distributional properties of Lévy processes and Lévy-Itô decomposition, that describes the structure of their sample paths.

First, we begin with introducing a fundamental process that leads us to building more complex stochastic processes.

3.1 Poisson process

The Poisson process is a fundamental example of a stochastic process with discontinuous trajectories and will be used as a building block for constructing complex jump processes.

Definition 3.1.1 *Let $(\tau_i)_{i\geq 1}$ be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^{n} \tau_i$. The process $(N_t)_{t\geq 0}$ defined by*

$$N_t = \sum_{n\geq 1} \mathbb{1}_{t\geq T_n}$$

*is called a **Poisson process** with intensity λ.*

The Poisson process is therefore defined as a **counting process**: it counts the number of random times (T_n) which occur between time 0 and t, where $(T_n - T_{n-1})_{n\geq 1}$ is an i.i.d. sequence of exponential variables. If $T_1, T_2, ...$ is the sequence of jump times of N, then N_t is simply the number of jumps between 0 and t:

$$N_t = \#\{i \geq 1, T_i \in [0,t]\}.$$

If $t > s$ then

$$N_t - N_s = \#\{i \geq 1, T_i \in (s,t]\}.$$

The jump times $T_1, T_2, ...$ form a random configuration of points on $[0,\infty)$ and the Poisson process N_t counts the number of such points in the interval $[0,t]$. $\#$ denotes the number of something.

Definition 3.1.2 *The counting procedure defines a **measure** M on $[0,\infty)$: for any measurable set $A \subseteq \mathbb{R}^+$, let*

$$M(\omega, A) = \#\{i \geq 1, T_i(\omega) \in A\}. \tag{3.1}$$

*Then $M(\omega, .)$ is a positive integer valued measure and $M(A)$ is finite with probability 1 for any bounded set A. The measure $M(\omega, .)$ depends on ω, it is thus a **random measure**.*

There is one more definition that is related to random measures, the so-called Poisson random measures, which will be introduced in the next section.

3.2 Lévy process

The term "Lévy process" honours the work of the French mathematician Paul Lévy who, although not alone in his contribution, played a ground-breaking role in bringing together an understanding and characterisation of processes with stationary and independent increments.

Definition 3.2.1 *(Lévy process) A cadlag stochastic process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$ such that $X_0 = 0$ is called **Lévy process** if it possesses the following properties:*

(i) X_t has independent increments: for every increasing sequence of times $t_0, ..., t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent.

(ii) X_t has stationary increments: the law of $X_{t+h} - X_t$ does not depend on t.

(iii) X_t has stochastic continuity: $\forall \varepsilon > 0, \lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0$.

(iv) X_t has cadlag (right-continuity and left limits) paths.

Denote: $\Delta X_t = X_t - X_{t-}$. We can associate the counting measure N to X_t in the following way: for $A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$, we put

$$N(t, A) = \textstyle\sum_{0<s\leq t} \mathbb{1}_A(\Delta X_s), \text{ for } t > 0.$$

Note this is a counting measure of jumps of X in A up to the time t. As the path is cadlag, for $A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$ such that $\overline{A} \subseteq \mathbb{R}^d \backslash \{0\}$, we have $N(t, A) < \infty$ a.s.

Definition 3.2.2 *A **random measure** on $\mathcal{T} \times (\mathbb{R}^d \backslash \{0\})$ is determined by*

$$N((a, b] \times A) = N(b, A) - N(a, A),$$

*where $a \leq b$ and $\mathcal{T} = [0, T]$, is called a **Poisson random measure** if it follows the Possion distribution with mean measure $\mathbb{E}[N((a, b] \times A)]$, and it for disjoint $(a_1, b_1] \times A_1, ..., (a_r, b_r] \times A_r \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$, $N((a_1, b_1] \times A_1), ..., N((a_r, b_r] \times A_r)$ are independent.*

Definition 3.2.3 *(Infinite divisibility) A probability distribution $\mathcal{F}$ on $\mathbb{R}^d$ is said to be **infinitely divisible** if for any integer $n \geq 2$, there exist n i.i.d. random variables $X_1, X_2, ..., X_n$ such that $X_1 + ... + X_n$ has distribution $\mathcal{F}$.*

Since the distribution of i.i.d. sums is given by convolution of the distribution of the summands, denote by μ the distribution of the X_k-s above, then $F = \mu * \mu * \mu$ is the n-th convolution of μ.

Therefore, for any $t > 0$ the distribution of a Lévy process $(X_t)_{t\geq 0}$ is infinitely divisible. This puts a constraint on the possible choices of distributions for (X_t): the distribution of increments of a Lévy process has to be infinitely divisible.

Proposition 3.2.1 *(Infinite divisibility and Lévy process)*
Let $(X_t)_{t\geq 0}$ be a Lévy process. Then for every t, X_t has an infinitely divisible distribution. Conversely, if $\mathcal{F}$ is an infinitely divisible distribution then there exists a Lévy process (X_t) such that the distribution of X_t is given by $\mathcal{F}$.

Proof: The direct implication was shown above. For the proof of the converse statement, we can refer to [61, Corollary 11.6].

Define the characteristic function of X_t,

Definition 3.2.4 *(Characteristic function) The* ***characteristic function*** *of Lévy process X_t:*

$$\Phi_t(z) \equiv \Phi_{X_t}(z) \equiv \mathbb{E}[e^{i\langle z, X_t\rangle}], \quad z \in \mathbb{R}^d$$

For $t \geq 0, s \in \mathbb{R}$, by writing $X_{t+s} = X_s + (X_{t+s} - X_s)$ and using the fact that $X_{t+s} - X_s$ is independent of X_s, we obtain that $t \mapsto \Phi_t(z)$ is a multiplicative function:

$$\begin{aligned} \Phi_{t+s}(z) = \Phi_{X_{t+s}}(z) &= \Phi_{X_s}(z)\Phi_{X_{t+s}-X_s}(z) \\ &= \Phi_{X_s}(z)\Phi_{X_t}(z) = \Phi_s\Phi_t \end{aligned}$$

The stochastic continuity of $t \mapsto X_t$ implies in particular that $X_t \mapsto X_s$ in distribution when $s \mapsto t$. Therefore, $\Phi_{X_s}(z) \mapsto \Phi_{X_t}(z)$ when $s \mapsto t$ so $t \mapsto \Phi_t(z)$ is a continuous function of t. Together with the multiplicative property $\Phi_{t+s}(z) = \Phi_s(z)\Phi_t(z)$ this implies that $t \mapsto \Phi_t(z)$ is an exponential function:

Proposition 3.2.2 *(Characteristic function of a Lévy process) Let X_t be a Lévy process on $\mathbb{R}^d$. There exists a continuous function $\psi : \mathbb{R}^d \mapsto \mathbb{R}$ called the characteristic exponent of X, such that,*

$$\mathbb{E}[e^{i\langle z, X_t\rangle}] = e^{t\psi(z)}, \quad z \in \mathbb{R}^d.$$

Recall the definition of the cumulant generating function of a random variable, we can see that ψ is the cumulant generating function of $X_1 : \psi = \Psi_{X_1}$ and that the cumulant generating function of X_1 varies linear in $t : \Psi_{X_t} = t\Psi_{X_1} = t\psi$. The law of X_t is therefore determined by the law of X_1.

3.3 Lévy-Itô decomposition

The Lévy-Itô decomposition entails that for every Lévy process there exists a vector c, a positive definite matrix σ and a positive measure ν that uniquely determine its distribution.

Theorem 3.3.1 *(Lévy-Itô decomposition theorem, [58] Theorem I.42). Let X_t be a Lévy process. Then, X_t admits the following formula:*

$$X_t = tc + \sigma W_t + \int_0^t \int_{|x|<1} x\widetilde{N}(ds\ dx) + \int_0^t \int_{|x|\geq 1} xN(ds\ dx)$$

for a.e.w for all $t \in \mathcal{T}$. Here, $c \in \mathbb{R}^d, \sigma$ is an $d \times d$ matrix W_t is a d-dimensional standard Brownian motion, $N(dt\ dx)$ is a Poisson random measure with the mean $\mathbb{E}[N(dt\ dx)]$, and $\widetilde{N}(dt\ dx) = N(dt\ dx) - \mathbb{E}[N(dt\ dx)]$. The process W_t and $t \mapsto (\int_0^t \int_{|x|<1} x\widetilde{N}(ds\ dx) + \int_0^t \int_{|x|\geq 1} xN(ds\ dx))$ are independent. Also the this decomposition presentation is unique.

Proof: See [13] Proposition 3.7.

By this theorem, $N(.\ .)$ derived from X_t defines a Poisson random measure on $\mathcal{T} \times (\mathbb{R}^d \backslash \{0\})$. Here we use the notation of stochastic integrals $\int_0^t \int xN(ds\ dx)$ and $\int_0^t \int x\widetilde{N}(ds\ dx)$.

We take the mean measure

$$\nu(A) = \mathbb{E}[N(1\ A)], \quad A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\}).$$

This (deterministic) measure is called the **Lévy measure** associated to z or to N. Note that ν enjoys

$$\int_{\mathbb{R}^d \backslash \{0\}} (1 \wedge |x|^2)\nu(dx) < \infty. \tag{3.2}$$

The compensated Poisson random measure associated to N is defined by

$$\widetilde{N}(dt\ dx) = N(dt\ dx) - dt\nu(dx).$$

In particular, if X_t has a finite mean that is if $\nu(dx)$ satisfies

$$\int_{|x|\geq 1} |x|\nu(dx) < \infty \Leftrightarrow \mathbb{E}[X_t] < \infty,$$

then X_t can be written in the compact form

$$X_t = t(c + \int_{|x|\geq 1} |x|\nu(dx)) + \sigma W_t + \int_0^t \int_{\mathbb{R}^d \backslash \{0\}} x\widetilde{N}(ds\ dx).$$

A measure ν on $\mathbb{R}^d \backslash \{0\}$ is a Lévy measure associated to some Lévy process if and only if it enjoys property (3.2). Indeed, we have the following Lévy-Khintchine representation.

Proposition 3.3.1 ***(Lévy-Khintchine representation)***

(i) Let X be a Lévy process on $\mathbb{R}^d\backslash\{0\}$. Then,

$$\mathbb{E}[e^{i\langle z,X_t\rangle}] = e^{t\psi(z)}, \quad z \in \mathbb{R}^d, \tag{3.3}$$

where

$$\psi(z) = i\langle c,z\rangle - \frac{1}{2}\langle z,\sigma\sigma^T z\rangle + \int (e^{i\langle z,X\rangle} - 1 - i\langle z,X\rangle \mathbb{1}_{\{|x|\leq 1\}})\nu(dx). \tag{3.4}$$

Here, $c \in \mathbb{R}^d$, $\sigma\sigma^T$ is a nonnegative definite matrix and ν is a measure which satisfies (3.2).

(ii) Given $c \in \mathbb{R}^d$, a matrix $\sigma\sigma^T \geq 0$ and a σ-finite measure ν on $\mathcal{B}(\mathbb{R}^d\backslash\{0\})$ satisfying (3.2), there exists a process X for which (3.3) and (3.4) hold. This process X is a Lévy process.

Proof: We use formula

$$\begin{aligned}
\mathbb{E}[e^{i\langle z,W_t\rangle}] &= \exp\{-\frac{1}{2}t\langle z,\sigma\sigma^T z\rangle\}, \\
\mathbb{E}[e^{i\langle z,\int_0^t\int_{|x|<1} x\tilde{N}(ds\ dx)\rangle}] &= \exp t\left[\int_{|z|<1} (e^{i\langle z,X\rangle} - 1 - i\langle z,X\rangle)\nu(dx)\right], \\
\mathbb{E}[e^{i\langle z,\int_0^t\int_{|z|\geq 1} xN(ds\ dx)\rangle}] &= \exp t\left[\int_{|z|\geq 1} (e^{i\langle z,X\rangle} - 1)\nu(dx)\right].
\end{aligned}$$

and we refer to Theorem 8.1 in [61], and Section 0 in [38].

Let $D_p = \{t \in T; \Delta X_t \neq 0\}$. Then, it is a countable subset of T a.s. Let $A \subset \mathbb{R}^m\backslash\{0\}$. In case $\nu(A) < \infty$, the process $D_p \ni t \mapsto \sum_{s\leq t,\Delta X_s\in A} \delta_{(s,\Delta X_s)}$ is called a **Poisson counting measure** associated to the Lévy process X_t taking values in A. The function $D_p \ni t \mapsto p_t = \Delta X_t$ is called a **Poisson point process** associated to the Lévy process X_t.

The notion of Poisson point process is defined in a general setting using point functions. A point function p is a mapping from D_p to $\mathbb{R}^m\backslash\{0\}$, where D_p is a countable subset of T. The function p defines a counting measure N_p on $T \times (\mathbb{R}^m\backslash\{0\})$ by

$$N_p((0,t] \times A) = \#\{s \in D_p; s \leq t, p_s \in A\}, t > 0, A \in \mathcal{B}(\mathbb{R}^m\backslash\{0\}).$$

Now we proceed by presenting Itô's formula.

Proposition 3.3.2 ***(Itô's formula)***

(i) Let X_t be a real-valued process given by

$$X_t = x + tc + \sigma W_t + \int_0^t \int_{\mathbb{R}\backslash\{0\}} \gamma(x)\widetilde{N}(ds\ dx), \quad t \geq 0,$$

where $\gamma(x)$ is such that $\int_{\mathbb{R}\backslash\{0\}} \gamma(x)^2\nu(dx) < \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function in $C^2(\mathbb{R})$ and $Y_t = f(X_t)$. Then, the process $Y_t, t \geq 0$ is a real-valued stochastic process which satisfies

$$\begin{aligned} dY_t = &\frac{df}{dx}(X_t)cdt + \frac{df}{dx}(X_t)\sigma dW_t + \frac{1}{2}\frac{d^2f}{dx^2}(X_t)\sigma^2 dt \\ &+ \int_{\mathbb{R}\backslash\{0\}} \left[f(X_t+\gamma(x)) - f(X_t) - \frac{df}{dx}(X_t)\gamma(x)\right]\nu(dx)dt \\ &+ \int_{\mathbb{R}\backslash\{0\}} \left[f(X_{t-}+\gamma(x)) - f(X_{t-})\right]\tilde{N}(dt\ dx). \end{aligned}$$

(ii) Let $X_t = (X_t^1, ..., X_t^d)$ be a d-dimensional process given by

$$X_t = x + tc + \sigma W_t + \int_0^t \int \gamma(x)\widetilde{N}(ds\ dx), \quad t \geq 0.$$

where $c \in \mathbb{R}^d, \sigma$ is a $d \times m$-matrix, $\gamma(x) = [\gamma_{ij}(x)]$ is a $d \times m$-matrix-valued function such that the integral exists, $W_t = (W_t^1, W_t^2, ..., W_t^d)^T$ is an m-dimensional standard Brownian motion, and

$$\widetilde{N}(dt\ dx) = (N_1(dt\ dx_1) - \mathbb{1}_{|x_1|<1}\nu(dx_1)dt, ..., N_m(dt\ dx_m) - \mathbb{1}_{|x_m|<1}\nu(dx_m)dt,$$

where N_j's are independent Poisson random measures with Lévy measures $\nu_j, j = 1, ..., m$. That is, X_t^i is given by

$$X_t^i = x_i + tc_i + \sum_{j=1}^m \sigma_{ij}W_{j,t} + \sum_{j=1}^m \int_0^t \int_{\mathbb{R}\backslash\{0\}} \gamma_{ij}(x)\widetilde{N}_j(ds\ dx_j), \quad i = 1, ..., d.$$

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function in $C^2(\mathbb{R}^d)$, and let $Y_t = f(X_t)$. Then, the process $Y_t, t \geq 0$ is real-valued stochastic process which satisfies

$$\begin{aligned} dY_t = &\sum_{i=1}^d \frac{\partial f}{\partial x_i}X_t c_i dt + \sum_{i=1}^d\sum_{j=1}^m \frac{\partial f}{\partial x_i}X_t\sigma_{ij}dW_{j,t} \\ &+ \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}X_t(\sigma\sigma^T)_{ij}dt + \sum_{j=1}^m \int_{\mathbb{R}\backslash\{0\}} [f(X_t+\gamma^j(x_j)) - f(X_t) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}X_t\gamma_{ij}(x)]\nu_j(dx_j)dt \\ &+ \sum_{j=1}^m \int_{\mathbb{R}\backslash\{0\}} [f(X_{t-}+\gamma^j(x)) - f(X_{t-})]\widetilde{N}_j(dt\ dx_j). \end{aligned}$$

Here, γ^j denotes the j-th column of the matrix $\gamma = [\gamma_{ij}]$.

Example 3.3.1 *Let $b = 0, \gamma(x) = 0, \sigma = 1$ and $f(x) = x^2$. Then Itô's formula leads to*

$$\int_0^T W_t dW_t = \frac{1}{2}(W_T^2 - T).$$

3.4 Example of Lévy process

Compound Poisson process

Definition 3.4.1 *(Compound Poisson processes) A* ***compound Poisson process*** *is defined as*

$$Y_t = \sum_{k=1}^{N_t} Y_k,$$

where jumps sizes Y_k, $k = 1, 2, ...$ are i.i.d. random variables with a common finite distribution ν on $\mathbb{R}^d \backslash \{0\}$ and N_t denotes a Poisson process with the intensity $\lambda > 0$, independent of Y_k. Then, Y_t has a representation

$$Y_t = \int_0^t \int_{\mathbb{R}^d \backslash \{0\}} xN(ds\ dx),$$

where $N(ds\ \ dx)$ denotes a Poisson random measure on $\mathcal{T} \times \mathbb{R}^d \backslash \{0\}$ with mean measure $\lambda ds \nu(dx)$.

The following properties of a compound Poisson process can be implied from the definition:

(i) The sample paths of X are cadlag piecewise constant functions.

(ii) The jump sizes $(Y_k)_{k \geq 1}$ are independent and identically distributed with law f.

Proposition 3.4.1 *Y_t is a compound Possion process if and only if it is a Lévy process and its sample paths are piecewise constant functions.*

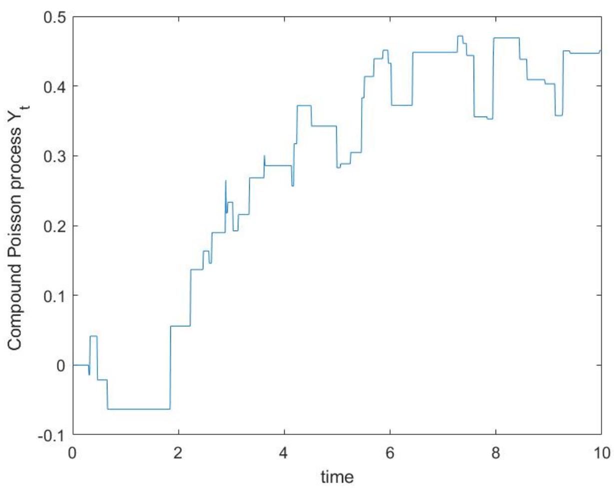


Figure 3.1: A simulated compound Poisson process path with intensity 5 and a normal jump height distribution $\mathcal{N}(0.005, 0.0025)$

3.5 Subordinators and Subordination

An important class of Lévy processes are subordinators. These are a subclass of Lévy processes with finite variation and paths that are almost surely non-decreasing. A subordinator is itself a stochastic process of the evolution of time within another stochastic process, the subordinated stochastic process. A subordinator will determine the random number of time steps occurring within subordinated process for a given unit of chronological time.

3.5.1 Subordinators

Definition 3.5.1 *(Total variation) The* ***total variation*** *of a function* $f : [a, b] \to \mathbb{R}^d$ *is defined by*

$$TV(f) = \sup \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|,$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ *of the interval* $[a, b]$*. In particular, in one dimension every increasing or decreasing function is of finite variation and every function of finite variation is a difference of two increasing functions.*

A Lévy process is said to be of ***finite variation*** *if its trajectories are functions of finite variation with probability 1.*

Notation: In Lévy-Khintchine representation Proposition (3.3.1), we denote $A = \sigma\sigma^T$ as the symmetric positive matrix. We call the representation as a Lévy process with its characteristic triplet (A, ν, γ).

Proposition 3.5.1 *(Finite variation Lévy process) A Lévy process is of finite variation if and only if its characteristic triplet* (A, ν, γ) *satisfies:*

$$A = 0 \quad \textit{and} \quad \int_{|x| \le 1} |x| \nu(dx) < \infty \tag{3.5}$$

Proof: See [13] Proposition 3.9.

Remark 3.5.1 ***Subordinators*** *can be regarded as increasing Lévy processes since they can be used as time changes for other Lévy processes. They serve as important blocks for constructing Lévy-based models.*

Proposition 3.5.2 *Let* $(X_t)_{t \ge 0}$ *be a Lévy process on* $\mathbb{R}$*. Then the following conditions are equivalent:*

(i) $X_t \ge 0$ *a.s. for some* $t > 0$*.*

(ii) $X_t \ge 0$ *a.s. for every* $t > 0$*.*

(iii) Sample paths of (X_t) are almost surely non decreasing: $t \geq s \Rightarrow X_t \geq X_s$ a.s.

(iv) The characteristic triplet of (X_t) satisfies $A = 0, \nu((-\infty, 0]) = 0, \int_0^\infty (x \wedge 1)\nu(dx) < \infty$ and $b \geq 0$ (positive drift), that is, (X_t) has no diffusion component, only positive jumps of finite variation and positive drift.

Proof: See [13] Proposition 3.10.

Proposition 3.5.3 *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ and let $f : \mathbb{R}^d \to [0, \infty)$ be a positive function such that $f(x) = O(|x|^2)$ as $x \to 0$. Then the process $(S_t)_{t\geq 0}$ defined by*

$$S_t = \sum_{\substack{s \leq t \\ \Delta X_s \neq 0}} f(\Delta X_s) \tag{3.6}$$

is a subordinator.

Proof: See [13] Proposition 3.11.

3.5.2 Subordination

Let $(S_t)_{t\geq 0}$ be a subordinator, a Lévy process satisfying any one of the equivalent conditions in Proposition 3.5.2, which means in particular that its paths are almost surely increasing. Since S_t is a positive random variable for all t, we describe it using Laplace transform. Let the characteristic triplet of S be $(0, \rho, b)$. Then the moment generating function of S_t is

$$\mathbb{E}[e^{uS_t}] = e^{tl(u)}, \quad \forall u \leq 0, \quad \text{where} \quad l(u) = bu + \int_0^\infty (e^{ux} - 1)\rho(dx). \tag{3.7}$$

We call $l(u)$ the Laplace exponent of S. The following theorem shows that the process S can be used to " time change" other Lévy processes and is interpreted as a time deformation.

Theorem 3.5.1 *Under a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ with characteristic exponent $\psi(u)$ and triplet (A, ν, γ) and let $(S_t)_{t\geq 0}$ be a subordinator with Laplace exponent $l(u)$ and triplet $(0, \rho, b)$. Then the process $(Y_t)_{t\geq 0}$ defined for each $\omega \in \Omega$ by $Y(t, \omega) = X(S(t, \omega), \omega)$ is a Lévy process. Its characteristic function is*

$$\mathbb{E}[e^{i\langle u, Y_t\rangle}] = e^{tl(\psi(u))}, \tag{3.8}$$

the characteristic exponent of Y is obtained by composition of the Laplace exponent of S with the characteristic exponent of X. The triplet (A^Y, ν^Y, c^Y) of Y is given by

$$A^Y = bA, \tag{3.9}$$

$$\nu^Y(B) = b\nu(B) + \int_0^\infty p_s^X(B)\rho(ds), \quad \forall B \in \mathcal{B}(\mathbb{R}^d) \tag{3.10}$$

$$\gamma^Y = b\gamma + \int_0^\infty \rho(ds) \int_{|x|\leq 1} x p_s^X(dx), \tag{3.11}$$

where p_t^X is the probability distribution of X_t.

$(Y_t)_{t\geq 0}$ *is said to be subordinate to the process* $(X_t)_{t\geq 0}$.

Proof: See [13] Theorem 4.2.

The purpose of introducing the concepts of subordinator and subordination is to construct a Lévy process by Brownian subordination.

3.5.3 Constructing Lévy processes by Brownian subordination

In this subsection, we consider subordinating a Brownian motion to get a new Lévy process. Let $(Z_t)_{t\geq 0}$ be a subordinator with Laplace exponent $l(u)$ and let $(W_t)_{t\geq 0}$ be a standard Brownian motion independent of (Z_t). Subordinating Brownian motion with drift μ by the process Z, we get a new Lévy process

$$X_t = \mu Z_t + \sigma W(Z_t).$$

Presentation (3.8) implies that X has characteristic exponent $\psi(u) = l(\frac{-u^2\sigma^2}{2} + i\mu u)$. The following theorem characterizes Lévy measures of processes that can be represented as subordinated Brownian motion with drift.

Theorem 3.5.2 *Let ν be a Lévy measure on $\mathbb{R}$ and $\mu \in \mathbb{R}$. There exists a Lévy process $(X_t)_{t\geq 0}$ with Lévy measure ν such that $X_t = W(Z_t) + \mu Z_t$ for some subordinator $(Z_t)_{t\geq 0}$ and Brownian motion $(W_t)_{t\geq 0}$ independent from Z if and only if the following conditions are satisfied:*

(i) ν is absolutely continuous with density $\nu(x)$.

(ii) $\nu(x)e^{-\mu x} = \nu(-x)e^{\mu x}$ for all x.

(iii) $\nu(\sqrt{u})e^{-\mu\sqrt{u}}$ is completely monotonic function on $(0, \infty)$.

This theorem allows to describe the jump structure of a process, that can be represented as time changed Brownian motion with drift.

Let ν be a Lévy measure on $\mathbb{R}^d$. It can be the Lévy measure of a subordinated Brownian motion (without drift) if and only if it is symmetric and $\nu(\sqrt{u})$ is a completely monotonic function on $(0, \infty)$. Consider a subordinator with zero drift and Lévy measure ρ. Formula (3.10) gives its Lévy density $\nu(x)$:

$$\nu(x) = \int_0^\infty e^{-\frac{(x-\mu t)^2}{2t}} \frac{\rho(dt)}{\sqrt{2\pi t}}$$

Then the formula allows us to write:

$$BS_\mu^{-1}(\nu) = e^{\mu^2 t/2} BS_0^{-1}(\nu e^{-\mu x}).$$

BS^{-1} denotes inverse transform of Brownian subordination. Hence we can deduce the time changed Brownian motion representation for an exponentially tilted Lévy measure from its representation for its symmetric modification.

Chapter 4

Option pricing with Lévy processes

The most important motivation for departing traditional Gaussian models in financial modelling has been to take into account a few of the empirical properties of asset returns which show some degree of discrepancy with these models. Lévy processes came into financial modelling when Mandelbrot (1963) [50] proposed $\alpha-$stable Lévy processes as models for cotton prices. Since then a variety of models based on Lévy processes have been developed as models for stock prices and tested on empirical data.

Market prices are observed in the form of time series of prices at a discrete set of dates. In this section we will look into exp-Lévy model, which plays one of the most used roles in pricing options,

$$S_t = S_0 \exp(X_t),$$

where X is a Lévy process.

But first we start with the very beginning model of pricing options with Lévy process. French mathematician L. Bachelier was the first to analyse Brownian motion mathematically, and that he did so in order to develop a theory of option pricing [1]. The Bachelier model is a mathematical pricing model considered to be particularly useful in pricing options when the value of the underlying becomes or may become negative. It is an alternative to the Black-Merton-Scholes and other option pricing models and is attractive because it does not rely on logarithms which cannot represent negative values. To derive the Bachelier model, we follow the procedure from S.Terakado (2019) [69].

4.1 Bachelier model

We assume that , under the risk-neutral measure, the stock process $\{S_t, t \geq 0\}$ satisfies an SDE of the form

$$dS_t = rS_t dt + \sigma dW_t,$$

where r is the constant interest rate, σ is the constant volatility, and W_t is standard Brownian motion. For $0 \leq t \leq T$,

$$S_T = S_t e^{r(T-t)} + \sigma \int_t^T e^{r(T-s)} dW_s.$$

That is,

$$\begin{aligned} S_T \mid S_t &\sim N\left(S_t e^{r(T-t)}, \frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)\right) \\ &\sim S_t e^{r(T-t)} + \sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}\,\xi, \end{aligned}$$

where ξ is standard normal random variable. The following Bachelier formula can be derived from the above Bachelier model based on the same argument of Black-Scholes and Merton.

Formula 4.1.1 ***(The Bachelier formula)*** *The price of a European call option at time t is given by*

(i) $r \neq 0$*,*

$$\begin{aligned} C_t &= (S_t - Ke^{-r(T-t)}\Phi(z) + \sigma\sqrt{\frac{1 - e^{-2r(T-t)}}{2r}}\phi(z), \\ z &= \frac{S_t - Ke^{-r(T-t)}}{\sigma\sqrt{\frac{1-e^{-2r(T-t)}}{2r}}}. \end{aligned}$$

(ii) $r = 0$*,*

$$\begin{aligned} C_t &= (S_t - K)\Phi(z) + \sigma\sqrt{T-t}\phi(z), \\ z &= \frac{S_t - K}{\sigma\sqrt{T-t}}. \end{aligned}$$

Derivation of Bachelier formula

$$\begin{aligned}
C_t &= e^{-r(T-t)} E\left((S_T - K)^+ \mid \mathcal{F}_t\right) \\
&= e^{-r(T-t)} E\left(\left(S_t e^{r(T-t)} + \sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}\,\xi - K\right)^+ \mid \mathcal{F}_t\right) \\
&= e^{-r(T-t)} \sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)} E\left(\left(\xi - \frac{K - S_t e^{r(T-t)}}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}}\right)^+ \mid \mathcal{F}_t\right) \\
&= e^{-r(T-t)} \left(S_t e^{r(T-t)} - K\right) \Phi\left(\frac{S_t e^{r(T-t)} - K}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}}\right) \\
&\quad + e^{-r(T-t)} \sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}\,\phi\left(\frac{S_t e^{r(T-t)} - K}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}}\right),
\end{aligned}$$

where Φ is the cumulative distribution function of a standard normal random variable, and ϕ is the corresponding density function.

Let $K^* = e^{-r(T-t)}K$, and

$$v^2(t,T) = \frac{\sigma^2}{2r}\left(1 - e^{-2r(T-t)}\right).$$

Then, we can re-express the price as

$$C_t = (S_t - K^*)\,\Phi\left(\frac{S_t - K^*}{v(t,T)}\right) + v(t,T)\,\phi\left(\frac{S_t - K^*}{v(t,T)}\right).$$

The model gives nice closed-formulas for pricing interest rate plain vanilla options and it is suitable especially in the today negative yield environment, since the forward rate can assume all the possible values, positive or negative on the whole real line. In some scenarios, it allows the underlying of IR derivatives, namely the forward rate, to be negative. This has really important application in real market.

On the day of 8th of April, 2020, CME Group posted the note CME Clearing Plan to Address the Potential of a Negative Underlying in Certain Energy Options Contracts, saying that after a threshold on price, it would change energy options model from Geometric Brownian Motion model and Black–Scholes model to Bachelier model. On that day, oil prices reached for first time in history negative values, where Bachelier model took an important role in option pricing and risk management.

4.2 Exponential of a Lévy process

Let $(X_t)_{t\geq 0}$ be a Lévy process with jump measure J_X. Applying the Itô's formula to $Y_t = \exp X_t$ yields:

$$\begin{aligned} Y_t &= 1 + \int_0^t Y_{s-} dX_s + \frac{\sigma^2}{2}\int_0^t Y_{s-} ds + \sum_{0\leq s\leq t;\Delta X_s\neq 0} (e^{X_{s-}+\Delta X_s} - e^{X_{s-}} - \Delta X_s e^{X_{s-}}) \\ &= 1 + \int_0^t Y_{s-} dX_s + \frac{\sigma^2}{2}\int_0^t Y_{s-} ds + \int_{[0,t]\times\mathbb{R}} Y_{s-}(e^z - 1 - z) J_X(ds\, dz) \end{aligned}$$

or, in other expression:

$$\frac{dY_t}{Y_{t-}} = dX_t + \frac{\sigma^2}{2} dt + (e^{\Delta X_t} - 1 - \Delta X_t) \tag{4.1}$$

Making an additional assumption that $\mathbb{E}[|Y_t|] = \mathbb{E}[\exp(X_t)] < \infty$, which is equivalent to saying that $\int_{|y|\geq 1} e^y \nu(dy) < \infty$, we can decompose Y_t into a martingale part and a drift part, where the martingale part is the sum of an integral with respect to the Brownian component of X and a compensated sum of jump terms:

$$M_t = 1 + \int_0^t Y_{s-}\sigma dW_s + \int_{[0,t]\times\mathbb{R}} Y_{s-}(e^z - 1)\widetilde{J_X}(ds\, dz), \tag{4.2}$$

while the drift term is given by:

$$\int_0^t Y_{s-}\left[\gamma + \frac{\sigma^2}{2} + \int_{-\infty}^{\infty} (e^z - 1 - z\mathbb{1}_{|z|\leq 1})\nu(dz)\right] ds. \tag{4.3}$$

Therefore, $Y_t = \exp(X_t)$ is a martingale of and only if the drift term vanishes, which is,

$$\gamma + \frac{\sigma^2}{2} + \int_{-\infty}^{\infty} (e^z - 1 - z\mathbb{1}_{|z|\leq 1})\nu(dz) = 0.$$

These properties are summarized in the following proposition:

Proposition 4.2.1 *(Exponential of a Lévy process)*

Let $(X_t)_{t\geq 0}$ be a Lévy process with Lévy triplet (σ^2, ν, γ) verifying

$$\int_{|y|\geq 1} e^y \nu(dy) < \infty.$$

Then $Y_t = \exp X_t$ is a semimartingale with decomposition $Y_t = M_t + A_t$ where the martingale part is given by

$$M_t = 1 + \int_0^t Y_{s-}\sigma dW_s + \int_{[0,t]\times\mathbb{R}} Y_{s-}(e^z - 1)\widetilde{J_X}(ds\, dz),$$

and the continuous finite variation drift part is given by

$$A_t = \int_0^t Y_{s-}\left[\gamma + \frac{\sigma^2}{2} + \int_{-\infty}^{\infty}(e^z - 1 - z\mathbb{1}_{|z|\leq 1})\nu(dz)\right] ds.$$

(Y_t) is a martingale if and only if

$$\gamma + \frac{\sigma^2}{2} + \int_{-\infty}^{\infty}(e^z - 1 - z\mathbb{1}_{|z|\leq 1})\nu(dz) = 0.$$

4.2.1 European options in exp-Lévy models

Consider an European call option on an asset S with maturity date T and strike price K, where the payoff of the option is given by $f_T = (S_T - K)^+$ at date T.

The price of a call option may be expressed as the risk-neutral conditional expectation of the payoff:

$$C_t(T, K) = e^{-r(T-t)}\widehat{\mathbb{E}}[(S_T - K)^+|\mathcal{F}_t] \tag{4.4}$$

In an exponential-Lévy model, the expression (4.4) can be simplified further. By stationary and independence of increments, the conditional expectation in (4.4) may be written as an expectation of the process at time $\tau = T - t$:

$$\begin{aligned} C(t, S, T = t + \tau, K) &= e^{-r\tau}\mathbb{E}[(S_T - K)^+|S_t = S] = e^{-r\tau}\mathbb{E}[(Se^{r\tau + X_\tau} - K)^+] \\ &= Ke^{-r\tau}\mathbb{E}(e^{x + X_\tau} - 1)^+, \end{aligned}$$

where x is defined by $x = \ln(S/K) + r\tau$. We see that similar to the Black-Scholes model, in all exp-Lévy models call option price depends on the time remaining until maturity but not on the actual date and the maturity date and is a homogeneous function of order 1 of S and K.

Defining the relative forward option price in terms of the relative variables (x, τ):

$$v(\tau, x) = \frac{e^{r\tau}C(t, S; T = t + \tau, K)}{K} \tag{4.5}$$

we conclude that the entire structure of option prices in exponential-Lévy models is parametrised by two variables:

$$v(\tau, x) = \mathbb{E}[(e^{x + X_\tau} - 1)^+]$$

This is a consequence of temporal and spatial homogeneity of Lévy processes. $u(\tau, .)$ can also be written as a convolution product :$u(\tau, .) = \rho_\tau * h$, where ρ_τ is the transition density of the Lévy process. Therefore if the process has smooth transition densities, $u(\tau, .)$ will be smooth, even if the payoff function h is not.

4.3 Integro-differential equations

In option pricing under Black-Scholes model, the value $C(t, S_t)$ of a European option can be derived by solving a partial differential equation:

$$\frac{\partial C}{\partial t}(t,S) + rS\frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} - rC(t,s) = 0.$$

with boundary conditions depending on the type of option considered. In reality, there are a few numerical methods for solving such equations. We will introduce partial integro-differential equation for $C(t, s)$ for exp-Lévy model.

4.3.1 Partial integro-differential equations for option prices

Consider an asset whose price process under risk neutral measure is given by an exp-Lévy model: $S_t = S_0 e^{rt+X_t}$. X is a Lévy process with characteristic triplet (σ^2, ν, γ) under risk-neutral measure $\mathbb{Q}$ such that the discounted price process $\widehat{S}_t = e^{-rt}S_t = e^{X_t}$ is a martingale.

Given assumptions that satisfy all regular conditions, we have the risk-neutral dynamics of S_t:

$$S_t = S_0 + \int_0^t rS_{s-}ds + \int_0^t S_{s-}\sigma dW_s + \int_0^t \int_{-\infty}^{\infty} (e^x - 1)S_{s-}\widetilde{J_X}(ds\ dx),$$

where $\widetilde{J_X}$ denotes the compensated jump measure of the Lévy process X and $\widehat{S}_t$ is a square-integrable martingale satisfies:

$$\frac{d\widehat{S}_t}{\widehat{S}_{t-}} = \sigma dW_t + \int_{-\infty}^{\infty}(e^x - 1)\widetilde{J_X}(dt\ dx), \quad \sup \mathbb{E}[\widehat{S}_t^2] < \infty.$$

The value of a European option is given by

$$C_t = \mathbb{E}[e^{-r(T-t)}H(S_T) \mid \mathcal{F}_t]$$

where H is the payoff of European option as a function of underlying asset. Let $\epsilon = T - t, x = \ln\frac{S}{K} + r\epsilon$ and define $h(x) = \frac{H(Ke^x)}{K}$ and $\vartheta(\epsilon, x) = e^{r\epsilon}\frac{C(t,S)}{K}$, we can then rewrite the above expression as

$$\vartheta(\epsilon, x) = \mathbb{E}[f(x + X_\epsilon)]$$

Differentiate with respect to ϵ to get the following integro-differential equation:

$$\frac{\partial \vartheta}{\partial \epsilon} = \eta^X \vartheta \quad \text{on} \quad [0,T]\times\mathbb{R}, \quad \vartheta(0,x) = h(x).$$

η is the infinitesimal generator of X:

$$\eta^X f(x) = \gamma\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \int \nu(dy)[f(x+y) - f(x) - \mathbb{1}_{|y|<1}\frac{\partial f}{\partial x}]. \tag{4.6}$$

Then if we replace $f(x)$ by $C(t,s)$ we obtain:

$$\frac{\partial C}{\partial t}(t,S)+rS\frac{\partial C}{\partial S}+\frac{\sigma^2S^2}{2}\frac{\partial^2 C}{\partial S^2}-rC(t,s)+\int\nu(dy)[C(t,Se^y)-C(t,s)-S(e^y-1)\frac{\partial C}{\partial S}(t,S)]=0$$

We have used the concept of infinitesimal generator in the above derivation. Now, let us review the definition of infinitesimal generator.

Remark 4.3.1 ***(Infinitesimal generator of a Lévy process)*** *Let* $(X_t)_{t\geq 0}$ *be Lévy process on* $\mathbb{R}^d$ *with characteristic triplet* (A,ν,γ). *Then the infinitesimal generator of* X *is defined for any* $f\in C_0^2(\mathbb{R})$ *as*

$$\eta^X f(x)=\frac{1}{2}\sum_{j,k=1}^{d}A_{jk}\frac{\partial^2 f}{\partial x_j\partial x_k}(x)+\sum_{j=1}^{d}\gamma_j\frac{\partial f}{\partial x_j}(x)+\int_{\mathbb{R}^d}\left(f(x+y)-f(x)-\sum_{j=1}^{d}y_j\frac{\partial f}{\partial x_j}(x)\mathbb{1}_{|y|\leq 1}\right)\nu(dy).$$

4.3.2 Feynman-Kac representation

The classical Feynman-Kac formula states the connection between linear parabolic partial differential equations (PDE) and expectation of stochastic processes driven by Brownian motion. It gives a method for solving linear PDEs by Monte Carlo simulations of random processes. The extension to (fully) nonlinear PDEs led to important developments in stochastic analysis and the emergence of the theory of backward stochastic differential equations (BSDE), which can be viewed as nonlinear Feynman-Kac formulas.

Consider a bounded function $h\in\eta^\infty(\mathbb{R})$. If

$$\exists a,b>0,\forall t\in[0,T],a\leq\sigma_t\leq b$$

then the Cauchy problem $\forall x\in\mathbb{R}$, $f(T,x)=h(x)$,

$$\frac{\partial f}{\partial t}(t,x)+\frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}(t,x)+\gamma_t\frac{\partial f}{\partial x}(t,x)+\int\nu(dy)[f(t,x+y)-f(t,x)-y\mathbb{1}_{|y|\leq 1}\frac{\partial f}{\partial x}(t,x)]=0,$$

has a unique solution given by

$$f(t,x)=\mathbb{E}[h(X_T^{t,x})],$$

where $X^{t,x}$ is given by

$$X_t^{s,x}=x+\int_s^t\gamma_\vartheta d\vartheta+\int_s^t\sigma_\vartheta dW_\vartheta+\int_s^t\int_{|y|\geq 1}yJ_X(d\vartheta\ dy)+\int_s^t\int_{|y|\leq 1}y\widetilde{J_X}(d\vartheta\ dy).$$

for all $t>s$. J_X denotes a Poisson random measure on $[0,T]\times\mathbb{R}$ with intensity $\mu(dy\ dt)=\nu(dy)dt$ and $\widetilde{J_X}$ is compensated Poisson measure.

$X_t^{s,x}$ is the position at time t of a jump process starting at x with drift γ and a time dependent volatility σ and a jump component described by a Lévy process.

4.4 Fourier transform methods for option pricing

Unlike the classical Black-Scholes case, in exponential-Lévy models there are no explicit expression for call option prices, since the probability density of a Lévy process is not known in closed form. This has intrigued the development of Fourier-based option pricing models for exponential-Lévy models.

We will describe one Fourier-based method for option pricing in exponential-Lévy models. The method developed by Carr and Madan [11] is easy to implement but has relatively lower convergence rates.

Definition 4.4.1 *The **Fourier transform** of a function f is defined by:*

$$\boldsymbol{F}f(v) = \int_{-\infty}^{\infty} e^{ixv} f(x)dx.$$

*The **inverse Fourier transform** is given by:*

$$\boldsymbol{F}^{-1}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixv} f(v)dx.$$

For $f \in L^2(\mathbb{R}), \mathbf{F}^{-1}\mathbf{F}f = f$. In what we follows, we denote $k = \ln K$ the log of strike price and assume without loss of generality that $t = 0$.

4.4.1 Method of Carr and Madan

In this section we set $S_0 = 1$, i.e., at time 0 all prices are expressed in units of the underlying. Let k be the log value of strike price K. An assumption necessary in this method is that the stock price have a moment of order $1 + \alpha$ for some $\alpha > 0$:

$$\exists \alpha > 0 : \int_{-\infty}^{\infty} \rho_T(s) e^{(1+\alpha)s} ds < \infty, \tag{4.7}$$

where ρ_T is the risk-neutral density of X_T. In terms of the Lévy density it is equivalent to the condition:

$$\exists \alpha > 0 \quad \int_{|y|\geq 1} \nu(dy) e^{(1+\alpha)y} < \infty.$$

We want to calculate the price of a call option:

$$C(k) = e^{-rT}\mathbb{E}[(e^{rT+X_T} - e^k)^+].$$

We express the Fourier transform in strike in terms of the characteristic function $\Phi_T(v)$ of X_T and then find the prices range of strikes by Fourier inversion. However, due to $C(k)$ is not integrable (it tends to a positive constant as $k \to -\infty$), we cannot proceed with characteristic function. The core step of this method is to compute the Fourier transform of the time value of the option:

$$f_T(k) = e^{-rT}\mathbb{E}[(e^{rT+X_T} - e^k)^+] - (1 - e^{k-rT})^+. \tag{4.8}$$

Let $\zeta_T(v)$ denote the Fourier transform of the time value:

$$\zeta_T(v) = \mathbf{F}f_T(v) = \int_{-\infty}^{+\infty} e^{ivk} f_T(k)dk. \tag{4.9}$$

Since the discounted price process is a martingale, we write

$$f_T(k) = e^{-rT}\int_{-\infty}^{+\infty} \rho_T(x)dx(e^{rT+x} - e^k)(\mathbb{1}_{k\leq x+rT} - \mathbb{1}_{k\leq rT}).$$

Assumption (4.7) allows us to compute $\zeta_T(v)$ by interchanging integrals, and conclude that

$$\zeta_T(v) = e^{ivrT}\frac{\Phi_T(v-i) - 1}{iv(1+iv)}. \tag{4.10}$$

By assumption (4.7), we can observe that the numerator becomes an analytic function and the fraction has a finite limit for $v \to 0$. Option prices can then be computed by inverting the Fourier transform,

$$f_T(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ivk}\zeta_T(v)dv.$$

In this method we need assumption (4.7) to derive the formulae but it is not necessary to know the exact value of α to compute, which makes this method easier to implement. However, a drawback of this approach, is the slower convergence rate of the algorithm. We can replace the time value with a smooth function of strike to improve the convergence rate. That is, instead of subtracting the intrinsic value of the option from its price, we can subtract the Black-Scholes call price with a fixed volatility. Denote

$$\tilde{f}_T(k) = e^{-rT}\mathbb{E}[(e^{rT+X_T} - e^k)^+] - C^\sigma_{BS}(k),$$

where $C^\sigma_{BS}(k)$ is the Black-Scholes of a call option with volatility σ and log-strike k. It can be shown that the Fourier transform of $\tilde{f}_T(k)$, denoted by $\tilde{\zeta}_T(v)$ has form

$$\tilde{\zeta}_T(v) = e^{ivrT}\frac{\Phi_T(v-i) - \Phi^\sigma_T(v-i)}{iv(1+iv)},$$

where $\Phi^\sigma_T(v) = \exp(-\frac{\sigma^2 T}{2}(v^2 + iv))$.

4.4.2 Computing Fourier transforms using FFT

To implement the algorithms using Fourier transforms, we need to numerically compute Fourier transforms using the fast Fourier transform (FFT) [14].

The FFT is an efficient algorithm for computing the sum,

$$F_n = \sum_{j=0}^{N-1} f_k e^{-\frac{2\pi inj}{N}}, \quad n = 0, ..., N-1.$$

To compute $F_0, ..., F_{N-1}$, where N is typically a power of 2. The algorithm reduces the number of multiplications in the required N summations from an order of N^2 to that of $N \ln_2(N)$. Suppose we want to approximate the inverse Fourier transform of a function $f(x)$ with a discrete Fourier transform. The integral must be truncated as,

$$\int_{-\infty}^{\infty} e^{-iux} f(x)dx \approx \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} e^{-iux} f(x)dx \approx \frac{\eta}{N} \sum_{j=0}^{N-1} w_j f(x_j) e^{-iux_j}, \tag{4.11}$$

where $x_j = -\frac{\eta}{2} + \frac{k\eta}{N-1}$ is the discretisation step and w_j are weights corresponding to the chosen integration rule (e.g. $w_0 = w_{N-1} = \frac{1}{2}$ and other weights are equal to 1).

Set $u_n = \frac{2\pi n(N-1)}{N\eta}$, sum in the last term of (4.11) becomes a discrete Fourier transform,

$$\mathbf{F}f(u_n) \approx \frac{\eta}{N} e^{iu\frac{\eta}{2}} \sum_{j=0}^{N-1} w_j f(x_j) e^{-2\pi \frac{inj}{N}}.$$

Hence, the FFT algorithm allows to compute $\mathbf{F}f(u_n)$ at points $u_n = \frac{2\pi n(N-1)}{N\eta}$. It is useful to notice that the grid step d in the Fourier space is related to the initial grid step Δ:

$$\frac{d\eta}{N-1} = \frac{2\pi}{N}.$$

This implies if we want to compute option prices on a finite grid of strikes and at the same time keep the discretisation error low. One of the limitation of the FFT method is that the grid must always be uniform and the grid size of power of 2. If we want to price a single option, please see [9], there they transform the contour of integration in the complex plane to achieve a better convergence. Below we have generated a diagram of an option surface by Heston model using FFT [Appendix B1].

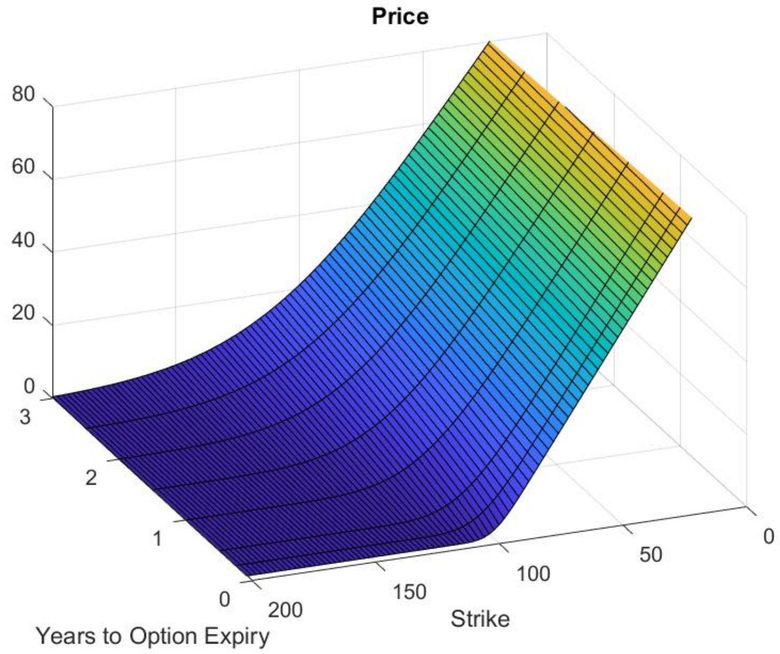


Figure 4.1: The figure shows an option price surface by Heston model using FFT using $r = 0.05, S_0 = 100, V_0 = 0.04, \kappa = 2, V = 0.04, \sigma = 0.2$ and $\rho = -0.8$.

4.5 Equivalence of measures for Lévy processes

We have introduced equivalent changes of measure in defining arbitrage-free pricing models, we now will focus on changes of measure in the case where a Lévy process is the source of randomness.

If the $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures then there exists a positive random variable, the density of $\mathbb{Q}$ with respect to $\mathbb{P}$ and denoted $\frac{d\mathbb{Q}}{d\mathbb{P}}$ such that for any random variable Z we have

$$\mathbb{E}^{\mathbb{Q}}[Z] = \mathbb{E}^{\mathbb{P}}[Z\frac{d\mathbb{Q}}{d\mathbb{P}}]$$

In this section we present a general result of equivalence of measures for Lévy processes. An significant finding of this result is that in presence of jumps, if we restrict our attention to structure preserving measures, the class of probabilities equivalent to a given one is surprisingly large.

Proposition 4.5.1 *(See [61] Theorems 33.1 and 33.2) Let $(X_t, \mathbb{P})$ and $(X_t, \mathbb{P}')$ be two Lévy processes on $\mathbb{R}$ with characteristic triplets (σ^2, ν, γ) and $(\sigma'^2, \nu', \gamma')$. Then $\mathbb{P}|_{\mathcal{F}_t}$ and $\mathbb{P}'|_{\mathcal{F}_t}$ are equivalent for all t if and only if the following conditions are satisfied:*

(i) $\sigma = \sigma'$

(ii) The Lévy measures are equivalent with

$$\int_{-\infty}^{\infty} (e^{\phi(x)/2} - 1)^2 \nu(dx) < \infty, \tag{4.12}$$

where $\phi(x) = \ln(\frac{dv'}{dv})$.

(iii) If $\sigma = 0$ *then we must in addition have*

$$\gamma' - \gamma = \int_{-1}^{1} x(\nu' - \nu)(dx). \tag{4.13}$$

When $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, the Radon-Nikodym derivative is

$$\frac{d\mathbb{P}'|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = e^{U_t} \tag{4.14}$$

with

$$U_t = \eta X_t^c - \frac{\eta^2\sigma^2 t}{2} - n\eta t$$
$$+ \lim_{\epsilon \downarrow 0} \left(\sum_{s \le t, |\Delta X_s| > \epsilon} \phi(\Delta X_s) - t \int_{|x|>\epsilon} (e^{\phi(x)} - 1)\nu(dx) \right).$$

Here (X_t^c) is the continuous part of (X_t) and η is such that

$$\gamma' - \gamma - \int_{-1}^{1} x(\nu' - \nu)(dx) = \sigma^2 \eta$$

if $\sigma > 0$ and 0 if $\sigma = 0$.

U_t is a Lévy process with characteristic triplet (a_U, ν_U, γ_U) given by:

$$a_U = \sigma^2 \eta^2 \tag{4.15}$$

$$\nu_U = \nu\phi^{-1} \tag{4.16}$$

$$\gamma_U = -\frac{1}{2}a\eta^2 - \int_{-\infty}^{\infty} (e^y - 1 - y\mathbb{1}_{|y|\le 1})(\nu\phi^{-1})(dy). \tag{4.17}$$

The proposition shows a feature of models with jumps compared to diffusion models: we have considerable freedom in changing the Lévy measure, while retaining the equivalence of measures, unless a diffusion component is present, we cannot freely change the drift.

4.6 The Esscher transform

Let X be a Lévy process with characteristic triplet $(\sigma^2, \nu, \gamma), \theta$ a real number and assume that the Lévy measure ν is such that $\int_{|x|\geq 1} e^{\theta x}\nu(dx) < \infty$.

Applying a measure transformation with the function $\phi(x)$ in Proposition (4.5.1) given by $\phi(x) = \theta x$ we obtain an equivalent probability under which X is a Lévy process with zero Gaussian component, Lévy measure $\tilde{\nu}(dx) = e^{\theta x}\nu(dx)$ and drift $\tilde{\gamma} = \gamma + \int_{-1}^{1} x(e^{\theta x} - 1)\nu(dx)$.

This transformation is known as the **Esscher transform**. By Proposition (4.5.1), the Radon-Nikodym derivaive [Appendix A1] corresponding to this measure change is:

$$\frac{d\mathbb{Q}|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = \frac{e^{\theta X_t}}{\mathbb{E}[e^{\theta X_t}]} = \exp(\theta X_t + \gamma(\theta)t),$$

where $\gamma(\theta) = -\ln \mathbb{E}[\exp(\theta X_1)]$ is the log of the moment generating function of X_1 which, up to the change of variable $\theta \leftrightarrow -i\theta$ is given by the characteristic exponent of the Lévy process X.

The Esscher transform can be used to construct equivalent martingale measures in exponential-Lévy models.

Proposition 4.6.1 *(Absence of arbitrage in exp-Lévy models) Let $(X, \mathbb{P})$ be a Lévy process. If the trajectories of X are neither almost surely increasing nor almost surely decreasing, then exp-Lévy model given by $S_t = e^{rt+X_t}$ is arbitrage-free: there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $(e^{-rt}S_t)_{t\in[0,T]}$ is a $\mathbb{Q}$-martingale, where r is the risk-free interest rate.*

The exponential-Lévy model is arbitrage-free in the following cases:

(i) X has a nonzero Gaussian component: $\sigma > 0$.

(ii) X has infinite variation $\int_{-1}^{1} |x|\nu(dx) = \infty$.

(iii) X has both positive and negative jumps.

Proof: See [13] Proposition 9.9.

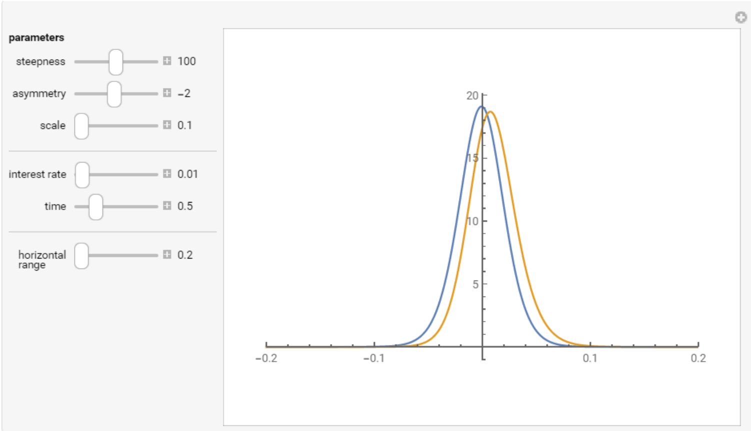


Figure 4.2: The Esscher Transform of the Densities of a Symmetric exp-Lévy Process

4.7 Relative entropy for Lévy processes

The notion of relative entropy or Kullback-Leibler distance is often used as measure of proximity of two equivalent probability measures. In this section, we will discuss relative entropy of the measures generated by two risk-neutral exp-Lévy processes.

Define $\mathbb{P}$ and $\mathbb{Q}$ to be two equivalent probability measures on space $(\Omega, \mathcal{F})$. The relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined as

$$\xi(\mathbb{Q}, \mathbb{P}) = \mathbb{E}^{\mathbb{Q}}\left[\ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right].$$

Introduce the strictly convex function $f(x) = x \ln x$, we can write the relative entropy as

$$\xi(\mathbb{Q}, \mathbb{P}) = \mathbb{E}^{\mathbb{P}}\left[f\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right].$$

It is observed that the relative entropy is a convex functional of $\mathbb{Q}$. Jensen's inequality shows that $\xi(\mathbb{Q}, \mathbb{P}) \geq 0$, with $\xi(\mathbb{Q}, \mathbb{P}) = 0$ if and only if $\dfrac{d\mathbb{Q}}{d\mathbb{P}} = 1$ almost surely. The following proposition shows that relative entropy can be expressed in terms of the Lévy measures if the measures are generated under exp-Lévy models.

Proposition 4.7.1 *(Relative entropy of Lévy processes) Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent measures on $(\Omega, \mathcal{F})$ generated by exponential-Lévy models with Lévy triplets $(\sigma^2, \nu^{\mathbb{P}}, \gamma^{\mathbb{P}})$ and $(\sigma^2, \nu^{\mathbb{Q}}, \gamma^{\mathbb{Q}})$. Assume $\sigma > 0$. The relative entropy $\xi(\mathbb{Q}, \mathbb{P})$ is given by:*

$$\xi(\mathbb{Q}|\mathbb{P}) = \frac{T}{2\sigma^2}\left\{\gamma^{\mathbb{Q}} - \gamma^{\mathbb{P}} - \int_{-1}^{1} x(\nu^{\mathbb{Q}} - \nu^{\mathbb{P}})(dx)\right\}^2 + \\ T\int_{-\infty}^{\infty}\left(\frac{d\nu^{\mathbb{Q}}}{d\nu^{\mathbb{P}}} \ln \frac{d\nu^{\mathbb{Q}}}{d\nu^{\mathbb{P}}} + 1 - \frac{d\nu^{\mathbb{Q}}}{d\nu^{\mathbb{P}}}\right)\nu^{\mathbb{P}}(dx).$$

If $\mathbb{P}$ and $\mathbb{Q}$ correspond to risk-neutral exponential-Lévy models, the relative entropy reduces to:

$$\xi(\mathbb{Q}|\mathbb{P}) = \frac{T}{2\sigma^2}\Big\{\int_{-\infty}^{\infty}(e^x-1)(\nu^{\mathbb{Q}}-\nu^{\mathbb{P}})(dx)\Big\}^2$$
$$+T\int_{-\infty}^{\infty}\left(\frac{d\nu^{\mathbb{Q}}}{d\nu^{\mathbb{P}}}\ln\frac{d\nu^{\mathbb{Q}}}{d\nu^{\mathbb{P}}}+1-\frac{d\nu^{\mathbb{Q}}}{d\nu^{\mathbb{P}}}\right)\nu^{\mathbb{P}}(dx).$$

Proof:

Let (X_t) be a Lévy process and $S_t = \exp(X_t)$. It is straightforward to see that histories generated by (X_t) and (S_t) are the same. Then we can equivalently compute the relative entropy if the log-price processes. We use formula (4.14) for Radon-Nikodym derivative to compute relative entropy of the two processes:

$$\xi = \int \frac{d\mathbb{Q}}{d\mathbb{P}}\ln\frac{d\mathbb{Q}}{d\mathbb{P}}d\mathbb{P} = \mathbb{E}^{\mathbb{P}}[U_T e^{U_T}].$$

where (U_t) is a Lévy process with characteristic triplet (a_U, ν_U, γ_U) given by presentations (4.15-4.17). Let $\phi_t(z)$ denote its characteristic function and $\psi(z)$ its characteristic exponent:

$$\phi_t(z) = \mathbb{E}^{\mathbb{P}}[e^{i(z,U_t)}] = e^{t\psi(z)}.$$

Then we have,

$$\mathbb{E}^{\mathbb{P}}[U_T e^{U_T}] = -i\frac{d}{dz}\phi_T(-i) = -iTe^{T\psi(-i)}\psi'(-i)$$
$$= -iT\psi'(-i)\mathbb{E}^{\mathbb{P}}[e^{U_T}] = -iT\psi'(-i).$$

From the Lévy-Khinchin representation in proposition (3.3.1),

$$\psi'(z) = -a^U z + i\gamma^U + \int_{-\infty}^{\infty}(ixe^{i(z,x)} - ix\mathbb{1}_{|x|\le 1})\nu^U(dx).$$

The relative entropy can be computed as:

$$\xi = a^U T + \gamma^U T + T\int_{-\infty}^{\infty}(xe^x - x\mathbb{1}_{|x|\le 1})\nu^U(dx)$$
$$= \frac{\sigma^2 T}{2}\eta^2 + T\int\left(\frac{d\nu^{\mathbb{Q}}}{d\nu^{\mathbb{P}}}\ln\frac{d\nu^{\mathbb{Q}}}{d\nu^{\mathbb{P}}}+1-\frac{d\nu^{\mathbb{Q}}}{d\nu^{\mathbb{P}}}\right)\nu^{\mathbb{P}}(dx),$$

where $\sigma^2\eta = \gamma^{\mathbb{Q}} - \gamma^{\mathbb{P}} - \int_{-1}^{1} x(\nu^{\mathbb{Q}}-\nu^{\mathbb{P}})(dx)$.

As $\sigma > 0$, we have

$$\frac{1}{2}\sigma^2\eta^2 = \frac{1}{2\sigma^2}\Big\{\gamma^{\mathbb{Q}} - \gamma^{\mathbb{P}} - \int_{-1}^{1} x(\nu^{\mathbb{Q}}-\nu^{\mathbb{P}})(dx)\Big\}^2.$$

This gives the proof of the first formula.

If $\mathbb{P}$ and $\mathbb{Q}$ correspond to risk-neutral exponential-Lévy model, then $\mathbb{Q}$ and $\mathbb{P}$ are martingale measures, where we can express the drift γ using σ and ν:

$$\frac{1}{2}\sigma^2\eta^2 = \frac{1}{2\sigma^2}\Big\{\int_{-\infty}^{\infty}(e^x-1)(\nu^{\mathbb{Q}}-\nu^{\mathbb{P}})(dx)\Big\}^2.$$

Substitute this in the first formula gives the proof of the second formula as required. □

4.8 Pricing in incomplete markets

The value of an option is defined as the cost of replicating strategy in risk neutral measure under arbitrage free setting in complete markets. In real markets, perfect hedges do not exist and options are not redundant: the method of pricing by replicating portfolio does not work because there are risks that one cannot hedge by continuous trading.

4.8.1 Merton's approach

The first application of jump processes in option pricing was introduced by Robert Merton [53]. He considered the jump diffusion model:

$$\text{under } \mathbb{P} : S_t = S_0 \exp[\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i], \tag{4.18}$$

where W_t is a standard Brownian motion, N_t is a Poisson process with intensity λ and $Y_i \sim N(m, \delta^2)$ are i.i.d. random variables, where W, N and Y are independent from each other. We can see that such a model is incomplete: there are many possible choices for a risk-neutral measure. Merton proposed the following choice, obtained as in the Black-Scholes model by changing the drift of the standard Brownian motion but leaving others unchanged:

$$\mathbb{Q}_M : S_t = S_0 \exp[\mu^M t + \sigma W_t^M + \sum_{i=1}^{N_t} Y_i], \tag{4.19}$$

μ^M is chosen such that $\hat{S}_t = S_t e^{-rt}$ is a martingale under $\mathbb{Q}^M$:

$$\mu^M = r - \frac{\sigma^2}{2} - \lambda \mathbb{E}[e^{Y_i} - 1] = r - \frac{\sigma^2}{2} - \lambda[\exp(m + \frac{\delta^2}{2}) - 1].$$

$\mathbb{Q}_M$ is the equivalent martingale measure obtained by shifting drift of the Brownian motion but leaving the jump part unchanged. A European option with payoff $f(S_T)$ can then be priced according to:

$$\begin{aligned} C_t^M &= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}_M}[f(S_T)|\mathcal{F}_t] \\ &= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}_M}[(S_T - K)^+ | S_t = S] \\ &= e^{-r(T-t)} \mathbb{E}[f(S e^{\mu^M (T-t) + \sigma W_{T-t}^M + \sum_{i=1}^{N_{T-t}} Y_i}] \end{aligned}$$

By conditioning on the number of jumps N_t, we can express $C(t, S_t)$ as a weighted sum of Black-Scholes prices: denoting the time to maturity by $\tau = T - t$ we can obtain:

$$C_m(t, S) = e^{-r\tau} \sum_{n \geq 0} \frac{e^{-\lambda \tau} (\lambda \tau)^n}{n!} C_{BS}(\tau, S_n; \sigma_n),$$

where $\sigma_n^2 = \sigma^2 + n\delta^2/\tau$,

$$S_n = S\exp[nm + \frac{n\delta^2}{2} - \lambda\exp(m + \frac{\delta^2}{2}) + \lambda\tau]$$

and

$$C_{BS}(\tau, S; \sigma) = e^{-r\tau}\mathbb{E}[f(Se^{r-\frac{\delta^2}{2})\tau+\sigma W_\tau})]$$

is the value of a European option with time to maturity τ and payoff f in a Black-Scholes model with volatility σ.

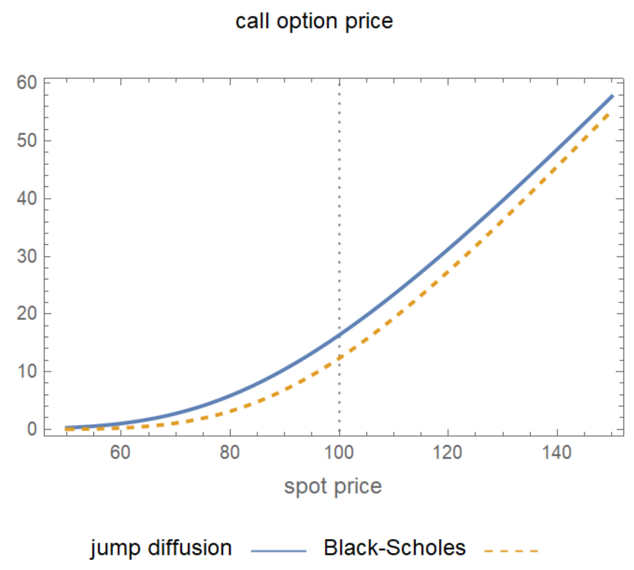


Figure 4.3: The diagram shows a simulation of call option's price paths related to its strike price under two different models, namely Merton and Black-Scholes

4.8.2 Simulations for the Merton jump-diffusion models

In general, from expression (4.18), we can easily derive the log return of stock price of MJD stock price as

$$R_{\Delta_t} = \ln(\frac{S_{t+\Delta t}}{S_t}) = (\mu - \frac{\sigma^2}{2})\Delta t + \sigma(W_{t+\Delta t} - W_t) + \sum_{i=N_t}^{N_{t+\Delta t}} Y_i \tag{4.20}$$

where $\Delta W_t = W_{t+\Delta t} - W_t$ is a standard Brownian motion increments and compound Poisson process increments $\sum_{i=N_t}^{N_{t+\Delta t}} Y_i$, which can be easily estimated. Then according to formula (4.20), log-return can be easily estimated, as well as stock prices.

For the given parameters $\mu = 0.16, \sigma = 0.3, \lambda = 5, m = 0.005$, and $\delta = 0.05$. Five paths of simulated stock prices [Appendix B2] are displayed where the initial share price $S_0 = 10$.

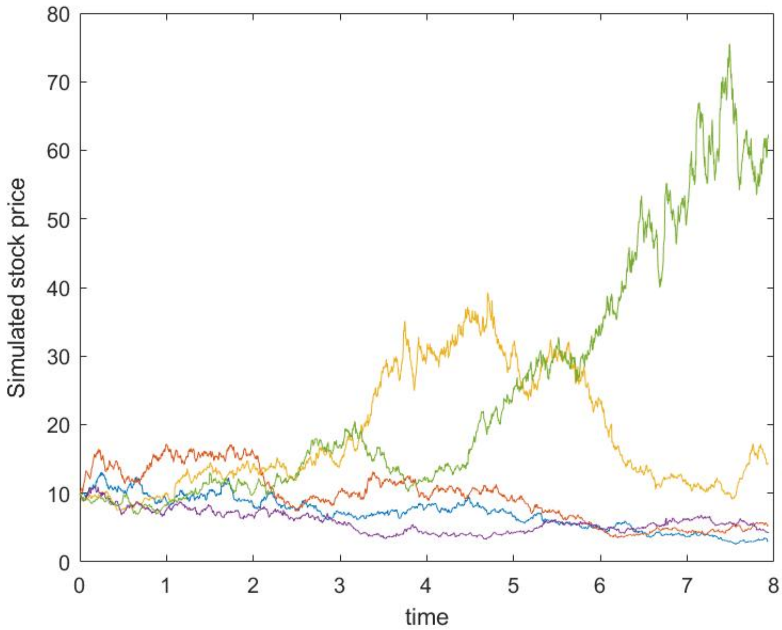


Figure 4.4: Five paths of JMD stock simulation with parameters $\mu = 0.16, \sigma = 0.3, \lambda = 5, m = 0.005$, and $\delta = 0.05$.

4.8.3 Estimate of the jump-diffusion parameters

The expectation and variance of log-return $R_{\Delta t}$ of JD modelled stock price are given [67],

$$\mathbb{E}(R_{\Delta t}) = (\mu - \frac{\sigma^2}{2})\Delta t + m\lambda\Delta t,$$
$$Var(R_{\Delta t}) = \sigma^2\Delta t + (m^2 + \delta^2)\lambda\Delta t.$$

To estimate the five parameters, we can use the maximum likelihood estimation method. We can use the MATLAB code *fminsearch* to estimate optimal parameters. Firstly, we will find an initial estimate of the parameters based on empirical data. The empirical log-returns $R_{\Delta t}$ of Apple (AAPL) are displayed in the next figure.

If the absolute value of the log-return is larger than some fixed positive value ϵ, then we can say there is a jump occurring. The parameter λ is estimated as

$$\begin{aligned}\hat{\lambda} &= \text{the number of jumps per year}\\ &= \frac{\text{total number of jumps}}{\text{total length in years}}\end{aligned}$$

For the value ϵ, dividing the empirical log-return data into two groups $\mathcal{D}$ and $\mathcal{M}$, the group $\mathcal{D}$ includes log-returns with absolutely value of log-returns less than ϵ, where there is no jump. Oppositely, the group $\mathcal{M}$ includes log-returns with absolute value larger than ϵ, where jumps have occurred.

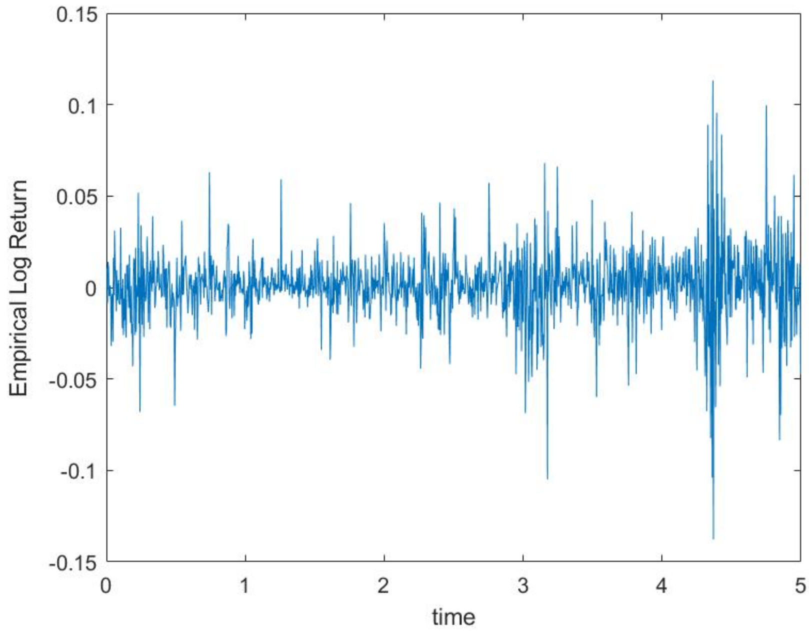


Figure 4.5: log-return of AAPL from 28/10/2015 to 29/10/2020

The parameters m and δ can estimated from equations:

$$\hat{m} = \hat{\mathbb{E}}(R^M_{\Delta t}) - (\hat{\mu} - \frac{\sigma^2}{2})\Delta t$$
$$\hat{\delta^2} = \widehat{Var}(R^M_{\Delta t}) - \sigma^2 \Delta t,$$

where $\hat{\mathbb{E}}(R^M_{\Delta t})$ and $\widehat{Var}(R^M_{\Delta t})$ are the sample mean and the sample variance of the empirical log-returns in group $\mathcal{M}$.

The parameters μ and σ can be estimated from the above formulas of $\mathbb{E}(R^D_{\Delta t})$ and $Var(R^D_{\Delta t})$,

$$\hat{\mu} = \frac{2\hat{\mathbb{E}}(R^D_{\Delta t}) + \widehat{Var}(R^D_{\Delta t})\Delta t}{2\Delta t}$$
$$\hat{\sigma^2} = \frac{\widehat{Var}(R^D_{\Delta t})}{\Delta t}$$

where $\hat{\mathbb{E}}(R^D_{\Delta t})$ and $\widehat{Var}(R^D_{\Delta t})$ are the sample mean and the sample variance of the empirical log-returns in group $\mathcal{D}$.

By reading off from figure 4.4, we can choose ϵ=0.03, we obtain the initial estimator of the parameters [Appendix B3] $\hat{\mu} = 0.3264, \hat{\lambda} = 21.2168, \hat{\sigma} = 0.1871, \hat{\delta} = 0.051, \hat{m} = -0.0014$.

Finally, we want to use our estimates of parameters from historical data to get stock price model for AAPL, and compare with realised stock prices [Appendix B4].

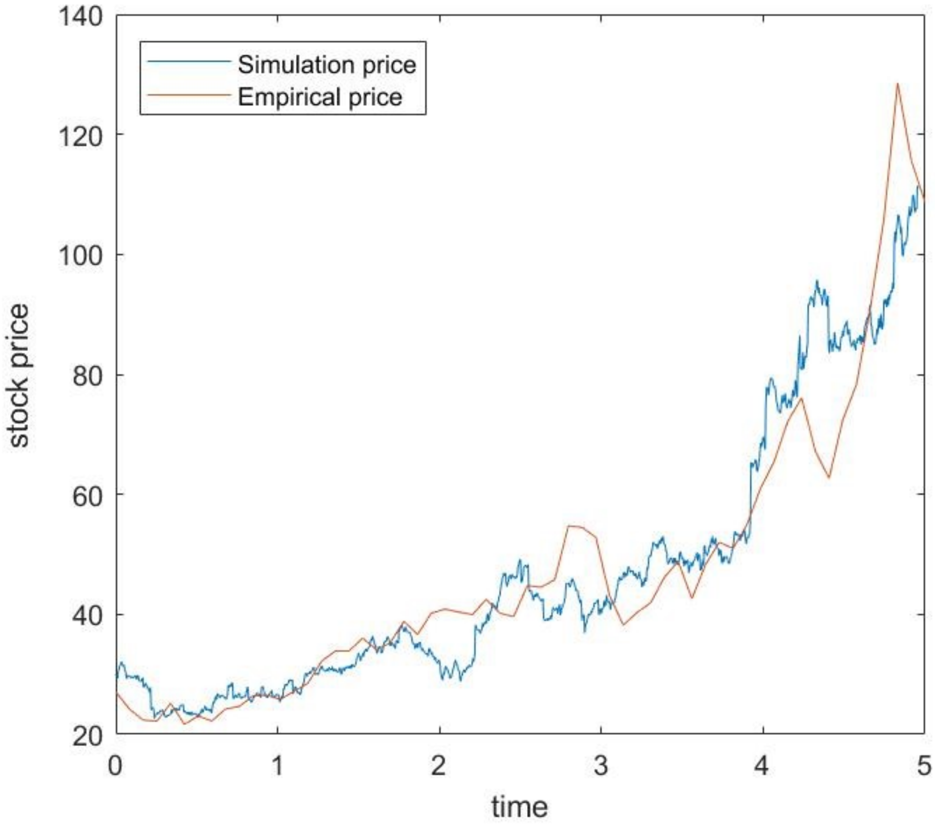


Figure 4.6: Simulated of AAPL prices from 11/2015 to 10/2020 compared with empirical prices within same period.

Comment: From our results graph, we can clearly see that our simulation prices follow very close trend as empirical price except a few volatile movements during a short period of time, which indicates that the jump diffusion model could be a good fit model under regular conditions (except extreme events). However, even though we get a reasonable result, there are still limitations behind this model. This model does not include any firm-specific factors such as dividends, merger & acquisitions, and quarterly financial reports nor any macroeconomic factors including adjust interest rates and macroeconomic indicators.

4.8.4 Monte Carlo method for exotic option in a jump-diffusion model

In this subsection, we will discuss the pricing of an up-and-out call option in a jump-diffusion model. A up-and-out call option is a type of knock-out barrier option that ceases to exist when the price of the underlying security rises above a specific price level, called the barrier price. If the price of the underlying does not rise above the barrier level, the option acts like any other option giving the holder the right but not the obligation to exercise their call option at the strike price on or before the expiration date specified in the contract.

The option pricing problem reduces to computing the following expectation if we suppose the zero interest rates:

$$C = \mathbb{E}[(e^{X_T} - K)^+ \mathbb{1}_{M_T < b}], \tag{4.21}$$

where $(X_t)_{t \geq 0}$ is a jump-diffusion process: $X_t = \gamma t + \sigma W_t + N_t$, such that (e^{X_t}) is a martingale and $M_t = \max_{0 \leq s \leq t} X_s$ is the maximum process associated to X. Now, we can use Monte Carlo method to compute this expectation.

(i) Simulate the jump times τ_i of compound Poisson part, the jump sizes $X_{\tau_i} - X_{\tau_{i-}}$ and the values of X at the jump times τ_i and at T. If any of these values is beyond the barrier price, the payoff for this trajectory is zero. Otherwise, we can analytically compute the probability that this trajectory has exceeded the barrier and come back between two consecutive jump times. The payoff of this trajectory will then be $(e^{X_T} - K)^+$ multiplied by this probability.

(ii) Let $\mathcal{F}^* = \sigma\{N_T, 0 \leq t \leq T; W_{\tau_i}, 0 \leq i \leq N\}$ where $\tau_i, 0 \leq i \leq N-1$ are the jump times of the compound Poisson part and $\tau_N = T$. Then (4.21) can be rewritten as

$$C = \mathbb{E}[(e^{X_T} - K)^+ \mathbb{E}[\mathbb{1}_{M_T < b} | \mathcal{F}^*]] \tag{4.22}$$

since X_T is $\mathcal{F}^*$-measurable.

(iii) The outer expectation in (4.22) will be computed by the Monte Carlo method, and the inner conditional expectation will be computed analytically. Using the Markov property of Brownian motion, we can find the analytic solution of inner expectation and substitute in expression (4.22) to get final formula:

$$C = \mathbb{E}\left[(e^{X_T} - K)^+ \prod_{i=1}^{N} \mathbb{1}_{X_{\tau_i} < b} \left\{ 1 - \exp\left(-\frac{2(X_{\tau_{i-}} - b)(X_{\tau_{i-1}} - b)}{(\tau_i - \tau_{i-1})\sigma^2} \right) \right\} \right] \tag{4.23}$$

(4.23) will be evaluated using the Monte Carlo method illustrated as following:

Remark 4.8.1 ***(Numerical method)***

(i) Simulate jump times $\{\tau_i\}$ and values $\{N_{\tau_i}\}$ of the compound Poisson part.

(ii) Simulate the values W_{τ_i} of the Brownian part at the points $\{\tau_i\}$.

(iii) Evaluate the functional under the expectation in 4.23.

(iv) Repeat the first three steps a sufficient number of times to compute the average value of the functional with the desired precision.

4.8.5 Utility indifference price

After addressing Merton's approach towards incomplete markets, we now turn to another approach on pricing the non-attainable contingent claims. We assume the market is free of arbitrage, in the sense that there exists equivalent martingale measures, but it contains non-attainable contingent claims, i.e. there are cash flows that cannot be replicated by self-financing trading strategies. This means we do not have a unique equivalent martingale measure.

For further approach to the problems of pricing and hedging contingent claims in incomplete markets, e.g.construction and use of super-replicating strategies, embedding in complete markets, use of the numeraire portfolio etc., are treated in e.g. Duffie and Skiadas (1994) [18]. The general theory of incomplete markets is developed in Magill and Quinzii (1996) [49].

Here we will follow the approach by Davis (1997) [16]. If the market is incomplete, we have several choices of equivalent martingale measures to price contingent claims. Now, we introduce a function that is widely used by economists-the utility function.

Definition 4.8.1 *A continuous function $U : (0, \infty) \to \mathbb{R}$ that is strictly increasing , strictly concave and continuously differentiable with $\lim_{x\to\infty} U'(x) = 0$ and $\lim_{x\to 0} U'(x) = \infty$ is called a **utility function**.*

Definition 4.8.2 ***(Certainty equivalent)*** *Consider now an investor with a utility function U and an initial wealth x. The **certainty equivalent** $c(x, f)$ of an uncertain payoff f is defined as the amount of wealth added to the initial wealth, results in the same level of expected utility:*

$$U(x + c(x, f)) = \mathbb{E}[U(x + f)] \Rightarrow c(x, f) = U^{-1}(\mathbb{E}[U(x + f)]) - x$$

An investor with such a utility function U and initial endowment x trading only in underlying assets $S_0, ..., S_d$ forms a dynamic portfolio ϕ, whose value at time t is $V_{\phi,x}(t)$ (keep tracking of the initial endowment). The investor's objective is to maximise expected utility under the original probability measure of his final wealth at time T given that he is allowed to choose his trading strategy ϕ from a suitable subset Φ. We write

$$\tilde{U}(x) = \sup_{\phi\in\Phi} \mathbb{E}[U(V_{\phi,x}(T))]$$

for the maximal utility. Now suppose the contingent claim X (a sufficiently integrable random variable) is made available for trading with current purchase price p. To find a fair price $\hat{p}$ for a contingent claim we follow a marginal rate of substitution argument commonly used in pricing.

$\hat{p}$ is a fair price for the contingent claim if diverting a little of his funds into it at time zero has a neutral effect on the investor's achievable utility. More precisely,

$$W(\delta, x, p) = \sup_{\phi\in\Phi} \mathbb{E}[U(V_{\phi,x-\delta}(T) + \frac{\delta}{p} X)],$$

then we can state:

Suppose that for each fixed $(x, p), W(\delta, x, p)$ is differentiable as a function of δ for $\delta = 0$, and that there is a unique solution $\hat{p}(x)$ of the equation

$$\frac{\partial W}{\partial \delta}(0, p, x) = 0.$$

Then $\hat{p}(x)$ is the fair option price at time $t = 0$.

Theorem 4.8.1 *(Davis). Suppose that $\widehat{U}$ is differentiable at each $x \in \mathcal{R}_+$ and that $\widehat{U}'(x) > 0$. Then the fair price of the definition is given by*

$$\widehat{p} = \frac{\mathbb{E}[U'(V_{\phi^*,x}(T))X]}{\widehat{U}'(x)}$$

Proof: See [5] Theorem 7.1.1.

The utility indifference price of an option depends on the initial wealth x of the investor. This implies that investors with same utility function but different initial wealths do not agree on the value of option. There is special category of utility function-exponential utility, where the initial wealth cancels out and obtains an indifference price independent of initial wealth.

Proposition 4.8.1 *Let $p_\alpha(f)$ be the utility indifference price for an exponential utility function $U_\alpha(x) = 1 - \exp(-\alpha x)$. Then:*

(i) $\lim_{\alpha\to\infty} p_\alpha(f) = \sup_{\mathbb{Q}\in M_e(S)} \mathbb{E}^{\mathbb{Q}}[f]$, *where $M_a(S)$ is the set of martingale measures absolutely continuous w.r.t $\mathbb{P}$.*

(ii) As $\alpha \to 0$ the utility indifference price defines a linear pricing rule given by
$\lim_{\alpha\to 0} p_\alpha(f) = \mathbb{E}^{\mathbb{Q}^*}[f]$,
where $\mathbb{Q}^*$ is a martingale measure equivalent to $\mathbb{P}$ which minimizes the relative entropy with respect to $\mathbb{P}$:

$$\xi(\mathbb{Q}^* \mid \mathbb{P}) = \inf_{\mathbb{Q}\in M_a(S)} \xi(\mathbb{Q} \mid \mathbb{P}).$$

The results shows that in the case of exponential utility, we have obtained a linear pricing rule based on a martingale measure $\mathbb{Q}^*$ which minimizes the relative entropy with respect to $\mathbb{P}$. These results shows that the indifference price is not robust to changes in the risk aversion parameter α, since the parameter is unobservable and raise some uncertainty of this method of pricing.

Chapter 5

Pricing options with geometric Lévy processes

In this chapter, we consider the problem of pricing contingent claims on a stock whose price process is modelled by a geometric Lévy process, with exact analogy with the geometric Brownian motion model. Consider a market that is incomplete and there is not a unique equivalent martingale. Therefore, it is not possible simply to use the martingale measure to price a contingent claim in the manner as in Black-Scholes model. We will follow Chan (1999) procedure [12] to derive solution for option prices under geometric Lévy processes.

5.1 Geometric Lévy processes

Consider the problem of pricing contingent claims on a stock whose price at t, S_t, is modelled by a geometric Lévy process, which has SDE:

$$dS_t = \mu_t S_{t-} dt + \sigma_t S_{t-} dX_t, \tag{5.1}$$

where X_t is a Lévy process. As we have introduced in the first chapter, the classical option pricing theory of Black and Scholes relies on the fact that the payoff of every contingent claim can be duplicated by a self-financing strategy. In such a complete market, there is a unique measure which makes the discounted price process a martingale. For the stock prices modelled above, there are many equivalent measures under which the discounted price process is a martingale, in contrast to Black-Scholes formula.

Additional criteria must be used to select an appropriate martingale measure from many measures with which to price a contingent claim. In the particular model, we will concentrate on various approaches to pricing options which the main ones being the Föllmer-Schweizer minimal measure and the martingale measure which has minimum relative entropy with respect to the canonical measure. We shall introduce some details in minimal measure in the later sections.

5.2 Decomposition of Lévy processes

From the Lévy-Khintchine representation (Proposition 3.3.1), we can deduce that X must be a linear combination of a Brownian motion and a quadratic pure jump process Y which is independent of the Brownian motion.

Definition 5.2.1 *A process Y is said to be* ***quadratic pure jump*** *if the continuous part of its quadratic variation $\langle Y \rangle \equiv 0$, in which its quadratic variation becomes simply:*

$$\langle Y \rangle_t = \sum_{0<s\le t} (\Delta Y_s)^2,$$

where $\Delta Y_s = Y_s - Y_{s-}$ is the jump size at time s.

It is straightforward if we separate out the Brownian component from the quadratic pure jump component Y and write

$$X_t = cW_t + Y_t. \tag{5.2}$$

From Section 3.3 Lévy-Itô decomposition, let $N(dt\ dx)$ be a Poisson measure on $\mathbb{R}^+ \times \mathbb{R}\backslash\{0\}$ with expectation measure $dt \times \nu$, where ν is the Lévy measure. Then the Lévy decomposition of Y says that

$$\begin{aligned} Y_t &= \int_{|y|\le 1} y(N((0,t]\ dy) - t\nu(dy)) + \int_{|y|\ge 1} yN((0,t]\ dy) + t\mathbb{E}\left[Y_1 - \int_{|y|\ge 1} y\nu(dy)\right] \\ &= \int_{|y|\le 1} y(N((0,t]\ dy) - t\nu(dy)) + \int_{|y|\ge 1} yN((0,t]\ dy) + \alpha t, \end{aligned} \tag{5.3}$$

where $\alpha = \mathbb{E}\left[Y_1 - \int_{|y|\ge 1} y\nu(dy)\right]$, the drift of the Lévy process.

In this model, it is required that the process X to satisfy numerous conditions. We require X:

$$\mathbb{E}[\exp(-hX_1)] < \infty \quad \text{for} \quad h \in (-h_1, h_2), \tag{5.4}$$

where $0 < h_1, h_2 < \infty$. And Y_1 has the finite expectation, $\mathbb{E}[Y_1] < \infty$. In terms of Lévy measure ν of Y, we have [61]

$$\int_{|y|\ge 1} e^{-hy}\nu(dy) < \infty, \tag{5.5}$$

$$\forall \gamma > 0, \quad \int_{|y|\ge 1} y^\gamma e^{-hy}\nu(dy) < \infty, \tag{5.6}$$

$$\int_{|y|\ge 1} y\nu(dy) < \infty, \tag{5.7}$$

for $h \in (-h_1, h_2)$.

Under these assumptions, (5.3) can be rewritten as

$$\begin{aligned} Y_t &= \int_{\mathbb{R}} y(N((0,t]\ dy) - t\nu(dy)) + t\mathbb{E}[Y_1] \\ &= M_t + \beta t, \end{aligned} \tag{5.8}$$

where $M_t = \int_{\mathbb{R}} y(N((0,t]\ dy) - t\nu(dy))$ is a martingale and let $\beta = \mathbb{E}[Y_1]$.

Notice that (5.7) gives the Doob decomposition of X as the sum of a martingale and a predictable process of finite variation.

Define a probability space $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$, substitute (5.8) in (5.2) we obtain:

$$X_t = cW_t + M_t + \beta t, \tag{5.9}$$

where X_t is a Lévy process satisfying the condition (5.4). Suppose that the filtration $\{\mathcal{F}_t\}$ is the minimal generated by X. Then the stock price S_t is the solution of the SDE, obtained by substituting (5.9) back into (5.1) :

$$dS_t = (\beta\sigma_t + \mu_t)S_{t-}dt + \sigma_t S_{t-}(cdW_t + dM_t), \tag{5.10}$$

where the coefficients σ_t and μ_t are deterministic continuous functions. This equation has an explicit solution [5.8] given by

$$\begin{aligned} S_t = S_0 \exp\Big\{ &\int_0^t c\sigma_s dW_s + \int_0^t \sigma_s dM_s + \int_0^t \left(\beta\sigma_s + \mu_s - \frac{c^2\sigma_s^2}{2}\right) ds \Big\} \\ &\times \prod_{0<s\leq t} (1 + \sigma_s \Delta M_s) \exp(-\sigma_s \Delta M_s). \end{aligned}$$

We can observe that $\sigma\{S_u : u \leq t\} = \mathcal{F}_t$ and so a contingent claim f_T expiring at time T could be regarded as a non-negative $\mathcal{F}_T$-measurable random variable.

In order to ensure that $S_t \geq 0$ for all t almost surely, we need $\sigma_t \Delta M_t \geq -1$ for all t. This in turn implies that the jumps of X must be bounded on at least one side. Suppose $\Delta X_t = \Delta M_t \in [-c_1, c_2]$, in other words, Lévy measure ν is supported on $[-c_1, c_2]$ where $c_1, c_2 \geq 0$ and one of them may be infinite. This implies that at least one of h_1, h_2 in (5.4) must be infinite. Thus to ensure that $S_t \geq 0$ we must have:

$$\forall t, \quad -\frac{1}{c_2} \leq \sigma_t \leq \frac{1}{c_1}. \tag{5.11}$$

The riskless rate of interest is given by a deterministic continuous function r_t and the value of B_t of a bond or bank account paying this rate has the process:

$$\frac{dB_t}{dt} = r_t B_t.$$

The discounted stock price is given by,

$$\widehat{S}_t = e^{-rt} S_t. \tag{5.12}$$

In later sections, we will introduce different measures equivalent to the underlying canonical measure $\mathbb{P}$, which makes $\widehat{S}_t$ a martingale.

5.3 Equivalent martingale measures in pricing formulas

We characterize all equivalent martingale measures $\mathbb{Q}$ under which the discounted price process $\widehat{S}_t$ defined in (5.12) is $\{\mathcal{F}_t\}$-martingale.

Continue with our model structure, let $M(dt\ dy) = N(dt\ dy) - dt\nu(dy)$ be the compensated measure, where $N(dt\ dy)$ is the Poisson measure associated with Y. Therefore the martingale part of Y can be written as $M_t = \int_0^t \int_{\mathbb{R}} yM(ds\ dy)$.

Note: We use $\mathbb{E}[\cdot]$ to denote expectations under canonical measure $\mathbb{P}$ while $\mathbb{E}^{\mathbb{Q}}[\cdot]$ denotes expectations with respect to any other measure $\mathbb{Q}$.

Let $\mathcal{P}$ denote the previsible σ-algebra on $\Omega \times \mathbb{R}^+$ associated with the filtration $\{\mathcal{F}_t\}$ and let $\widetilde{\mathcal{P}} = \mathcal{P} \times \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. A function $H(\omega, t, x)$ which is $\widetilde{\mathcal{P}}$-measurable will be called Borel previsible.

Remark 5.3.1 *A Borel previsible function $H(t, y)$ is one such that the process $t \mapsto H(t, y)$ is previsible for fixed y and the function $y \mapsto H(t, y)$ is Borel-measurable for fixed t.*

Lemma 5.3.1 *Let G_t and $H(t, y)$ be previsible and Borel previsible processes respectively. Suppose*

$$\mathbb{E}\left[\int_0^t G_s^2 ds\right] < \infty$$

and $H \geq 0, H(t, 0) = 1$ for all $t \geq 0$. Let $h(t, y)$ be another Borel previsible process such that

$$\int_{\mathbb{R}} [H(t, y) - h(t, y) - 1]\nu(dy) < \infty. \tag{5.13}$$

Define a process Z_t by

$$\begin{aligned} Z_t = \exp\Big\{ & \int_0^t G_s dW_s - \frac{1}{2}\int_0^t G_s^2 ds + \int_0^t \int_{\mathbb{R}} h(s, y)M(ds\ dy) \\ & - \int_{[0,t)\times\mathbb{R}} [H(s, y) - h(s, y) - 1]\nu(dy)ds\Big\} \\ & \times \prod_{0<s\leq t} H(s, \Delta Y_s)\exp(-h(s, \Delta Y_s)). \end{aligned} \tag{5.14}$$

Then Z is a nonnegative local martingale with $Z_0 = 1$ and Z is positive if and only if $H > 0$. The processes G, H and h can be chosen so that $\mathbb{E}[Z_t] = 1$ for all t, in which case Z is a martingale.

Proof: See [12] Lemma 3.1.

Now, we introduce an important theorem we will use later that is based on Lemma 5.3.1.

Theorem 5.3.1 *Let $\widetilde{\mathbb{P}}$ be a measure which is absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{F}_T$. Then*

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_T} = Z_T,$$

where Z in in lemma (5.3.1), for some G, H and h for which $\mathbb{E}[Z_T] = 1$. Under $\widetilde{\mathbb{P}}$, the process

$$\widetilde{W}_t = W_t - \int_0^t G_s ds \tag{5.15}$$

is a Brownian motion and the process Y is a quadratic pure jump process with compensator measure given by $\tilde{\nu}(dt\ dy) = dt\widetilde{\nu}_t(dy)$, where

$$\widetilde{\nu}_t(dy) = H(t, y)\nu(dy), \tag{5.16}$$

and previsible part is given by

$$\widetilde{\beta}_t = \mathbb{E}[Y_t] = \beta t + \int_0^t \int_{\mathbb{R}} y(H(s, y) - 1)\nu(dy)ds. \tag{5.17}$$

Proof: See [39] Theorems (3.24) and (5.19).

Now focusing on pricing a contingent claim f_T, we need to find an equivalent measure $\mathbb{Q}$ under which the discounted price process is a martingale. Then the price of the claim is given by

$$f = \mathbb{E}^{\mathbb{Q}}[e^{-rT} f_T]. \tag{5.18}$$

By Theorem 5.3.1, under probability measure $\mathbb{Q}$, Y has Doob-Meyer decomposition [Appendix A4],

$$Y_t = \widetilde{M}_t + \beta t + \int_0^t \int_{\mathbb{R}} y(H(s, y) - 1)\nu(dy)ds, \tag{5.19}$$

where $\widetilde{M}$ is a $\mathbb{Q}$-martingale:

$$\widetilde{M}_t = M_t - \int_0^t \int_{\mathbb{R}} y(H(s, y) - 1)\nu(dy)ds, \tag{5.20}$$

where M is the $\mathbb{P}$-martingale in the Doob-Meyer decomposition of Y under $\mathbb{P}$. Notice that $\Delta\widetilde{M}_t = \Delta M_t$.

Now we write the discounted stock price $\widehat{S}_t$ in terms of the $\mathbb{Q}$-martingale $\tilde{M}$ and $\mathbb{Q}$-Brownian motion $\widetilde{W}$, we have

$$
\begin{aligned}
\widehat{S}_t = S_0 \exp\Big\{ \int_0^t c\sigma_s d\widetilde{W}_s + \int_0^t c\sigma_s d\widetilde{M}_s + \int_0^t (\beta\sigma_s + \mu_s - r_s - \frac{c^2\sigma_s^2}{2} + c\sigma_s G_s) ds \\
+ \int_0^t \sigma_s \int_{\mathbb{R}} y(H(s,y)-1)\nu(dy) ds \Big\} \times \prod_{0<s\le t} (1+\sigma_s \Delta\widetilde{M}_s) e^{-\sigma_s \Delta\widetilde{M}_s}.
\end{aligned}
$$

Since

$$
\exp\Big\{ \int_0^t c\sigma_s d\widetilde{W}_s + \int_0^t c\sigma_s d\widetilde{M}_s - \int_0^t \frac{c^2\sigma_s^2}{2} ds \Big\} \prod_{0<s\le t} (1+\sigma_s \Delta\widetilde{M}_s) e^{-\sigma_s \Delta\widetilde{M}_s}
$$

is a $\mathbb{Q}$-martingale, a necessary and sufficient condition for $\hat{S}$ to be a martingale under $\mathbb{Q}$ is the existence of G and H for which the process Z in Lemma 5.3.1 is a positive martingale and such that

$$
c\sigma_s G_s + \beta\sigma_s + \mu_s - r_s + \int_{\mathbb{R}} \sigma_s y (H(s,y)-1)\nu(dy) = 0 \tag{5.21}
$$

for all s, almost surely. However (5.21) does not specify G and H, and hence the equivalent martingale measure $\mathbb{Q}$ is unique. Next, we will examine various approaches to choose G and H based on other criteria in addition to (5.21).

5.3.1 Föllmer–Schweizer minimal measure

When the noise X in (5.1) is a standard Brownian motion, the unique equivalent martingale measure $\mathbb{Q}$ is obtained by

$$
\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_T} = Z_T, \tag{5.22}
$$

where $dZ_t = \gamma_t Z_t dW_t$.

The process γ is chosen so as to make $\hat{S}$ a martingale under $\mathbb{Q}$. Now we can use the martingale measure $\mathbb{Q}$ defined by (5.22), where the Radon-Nikodym derivative Z is now given by

$$
Z_t = 1 + \int_0^t \gamma_s Z_{s-}(c dW_s + dM_s). \tag{5.23}
$$

We can see from here that the Brownian motion in the Black-Scholes setting has been replaced by the martingale part of the noise process X.

In the Lemma 5.3.1, we see that, in general,

$$
Z_t = 1 + \int_0^t G_s Z_{s-} dW_s + \int_0^t \int_{\mathbb{R}} Z_{s-}[H(s,y)-1] M(ds\, dy).
$$

Compare this with expression (5.23), we see that we require

$$H(s,y) - 1 = \frac{G_s y}{c} = h(s,y), \tag{5.24}$$

so that $\gamma_s = \frac{G_s}{c}$. To obtain a martingale measure, we will use the martingale condition (5.21) together with (5.24).

Let $\nu = \int_{\mathbb{R}} y^2 \nu(dy)$, we find solution to (5.21) and (5.24):

$$\begin{aligned} G_s &= \frac{c(r_s - b_s - \alpha\sigma_s)}{\sigma_s(c^2 + \nu)}, \\ H(s,y) - 1 &= \left(\frac{r_s - b_s - \alpha\sigma_s}{\sigma_s(c^2 + \nu)} \right) y \end{aligned} \tag{5.25}$$

Substitute back to (5.23), we obtain:

$$\gamma_s = \frac{r_s - b_s - \alpha\sigma_s}{\sigma_s(c^2 + \nu)} \tag{5.26}$$

Next, there are conditions to ensure that $H(s, \Delta Y_s) > 0$ or the measure we obtained will not be a probability measure. Based on assumptions that the jump size $\Delta y \in [-c_1, c_2]$, therefore we require

$$-\frac{1}{c_2} < \frac{r_s - b_s - \alpha\sigma_s}{\sigma_s(c^2 + \nu)} < \frac{1}{c_1}. \tag{5.27}$$

In turns out that the martingale measure given by (5.22), (5.23) and (5.26) is the Föllmer–Schweizer minimal measure introduced by Föllmer and Schweizer (1991) [21]. The minimal measure is closely connected to a hedging portfolio, which minimizes the risk involved in trying to duplicate a contingent claim f_T. We will briefly sketch the main idea of Föllmer–Schweizer minimal measure.

5.3.2 Minimizing risk in an incomplete market

Consider a contingent claim at time T given by a random variable $f \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. In order to hedge against the claim, we want to use a portfolio strategy which involves the stock S and a riskless bond $Y \equiv 1$, and which yields the random payoff f at the terminal time T. Let ξ_t and η_t denote the amounts of stock and bond respectively, held at time t. Assume that the process $\xi = (\xi_t)_{0 \le t \le T}$ is predictable while $\eta = (\eta_t)_{0 \le t \le T}$ is allowed to be adapted. The discounted quantity will be denoted by $\widehat{f}_t = e^{-rt} f_t$. Then the discounted value of the portfolio at time t can be expressed as:

$$\widehat{V}_t = \xi_t \widehat{S}_t + \eta_t \quad (0 \le t \le T)$$

We define the cost accumulated up to time t by

$$C_t = \widehat{V}_t - \int_0^t \xi_s d\widehat{S}_s$$

and the remaining risk by

$$\mathbb{E}[(C_T - C_t)^2|\mathcal{F}_t].$$

We look for an admissible strategy which minimizes the remaining risk over all admissible continuations of this strategy from time t on.

f is attainable if and only if the remaining risk can be reduced to zero. But for a general contingent claim, the cost process associated to a risk-minimizing strategy will no longer be self-financing. Instead, it will be mean-self-financing in the sense that

$$\mathbb{E}[C_T - C_t|\mathcal{F}_t] = 0 \quad (0 \leq t \leq T).$$

In other words, the cost process C associated to a risk-minimizing strategy is a martingale.

To proceed, we need to define the concept of optimal strategy.

Definition 5.3.1 *An admissible strategy (ξ, η) is called **optimal** if the associated cost C is a square-integrable martingale orthogonal to the martingale part (Doob decomposition) of $\hat{S}$ under $\mathbb{P}$.*

In the existence of a unique risk-minimizing strategy is shown. In order to describe it, consider the *Kunita-Watanabe decomposition* [Appendix A5], where the contingent claim is represented by f_T:

$$\widehat{f_T} = f_0 + \int_0^T \xi_s d\widehat{S}_s + L_T \tag{5.28}$$

for some ξ, where L_t is a square-integrable martingale orthogonal to the martingale part of $\hat{S}$ under $\mathbb{P}$. Then risk-minimizing strategy is now given by

$$\widehat{V}_t = f_0 + \int_0^t \xi_s d\widehat{S}_s + L_t,$$
$$\eta_t = \widehat{V}_t - \xi_t \widehat{S}_t,$$

In the present martingale case, the process V can also be computed directly as a right-continuous version of the martingale

$$V_t = \mathbb{E}[f|\mathcal{F}_t] \quad (0 \leq t \leq T)$$

Thus, the problem is solved by using a well-known projection technique in space $\mathcal{M}^2$ of square-integrable martingales: simply project the martingale V associated to Γ on the martingale S. From the risk-minimizing strategy, we see that (ξ, η) is an optimal admissible strategy. Conversely, an optimal admissible strategy (ξ, η) gives a decomposition of the form (5.28) with $L_t = C_t - C_0$. Thus, the existence of an optimal strategy is equivalent to a decomposition of the form (5.28).

5.3.3 Minimal martingale measure

The notion of a martingale measure $\mathbb{P}^* \approx \mathbb{P}$ was defined [20] following properties:

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} \in L^2(\Omega, \mathcal{F}, \mathbb{P}),$$
$$S \text{ is a martingale under } \mathbb{P}^*.$$

Such a martingale measure is determined by the right-continuous square-integrable martingale $G^* = (G_t^*)_{0\le t\le T}$ with

$$G_t^* = \mathbb{E}\left[\frac{d\mathbb{P}^*}{d\mathbb{P}}|\mathcal{F}_t\right] \quad (0 \le t \le T).$$

Under $\mathbb{P}^*$, the Doob-Meyer decomposition of M is given by $M = X - X_0 + (-A)$. But the theory of the Girsanov transformation [Appendix A2] shows that the predictable process of bound variation can also be computed in terms of G^*:

$$-A_t = \int_0^t \frac{1}{G_{s-}^*} d\langle M, G^*\rangle_s \quad (0 \le t \le T);$$

Since $\langle M, G^*\rangle \ll \langle M\rangle = \langle X\rangle$, the process A must be absolutely continuous with respect to the variance process $\langle X\rangle$ of X, i.e.,

$$A_t = \int_0^t \tau_s d\langle X\rangle_s \quad (0 \le t \le T),$$

for some predictable process $(\tau_t)_{0\le t\le T}$.

Definition 5.3.2 *A martingale measure $\hat{\mathbb{P}} \approx \mathbb{P}$ will be called minimal if*

$$\hat{\mathbb{P}} = \mathbb{P} \quad \text{on } \mathcal{F}_0, \tag{5.29}$$

and if any square-integrable $\mathbb{P}-$martingale which is orthogonal to M under $\mathbb{P}$ remains a martingale under $\hat{\mathbb{P}}$:

$$L \in \mathcal{M}^2 \text{ and } \langle L, M\rangle \Longrightarrow L \text{ is a martingale under } \hat{\mathbb{P}} \tag{5.30}$$

Now, let us focus on the existence and uniqueness of minimal martingale measure. We demonstrate this from the following theorem:

Theorem 5.3.2 *(i) The minimal martingale measure $\hat{\mathbb{P}}$ is uniquely determined.*

(ii) $\hat{\mathbb{P}}$ exists if and only if

$$\widehat{G_t} = \exp\left(-\int_0^t \tau_s dM_s - \frac{1}{2}\int_0^t \tau_s^2 d\langle x\rangle_s\right) \quad (0 \le t \le T)$$

is a square-integrable martingale under $\mathbb{P}$; in that case, $\hat{\mathbb{P}}$ is given by $\dfrac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \widehat{G_T}$

(iii) The minimal martingale measure preserves orthogonality: Any $L \in \mathcal{M}^2$ *with* $\langle L, M \rangle = 0$ *under* $\mathbb{P}$ *satisfies* $\langle L, X \rangle = 0$ *under* $\hat{\mathbb{P}}$.

We will show the proof of the first theorem that the minimal martingale measure is uniquely determined.

Proof: (i) Let $G^* = (G_t^*)_{0 \le t \le T}$ be the square-integrable martingale associated to a martingale measure $\mathbb{P}^* \approx \mathbb{P}$. Then

$$G_t^* = G_0^* + \int_0^t \beta_s dM_s + L_t \quad (0 \le t \le T)$$

where L is a square-integrable martingale under $\mathbb{P}$ orthogonal to M, and $D = (D_t)_{0 \le t \le T}$ is a predictable process with

$$\mathbb{E}\left[\int_0^T D_s^2 d\langle M \rangle\right] < \infty$$

Under $\mathbb{P}^*$, the predictable process of bounded variation in the Doob-Meyer decomposition of M is given by

$$\int_0^t \frac{1}{G_{s-}^*} d\langle G^*, M \rangle_s = \int_0^t \frac{1}{G_{s-}^*} D_s d\langle X \rangle_s.$$

$X = X_0 + M + A$ is assumed to be a martingale under $\mathbb{P}^*$, and we get

$$\tau = -\frac{D}{G_-^*};$$

since $G^* > 0$ $\mathbb{P}$-a.s. due to $\mathbb{P}^* \approx \mathbb{P}$ and since $\langle M \rangle = \langle X \rangle$, plus the condition for finite expectation above, we get

$$\int_0^T \tau_s^2 d\langle X \rangle_s < \infty \quad \mathbb{P} - \text{a.s.}$$

Now suppose that $\mathbb{P}*$ is minimal. Then $G_0^* = 1$ due to (5.29), and L is a martingale under $\mathbb{P}^*$ due to (5.30). This implies $\langle L, G^* \rangle = 0$, and so we get

$$\langle L \rangle = 0 = \langle L, G^* \rangle = 0,$$

hence $L \equiv 0$. Thus, G^* solves the stochastic equation

$$G_t^* = 1 + \int_0^t G_{s-}^* (-\tau_s) dM_s.$$

Since M is continuous and $\langle M \rangle = \langle X \rangle$, we obtain $G^* = \hat{G}$, hence uniqueness. □

Clearly, if an optimal strategy and a minimal equivalent martingale measure $\mathbb{Q}$ exists, we have $\widehat{V}_t = \mathbb{Q}[\widehat{f_T} | \mathcal{F}_t]$, thus taking $V_0 = \mathbb{Q}[\widehat{f_T}]$ as the price of the contingent claim.

Then we want to show that measure given by (5.22), (5.23), (5.26) is minimal. It follows from that in [21] for the continuous case. Under measure $\mathbb{P}$, $\hat{S}$ satisfies

$$\begin{aligned}\widehat{S}_t &= \widehat{S}_0 + \int_0^t \sigma_s \widehat{S}_{s-}(cdW_s + dM_s) + \int_0^t (\beta\sigma_s + \mu_s - r_s)\widehat{S}_{s-}ds \\ &= \widehat{S}_0 + Z_t + A_t,\end{aligned} \tag{5.31}$$

where

$$Z_t = \int_0^t \sigma_s \widehat{S}_{s-}(cdW_s + dM_s)$$

is a $\mathbb{P}$-martingale and

$$A_t = \int_0^t (\beta\sigma_s + \mu_s - r_s)\widehat{S}_{s-}ds$$

is continuous adapted and hence a previsible process. Therefore (5.29) gives the Doob decomposition of $\hat{S}$ under $\mathbb{P}$.

Definition (5.3.3) means that $\hat{\mathbb{P}}$ preserves the martingale property as far as possible under the restriction that X is a martingale under $\mathbb{P}^*$. This minimal departure from the given measure $\mathbb{P}$ can also be expressed in terms of the **relative entropy**:

$$H(\mathbb{Q} \mid \mathbb{P}) = \begin{cases} \int \log \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{Q} & \text{if} \quad \mathbb{Q} \ll \mathbb{P} \\ +\infty & \text{otherwise.} \end{cases}$$

Since the relative entropy is always nonnegative, and that $H(\mathbb{Q} \mid \mathbb{P}) = 0$ is equivalent to $\mathbb{Q} = \mathbb{P}$.

Theorem 5.3.3 *In the class of all martingale measures $\mathbb{P}^*$, the minimal martingale measure $\hat{\mathbb{P}}$ is characterized by the fact that it minimizes the functional*

$$H(\mathbb{P}^*|\mathbb{P}) - \frac{1}{2}\mathbb{E}^*\left[\int_0^T \tau_s^2 d\langle X\rangle_s\right]. \tag{5.32}$$

In particular, $\hat{\mathbb{P}}$ minimizes the relative entropy $H(.|\mathbb{P})$ among all martingale measures $\mathbb{P}^*$ with fixed expectation

$$\mathbb{E}^*\left[\int_0^T \tau_s^2 d\langle X\rangle_s\right]. \tag{5.33}$$

Proof: If $\mathbb{P}^*$ is a martingale measure, then M has the Doob-Meyer decomposition

$$M_t = X_t - X_0 + \left(-\int_0^t \tau_s d\langle X\rangle_s\right)$$

under $\mathbb{P}^*$. Due to the first property of 5.3.3 that $\dfrac{d\mathbb{P}^*}{d\mathbb{P}} \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, we have

$$G_T^* := \frac{d\mathbb{P}^*}{d\mathbb{P}} \in L^2(\Omega, \mathcal{F}, \mathbb{P});$$

in particular, the relative entropy in finite:

$$H(\mathbb{P}^* \mid \mathbb{P}) = \int G_T^* \log G_T^* d\mathbb{P} < \infty.$$

Now suppose that $\hat{\mathbb{P}} \approx \mathbb{P} \approx \mathbb{P}^*$ is the minimal martingale measure. Then we have

$$\begin{aligned} H(\mathbb{P}^* \mid \mathbb{P}) &= H(\mathbb{P}^* \mid \hat{\mathbb{P}}) + \int \log \widehat{G_T} d\mathbb{P}^* \\ &= H(\mathbb{P}^* \mid \hat{\mathbb{P}}) + \int \left(- \int_0^T \tau_s dM_s - \frac{1}{2} \int_0^T \tau_s^2 d\langle X \rangle_s \right) d\mathbb{P}^* \\ &= H(\mathbb{P}^* \mid \hat{\mathbb{P}}) + \frac{1}{2} \mathbb{E}^* \left[\int_0^T \tau_s^2 d\langle X \rangle_s \right] \end{aligned}$$

(localise first, then pass to the limit using $H(\mathbb{P}^* \mid \mathbb{P} < \infty)$. In particular, the expectation in (5.33) is finite. Thus,

$$H(\mathbb{P}^* \mid \mathbb{P}) - \frac{1}{2} \mathbb{E}^* \left[\int_0^T \tau_s^2 d\langle X \rangle_s \right] = H(\mathbb{P}^* \mid \hat{\mathbb{P}}) \geq 0,$$

and the minimal value 0 is assumed if and only if $\mathbb{P}^* = \hat{\mathbb{P}}$. $\square$

5.3.4 Pricing by martingale decompositions

In the continuous case, an important property of the minimal measure is that it gives $\hat{S}$ the law of its martingale part under the Doob decomposition. The minimal measure can be uniquely characterized by this property. It is important to find which equivalent martingale measure will give $\hat{S}$ the law of its martingale part and to compare it to the minimal measure.

Consider the Doob decomposition of $\hat{S}$ under $\mathbb{P}$ given by (5.31). Let $\mathbb{Q}$ be a martingale measure as described in Theorem (5.3.1), satisfying the martingale condition (5.21). Under $\mathbb{Q}$, $\hat{S}$ satisfies

$$\widehat{S}_t = \widehat{S}_0 + \int_0^t \sigma_s \widehat{S}_{s-}(cd\widetilde{W}_s + d\widetilde{M}_s), \tag{5.34}$$

where $\widetilde{M}$ is the $\mathbb{Q}$-martingale and $\widetilde{W}$ is the $\mathbb{Q}$-Brownian motion. More specifically,

$$\widetilde{M}_t = \int_0^t \int_{\mathbb{R}} y(\tilde{N}(ds\ dy) - \widetilde{\nu}_s(dy)ds),$$

where $\tilde{N}(ds\ dy)$ is a Poisson measure with compensator measure $\widetilde{\nu}_s(dy)$ in Theorem (5.3.1). Comparing (5.34) with the form of Z, we can see that the only way in which $\hat{S}$ can have the law of Z under $\mathbb{Q}$ is to have $\tilde{\nu} \equiv \nu$, which in turn implies that $H \equiv 1$ and $h = 0$ and $\mathbb{Q}$ is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} |_{\mathcal{F}_T} = \exp\left\{ \int_0^T G_s dW_s - \frac{1}{2}\int_0^T G_s^2 ds \right\} \tag{5.35}$$

where,

$$G_s = \frac{r_s - \mu_s - \beta\sigma_s}{c\sigma_s}. \tag{5.36}$$

From (5.35) and (5.36) we can see about the measure that it corresponds to the classical Black-Scholes formula, if we regard $b + \alpha\sigma$ as an overall drift which (5.35) removes by only changing the drift of the underlying Brownian motion W while leaving the jump part of the noise alone.

5.3.5 Pricing by minimum relative entropy and Esscher transform

Gerber and Shiu (1995) [23] proposed pricing contingent claims by Esscher transform. We apply Esscher transform definition in Section 4.5. Let $\theta \in \mathbb{R}$ be fixed. Then the Esscher transform of a Lévy process X is defined to be the process whose law $\mathbb{Q}_\theta$ is given by

$$\frac{d\mathbb{Q}_\theta}{d\mathbb{P}} |_{\mathcal{F}_t} = \exp\{-\theta X_t + \gamma(\theta)t\},$$

where $\gamma(\theta) = -\log \mathbb{E}[\exp(-\theta X_1)]$ is the Lévy exponent of X.

Note: We replace θ in Section 4.3 by $-\theta$ to be consistent with conditions (5.4)-(5.7).

If the stock price process has constant coefficients, the value of θ can be chosen so as to make the discounted price process $\hat{S}$ a martingale under $\mathbb{Q}_\theta$.

When the stock price process has time-dependent coefficients as in this model, considering generalized Esscher transforms of the form

$$\frac{d\mathbb{Q}_\theta}{d\mathbb{P}} |_{\mathcal{F}_t} = \exp\Big\{ -\int_0^t \theta_s dX_s + \int_0^t \gamma(\theta_s) ds \Big\} \tag{5.37}$$

and choose θ_s to satisfy the martingale condition. From results of [12], we see that the Esscher transform corresponds to the choices $H(t,y) = \exp(-\theta_t y), h(t,y) = -\theta_t y$ and $G \equiv -c\theta$. Then the martingale condition (5.21) can then be used to specify θ as follows:

$$-c^2\sigma_s\theta_s + \beta\sigma_s + \mu_s - r_s + \int_{\mathbb{R}} \sigma_s x(\exp(-\theta_s x) - 1)\nu(dx) = 0. \tag{5.38}$$

Note: This equation has unique solution θ for which $\gamma(\theta_s) < \infty$ for all s. For an equivalent martingale measure $\mathbb{Q}$ given by Theorem 5.3.2 and the martingale condition (5.21), the relative entropy in terms of the $\mathbb{Q}$-martingales $\widetilde{W}$ and $\widetilde{M}$ is therefore

$$\begin{aligned} I_{\mathbb{P}}(\mathbb{Q}) &= \mathbb{E}^{\mathbb{Q}}\left[\log \frac{d\mathbb{Q}}{d\mathbb{P}} |_{\mathcal{F}_T}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2}\int_0^T G_s^2 ds + \int_0^T \int_{\mathbb{R}} [H(s,y)(\log H(s,y) - 1) + 1]\nu(dy) ds\right] \end{aligned}$$

From this expression, we can clearly deduce that finding equivalent martingale measure of minimum relative entropy is to minimize

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2}G_s^2 + \int_{\mathbb{R}} [H(s,y)(\log H(s,y) - 1) + 1]\nu(dy)\right]$$

for fixed s subject to (5.21). Since measure $\mathbb{Q}$ varies with G and H, we need to discuss if the problem can be reduced to minimize:

$$\frac{1}{2}G_s^2 + \int_{\mathbb{R}} [H(s,y)(\log H(s,y) - 1) + 1]\nu(dy) \tag{5.39}$$

This optimization problem has a deterministic solution of G and H since all the coefficients are deterministic. Therefore, we need to find optimal choices of G and H to minimize the expression. First, we fix G and choose H to minimize the function by setting up Lagrange function. The solution to this optimization problem:

$$H^*(s,y) = \exp(-\lambda_s\sigma_s y), \quad G^* = -c\sigma\lambda.$$

where λ_s is the solution of equation:

$$-c^2\sigma_s^2\lambda_s + \beta\sigma_s + \mu_s - r_s + \int_{\mathbb{R}} \sigma_s y(\exp(-\lambda_s\sigma_s y) - 1)\nu(dy) = 0.$$

Comparing this equation with equation (5.38), we can see that this is precisely the measure constructed via Esscher transform, with $\theta \equiv \sigma\lambda$.

5.4 Integro-differential equations for stock prices

Consider any contingent claim whose payoff depends only on the value at maturity of the underlying security. Write the payoff as $f(S_t)$. A European call with strike K, for which $f(S_t) = (S_t - K)^+$, let

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}f(S_t) \mid \mathcal{F}_t\right]$$

be the value of the claim at time t. Recall in the Section 4.3, the valuation process V admits a Feynman-Kac type presentation $V_t = g(t, S_t)$, where g is the solution to the Cauchy problem associated with a linear PDE. The resulting integro-differential equation has the integral term associated with the jumps in Lévy process.

Following from previous model, under $\mathbb{Q}$, the price of the underlying stock satisfies

$$dS_t = r_t S_{t-}dt + \sigma_t S_{t-}(cd\widetilde{W}_t + d\widetilde{M}_t).$$

Let η be the following integro-differential operator:

$$\eta f(x) = \frac{1}{2}c^2\sigma_t^2 x^2 \frac{\partial f}{\partial x}(x) + r_t x\frac{\partial f}{\partial x}(x) + \int_{\mathbb{R}}[f(x+\sigma_t xy) - f(x) - \sigma_t xy\frac{\partial f}{\partial x}(x)]\widetilde{\nu}_t(dy).$$

Theorem 5.4.1 *Let $g(t,x)$ be the solution to the Cauchy problem:*

$$\frac{\partial g}{\partial t} + \eta g - r_t g = 0, \quad g(T,x) = f(x). \tag{5.40}$$

Then g admits the representation

$$g(t,x) = \mathbb{E}^{\mathbb{Q}}_{t,x}\left[e^{-r(T-t)}f(S_T)\right].$$

Proof: This is Feynman-Kac presentation under our setting. For any fixed t, apply Itô's formula to the process $t' \mapsto e^{-r(T-t')}g(t', S_{t'})$, we can show that it is a $\mathbb{Q}$-martingale then we can take its $\mathbb{Q}$-expectation. □

The Markov property with the above theorem shows that $V_t = g(t, S_t)$. Then we can solve the Cauchy problem equation numerically to compute the price of option.

5.5 Numerical simulation of Esscher transform measure

In this section, we will directly apply simulation results to compare simulated option prices under different measures. Let $\sigma \equiv b \equiv c = 1$ and $r = 0$, where we suppose risk free interest is zero. We want to calculate the price of a European call option with strike price $K = 1$ in unit and maturity time at $t = 1$ in unit for various values of the initial share price S_0, using martingale measures discussed above.

If we let Y in the above model follows a Gamma (1,1) process whose law is given by

$$\mathbb{E}[\exp(-\lambda Y_t)] = \left(\frac{1}{1+\lambda}\right)^t.$$

Then the Lévy measure is $\nu(dy) = y^{-1}e^{-y}\mathbb{1}_{[0,\infty)}(y)dy$ and the previsible part is $\beta = \mathbb{E}[Y_1] = 1$. For Esscher transform measure, we submit our parameters in equation (5.38), then the equation can be simplified to:

$$-\theta + 2\int_0^\infty e^{-(\theta+1)y} - e^{-y}dy = 0$$

Solve for the equaiton we get $\theta = \sqrt{2}$. Under this measure, Y is still a Gamma process, but with shape parameter 1 and scale parameter $\theta + 1$,

$$\mathbb{E}^{\mathbb{Q}_\theta}[\exp(-\lambda Y_t)] = \left(\frac{\theta+1}{\theta+1+\lambda}\right)^t.$$

Now we have two ways to obtain option prices based on Gamma process model. One way is to simulate Gamma process and find option prices by Monte Carlo method. In our setting, we will obtain the price by numerically solving equation in (5.40). Here is result of simulation.

Table 5.1: Price based on Gamma process model

S_0	**Black-Scholes measure**	**Esscher transform measure**
0.50	0.149	0.107
0.75	0.295	0.240
1.25	0.653	0.587
1.50	0.852	0.785

Table 5.1 gives the value of $g(0, S_0)$, where g is the solution to (5.4.1).

Chapter 6

Variance gamma process and its extensions

The variance gamma model completed by Madan, Carr and Chang (1998) [48], was introduced as an extension of geometric Brownian motion to solve some imperfections that the Black-Scholes model has in pricing option. The process is obtained by evaluating Brownian motion with drift at a random time given by a gamma process. The two additional parameters are the drift of the Brownian motion and the volatility of the time change. These additional parameters is able to control over the skewness and kurtosis of stock price return distribution. Closed forms are obtained for the return density and the prices of European options. The additional parameters successfully correct for pricing biases in the Black-Scholes model that is a parametric special case of the option pricing model.

We then extend this model to a multivariate case. We use the model derived by Semeraro (2006) [64]. The model is constructed as a multivariate Lévy process defined by subordination of a Brownian motion with independent components by a multivariate gamma subordinator.

6.1 Variance gamma process

In a continuous time economy setting, over the interval $[0, \Theta]$, in which are traded a stock, a money market account, and options on the stock for all strikes and maturities $0 < T \leq \Theta$. Suppose a constant continuous compounded interest rate r with money market account value of $\exp(rt)$, stock prices of S_t and European call option prices $c(t, K, T)$ with strike K and maturity $T > t$, at time t.

The VG process is obtained by evaluating Brownian motion with drift at a random time given by a gamma process. Let

$$B(t, \theta, \sigma) = \theta t + \sigma W_t$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion. The process $(B_t)_{t\geq 0}$ is a Brownian motion with drift θ and volatility σ.

The gamma process $\gamma(t,\mu,\nu)$ with mean rate μ and variance rate ν is the process of independent gmma increments over non-overlapping intervals of time $(t, t+h)$. The **density** $f_h(g)$, of the increment $g = \gamma(t+h,\mu,\nu) - \gamma(t,\mu,\nu)$ is given by the gamma density function with mean μh and variance νh:

$$f_h(g) = \left(\frac{\mu}{\nu}\right)^{\frac{\mu^2 h}{\nu}} \frac{g^{\frac{\mu^2 h}{\nu}-1}\exp(-\frac{\mu}{\nu}g)}{\Gamma(\frac{\mu^2 h}{\nu})}, \quad g > 0, \tag{6.1}$$

where $\Gamma(x)$ is the gamma function. The gamma density has a **characteristic function** given by

$$\phi_{\gamma_t}(u) = \left(\frac{1}{1 - iu\frac{\nu}{\mu}}\right)^{\frac{\mu^2 t}{\nu}} \tag{6.2}$$

The ***VG* process** $X(t,\sigma,\nu,\theta)$ is defined in terms of the Brownian motion with drift $B(t,\theta,\sigma)$ and the gamma process with unit mean rate:

$$X(t,\sigma,\nu,\theta) = B(\gamma(t,1,\nu),\theta,\sigma). \tag{6.3}$$

The VG process has three parameters:σ the volatility of the Brownian motion; ν the variance rate of the gamma time change; θ the drift in the Brownian motion with drift.

The **density function** for the VG process at time t can be expressed conditional on the realisation of the gamma time change g as a normal density function:

$$f_{X_t}(X) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}} \exp\left(-\frac{(X-\theta g)^2}{2\sigma^2 g}\right) \frac{g^{\frac{t}{\nu}-1}\exp(-\frac{g}{\nu})}{\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})} dg. \tag{6.4}$$

The **characteristic function** for the VG process:

$$\phi_{X_t}(u) = \left(\frac{1}{1 - i\theta u + (\sigma^2\nu/2)u^2}\right)^{\frac{t}{\nu}}. \tag{6.5}$$

The dynamics of the continuous time gamma process is best explained by describing a simulation of the process. As the process is an infinitely divisible one, of independent and identically distributed increments over non-overlapping intervals of equal length, the simulation may be described in terms of the **Lévy measure** explicitly given by

$$\kappa_\gamma(dx) = \frac{\mu^2\exp(-\frac{\mu}{\nu}x)}{\nu x}dx, \quad \text{for } x > 0 \text{ and } 0 \text{ otherwise.} \tag{6.6}$$

Since the Lévy measure has an infinite integral, we see that the gamma process has an infinite arrival rate of jumps, most of which are small, as indicated by the concentration of the Lévy measure at the origin.

The VG process can also be expressed as the difference of two independent increasing gamma processes,

$$X(t,\sigma,\nu,\theta) = \gamma_p(t,\mu_p,\nu_p) - \gamma_n(t,\mu_n,\nu_n). \tag{6.7}$$

The Lévy measure for the VG process is in terms of a symmetric VG process subjected to a measure change induced by a constant relative risk aversion utility function as in Madan and Milne (1991) [47]. We view as the difference of two gamma processes we may write the Lévy measure for (X_t), employing as,

$$\kappa_X(dx) = \begin{cases} \frac{\mu_n^2}{\nu_n}\frac{\exp(-\frac{\mu_n}{\nu_n}|x|)}{|x|}dx & \text{for} \quad x<0, \\ \frac{\mu_p^2}{\nu_p}\frac{\exp(-\frac{\mu_p}{\nu_p}x)}{|x|}dx & \text{for} \quad x>0. \end{cases} \tag{6.8}$$

The explicit relation between parameters of the gamma processes differenced in (6.7) and the original parameters of the VG process (6.3) is given by

$$\mu_p = \frac{1}{2}\sqrt{\theta^2+\frac{2\sigma^2}{\nu}}+\frac{\theta}{2}, \quad \mu_n = \frac{1}{2}\sqrt{\theta^2+\frac{2\sigma^2}{\nu}}-\frac{\theta}{2}, \tag{6.9}$$

$$\nu_p = \left(\frac{1}{2}\sqrt{\theta^2+\frac{2\sigma^2}{\nu}}+\frac{\theta}{2}\right)^2\nu, \quad \nu_n = \left(\frac{1}{2}\sqrt{\theta^2+\frac{2\sigma^2}{\nu}}-\frac{\theta}{2}\right)^2\nu. \tag{6.10}$$

The parameters of the VG process, directly reflect the skewness and kurtosis of the return distribution. Explicit expressions for the first four central moments of the return distribution over an interval of length t are given:

$$\begin{aligned} \mathbb{E}[X_t] = \theta t, \quad \mathbb{E}[(X_t-\mathbb{E}[X_t])^2] &= (\theta^2+\sigma^2)t, \\ \mathbb{E}[(X_t-\mathbb{E}[X_t])^3] &= (2\theta^3\nu^2+3\sigma^2\theta\nu)t, \\ \mathbb{E}[(X_t-\mathbb{E}[X_t])^4] &= (3\sigma^4\nu+12\sigma^2\theta^2\nu^2+6\theta^4\nu^3)t+(3\sigma^4+6\sigma^2\theta^2\nu+3\theta^4\nu^2)t^2. \end{aligned}$$

6.1.1 Variance gamma for stock price processes and option pricing

In this subsection, we will describe the risk neutral dynamics of the stock price in terms of the VG process and derives the prices of European options on the stock. The new specification for the statistical stock price dynamics is obtained by replacing the role of Brownian motion in the Black-Scholes geometric Brownian motion model by the VG process. Then the s.p.p is given by

$$S_t = S_0 \exp\left[mt + X(t,\sigma_S,\nu_S,\theta_S)+\omega_S t\right] \tag{6.11}$$

where $\omega_S = \frac{1}{\nu_S}\ln(1-\theta_S\nu_S-\sigma_S^2\nu_S/2)$, and m is the mean rate of return on the stock under the statistical probability measure.

Under the risk neutral process, discounted stock prices are martingales and then the risk neutral process is given by

$$S_t = S_0 \exp\left[rt + X(t, \sigma_{RN}, \nu_{RN}, \theta_{RN}) + \omega_{RN} t\right], \tag{6.12}$$

where subscript RN on the VG parameters indicates that these are the risk neutral parameters, and $\omega_{RN} = \frac{1}{\nu_{RN}} \ln(1 - \theta_{RN}\nu_{RN} - \frac{\theta_{RN}^2 \nu_{RN}}{2})$.

The density of the log stock price relative over an interval of length t is, conditional on the realization of the gamma time change, a normal density function.

Theorem 6.1.1 *The density for the log price relative $z = \ln \frac{S_t}{S_0}$ when prices follow the VG process dynamics of equation (6.11) is given by*

$$f(z) = \frac{2\exp(\theta x/\sigma^2)}{\nu^{\frac{t}{\nu}} \sqrt{2\pi}\sigma\Gamma(\frac{t}{\nu})} \left(\frac{x^2}{2\sigma^2/\nu + \theta^2}\right)^{\frac{t}{2\nu} - \frac{1}{4}} K_{\frac{t}{\nu} - \frac{1}{2}} \left(\frac{1}{\sigma^2}\sqrt{x^2(2\sigma^2/\nu + \theta^2)}\right), \tag{6.13}$$

where

$$x = z - mt - \frac{t}{\nu}\ln(1 - \theta\nu - \sigma^2\nu/2).$$

The price of a European call option, $c(S_0, K, t)$, for a strike of K and maturity t, is given by

$$c(S_0, K, t) = e^{-rt}\mathbb{E}[(S_t - K)^+]$$

This evaluation of the option price proceeds by first conditioning on a knowledge of the random time change g that has an independent gamma distribution. The European option price for VG risk-neutral valuation is obtained on integrating the conditional Black-Scholes formula over g with respect the gamma density. The price formula is given by the following theorem:

Theorem 6.1.2 *The European call option price on a stock, where the risk neutral dynamics of the stock price is given by the VG process is*

$$c(S_0, K, t) = S_0\Psi\left(d\sqrt{\frac{1 - c_1}{\nu}}, (\alpha + s)\sqrt{\frac{\nu}{1 - c_1}}, \frac{t}{\nu}\right)$$
$$- K\exp(-rt)\Psi\left(d\sqrt{\frac{1 - c_2}{\nu}}, (\alpha + s)\sqrt{\frac{\nu}{1 - c_2}}, \frac{t}{\nu}\right)$$

where

$$d = \frac{1}{s}\left[\ln(\frac{S_0}{K}) + rt + \frac{t}{\nu}\ln(\frac{1 - c_1}{1 - c_2})\right] \text{ and } \zeta = -\frac{\theta}{\sigma^2}, s = \frac{\sigma}{\sqrt{1 + (\frac{\theta}{\sigma})^2\frac{\nu}{2}}}, \alpha = \zeta s.$$

$$c_1 = \frac{\nu(\alpha + s)^2}{2}, c_2 = \frac{\nu\alpha^2}{2}$$

and the function Ψ is defined in terms of the modified bessel function of the second kind [Appendix A3] and degenerate hypergeometric function of two variables [48, A11].

We expect the additional parameters of the VG model to be important for option pricing. Risk aversion implies risk-neutral density of returns is negatively skewed ($\theta < 0$ or $\alpha > 0$), a feature that is missed by the Black Scholes model where symmetry is essentially a consequence of continuity coupled with continuous rebalancing.

Below the table, we compare the analytic results of European call option price given by Carr-Madan formula with option prices obtained from Monte Carlo simulation of 10^7 times [Appendix B5].

Table 6.1: Option prices based on VG analytic solutions and Monte Carlo simulation

S_0	**Analytic price**	**Monte Carlo price**
0.50	0.023	0.022
0.75	0.087	0.086
1.00	0.211	0.209
1.25	0.389	0.389

This table gives the European call prices at different initial stock prices under two methods, where $\nu = 0.5, \theta = -0.02, \sigma = 0.5, r = 0.05$ and $K = 1$.

6.2 Sato's model for multivariate option pricing

This section provides a multivariate Sato model for multivariate option pricing where the asset log-returns are expressed as Sato time-changed Brownian motions and where the time change is the weighted sum of a common and an idiosyncratic component. The main advantage of this model is that it allows us to replicate univariate option prices in both the strike and time-to-maturity dimensions.

Followed by introduction of Sato process, we will focus on the model proposed by Semeraro, the so-called α variance gamma model, which rests on a multivariate subordinator process composed of the weighted sum of two independent gamma processes: an idiosyncratic and a common component. Finally, the VG Sato model is obtained by replacing the Lévy time-changed Brownian motions in the setting of Semeraro by Sato time-changed Brownian motions and lead to marginal characteristic functions of the Sato type.

6.2.1 Sato processes

Sato processes are closely linked to the class of self-decomposable laws. Details we refer to []. Here, we will just provide some main definitions and properties.

Definition 6.2.1 *The distribution of a random variable X is self-decomposable if, for any constant $c, 0 < c < 1, X$ has the same probability law as the sum of downscaled version of itself and an independent random variable X_c:*

$$X \overset{\mathrm{d}}{=} cX + X_c$$

Self-decomposable distributions are a subclass of infinitely divisible distributions with a Lévy-Khintchine representation of the form:

$$\log(\phi_X(u)) = i\langle\gamma, u\rangle - \frac{1}{2}\langle u, \sigma\sigma^T u\rangle + \int_{-\infty}^{\infty}(\exp i\langle u, x\rangle - 1 - i\langle u, x\rangle \mathbb{1}_{|x|<1})\frac{h(x)}{|x|}dx \quad (6.14)$$

where $h(x) \geq 0$ is decreasing for positive x and increasing for negative x. Hence, self-decomposable laws are necessarily of infinite activity

A Sato process can be constructed from any self-decomposable distribution as follows. The probability law of the Sato process at time t is obtained by scaling the self-decomposable law of X at unit time: $X_t \overset{\mathrm{d}}{=} t^\gamma X$. γ is the self-similarity exponent.

6.2.2 The VG Sato process

From (6.14) and Lévy measure of the VG process (6.8) it is clear that the VG probability law at unit time is self-decomposable for all acceptable VG parameter sets $\{\sigma, \nu, \theta\}$. By making use of the space-scaling property of VG random variables, the characteristic function of the VG Sato process at time t is thus given by:

$$\begin{aligned}\phi_{VG\ Sato}(u, t, \sigma, \nu, \theta, \gamma) &= \phi_{VG}(u, 1, t^\gamma\sigma, \nu, t^\gamma\theta) \\ &= (1 - i\langle u, \nu\theta t^\gamma\rangle + \frac{1}{2}\sigma^2\nu t^{2\gamma}u^2)^{-1/\nu}\end{aligned}$$

6.2.3 Multi-parameter process

Before we move to $\alpha - VG$ model, there is one more theorem that plays a fundamental role in the characterisation in terms of Lévy triplet of the process we are going to construct. The univariate version is Theorem 30.1 [61]. The general version and its proof are in [3, Theorem 3.3].

Consider n independent Lévy processes $X_1(t), ..., X_n(t)$. The stacked process $\mathbf{X}_t = (X_1(t), ..., X_n(t))^T$ is a Lévy process on $\mathbb{R}^n$. Consider the multi-parameter $\mathbf{s} = (s_1, ..., s_n)^T \in \mathbb{R}^n_+$ and the partial order on $\mathbb{R}^n_+$:

$$\mathbf{s}^1 \preceq \mathbf{s}^2 \Leftrightarrow s^1_j \leq s^2_j, \quad j = 1, ...n.$$

Define the multi-parameter process $\{\mathbf{X}(\mathbf{s}), \mathbf{s} \in \mathbb{R}^n_+\}$ by

$$\mathbf{X}(\mathbf{s}) = (X_1(s_1), ..., X_n(s_n))^T.$$

Theorem 6.2.1 *Let $\boldsymbol{G}$ be a multivariate subordinator with triplet $(\gamma_{\boldsymbol{G}}, 0, \nu_{\boldsymbol{G}})$ and let $\lambda_t = \mathcal{L}(\boldsymbol{G}(t))$. Let $\boldsymbol{X}_t$ be a Lévy process on $\mathbb{R}^n_+$ independent from $\boldsymbol{G}$ with independent components and triplet $(\gamma_{\boldsymbol{X}}, \Sigma_{\boldsymbol{X}}, \nu_{\boldsymbol{X}})$, where $\Sigma_{\boldsymbol{X}} = diag(\sigma_1, ..., \sigma_n) := \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{pmatrix}$, and let $\rho_s = \mathcal{L}(\boldsymbol{X}(\boldsymbol{s}))$. Define the process $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t \geq 0\}$ by the following*

$$\boldsymbol{Y}(t) = (X_1(G_1(t)), ..., X_n(G_n(t)))^T, \quad t \geq 0$$

then the process $\boldsymbol{Y}$ is a Lévy process and

$$\mathbb{E}[e^{i\langle z, \boldsymbol{Y}(t)\rangle}] = \exp(t\Psi_{\boldsymbol{G}}(\log \psi_{\boldsymbol{X}}(\boldsymbol{z}))), \quad \boldsymbol{z} \in \mathbb{R}^n_+,$$

where for any $\boldsymbol{\omega} = (\omega_1, ..., \omega_n)^T \in \mathbb{C}^n$ with $Re(\omega_j) \leq 0, j = 1, ..., n$, we let

$$\Psi_{\boldsymbol{G}}(\boldsymbol{\omega}) = \langle \boldsymbol{m} \cdot \boldsymbol{\omega} \rangle + \int_{\mathbb{R}^n} (e^{\langle \boldsymbol{\omega}, x\rangle} - 1)\nu(dx).$$

More, the characteristic triplet $(\gamma_{\boldsymbol{Y}}, \Sigma_{\boldsymbol{Y}}, \nu_{\boldsymbol{Y}})$ of $\boldsymbol{Y}$ is as follows

$$\begin{aligned} \gamma_{\boldsymbol{Y}} &= \int_{\mathbb{R}^n} \gamma_{\boldsymbol{G}}(ds) \int_{|x| \leq 1} x\rho_s(dx) + \langle \boldsymbol{m}, \gamma_{\boldsymbol{X}} \rangle. \\ \Sigma_{\boldsymbol{Y}} &= diag(m_1\sigma_1, ..., m_n\sigma_n) \\ \nu_{\boldsymbol{Y}}(\boldsymbol{B}) &= \nu_1(\boldsymbol{B}) + \nu_2(\boldsymbol{B}) \end{aligned}$$

where ν_1 and ν_2 are defined by $\nu_1(0) = 0, \nu_2(0) = 0$ and for $\boldsymbol{B} \in \mathcal{B}(\mathbb{R}^n \backslash 0)$,

$$\begin{aligned} \nu_1(\boldsymbol{B}) &= \int_{\mathbb{R}^n_+} \rho_{\boldsymbol{s}}(\boldsymbol{B})\nu_{\boldsymbol{G}}(ds), \\ \nu_2(\boldsymbol{B}) &= \int_{\boldsymbol{B}} m_1 \mathbb{1}_{A_1}(x)\nu_{X_1}(dx) + \cdots + m_n \mathbb{1}_{A_n}(x)\nu_{X_n}(dx). \end{aligned}$$

where $x \in \mathbb{R}, \nu_{X_i}, i = 1, ..., n$ are the Lévy measures of the independent marginal processes of $\boldsymbol{X}$ and finally $A_i = \{\boldsymbol{x} = (x_1, ..., x_n)^T \in \mathbb{R}^n : x_k = 0$ for $k \neq i, k = 1, ..., n\}, i = 1, ..., n$.

6.2.4 The α VG Lévy two-factor models

Under αVG models, the $N-$dimensional stock return is modelled by the exponential of a multivariate time-changed Brownian motion:

$$\mathbf{S}_t = \begin{pmatrix} S_t^{(1)} \\ S_t^{(2)} \\ \vdots \\ S_t^{(N)} \end{pmatrix} = \begin{pmatrix} \frac{S_0^{(1)} \exp((r-q_1)t + Y_t^{(1)})}{\mathbb{E}[\exp(Y_t^{(1)})]} \\ \frac{S_0^{(2)} \exp((r-q_2)t + Y_t^{(2)})}{\mathbb{E}[\exp(Y_t^{(2)})]} \\ \vdots \\ \frac{S_0^{(N)} \exp((r-q_N)t + Y_t^{(N)})}{\mathbb{E}[\exp(Y_t^{(N)})]} \end{pmatrix}$$

where $S_0^{(i)}$ is the spot price of the ith underlying, r is the risk-free interest rate, q_i denotes the dividend yield of the ith stock and $\mathbf{Y}$ is an $N-$dimensional time-changed Brownian motion. More particularly, under the Lévy setting, the process $\mathbf{Y}$ is given by:

$$\mathbf{Y}_t = \begin{pmatrix} Y_t^{(1)} \\ Y_t^{(2)} \\ \vdots \\ Y_t^{(N)} \end{pmatrix} = \begin{pmatrix} \theta_1 G_t^{(1)} + \sigma_1 W^{(1)}_{G_t^{(1)}} \\ \theta_2 G_t^{(1)} + \sigma_2 W^{(2)}_{G_t^{(2)}} \\ \vdots \\ \theta_N G_t^{(N)} + \sigma_N W^{(N)}_{G_t^{(N)}} \end{pmatrix}$$

where $W^{(i)}, i = 1, ..., N$, are independent standard Brownain motions and where the subordinator $G_t^{(i)}$ are the weighted sum of two gamma processes:

$$\mathbf{G}_t = \begin{pmatrix} G_t^{(1)} \\ G_t^{(2)} \\ \vdots \\ G_t^{(N)} \end{pmatrix} = \begin{pmatrix} X_t^{(1)} + \alpha_1 Z_t \\ X_t^{(2)} + \alpha_2 Z_t \\ \vdots \\ X_t^{(N)} + \alpha_N Z_t \end{pmatrix}$$

where $\alpha_i > 0, Z_1 \sim \Gamma(c_1, c_2), c_1, c_2 > 0$ and $X_1^{(i)} \sim \Gamma(a_i, b_i), a_i, b_i > 0$ are independent random variables and are independent on the $W^{(i)}$.

6.2.5 The α VG Sato two-factor models

Under Sato setting, the asset log-returns are built by space scaling the time-changed Brownian motions taken at unit time:

$$\mathbf{Y}_t = \begin{pmatrix} Y_t^{(1)} \\ Y_t^{(2)} \\ \vdots \\ Y_t^{(N)} \end{pmatrix} = \begin{pmatrix} \theta_1 t^{\gamma_1} G^{(1)} + \sigma_1 t^{\gamma_1} W^{(1)}_{G^{(1)}} \\ \theta_2 t^{\gamma_2} G^{(2)} + \sigma_2 t^{\gamma_2} W^{(2)}_{G^{(2)}} \\ \vdots \\ \theta_N t^{\gamma_N} G^{(N)} + \sigma_N t^{\gamma_N} W^{(N)}_{G^{(N)}} \end{pmatrix}$$

where

$$\mathbf{G} = \begin{pmatrix} G^{(1)} \\ G^{(2)} \\ \vdots \\ G^{(N)} \end{pmatrix} = \begin{pmatrix} X^{(1)} + \alpha_1 Z \\ X^{(2)} + \alpha_2 Z \\ \vdots \\ X^{(N)} + \alpha_N Z \end{pmatrix}$$

where $\alpha_i > 0, Z \sim \Gamma(c_1, c_2), c_1, c_2 > 0$ and $X^{(i)} \sim \Gamma(a_i, b_i), a_i, b_i > 0$ are independent random variables and are independent on the $W^{(i)}$.

Lemma 6.2.1 *The characteristic function of the process $\boldsymbol{Y}_t$ is given by*

$$\begin{aligned}\phi_{\boldsymbol{Y}}(u,t) &= \mathbb{E}[\exp(i\boldsymbol{u}'\boldsymbol{Y}_t)] \\ &= \prod_{i=1}^{N} \phi_{\boldsymbol{X}^{(i)}}(u_i\theta_i t^{\gamma_i} + \frac{1}{2} i\sigma_i^2 t^{2\gamma_i} u_i^2)\phi_{Z_1}\left(\sum_{i=1}^{N} \alpha_i (u_i\theta_i t^{\gamma_i} + \frac{1}{2} i\sigma_i^2 t^{2\gamma_i} u_i^2)\right)\end{aligned}$$

Proof: See [26] Lemma 3.1.

The marginal characteristic functions are directly obtained from Lemma (6.2.1):

$$\begin{aligned}\phi_{Y^{(i)}}(u,t) &= \mathbb{E}[\exp(iuY_t^{(i)})] \\ &= \left(1 - i\frac{u\theta_i t^{\gamma_i} + \frac{1}{2}i\sigma_i^2 t^{2\gamma_i}u^2}{b_i}\right)^{-a_i}\left(1 - i\frac{\alpha_i}{c_2}(u\theta_i t^{\gamma_i} + \frac{1}{2}i\sigma_i^2 t^{2\gamma_i}u^2)\right)^{-c_1}\end{aligned}$$

As we have obtained the joint and the marginal characteristic functions are known in closed form, we have the following lemma:

Lemma 6.2.2 *The linear correlation between the asset log-return processes $Y_t^{(i)}$ and $Y_t^{(j)}$ is time independent and equal to the correlation under the corresponding Lévy models.*

$$\rho_{ij} = \frac{cov(Y_t^{(i)}, Y_t^{(j)})}{\sqrt{var[Y_t^{(i)}]var[Y_t^{(j)}]}}$$

where $cov(Y_t^{(i)}, Y_t^{(j)}) = \theta_i\theta_j\alpha_i\alpha_j\frac{c_1}{c_2^2}t^{\gamma_i+\gamma_j}$ *and*

$$var[Y_t^{(i)}] = \left(\theta_i^2(\frac{a_i}{b_i^2} + \alpha_i^2\frac{c_1}{c_2^2}) + \sigma_i^2(\frac{a_i}{b_i} + \alpha_i\frac{c_1}{c_2})\right)t^{2\gamma_i}$$

Proof: See [27].

Let N be the number of underlying stocks, $M^{(i)}$ be the number of quoted options for the ith stock. The model vanilla option prices of the ith underlying are computed by using the Carr-Madan formula,

Formula 6.2.1 ***(Carr-Madan formula)***

$$C^{(i)}(K,T) = \frac{\exp(-\alpha\log(K))}{\pi}\int_0^{+\infty}\exp(-i\nu\log(K))\varphi^{(i)}(\nu)d\nu$$

where

$$\varphi^{(i)}(\nu) = \frac{\exp(-rT)\Phi^i(\nu-(\alpha+1)i,T)}{\alpha^2+\alpha-\nu^2+i(2\alpha+1)\nu}$$

where Φ^i is the risk-neutral characteristic function of the ith log stock price process at maturity T:

$$\begin{aligned}\Phi^i(u,T) &= \mathbb{E}_{\mathbb{Q}}[\exp(iu\log(S_T^{(i)})) \mid S_0^{(i)}] \\ &= \frac{\exp(iu(\log(S_0^{(i)}) + (r-q_i)T))\phi_{Y^{(i)}}(u,T}{\phi_{Y^{(i)}}(-i,T)^{iu}}\end{aligned}$$

The Carr-Madan formula can be used for all the models under investigation since we now in closed form the marginal characteristic function of log return of stocks.

6.2.6 Monte Carlo for multivariate option pricing

The multivariate Lévy and Sato two-factor models allow for a straightforward Monte Carlo simulation for the multivariate asset log-return process, which can be used to price multivariate derivatives. For numerical examples, we can consider 252 trading days a year, a daily discretization and ten thousands Monte Carlo simulations. One realization of the multivariate Lévy process in Section 6.2.4 at the set of times $\{0, \Delta t, 2\Delta t, ..., T = M\Delta t\}$ is simulated below.

(i) Simulate M independent outcomes of the $N+1$ gamma subordinator at time Δt:

$$X^{(i)}_{\Delta t}(m) \sim Gamma(a_i \Delta t, b_i), \quad Z_{\Delta t}(m) \sim Gamma(c_1 \Delta t, c_2), \quad m = 1, ..., M.$$

The subordinator processes are given by:

$$X^{(i)}_0 = 0, \quad X^{(i)}_{j\Delta t} = \sum_{m=1}^{j} X^{(i)}_{\Delta t}(m), \quad j = 1, ..., M$$

and

$$Z_0 = 0, \quad Z_{j\Delta t} = \sum_{m=1}^{j} Z_{\Delta t}(m), \quad j = 1, ..., M$$

(ii) Then G is given by

$$G^{(i)}_{j\Delta t} = X^{(i)}_{j\Delta t} + \alpha_i Z_{j\Delta t}$$

(iii) We then simulate one realization of the multivariate time-changed Brownian motion by:

$$Y j\Delta t^{(i)} = \theta_i G^{(i)}_{j\Delta t} + V^{(i)}_{j\Delta t}, \quad V^{(i)}_{j\Delta t} \sim N(0, \sigma_i^2 G^{(i)}_{j\Delta t}).$$

The price of options is then computed by considering the mean of the discounted payoff over a large number of realizations. If we follow Sato two-factor modes, as above, we will adjust (i) as

$$\begin{aligned} X^{(i)}_0 = 0, \quad X^{(i)}_{j\Delta t} &\sim \Gamma(a_i, b_i), \quad j = 1, ..., M \\ Z_0 = 0, \quad Z_{j\Delta t} &\sim \Gamma(c_1, c_2), \quad j = 1, ..., M \end{aligned}$$

One sample path of the multivariate asset log-return is then obtained by

$$Y j\Delta t^{(i)} = \theta_i (j\Delta t)^{\gamma_i} G^{(i)}_{j\Delta t} + V^{(i)}_{j\Delta t}, \quad V^{(i)}_{j\Delta t} \sim N(0, \sigma_i^2 (j\Delta t)^{2\gamma_i} G^{(i)}_{j\Delta t}).$$

In general, this section introduces multivariate Sato models for asset pricing, built on a Sato time-changed Brownian motion where the time change consists of a weighted sum of an idiosyncratic and a common component.

Appendix A

A1. Radon-Nikodym Derivative

Let $(\Omega, \mathcal{F}, \nu)$ be a measure space and f be a nonnegative Borel function. Note that

$$\lambda(A) = \int_A f d\nu, \quad A \in \mathcal{F}$$

is a measure satisfying

$$\nu(A) = 0 \;\Rightarrow \lambda(A) = 0.$$

We say λ is absolutely continuous w.r.t. ν. Computing $\nu(A)$ can be done through integration w.r.t. a well-known measure $\lambda \ll \nu$ is also most sufficient.

Theorem. (Radon-Nikodym theorem) Let ν and λ be two measures on $(\Omega, \mathcal{F})$ and ν be $\sigma-$finite. If $\lambda \ll \nu$, then there exists a nonnegative Borel function f on Ω such that

$$\lambda(A) = \int_A f d\nu, \quad A \in \mathcal{F}$$

More, f is unique a.e. ν, i.e., if $\lambda(A) = \int_A h d\nu$ for any $A \in \mathcal{F}$, then $f = h$ a.e.ν. The function f is called the **Radon-Nikodym derivative** of λ w.r.t. ν and is denoted by $\frac{d\lambda}{d\nu}$.

If f is Borel on $(\Omega, \mathcal{F})$ and $\int_A f d\nu = 0$ for any $A \in \mathcal{F}$, then $f = 0$ a.e.

If $\int f d\nu = 1$ for an $f \geq 0$ a.e. ν, then λ is a probability measure and f is called its probability density function w.r.t ν.

A2. Girsanov Transformation

Let $W = (W_1, ..., W_d)$ be a d-dimensional Brownian motion defined on a filtered probability space with filtration satisfying regular conditions. Let $(\gamma_t)_{0 \leq t \leq T}$ be a measurable, adapted d-dimensional process with $\int_0^T \gamma_{i,t}^2 dt < \infty$ a.s., $i = 1, ..., d$, and define process $(G_t)_{0 \leq t \leq T}$ by

$$G_t = \exp\Big\{ - \int_0^t \gamma_s' dW_s - \frac{1}{2} \int_0^t ||\gamma_s||^2 ds \Big\}$$

Then G is continuous and being the stochastic exponential of $-\int_0^t \gamma_s' dW_s$, is a local martingale. Given sufficient integrability on the process γ, G will be a continuous martingale. The **Novikov's condition** is stated as,

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^t ||\gamma_s||^2 ds\right\}\right] < \infty$$

Theorem. (Girsanov) Let γ be as above and satisfy Novikov's condition; let G be the corresponding continuous martingale. Define the processes $\widetilde{W_i}, i = 1, ..., d$ by

$$\widetilde{W_i} := W_{i,t} + \int_0^t \gamma_{i,s} ds, \quad 0 \leq t \leq T.$$

Then under the equivalent probability measure $\tilde{\mathbb{P}}$ with Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = G_T,$$

the process $W = (W_1, ..., W_d)$ is d-dimensional Brownian motion.

A3. Bessel Functions

Bessel's equation of order k is in the form

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - k^2)y = 0$$

for every $x \in \mathbb{R}$. The general solution is given by

$$y(x) = AJ_k(x) + BY_k(x),$$

where A, B are arbitrary constants and J_k is a **Bessel function of the first kind**,

$$J_k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{k+2n}}{n!\Gamma(k+n+1)}.$$

Consider a modified Bessel equation of order k,

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} - (x^2 + k^2)y = 0,$$

which has a general solution in the form

$$y(x) = AJ_k(ix) + BJ_k(ix).$$

Real-valued modified Bessel functions is given by

$$V_k(x) = e^{-ik\pi/2} J_k(ix).$$

$Y_k(x)$ in the solution to Bessel's equation is referred to as a **Bessel function of the second kind** or sometimes the Weber function. It is introduced by H. M. MacDonald, in the form

$$Y_k(x) = (\frac{\pi}{2})\frac{V_{-k}(x) - V_k(x)}{\sin(k\pi)}.$$

More, there is a recurrence relation,

$$Y_{k+1}(x) = Y_{k-1}(x) + \frac{2k}{x}Y_k(x).$$

A4. Doob-Meyer decomposition

J. L. Doob noticed that in discrete time an integrable process can be uniquely presented as the sum of a martingale M and a predictable process A starting at 0; in addition, the process A is increasing iff S is a submartingale. The **Doob-Meyer decomposition** is defined as

$$X = X_0 + M + A,$$

of X into a local martingale $M = (M_t)_{0\leq t\leq T}$ and a predictable process $A = (A_t)_{0\leq t\leq T}$ with paths of bounded variation, this amounts to the integrability condition

$$\mathbb{E}[X_0^2 + \langle X\rangle_T + |A|_T^2] < \infty.$$

Here $\langle X\rangle = \langle M\rangle$ denotes the pathwise defined quadratic variation process of X resp. M, and $|A|$ is the total variation of A. In particular, M is a square-intergrable martingale under $\mathbb{P}$.

A5. Galtchouk-Kunita-Watanabe decomposition

Let $(\widehat{S}_t)_{0\leq t\leq T}$ be a square-integrable martingale with respect to $\mathbb{Q}$. Any random variable $\widehat{Y}$ with finite variance depending on the history $(\mathcal{F}_t^S)_{0\leq t\leq T}$ of $\hat{S}$ can be represented as the sum of a stochastic integral with respect to $\hat{S}$ and a random variable Z orthogonal to a set of attainable claims Γ: there exists a square integrable predictable strategy $(\phi_t^Y)_{0\leq t\leq T}$ such that,

$$\widehat{Y} = \mathbb{E}^{\mathbb{Q}}[\widehat{Y}] + \int_0^T \phi_t^Y d\widehat{S}_t + Z^Y,$$

where Z^Y is orthogonal to all stochastic integrals with respect to $\hat{S}$. Moreover, the martingale defined by $Z_t^Y = \mathbb{E}^{\mathbb{Q}}[Z^Y \mid \mathcal{F}_t]$ is strongly orthogonal to Γ: for any square integrable predictable process $(\gamma_t)_{0\leq t\leq T}$, $Z_t \int_0^t \gamma dS$ is again a martingale.

Appendix B

B1. Generate option price surface using FFT (Matlab)

```
AssetPrice = 100;
Rate = 0.05;
DividendYield = 0;
OptSpec = 'call';

V0 = 0.04;
ThetaV = 0.04;
Kappa = 2.0;
SigmaV = 0.2;
RhoSV = -0.8;

Settle = datenum('05-Nov-2018');
Maturity = datemnth(Settle, 12*[1/12 0.25 (0.5:0.5:3)]');
Times = yearfrac(Settle, Maturity);
Strike = (2:2:200)';

% Increase 'NumFFT' to support a wider range of strikes
NumFFT = 2^13;

Call =  optByHestonFFT(Rate, AssetPrice, Settle, Maturity, OptSpec,
Strike, ... V0, ThetaV, Kappa, SigmaV, RhoSV, 'DividendYield', DividendYi
'NumFFT', NumFFT, ... 'CharacteristicFcnStep', 0.050, 'LogStrikeStep', 0.
'ExpandOutput', true);
[X,Y] = meshgrid(Times,Strike);

figure;
surf(X,Y,Call);
title('Price');
```

```
xlabel('Years to Option Expiry');
ylabel('Strike');
view(-112,34);
xlim([0 Times(end)]);
zlim([0 80]);
```

B2. Simulate stock price paths under Jump-Diffusion model (Matlab)

```
Ns=5; %number of simulation
dt=1/252; %the per-time in one year
t=linspace(0,(1250-1)*dt,1250)';
mu=0.16;sigma=0.3;lambda=5;m=0.005;delta=0.05; S0=10;
Rsim=lognrnd(dt,mu,sigma,lambda,m,delta,t,Ns);
Ssim=S0*exp(cumsum([zeros(1,size(Rsim,2));Rsim]));
plot(t,Ssim)
hold on
plot(t_2,empiri cal_price)
xlabel('time')
ylabel('stock_ price')
function Rsim=lognrnd(dt,mu,sigma,lambda,m,delta,t,Ns) %define simulated
function
dN=poissrnd(lambda*dt,length(t)-1,Ns);
Y=m*dN+delta*sqrt(dN).*randn(length(t)-1,Ns);%sum of normal jumps
dW=sqrt(dt).*normrnd(0,1,length(t)-1,Ns); % SBM
Rsim=(mu-sigma^2/2)*dt+sigma*dW+Y;
end
```

B3. Estimate parameters (Matlab)

```
S=csvread('AAPL.csv',1,5,[1,5,1260,5]);% historical stock price
dt=1/252;
R=diff(log(S),1);% the return of logarithmic stock price
epsilon=0.03; %values to verify jump
jumpindex=find(abs(R)>epsilon); %if ture considered as jump
lambdahat=length(jumpindex)/((length(S)-1)*dt);%jump intensity
```

```
Rjumps=R(jumpindex);%the data of 'jumpindex'
diffusionindex=find(abs(R)<=epsilon);%without jumps
Rdiffusion=R(diffusionindex);%data of diffusion index
sigmahat=std(Rdiffusion)/sqrt(dt);
muhat=(2*mean(Rdiffusion)+(sigmahat^2)*dt )/(2*dt) ;
deltahat=sqrt((var (Rjumps)-sigmahat^2*dt)) ;
mhat=mean(Rjumps)-(muhat-sigmahat^2/2 )*dt ;
plot(S)
```

B4. Simulate AAPL stock prices compared with empirical prices (Matlab)

```
empirical_price=csvread('aapl_5.csv');
t_2= linspace(0,5,60);

Ns=1; %number of simulation
dt=1/252; %the per-time in one year
t=linspace(0,(1250-1)*dt,1250)';
mu=0.3264;sigma=0.1871;lambda=21.2168;m=-0.0014;delta=0.051; S0=30;
Rsim=lognrnd(dt,mu,sigma,lambda,m,delta,t,Ns);
Ssim=S0*exp(cumsum([zeros(1,size(Rsim,2));Rsim]));
plot(t,Ssim)
hold on
plot(t_2,empirical_price)
xlabel('time')
ylabel('stock_ price')
function Rsim=lognrnd(dt,mu,sigma,lambda,m,delta,t,Ns) %define simulated
return function
dN=poissrnd(lambda*dt,length(t)-1,Ns);
Y=m*dN+delta*sqrt(dN).*randn(length(t)-1,Ns);%sum of normal jumps
dW=sqrt(dt).*normrnd(0,1,length(t)-1,Ns); % SBM
Rsim=(mu-sigma^2/2)*dt+sigma*dW+Y;
end
```

B5. Monte Carlo simulation of option price under VG model (R)

```
v=c(2,-0.02, 0.5)
N=10^7
T=1
g=rgamma(N,shape=v[1]*T,scale=1/v[1])
V=c()
V=rnorm(N,v[2]*g,((v[3])^2*g)^0.5)

S0=0.5 #S_0=0.5,0.75,1,1.25
r=0.05
omega=r+v[1]*log(1-v[2]/v[1]-v[3]^2/(2*v[1]))
K=1

c=c()
for (i in 1:N) {
  c[i]=max(S0*exp(omega*T+V[i])-K,0)
}

c0=(exp(-r*T)/N)*sum(c)
```

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