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FRACTIONAL BOUNDARY VALUE PROBLEMS: ANALYSIS AND NUMERICAL METHODS

Neville J. Ford [1], **M. Luísa Morgado** [2]

Abstract

In this paper we consider nonlinear boundary value problems of fractional order α, $0 < \alpha < 1$. We study the existence and uniqueness of the solution and extend existing published results. In the last part of the paper we study a class of prototype methods to determine their numerical solution.

MSC 2010: Primary 65L05:

Key Words and Phrases: Fractional Calculus, fractional ordinary differential equations, Caputo derivative, numerical methods.

1. Introduction

In this paper we focus on problems of the form

$$D_*^\alpha y(t) = f(t, y(t)), \quad t \in [0, T] \tag{1.1}$$

$$y(a) = y_a, \tag{1.2}$$

where $0 < a < T$, f is a suitably behaved function and D_*^α denotes the Caputo differential operator of order $\alpha \notin \mathbb{N}$ ([1]).

The Caputo differential operator may be defined by

$$D_*^\alpha y(t) := D^\alpha (y - T[y])(t)$$

where $T[y]$ is the Taylor polynomial of degree $\lfloor \alpha \rfloor$ for y, centered at 0, and D^α is the Riemann-Liouville derivative of order α [9]. The latter is defined by $D^\alpha := D^{\lceil \alpha \rceil} J^{\lceil \alpha \rceil - \alpha}$, with J^β representing the Riemann-Liouville integral operator,

$$J^\beta y(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} y(s) ds$$

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and $D^{\lceil\alpha\rceil}$ as the classical integer order derivative. Here, $\lfloor\alpha\rfloor$ denotes the biggest integer smaller than α, and $\lceil\alpha\rceil$ represents the smallest integer greater than or equal to α.

In [5], Diethelm and Ford studied problem (1.1)-(1.2) in the case where $a = 0$, that is they considered an initial value problem and they analysed not only the issues of existence and uniqueness of the solution, but also the dependence of the solution on the parameters in the differential equation.

Since the case where the conditions are given at $a = 0$ is well-understood, here we consider the boundary value problem where $a \neq 0$ and we seek solutions over a finite interval $[0, T]$ where $0 < a < T$.

Concerning the case where $a > 0$, Diethelm and Ford recently investigated the uniqueness of the solution. They proved that, under some simple natural conditions on f, there is at most one initial value y_0 for which the solution of the problem

$$D_*^\alpha y(t) = f(t, y(t)), \quad t \in [0, T] \tag{1.3}$$

$$y(0) = y_0, \tag{1.4}$$

satisfies $y(a) = y_a$. The fundamental Theorem that shows that the problem under consideration is well-posed is the following:

THEOREM **1.1** ([4]). *Let $0 < \alpha < 1$ and assume $f : [0, b] \times [c, d] \to \mathbb{R}$ to be continuous and satisfy a Lipschitz condition with respect to the second variable. Consider two solutions y_1 and y_2 to the differential equation*

$$D_{*0}^\alpha y_j(t) = f(t, y_j(t)) \qquad (j = 1, 2) \tag{1.5}$$

subject to the initial conditions $y_j(0) = y_{j0}$, respectively, where $y_{10} \neq y_{20}$. Then, for all t where both $y_1(t)$ and $y_2(t)$ exist, we have $y_1(t) \neq y_2(t)$.

In other words, if we know the value of a solution y to the equation (1.1) at $t = a$ then there is at most one corresponding value of $y(s)$ at any $s \in [0, a]$.

But in that paper, the authors did not provide results on the existence of solutions and, to our knowledge, there is still no complete proof of the existence of the solution of problem (1.1)-(1.2) in the literature. This will be one of our main goals in this paper.

The paper is organised as follows: In section **2** we discuss the existence and uniqueness of solutions to the FBVP (1.1)-(1.2).

The authors of [4] also proposed a numerical scheme for the solution of FBVPs. Such scheme, based on a shooting algorithm to find the appropriate initial value corresponding to a particular boundary value, provides a useful prototype approach. In section **3** we shall investigate more fully this type of approach when a range of basic numerical schemes are utilised. We also compare the efficiency of the numerical methods by considering their performance on problems with non-smooth solutions.

Finally, in the last section we present some conclusions and some suggestions for future work.

2. Existence and uniqueness of the solution

In this section our main aim is to establish a new basic existence theorem for the boundary value problem (1.1)-(1.2). This will combine with existing known results to provide a comprehensive existence and uniqueness theory.

First we recall a well known result from Fractional Calculus:

LEMMA **2.1**. *If the function f is continuous, the initial value problem (1.3)-(1.4) is equivalent to the following Volterra integral equation of the second kind:*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds. \tag{2.6}$$

Assume $a > 0$ is fixed, $y(a) = y_a$ and $0 \le t \le a$, we have, taking (2.6) into account

$$\begin{aligned} y(t) &= y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds \\ &= y(a) - \frac{1}{\Gamma(\alpha)} \int_0^a (a-s)^{\alpha-1} f(s, y(s))ds + \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds \end{aligned} \tag{2.7}$$

It follows that any solution of (2.7) for $t \in [0, a]$ also satisfies (1.1)-(1.2). We use this observation as the motivation for the following Theorem:

THEOREM **2.1**. *Let $a > 0$ and y_a be constant, $0 < \alpha < 1$. For the equation*

$$y(t) = y_a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds - \frac{1}{\Gamma(\alpha)} \int_0^a (a-s)^{\alpha-1} f(s, y(s))ds \tag{2.8}$$

assume the following conditions hold:

(1) *f satisfies a Lipschitz condition with Lipschitz constant $L > 0$ with respect to its second argument*

(2) $\frac{2La^{\alpha}}{\Gamma(\alpha+1)} < 1$.

Then (2.8) has a unique solution $y(t)$ for $0 \leq t \leq a$.

Proof:

We rewrite (2.8) as the Fredholm equation:

$$\begin{aligned} y(t) = y_a \quad &+ \quad \frac{1}{\Gamma(\alpha)} \int_0^a (t-s)^{\alpha-1} \Xi_{[0,t]}(s) f(s, y(s)) ds \\ &- \quad \frac{1}{\Gamma(\alpha)} \int_0^a (a-s)^{\alpha-1} f(s, y(s)) ds \end{aligned} \tag{2.9}$$

where Ξ is the indicator function of the interval $[0,t]$.

As usual, we set up the recurrence

$$\begin{aligned} \hat{y}_0(t) &= y_a \\ \hat{y}_n(t) &= y_a + \frac{1}{\Gamma(\alpha)} \int_0^a (t-s)^{\alpha-1} \Xi_{[0,t]}(s) f(s, \hat{y}_{n-1}(s)) ds \\ &- \frac{1}{\Gamma(\alpha)} \int_0^a (a-s)^{\alpha-1} f(s, \hat{y}_{n-1}(s)) ds \end{aligned}$$

It follows that

$$\begin{aligned} \| & \hat{y}_{n+1} - \hat{y}_n \| \\ \leq & \frac{1}{\Gamma(\alpha)} L \| \hat{y}_n - \hat{y}_{n-1} \| \left(| \int_0^t (t-s)^{\alpha-1} - (a-s)^{\alpha-1} ds | + | \int_t^a (a-s)^{\alpha-1} ds | \right) \\ = & \frac{1}{\Gamma(\alpha)} \| \hat{y}_n - \hat{y}_{n-1} \| \frac{L}{\alpha} \left(|0 - t^\alpha - (a-t)^\alpha + a^\alpha| + |0 - (a-t)^\alpha| \right) \\ \leq & \frac{2La^\alpha}{\Gamma(\alpha+1)} \| \hat{y}_n - \hat{y}_{n-1} \| \end{aligned}$$

for $t \in [0, a]$, and so the sequence $\{\hat{y}_n\}$ converges absolutely and uniformly to a solution of (2.9) and hence of (2.8).

Uniqueness also follows in the usual way.

As a consequence of Theorem **2.1**, we find that, subject to the conditions of the Theorem, every FBVP (1.1)-(1.2) coincides with a unique initial condition y_0. It follows that there is an exact correspondence between fractional boundary value problems and initial value problems. In other words, we may conclude the following:

LEMMA **2.2**. *If the function f is continuous and satisfies a Lipschitz condition with Lipschitz constant $L > 0$ with respect to its second argument, and if $\frac{2La^\alpha}{\Gamma(\alpha+1)} < 1$, then the boundary value problem (1.1)-(1.2) is equivalent to the integral equation (2.7).*

Existence and uniqueness theory for the FBVP for $t > a$ is inherited from the corresponding initial value problem theory. For details, see [5].

3. Numerical methods and results

If f is continuous and satisfies a Lipschitz condition with respect to the second variable, from the results in [4] we know that for the solution of (1.1) that passes through the point (a, y_a), we are able to find at most one point $(0, y_0)$ that also lies on the same solution trajectory. According to Theorem **2.1** we now know that if, in addition, $\frac{2La^\alpha}{\Gamma(\alpha+1)} < 1$ then such a point $(0, y_0)$ will in fact exist. In paper [4] the solution was found by using a shooting algorithm based on the bisection method. In what follows we will use a different approach where the bisection is replaced by the secant method. Also other numerical methods for solving the initial value problems than the one used by the authors of that paper, will be considered.

To be more precise, our first step will be to find two initial guesses for $y(0)$, say y_{01} and y_{02}, satisfying $y(a)|_{y(0)=y_{01}} < w < y(a)|_{y(0)=y_{02}}$. Next, iterate by the secant method to provide successive approximations for y_0 until the distance between the two last approximations does not exceed a given tolerance ϵ. In our numerical experiments we have used $\epsilon = 10^{-8}$.

Note that the evaluation of $y(a)$ may require the use of a IVP numerical solver. The methods that we used to solve the initial value problem

$$\begin{aligned} D_*^\alpha(y(t)) &= f(t, y(t)), & (3.10)\\ y(0) &= y_0, & (3.11) \end{aligned}$$

are listed bellow:

Method 1: The first method we have considered was the fractional Adams scheme of [6], the one also used in [4];

Method 2: This is a finite difference method based on the definition of the Grunwald-Letnikov operator (see, for example, [7]);

Method 3: Fractional backward difference based on quadrature (see, for example, [2], [7]);

Method 4: A higher order method proposed initially by Lubich with convergence order $p = 3$ ([3], [8]).

In order to compare the efficiency of these methods, let us begin with the following example:

$$\begin{aligned} D_*^\alpha(y(t)) &= -\frac{1}{2}y(t) + \frac{1}{2}t^2 + 2\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}, \quad 0 < t \le 1, & (3.12)\\ y(0.5) &= 0.25, \end{aligned}$$

whose analytical solution is known and is given by $y(t) = t^2$.

Since in our numerical methods we begin by determining the value of $y(0)$ for which the solution of the initial value problem matches the given boundary condition, it is natural, in order to test its accuracy, to evaluate the absolute error at the point where that boundary condition is imposed (generally we do not have an analytical solution to compare the obtained numerical results). For this example, with $\alpha = \frac{1}{4}$, the absolute errors at $t = 0.5$ and $t = 1$ and the obtained values of $y(0)$ are presented in tables 1, 2, 3 and 4.

h	$y(0)$	Absolute error at $t = 0.5$	Absolute error at $t = 1$
1/10	-2.60518×10^{-2}	$< 10^{-25}$	1.57×10^{-2}
1/20	-1.01031×10^{-2}	$< 10^{-25}$	5.97×10^{-3}
1/40	-3.91042×10^{-3}	$< 10^{-25}$	2.32×10^{-3}
1/80	-1.52436×10^{-3}	$< 10^{-25}$	9.12×10^{-4}
1/160	-5.99889×10^{-4}	$< 10^{-25}$	3.63×10^{-4}
1/320	-2.38312×10^{-4}	$< 10^{-25}$	1.46×10^{-4}

TABLE 1. Comparison with the exact solution (shooting algorithm with method 1 to solve the IVP)

h	$y(0)$	Absolute error at $t = 0.5$	Absolute error at $t = 1$
1/10	-1.28536×10^{-2}	$< 10^{-25}$	8.83×10^{-3}
1/20	-6.55284×10^{-3}	$< 10^{-25}$	4.39×10^{-3}
1/40	-3.31168×10^{-3}	$< 10^{-25}$	2.20×10^{-3}
1/80	-1.66558×10^{-3}	$< 10^{-25}$	1.09×10^{-3}
1/160	-8.35479×10^{-4}	$< 10^{-25}$	5.46×10^{-4}
1/320	-4.18481×10^{-4}	$< 10^{-25}$	2.73×10^{-4}

TABLE 2. Comparison with the exact solution (shooting algorithm with method 2 to solve the IVP)

The numerical errors at the discretisation points are plotted in figures 1, 2, 3 and 4, where once again we have considered $\alpha = \frac{1}{4}$.

Analysing tables 1-4 we observe that the absolute error at $t = 0.5$, the point where the boundary condition is imposed, does not decrease as the step-size goes smaller, although we are comparing very small quantities. This is not surprising since for each value of h we obtain a different value for $y(0)$ and the solution of the boundary value problem is obtained considering that value of $y(0)$ in the respective initial value problem solver. If for all the considered values of h we determine the solution of the boundary value problem for the same value of $y(0)$, then we expect to obtain small absolute

h	$y(0)$	Absolute error at $t = 0.5$	Absolute error at $t = 1$
1/10	-3.20261×10^{-3}	$< 10^{-25}$	6.24×10^{-4}
1/20	-1.03931×10^{-3}	5.55×10^{-17}	1.83×10^{-4}
1/40	-3.29896×10^{-4}	$< 10^{-25}$	5.40×10^{-5}
1/80	-1.03182×10^{-4}	$< 10^{-25}$	1.60×10^{-5}
1/160	-3.19341×10^{-5}	$< 10^{-25}$	4.75×10^{-6}
1/320	-9.80596×10^{-6}	5.55×10^{-17}	1.41×10^{-6}

TABLE 3. Comparison with the exact solution (shooting algorithm with method 3 to solve the IVP)

h	$y(0)$	Absolute error at $t = 0.5$	Absolute error at $t = 1$
1/10	-3.53309×10^{-6}	1.67×10^{-16}	1.24×10^{-6}
1/20	-4.14828×10^{-7}	$< 10^{-25}$	2.18×10^{-7}
1/40	-2.15950×10^{-8}	$< 10^{-25}$	2.55×10^{-8}
1/80	4.02715×10^{-9}	$< 10^{-25}$	1.85×10^{-9}
1/160	1.65614×10^{-9}	5.55×10^{-17}	5.29×10^{-11}
1/320	3.84180×10^{-10}	$< 10^{-25}$	5.46×10^{-11}

TABLE 4. Comparison with the exact solution (shooting algorithm with method 4 to solve the IVP)

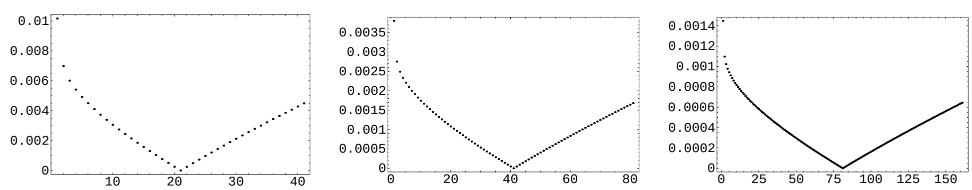

FIGURE 1. Absolute errors at the discretisation points using method 1 with $h = 1/40$, $h = 1/80$ and $h = 1/160$, respectively.

errors as the step-size decreases. This, in fact, can be observed in tables 5, 6, 7 and 8, where for each value of the step-size h and for each initial value solver, the numerical solution is determined with $y(0)$ obtained by shooting with $h = 1/320$. As an alternative, and taking figures 1-4 into account, we could also determine the infinity norm of the error at the discretisation points ($\|E\|_\infty = \max |E_i|$, where E_i is the error at point x_i) and observe corresponding behaviour (see table 9).

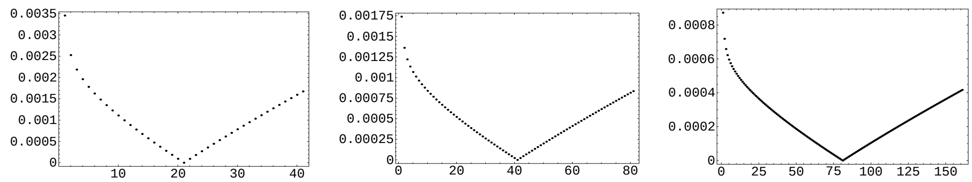

FIGURE 2. Absolute errors at the discretisation points using method 2 with $h = 1/40$, $h = 1/80$ and $h = 1/160$, respectively.

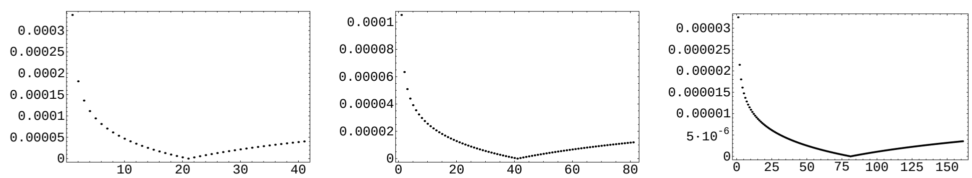

FIGURE 3. Absolute errors at the discretisation points using method 3 with $h = 1/40$, $h = 1/80$ and $h = 1/160$, respectively.

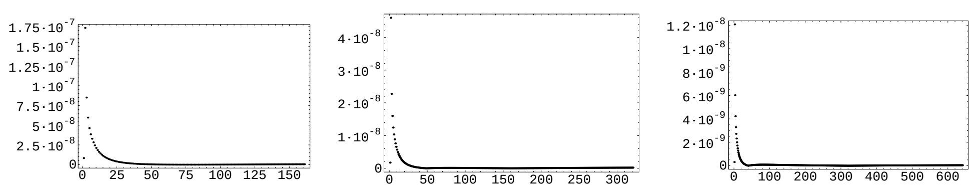

FIGURE 4. Absolute errors at the discretisation points using method 4 with $h = 1/40$, $h = 1/80$ and $h = 1/160$, respectively.

Next we consider another example:

$$\begin{aligned} D_*^{\frac{1}{2}}(y(t)) &= \frac{1}{4}y(t) - \frac{1}{4}t^{\frac{3}{2}} + \frac{3\sqrt{\pi}}{4}t, \quad 0 < t \leq 1, \qquad (3.13) \\ y(0.5) &= \frac{1}{2\sqrt{2}}, \end{aligned}$$

whose analytical solution is $y(t) = t^{\frac{3}{2}}$. Note that in this case the solution is no longer smooth. In the previous example we have considered a problem with a smooth solution but whose fractional derivative $D^\alpha y$ was

h	Absolute error at $t = 0.5$	Absolute error at $t = 1$
1/10	1.75×10^{-2}	3.21×10^{-2}
1/20	6.68×10^{-3}	1.23×10^{-2}
1/40	2.49×10^{-3}	4.66×10^{-3}
1/80	8.71×10^{-4}	1.73×10^{-3}
1/160	2.45×10^{-4}	5.93×10^{-4}

TABLE 5. Comparison with the exact solution (method 1 to solve the IVP with $y_0 = -0.000238312$.)

h	Absolute error at $t = 0.5$	Absolute error at $t = 1$
1/10	8.69×10^{-3}	1.69×10^{-2}
1/20	4.23×10^{-3}	8.35×10^{-3}
1/40	1.98×10^{-3}	4.05×10^{-3}
1/80	8.50×10^{-4}	1.89×10^{-3}
1/160	2.83×10^{-4}	8.13×10^{-4}

TABLE 6. Comparison with the exact solution (method 2 to solve the IVP with $y_0 = -0.000418481$.)

h	Absolute error at $t = 0.5$	Absolute error at $t = 1$
1/10	2.18×10^{-3}	2.67×10^{-3}
1/20	7.01×10^{-4}	8.41×10^{-4}
1/40	2.17×10^{-4}	2.58×10^{-4}
1/80	6.33×10^{-5}	7.56×10^{-5}
1/160	1.50×10^{-5}	1.89×10^{-5}

TABLE 7. Comparison with the exact solution (method 3 to solve the IVP with $y_0 = -9.80596 \times 10^{-6}$.)

not smooth. In tables 10, 11, 12 and 13 we present the values of $\|E\|$ for example (3.12) with $\alpha = \frac{1}{2}$ and example (3.13).

Finally, we consider a nonlinear example:

$$\begin{aligned} D_*^{\frac{1}{2}}(y(t)) &= \sin(y(t)), \quad 0 < t \leq 1, \\ y(0.1) &= 1, \end{aligned} \tag{3.14}$$

For this example, the approximations for the initial value $y(0)$ and for the approximate value of $y(1)$, are presented for different values of the step-size h in Tables 14 and 15. With respect to table 15, we observe that

h	Absolute error at $t = 0.5$	Absolute error at $t = 1$
1/10	2.35×10^{-6}	9.91×10^{-7}
1/20	2.79×10^{-7}	4.54×10^{-8}
1/40	1.48×10^{-8}	1.15×10^{-8}
1/80	2.46×10^{-9}	4.17×10^{-9}
1/160	8.61×10^{-10}	7.58×10^{-10}

TABLE 8. Comparison with the exact solution (method 4 to solve the IVP with $y_0 = 3.8418 \times 10^{-10}$.)

h	Method 1	Method 2	Method 3	Method 4
1/10	2.61×10^{-2}	1.29×10^{-2}	3.20×10^{-3}	9.03×10^{-6}
1/20	1.01×10^{-2}	6.55×10^{-3}	1.04×10^{-3}	2.73×10^{-6}
1/40	3.91×10^{-3}	3.31×10^{-3}	3.30×10^{-4}	7.80×10^{-7}
1/80	1.52×10^{-3}	1.67×10^{-3}	1.03×10^{-4}	2.11×10^{-7}
1/160	6.00×10^{-4}	8.35×10^{-4}	3.19×10^{-5}	5.53×10^{-8}
1/320	2.38×10^{-4}	4.18×10^{-4}	9.81×10^{-6}	1.43×10^{-8}

TABLE 9. Values of $\|E\|_\infty$ for example (3.12) with $\alpha = \frac{1}{4}$

h	Example (3.12)	EOC	Example (3.13)	EOC
1/10	1.05×10^{-2}		3.33×10^{-3}	
1/20	3.33×10^{-3}	1.66	1.15×10^{-3}	1.53
1/40	1.08×10^{-3}	1.62	3.99×10^{-4}	1.53
1/80	3.57×10^{-4}	1.60	1.39×10^{-4}	1.52
1/160	1.20×10^{-4}	1.57	4.86×10^{-5}	1.52
1/320	4.08×10^{-5}	1.56	1.70×10^{-5}	1.52

TABLE 10. Values of $\|E\|_\infty$ for example (3.12) with $\alpha = \frac{1}{2}$ and example (3.13) (shooting method with method 1 to solve the IVP).

the estimates of the convergence orders decrease, especially for methods 2 and 3, in which it is necessary to use also a method for solving nonlinear equations. Here we have used the Newton's method. Moreover, when using method 2 we could not observe convergence for non small stepsizes.

4. Conclusions

For a class of boundary value problems for fractional differential equations with order between 0 and 1, we have established sufficient conditions

h	Example (3.12)	EOC	Example (3.13)	EOC
1/10	2.65×10^{-2}		2.28×10^{-2}	
1/20	1.35×10^{-2}	0.97	1.16×10^{-2}	0.97
1/40	6.85×10^{-3}	0.98	5.85×10^{-3}	0.99
1/80	3.46×10^{-3}	0.99	2.95×10^{-3}	0.99
1/160	1.76×10^{-3}	0.98	1.48×10^{-3}	1.00
1/320	8.97×10^{-4}	0.97	7.56×10^{-4}	0.97

TABLE 11. Values of $\|E\|_\infty$ for example (3.12) with $\alpha = \frac{1}{2}$ and example (3.13) (shooting method with method 2 to solve the IVP).

h	Example (3.12)	EOC	Example (3.13)	EOC
1/10	1.07×10^{-2}		7.92×10^{-3}	
1/20	4.02×10^{-3}	1.41	3.00×10^{-3}	1.40
1/40	1.48×10^{-3}	1.44	1.11×10^{-3}	1.43
1/80	5.36×10^{-4}	1.47	4.05×10^{-4}	1.45
1/160	1.93×10^{-4}	1.47	1.46×10^{-4}	1.47
1/320	6.88×10^{-5}	1.49	5.26×10^{-5}	1.47

TABLE 12. Values of $\|E\|_\infty$ for example (3.12) with $\alpha = \frac{1}{2}$ and example (3.13) (shooting method with method 3 to solve the IVP).

h	Example (3.12)	EOC	Example (3.13)	EOC
1/10	1.24×10^{-16}		1.67×10^{-16}	
1/20	1.69×10^{-16}	-	2.22×10^{-16}	-
1/40	2.53×10^{-16}	-	4.44×10^{-16}	-
1/80	3.46×10^{-16}	-	4.44×10^{-16}	-
1/160	3.86×10^{-16}	-	6.66×10^{-16}	-
1/320	6.79×10^{-16}	-	1.22×10^{-15}	-

TABLE 13. Values of $\|E\|_\infty$ for example (3.12) with $\alpha = \frac{1}{2}$ and example (3.13) (shooting method with method 4 to solve the IVP).

for the existence and uniqueness of the solution. As mentioned before, uniqueness results have already been obtained in [4], but in that paper the authors did not consider the existence problem. Here we have shown that both existence and uniqueness results can be obtained.

h	Method 1	Method 2	Method 3	Method 4
1/10	0.725819	0.689636	0.764178	0.605204
1/20	0.717985	0.685478	0.742505	0.651816
1/40	0.715461	0.687812	0.729214	0.683377
1/80	0.714580	0.692456	0.721914	0.698189
1/160	0.714269	0.697277	0.718091	0.705968
1/320	0.714159	0.701451	0.716125	0.709982

TABLE 14. Values of $y(0)$ for example (3.14) (shooting method with methods 1,2,3,4 to solve the IVP).

h	Method 1	Method 2	Method 3	Method 4
1/10	1.813738	1.794170	1.831123	1.729782
1/20	1.807430	1.788349	1.821275	1.761414
1/40	1.805464	1.788290	1.813783	1.784294
1/80	1.804784	1.790668	1.809352	1.794027
1/160	1.804544	1.793572	1.806962	1.799111
1/320	1.804459	1.796210	1.805714	1.801731
EOC	1.55	-	0.75	0.90

TABLE 15. Values of $y(1)$ for example (3.14) (shooting method with methods 1,2,3,4 to solve the IVP).

With respect to the numerical methods, we have considered a prototype method for solving the boundary value problems based on a shooting argument. We have considered four standard methods for solving the initial value problems. Comparing the obtained numerical results, we see that the low order methods behave quite similarly for linear problems, in the sense that the expected estimated convergence orders (EOC) are observed. Concerning the higher order method used here (the Lubich method with convergence order $p = 3$), we conclude by analysing the obtained results that although it is very accurate, it does not reveal the expected theoretical convergence order. We believe that these results have an easy explanation: it is known that these higher numerical methods should be used with some prudence, because instability may occur due to the cancellation of digits in the linear system for the determination of the so-called starting weights ([3]). Besides, as explained before, since, for each step size h we are shooting on the initial value $y(0)$ and the final approximate solution is obtained by solving the initial value problem for that value of $y(0)$, for different values of h the computed solutions of the BVP are obtained with different initial values.

For all these reasons we believe that method 1 ([6]) to solve the initial value problems when shooting on the unknown value of $y(0)$ is the most competitive method, since it is easy to implement, for linear and nonlinear problems and the obtained numerical results illustrate that this method performs well when dealing with non-smooth solutions.

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