Analytical and numerical treatment of oscillatory mixed differential equations with differentiable delays and advances

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Abstract

In this work we study the oscillatory behaviour of the differential equation of mixed type

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\, d\eta(\theta)$$

with delays, $r(\theta)$, and advances, $\tau(\theta)$, both differentiable. Some analytical and numerical criteria are obtained in order to guarantee that all solutions are oscillatory.

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1. Introduction

The aim of this work is to study the oscillatory behaviour of the differential equation of mixed type

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\, d\eta(\theta) \tag{1}$$

where $x(t) \in \mathbb{R}$, $\nu(\theta)$ and $\eta(\theta)$ are real functions of bounded variation on $[-1, 0]$ normalized so that $\nu(-1) = \eta(-1) = 0$, $r(\theta)$ and $\tau(\theta)$ are nonnegative real continuous functions on $[-1, 0]$. Taking

$$\|\tau\| = \max\{\tau(\theta) : \theta \in [-1, 0]\},$$

the advance $\tau(\theta)$ will be assumed to satisfy

$$\tau(\theta_0) = \|\tau\| > \tau(\theta), \quad \forall \theta \neq \theta_0. \tag{2}$$

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In case of having $\tau(\theta_0) > 0$, the function $\eta(\theta)$ is supposed to be atomic at θ_0, that is, such that

$$\eta\left(\theta_0^+\right) - \eta\left(\theta_0^-\right) \neq 0. \tag{3}$$

The equation (1) represents the wider class of linear functional differential equations of mixed type and is considered by Krisztin [8] as a basis for some mathematical applications appearing in the literature, such as in [3] and [12].

Letting $R = \max\{\|r\|, \|\tau\|\}$, by a solution of (1) we will mean any differentiable function $x : [-R, +\infty) \to \mathbb{R}$ which satisfies (1) for every $t \in [0, +\infty)$.

As usual, we will say that a solution $x(t)$ of (1) oscillates if it has arbitrarily large zeros. In [8] $x(t)$ is called oscillatory if there is no cone, $\mathcal{K}$, such that $x(t) \in \mathcal{K}$, eventually. Notice that for equations, both definitions coincide. When all solutions oscillate (1) will be said to be oscillatory.

By assuming that delays and advances are positive and differentiable on $[-1, 0]$, one can obtain some special criteria for having (1) oscillatory. In this paper we will analyze this case, complementing the results in [9] for the case where delays and advances are only continuous. Further theoretical results for delay equations are obtained in [10] and these can be extended in a natural way to the mixed equation.

The two main ingredients in theory of linear delay equations (see [7]) are the existence of a unique solution, for any given initial condition, and the exponential boundeness on those solutions. As is shown in [11], this is not at all the situation of a differential equation of mixed type like (1). However, under the atomicity assumption (3), one has that every oscillatory solution is exponentially bounded as $t \to \infty$ ([8, Proposition 4]). This fact enables the oscillatory behaviour of (1) to be studied through the analysis of the zeros of the characteristic equation

$$\lambda = \int_{-1}^{0} \exp(-\lambda r(\theta))\, d\nu(\theta) + \int_{-1}^{0} \exp(\lambda \tau(\theta))\, d\eta(\theta). \tag{4}$$

In fact, if we let

$$M(\lambda) = \int_{-1}^{0} \exp(-\lambda r(\theta))\, d\nu(\theta) + \int_{-1}^{0} \exp(\lambda \tau(\theta))\, d\eta(\theta),$$

by [8, Corollary 5] the equation (1) is oscillatory if and only if $M(\lambda) \neq \lambda$, for every real λ. Therefore, if either

$$M(\lambda) > \lambda, \quad \forall \lambda \in \mathbb{R} \tag{5}$$

or

$$M(\lambda) < \lambda, \quad \forall \lambda \in \mathbb{R} \tag{6}$$

we can conclude that equation (1) is oscillatory.

2. Differentiable delays and advances

By an increasing (decreasing) function on an interval $[a, b]$ we will mean any nondecreasing (respectively nonincreasing) function, ϕ, such that $\phi(a) < \phi(b)$ (respectively, $\phi(a) > \phi(b)$). Assuming that $-1 \leqslant \theta_1 \leq 0$, let $D^+(\theta_1)$ be the family of all positive differentiable functions, which are increasing on $[-1, \theta_1]$ and decreasing on $[\theta_1, 0]$. If $\theta_1 = 0$, we obtain the set, D_i^+ of all positive increasing differentiable functions on the interval $[-1, 0]$. In the case where $\theta_1 = -1$, we obtain the class D_d^+ of all decreasing positive differentiable functions on $[-1, 0]$.

For $r \in D^+(\theta_1)$ and $\tau \in D^+(\theta_0)$ with θ_0 as in (2), we define the value

$$S_1 = e^{-1}\left(\int_{-1}^{0} \nu(\theta)\, d\ln r(\theta) + \int_{-1}^{0} \eta(\theta)\, d\ln \tau(\theta)\right).$$

Through (5) we obtain the following theorems.

Theorem 2.1. *For $r \in D^{+}(\theta_1)$ and $\tau \in D^{+}(\theta_0)$, let*

$$\begin{aligned} \nu(\theta) &\leqslant 0 \text{ for } \theta \in [-1, \theta_1[\,,\ \nu(\theta) \geqslant 0 \text{ for } \theta \in [\theta_1, 0] && (7)\\ \eta(\theta) &\leqslant 0 \text{ for } \theta \in [-1, \theta_0[\,,\ \eta(\theta) \geqslant 0 \text{ for } \theta \in [\theta_0, 0]\,, && (8)\end{aligned}$$

such that $\eta(0) > 0$. If

$$1 + \ln(\tau(0)\,\eta(0)) + \tau(0)\, S_1 > 0 \tag{9}$$

then the equation (1) is oscillatory.

Proof: For $\lambda = 0$, we have $M(0) = \nu(0) + \eta(0) > 0$. Let $\lambda \neq 0$. Using integration by parts we obtain

$$\begin{aligned} M(\lambda) &= \exp(-\lambda r(0))\,\nu(0) + \exp(\lambda\tau(0))\,\eta(0) + \\ &\quad +\lambda \int_{-1}^{0} \exp(-\lambda r(\theta))\,\nu(\theta)\, dr(\theta) - \lambda \int_{-1}^{0} \exp(\lambda\tau(\theta))\,\eta(\theta)\, d\tau(\theta)\,. \end{aligned} \tag{10}$$

Since $\nu(\theta)\, r'(\theta) \leqslant 0$ and $\eta(\theta)\,\tau'(\theta) \leqslant 0$ for $\theta \in [-1, 0]\,$, and $u \exp(-u) \leqslant 1/e$, for every real u, we have

$$M(\lambda) \geqslant \exp(-\lambda r(0))\,\nu(0) + \exp(\lambda\tau(0))\,\eta(0) + S_1.$$

Therefore

$$\begin{aligned} M(\lambda) - \lambda &\geqslant \exp(-\lambda r(0))\,\nu(0) + \exp(\lambda\tau(0))\,\eta(0) - \lambda + S_1 \\ &\geqslant \exp(\lambda\tau(0))\,\eta(0) - \lambda + S_1. \end{aligned} \tag{11}$$

As $\eta(0) > 0$, the function $f(\lambda) = \exp(\lambda\tau(0))\,\eta(0) - \lambda$ attains an absolute minimum at

$$\lambda_0 = -\frac{\ln(\tau(0)\,\eta(0))}{\tau(0)}$$

and consequently

$$M(\lambda) - \lambda \geqslant \frac{1}{\tau(0)} + \frac{1}{\tau(0)} \ln(\tau(0)\,\eta(0)) + S_1 > 0.$$

Thus (5) is satisfied, which completes the proof.

Example 2.1. *Consider the equation (1) for $\nu(\theta) = (3\theta + 1)(\theta + 1)\,,\ \eta(\theta) = (\theta + 1)(2\theta + 1)\,,$*

$$r(\theta) = -\frac{3}{2}\theta^2 - \theta + 1$$

and

$$\tau(\theta) = -\theta^2 - \theta + 2$$

As

$$\begin{aligned} S_1 &= e^{-1} \int_{-1}^{0} (3\theta + 1)(\theta + 1)\, \frac{-3\theta + 1}{-\frac{3}{2}\theta^2 - \theta + 1} d\theta + e^{-1} \int_{-1}^{0} (\theta + 1)(2\theta + 1)\, \frac{-2\theta - 1}{-\theta^2 - \theta + 2} d\theta \\ &\approx -0.1421, \end{aligned}$$

$$1 + \ln(\tau(0)\,\eta(0)) + \tau(0)\, S_1 = 1 + \ln 2 + 2S_1 \approx 1.4089,$$

the corresponding equation (1) is oscillatory.

Example 2.2. *Consider the equation (1) with*

$$\nu(\theta) = \begin{cases} -\theta - 1, & \text{if } \theta \in [-1, 0[\\ 1, & \text{if } \theta = 0 \end{cases},$$

$$\eta(\theta) = \theta + 1, \quad r(\theta) = \theta + 2 \quad \text{and} \quad \tau(\theta) = -\theta + 1.$$

The corresponding equation is oscillatory since

$$S_1 = e^{-1} \int_{-1}^{0} \frac{-\theta - 1}{\theta + 2} d\theta + e^{-1} \int_{-1}^{0} \frac{-(\theta + 1)}{-\theta + 1} d\theta \approx -0.25499$$

and

$$1 + \ln(\tau(0)\eta(0)) + \tau(0) S_1 = 1 + \ln 1 + S_1 \approx 0.74501.$$

Now let

$$S_2 = \int_{-1}^{0} \nu(\theta)\, dr(\theta) - \int_{-1}^{0} \eta(\theta)\, d\tau(\theta).$$

Theorem 2.2. *Let $r \in D^{+}(\theta_1)$, $\tau \in D^{+}(\theta_0)$. If (7)-(8) hold such that $\nu(0) + \eta(0) > 0$ and*

$$1 - e\tau(0)\eta(0) < S_2 < 1 + er(0)\nu(0) \tag{12}$$

then equation (1) is oscillatory.

Proof: The case where $\lambda = 0$, follows as in the proof of Theorem 2.1.
For $\lambda \neq 0$, by (10) we have

$$\begin{aligned} \frac{M(\lambda)}{\lambda} &= \frac{\exp(-\lambda r(0))}{\lambda}\nu(0) + \frac{\exp(\lambda\tau(0))}{\lambda}\eta(0) \\ &+ \int_{-1}^{0} \exp(-\lambda r(\theta))\,\nu(\theta)\, dr(\theta) - \int_{-1}^{0} \exp(\lambda\tau(\theta))\,\eta(\theta)\, d\tau(\theta). \end{aligned} \tag{13}$$

Let $\lambda > 0$. Since $\exp(-u) < 1$, $\exp u > 1$, $\frac{\exp(-u)}{u} > 0$ and $\frac{\exp u}{u} \geq e$, for $u > 0$, we obtain

$$\frac{M(\lambda)}{\lambda} > e\tau(0)\eta(0) + S_2 > 1$$

and so $M(\lambda) > \lambda$.
For $\lambda < 0$, the same arguments imply that

$$\frac{M(\lambda)}{\lambda} < -er(0)\nu(0) + S_2 < 1$$

and $M(\lambda) > \lambda$. Hence (5) is again satisfied and (1) is oscillatory.

Example 2.3. *Consider the equation (1) with*

$$\nu(\theta) = (5\theta + 4)(\theta + 1), \quad \eta(\theta) = (10\theta + 9)(\theta + 1),$$

$$r(\theta) = -\frac{5}{2}\theta^2 - 4\theta + 5$$

and

$$\tau(\theta) = -5\theta^2 - 9\theta + 1.$$

We have

$$S_2 = -\int_{-1}^{0} (5\theta + 4)^2 (\theta + 1)\, d\theta + \int_{-1}^{0} (10\theta + 9)^2 (\theta + 1)\, d\theta \approx 15.417$$

and

$$-23.465 \approx 1-9e = 1-e\tau(0)\eta(0) < S_2 < 1+er(0)\nu(0) = 1+20e \approx 55.366.$$

So, the corresponding equation is oscillatory. Notice that in this case as

$$\begin{aligned} S_1 &= e^{-1}\left(\int_{-1}^{0} \frac{-(5\theta+4)^2(\theta+1)}{-\frac{5}{2}\theta^2-4\theta+5}d\theta + \int_{-1}^{0} \frac{-(10\theta+9)^2(\theta+1)}{-5\theta^2-9\theta+1}d\theta\right) \\ &\approx -3.6737 \end{aligned}$$

and

$$1+\ln(\tau(0)\eta(0)) + \tau(0)S_1 = 1+\ln 9 + S_1 = -0.47648 < 0.$$

we cannot apply Theorem 2.1.

With respect to condition (6) we obtain the following theorem.

Theorem 2.3. *Let $r \in D^+(\theta_1)$, $\tau \in D^+(\theta_0)$ and*

$$\begin{aligned} \nu(\theta) &\geqslant 0 \text{ for } \theta \in [-1,\theta_1[\,,\ \nu(\theta) \leqslant 0 \text{ for } \theta \in [\theta_1, 0]\,, \quad (14) \\ \eta(\theta) &\geqslant 0 \text{ for } \theta \in [-1,\theta_0[\,,\ \eta(\theta) \leqslant 0 \text{ for } \theta \in [\theta_0, 0] \quad (15) \end{aligned}$$

such that $\nu(0) < 0$. If

$$1+\ln(r(0)|\nu(0)|) - r(0)S_1 > 0 \quad (16)$$

then the equation (1) is oscillatory.

Proof: For $\lambda = 0$, we have $M(0) = \nu(0) + \eta(0) < 0 = \lambda$.

Let $\lambda \neq 0$. Applying (10) and taking into account that now $\nu(\theta)r'(\theta) \geqslant 0$ and $\eta(\theta)\tau'(\theta) \geqslant 0$ for $\theta \in [-1,0]$, and $u\exp(-u) \leqslant 1/e$, for every real u, we have

$$M(\lambda) \leqslant \exp(-\lambda r(0))\nu(0) + \exp(\lambda\tau(0))\eta(0) + S_1.$$

Notice that, in this case, $M(\lambda) \to -\infty$, as $\lambda \to \pm\infty$.

Therefore

$$M(\lambda) - \lambda \leq \exp(-\lambda r(0))\nu(0) - \lambda + S_1. \quad (17)$$

The function $g(\lambda) = \exp(-\lambda r(0))\nu(0) - \lambda$ has a maximum at

$$\lambda_0 = \frac{\ln(r(0)|\nu(0)|)}{r(0)}$$

and consequently by (16)

$$M(\lambda) - \lambda \leqslant -\frac{1}{r(0)} - \frac{1}{r(0)}\ln(r(0)|\nu(0)|) + S_1 < 0,$$

for every $\lambda \in \mathbb{R}$.

Thus (6) is satisfied and (1) is oscillatory.

Remark 2.1. *Notice that conditions (7) and (8) of Theorem 2.1, by (11), imply that $M(\lambda) - \lambda \to +\infty$, as $\lambda \to \pm\infty$. Analogously to (14) and (15) of Theorem 2.3, by (17), one has $M(\lambda) - \lambda \to -\infty$, as $\lambda \to \pm\infty$. This means that in such situations, the real roots of the characteristic equation (4) are bounded.*

Example 2.4. *Consider the equation (1) with*

$$\nu(\theta) = (-\theta - 1)(4\theta + 3), \quad \eta(\theta) = -8\theta - 8,$$

$$r(\theta) = -2\theta^2 - 3\theta + 1,$$

and

$$\tau(\theta) = -\theta + 1.$$

Notice that

$$S_1 = e^{-1} \int_{-1}^{0} \frac{(-\theta - 1)(4\theta + 3)(-4\theta - 3)}{-2\theta^2 - 3\theta + 1} d\theta + e^{-1} \int_{-1}^{0} \frac{8\theta + 8}{-\theta + 1} d\theta \approx 1.6372.$$

and

$$1 + \ln(r(0)|\nu(0)|) - r(0) S_1 = 1 + \ln 3 - S_1 \approx 0.4614.$$

By Theorem 2.3, the corresponding equation (1) is oscillatory.

Example 2.5. *Consider*

$$\nu(\theta) = \begin{cases} \theta + 1, & \text{if } \theta \in [-1, 0[\\ -1, & \text{if } \theta = 0 \end{cases},$$

$$\eta(\theta) = -\theta - 1, \quad r(\theta) = -\theta^2 + 2 \quad \text{and} \quad \tau(\theta) = -\theta + 3.$$

The equation (1) is oscillatory since

$$S_1 = e^{-1} \left(\int_{-1}^{0} \frac{-2\theta(\theta + 1)}{-\theta^2 + 2} d\theta + \int_{-1}^{0} \frac{\theta + 1}{-\theta + 3} d\theta \right) \approx 0.1291,$$

and

$$1 + \ln(r(0)|\nu(0)|) - r(0) S_1 = 1 + \ln 2 - 2S_1 \approx 1.4349.$$

Theorem 2.4. *Let $r \in D^+(\theta_1)$, $\tau \in D^+(\theta_0)$ and assume that (14)-(15) are satisfied such that $\nu(0) + \eta(0) < 0$. If*

$$1 + er(0)\nu(0) < S_2 < 1 - e\tau(0)\eta(0) \tag{18}$$

then the equation (1) is oscillatory.

Proof: When $\lambda = 0$, as before one has $M(0) = \nu(0) + \eta(0) < 0$.
Let $\lambda > 0$. Using (13) and the arguments as in Theorem 2.2, we obtain

$$\frac{M(\lambda)}{\lambda} < e\tau(0)\eta(0) + S_2,$$

and by (18) follows that $M(\lambda) < \lambda$.
For $\lambda < 0$, the same arguments as before enable us to conclude that

$$\frac{M(\lambda)}{\lambda} > er(0)|\nu(0)| + S_2 > 1.$$

So, by (18) one has also $M(\lambda) < \lambda$, which achieves the proof.

For the case where $\theta_0 = \theta_1 = -1$, the delays and advances are in $D_{d.}^+$. When $\theta_0 = \theta_1 = 0$, the delays and advances are in D_i^+. The following example illustrates this situation for Theorem 2.4.

Example 2.6. *Let the equation (1)*

$$\nu(\theta) = -(5\theta+1)(\theta+1), \quad \eta(\theta) = -(6\theta+1)(\theta+1),$$

$$r(\theta) = -10\theta^2 - 4\theta + 10,$$

and

$$\tau(\theta) = -3\theta^2 - \theta + 1.$$

We have

$$\begin{aligned} S_2 &= \int_{-1}^{0} (5\theta+1)(\theta+1)(20\theta+4)\,d\theta - \int_{-1}^{0} (6\theta+1)^2(\theta+1)\,d\theta \\ &\approx 2.1667 \end{aligned}$$

$$-26.138 \approx 1 - 10e = 1 + er(0)\nu(0) < S_2 < 1 - e\tau(0)\eta(0) = 1 + e \approx 3.7183,$$

so, the corresponding equation is oscillatory.

3. Numerical experiments

In this section, we show how numerical approximations can be used to derive information about oscillation or non-oscillation of solutions to a mixed-type equation. To begin, we give an overview of the approach, which builds on that adopted in [4]. We give more details later.

The general approach is to derive a discrete system that approximates the underlying mixed-type equation and to analyze the behaviour of solutions of the discrete scheme. The approach we have adopted here is to use a very simple discretization, based on an Euler rule to approximate the derivative on the left hand side of the equation, and a trapezoidal rule to approximate the integrals on the right hand side. In principle, one could use a more complicated approach, but the results we obtain here are very good and the method is already effective in our view.

As a general principle, we shall use a fixed step length $h > 0$ and the resulting system of discrete equation will take the form of difference equations or a recurrence relation. This can be analyzed using its characteristic equation and (for no oscillatory solutions) we are looking for the case when there are no non-negative real characteristic roots.

The root counting method we have adopted (see [4] for further discussion) is based on an application of the argument principle and Rouché's Theorem to count zeros of a polynomial function inside a closed path. We choose a rectangular path with vertices at $0 \pm \frac{1}{M}i, M \pm \frac{1}{M}i$ for large positive values of $M \in \mathbb{R}$ and count the zeros inside the rectangle as $M \to \infty$. As we saw in [4], one can show that the characteristic polynomial of the discrete problem has zeros close to the positive real axis only if the characteristic equation of the underlying continuous problem has characteristic values close to the real axis. Further details of the analytical results will be found in [4] (see also [1, 2, 5, 6].)

For the detail, consider the numerical scheme for the equation (1)

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\,d\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\,d\eta(\theta)$$

where, $x(t) \in \mathbb{R}$, $\nu(\theta)$ and $\eta(\theta)$ are real functions of bounded variation on $[-1, 0]$ normalized in manner that $\nu(-1) = \eta(-1) = 0$, $r(\theta)$ and $\tau(\theta)$ are nonnegative real continuous functions on $[-1, 0]$. We shall use the backward Euler method to approximate the time derivative and use the trapzoidal method to approximate the integral. Then we obtain the corresponding discrete characteristic polynomial. Further we use Rouché's Theorem to find the numbers of real roots of the discrete characteristic polynomial. We observe that the equation (1) is oscillatory if and only if the characteristic polynomial has no real roots, which is consistent with the theoretic results.

Below we will describe how to find the discrete characteristic polynomial of (1). In all the numerical examples, we assume that $r(\theta), \nu(\theta), \tau(\theta)$ and $\eta(\theta)$ are quadratic polynomials.

Let us consider how to discretize the integral $\int_{-1}^{0} x(t - r(\theta))\, d\nu(\theta)$. A similar idea can be applied to the integral $\int_{-1}^{0} x(t + \tau(\theta))\, d\eta(\theta)$

We first need to find the critical point θ_r of $r(\theta)$ on $[-1, 0]$, i.e., $r'(\theta_r) = 0$. Assume that $r(\theta)$ attains its maximum value at θ_r, i.e., $r(\theta)$ is increasing on $[-1, \theta_r]$ and decreasing on $[\theta_r, 0]$. We also assume that

$$r(-1) = r_{-1} > 0, \qquad r(0) = r_0 > 0.$$

Obviously, in this case $r(\theta_r) = r_c \geq \max\{r_{-1}, r_0\}$.

We write the integral in two parts:

$$\int_{-1}^{0} x(t - r(\theta))\, d\nu(\theta) = \int_{-1}^{\theta_r} x(t - r(\theta))\, d\nu(\theta) + \int_{\theta_r}^{0} x(t - r(\theta))\, d\nu(\theta).$$

Let $0 = t_0 < t_1 < t_2 < \cdots < t_n < \ldots$ be time points and let $h = t_{j+1} - t_j$ be the time step.

The idea of the discretization of the integral $\int_{-1}^{\theta_r} x(t + r(\theta))\, d\nu(\theta)$ is to find two nonnegative integers $N_1,\ N_2,\ N_1 > N_2$ such that

$$-1 = \theta_{-N_1} < \theta_{-N_1+1} < \cdots < \theta_{-N_2} = \theta_r,$$

is a partition of $[-1, \theta_r]$ and

$$r(\theta_{-N_1}) = r(-1) = r_{-1} = N_1 h, \tag{19}$$
$$r(\theta_j) = N_1 h + m_r(N_1 + j)h, \quad j = -N_1 + 1, -N_1 + 2, \ldots, -N_1 + (N_1 - N_2 - 1), \tag{20}$$
$$r(\theta_{-N_2}) = r(\theta_r) = r_c = N_1 h + m_r(N_1 - N_2)h. \tag{21}$$

Here m_r is some positive integer which guarantees that $N_2 \geq 0$. Such N_1 and N_2 can be obtained by (19) and (21),

$$N_1 = \frac{r_{-1}}{h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r}. \tag{22}$$

Note that $\theta_j,\ j = -N_1, -N_1 + 1, \ldots, -N_1 + (N_1 - N_2)$ can be obtained by solving (19) - (21) for the given $r(\theta)$.

The idea of the discretization of the integral $\int_{\theta_r}^{0} x(t + r(\theta))\, d\nu(\theta)$ is to find two nonnegative integers $N_3,\ N_4$ such that

$$\theta_r = \theta_{-N_3} < \theta_{-N_3+1} < \cdots < \theta_{-1} < \theta_0 = 0,$$

is a partition of $[\theta_r, 0]$ and

$$r(\theta_0) = r(0) = r_0 = N_4 h, \tag{23}$$
$$r(\theta_l) = (N_3 h + N_4 h) - (N_3 + l)h, \quad l = -N_3 + 1, -N_3 + 2, \ldots, -1, \tag{24}$$
$$r(\theta_{-N_3}) = r(\theta_r) = r_c = N_3 h + N_4 h. \tag{25}$$

Such N_3 and N_4 can be obtained by (23) and (25),

$$N_4 = \frac{r_0}{h}, \qquad N_3 = \frac{r_c}{h} - N_4. \tag{26}$$

Note that $\theta_l,\ l = -N_3, -N_3 + 1, \ldots, -1, 0$ can be obtained by solving (23) - (25) for the given $r(\theta)$.

Now we can discretize the integral $\int_{-1}^{0} x(t+r(\theta))\,d\nu(\theta)$ at $t=t_n$. We have

$$\begin{aligned}\int_{-1}^{0} x(t_n-r(\theta))\,d\nu(\theta) &= \int_{-1}^{\theta_r} x(t_n-r(\theta))\,d\nu(\theta)+\int_{\theta_r}^{0} x(t_n-r(\theta))\,d\nu(\theta)\\
&\approx \sum_{j=-N_1}^{-N_2-1} x(t_n-r(\theta_j))\Big(\nu(\theta_{j+1})-\nu(\theta_j)\Big)+\sum_{l=-N_3}^{-1} x(t_n-r(\theta_l))\Big(\nu(\theta_{l+1})-\nu(\theta_l)\Big)\\
&= \sum_{j=-N_1}^{-N_2-1} x(nh-[N_1h+m_r(N_1+j)h])\Big(\nu(\theta_{j+1})-\nu(\theta_j)\Big)\\
&\quad+\sum_{l=-N_3}^{-1} x(nh-[N_3h+N_4h-(N_3+l)h])\Big(\nu(\theta_{l+1})-\nu(\theta_l)\Big)\end{aligned}$$

Similarly we can discretize the integral $\int_{-1}^{0} x(t+\tau(\theta))\,d\eta(\theta)$. Now let us summarize the steps to find the characteristic polynomial of (1).

Step 1. Find the critical point θ_r of $r(\theta)$ on $[-1,0]$. Without loss of the generality, we assume that $r(\theta)$ is increasing on $[-1,\theta_r]$ and decreasing on $[\theta_r,0]$ and $r(-1)=r_{-1}>0,\ r(0)=r_0>0$.

Step 2. Find the nonnegative integers $N_1,N_2,\ N_1>N_2$ by

$$N_1=\frac{r_{-1}}{h},\qquad N_2=\frac{(m_r+1)N_1-\frac{r_c}{h}}{m_r},$$

where $r_c=r(\theta_r)$ and m_r is some positive integer such that $N_2\geq 0$.

Find the nonnegative integers N_3 and N_4 by

$$N_4=\frac{r_0}{h},\qquad N_3=\frac{r_c}{h}-N_4.$$

Step 3. Find the critical point θ_τ of $\tau(\theta)$ on $[-1,0]$. Without loss of the generality, we assume that $\tau(\theta)$ is increasing on $[-1,\theta_\tau]$ and decreasing on $[\theta_\tau,0]$ and $\tau(-1)=\tau_{-1}>0,\ \tau(0)=\tau_0>0$.

Step 4. Find the nonnegative integers $M_1,M_2,\ M_1>M_2$ by

$$M_1=\frac{\tau_{-1}}{h},\qquad M_2=\frac{(m_\tau+1)M_1-\frac{\tau_c}{h}}{m_\tau},$$

where $\tau_c=\tau(\theta_\tau)$ and m_τ is some positive integer such that $M_2\geq 0$.

Find the nonnegative integers M_3 and M_4 by

$$M_4=\frac{\tau_0}{h},\qquad M_3=\frac{\tau_c}{h}-M_4.$$

Step 5. Approximating the time derivative in (1) by the backward Euler method and approximating the integral in (1) by the Trapezoidal method, we obtain, at time t_n,

$$\begin{aligned}\frac{x(t_{n+1})-x(t_n)}{h} &\approx \sum_{j=-N_1}^{-N_2-1} x\big(nh-[N_1h+m_r(N_1+j)h]\big)\big(\nu(\theta_{j+1})-\nu(\theta_j)\big)\\
&\quad+\sum_{l=-N_3}^{-1} x\big(nh-[N_3h+N_4h-(N_3+l)h]\big)\big(\nu(\theta_{l+1})-\nu(\theta_l)\big)\\
&\quad+\sum_{j=-M_1}^{-M_2-1} x\big(nh+[M_1h+m_\tau(M_1+j)h]\big)\big(\eta(\theta'_{j+1})-\eta(\theta'_j)\big)\\
&\quad+\sum_{l=-M_3}^{-1} x\big(nh+[M_3h+M_4h-(M_3+l)h]\big)\big(\eta(\theta'_{l+1})-\eta(\theta'_l)\big).\end{aligned}$$

Here θ_j and θ_l are determined by

$$r(\theta_j) = N_1 h + m_r(N_1 + j)h, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

and

$$r(\theta_l) = (N_3 h + N_4 h) - (N_3 + l)h, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0.$$

Similarly, θ'_j and θ'_l are determined by

$$r(\theta'_j) = M_1 h + m_\tau(M_1 + j)h, \quad j = -M_1, -M_1 + 1, \ldots, -M_2,$$

and

$$r(\theta'_l) = (M_3 h + M_4 h) - (M_3 + l)h, \quad l = -M_3, -M_3 + 1, \ldots, -1, 0.$$

Denote $x^n \approx x(t_n),\ n = 0, 1, 2, \ldots$. We have

$$\begin{aligned}\frac{x^{n+1} - x^n}{h} &= \sum_{j=-N_1}^{-N_2-1} x^{n-[N_1+m_r(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) \\ &+ \sum_{l=-N_3}^{-1} x^{n-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big) \\ &+ \sum_{j=-M_1}^{-M_2-1} x^{n+[M_1+m_\tau(M_1+j)]}\big(\eta(\theta'_{j+1}) - \eta(\theta'_j)\big) \\ &+ \sum_{l=-M_3}^{-1} x^{n+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big).\end{aligned}$$

Denote $N = \max\{N_1 + m_r(N_1 - N_2 - 1), N_3 + N_4\}$. Choosing $n = N$ and replacing x by z, we get the following discrete characteristic equation of (1)

$$\begin{aligned}P(z) = -z^{N+1} + z^N + h\Big[&\sum_{j=-N_1}^{-N_2-1} z^{n-[N_1+m_r(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) \\ &+ \sum_{l=-N_3}^{-1} z^{n-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big) \\ &+ \sum_{j=-M_1}^{-M_2-1} z^{n+[M_1+m_\tau(M_1+j)]}\big(\eta(\theta'_{j+1}) - \eta(\theta'_j)\big) \\ &+ \sum_{l=-M_3}^{-1} z^{n+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Big].\end{aligned}$$

Step 6. Apply Rouché's Theorem to determine the existence of the positive real roots of the characteristic polynomial $P(z)$.

$$\frac{1}{2\pi i}\int_C \frac{P'(z)}{P(z)}\,dz = \text{Number of zeros of } P(z) \text{ inside the closed curve } C.$$

In our numerical simulation, we chose the curve C as the boundary of a rectangle with vertices $A(0, \frac{1}{M})$, $B(0, -\frac{1}{M})$, $C(M, -\frac{1}{M})$ and $D(M, \frac{1}{M})$ for some sufficiently large M.

Remark 3.1. *We can use a similar idea to work on the case where $r(-1) = r_{-1} < 0$ and $r(0) = r_0 < 0$, or $r(-1) \cdot r(0) < 0$.*

Remark 3.2. *We can also use a similar idea to work on the case where $r(\theta)$ (or $\tau(\theta)$) is decreasing on $[-1,\theta_c]$ (or $[-1,\theta_\tau]$) and increasing on $[\theta_c,0]$ (or $[\theta_\tau,0]$).*

Below we will consider how to construct the discrete characteristic polynomials for some examples.

Example 3.1. *Consider the equation (1) with the conditions of example 2.1*

$$x'(t)=\int_{-1}^{0}x(t-r(\theta))\,d\nu(\theta)+\int_{-1}^{0}x(t+\tau(\theta))\,d\eta(\theta). \tag{27}$$

Here

$$\nu(\theta)=(3\theta+1)(\theta+1),\qquad \eta(\theta)=(\theta+1)(2\theta+1),$$

and

$$r(\theta)=-\frac{3}{2}\theta^2-\theta+1,\qquad \tau(\theta)=-\theta^2-\theta+2.$$

Let us find the discrete characteristic polynomial of (27). We first find the critical point θ_r of $r(\theta)$ on $[-1,0]$. Let $r'(\theta)=-3\theta-1=0$. we get $\theta_r=-\frac{1}{3}$. Further it is easy to find that $r(\theta)$ is increasing on $[-1,\theta_r]$ and decreasing on $[\theta_r,0]$ and $r(-1)=r_{-1}=\frac{1}{2}>0$, $r(0)=r_0=1>0$ and $r(\theta_r)=r_c=\frac{7}{6}$.

The nonnegative integers N_1,N_2, $N_1>N_2$ can be determined by

$$N_1=\frac{r_{-1}}{h}=\frac{1}{2h},\qquad N_2=\frac{(m_r+1)N_1-\frac{r_c}{h}}{m_r}=\frac{1}{6h},$$

where we choose $m_r=2$ which guarantees that $N_2>0$.

The nonnegative integers N_3 and N_4 can be determined by

$$N_4=\frac{r_0}{h}=\frac{1}{h},\qquad N_3=\frac{r_c}{h}-N_4=\frac{1}{6h}.$$

Next we will find the critical point θ_τ of $\tau(\theta)$ on $[-1,0]$. Let $\tau'(\theta)=-2\theta-1=0$. we get $\theta_\tau=-\frac{1}{2}$. Further it is easy to find that $\tau(\theta)$ is increasing on $[-1,\theta_\tau]$ and decreasing on $[\theta_\tau,0]$ and $\tau(-1)=\tau_{-1}=2>0$, $\tau(0)=\tau_0=2>0$ and $\tau(\theta_\tau)=\tau_c=2.25$.

The nonnegative integers M_1,M_2, $M_1>M_2$ can be determined by

$$M_1=\frac{\tau_{-1}}{h}=\frac{2}{h},\qquad M_2=\frac{(m_\tau+1)M_1-\frac{\tau_c}{h}}{m_\tau}=\frac{7}{4h},$$

where we choose $m_\tau=1$ which guarantees that $M_2\geq 0$.

The nonnegative integers M_3 and M_4 can be determined by

$$M_4=\frac{\tau_0}{h}=\frac{2}{h},\qquad M_3=\frac{\tau_c}{h}-M_4=\frac{1}{4h}.$$

Finally we denote $N=\max\{N_1+2(N_1-N_2-1),N_3+N_4\}$. Then we obtain the following discrete characteristic equation of (27)

$$\begin{aligned}P(z)=-z^{N+1}+z^N+h\Big[&\sum_{j=-N_1}^{-N_2-1}z^{N-[N_1+2(N_1+j)]}\big(\nu(\theta_{j+1})-\nu(\theta_j)\big)\\&+\sum_{l=-N_3}^{-1}z^{N-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1})-\nu(\theta_l)\big)\\&+\sum_{j=-M_1}^{-M_2-1}z^{N+[M_1+(M_1+j)]}\big(\eta(\theta'_{j+1})-\eta(\theta'_j)\big)\\&+\sum_{l=-M_3}^{-1}z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1})-\eta(\theta'_l)\big)\Big].\end{aligned}$$

Here θ_j and θ_l are determined by

$$r(\theta_j) = -\frac{3}{2}\theta_j^2 - \theta_j + 1 = (3N_1 + 2j)h, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

and

$$r(\theta_l) = -\frac{3}{2}\theta_l^2 - \theta_l + 1 = N_4 h - lh, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0.$$

which implies that

$$\theta_j = \frac{1 + \sqrt{1 + 6\big(1 - (3N_1 + 2j)h\big)}}{2 \times (-3/2)}, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

and

$$\theta_l = \frac{1 - \sqrt{1 + 6\big(1 - N_4 h + lh\big)}}{2 \times (-3/2)}, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0.$$

Similarly, θ'_j and θ'_l are determined by

$$\theta'_j = \frac{1 + \sqrt{1 + 4\big(2 - (2M_1 + j)h\big)}}{2 \times (-1)}, \quad j = -M_1, -M_1 + 1, \ldots, -M_2,$$

and

$$\theta'_l = \frac{1 - \sqrt{1 + 4\big(2 - M_4 h + lh\big)}}{2 \times (-1)}, \quad l = -M_3, -M_3 + 1, \ldots, -1, 0.$$

Applying Rouché's Theorem, we find that $P(z)$ has no positive real roots and so this satisfies the conditions for discrete equation to be oscillatory. Hence the numerical results are consistent with the theoretical results about the oscillatory property of the equation (27)*. See Figure* 1 *and Table* 1*.*

Step Length h	Length of Rectangle M	Number of Zeros N_P
0.05	2	12
0.05	4	6
0.05	10	2
0.05	20	2
0.05	large	0

Table 1: cf Fig 1: Number of zeros of the polynomial by Rouché's Theorem

Example 3.2. *Consider the equation (1) for the example 2.2*

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\, d\eta(\theta). \tag{28}$$

Here

$$\nu(\theta) = \begin{cases} -\theta - 1, & -1 \le \theta < 0, \\ 1, & \theta = 0, \end{cases}$$

and

$$\eta(\theta) = \theta + 1, \qquad r(\theta) = \theta + 2, \qquad \tau(\theta) = -\theta + 1.$$

We now find the discrete characteristic polynomial of (28)*. We first find the critical point θ_r of $r(\theta)$ on $[-1, 0]$. We get $\theta_r = 0$. Further it is easy to see that $r(\theta)$ is increasing on $[-1, \theta_r]$ and $r(-1) = r_{-1} = 1 > 0$, $r(0) = r_0 = 2 > 0$ and $r(\theta_r) = r_c = 2$.*

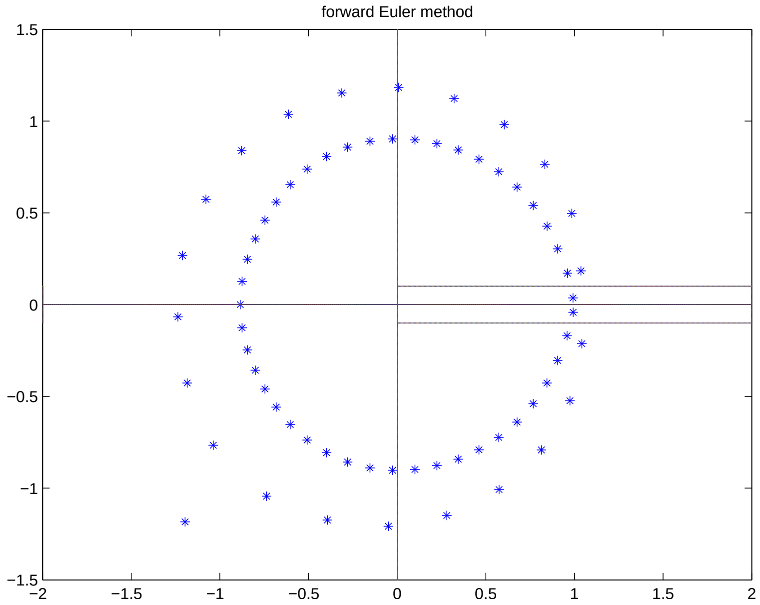

Figure 1: Characteristic Plot for h=0.05, M=10

The nonnegative integers $N_1, N_2,\ N_1 > N_2$ can be determined by

$$N_1 = \frac{r_{-1}}{h} = \frac{1}{h}, \qquad N_2 = \frac{(m_r+1)N_1 - \frac{r_c}{h}}{m_r} = 0,$$

where we choose $m_r = 1$ which guarantees that $N_2 \geq 0$.

Next we will find the critical point θ_τ of $\tau(\theta)$ on $[-1,0]$. We get $\theta_\tau = -1$. Further it is easy to see that $\tau(\theta)$ is decreasing on $[\theta_\tau, 0]$ and $\tau(-1) = \tau_{-1} = 2 > 0,\ \tau(0) = \tau_0 = 1 > 0$ and $\tau(\theta_\tau) = \tau_c = 2$.

The nonnegative integers M_3, M_4 can be determined by

$$M_4 = \frac{\tau_0}{h} = \frac{1}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{1}{h}.$$

Finally we denote $N = N_1 + (N_1 - N_2 - 1)$. Then we obtain the following discrete characteristic equation of (28)

$$\begin{aligned} P(z) = -z^{N+1} + z^N + h\Big[&\sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) \\ &+ \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Big]. \end{aligned} \tag{29}$$

Here θ_j are determined by

$$r(\theta_j) = \theta_j + 2 = N_1 h + (N_1 + j)h, \quad j = -N_1, -N_1+1, \ldots, -N_2,$$

which implies that

$$\theta_j = N_1 h + (N_1 + j)h - 2, \quad j = -N_1, -N_1+1, \ldots, -N_2.$$

Similarly, θ'_l are determined by

$$\tau(\theta'_l) = -\theta'_l + 1 = (M_3 h + M_4 h) - (M_3 + l)h, \quad j = -M_3, -M_3+1, \ldots, -1, 0.$$

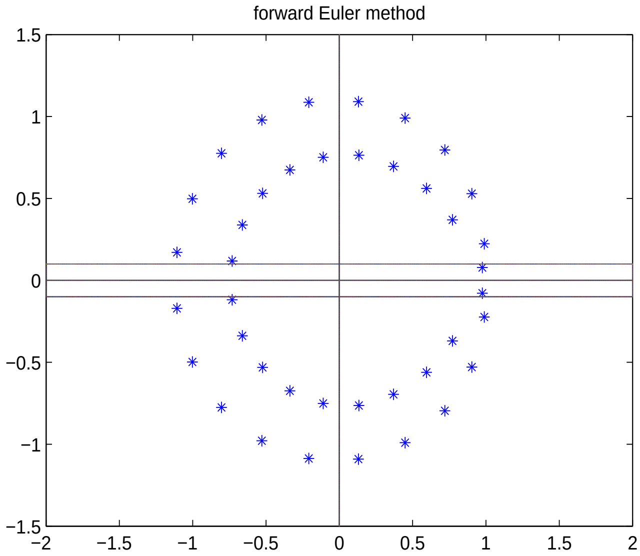

Figure 2: Characteristic Plot for h=0.01, M=8

which implies that

$$\theta'_l = -(M_3h + M_4h) + (M_3 + l)h + 1, \quad l = -M_3, -M_3 + 1, \ldots, -1, 0.$$

Remark 3.3. *Note that $\nu(\theta)$ has a jump at $\theta = 0$, therefore we have, in (29),*

$$\begin{aligned}\nu(\theta_{-N_2}) - \nu(\theta_{-N_2-1}) &= \nu(0) - \nu(N_1h + N_1h - N_2h - h)\\ &= \nu(0) - \nu(2-h) = 1 - \big(-(2-h) - 1\big)\\ &= 4 - h,\end{aligned}$$

Applying Rouché's Theorem, we find that $P(z)$ has no positive real roots and therefore satisfies the conditions for the discrete equation to be oscillatory. Hence the numerical results are consistent with the theoretical results about the oscillation proprty for the equation (28)*. See Figure* 2 *and Table* 2*.*

Example 3.3. *Consider the equation (1) for the example 2.3*

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\, d\eta(\theta). \tag{30}$$

Here

$$\nu(\theta) = (5\theta + 4)(\theta + 1), \qquad \eta(\theta) = (10\theta + 9)(\theta + 1),$$

and

$$r(\theta) = -\frac{5}{2}\theta^2 - 4\theta + 5, \qquad \tau(\theta) = -5\theta^2 - 9\theta + 1.$$

We now find the discrete characteristic polynomial of (30)*. We first find the critical point θ_r of $r(\theta)$ on $[-1, 0]$. Let $r'(\theta) = -5\theta - 4 = 0$. We get $\theta_r = -\frac{4}{5}$. Further it is easy to find that $r(\theta)$ is increasing on $[-1, \theta_r]$ and decreasing on $[\theta_r, 0]$ and $r(-1) = r_{-1} = 6.5 > 0$, $r(0) = r_0 = 5 > 0$ and $r(\theta_r) = r_c = 6.6$.*

The nonnegative integers N_1, N_2, $N_1 > N_2$ can be determined by

$$N_1 = \frac{r_{-1}}{h} = \frac{6.5}{h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r} = 2N_1 - \frac{6.6}{h},$$

where we choose $m_r = 1$ *which guarantees that* $N_2 \geq 0$.

The nonnegative integers N_3 *and* N_4 *can be determined by*

$$N_4 = \frac{r_0}{h} = \frac{5}{h}, \qquad N_3 = \frac{r_c}{h} - N_4 = \frac{6.6}{h} - N_4.$$

Next we will find the critical point θ_τ *of* $\tau(\theta)$ *on* $[-1, 0]$. *Let* $\tau'(\theta) = -10\theta - 9 = 0$. *we get* $\theta_\tau = -\frac{9}{10}$. *Further it is easy to find that* $\tau(\theta)$ *is increasing on* $[-1, \theta_\tau]$ *and decreasing on* $[\theta_\tau, 0]$ *and* $\tau(-1) = \tau_{-1} = 5 > 0$, $\tau(0) = \tau_0 = 1 > 0$ *and* $\tau(\theta_\tau) = \tau_c = 5.05$.

The nonnegative integers M_1, M_2, $M_1 > M_2$ *can be determined by*

$$M_1 = \frac{\tau_{-1}}{h} = \frac{5}{h}, \qquad M_2 = \frac{(m_\tau + 1)M_1 - \frac{\tau_c}{h}}{m_\tau} = 2M_1 - \frac{5.05}{h},$$

where we choose $m_\tau = 1$ *which guarantees that* $M_2 \geq 0$.

The nonnegative integers M_3 *and* M_4 *can be determined by*

$$M_4 = \frac{\tau_0}{h} = \frac{2}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{1}{4h}.$$

Finally we denote $N = \max\{N_1 + 2(N_1 - N_2 - 1), N_3 + N_4\}$. *Then we obtain the following discrete characteristic equation of* (30)

$$\begin{aligned} P(z) = -z^{N+1} + z^N + h\Big[& \sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+2(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) \\ & + \sum_{l=-N_3}^{-1} z^{N-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big) \\ & + \sum_{j=-M_1}^{-M_2-1} z^{N+[M_1+(M_1+j)]}\big(\eta(\theta'_{j+1}) - \eta(\theta'_j)\big) \\ & + \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Big]. \end{aligned}$$

Here θ_j *and* θ_l *are determined by*

$$r(\theta_j) = -\frac{5}{2}\theta_j^2 - 4\theta_j + 5 = (2N_1 + j)h, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

and

$$r(\theta_l) = -\frac{5}{2}\theta_j^2 - 4\theta_j + 5 = N_4 h - lh, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0.$$

which implies that

$$\theta_j = \frac{4 + \sqrt{16 + 10\big(5 - (2N_1 + j)h\big)}}{2 \times (-5/2)}, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

and

$$\theta_l = \frac{4 - \sqrt{16 + 10\big(5 - N_4 h + lh\big)}}{2 \times (-5/2)}, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0.$$

Similarly, θ'_j *and* θ'_l *are determined by*

$$\theta'_j = \frac{9 + \sqrt{81 + 20\big(1 - (2M_1 + j)h\big)}}{2 \times (-5)}, \quad j = -M_1, -M_1 + 1, \ldots, -M_2,$$

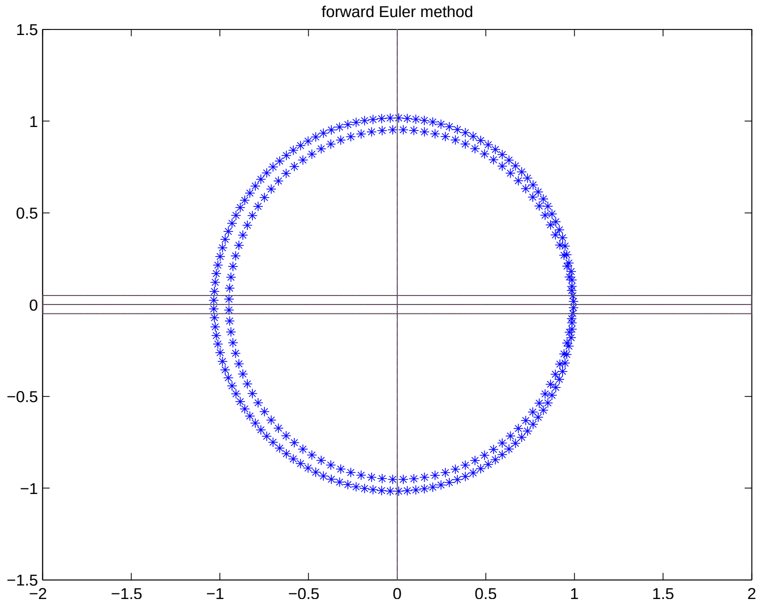

Figure 3: Characteristic Plot for h=0.05, M=20

and

$$\theta'_l = \frac{9 - \sqrt{81 + 20\big(1 - M_4 h + lh\big)}}{2 \times (-5)}, \qquad l = -M_3, -M_3 + 1, \dots, -1, 0.$$

Applying Rouché's Theorem, we find that $P(z)$ has no positive real roots which satisfies the conditions for oscillation of the discrete equation. Hence the numerical results are consistent with the theoretical results about the oscillation of the equation (30)*. See Figure* 3 *and Table* 3*.*

Step Length h	Length of Rectangle M	Number of Zeros N_p	
0.05	2	78	
0.05	4	38	
0.05	10	14	
0.05	20	8	
0.05	large	0	

Table 2: cf Fig 3 : Number of zeros of the polynomial by Rouché's Theorem

Example 3.4. *Consider the equation (1) for the example 2.4*

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\, d\eta(\theta). \tag{31}$$

Here

$$\nu(\theta) = (-\theta - 1)(4\theta + 3), \qquad \eta(\theta) = -8\theta - 8,$$

and

$$r(\theta) = -2\theta^2 - 3\theta + 1, \qquad \tau(\theta) = -\theta + 1.$$

We now find the discrete characteristic polynomial of (31)*. We first find the critical point θ_r of $r(\theta)$ on* $[-1, 0]$*. Let $r'(\theta) = -4\theta - 3 = 0$. We get $\theta_r = -\frac{3}{4}$. Further it is easy to find that $r(\theta)$ is increasing on* $[-1, \theta_r]$ *and decreasing on* $[\theta_r, 0]$ *and* $r(-1) = r_{-1} = 2 > 0$*,* $r(0) = r_0 = 1 > 0$ *and* $r(\theta_r) = r_c = \frac{17}{8}$*.*

The nonnegative integers $N_1, N_2,\ N_1 > N_2$ can be determined by

$$N_1 = \frac{r_{-1}}{h} = \frac{2}{h}, \qquad N_2 = \frac{(m_r+1)N_1 - \frac{r_c}{h}}{m_r} = 2N_1 - \frac{17}{8h},$$

where we choose $m_r = 1$ which guarantees that $N_2 \geq 0$.

The nonnegative integers N_3 and N_4 can be determined by

$$N_4 = \frac{r_0}{h} = \frac{1}{h}, \qquad N_3 = \frac{r_c}{h} - N_4 = \frac{17}{8h} - N_4.$$

Next we will find the critical point θ_τ of $\tau(\theta)$ on $[-1, 0]$. We get $\theta_\tau = -1$. Further it is easy to find that $\tau(\theta)$ is decreasing on $[\theta_\tau, 0]$ and $\tau(-1) = \tau_{-1} = 2 > 0,\ \tau(0) = \tau_0 = 1 > 0$ and $\tau(\theta_\tau) = \tau_c = 2$.

The nonnegative integers M_3 and M_4 can be determined by

$$M_4 = \frac{\tau_0}{h} = \frac{1}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{2}{h} - M_4.$$

Finally we denote $N = \max\{N_1 + (N_1 - N_2 - 1), N_3 + N_4\}$. Then we obtain the following discrete characteristic equation of (31)

$$\begin{aligned} P(z) = -z^{N+1} + z^N + h\Big[& \sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) \\ & + \sum_{l=-N_3}^{-1} z^{N-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big) \\ & + \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Big]. \end{aligned}$$

Here θ_j and θ_l are determined by

$$r(\theta_j) = -2\theta_j^2 - 3\theta_j + 1 = (2N_1 + j)h, \quad j = -N_1, -N_1+1, \ldots, -N_2,$$

and

$$r(\theta_l) = -2\theta_l^2 - 3\theta_l + 1 = N_4 h - lh, \quad l = -N_3, -N_3+1, \ldots, -1, 0.$$

which implies that

$$\theta_j = \frac{3 + \sqrt{9 + 8\big(1 - (2N_1 + j)h\big)}}{2 \times (-2)}, \quad j = -N_1, -N_1+1, \ldots, -N_2,$$

and

$$\theta_l = \frac{3 - \sqrt{9 + 8\big(1 - N_4 h + lh\big)}}{2 \times (-2)}, \quad l = -N_3, -N_3+1, \ldots, -1, 0.$$

Similarly, θ'_l are determined by

$$\theta'_l = -M_4 h + lh + 1, \quad l = -M_3, -M_3+1, \ldots, -1, 0.$$

Applying Rouché's Theorem, we find that $P(z)$ has no positive real roots which satisfies the conditions for oscillation of the discrete equation. Hence the numerical results are consistent with the theoretical results about the oscillation of the equation (31)*. See Figure* 4 *and Table* 4*.*

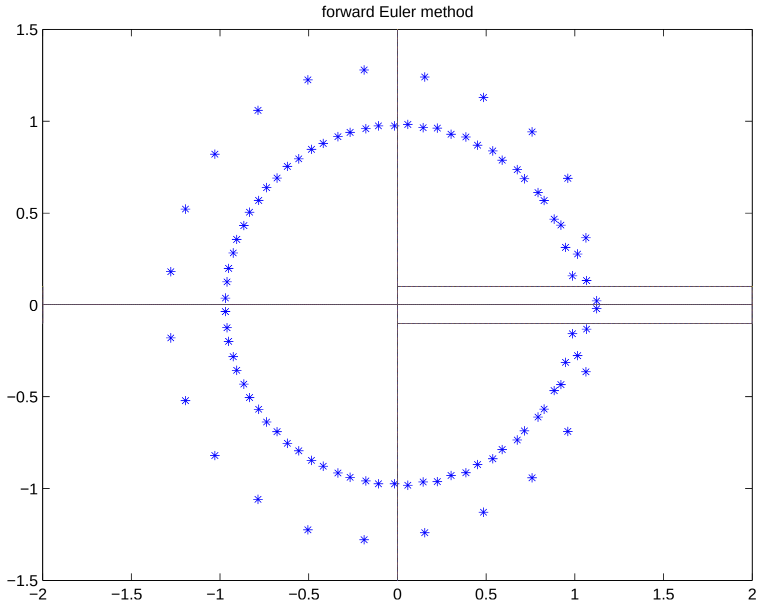

Figure 4: Characteristic Plot for h=0.05, M=10

Step Length h	Length of Rectangle M	Number of Zeros N_p
0.05	2	6
0.05	8	2
0.05	10	2
0.05	20	2
0.05	30	2
0.05	large	0

Table 3: cf Fig 4 : Number of zeros of the polynomial by Rouché's Theorem

Example 3.5. *Consider the equation (1) for the example 2.5*

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\, d\eta(\theta). \tag{32}$$

Here

$$\nu(\theta) = \begin{cases} \theta + 1, & -1 \le \theta < 0, \\ 0, & \theta = 0, \end{cases}$$

and

$$\eta(\theta) = -\theta - 1, \qquad r(\theta) = -\theta^2 + 2, \qquad \tau(\theta) = -\theta + 3.$$

We now find the discrete characteristic polynomial of (32)*. We first find the critical point* θ_r *of* $r(\theta)$ *on* $[-1, 0]$*. Let* $r'(\theta) = -2\theta = 0$*. We get* $\theta_r = 0$*. Further it is easy to see that* $r(\theta)$ *is increasing on* $[-1, \theta_r]$ *and* $r(-1) = r_{-1} = 1 > 0,\ r(0) = r_0 = 2 > 0$ *and* $r(\theta_r) = r_c = 2$*.*

The nonnegative integers $N_1, N_2,\ N_1 > N_2$ *can be determined by*

$$N_1 = \frac{r_{-1}}{h} = \frac{1}{h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r} = 2N_1 - \frac{2}{h},$$

where we choose $m_r = 1$ which guarantees that $N_2 \geq 0$.

Next we will find the critical point θ_τ of $\tau(\theta)$ on $[-1, 0]$. We get $\theta_\tau = -1$. Further it is easy to see that $\tau(\theta)$ is decreasing on $[\theta_\tau, 0]$ and $\tau(-1) = \tau_{-1} = 4 > 0$, $\tau(0) = \tau_0 = 3 > 0$ and $\tau(\theta_\tau) = \tau_c = 4$.

The nonnegative integers M_3 and M_4 can be determined by

$$M_4 = \frac{\tau_0}{h} = \frac{3}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{1}{h}.$$

Finally we denote $N = N_1 + (N_1 - N_2 - 1), N_3 + N_4\}$. Then we obtain the following discrete characteristic equation of (32)

$$P(z) = -z^{N+1} + z^N + h\Big[\sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+2(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big)$$
$$+ \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Big].$$

Here θ_j are determined by

$$r(\theta_j) = -\theta_j^2 + 2 = (2N_1 + j)h, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

which implies that

$$\theta_j = -\sqrt{2 - (2N_1 + j)h}, \quad j = -N_1, -N_1 + 1, \ldots, -N_2.$$

Similarly, θ'_l are determined by

$$\tau(\theta'_l) = -\theta'_l - 1 = M_4 h - lh,$$

which implies that

$$\theta'_l = -M_4 h + lh - 1, \quad l = -M_3, -M_3 + 1, \ldots, -1, 0.$$

Applying Rouché's Theorem, we find that $P(z)$ has no positive real roots which satisfies the conditions for oscillation of the discrete equation. Hence the numerical results are consistent with the theoretical results about the oscillation of the equation (32). *See Figure* 5 *and Table* 5.

Step Length h	Length of Rectangle M	Number of Zeros N_p
0.01	2	10
0.01	4	4
0.01	8	2
0.01	10	2
0.01	20	2
0.01	Large	0

Table 4: cf Fig 5 : Number of zeros of polynomial by Rouché's Theorem

Example 3.6. *Consider the equation (1) for the example 2.6*

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\, d\eta(\theta). \tag{33}$$

Here

$$\nu(\theta) = -(5\theta + 1)(\theta + 1), \qquad \eta(\theta) = -(6\theta + 1)(\theta + 1),$$

and

$$r(\theta) = -10\theta^2 - 4\theta + 10, \qquad \tau(\theta) = -3\theta^2 - \theta + 1.$$

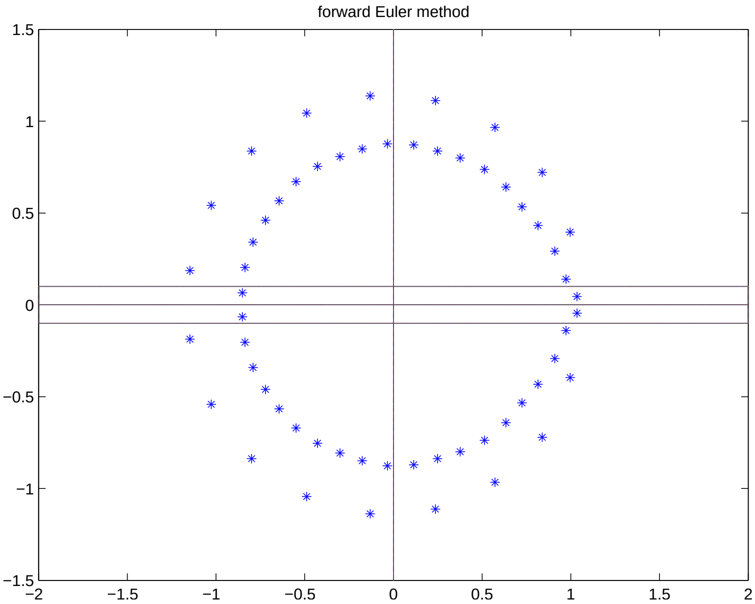

Figure 5: Characteristic Plot for h=0.01, M=10

We now find the discrete characteristic polynomial of (33)*. We first find the critical point* θ_r *of* $r(\theta)$ *on* $[-1,0]$*. Let* $r'(\theta) = -20\theta - 4 = 0$*. We get* $\theta_r = -\frac{1}{5}$*. Further it is easy to find that* $r(\theta)$ *is increasing on* $[-1,\theta_r]$ *and decreasing on* $[\theta_r, 0]$ *and* $r(-1) = r_{-1} = 4 > 0,\ r(0) = r_0 = 10 > 0$ *and* $r(\theta_r) = r_c = 10.4$*.*

The nonnegative integers $N_1, N_2,\ N_1 > N_2$ *can be determined by*

$$N_1 = \frac{r_{-1}}{h} = \frac{4}{h}, \qquad N_2 = \frac{(m_r+1)N_1 - \frac{r_c}{h}}{m_r} = \frac{3N_1 - \frac{10.4}{h}}{2},$$

where we choose $m_r = 2$ *which guarantees that* $N_2 \geq 0$*.*

The nonnegative integers N_3 *and* N_4 *can be determined by*

$$N_4 = \frac{r_0}{h} = \frac{10}{h}, \qquad N_3 = \frac{r_c}{h} - N_4 = \frac{10.4}{h} - N_4.$$

Next we will find the critical point θ_τ *of* $\tau(\theta)$ *on* $[-1,0]$*. We get* $\theta_\tau = -\frac{1}{6}$*. Further it is easy to find that* $\tau(\theta)$ *is increasing on* $[-1,\theta_\tau]$ *and decreasing on* $[\theta_\tau, 0]$ *and* $\tau(-1) = \tau_{-1} = -1 < 0,\ \tau(0) = \tau_0 = 1 > 0$ *and* $\tau(\theta_\tau) = \tau_c = \frac{13}{12}$*.*

Note that here $\tau(-1) = \tau_{-1} = -1 < 0$*. Let us discretize the integral* $\int_{-1}^{\theta_\tau} x(t+\tau(\theta))\, d\eta(\theta)$*. We need to find some nonnegative integers* $M_1, M_2,\ M_1 > M_2$ *such that* $-1 = \theta_{-M_1} < \theta_{-M_1+1} < \cdots < \theta_{-M_2} = \theta_\tau$ *is a partition of* $[-1,\theta_\tau]$ *and*

$$\begin{aligned}
&\tau(\theta_{-M_1}) = \tau(-1) = \tau_{-1} = -M_1 h,\\
&\tau(\theta_j) = -M_1 h + m_\tau(M_1 + j)h, \quad j = -M_1+1, -M_1+2, \ldots, -M_2-1,\\
&\tau(\theta_{-M_2}) = \tau(\theta_\tau) = \tau_c = -M_1 h + m_\tau(M_1 - M_2)h,
\end{aligned}$$

Here m_τ *is some positive integer which guarantees that* $M_2 \geq 0$*. In fact, we can determine* $M_1, M_2,\ M_1 > M_2$ *by the following:*

$$M_1 = -\frac{\tau_{-1}}{h} = \frac{1}{h},$$

and, with $m_\tau = 3$,

$$-M_1 h + m_\tau(M_1 - M_2)h = \frac{13}{12},$$

which implies that $M_2 = \frac{2M_1 - \frac{13}{12h}}{3} > 0$. *We remark that the bigger* m_r *is, the faster* $\tau(\theta_j)$, $j = -M_1 + 1, -M_1 + 2, \ldots, -M_2$ *increase. In order to guarantee* M_2 *is nonnegative, we need to choose* $m_r \geq 3$.

The nonnegative integers M_3 *and* M_4 *can be determined by*

$$M_4 = \frac{\tau_0}{h} = \frac{21}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{1}{12h}.$$

Finally we denote $N = \max\{N_1 + m_r(N_1 - N_2 - 1), N_3 + N_4\}$. *Then we obtain the following discrete characteristic equation of* (33)

$$\begin{aligned} P(z) = -z^{N+1} + z^N + h\Big[& \sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+2m_r(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) \\ & + \sum_{l=-N_3}^{-1} z^{N-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big) \\ & + \sum_{j=-M_1}^{-M_2-1} z^{N+[M_1+m_\tau(M_1+j)]}\big(\eta(\theta'_{j+1}) - \eta(\theta'_j)\big) \\ & + \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Big]. \end{aligned}$$

Here θ_j *and* θ_l *are determined by*

$$r(\theta_j) = -10\theta_j^2 - 4\theta_j + 10 = N_1 h + m_r(N_1 + j)h, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

and

$$r(\theta_l) = -10\theta_l^2 - 4\theta_l + 10 = N_4 h - lh, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0.$$

which implies that

$$\theta_j = \frac{4 + \sqrt{16 + 40\big(10 - (N_1 h + m_r(N_1 + j)h\big)}}{2 \times (-10)}, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

and

$$\theta_l = \frac{4 - \sqrt{16 + 40\big(10 - N_4 h + lh\big)}}{2 \times (-10)}, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0.$$

Similarly, θ'_j *and* θ'_l *are determined by*

$$\theta'_j = \frac{1 + \sqrt{1 + 12\big(1 + M_1 h - m_\tau(M_1 + j)h\big)}}{2 \times (-3)}, \quad j = -M_1, -M_1 + 1, \ldots, -M_2,$$

and

$$\theta'_l = \frac{1 - \sqrt{1 + 12\big(1 - M_4 h + lh\big)}}{2 \times (-3)}, \quad l = -M_3, -M_3 + 1, \ldots, -1, 0.$$

Applying Rouché's Theorem, we find that $P(z)$ *has no positive real roots which satisfies the conditions for oscillation of the discrete equation. Hence the numerical results are consistent with the theoretical results about the oscillation of the equation* (33). *See Figure* 6 *and Table* 6.

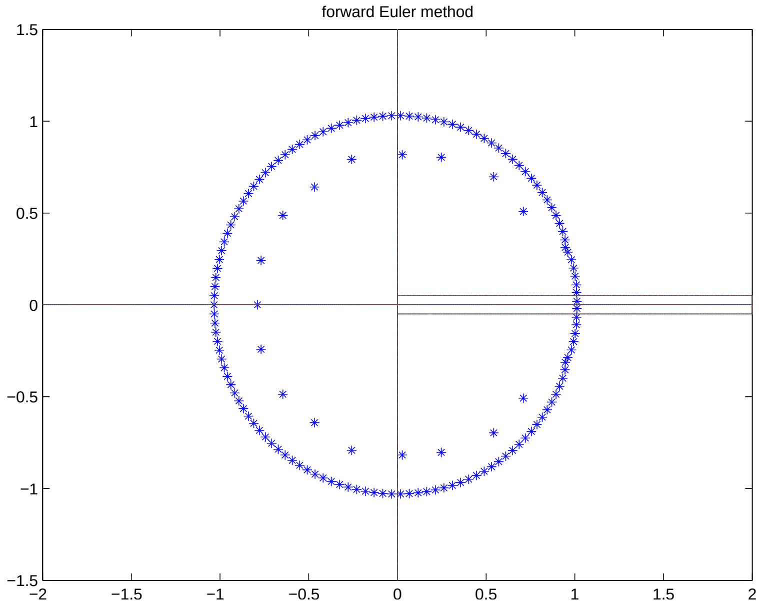

Figure 6: Characteristic Plot for h=0.05, M=20

Step Length h	Length of Rectangle M	Number of Zeros N_p
0.05	2	24
0.05	4	12
0.05	8	6
0.05	10	4
0.05	20	2
0.05	30	2
0.05	large	0

Table 5: cf Fig 6 : Number of zeros of the polynomial by Rouché's Theorem

4. Conclusions

As we have seen, the numerical approach introduced here does provide a reliable method for determining whether or not linear mixed functional differential equations are oscillatory. Based on the experiments we have tried, the technique works also for non-linear problems, but there is a need for further analytical results in this case.

References:

[1] C.T.H. Baker and N.J. Ford, Some applications of the boundary-locus method and the method of D-partitions, IMA J. Numer. Anal. 11. 143–158, 1991.

[2] J.B. Conway, Functions of One Complex Variable, Springer, New York, 1973.

[3] H. Chi, J. Bell and B. Hassard, Numerical solution of a nonlinear advance-delay-differential equation from nerve conduction theory, J. Math. Biol. 583-601, 1986.

[4] N.J. Ford, Y. Yan, and M.A. Malique, Numerical treatment of oscillatory functional differential equations, Journal of Computational and Applied Mathematics, 234(2010), 2757-2767.

[5] N.J. Ford, Numerical approximation of the characteristic values for a delay differential equation Proceedings of the IMACS World Congress on Mathematics and Computers in Simulation, Lausanne, 2000, 213-7.

[6] N.J. Ford and S.M. Verduyn Lunel, Characterising small solutions in delay differential equations through numerical approximations, Applied Mathematics and Computation, 131, 253–270, 2002.

[7] J.K. Hale, S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer, New York, 1993.

[8] T. Krisztin, Nonoscillation for functional differential equations of mixed type, Journal of Math. Anal. Appl., 245, 326–345, 2000.

[9] S. Pinelas, Oscillations and nonoscillations in mixed differential equations with monotonic delays and advances, Advances in Dyn. Syst. and Appl., 4, 107–121, 2009.

[10] D. Roose and K. Engelborghs, On Stability of LMS Methods and Characteristic Roots of Delay Differential Equations, SIAM Journal on Numerical Analysis, 40, 629 - 650, 2002.

[11] A. Rustichini, Functional differential equations of mixed type: the linear autonomous case, J. Dynamics Differential Equations 1, 121-143, 1989.

[12] A. Rustichini, Hopf bifurcation for functional differential equations of mixed type, J. Dynamics Differential Equations 1, 145-117, 1989.