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The numerical solution of forward-backward differential equations: Decomposition and related issues

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Abstract

This paper focuses on the decomposition, by numerical methods, of solutions to mixed-type functional differential equations (MFDEs) into sums of “forward” solutions and “backward” solutions. We consider equations of the form $x'(t) = ax(t) + bx(t-1) + cx(t+1)$ and develop a numerical approach, using a central difference approximation, which leads to the desired decomposition and propagation of the solution. We include illustrative examples to demonstrate the success of our method, along with an indication of its current limitations.

Key words:

Mixed-type functional differential equations, decomposition of solutions, central differences

AMS Subject Classification.: 34K06, 34K10, 34K28, 65Q05

1. Introduction

Interest in the study of mixed-type functional equations (MFDEs), or forward-backward equations, developed following the pioneering work of Rustichini in 1989 [19, 20]. The analysis of such equations, with both advanced and delayed arguments, presents a significant challenge to both analysts and numerical analysts alike. We are reminded in the opening section of [12] that “the dichotomy of insight and numbers is specific to numerical analysis”, that “computation should not wait until analysis has run out of steam” but that we should “employ computational algorithms that reflect known qualitative features of the underlying system”. The analytical decomposition of solutions of mixed-type equations as sums of “forward” solutions and “backward” solutions has been studied by J. Mallet-Paret and S. M. Verduyn Lunel in [18]. It is our aim in this paper to present an algorithm to decompose the solution of a particular class of MFDE into *growing* and *decaying* components and to provide further insight into issues related to the success or otherwise of this approach. We choose not to provide a more detailed review of current literature here. Instead we refer the reader to [1, 17, 19, 20] and for further examples of applications of MFDEs to [2, 3].

We focus our attention on the linear autonomous functional equation given by

$$x'(t) = ax(t) + bx(t-1) + cx(t+1), \tag{1}$$

a particular case of the nonautonomous equation

$$x'(t) = a(t)x(t) + b(t)x(t-1) + c(t)x(t+1). \tag{2}$$

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We consider the boundary value problem where we seek a function x , defined on $[-1, k]$, that satisfies equation (1) for almost all $t \in (0, k - 1]$ and also satisfies the boundary conditions

$$x(t) = \phi_1(t) \text{ for } t \in [-1, 0] \quad (3)$$

$$x(t) = f(t) \text{ for } t \in (k - 1, k], \quad (4)$$

where the constant $k \in \mathbb{N}$. Usually we shall require that x is continuous on $[-1, k]$. In earlier work [6, 7] we apply linear θ -methods to (1) with (3) and (4) with a fixed step length $h = \frac{1}{N}$. This leads to an equation of the form

$$y_{n+N} = Ay_{n+N-1} \text{ with } A, \text{ a block structured matrix} = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}, \quad (5)$$

where M_1 takes the form $\begin{pmatrix} -\frac{(1-\theta)}{\theta} & 0 & \dots & 0 & \frac{(1-h\theta a)}{h\theta c} & \frac{-[1+h(1-\theta)a]}{h\theta c} & 0 & \dots & 0 & \frac{-b}{c} \end{pmatrix}$, $M_2 = \begin{pmatrix} -\frac{b(1-\theta)}{\theta c} \end{pmatrix}$, M_3 is an appropriate identity matrix and $M_4 = (0 \dots 0)^T$. We obtain $y_{kN} = A^{(k-1)N} y_N$ which, together with ϕ_1 on $[-1, 0]$ and the boundary condition on $(k - 1, k]$, enables the calculation of a set of solution points on $(0, 1]$. We compare these with the known true solution and then, using $\phi_1(t)$ on $[-1, 0]$ and the approximate initial conditions on $(0, 1]$, we find the solution on $(1, k - 1]$.

Here we use a central difference approximation, an approach recommended for equations involving both advanced and delayed terms. Motivated by the successful analytical decomposition of the solution into ‘forward’ solutions and ‘backward’ solutions [18] we show that:-

1. It is possible to achieve a decomposition of the solution by numerical methods;
2. Following successful decomposition of the solution we can use the computed solution points on $(0, 1]$, along with $\phi_1(t)$ on $[-1, 0]$, to propagate the solution on $(1, k - 1]$
3. The change in methodology leads to an improvement on our earlier numerical scheme in terms of applicability and accuracy.

In Section 3 we consider the zeros of the characteristic equation and indicate a potential classification of the eigenspectra dependent on the coefficients of the equation. In Section 4.1 we present our numerical scheme. We again test our approach using equations with a known exact solution. Illustrative examples can be found in Section 5.

2. Relevant analytical theory

It is natural to begin with a brief overview of existence and uniqueness theory. As is well known, the existence problem for mixed type equations is not straightforward. Based on the boundary conditions defined on $[-1, 0]$ and $(k - 1, k]$ one needs additional conditions to be satisfied to ensure that there is a solution to the problem on $(0, k - 1]$. We refer to [14] for further discussion.

The solution of a corresponding delay equation (the case where the advanced term has zero coefficient) is known to have, in principle, discontinuities in derivatives at the origin and at integer multiples of the delay. The situation for mixed equations is somewhat more complicated. Derivative discontinuities may arise, but are usually associated with discontinuities in the solution at neighbouring integer values. This can be seen directly by considering (1) and the relationship between the derivative at t and the function values at $t, t - 1, t + 1$. This means that we need to consider solutions from some general function space, such as a Sobolev space, where such discontinuities in the function and its derivatives are allowed.

For uniqueness, we consider the operator Λ^τ , with kernel κ , and mapping $W_0^{1,p}(-\tau, \tau) \rightarrow L^p(-\tau, \tau)$ defined in [18] as

$$(\Lambda^\tau x)(t) = \dot{x}(t) - a(t)x(t) - b(t)x(t - 1) - c(t)x(t + 1), |t| \leq \tau, \quad (6)$$

where $W_0^{1,p}(-\tau, \tau) \subseteq W^{1,p}(-\tau, \tau)$ is the subspace of $x \in W^{1,p}(-\tau, \tau)$ for which $x(-\tau) = x(\tau) = 0$. When evaluating (6) $x(t) = 0$ is extended to $[-\tau - 1, -\tau] \cup [\tau, \tau + 1]$.

Theorem 2.1. *Theorem 6 in [18]*

With $a(t)$ real-valued, assume that either $b(t) > 0$ and $c(t) > 0$ for almost every $t \in \mathbb{R}$, or else that $b(t) < 0$ and $c(t) < 0$ for almost every $t \in \mathbb{R}$. Then for equation (2) we have that $\kappa(\Lambda^\tau) = \{0\}$ for every $\tau > 0$.

In relation to (1): Theorem 2.1 states that if $bc > 0$ then $\kappa(\Lambda^\tau) = \{0\}$ for every $\tau > 0$. If this condition is satisfied we conclude that the problem under consideration has at most one solution in $W^{1,p}(0, k-1)$.

For practical purposes and in real physical applications it is reasonable to assume that the solution of a mixed type equation is at least continuous. In many of the examples, we shall consider smooth solutions, but we shall also consider the effect of derivative discontinuities on our numerical scheme.

3. The spectrum of the operator

The characteristic quasipolynomial for equation (1) takes the form

$$\lambda = a + be^{-\lambda} + ce^\lambda. \quad (7)$$

The spectrum of the operator defined by $(\Lambda x)(t) = \dot{x}(t) - ax(t) - bx(t-1) - cx(t+1)$ is an “infinite sequence of eigenvalues, with unbounded real part, in the positive and negative halfspaces” [19].

Introducing $\lambda = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \frac{1}{2} \ln \left| \frac{b}{c} \right|$ leads to

$$\cos \beta = \frac{(\alpha - a)}{(be^{-\alpha} + ce^\alpha)}, \quad (8)$$

$$\beta = \pm (ce^\alpha - be^{-\alpha}) \sqrt{1 - \frac{(\alpha - a)^2}{(be^{-\alpha} + ce^\alpha)^2}}. \quad (9)$$

We are interested in the case when the solution to equation (1) with (3) and (4), if one exists, is unique. By Theorem 2.1 a sufficient condition is that the coefficients b and c satisfy $bc > 0$. Figures 1 to 5 involve equation (1) with $a = \frac{(k_1 e^{k_1} - k_2 e^{k_2})}{(e^{k_1} - e^{k_2})} - c(e^{k_1} + e^{k_2})$, $b = k_1 e^{k_1} - a e^{k_1} - c e^{2k_1}$, $k_1 = 2$, $k_2 = -3$. The coefficients (a , b and c) have been chosen so that the equation has exact solutions of the form $x(t) = C_1 e^{k_1 t} + C_2 e^{k_2 t}$, where C_1 and C_2 are constants. We choose $k_1 > 0$ and $k_2 < 0$ to ensure that the solution consists of both growth and decay terms. We can show that $bc > 0$ if $c > \frac{(k_1 - k_2)}{(e^{k_1} - e^{k_2})}$ or $c < 0$.

In the left-hand diagrams in Figures 1 to 5 we plot β against α given by equations (8) and (9) for five different values of c . In the right-hand diagrams of Figures 1 to 5 we plot the corresponding graphs for $\cos \beta$ against α . In each figure the characteristic values λ are given by the points of intersection of the two graphs. Figures 1, 3 and 5 clearly show the existence of zeros of the characteristic polynomial with unbounded positive and negative real parts (as expected [19]). In Figure 2, corresponding to a pure advance equation, there is an unbounded set of roots with positive real part ($\alpha > 0$) and a unique root with negative real part ($\alpha = -3$). Figure 4 shows an unbounded set of roots with negative real part ($\alpha < 0$) and a unique root with positive real part ($\alpha = 2$); this corresponds to a pure delay equation.

The values of c , namely $c = -2$, $\frac{5}{(e^2 - e^{-3})}$, 0.6, 0 and 0.9, have been chosen to demonstrate a potential classification of the problem according to whether or not the sufficient condition of Theorem 2.1, $bc > 0$, is satisfied. Figures 1 and 5 are illustrative of the case when $bc > 0$. In Figure 2 we use $c = \frac{(k_1 - k_2)}{e^{k_1} - e^{k_2}}$, leading to $b = 0$ and hence giving an advanced equation, whilst in Figure 4 we take $c = 0$, leading to a delay equation. The trajectories in Figure 3, illustrating the case when $bc < 0$, differ from those in Figures 1 and 5. We observe that the eigenvalues do not all lie on the same trajectory in the right-hand figure and that a deviation in the trajectory is visible in the left-hand diagram. The features of the diagrams identified here, in relation to the value of bc , have been observed for other equations and may provide further insight in our future work with MFDEs.

4. Numerical approach using central differences

We introduce

$$y_{n+N} = \begin{pmatrix} x_{n+N} & x_{n+N-1} & x_{n+N-2} & \dots & \dots & x_{n-N+1} \end{pmatrix}^T,$$

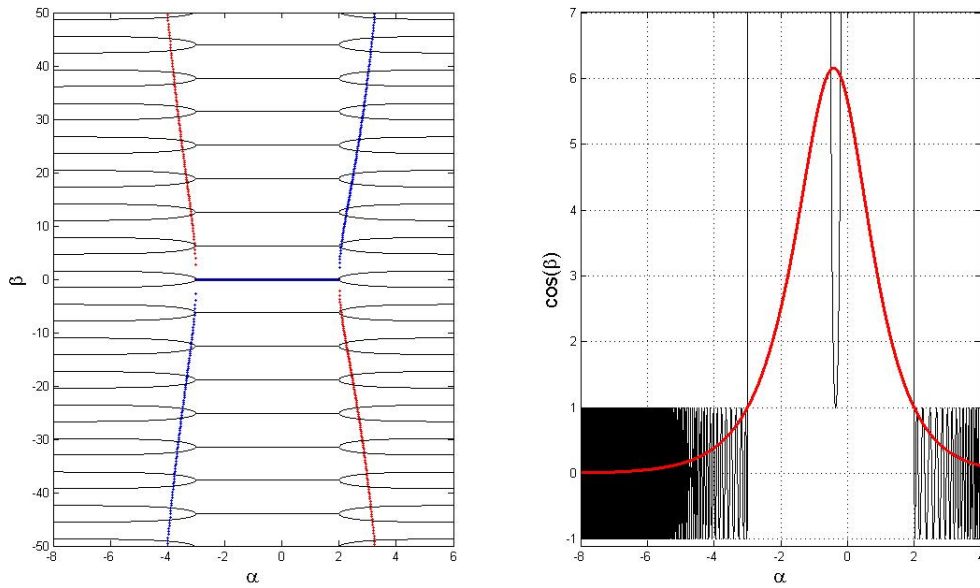


Figure 1: Equation (1) with $k_1 = 2, k_2 = -3, c = -2.bc > 0$.

Sufficient condition given by Theorem 2.1 is satisfied.

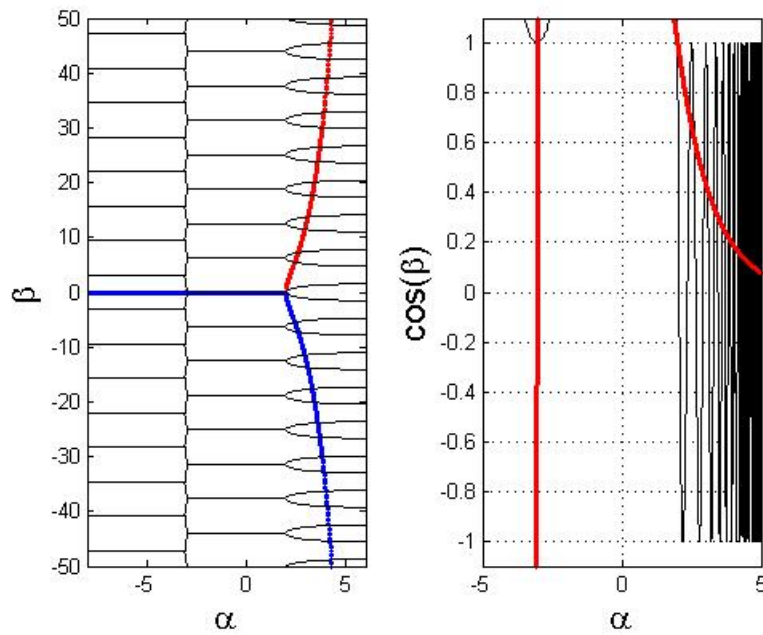


Figure 2: Equation (1) with $k_1 = 2, k_2 = -3, b = 0$.

$bc = 0$. Sufficient condition given by Theorem 2.1 is not satisfied. Equation is of the advanced type.

where $x_i \approx x(ih)$, with h the fixed step length. Using central differences, given by $\frac{x_{n+1} - x_{n-1}}{2h} = a_n x_n + b_n x_{n-N} + c_n x_{n+N}$, with

$$x'(t) = a(t)x(t) + b(t)x(t-1) + c(t)x(t+1), \quad (10)$$

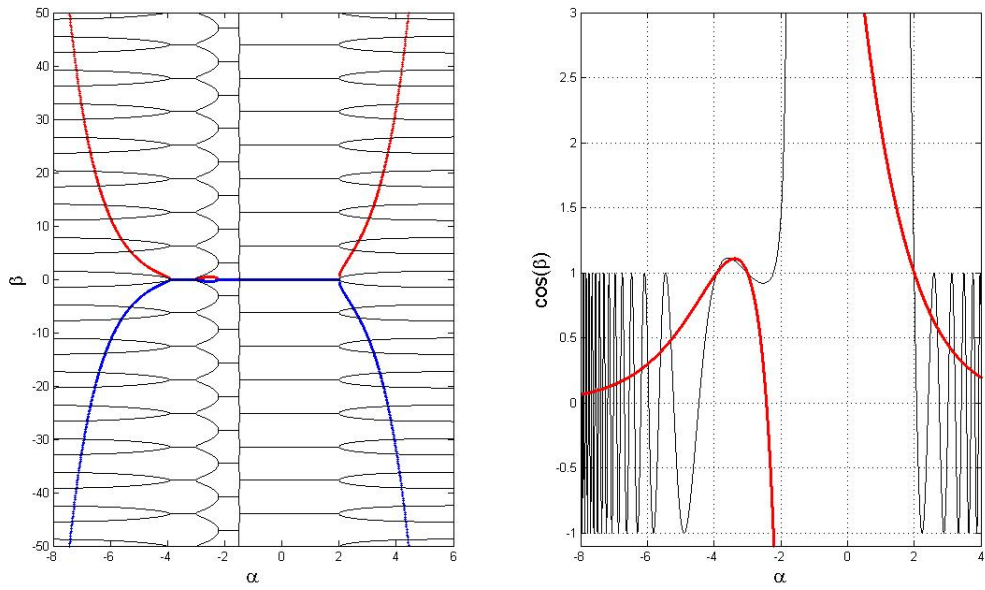


Figure 3: Equation (1) with $k_1 = 2, k_2 = -3, c = 0.6$. $bc < 0$. Sufficient condition (for a null kernel) given by Theorem 2.1 is not satisfied.

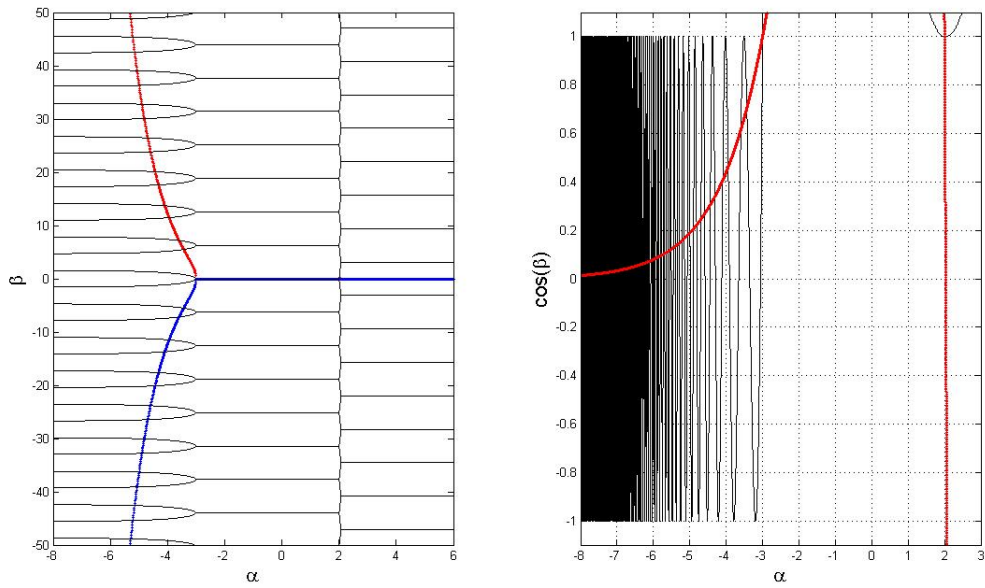


Figure 4: Equation (1) with $k_1 = 2, k_2 = -3, c = 0$. $bc = 0$. Sufficient condition given by Theorem 2.1 is not satisfied. Equation is of the delay type.

we obtain

$$x_{n+N} = \frac{1}{2hc_n}x_{n+1} - \frac{a_n}{c_n}x_n - \frac{1}{2hc_n}x_{n-1} - \frac{b_n}{c_n}x_{n-N}.$$

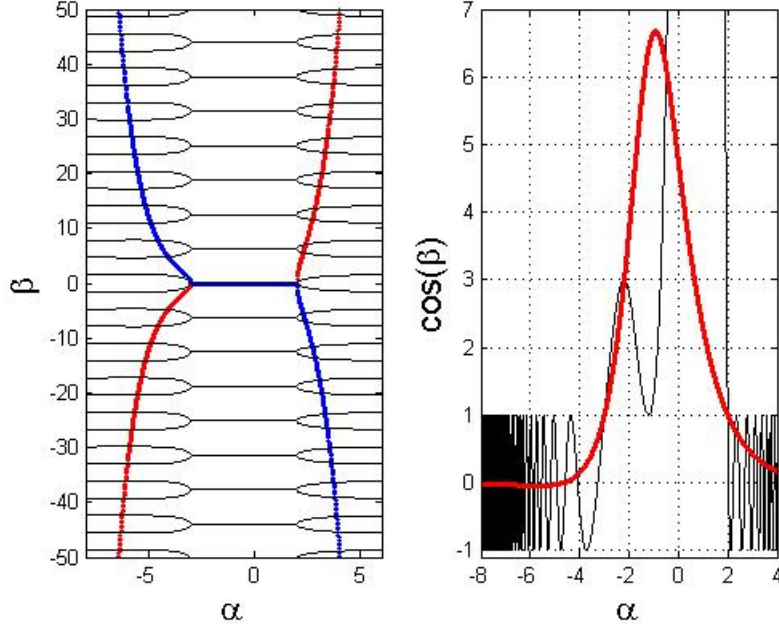


Figure 5: Equation (1) with $k_1 = 2, k_2 = -3, c = 0.9$. $bc > 0$. Sufficient condition given by Theorem 2.1 is satisfied.

This leads to

$$y_{n+N} = A(n)y_{n+N-1} \tag{11}$$

where $A(n)$ is a $2N \times 2N$ matrix given by

$$A(n) = \begin{pmatrix} 0 & \dots & 0 & \frac{1}{2hc_n} & -\frac{a_n}{c_n} & -\frac{1}{2hc_n} & 0 & \dots & 0 & -\frac{b_n}{c_n} \\ 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & \ddots & & & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & & & \ddots & \ddots & \vdots \\ \vdots & & & & & & & & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 0 \end{pmatrix}.$$

4.1. Decomposition of solutions: Our method

Following the results of J. Mallet-Paret and S. M. Verduyn Lunel [18] we assume that the exact solution function can be decomposed into a growing component and a decaying component. We are then able to form a stable forward numerical approximation of the decaying function as a sum of eigenfunctions corresponding to eigenvalues, λ , of the matrix A smaller than unity, and to construct a stable backwards approximation of the growing function using the eigenvalues, λ , of A which are greater than unity in magnitude. In this way our approach provides stable approximations of both components of the exact solution.

For the autonomous equation (1) $A(n)$ is a constant matrix A and equation (11) takes the form $y_{n+N} = Ay_{n+N-1}$ leading to

$$y_{n+N} = A^N y_n. \quad (12)$$

The matrix A is a companion matrix whose eigenvalues approximate elements of $\{e^{\lambda h} : \lambda \text{ satisfies (7)}\}$ as $h \rightarrow 0$ (see, for example, [4]). Since the zeros of (7) are distinct, the matrix A will be diagonalisable for small enough $h > 0$. (Note that, in any case, the following argument still applies with the Jordan canonical forms in place of the diagonal sub-matrices). We decompose the matrix A , writing $A = A_1 + A_2$, and deriving A_1 and A_2 as follows:

1. Find matrix $D = \text{diagonal}(\lambda_1, \lambda_2, \dots, \lambda_{2N})$ such that λ_i are the eigenvalues of A and $(|\lambda_1| > |\lambda_2| > \dots > |\lambda_{2N}|)$. Find the associated matrix of eigenvectors V . Hence $D = V^{-1}.A.V$.
2. Define $\Lambda_1 = \{\lambda : |\lambda| < 1\}$, $\Lambda_2 = \{\lambda : |\lambda| > 1\}$. Assume that $\Lambda_1 = \{\lambda_1, \lambda_2, \dots, \lambda_N\}$, $\Lambda_2 = \{\lambda_{N+1}, \lambda_{N+2}, \dots, \lambda_{2N}\}$.
3. Define $D_1 = \text{diag}(0, \dots, 0, \lambda_{N+1}, \lambda_{N+2}, \dots, \lambda_{2N})$, $D_2 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N, 0, \dots, 0)$. We see that $D = D_1 + D_2$ and note that if a matrix B is such that $B = \text{diag}(b_1, b_2, \dots, b_n)$ then $B^k = \text{diag}(b_1^k, b_2^k, \dots, b_n^k)$. This gives $D^N = D_1^N + D_2^N$.
4. Define A_1 and A_2 by $A_1 = V.D_1.V^{-1}$ and $A_2 = V.D_2.V^{-1}$.

In common with Rustichini [19] we have partitioned our eigenvalues into two sets, depending on whether or not their magnitude is greater than 1. With A_1 and A_2 defined as above

$$\begin{aligned} A^N &= (V.D.V^{-1})^N \\ &= V.D^N.V^{-1} \\ &= V.(D_1^N + D_2^N).V^{-1} \\ &= V.(D_1^N).V^{-1} + V.(D_2^N).V^{-1} \\ &= (V.D_1.V^{-1})^N + (V.D_2.V^{-1})^N \\ &= A_1^N + A_2^N. \end{aligned}$$

Hence

$$y_{n+N} = A^N y_n = (A_1 + A_2)^N y_n = ((A_1)^N + (A_2)^N) y_n.$$

This can be extended to

$$y_{n+kN} = (A_1 + A_2)^{kN} y_n = ((A_1)^{kN} + (A_2)^{kN}) y_n.$$

To find the solution on $(0, 1]$ we define the matrices IC , BC , SM and X , where $IC, BC, X \in \mathbb{R}^{N \times 1}$, and $SM \in \mathbb{R}^{2N \times 2N}$. IC is obtained from the initial condition on $[-1, 0]$ and BC from the boundary condition on $(k-1, k]$. We introduce SM as the solution matrix $(A_1)^{(k-1)N} + (A_2)^{(k-1)N}$ and X as the solution set on $(0, 1]$. We find that

$$\begin{pmatrix} BC \\ ** \end{pmatrix} = \begin{pmatrix} SM1 & SM2 \\ SM3 & SM4 \end{pmatrix} \begin{pmatrix} X \\ IC \end{pmatrix}, \quad (13)$$

where $SM1, SM2, SM3, SM4 \in \mathbb{R}^{N \times N}$. We see that $BC = SM1.X + SM2.IC$ giving $X = (SM1)^{-1}(BC - SM2.IC)$. Results of applying this method are illustrated in example 5.1.

Here we identify some potential limitations of our approach:

- The method depends on having equal numbers of eigenvalues with magnitudes < 1 and > 1 . This means that an extension to problems where the delays and advances are different will not be immediate.
- The underlying problem is ill-posed but, given any set of boundary conditions and an MFDE of the form being considered here, the method will find a solution even if one does not exist! [The solution will be to a perturbation of the original problem.]
- Increasing the dimension of the problem leads eventually to a near singular matrix.

4.2. Propagation of the solution

Having obtained a set of solution values on $(0, 1]$ using the method presented in Section 4, we now use this, along with the boundary condition on $[-1, 0]$, to solve the initial value problem on $(1, k - 1]$. We find the growing and decaying components of the solution on $(1, k - 1]$, using $(A_1)^{(k-1)N}$ and $(A_2)^{(k-1)N}$, and the solution given by $(A_1)^{(k-1)N} + (A_2)^{(k-1)N}$, each with the approximate initial conditions on $[-1, 1]$. We also calculate the true solution on $(1, k - 1]$. In our numerical investigations we are able to observe that the sum of the solutions using $(A_1)^{(k-1)N}$ and $(A_2)^{(k-1)N}$ is indeed a good approximation to the true solution of the equation. We illustrate this in example 5.2.

5. Numerical examples

In our examples we use equations with known exact solutions. First we focus on equations with smooth solutions containing both a growth and a decay term in the solution. In our final example, we shall consider the effect of derivative discontinuities. In examples 5.1 to 5.4 we consider equation (1) with $a = \frac{(k_1 e^{k_1} - k_2 e^{k_2})}{(e^{k_1} - e^{k_2})} - c(e^{k_1} + e^{k_2})$, $b = k_1 e^{k_1} - a e^{k_1} - c e^{2k_1}$, chosen so that any function of the form $x(t) = C_1 e^{k_1 t} + C_2 e^{k_2 t}$ with $C_1, C_2 \in \mathbb{R}$ is a solution of the equation. We impose the condition that $x(t)$ is equal to this function on $[-1, 0]$ and $(k - 1, k]$. In our numerical examples we take $C_1 = 1$ and $C_2 = 1$. [Including C_1 and C_2 enables either the growth term or the decay term to be ‘switched off’.] We discretise the equation using the method outlined in Section 4.1. We use X as the computed sequence and X_E as the sequence derived from the exact solution (at the same grid points). In examples 5.1, 5.3 and 5.4 we estimate the 2-norm of the error by calculating $\mathcal{E} = h \times \|X - X_E\|_2$ and the order of convergence, p , as $h \rightarrow 0$, of X to X_E . We compare the results of applying the method presented in section 4.1 with those obtained using methods from [6], [21] and [22] in example 5.1, demonstrate the decomposition of the solution into growing and decaying components in example 5.2 and consider the effects on the error of varying N , k and c in examples 5.3 and 5.4.

Example 5.1. *In this example we use a step length $h_i = \frac{1}{N_i}$, with parameter values $k_1 = 2$, $k_2 = -3$, and $c = -5$. A set of solution points on $(0, 1]$ is obtained by the method given in Section 4 for equation (1) with these parameters. We compare this with the set of true solution values and in Table 1 we give the mean squared error for a range of values of k . We compute an approximation for the order of the method and observe that, providing the matrix remains well-conditioned, the method achieves order 2.*

Tables 2 and 3 relate to the same problem as Table 1. However, Table 2 gives values arising from using the method in [6] (using the trapezium rule and without decomposition of the solution) while the results reported in Table 3 arise from using a collocation method and a least squares approach (see [21, 22] for further details).

In example 5.1 we observe that the method using decomposition is an improvement on the method of [6] in that the errors are generally smaller, the method achieves its expected order of 2 for higher values of k and we are able to solve the problem for larger values of k before the matrix becomes ill-conditioned (as identified by Matlab). The derivations of A_1 and A_2 have resulted in two matrices with ranges of eigenvalues both smaller than the original matrix A , leading to spectral condition numbers both smaller than that of A . Hence, each of A_1 and A_2 is better conditioned than A . We also observe that both the collocation method and the least squares method result in smaller errors and a higher order than the method using decomposition.

Example 5.2. *We choose parameter values $k_1 = 0.3$, $k_2 = -0.6$, and $c = -3$ and step length $h = \frac{1}{16}$. We illustrate successful decomposition and propagation of the solution for $k = 4$ (see Figure 6) and for $k = 9$ (see figure 7). We identify the trajectories as follows:*

- (diamonds) *The growing solution propagated using A_2^i with $i = 1$ to $(k - 1) * N$.*
- (triangles) *The decaying solution propagated using A_1^i with $i = 1$ to $(k - 1) * N$.*

		\mathcal{E}^2 and estimates of order p					
		k=3		k=4		k=5	
i	N_i	\mathcal{E}^2	p	\mathcal{E}^2	p	\mathcal{E}^2	p
1	8	1.734×10^{-5}		4.028×10^{-5}		7.282×10^{-5}	
2	16	9.342×10^{-7}	2.107	2.166×10^{-6}	2.109	3.9117×10^{-6}	2.109
3	32	5.358×10^{-8}	2.062	1.243×10^{-7}	2.062	2.252×10^{-7}	2.059
4	64	3.188×10^{-9}	2.036	7.429×10^{-9}	2.032	1.353×10^{-8}	2.028
5	128	1.940×10^{-10}	2.019	4.540×10^{-10}	2.016	8.287×10^{-10}	2.015
6	256	1.196×10^{-11}	2.010	2.804×10^{-11}	2.009	5.124×10^{-11}	2.008

		\mathcal{E}^2 and estimates of order p					
		k=10		k=15		k=20	
i	N_i	\mathcal{E}^2	p	\mathcal{E}^2	p	\mathcal{E}^2	p
1	8	3.819×10^{-4}		9.476×10^{-4}		1.8×10^{-3}	
2	16	2.070×10^{-5}	2.103	5.197×10^{-5}	2.094	9.822×10^{-5}	2.098
3	32	1.213×10^{-6}	2.046	3.045×10^{-6}	2.046	5.752×10^{-6}	2.047
4	64	7.332×10^{-8}	2.024	1.842×10^{-7}	2.024	3.481×10^{-7}	2.023
5	128	4.504×10^{-9}	2.013	1.133×10^{-8}	2.012	1.545×10^{-8}	2.246
6	256	2.794×10^{-10}	2.005	1.564×10^{-8}	***	***	***

Table 1: Example 5.1. Errors in the solution on $(0, 1]$ and estimates of p using decomposition method presented in Section 4.
*** Warning given by Matlab - Matrix is near to being singular

		\mathcal{E}^2 and estimates of order p					
		k=3		k=4		k=5	
i	N_i	\mathcal{E}^2	p	\mathcal{E}^2	p	\mathcal{E}^2	p
1	8	6.517×10^{-6}		8.630×10^{-6}		9.164×10^{-6}	
2	16	4.223×10^{-7}	1.974	4.838×10^{-7}	2.078	6.216×10^{-7}	1.941
3	32	3.962×10^{-8}	1.707	2.910×10^{-8}	2.028	3.212×10^{-7}	0.476
4	64	6.728×10^{-9}	1.279	1.750×10^{-9}	2.028	8.497×10^{-7}	
5	128	2.369×10^{-9}	0.753	9.623×10^{-11}	2.093	3.933×10^{-6}	
6	256	1.405×10^{-9}	0.377	4.310×10^{-11}	0.579	***	

Table 2: Example 5.1. Errors in the solution on $(0, 1]$ and estimates of p using method presented in [6].
*** Warning given by Matlab - Matrix is near to being singular

- (circles) The solution propagated using $A_1^i + A_2^i$ with $i = 1$ to $(k - 1) * N$, that is the sum of the decomposed components of the solution.
- (solid line) The true solution.

We observe the growing solution and the decaying solution. We also see that the sum of these components is a good approximation to the true solution. The features are more clearly seen in Figure 6.

Example 5.3. We choose parameter values $k_1 = 0.9$, $k_2 = -0.3$, and $c = -3$ and step length $h_i = \frac{1}{N_i}$. In Table 4 we present the mean squared errors, \mathcal{E}^2 and an estimate of the order of convergence. We observe that the errors decrease as N increases and increase as k increases and that the method becomes less reliable as k and N increase together.

Example 5.4. In this example we consider the effect of varying c on the mean squared error. We choose parameter values $k_1 = 1.5$, $k_2 = -0.6$ and step length $h = \frac{1}{128}$. In Table 5 we present the mean squared errors, \mathcal{E}^2 , for the stated values of c . The values $c = -10, -3, 4$ and 10 have been chosen to illustrate cases

Collocation method		k=3		k=4		k=5	
i	N_i	\mathcal{E}^2	p	\mathcal{E}^2	p	\mathcal{E}^2	p
1	8	1.2201×10^{-12}		8.1148×10^{-15}		2.2983×10^{-17}	
2	16	2.4607×10^{-14}	2.8159	3.2831×10^{-18}	5.6356	5.7395×10^{-21}	5.9837
3	32	6.5301×10^{-17}	4.2789	3.8173×10^{-20}	3.2132	5.3640×10^{-24}	5.0317
4	64	1.9973×10^{-18}	2.5155	1.2414×10^{-21}	2.4713	1.8887×10^{-25}	2.4139
5	128	6.2731×10^{-20}	2.4964	3.9122×10^{-23}	2.4939	6.1379×10^{-27}	2.4717
6	256	1.9699×10^{-21}	2.4965	1.2238×10^{-24}	2.4993	2.0350×10^{-28}	2.4573
Least Squares method		k=3		k=4		k=5	
i	N_i	\mathcal{E}^2	p	\mathcal{E}^2	p	\mathcal{E}^2	p
1	8	1.1372×10^{-12}		6.5935×10^{-15}		2.3054×10^{-17}	
2	16	2.0699×10^{-14}	2.8899	4.8178×10^{-18}	5.2092	5.4548×10^{-21}	6.0226
3	32	3.3638×10^{-18}	6.2936	6.3745×10^{-21}	4.7809	3.5137×10^{-24}	5.3002
4	64	1.2158×10^{-20}	4.0560	1.8959×10^{-23}	4.1967	7.7212×10^{-27}	4.4150
5	128	6.5160×10^{-23}	3.7719	9.3209×10^{-26}	3.8341	4.4973×10^{-29}	3.7118

Table 3: Example 5.1. Errors in the solution on $(0, 1]$ and estimates of p using Collocation method and a least squares method presented in [21, 22].

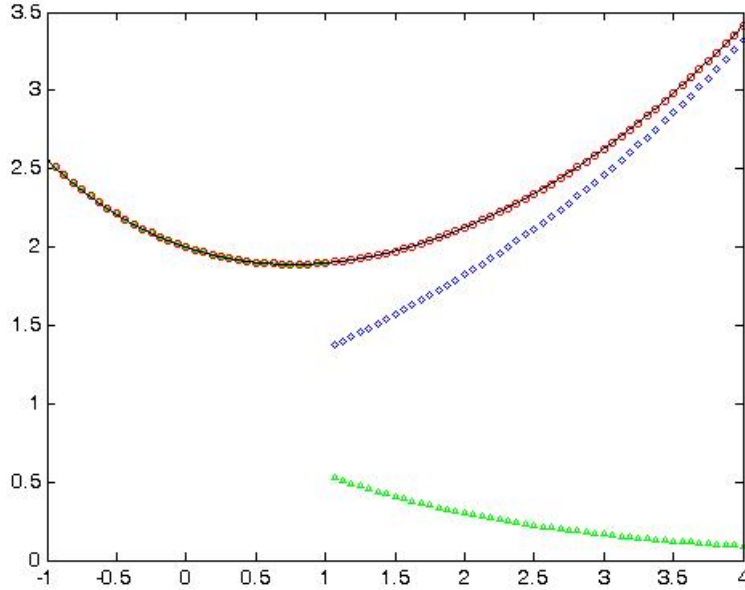


Figure 6: Numerical evidence of successful decomposition. Equation (1) with $k = 4, k_1 = 0.3, k_2 = -0.6, N = 16, c = -3$.

when $bc > 0$, whilst we use $c = 0.05$ and $c = 0.6$ as representative of the case when $bc < 0$. We observe that the errors increase as c increases towards 0 and decrease as c increases beyond $\frac{(k_1 - k_2)}{(e^{k_1} - e^{k_2})}$. For values of c in the interval $\left[0, \frac{(k_1 - k_2)}{(e^{k_1} - e^{k_2})}\right]$ we observe that the method quickly becomes unreliable as k increases and the matrix approaches a singular matrix. In Figure 8 we plot $\log_{10}(\text{Mean squared error})$ as c varies for equation (1) with $k_1 = 0.8, k_2 = -0.05, k = 3, N = 32$. In Figure 9 we plot $\log_{10}(\text{Mean squared error})$ as c and k vary. We observe increased errors as c approaches, lies in, and leaves the interval $\left[0, \frac{(k_1 - k_2)}{(e^{k_1} - e^{k_2})}\right]$ and as k

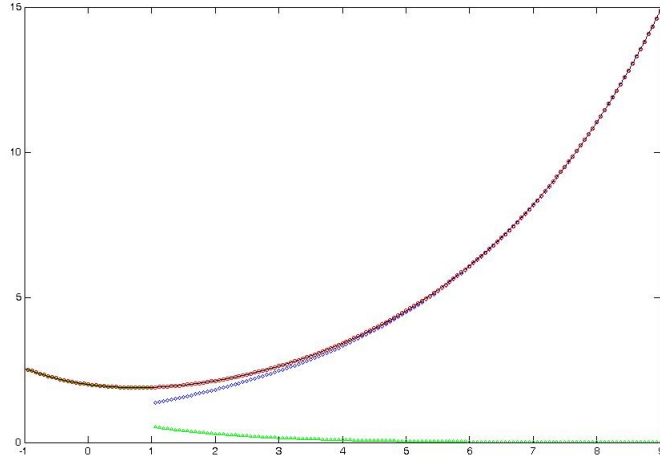


Figure 7: Numerical evidence of successful decomposition. Equation (1) $k = 9, k_1 = 0.3, k_2 = -0.6, N = 16, c = -3$.

		\mathcal{E}^2 and estimates of order p							
i	N_i	k=3		k=5		k=7		k=9	
		\mathcal{E}^2	p	\mathcal{E}^2	p	\mathcal{E}^2	p	\mathcal{E}^2	p
1	8	4.019×10^{-7}		1.976×10^{-5}		6.168×10^{-4}		1.84×10^{-2}	
2	16	2.314×10^{-8}	2.059	1.139×10^{-6}	2.058	3.565×10^{-5}	2.057	1.1000×10^{-2}	2.032
3	32	1.372×10^{-9}	2.038	6.791×10^{-8}	2.034	2.130×10^{-6}	2.032	6.3670×10^{-5}	2.055
4	64	8.307×10^{-9}	2.023	4.134×10^{-9}	2.019	1.794×10^{-7}	1.785	6.950×10^1	2.055
5	128	5.101×10^{-12}	2.013	2.557×10^{-10}	2.007	1.553	-	8.177×10^{10}	-
6	256	3.159×10^{-9}	2.007	3.244×10^{-8}	-	2.998×10^6	-	1.813×10^{21}	-

Table 4: Example 5.3. Errors in the solution on $(-1, k]$ and estimates of p as k varies.

increases.

		\mathcal{E}^2					
k	$c = -10$	$c = -3$	$c = 0.05$	$c = 0.6$	$c = 4$	$c = 10$	
2	8.8296×10^{-13}	7.8947×10^{-12}	3.7662×10^{-10}	4.5718×10^{-10}	6.7488×10^{-12}	1.0161×10^{-12}	
3	2.3467×10^{-11}	2.0962×10^{-10}	5.3568×10^{-7}	2.6135×10^{-8}	1.7579×10^{-10}	2.6831×10^{-11}	
4	4.2716×10^{-10}	3.8205×10^{-9}	1.3906×10^8	3.9983×10^{-4}	3.1786×10^{-9}	4.8715×10^{-10}	
5	7.3236×10^{-9}	6.5543×10^{-8}	***	3.1831×10^4	5.4416×10^{-8}	8.3467×10^{-9}	
6	1.2660×10^{-7}	1.1477×10^{-6}	***	1.7693×10^{16}	9.4072×10^{-7}	1.4426×10^{-7}	
7	2.2264×10^{-6}	1.7193×10^{-3}	***	***	4.0014×10^{-5}	2.5369×10^{-6}	
8	3.9753×10^{-5}	4.7808×10^{-1}	***	***	2.8207×10^{-1}	4.5298×10^{-5}	
9	7.1863×10^{-4}	1.8376×10^7	***	***	3.7569×10^4	8.1888×10^{-4}	
10	1.3110×10^{-2}	3.1562×10^{11}	***	***	2.1987×10^8	1.4990×10^{-2}	
11	2.629×10^{-1}	2.3587×10^{17}	***	***	1.2752×10^{14}	3.5605×10^{-1}	
12	4.1288×10^1	2.6717×10^{20}	***	***	1.4778×10^{18}	2.4791×10^2	

Table 5: Equation (1) with $k_1 = 1.5, k_2 = -0.6$ and $h = \frac{1}{128}$. Errors in the solution on $(-1, k]$ as c and k vary.

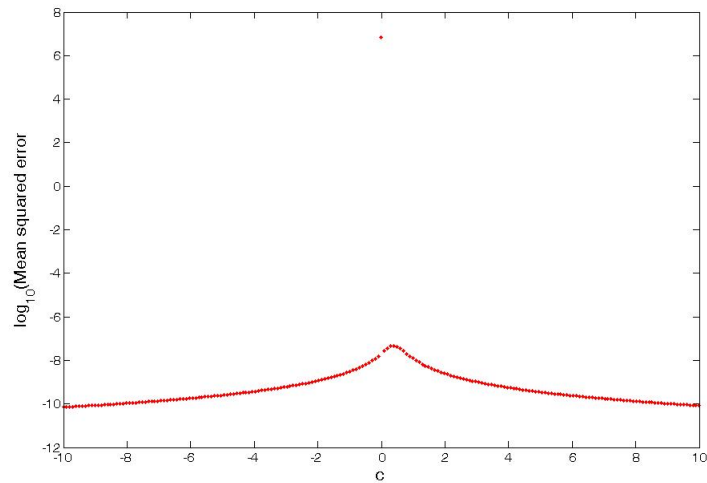


Figure 8: $\log_{10}(\text{Mean squared error})$ as c varies for Equation (11) with $k = 3, k_1 = 0.8, k_2 = -0.05, N = 32$.

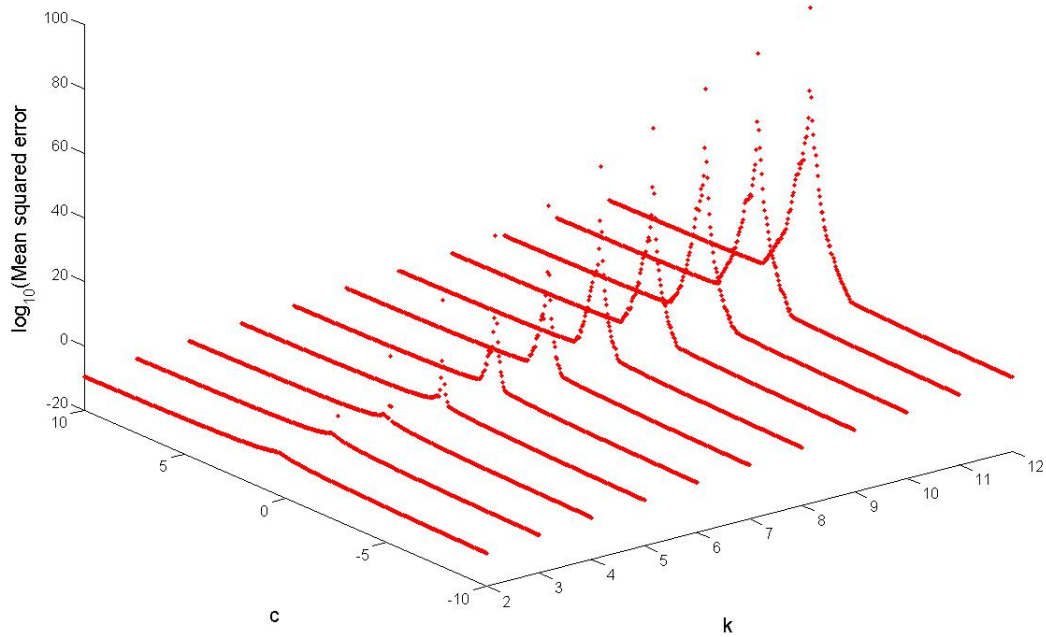


Figure 9: $\log_{10}(\text{Mean squared error})$ as c and k vary for Equation (11) with $k = 3, k_1 = 1.5, k_2 = -0.6, N = 32$.

5.1. Effect of derivative discontinuities on the method

Finally we consider problems whose solution is continuous but where there are discontinuities in derivatives at integer values.

Example 5.5. For $k = 2$ we consider the boundary value problem given by (1) subject to the conditions

$$x(t) = 1 \text{ for } t \in [-1, 0] \quad (14)$$

$$x(t) = \frac{1}{c}(1-b) - \frac{a}{c}t \text{ for } t \in (1, 2]. \quad (15)$$

The problem has been constructed such that on $(0, 1]$ the solution, $x(t)$, is given by $x(t) = 1 + t$. We choose values of a, b and c to illustrate the following three cases: (i) the solution is continuous and differentiable at $t = 1$, (ii) the solution is continuous at $t = 1$ but the derivative 'jumps' at $t = 1$ (iii) both the solution and its derivative are discontinuous at $t = 1$. We use a step length $h_i = \frac{1}{N_i}$ and obtain a set of solution points on $(0, 1]$ using our method (given in Section 4) for equation (1) with these values. The results are presented in Table 6. As expected we observe a reduction in the order obtained when break points exist in the solution.

		\mathcal{E}^2 and estimates of order p					
		$a = 0.2, b = 1.2, c = -0.2$		$a = 0.1, b = 0.4, c = 0.25$		$a = 0.3, b = -0.1, c = 0.5$	
$x(t)$ at $t = 1$		Continuous		Continuous		Discontinuous	
$x'(t)$ at $t = 1$		Continuous		Discontinuous		Discontinuous	
i	N_i	\mathcal{E}^2	p	\mathcal{E}^2	p	\mathcal{E}^2	p
1	8	4.618×10^{-28}		1.524×10^{-2}		1.727×10^{-1}	
2	16	4.283×10^{-27}	***	3.813×10^{-3}	0.99943	1.207×10^{-1}	0.2584
3	32	3.428×10^{-27}	***	9.536×10^{-4}	0.99977	9.804×10^{-2}	0.1500
4	64	1.166×10^{-26}	***	2.384×10^{-4}	0.99987	8.758×10^{-2}	0.0814
5	128	1.161×10^{-24}	***	5.961×10^{-5}	0.99996	8.256×10^{-2}	0.0426
6	256	2.085×10^{-24}	***	1.490×10^{-5}	0.99999	8.010×10^{-2}	0.0218

Table 6: Example 5.5. Errors in the solution on $(0, 1]$ and estimates of p .

*** indicates that machine accuracy has been exceeded

Example 5.6. For $k = 2, 3$ and 4 we consider the boundary value problem given by (1) subject to the conditions

$$x(t) = 1 - t \text{ for } t \in [-1, 0] \quad (16)$$

$$x(t) = f(t) \text{ for } t \in (k-1, k]. \quad (17)$$

The problem has been constructed such that on $(0, 1]$ the solution $x(t)$ is given by $x(t) = 1 + t$. We find that:

When $k = 2$, $f(t) = \frac{(1-3b)}{c} + \frac{(b-a)}{c}t$

When $k = 3$, $f(t) = \frac{1}{c^2}(b - 2a + 4ab - a^2 + bc) + \frac{t}{c^2}(a^2 - ab - bc)$

When $k = 4$, $f(t) = \frac{1}{c^3}(3a^2 - 2ab - bc - 5a^2b + 2a^3 - 2abc) + \frac{b}{c^2}(5b - 2a - 1) - \frac{t}{c^3}(a^3 - a^2b - 2abc + b^2c)$.

We choose values of a, b and c such that the solution is continuous at $t = 1$ and $t = 2$. We use a step length $h_i = \frac{1}{N_i}$ and obtain a set of solution points on $(0, 1]$ using our method (given in Section 4) for equation (1) with these values. We present the results of comparing this with the set of true solution values for the case when $a = 5/3, b = 2/3$ and $c = -1$ in Table 7. We observe a reduction in the order achieved as the value of k increases.

These results indicate that, while the method does still converge in the presence of derivative discontinuities, the order of convergence is reduced. This is in accordance with experience in similar situations elsewhere.

		\mathcal{E}^2 and estimates of order p					
		k=2		k=3		k=4	
i	N_i	\mathcal{E}^2	p	\mathcal{E}^2	p	\mathcal{E}^2	p
1	8	8.187×10^{-29}		1.418×10^{-3}		9.095×10^{-2}	
2	16	1.143×10^{-28}	***	3.904×10^{-4}	0.921	7.426×10^{-2}	0.145
3	32	6.735×10^{-27}	***	1.032×10^{-4}	0.960	6.620×10^{-2}	0.083
4	64	4.889×10^{-26}	***	2.658×10^{-5}	0.979	6.223×10^{-2}	0.045
5	128	9.827×10^{-26}	***	6.747×10^{-6}	0.989	6.026×10^{-2}	0.022
6	256	1.030×10^{-24}	***	1.700×10^{-6}	0.994	5.928×10^{-2}	0.012

Table 7: Example 5.6. Errors in the solution on $(0, 1]$ and estimates of p .
*** indicates that machine accuracy has been exceeded

6. Summary, observations and conclusions

Motivated by analytical results we have achieved a numerical decomposition of the solution into ‘growing’ and ‘decaying’ components. This decomposition technique has led to an improvement in the accuracy of the solution when propagated on $(1, k - 1]$ when compared with earlier approaches. We have indicated some limitations of our approach.

Of course, the approach in this paper depends upon the ability to express the solution of an MFDE in terms of the eigenfunctions of the differential operator. It is well known that this may not apply to some delay equations (those which have so-called ‘small solutions’) and there is a need to check some non-degeneracy property of the problem. The same applies in the mixed-type problem. This time the equation is degenerate when there exist either super-exponential solutions (that increase faster than any exponential), or small solutions (that decay faster than any exponential). One such non-degeneracy property is given in the conditions of Theorem 2.1. Further information on this theme may be found in [8, 9, 11, 16, 23, 24] and the references therein and will be the focus of a future work.

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